Cone-Beam Tomography

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January 10, 2019

1 Introduction

This paper describes some of the theory behind the inversion of the X-ray Transform which leads to inversion formulas of filtered backprojection type. The theory is inspired by the Katsevich algorithms, but mostly follows [1].

2 Background

In the following note we define some popular integral transforms and image reconstruction theorems for cone-beam tomography. We shall denote vectors in bold type. We start with the following transform identities.

Fourier Transform: For $f \in L^1(\mathbb{R}^n)$ we define the Fourier transform by

$$\mathcal{F}\{f\}(\mathbf{X}) = \int_{\mathbb{R}^n} f(\mathbf{x}) e^{-2\pi i <\mathbf{x},\mathbf{X}>} \, d\mathbf{x}.$$

We will also denote the Fourier transform of f by $\widehat{f}$, i.e., $\mathcal{F}\{f\}(\mathbf{X}) = \widehat{f}(\mathbf{X})$.

Radon Transform: For $f \in L^1(\mathbb{R}^n)$ we define the Radon transform by

$$Rf(\rho, \theta) = \int_{\mathbf{x}\cdot\theta=\rho} f(\mathbf{x}) \, d\mathbf{x}$$

for $\rho \in \mathbb{R}$ and $\theta \in \mathbb{R}^n$.

X-ray Transform: For $f \in L^1(\mathbb{R}^n)$ we define the X-ray transform by

$$Pf(\mathbf{y}, \theta) = \int_{\mathbb{R}} f(\mathbf{y} + t\theta) \, dt$$

for $\mathbf{y} \in \Theta^\perp = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \cdot \theta = 0\}$ and $\theta \in \mathbb{R}^n$.

Divergent-Beam Transform: For $f \in L^1(\mathbb{R}^n)$ we define the Divergent-Beam transform by

$$Df(\mathbf{y}, \theta) = \int_0^\infty f(\mathbf{y} + t\theta) \, dt$$

for $\mathbf{y}, \theta \in \mathbb{R}^n$.

The adjoints of the Radon and X-ray transforms are given by

$$\begin{aligned} R^* g(\mathbf{x}) &= \int_{S^{n-1}} g(\mathbf{x} \cdot \theta, \theta)\, d\theta \\ P^* g(\mathbf{x}) &= \int_{S^{n-1}} g(E_\theta \mathbf{x}, \theta)\, d\theta, \end{aligned}$$

where S^{n-1} is the unit hyper-sphere in $\mathbb{R}^n$, $x \in \mathbb{R}^n$, and E_θ is the orthogonal projection on $\Theta^\perp$.

Now we state some preliminary, well-known theorems in tomography. The Radon inversion formula is stated below.

Theorem 2.1. *Let $f \in C^2(\mathbb{R}^3)$ and $\omega = (\cos\varphi \sin\vartheta, \sin\varphi \sin\vartheta, \sin\vartheta)^T$. Then*

$$\begin{aligned} f(\boldsymbol{x}) &= -\frac{1}{8\pi^2} \int_0^{2\pi} \int_0^{\pi} R'' f(\boldsymbol{x} \cdot \omega, \omega) \sin\vartheta d\vartheta\, d\varphi \\ &= -\frac{1}{8\pi^2} \int_{S^2} R'' f(\boldsymbol{x} \cdot \omega, \omega)\, d\Omega \\ &= -\frac{1}{8\pi^2} R^* R'' f(\boldsymbol{x}), \end{aligned} \tag{1}$$

where $d\Omega = \sin\vartheta\, d\vartheta\, d\varphi$ and $R''(\rho, \omega) = \frac{\partial^2}{\partial \rho^2} Rf(\rho, \omega)$.

See [2] for a proof.

The next theorem is known as the projection-slice or central section theorem.

Theorem 2.2. *Let $f \in L^1(\mathbb{R}^n)$. Then*

$$\begin{aligned} \widehat{R_\theta f}(\sigma) &= \widehat{f}(\sigma\theta), \quad \sigma \in \mathbb{R}, \\ \widehat{P_\theta f}(\eta) &= \widehat{f}(\eta), \quad \eta \in \Theta^\perp. \end{aligned}$$

See [2] for a proof.

Proposition 2.1. *Let $f \in C^2(\mathbb{R}^3)$. Then for $\theta \in S^2$, $a \in \mathbb{R}^3$,*

$$\frac{\partial}{\partial \rho}(Rf)(\boldsymbol{a} \cdot \theta, \theta) = \int_{\omega \in \Theta^\perp \cap S^2} \frac{\partial}{\partial \theta}(Df)(\boldsymbol{a}, \omega)\, d\omega, \tag{2}$$

where $\frac{\partial}{\partial \theta}$ is the directional derivative in the direction θ, acting on the second argument of Df. We also have

$$\begin{aligned} Df(\boldsymbol{y}, \theta) &= -\frac{1}{8\pi^2} \int_0^{\infty} \int_0^{\pi} \int_0^{2\pi} \frac{\partial^2}{\partial \rho^2} Rf(\rho, \omega) \Big|_{\rho = (\boldsymbol{y} + l\theta) \cdot \omega} d\varphi\, \sin\vartheta\, d\vartheta\, dl \\ &= -\frac{1}{8\pi^2} \int_0^{\infty} \int_{S^2} \frac{\partial^2}{\partial \rho^2} Rf(\rho, \omega) \Big|_{\rho = (\boldsymbol{y} + l\theta) \cdot \omega} d\Omega\, dl \end{aligned} \tag{3}$$

for $\omega = (\cos\varphi \sin\vartheta, \sin\varphi \sin\vartheta, \cos\vartheta)^T$.

See [2] for the proof of equation (2). For a proof of equation (3) simply insert equation (1) into the definition of the Divergent beam transform. The divergent beam transform most closely models the data acquired in cone-beam Computed Tomography.

3 Image Reconstruction with Cone-Beam Data

In this section we outline the theory of image reconstruction from cone beam data. We mostly follow [1]. Although this paper deals with Katsevich-type reconstructions from 3-PI acquisition, it provide some nice analysis with a general source trajectory.

The theory states that reconstruction is performed in three steps:

1. Derivative along parallel rays.

2. Hilbert transform along curves on the detector.

3. Weighted backprojection.

The next theorem outlines the main result of this section and states the relation between the reconstructed volume, f, and the filtered (derivative and Hilbert filter along curves) data.

Theorem 3.1. *Let $f \in C^2(\mathbb{R}^3)$ and $\boldsymbol{y} : \mathbb{R} \to \mathbb{R}^3$ be a continuously-differentiable source trajectory (e.g., helical or circular) that does not intersect the support of f. Define $\boldsymbol{\beta} = \boldsymbol{\beta}(s, \boldsymbol{x}) = \frac{\boldsymbol{x}-\boldsymbol{y}(s)}{|\boldsymbol{x}-\boldsymbol{y}(s)|}$ and $\boldsymbol{e} = \boldsymbol{e}(s, \boldsymbol{x})$ such that $\boldsymbol{\beta} \cdot \boldsymbol{e} = 0$ and $|\boldsymbol{e}| = 1$. Also define*

$$I(s, \boldsymbol{x}) \quad \equiv \quad \int_{-\pi}^{\pi} \frac{\partial}{\partial q} Df(\boldsymbol{y}(q), \cos\gamma\boldsymbol{\beta} + \sin\gamma\boldsymbol{e})\bigg|_{q=s} \frac{d\gamma}{\sin\gamma}.$$

Then

$$\begin{aligned} I(s, \boldsymbol{x}) \quad &= \quad \frac{1}{4}|\boldsymbol{x} - \boldsymbol{y}(s)| \int_{S^2} sgn(\omega \cdot \boldsymbol{e})(\omega \cdot \dot{\boldsymbol{y}}(s))\delta(\omega \cdot (\boldsymbol{x} - \boldsymbol{y}(s)))R''f(\omega \cdot \boldsymbol{x}, \omega)\, d\Omega \qquad (4) \\ &= \quad -2\pi^2|\boldsymbol{x} - \boldsymbol{y}(s)| \int_{\mathbb{R}^3} sgn(\xi \cdot \boldsymbol{e})(\xi \cdot \dot{\boldsymbol{y}}(s))\delta(\xi \cdot (\boldsymbol{x} - \boldsymbol{y}(s)))\widehat{f}(\xi)e^{2\pi i <\boldsymbol{x},\xi>}\, d\xi. \qquad (5) \end{aligned}$$

In the above, $I(s, \mathbf{x})$, is derived from the X-ray Transform data by taking a derivative along parallel rays and then applying a Hilbert filter along curves on the detector.

Proof. Let $\theta(s, \mathbf{x}, \gamma) = \cos\gamma\boldsymbol{\beta}(s, \mathbf{x}) + \sin\gamma\mathbf{e}(s, \mathbf{x})$. From equations (4) and (3) we have

$$\begin{aligned} I(s, \mathbf{x}) \quad &\equiv \quad \int_{-\pi}^{\pi} \frac{\partial}{\partial q} Df(\mathbf{y}(q), \cos\gamma\boldsymbol{\beta} + \sin\gamma\mathbf{e})\bigg|_{q=s} \frac{d\gamma}{\sin\gamma} \\ &= \quad \int_{-\pi}^{\pi} \frac{\partial}{\partial q}\left[-\frac{1}{8\pi^2}\int_0^{\infty}\int_{S^2} R''f(\omega \cdot [\mathbf{y}(q) + l\theta(s, \mathbf{x}, \gamma)], \omega)\, dl\, d\Omega\right]\bigg|_{q=s} \frac{d\gamma}{\sin\gamma} \\ &= \quad -\frac{1}{8\pi^2}\int_{-\pi}^{\pi} \frac{\partial}{\partial q}\left[\int_0^{\infty}\int_{S^2}\int_{\mathbb{R}} R''f(\xi, \omega)\delta(\omega \cdot [\mathbf{y}(q) + l\theta(s, \mathbf{x}, \gamma)] - \xi)\, d\xi\, dl\, d\Omega\right]\bigg|_{q=s} \frac{d\gamma}{\sin\gamma} \\ &= \quad -\frac{1}{8\pi^2}\int_{-\pi}^{\pi}\int_0^{\infty}\int_{S^2}\int_{\mathbb{R}} R''f(\xi, \omega)\int_{\mathbb{R}} \frac{\partial}{\partial q} e^{2\pi i\rho[\omega\cdot(\mathbf{y}(q)+l\theta(s,\mathbf{x},\gamma))-\xi]}\bigg|_{q=s} d\rho\, d\xi\, dl\, d\Omega\, \frac{d\gamma}{\sin\gamma} \\ &= \quad -\frac{1}{8\pi^2}\int_{\mathbb{R}}\int_{S^2} R''f(\xi, \omega)\int_{\mathbb{R}}\left\{\int_{-\pi}^{\pi}\int_0^{\infty}\frac{\partial}{\partial q} e^{2\pi i\rho[\omega\cdot(\mathbf{y}(q)+l\theta(s,\mathbf{x},\gamma))]}\bigg|_{q=s} dl\, \frac{d\gamma}{\sin\gamma}\right\} e^{-2\pi i\rho\xi}\, d\rho\, d\Omega\, d\xi. \end{aligned}$$

The derivative of the exponential is given by

$$\frac{\partial}{\partial q} e^{2\pi i\rho[\omega\cdot(\mathbf{y}(q)+l\theta(s,\mathbf{x},\gamma))]}\bigg|_{q=s} \quad = \quad 2\pi i\rho < \dot{\mathbf{y}}(s), \omega > e^{2\pi i\rho[\omega\cdot(\mathbf{y}(s)+l\theta)]}.$$

We make the substitution $u_1 = l\cos\gamma$, $u_2 = l\sin\gamma$. Then $\frac{\partial(u_1,u_2)}{\partial(l,\gamma)} = |l|$ and thus

$$\frac{dl\,d\gamma}{\sin\gamma} = \frac{du_1\,du_2}{l\sin\gamma} = \frac{du_1\,du_2}{u_2}.$$

Now the double integral within the curly braces is given by

$$\begin{aligned}
& 2\pi i\rho < \dot{\mathbf{y}}(s), \omega > \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{u_2} e^{2\pi i\rho[\mathbf{y}(s)+u_1\beta+u_2 e]\cdot\omega}\, du_1\, du_2 \\
= \; & 2\pi i\rho < \dot{\mathbf{y}}(s), \omega > e^{2\pi i\rho<\omega,\mathbf{y}(s)>} i\pi \mathrm{sgn}(\rho\omega\cdot\mathbf{e})\delta(\rho\omega\cdot\beta) \\
= \; & -2\pi^2 < \dot{\mathbf{y}}(s), \omega > \mathrm{sgn}(\omega\cdot\mathbf{e})\delta(\omega\cdot\beta) e^{2\pi i\rho<\omega,\mathbf{y}(s)>} \\
= \; & -2\pi^2 < \dot{\mathbf{y}}(s), \omega > \mathrm{sgn}(\omega\cdot\mathbf{e})|\mathbf{x}-\mathbf{y}(s)|\delta(\omega\cdot(\mathbf{x}-\mathbf{y}(s))) e^{2\pi i\rho<\omega,\mathbf{y}(s)>}.
\end{aligned}$$

And now we have

$$\begin{aligned}
I(s,\mathbf{x}) &= \frac{|\mathbf{x}-\mathbf{y}(s)|}{4} \int_{\mathbb{R}}\int_{S^2} R''f(\xi,\omega) < \dot{\mathbf{y}}(s), \omega > \mathrm{sgn}(\omega\cdot\mathbf{e})\delta(\omega\cdot(\mathbf{x}-\mathbf{y}(s))) \int_{\mathbb{R}} e^{2\pi i\rho[\omega\cdot\mathbf{y}(s)-\xi]}\, d\rho\, d\Omega\, d\xi \\
&= \frac{|\mathbf{x}-\mathbf{y}(s)|}{4} \int_{S^2} R''f(\omega\cdot\mathbf{x},\omega) < \dot{\mathbf{y}}(s), \omega > \mathrm{sgn}(\omega\cdot\mathbf{e})\delta(\omega\cdot(\mathbf{x}-\mathbf{y}(s)))\, d\Omega
\end{aligned}$$

and now we have established (4). Now from the Projection-Slice Theorem we have that

$$\mathcal{F}\{R''_\omega f\}(\sigma) = -4\pi\sigma^2 \widehat{R_\omega f}(\sigma) = -4\pi\sigma^2 \widehat{f}(\sigma\omega).$$

If we let

$$h(\omega,\mathbf{x},s) \equiv < \dot{\mathbf{y}}(s), \omega > \mathrm{sgn}(\omega\cdot\mathbf{e})\delta(\omega\cdot(\mathbf{x}-\mathbf{y}(s))).$$

Then we have

$$I(s,\mathbf{x}) = -\pi^2|\mathbf{x}-\mathbf{y}(s)| \int_{S^2} h(\omega,\mathbf{x},s) \int_{\mathbb{R}} \sigma^2 \widehat{f}(\sigma\omega) e^{2\pi i\rho<\mathbf{x},\omega>}\, d\sigma\, d\Omega.$$

Using the substitution $\xi = \sigma\omega$, we have $d\xi = \sigma^2\, d\sigma\, d\Omega$ and

$$\begin{aligned}
I(s,\mathbf{x}) &= -2\pi^2|\mathbf{x}-\mathbf{y}(s)| \int_{\mathbb{R}^3} h\left(\frac{\xi}{|\xi|},\mathbf{x},s\right) \widehat{f}(\xi) e^{2\pi i<\mathbf{x},\xi>}\, d\xi \\
&= -2\pi^2|\mathbf{x}-\mathbf{y}(s)| \int_{\mathbb{R}^3} h(\xi,\mathbf{x},s)\widehat{f}(\xi) e^{2\pi i<\mathbf{x},\xi>}\, d\xi.
\end{aligned}$$

□

Note that the above theorem holds for a general curve, $\mathbf{y}(s)$, and general filtering vector $\mathbf{e} = \mathbf{e}(s,\mathbf{x})$. We refer to the plane spanned by β and $\mathbf{e}$ as the *filtering plane* and the intersection of this plane with the detector as a *filtering line.*

Katsevich proves an exact inversion formula for a helix source trajectory which is summarized in the following theorem.

Theorem 3.2. *Let $I(s,\boldsymbol{x})$ be defined as in Theorem 3.1, $\boldsymbol{y}(s) = (R\cos(s), R\sin(s), hs)^T$, and $I_{PI}(\boldsymbol{x}) \equiv [s_a(\boldsymbol{x}), s_b(\boldsymbol{x})]$ be the unique interval such that $s_b - s_a < 2\pi$ and $\boldsymbol{x}$ lies along the line between $\boldsymbol{y}(s_a)$ and $\boldsymbol{y}(s_b)$. Also let $\boldsymbol{e}(s,\boldsymbol{x})$ be such that the plane spanned by $\beta(s,\boldsymbol{x})$ and $\boldsymbol{e}(s,\boldsymbol{x})$ and through $\boldsymbol{x}$ intersects the source trajectory at exactly three points equi-spaced in the source parameter s. Then*

$$f(\boldsymbol{x}) = -\frac{1}{2\pi^2}\int_{I_{PI}(\boldsymbol{x})} \frac{1}{|\boldsymbol{x}-\boldsymbol{y}(s)|} I(s,\boldsymbol{x})\, ds.$$

We would like to establish Katsevich type reconstruction formulas for other source trajectories. For a general source trajectory, we have

$$f_d(\mathbf{x}) \equiv -\frac{1}{2\pi^2}\int_{I_{BP}(\mathbf{x})} \frac{1}{|\mathbf{x}-\mathbf{y}(s)|} I(s,\mathbf{x})\,ds = \int_{\mathbb{R}^3} H(\xi,\mathbf{x})\widehat{f}(\xi)e^{2\pi i<\mathbf{x},\xi>}\,d\xi,$$

where

$$\begin{aligned} H(\xi,\mathbf{x}) &\equiv \int_{I_{BP}(\mathbf{x})} h(\xi,\mathbf{x},s)\,ds \\ &= \int_{I_{BP}(\mathbf{x})} \operatorname{sgn}(\xi\cdot\mathbf{e}(s,\mathbf{x}))(\xi\cdot\dot{\mathbf{y}}(s))\delta(\xi\cdot(\mathbf{x}-\mathbf{y}(s)))\,ds. \end{aligned}$$

and $I_{BP}(\mathbf{x})$ is the source interval for back-projection. If $H(\xi,\mathbf{x}) = 1$, then $f_d = f$. Note that $H(\xi,\mathbf{x}) = H(\xi/|\xi|,\mathbf{x})$.

4 Properties of $H(\mathbf{x},\xi)$

Let the source trajectory, $\mathbf{y}(s) = (R\cos(s), R\sin(s), hs)^T$ for $h \geq 0$ be defined for $s \in [-s_{max}, s_{max}]$. The support of f in contained within the set $\{(x_1,x_2,x_3)\in\mathbb{R}^3 : x_1^2 + x_2^2 \leq r^2,\ x_3\in\mathbb{R}\}$ for $r < R$. The detector is defined on the set

$$D(s) \equiv \left\{(x_1,x_2,x_3)\in\mathbb{R}^3 : (x_1 - R\cos(s))^2 + (x_2 - R\sin(s))^2 \leq 4R^2,\ |x_3 - hs| \leq H\right\}.$$

Now we let

$$H(\mathbf{x},\xi) \equiv \int_{-s_{max}}^{s_{max}} h(\mathbf{x},\xi,s)\mathbf{1}_{\mathbf{M}}(\mathbf{x},s)\,ds,$$

where the measured set is given by $\mathbf{M} = \left\{(\mathbf{x},s) : |x_3 - hs| \leq \frac{H}{2R}\sqrt{\|\mathbf{x}-\mathbf{y}(s)\|^2 - (x_3 - hs)^2}\right\}$ and

$$h(\mathbf{x},\xi,s) \equiv \operatorname{sgn}(\xi\cdot\mathbf{e}(s,\mathbf{x}))(\xi\cdot\dot{\mathbf{y}}(s))\delta(\xi\cdot(\mathbf{x}-\mathbf{y}(s))).$$

We also let

$$\{s_k(\mathbf{x},\xi)\}_{k=1}^{N(\mathbf{x},\xi)} \equiv \{s\in[-s_{max}, s_{max}] : \xi\cdot(\mathbf{x}-\mathbf{y}(s)) = 0,\ \mathbf{1}_{\mathbf{M}}(\mathbf{x},s) = 1\}.$$

Note that $N(\mathbf{x},\xi)\in\mathbb{N}$ and $N(\mathbf{x},\xi) \leq \lceil\frac{2s_{max}}{\pi}\rceil$.

We also have

$$\delta(\xi\cdot(\mathbf{x}-\mathbf{y}(s))) = \sum_{k=1}^{N(\xi,\mathbf{x})} \frac{\delta(s-s_k)}{|\xi\cdot\dot{\mathbf{y}}(s_k)|}$$

and thus for fixed $\mathbf{x},\xi\in\mathbb{R}^3$ we have that

$$H(\mathbf{x},\xi) = \sum_{k=1}^{N(\mathbf{x},\xi)} \operatorname{sgn}(\xi\cdot\mathbf{e}(s_k,\mathbf{x}))\operatorname{sgn}(\xi\cdot\dot{\mathbf{y}}(s_k)).$$

Now we take another look at $N(\mathbf{x},\xi)$. Without loss of generality assume that $\xi = (\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta)^T$. We also let $\mathbf{x} = (r\cos\psi, r\sin\psi, x_3)^T$. We wish to find s such that $\xi \cdot \mathbf{y}(s) = \xi \cdot \mathbf{x}$. Then

$$\begin{aligned} \xi \cdot \mathbf{y}(s) &= R\sin\theta\cos(\varphi - s) + hs\cos\theta \\ \xi \cdot \mathbf{x} &= r\sin\theta\cos(\varphi - \psi) + x_3\cos\theta \end{aligned}$$

and thus

$$R\cos(\varphi - s) = r\cos(\varphi - \psi) - (sh - x_3)\cot\theta.$$

4.1 Determination of Filtering Planes

In this section we discuss the determination of filtering planes which is equivalent to the determination of $\mathbf{e}(s,\mathbf{x})$. For our applications we wish to choose $\mathbf{e}(s,\mathbf{x})$ such that

$$\operatorname{sgn}(\xi \cdot \mathbf{e}(s,\mathbf{x}))\operatorname{sgn}(\xi \cdot \dot{\mathbf{y}}(s)) = 1$$

for $\mathbf{x} \in \operatorname{supp}(f)$ and $s \in I_{BP}(\mathbf{x})$ such that $\xi \cdot (\mathbf{x} - \mathbf{y}(s)) = 0$.

Let $\mathbf{e}(s,\mathbf{x}) = \frac{\widetilde{\mathbf{e}}(s,\mathbf{x})}{\|\widetilde{\mathbf{e}}(s,\mathbf{x})\|}$ where $\widetilde{\mathbf{e}}(s,\mathbf{x})$ is the orthogonal projection of $\dot{\mathbf{y}}(s)$ along β, i.e.,

$$\begin{aligned} \widetilde{\mathbf{e}}(s,\mathbf{x}) &= \dot{\mathbf{y}}(s) - (\dot{\mathbf{y}}(s) \cdot \beta)\beta \\ &= \beta \times (\dot{\mathbf{y}}(s) \times \beta). \end{aligned}$$

With this choice of $\mathbf{e}(s,\mathbf{x})$ we have that $\mathbf{e}(s,\mathbf{x}) \cdot \beta(s,\mathbf{x}) = 0$ and

$$\operatorname{sgn}(\xi \cdot \mathbf{e}(s,\mathbf{x}))\operatorname{sgn}(\xi \cdot \dot{\mathbf{y}}(s)) = \operatorname{sgn}(\xi \cdot \widetilde{\mathbf{e}}(s,\mathbf{x}))\operatorname{sgn}(\xi \cdot \dot{\mathbf{y}}(s)) = \operatorname{sgn}(\xi \cdot \dot{\mathbf{y}}(s))\operatorname{sgn}(\xi \cdot \dot{\mathbf{y}}(s)) = 1$$

and thus

$$H(\xi,\mathbf{x}) = N(\xi,\mathbf{x}).$$

The next proposition establishes an important property of the filtering lines that enables an efficient calculation of $I(s,\mathbf{x})$.

Proposition 4.1. *Let $\boldsymbol{x}, \boldsymbol{x}' \in supp(f)$ and $\beta = \beta(s,\boldsymbol{x})$, $\boldsymbol{e} = \boldsymbol{e}(s,\boldsymbol{x})$, $\beta' = \beta(s,\boldsymbol{x}')$, and $\boldsymbol{e}' = \boldsymbol{e}(s,\boldsymbol{x}')$. Suppose $\beta' \in span(\beta,\boldsymbol{e})$. Then $span(\beta,\boldsymbol{e}) = span(\beta',\boldsymbol{e}')$, i.e., the filtering step is translation-invariant.*

Proof. We have that

$$\mathbf{e}' \in \operatorname{span}(\dot{\mathbf{y}},\beta') = \operatorname{span}(\dot{\mathbf{y}},\beta,\mathbf{e}) = \operatorname{span}(\beta,\mathbf{e}).$$

□

We now calculate the cone-fan beam coordinates, $[\alpha, v]$, of the filtering lines for $h = 0$. Without loss of generality we may assume that $s = 0$. Since the filtering lines are translation invariant, we may assume that $\mathbf{x} = (-R, 0, x_3)$. Then

$$\beta = \begin{bmatrix} -\frac{2R}{\sqrt{4R^2+x_3^2}} \\ 0 \\ \frac{x_3}{\sqrt{4R^2+x_3^2}} \end{bmatrix}, \qquad \mathbf{e} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

and $\mathbf{y}(0) + t(\cos\gamma\beta + \sin\gamma\mathbf{e}) \in \mathbf{D}(0)$ for $t = \sqrt{4R^2 + x_3^2}$ and thus the filtering lines are given by

$$h(\alpha) = x_3 \cos\alpha.$$

Filtering lines are shown in figure **??**.

Now let $\mathbf{x} \in \mathrm{supp}(f)$. The filtering line for $\mathbf{x}$ is given by $c(\alpha; \mathbf{x}, s) = \frac{Rx_3}{(\mathbf{x}-\mathbf{y}(s))\cdot\ddot{\mathbf{y}}(s)} \cos\alpha$. Therefore we have that

$$I(s, \mathbf{x}) \quad = \quad \int_{-\pi}^{\pi} \frac{\partial}{\partial q} Df(\mathbf{y}(q), \Theta(s, \alpha, \mathbf{x}))\bigg|_{q=s} \frac{1}{\sin(\alpha - \gamma(\mathbf{x}, s))}\, d\alpha,$$

where

$$\begin{aligned}
\Theta(s, \alpha, \mathbf{x}) &= \frac{(\mathbf{x} - \mathbf{y}(s)) \cdot \ddot{\mathbf{y}}(s)}{\sqrt{((\mathbf{x} - \mathbf{y}(s)) \cdot \ddot{\mathbf{y}}(s))^2 + R^2 x_3^2 \cos^2\alpha}} \begin{bmatrix} \cos(s+\alpha) \\ \sin(s+\alpha) \\ -\frac{Rx_3 \cos\alpha}{(\mathbf{x}-\mathbf{y}(s))\cdot\ddot{\mathbf{y}}(s)} \end{bmatrix} \\
\gamma(\mathbf{x}, s) &= \tan^{-1}\left(\frac{< \dot{\mathbf{y}}(s), \beta >}{< \ddot{\mathbf{y}}(s), \beta >}\right)
\end{aligned}$$

In the next section we discuss the rebinning of the data into cone-parallel coordinates. This new coordinate system enables easier implementation and image reconstructed in this new coordinate system have more desirable noise characteristics.

5 Cone-Fan to Cone-Parallel Beam Rebinning

An LOR in the Cone-Fan geometry is given by

$$L_{CF}(s, \alpha, \theta) \quad = \quad \left\{ \begin{bmatrix} R\cos(s) \\ R\sin(s) \\ 0 \end{bmatrix} + t \begin{bmatrix} -\cos(s+\alpha)\sin\theta \\ -\sin(s+\alpha)\sin\theta \\ \cos\theta \end{bmatrix},\ t \in \mathbb{R} \right\},$$

where $v = \frac{\cos\theta}{\sin\theta}$ and $\frac{1}{\sqrt{1+v^2}} = |\sin\theta|$. With $r = R\sin\alpha$ and $\varphi = s + \alpha - \frac{\pi}{2}$ the Cone-Parallel LOR is given by

$$L_{CP}(r, \varphi, \theta) \quad = \quad \left\{ r \begin{bmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{bmatrix} + \sqrt{R^2 - r^2} \begin{bmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{bmatrix} + t \begin{bmatrix} \sin\varphi\sin\theta \\ -\cos\varphi\sin\theta \\ \cos\theta \end{bmatrix},\ t \in \mathbb{R} \right\}$$

$$
\begin{aligned}
\begin{bmatrix}\varphi\\ r\end{bmatrix} &= \begin{bmatrix}s+\alpha-\frac{\pi}{2}\\ R\sin\alpha\end{bmatrix}\\
\begin{bmatrix}s\\ \alpha\end{bmatrix} &= \begin{bmatrix}\varphi-\sin^{-1}(r/R)+\frac{\pi}{2}\\ \sin^{-1}(r/R)\end{bmatrix}\\
v(\varphi,\mathbf{x}) &= \frac{x_3}{\mathbf{x}\cdot p^{\perp}(\varphi)+\sqrt{R^2-(\mathbf{x}\cdot\mathbf{p}(\varphi))^2}}\\
r(\varphi,\mathbf{x}) &= \mathbf{x}\cdot\mathbf{p}(\varphi)\\
\alpha_{\mathbf{x}}(\varphi) &= \sin^{-1}\left(\frac{1}{R}\mathbf{x}\cdot\mathbf{p}(\varphi)\right)\\
\alpha_{\mathbf{x}}(s) &= \sin^{-1}\left(\frac{((\beta\cdot\widehat{\mathbf{z}})\widehat{\mathbf{z}}-\beta)\cdot\mathbf{p}^{\perp}(s)}{|(\beta\cdot\widehat{\mathbf{z}})\widehat{\mathbf{z}}-\beta|}\right)=\sin^{-1}\left(\frac{x_1\sin(s)-x_2\cos(s)}{\sqrt{(x_1-R\cos(s))^2+(x_2-R\sin(s))^2}}\right)\\
\mathbf{p}(\varphi) &= \begin{bmatrix}\cos\varphi\\ \sin\varphi\\ 0\end{bmatrix},\qquad \mathbf{p}^{\perp}(\varphi)=\begin{bmatrix}-\sin\varphi\\ \cos\varphi\\ 0\end{bmatrix}=\mathbf{p}(\varphi+\pi/2)\\
\dot{\mathbf{y}}(s) &= \begin{bmatrix}-R\sin(s)\\ R\cos(s)\\ 0\end{bmatrix},\qquad \ddot{\mathbf{y}}(s)=\begin{bmatrix}-R\cos(s)\\ -R\sin(s)\\ 0\end{bmatrix}=-\mathbf{y}(s)
\end{aligned}
$$

The following formulas may be useful:

$$
\begin{aligned}
\beta(\mathbf{x},s)\cdot\mathbf{p}^{\perp}(s) &= \frac{\mathbf{x}\cdot\mathbf{p}^{\perp}(s)}{|\mathbf{x}-R\mathbf{p}(s)|}\\
\mathbf{e}(\mathbf{x},s)\cdot\mathbf{p}^{\perp}(s) &= \sqrt{1-(\beta(\mathbf{x},s)\cdot\mathbf{p}^{\perp}(s))^2}\\
&= \frac{\sqrt{|\mathbf{x}-R\mathbf{p}(s)|^2-(\mathbf{x}\cdot\mathbf{p}^{\perp}(s))^2}}{|\mathbf{x}-R\mathbf{p}(s)|}.
\end{aligned}
$$

The cone-fan coordinates in terms of $\Theta(s,\mathbf{x},\gamma)=\cos\gamma\beta+\sin\gamma\mathbf{e}$ are given by

$$
\begin{aligned}
\cos\theta(s,\mathbf{x},\gamma) &= \Theta(s,\mathbf{x},\gamma)\cdot\widehat{\mathbf{z}}\\
&= \left(\frac{\beta\cdot\widehat{\mathbf{z}}}{\mathbf{e}\cdot\mathbf{p}^{\perp}}\right)\left[(\mathbf{e}\cdot\mathbf{p}^{\perp})\cos\gamma-(\beta\cdot\mathbf{p}^{\perp})\sin\gamma\right]\\
\sin\alpha(s,\mathbf{x},\gamma) &= \frac{((\Theta(s,\mathbf{x},\gamma)\cdot\widehat{\mathbf{z}})\widehat{\mathbf{z}}-\Theta(s,\mathbf{x},\gamma))\cdot\mathbf{p}^{\perp}(s)}{|(\Theta(s,\mathbf{x},\gamma)\cdot\widehat{\mathbf{z}})\widehat{\mathbf{z}}-\Theta(s,\mathbf{x},\gamma)|}\\
&= -\frac{(\beta\cdot\mathbf{p}^{\perp})\cos\gamma+(\mathbf{e}\cdot\mathbf{p}^{\perp})\sin\gamma}{\sqrt{1-\left(\frac{\beta\cdot\widehat{\mathbf{z}}}{\mathbf{e}\cdot\mathbf{p}^{\perp}}\right)^2\left[(\mathbf{e}\cdot\mathbf{p}^{\perp})\cos\gamma-(\beta\cdot\mathbf{p}^{\perp})\sin\gamma\right]^2}}\\
&= -\frac{(\beta\cdot\mathbf{p}^{\perp})\cos\gamma+(\mathbf{e}\cdot\mathbf{p}^{\perp})\sin\gamma}{\sqrt{1-\cos^2\theta}}\\
&= -\frac{(\beta\cdot\mathbf{p}^{\perp})\cos\gamma+(\mathbf{e}\cdot\mathbf{p}^{\perp})\sin\gamma}{\sin\theta}.
\end{aligned}
$$

Let $N(\varphi,r)$ for $\varphi\in[0,\pi)$ and $r\in[-r_{max},r_{max}]$, $r_{max}=R\sin\alpha_{max}$, be the number of measurements for $s\in[-s_{max},s_{max}]$ and $\alpha\in[-\alpha_{max},\alpha_{max}]$. Assume that $s_{max}<=\pi$.

Then

$$
\begin{aligned}
N\left(\varphi - \frac{\pi}{2}, r\right) &= \begin{cases} 0, & s_{max} < \frac{\pi}{2} + \alpha_{max} \text{ and } \left|\frac{r_{max}}{\alpha_{max}}\varphi - r\right| > \frac{r_{max}}{\alpha_{max}} s_{max}, \\ 2, & s_{max} > \frac{\pi}{2} - \alpha_{max} \text{ and } \left|\frac{r_{max}}{\alpha_{max}}\varphi + r\right| > \frac{r_{max}}{\alpha_{max}}(\pi - s_{max}), \\ 1, & \text{otherwise} \end{cases} \\
N(\varphi, \mathbf{x}) &= N(\varphi, r(\mathbf{x}, \varphi))\mathbf{1}_{\left[|v(\varphi,\mathbf{x})| < \frac{H}{2R}\right]}(\mathbf{x})
\end{aligned}
$$

Suppose that we have one full axial rotation worth of data. Let $\mathbf{x} = (r\cos\psi, r\sin\psi, x_3)$, $\widetilde{x}_3 = \frac{|x_3|}{H/2}$, and $\widetilde{r} = \frac{|r|}{R}$. Then we have measured projections through $\mathbf{x}$ for an azimuthal range of

$$\int_0^{2\pi} \mathbf{1}_{\left[|v(\varphi,\mathbf{x})| < \frac{H}{2R}\right]}(\varphi, \mathbf{x})\, d\varphi = 2\cos^{-1}\left(\frac{\widetilde{x}_3^2 + \widetilde{r}^2 - 1}{2\widetilde{x}_3\widetilde{r}}\right).$$

Moreover, the azimuthal angles are given by

$$\left\{\varphi \in [0, 2\pi) : |\varphi - \psi| \leq \cos^{-1}\left(\frac{\widetilde{x}_3^2 + \widetilde{r}^2 - 1}{2\widetilde{x}_3\widetilde{r}}\right)\right\}.$$

6 Cone Beam Reconstruction with the Ramp Filter

Let

$$
\begin{aligned}
\mathbf{y}(s, \varphi) &\equiv s\begin{bmatrix}\cos\varphi \\ \sin\varphi \\ 0\end{bmatrix} - \sqrt{R^2 - s^2}\begin{bmatrix}-\sin\varphi \\ \cos\varphi \\ 0\end{bmatrix} = s\mathbf{p}(\varphi) - \sqrt{R^2 - s^2}\mathbf{p}^{\perp}(\varphi), \\
\mathbf{u}(s, \varphi, v) &\equiv \begin{bmatrix} s\cos\varphi \\ s\sin\varphi \\ v\sqrt{R^2 - s^2}\end{bmatrix} = s\mathbf{p}(\varphi) + v\sqrt{R^2 - s^2}\widehat{\mathbf{z}}, \\
\Theta(\varphi, v) &\equiv \begin{bmatrix}-\sin\varphi \\ \cos\varphi \\ v\end{bmatrix} = \mathbf{p}^{\perp}(\varphi) + v\widehat{\mathbf{z}}, \\
\widehat{\Theta}(\varphi, v) &\equiv \frac{1}{\sqrt{1 + v^2}}\Theta(\varphi, v) \\
Pf(y, \Theta) &\equiv \int_{\mathbb{R}} f(y + t\Theta)\, dt.
\end{aligned}
$$

Note that $Pf(\mathbf{y}, \Theta) = \frac{1}{\sqrt{1+v^2}} Pf(\mathbf{y}, \widehat{\Theta})$ and $\mathbf{u}(s, \varphi, v) = \mathbf{y}(s, \varphi) + \sqrt{R^2 - s^2}\Theta(\varphi, v)$ and thus

$$g(s, \varphi, \delta) \equiv Pf(\mathbf{y}(s, \varphi), \Theta(\varphi, v)) = Pf(\mathbf{u}(s, \varphi, v), \Theta(\varphi, v)).$$

Let $s = <\mathbf{x}, \mathbf{p}(\varphi)>$ and $t = <\mathbf{x}, \mathbf{p}^{\perp}(\varphi)>$. Then for appropriate choice of v,

$$
\begin{aligned}
\mathbf{x} &= \mathbf{u}(s, \varphi, v) + t\Theta(\varphi, v) \\
&= <\mathbf{x}, \mathbf{p}>\mathbf{p} + <\mathbf{x}, \mathbf{p}^{\perp}>\mathbf{p}^{\perp} + \left(\sqrt{R^2 - s^2} + <\mathbf{x}, \mathbf{p}^{\perp}>\right) v\widehat{\mathbf{z}}
\end{aligned}
$$

and thus we define

$$v(\varphi, \mathbf{x}) \equiv \frac{x_3}{\sqrt{R^2 - (\mathbf{x}\cdot\mathbf{p})^2} + \mathbf{x}\cdot\mathbf{p}^{\perp}}.$$

We will use the convention $\Theta(\varphi, \mathbf{x}) = \Theta(\varphi, v(\varphi, \mathbf{x}))$; please excuse the abuse of notation. Now we may define the backprojection operator by

$$P^* g(\mathbf{x}) \equiv \int_0^{2\pi} g(\mathbf{y}(<\mathbf{x}, \mathbf{p}>, \varphi), \Theta(\varphi, \mathbf{x}))\, d\varphi.$$

Let

$$f_d(\mathbf{x}) \equiv P^* \mathcal{D}_s P f(\mathbf{x}).$$

where $\mathcal{D}_s = \frac{\partial}{\partial s}$. Then for any $\mathbf{y}, \Theta \in \mathbb{R}^3$.

$$\begin{aligned} Pf(\mathbf{y}, \Theta) &= \int_{\mathbb{R}} f(\mathbf{y} + t\Theta)\, dt \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^3} \widehat{f}(\xi) e^{2\pi i (\mathbf{y}+t\Theta)\cdot\xi}\, d\xi\, dt \\ &= \int_{\mathbb{R}^3} \widehat{f}(\xi) e^{2\pi i \mathbf{y}\cdot\xi} \delta(\Theta\cdot\xi)\, d\xi. \end{aligned}$$

and thus

$$f_d(\mathbf{x}) = i \int_{\mathbb{R}^3} \widehat{f}(\xi) \int_0^{2\pi} \delta(\Theta(\varphi, \mathbf{x})\cdot\xi)\, (\mathbf{y}_s(<\mathbf{x}, \mathbf{p}>, \varphi)\cdot\xi)\, e^{2\pi i \mathbf{y}(<\mathbf{x},\mathbf{p}>,\varphi)\cdot\xi}\, d\varphi\, d\xi.$$

Since the inner integral is only nonzero when $\Theta\cdot\xi = 0$, we have that

$$\begin{aligned} \mathbf{x}\cdot\xi &= \left(\mathbf{u} + <\mathbf{x}, \mathbf{p}^{\perp}> \Theta\right)\cdot\xi \\ &= \mathbf{u}\cdot\xi \\ &= \mathbf{y}\cdot\xi \end{aligned}$$

and thus

$$f_d(\mathbf{x}) = i \int_{\mathbb{R}^3} \widehat{f}(\xi) e^{2\pi i \mathbf{x}\cdot\xi} \int_0^{2\pi} \delta(\Theta(\varphi, \mathbf{x})\cdot\xi)\, (\mathbf{y}_s\cdot\xi)\; d\varphi\, d\xi.$$

Now we focus on this inner integral. Note that

$$\begin{aligned} \frac{d}{d\varphi}\Theta(\varphi, v(\mathbf{x}, \varphi)) &= \frac{d}{d\varphi}\begin{bmatrix} -\sin\varphi \\ \cos\varphi \\ v(\mathbf{x}, \varphi) \end{bmatrix} \\ &= -\mathbf{p}(\varphi) + v_{\varphi}(\mathbf{x}, \varphi)\widehat{\mathbf{z}} \end{aligned}$$

and

$$\begin{aligned}
\frac{d}{d\varphi}\left[v(\mathbf{x},\varphi)\right] &= \frac{d}{d\varphi}\left[\frac{x_3}{\sqrt{R^2-(\mathbf{x}\cdot\mathbf{p})^2}+\mathbf{x}\cdot\mathbf{p}^\perp}\right] \\
&= -\frac{x_3\frac{d}{d\varphi}\left[\sqrt{R^2-(\mathbf{x}\cdot\mathbf{p})^2}+\mathbf{x}\cdot\mathbf{p}^\perp\right]}{\left(\sqrt{R^2-(\mathbf{x}\cdot\mathbf{p})^2}+\mathbf{x}\cdot\mathbf{p}^\perp\right)^2} \\
&= \frac{x_3(x\cdot p)\left[1+\frac{x\cdot p^\perp}{\sqrt{R^2-(x\cdot p)^2}}\right]}{\left(\sqrt{R^2-(x\cdot p)^2}+x\cdot p^\perp\right)^2} \\
&= \frac{x_3}{\sqrt{R^2-(\mathbf{x}\cdot\mathbf{p})^2}+\mathbf{x}\cdot\mathbf{p}^\perp}\left(\frac{\mathbf{x}\cdot\mathbf{p}}{\sqrt{R^2-(\mathbf{x}\cdot\mathbf{p})^2}}\right) \\
&= \frac{\mathbf{x}\cdot\mathbf{p}}{\sqrt{R^2-(\mathbf{x}\cdot\mathbf{p})^2}}v(\mathbf{x},\varphi).
\end{aligned}$$

Therefore

$$\frac{d}{d\varphi}\Theta(\varphi,v(\mathbf{x},\varphi)) = -\mathbf{p}(\varphi)+\frac{\mathbf{x}\cdot\mathbf{p}}{\sqrt{R^2-(\mathbf{x}\cdot\mathbf{p})^2}}v(\mathbf{x},\varphi)\widehat{\mathbf{z}}.$$

We also have

$$\begin{aligned}
\mathbf{u}_s(s,\varphi,v)\big|_{s=\mathbf{x}\cdot\mathbf{p},v=v(\mathbf{x},\varphi)} &= \mathbf{p}(\varphi)-\frac{sv}{\sqrt{R^2-s^2}}\widehat{\mathbf{z}}\Bigg|_{s=\mathbf{x}\cdot\mathbf{p},v=v(\mathbf{x},\varphi)} \\
&= -\frac{d}{d\varphi}\Theta(\varphi,\mathbf{x}).
\end{aligned}$$

Moreover for ξ such that $\xi\cdot\Theta=0$, $\mathbf{u}\cdot\xi=\mathbf{y}\cdot\xi$ and thus $\mathbf{y}_s\cdot\xi=\mathbf{u}_s\cdot\xi$. Now we see that

$$\begin{aligned}
f_d(\mathbf{x}) &= -i\int_{\mathbb{R}^3}\widehat{f}(\xi)e^{2\pi i\mathbf{x}\cdot\xi}\int_0^{2\pi}\left(\frac{d}{d\varphi}\Theta(\varphi,\mathbf{x})\cdot\xi\right)\delta(\Theta(\varphi,\mathbf{x})\cdot\xi)\,d\varphi\,d\xi \\
&= -i\int_{\mathbb{R}^3}\widehat{f}(\xi)e^{2\pi i\mathbf{x}\cdot\xi}\int_0^{2\pi}\operatorname{sgn}\left(\frac{d}{d\varphi}\Theta(\varphi,\mathbf{x})\cdot\xi\right)\delta\left(\frac{\Theta(\varphi,\mathbf{x})\cdot\xi}{\frac{d}{d\varphi}\Theta(\varphi,\mathbf{x})\cdot\xi}\right)d\varphi\,d\xi \\
&= \int_{\mathbb{R}^3}\widehat{f}(\xi)e^{2\pi i\mathbf{x}\cdot\xi}\left\{-i\sum_k\operatorname{sgn}\left(\frac{d\Theta}{d\varphi}(\varphi_k(\xi),\mathbf{x})\cdot\xi\right)\right\}d\xi \\
&= \int_{\mathbb{R}^3}\widehat{f}(\xi)e^{2\pi i\mathbf{x}\cdot\xi}\left\{i\sum_k\operatorname{sgn}\left(\left[\mathbf{p}(\varphi_k)+\frac{\mathbf{x}\cdot\mathbf{p}(\varphi_k)}{\sqrt{R^2-(\mathbf{x}\cdot\mathbf{p}(\varphi_k))^2}}\mathbf{p}^\perp(\varphi_k)\right]\cdot\xi\right)\right\}d\xi
\end{aligned}$$

where $\{\varphi_k(\xi)\}$ are such that $\Theta(\varphi_k(\xi),\mathbf{x})\cdot\xi=0$.

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