Signal Processing - Definitions and Transform Tables

Contents

1 Basic Definitions

Fundamental Period (discrete): The fundamental period of a discrete signal is the smallest $N \in \mathbb{N}$ such that $x[n] = x[n+N]$ for all $n \in \mathbb{Z}$. The units are seconds.

Fundamental Frequency (discrete): The fundamental frequency of a discrete signal is $f = 1/N$, where N is defined above. The unit hertz (Hz) = cycles per second.

Angular Frequency (discrete): The angular frequency of a discrete signal is $\omega = 2\pi f = \frac{2\pi}{N}$, with N defined as above. The units are radians per second.

Fundamental Period (continuous): The fundamental period of a continuous signal is the smallest $T > 0$ such that $x(t) = x(t+T)$ for all $t \in \mathbb{R}$. The units of T are seconds.

Fundamental Frequency (continuous): The fundamental frequency of a continuous signal is $f = 1/T$, with T defined as above. The units of f are hertz (Hz) = cycles per second.

Angular Frequency (continuous): The angular frequency of a continuous signal is $\Omega = 2\pi f = \frac{2\pi}{T}$. The units are radians per second.

Causal System: A system is causal if, for every choice of n_0, the output sequence value at the index $n = n_0$ depends only on the input sequence values for $n \leq n_0$. A system is causal iff the system impluse response, $h[n]$, equals zero for $n < 0$. The z-transform of a causal system will have a region of convergence of the form $|z| > a$.

(BIBO) Stable System: A system is stable in the bounded-input, bounded-output (BIBO) sense if and only if every bounded input sequence produces a bounded output sequence. An LTI system is stable if and only if the system impulse response, h, is absolutely summable. The impulse response is absolutely summable if and only if the region of convergence of the Z-transform of h includes the unit circle.

Continuous-Time Convolution: The convolution of two functions $h(t)$ and $x(t)$ is given by

$$h(t) * x(t) = \int_{\mathbb{R}} h(t-s)x(s)\,ds.$$

Continuous-Time Periodic Convolution: The convolution of two functions $h(t) = h(t+T_1)$ and $x(t) = x(t+T_2)$ with period $T = \text{lcm}(T_1, T_2)$ is given by

$$h(t) * x(t) = \int_0^T h(t-s)x(s)\,ds.$$

Discrete-Time Convolution: The convolution of two sequences $h[n]$ and $x[n]$ is given by

$$h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[n-k]x[k].$$

Sinus Cardinal (sinc) Function: The sinc function is defined as $\text{sinc}(u) = \sin(\pi u)/(\pi u)$.

Power Series: We have that $\sum_{n=0}^{N} r^n = \frac{1-r^{N+1}}{1-r}$.

2 Transform Definitions

Z-Transform: Let $x[n]$ be a discrete-time signal. Then the z-Transform and inverse z-Transform of x are

$$\begin{aligned} X(z) &= \sum_{k=-\infty}^{\infty} x[k] z^{-k} \\ x[n] &= \frac{1}{2\pi j} \oint X(z) z^{n-1}\, dz. \end{aligned}$$

The term "z-transform" was coined by uneducated engineers who had never heard of the Laurent transform which is what mathematicians had been calling it for hundreds of years.

Discrete Time Fourier Series (DTFS): Suppose that $x[n] = x[n+N]$ for all $n \in \mathbb{Z}$. Then $\omega_0 = \frac{2\pi}{N}$ is the fundamental frequency of $x[n]$. Then the discrete time Fourier series and inverse Fourier series are

$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N} \qquad \hat{G} = \{0, 1, \ldots, N-1\} \\ x[n] &= \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j2\pi kn/N} \qquad G = \{0, 1, \ldots, N-1\}. \end{aligned}$$

Discrete Time Fourier Transform (DTFT): Let $x[n]$ be a discrete-time signal. Then the discrete time Fourier and inverse Fourier transform are

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \qquad \hat{G} = [0, 2\pi) \sim [-\pi, \pi) \\ x[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n}\, d\omega \qquad G = \mathbb{Z}. \end{aligned}$$

Note that since $n \in \mathbb{Z}$, $\omega \in [-\pi, \pi) \sim [0, 2\pi)$ and $X(e^{j\omega}) = X(z)|_{z=e^{j\omega}}$.

Fourier Series (FS): Suppose that $x(t) = x(t+T)$ for all $t \in \mathbb{R}$. Then $\Omega_0 = \frac{2\pi}{T}$ is the fundamental frequency of $x(t)$. Then the Fourier series and inverse Fourier series are

$$\begin{aligned} X[k] &= \frac{1}{T} \int_0^T x(t) e^{-j2\pi kt/T}\, dt \qquad \hat{G} = \mathbb{Z} \\ x(t) &= \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi kt/T} \qquad G = [0, T) \sim [-T/2, T/2). \end{aligned}$$

Fourier Transform (FT): Let $x(t)$ be a continuous-time signal. Then the Fourier and inverse Fourier transform of x are

$$\begin{aligned} X(j\Omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}\,dt \qquad \hat{G} = \mathbb{R} \\ x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t}\,d\Omega \qquad G = \mathbb{R}. \end{aligned}$$

Note that since $t \in \mathbb{R}$, $\Omega \in \mathbb{R}$.

(Unilateral) Laplace Transform: The Laplace transform, $X(s) = \mathcal{L}(x(t))$, and inverse Laplace transform are given by

$$\begin{aligned} X(s) &= \int_{0-}^{\infty} x(t)e^{-st}\,dt, \quad s \in \mathbb{C} \text{ with } \Re(s) > 0 \\ x(t) &= \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\,dt. \end{aligned}$$

Bilateral Laplace Transform: The bilateral Laplace transform is given by

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt.$$

3 Properties of the Z (Laurent) Transform

Properties of the Z-Transform: Let $x[n] \leftrightarrow^{ZT} X(z)$ with ROC R_x. Then
(i) $x[-n] \leftrightarrow^{ZT} X(1/z)$ with ROC $1/R_x$ (e.g. $R_x : a < |z| < b$, $1/R_x : 1/b < |z| < 1/a$)
(ii) $x[n-n_0] \leftrightarrow^{ZT} z^{-n_0}X(z)$
(iii) $\alpha^n x[n] \leftrightarrow^{ZT} X(z/\alpha)$ with ROC $|\alpha|R_x$
(iv) $x[n] * y[n] \leftrightarrow^{ZT} X(z)Y(z)$ with ROC $\supseteq R_x \cap R_y$
(v) $nx[n] \leftrightarrow^{ZT} -z\frac{d}{dz}X(z)$

Left-Sided Sequence (LSS): An LSS is a sequence such that $x[n] = 0$ for $n \geq N$. The ROC for an LSS is of the form $|z| < r_-$.

Right-Sided Sequence (RSS): An RSS is a sequence such that $x[n] = 0$ for $n < N$. The ROC for an RSS is of the form $|z| > r_+$.

Two-Sided Sequence (TSS): A TSS is a sequence that has infinite duration in both the positive and negative directions. The ROC for an TSS is of the form $r_+ < |z| < r_-$.

Causality: We have that $h[n]$ causal implies that $H(z)$ does not have a pole at infinity and thus $h[n]$ is not causal if $H(z)$ has a pole at infinity. If we assume that $H(z) = A(z)/B(z)$ is a rational function, then $h[n]$ causal implies that degree$(B(z)) \geq$ degree$(A(z))$.

Stable and Causal Sequence: A system is causal and stable if and only if all the poles of $H(z)$ are inside the unit circle.

Cauchy Residue Theorem: Suppose that X is an analytic function is a simply connected domain except for isolated singularities at $z_1, \ldots, z_m$. Let γ be a simple closed curve in the region where X is analytic and not intersecting any of the z_i's. then

$$\frac{1}{2\pi i}\int_\gamma X(z)\,dz = \sum_{k=1}^{m} Res(X; z_k).$$

Evaluation of Residues of Simple Poles: Suppose that $X(z)$ has a simple pole at $z = a$. Then

$$Res(X; a) = \lim_{z\to a}(z-a)X(z).$$

4 Sampling of Continuous-Time Signals

Poisson Summation Formula: The Poisson Summation formula is the basis for sampling theory and it is given by

$$\sum_{n=-\infty}^{\infty} x(nT)e^{-jn\Omega T} = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(j\left(\Omega - \frac{2\pi}{T}k\right)\right)$$

for an integrable continuous-time signal, $x(t)$.

Frequency-Domain Representation of Sampling: Let $x_c(t)$ be a continuous-time signal and $x_s(t) = x_c(t)s(t)$, where

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT).$$

Then

$$X_s(j\Omega) = \frac{1}{2\pi}X_c(j\Omega) * S(j\Omega) = \frac{1}{2\pi}X_c(j\Omega) * \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\Omega - k\Omega_s) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)),$$

where $\Omega_s = 2\pi/T$. Now let $x[n] = x_c(nT)$. Then

$$\begin{aligned}
X_s(j\Omega) &= X(e^{j\omega})\big|_{\omega=\Omega T} = X(e^{j\Omega T}) \\
X(e^{j\Omega T}) &= \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)) \\
X(e^{j\omega}) &= \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)
\end{aligned}$$

Aliasing: Suppose that $X(j\Omega) = 0$ for $|\Omega| > \Omega_N$. Aliasing will occur if $\Omega_s \leq 2\Omega_N$. In other words, we should sample our signal at a rate $T_N \leq \pi/\Omega_N$ to avoid aliasing. The frequency $2\Omega_N$

is referred to as the Nyquist frequency and $T_N = \pi/\Omega_N$ is referred to as the Nyquist rate.

Reconstruction of a Bandlimited Signal From its Samples: Suppose a continuous-time signal $x_c(t)$ was sampled at a rate of T which is above the Nyquist rate to produce a sequence $x[n]$. Then $x_c(t)$ can be recovered from its samples by

$$\begin{aligned} x_c(t) &= \sum_{n=-\infty}^{\infty} x[n]h_r(t-nT) = \sum_{n=-\infty}^{\infty} x[n]\frac{\sin(\pi(t-nT)/T)}{\pi(t-nT)/T} \\ H_r(j\Omega) &= T\begin{cases} 1, & |\Omega| \le \frac{\pi}{T}, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Error Bounds in the Reconstruction of an Undersampled Signal: Let $x_c(t)$ be a continuous-time signal and $x[n] = x_c(nT)$ be sampled version. Also let

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]\frac{\sin(\pi(t-nT)/T)}{\pi(t-nT)/T}.$$

Then

$$|x_c(t) - x_r(t)| \le \frac{1}{\pi}\int_{|\Omega|\ge\pi/T} |X(j\Omega)|\, d\Omega.$$

Non-integer Delay: Consider the non-integer delay system: $y_c(t) = x_c(t-\Delta)$, where $\Delta \in \mathbb{R}$. If the system is sampled at a rate of T, then

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} x[k]\frac{\sin(\pi(n-k-\Delta))}{\pi(n-k-\Delta)}.$$

Note that if $\Delta = n_0 \in \mathbb{Z}$, then $h[n] = \delta[n-n_0]$.

Down-Sampling by an Integer Factor: Let $x_d[n] = x[nM]$ for $M \in \mathbb{N}$. Then

$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{k=0}^{M-1} X(e^{j(\omega/M - 2\pi k/M)}).$$

Before down-sampling, one should pass the signal through a lowpass filter with gain 1 and cutoff frequency π/M to avoid aliasing.

Up-Sampling by an Integer Factor: Let $L \in \mathbb{N}$ and

$$x_e[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-kL] = \begin{cases} x[n/L], & n/L \in \mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$X_e(e^{j\omega}) = X(e^{j\omega L}).$$

To correctly recover the values of the signal at these new sampling points, we need to pass it through a lowpass filter of gain L and cutoff frequency π/L (this will act as an ideal interpolator), i.e.,

$$x_i[n] = x_e[n] * \frac{\sin(\pi n/L)}{\pi n/L} = \sum_{k=-\infty}^{\infty} x[k]\delta[n-kL] * \frac{\sin(\pi n/L)}{\pi n/L} = \sum_{k=-\infty}^{\infty} x[k]\frac{\sin(\pi(n-kL)/L)}{\pi(n-kL)/L}.$$

Sampling of Periodic Signals: Let $x_c(t) = x_c(t+T)$ and $x[n] = x_c(nT_s)$, where $NT_s = T$. Then

$$X[k] = \frac{T}{T_s} \sum_{l=-\infty}^{\infty} X_c[k - lN].$$

Note that

$$X_c[k] = \frac{1}{T} X_c(j\Omega)\Big|_{\Omega=2\pi k/T} \qquad X[k] = X\left(e^{j\omega}\right)\big|_{\omega=2\pi k/N}.$$

5 Phase Response

Phase: Let $z = x + jy$. Then the phase of z is $\angle z = \tan^{-1}(y/x)$. The phase of a complex-valued function is often denoted by $\theta(\omega)$.

Group Delay: The group delay of $H(e^{j\omega})$, with phase $\theta(\omega)$ is given by

$$grd[H(e^{j\omega})] = -\frac{d}{d\omega}\theta(\omega) = \mathrm{Re}\left[\frac{j\frac{d}{d\omega}H(e^{j\omega})}{H(e^{j\omega})}\right]$$

Phase and Group Delay of a Linear Function: Let $c = re^{j\theta}$ and

$$H(e^{j\omega}) = 1 - ce^{-j\omega} = 1 - re^{j\theta}e^{-j\omega} = 1 - re^{-j(\omega-\theta)}.$$

Then

$$\begin{aligned} \angle H(e^{j\omega}) &= \tan^{-1}\left(\frac{r\sin(\omega-\theta)}{1 - r\cos(\omega-\theta)}\right) \\ grd[H(e^{j\omega})] &= \frac{r^2 - r\cos(\omega-\theta)}{1 + r^2 - 2r\cos(\omega-\theta)}. \end{aligned}$$

We see that $grd[H(e^{j\omega})] < 0$ if $r < \cos(\omega - \theta)$. Thus if the zero of $H(e^{j\omega})$ is outside the unit circle, the group delay is always positive. This is key to understanding minimum-phase systems. If $r = 1$, then $\angle H(e^{j\omega}) = \frac{\pi}{2} - \frac{\omega-\theta}{2}$.

Phase Response and Group Delay of Rational Functions: Let

$$H(e^{j\omega}) = \frac{b_0}{a_0}\frac{\prod_{k=1}^{M}(1 - c_k e^{-j\omega})}{\prod_{k=1}^{N}(1 - d_k e^{-j\omega})}.$$

Then the phase of $H(e^{j\omega})$ is

$$\angle H(e^{j\omega}) = \angle\frac{b_0}{a_0} + \sum_{k=1}^{M}\angle[1 - c_k e^{-j\omega}] - \sum_{k=1}^{N}\angle[1 - d_k e^{-j\omega}]$$

and the group delay is

$$grd[H(e^{j\omega})] = grd\frac{b_0}{a_0} + \sum_{k=1}^{M} grd[1 - c_k e^{-j\omega}] - \sum_{k=1}^{N} grd[1 - d_k e^{-j\omega}].$$

All Pass System: An all pass system is given by

$$H(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}$$

for $|a| \neq 1$. Note that $\angle a = \angle 1/a^*$, $grd[H(e^{j\omega})] > 0$, and

$$|H(e^{j\omega})| = \left| \frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}} \right| = \left| e^{-j\omega} \frac{1 - a^* e^{j\omega}}{1 - ae^{-j\omega}} \right| = 1.$$

Minimum Phase System: A system $H(z)$ is a minimum phase system if all poles and zeros are within the unit circle. Note that if $H_1(z) = 1 - az^{-1}$ and $H_2(z) = 1 - (1/a^*)z^{-1}$ with $|a| < 1$, then $\angle H_1(e^{j\omega}) < \angle H_2(e^{j\omega})$ and $grd[H_1(e^{j\omega})] < grd[H_2(e^{j\omega})]$.

All Pass - Minimum Phase Decomposition: Suppose that $H(z)$ has a zero at $z = 1/c^*$ which is outside the unit circle. All other poles and zeros are inside the unit circle. Then we have

$$H(z) = H_1(z)(z^{-1} - c^*) = H_1(z)(z^{-1} - c^*)\frac{1 - cz^{-1}}{1 - cz^{-1}} = [H_1(z)(1 - cz^{-1})]\frac{z^{-1} - c^*}{1 - cz^{-1}} = H_{min}(z)H_{ap}(z).$$

Generalized Linear Phase: A system is referred to as a generalized linear phase system if its frequency response is of the form

$$H(e^{j\omega}) = A(e^{j\omega})e^{-j\alpha\omega + j\beta},$$

where $A(e^{j\omega})$ is a real-valued function of ω. Thus the phase is $\theta(\omega) = -\alpha\omega + \beta$.

Four Types of FIR Linear-Phase Systems: The family of causal FIR filters with constant group delay can be classified one of the four following classes. They are:

Type I: $h[n] = h[M - n] \iff H(e^{j\omega}) = e^{-j\omega M/2} \sum_{k=0}^{M/2} a[k] \cos(\omega k)$

$0 \leq n \leq M, M \in 2\mathbb{Z}$

Type II: $h[n] = h[M - n] \iff H(e^{j\omega}) = e^{-j\omega M/2} \sum_{k=1}^{(M+1)/2} b[k] \cos(\omega(k - 1/2))$

$0 \leq n \leq M, M \in 2\mathbb{Z} + 1$

Type III: $h[n] = -h[M - n] \iff H(e^{j\omega}) = je^{-j\omega M/2} \sum_{k=0}^{M/2} c[k] \sin(\omega k)$

$0 \leq n \leq M, M \in 2\mathbb{Z}$

Type IV: $h[n] = -h[M - n] \iff H(e^{j\omega}) = je^{-j\omega M/2} \sum_{k=1}^{(M+1)/2} d[k] \sin(\omega(k - 1/2))$

$0 \leq n \leq M, M \in 2\mathbb{Z} + 1$

Location of Zeros for FIR Linear-Phase Systems: Suppose that $h[n]$ is a real-valued impulse response with linear phase. Then each complex root of $H(e^{j\omega})$ will be part of a set of four conjugate reciprocal zeros. Each zero of $H(e^{j\omega})$ on the unit circle will be part of a set of two conjugate zeros. Each real root (not equal to ± 1) of $H(e^{j\omega})$ will be part of a set of two reciprocal zeros. Each zero at $z = \pm 1$ may appear by itself. Thus the cases are:

$$(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})(1 - r^{-1}e^{j\theta}z^{-1})(1 - r^{-1}e^{-j\theta}z^{-1})$$
$$(1 - e^{j\theta}z^{-1})(1 - e^{-j\theta}z^{-1})$$
$$(1 - rz^{-1})(1 - r^{-1}z^{-1})$$
$$(1 \pm z^{-1})$$

Type II filters have a zero at $z = -1$, type III filters have zeros at $z = \pm 1$, and type IV have a zero at $z = 1$.

6 Filter Design

Parameters for Filter Specification: Let $[0, \omega_p]$ be passband and $[\omega_s, \pi]$ be the stopband of the filter. Let $\delta_1 > 0$ and $\delta_2 > 0$ be such that

$$\begin{aligned} 1 - \delta_1 \leq |H(e^{j\omega})| \leq 1 + \delta_1, & \qquad |\omega| \leq \omega_p \\ |H(e^{j\omega})| \leq \delta_2, & \qquad \omega_s \leq |\omega| \leq \pi. \end{aligned}$$

Butterworth Filter: The magnitude response of the Butterworth filter of order N is given by

$$|H(j\Omega)|^2 = \frac{1}{1 + (\Omega/\Omega_c)^{2N}}.$$

Chebychev Filter: The magnitude response of the Chebychev filter is

$$|H(j\Omega)|^2 = \frac{1}{1 + \varepsilon^2 V_N^2(\Omega/\Omega_c)}, \quad V_N(x) = \cos(N \cos^{-1}(x)).$$

Elliptic Filter: The magnitude response of the Elliptic filter is

$$|H(e^{j\omega})|^2 = \frac{1}{1 + \varepsilon^2 V_N^2(\Omega)}, \quad V_N(\Omega) = \text{ Jacobi elliptic function.}$$

Filter Design by Impulse Invariance: Let $h_c(t)$ be the impulse response of the continuous-time system. Then we design the discrete-time filter by $h[n] = T_d h_c(nT_d)$. Note that

$$H_c(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k} \quad \Rightarrow \quad H(z) = \sum_{k=1}^{N} \frac{T_d A_k}{1 - e^{s_k T_d} z^{-1}}.$$

Filter Design with a Bilinear Transformation: Let $h_c(t)$ be the impulse response of the continuous-time system. Then we let

$$\begin{aligned} H(z) &= H_c\left[\frac{2}{T_d}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)\right] \\ H(e^{j\omega}) &= H_c\left(\frac{2}{T_d}\tan(\omega/2)\right). \end{aligned}$$

Filter Design by Windowing: Let $h_d[n]$ be some (ideal) IIR filter. Then we can obtain an FIR filter by $h[n] = h_d[n]w[n]$, where $w[n]$ is some windowing function of finite duration.

Common Windows: Some common windows (all with linear phase) used in filter design are

$$\text{Rectangular} \qquad w[n] = \begin{cases} 1, & 0 \le n \le M, \\ 0, & \text{otherwise} \end{cases}$$

$$\text{Bartlett (triangular)} \qquad w[n] = \begin{cases} 2n/M, & 0 \le n \le M/2, \\ 2 - 2n/M, & M/2 < n \le M, \\ 0, & \text{otherwise} \end{cases}$$

$$\text{Hann} \qquad w[n] = \begin{cases} 0.5 - 0.5\cos(2\pi n/M), & 0 \le n \le M, \\ 0, & \text{otherwise} \end{cases}$$

$$\text{Hamming} \qquad w[n] = \begin{cases} 0.54 - 0.46\cos(2\pi n/M), & 0 \le n \le M, \\ 0, \text{otherwise} \end{cases}$$

$$\text{Blackman} \qquad w[n] = \begin{cases} 0.42 - 0.5\cos(2\pi n/M) + 0.08\cos(4\pi n/M), & 0 \le n \le M, \\ 0, & \text{otherwise} \end{cases}$$

Note that in all the above windows, $w[n] = w[M-n]$ for $0 \le n \le M$ and $w[n] = 0$ otherwise. Thus $W(e^{j\omega}) = W_e(e^{j\omega})e^{-j\omega M/2}$, where $W_e(e^{j\omega})$ is a real, even function of ω. Below is a table of the main lobe widths and side lobe levels of the above windows.

Window	Side Lobe Level (dB)	3 dB BW $(\Delta\omega)_{3dB}$
Rectangular	-13	$0.89(2\pi/M)$
Bartlett	-27	$1.28(2\pi/M)$
Hann	-32	$1.44(2\pi/M)$
Hamming	-43	$1.30(2\pi/M)$
Blackman	-58	$1.68(2\pi/M)$

Kaiser Window: The Kaiser window is defined by

$$w[n] = \begin{cases} \frac{I_0[\beta(1-[(n-\alpha)/\alpha]^2)^{1/2}]}{I_0(\beta)}, & 0 \le n \le M, \\ 0, & \text{otherwise,} \end{cases}$$

where $\alpha = M/2$ and $I_0(\cdot)$ represents the zero-th order modified Bessel function of the first kind. Note that $w[n] = w[M-n]$. Let $\Delta\omega = \omega_s - \omega_p$ and $A = -20\log_{10}\delta$, then choose β and M

according to

$$\begin{aligned}
\beta &= \begin{cases} 0.1102(A-8.7), & A>50, \\ 0.5842(A-21)^{0.4}+0.07886(A-21), & 21 \le A \le 50, \\ 0.0, & A<21 \end{cases} \\
M &= \frac{A-8}{2.285\Delta\omega}.
\end{aligned}$$

7 The Discrete Fourier Transform

Periodicity of the DFT: The DFT of an N point sequence is periodic with period N. The inverse DFT of an N point DFT is also periodic with period N.

Twidle Factor: We define the twidle factor as $W_N = e^{-j(2\pi/N)}$. Note that $W_{N/k} = W_n^k$.

Periodic Extension: Suppose that $x[n] = 0$ for $n < 0$ and $n > M$. Then we shall denote the periodic extension of period $N > M$ of $x[n]$ by $\widetilde{x}[n]$ and thus

$$\widetilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n-rN].$$

Periodic Convolution: Let $x_1[n]$ and $x_2[n]$ be two sequences. Suppose that we take an N-point DFT of the two sequences to get $\widetilde{X}_1[k]$ and $\widetilde{X}_2[k]$. Let $\widetilde{X}_3[k] = \widetilde{X}_1[k]\widetilde{X}_2[k]$. Then $\widetilde{x}_3[n]$ is the periodic convolution of $\widetilde{x}_1[n]$ and $\widetilde{x}_2[n]$, where $\widetilde{x}_1[n]$ and $\widetilde{x}_2[n]$ are the periodic extensions (with period N) of $x_1[n]$ and $x_2[n]$, i.e.,

$$\widetilde{x}_3[n] = \sum_{m=0}^{N-1} \widetilde{x}_1[m]\widetilde{x}_2[n-m].$$

Circular Convolution: Let $((n))_N = n \bmod N$. Using the notation used above we have that

$$\begin{aligned}
x_3[n] &= \sum_{m=0}^{N-1} \widetilde{x}_1[m]\widetilde{x}_2[n-m], \quad 0 \le n \le N-1 \\
&= \sum_{m=0}^{N-1} x_1[((m))_N]x_2[((n-m))_N], \quad 0 \le n \le N-1, \\
&= x_1[n] *^N x_2[n],
\end{aligned}$$

where the second sum is referred to as circular convolution.

Fourier Transform of Periodic Signals: Let $\widetilde{x}[n]$ be a periodic signal with period N and $\widetilde{x}[n] \leftrightarrow^{DFT} \widetilde{X}[k]$. Then the Fourier transform of $\widetilde{x}[n]$ is defined as

$$\widetilde{X}(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{N}\widetilde{X}[k]\delta\left(\omega - \frac{2\pi k}{N}\right).$$

Relating the DTFS and DTFT: Let $x[n]$ be such that $x[n] = 0$ except for $0 \le n \le N-1$ and let $\widetilde{x}[n]$ be the periodic extension of $x[n]$ with period N, i.e.,

$$\widetilde{x}[n] = x[n] * \widetilde{p}[n] = \sum_{r=-\infty}^{\infty} x[n - rN],$$

where $\widetilde{p}[n] = \sum_{r=-\infty}^{\infty} \delta[n - rN]$. Then we find that

$$\widetilde{X}[k] = X(e^{j(2\pi/N)k}) = X(e^{j\omega})\big|_{\omega=(2\pi/N)k}.$$

Simply put, the DFT is obtained by periodically sampling the FT.

Circular Convolution as Linear Convolution with Possible Aliasing: Let $x_1[n]$ be a sequence of length L, $x_2[n]$ be a sequence of length P, and $x_3[n] = x_1[n] * x_2[n]$. Then $x_3[n]$ has a length of at most $L + P - 1$ and $x_3[n]$ has Fourier transform

$$X_3(e^{j\omega}) = X_1(e^{j\omega})X_2(e^{j\omega}).$$

If we take $N > 0$ point DFT's and define $X[k] = X(e^{j(2\pi/N)k})$, then

$$X_3[k] = X_1[k]X_2[k].$$

If we take the inverse DFT of $X_3[k]$, then we have

$$x_{3p}[n] = \begin{cases} \sum_{r=-\infty}^{\infty} x_3[n - rN], & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$$

and thus $x_{3p}[n] = x_1[n] *^N x_2[n]$. Now we see that $x_3[n] = x_{3p}[n]$, i.e. $x_1[n] * x_2[n] = x_1[n] *^N x_2[n]$ if $N \ge L + P - 1$.

Discrete Cosine Transform: The discrete cosine transform and its inverse are given by

$$\begin{aligned} X^c[k] &= 2 \sum_{n=0}^{N-1} x[n] \cos\left(\frac{\pi k(2n+1)}{2N}\right), \quad 0 \le k \le N-1, \\ x[n] &= \frac{1}{N} \sum_{k=0}^{N-1} \beta[k] X^c[k] \cos\left(\frac{\pi k(2n+1)}{2N}\right), \quad \beta[k] = \begin{cases} \frac{1}{2}, & k = 0, \\ 1, & 1 \le k \le N-1 \end{cases}. \end{aligned}$$

Just as the Fourier transform enforces periodicity on the signal, the discrete cosine transform enforces both periodicity and even symmetry.

8 Time-Frequency Analysis: The Spectrogram

Short-Time Fourier Transform (STFT): Let $w[n]$ be a finite duration "window". Then the short-time Fourier transform and its inverse are given by

$$\begin{aligned} X[n,\lambda) &= \sum_{m=-\infty}^{\infty} x[n+m]w[m]e^{-j\lambda m}, \quad n \in \mathbb{Z},\ \lambda \in [0,2\pi) \\ x[n+m]w[m] &= \frac{1}{2\pi}\int_0^{2\pi} X[n,\lambda)e^{j\lambda m}\,d\lambda \\ x[n] &= \frac{1}{2\pi w[0]}\int_0^{2\pi} X[n,\lambda)e^{j\lambda m}\,d\lambda, \quad \text{if } w[0] \neq 0. \end{aligned}$$

Note that

$$\begin{aligned} X[n,\lambda) &= \sum_{m=-\infty}^{\infty} x[m]w[-(n-m)]e^{j\lambda(n-m)} \\ &= x[n] * h_\lambda[n], \quad h_\lambda[n] = w[-n]e^{j\lambda n} \\ &= \frac{1}{2\pi}\int_0^{2\pi} X(e^{j\theta})W(e^{j(\lambda-\theta)})e^{j\theta n}\,d\theta. \end{aligned}$$

Short-Time DFT: If we sample the STFT at frequencies $\lambda_k = 2\pi k/N$ for $N \geq L$ where L is the length of the window, $w[n]$, then

$$\begin{aligned} X[n,k] &= X[n,\lambda_k) = \sum_{m=0}^{L-1} x[n+m]w[m]e^{-j(2\pi/N)km}, \quad 0 \leq k \leq N-1 \\ x[n+m] &= \frac{1}{Nw[m]}\sum_{k=0}^{N-1} X[n,k]e^{j(2\pi/N)km}, \quad 0 \leq m \leq L-1. \end{aligned}$$

The function $X[n,k]$ is often referred to as the spectrogram of $x[n]$. Note that we may decimate $X[n,k]$ in time by a factor of R (i.e. $X[nR,k]$) where $N \geq L \geq R$ with no loss of information. By no loss of information we mean that $X[nR,k]$ can be inverted to recover $x[n]$.

Wideband vs. Narrowband Spectrogram: Suppose that $x[n]$ is a non-stationary signal that we plan to analyze with the STFT. A wideband spectrogram representation results from a window that is relatively short in time and is characterized by poor resolution in frequency and good resolution in time. Dually, a narrowband spectrogram representation results from a window that is relatively long in time and is characterized by good resolution in frequency and poor resolution in time.

9 Homomorphic (Cepstral) Processing

Cepstrum: The term cepstrum is derived from reversing the letters of spectrum. It was termed by morons who don't like descriptive technical words like homomorphic.

Real Cepstrum: The real cepstral of a sequence $x[n]$ is defined as

$$c_x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|X(e^{j\omega})| e^{j\omega n}\, d\omega.$$

Complex Cepstrum: The complex cepstrum of a (real) sequence $x[n]$ is defined as

$$\widehat{x}[n] = \frac{1}{2\pi j}\int_{|z|=c} \log[X(z)] z^{n-1}\, dz,$$

where the region of convergence of the z-transform of x includes the unit circle. Then we have that

$$\begin{aligned}
\widehat{x}[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \log[X(e^{j\omega})] e^{j\omega n}\, d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left[\log|X(e^{j\omega})| + j\angle X(e^{j\omega})\right] e^{j\omega n}\, d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \widehat{X}(e^{j\omega}) e^{j\omega n}\, d\omega.
\end{aligned}$$

Note that $\widehat{x}[n] \quad \leftrightarrow^{DTFT} \quad \widehat{X}(e^{j\omega})$ and $c_x[n] = \frac{\widehat{x}[n]+\widehat{x}^*[-n]}{2}$. Also if $x[n] = v[n] * p[n]$, then $X(z) = V(z)P(z)$ and thus $\widehat{X}(z) = \widehat{V}(z) + \widehat{P}(z)$ and $\widehat{x}[n] = \widehat{v}[n] + \widehat{p}[n]$.

Recursive Definition of Cepstra: We have that

$$x[n] = \sum_{k=-\infty}^{\infty} \frac{k}{n}\widehat{x}[k] x[n-k].$$

Minimum-Phase Sequence: A minimum-phase sequence is defined as one whose complex ceptrum is zero for $n < 0$.

Maximum-Phase Sequence: A maximum-phase sequence is defined as one whose complex ceptrum is zero for $n > 0$.

10 Transform Identities

10.1 Basic Discrete-Time Fourier Series (DTFS) Pairs

Time Domain	Frequency Domain
$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi kn/N}$ period = N	$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$ $\omega_0 = \frac{2\pi}{N}$
$x[n-m]$	$e^{-j(2\pi/N)km}X[k]$
$e^{j(2\pi/N)ln}x[n]$	$X[k-l]$
$\sum_{m=0}^{N-1} x_1[((m))_N]x_2[((n-m))_N]$	$X_1[k]X_2[k]$
$x_1[n]x_2[n]$	$N\sum_{l=0}^{N-1} X_1[l]X_2[k-l]$
$x[n] = \begin{cases} 1, & \lvert n\rvert \leq M \\ 0, & M < \lvert n\rvert \leq N/2 \end{cases}$ $x[n] = x[n+N]$	$X[k] = \frac{\sin\left(k\frac{\omega_0}{2}(2M+1)\right)}{N\sin\left(k\frac{\omega_0}{2}\right)}$
$x[n] = e^{jp\omega_0 n}$	$X[k] = \begin{cases} 1, & k \in p + N\mathbb{Z} \\ 0, & \text{otherwise} \end{cases}$
$x[n] = \cos(p\omega_0 n)$	$X[k] = \begin{cases} \frac{1}{2}, & k \in \pm p + N\mathbb{Z} \\ 0, & \text{otherwise} \end{cases}$
$x[n] = \sin(p\omega_0 n)$	$X[k] = \begin{cases} \frac{1}{2j}, & k \in p + N\mathbb{Z} \\ \frac{-1}{2j} & k \in -p + N\mathbb{Z} \\ 0, & \text{otherwise} \end{cases}$
$x[n] = 1$	$X[k] = \begin{cases} N, & k \in N\mathbb{Z} \\ 0, & \text{otherwise} \end{cases}$
$x[n] = \sum_{p=-\infty}^{\infty} \delta[n-pN]$	$X[k] = 1$

10.2 Basic Fourier Series (FS) Pairs

Time Domain	Frequency Domain
$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi kt/T}$ period = T	$X[k] = \frac{1}{T}\int_0^T x(t) e^{-j2\pi kt/T}\, dt$ $\Omega_0 = \frac{2\pi}{T}$
$x(t) = \begin{cases} 1, & \lvert t\rvert \le T_0 \\ 0, & T_0 < \lvert t\rvert \le T/2 \end{cases}$	$X[k] = \frac{\sin(k\Omega_0 T_0)}{k\pi}$
$x(t) = e^{jp\Omega_0 t}$	$X[k] = \delta[k-p]$
$x(t) = \cos(p\Omega_0 t)$	$X[k] = \frac{1}{2}\delta[k-p] + \frac{1}{2}\delta[k+p]$
$x(t) = \sin(p\Omega_0 t)$	$X[k] = \frac{1}{2j}\delta[k-p] - \frac{1}{2j}\delta[k+p]$
$x(t) = \sum_{p=-\infty}^{\infty} \delta(t - pT)$	$X[k] = \frac{1}{T}$

$$
\begin{aligned}
x'(t) &\longleftrightarrow \frac{j2\pi k}{T} X[k] \\
x(t - t_0) &\longleftrightarrow e^{-j2\pi k t_0/T} X[k] \\
x_1 * x_2(t) &\longleftrightarrow T X_1[k] X_2[k] \\
x_1(t) x_2(t) &\longleftrightarrow X_1 * X_2[k]
\end{aligned}
$$

10.3 Basic Discrete-Time Fourier Transform (DTFT) Pairs

Time Domain	Frequency Domain
$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,d\omega$	$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$
$x[n] \in \mathbb{R}$	$X(e^{j\omega}) = X^*(e^{j\omega}),\ \lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$
$x[n-n_d], \quad nx[n]$	$e^{-j\omega n_d}X(e^{j\omega}), \quad -jX'(e^{j\omega})$
$x[n] * y[n], \quad x[n]y[n]$	$X(e^{j\omega})Y(e^{j\omega}), \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})\,d\theta$
$x[n] = \begin{cases} 1, & \lvert n\rvert \le M \\ 0, & \text{otherwise} \end{cases}$	$X(e^{j\omega}) = \frac{\sin\left[\omega\left(\frac{2M+1}{2}\right)\right]}{\sin\left(\frac{\omega}{2}\right)}$
$x[n] = \alpha^n u[n], \quad \lvert\alpha\rvert < 1$	$X(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega}}$
$x[n] = \delta[n]$	$X(e^{j\omega}) = 1$
$x[n] = u[n]$	$X(e^{j\omega}) = \frac{1}{1-e^{-j\omega}} + \pi\sum_{p=-\infty}^{\infty}\delta(\omega - 2\pi p)$
$x[n] = \frac{\sin(Wn)}{\pi n}, \quad 0 < W \le \pi$	$X(e^{j\omega}) = \begin{cases} 1, & \lvert\omega\rvert \le W \\ 0, & W < \lvert\omega\rvert \le \pi \end{cases}$ $X(e^{j\omega})$ is 2π periodic
$x[n] = (n+1)\alpha^n u[n]$	$X(e^{j\omega}) = \frac{1}{(1-\alpha e^{-j\omega})^2}$
$x[n] = \cos(\omega_1 n)$	$X(e^{j\omega}) = \pi\sum_{k=-\infty}^{\infty}\delta(\omega-\omega_1-k2\pi)+\delta(\omega+\omega_1-k2\pi)$
$x[n] = \sin(\omega_1 n)$	$X(e^{j\omega}) = \frac{\pi}{j}\sum_{k=-\infty}^{\infty}\delta(\omega-\omega_1-k2\pi)-\delta(\omega+\omega_1-k2\pi)$
$x[n] = e^{j\omega_1 n}$	$X(e^{j\omega}) = 2\pi\sum_{k=-\infty}^{\infty}\delta(\omega-\omega_1-k2\pi)$
$x[n] = \sum_{k=-\infty}^{\infty}\delta(n-kN)$	$X(e^{j\omega}) = \frac{2\pi}{N}\sum_{k=-\infty}^{\infty}\delta\left(\omega - \frac{k2\pi}{N}\right)$

10.4 Basic Fourier Transform (FT) Pairs

Time Domain	Frequency Domain
$x(t) = \frac{1}{2\pi}\int_{\mathbb{R}} X(j\Omega)e^{j\Omega t}\,dt$	$X(j\Omega) = \int_{\mathbb{R}} x(t)e^{-j\Omega t}\,d\Omega$
$x(t) = \begin{cases} 1, & \lvert t\rvert \le T_0 \\ 0, & \text{otherwise} \end{cases}$	$X(j\Omega) = 2\frac{\sin(\Omega T_0)}{\Omega}$
$x(t) = \frac{\sin(Wt)}{\pi t}$	$X(j\Omega) = \begin{cases} 1, & \lvert\Omega\rvert \le W \\ 0, & \text{otherwise} \end{cases}$
$x(t) = \delta(t), \quad x(t) = 1$	$X(j\Omega) = 1, \quad X(j\Omega) = 2\pi\delta(\Omega)$
$x(t) = u(t)$	$X(j\Omega) = \frac{1}{j\Omega} + \pi\delta(\Omega)$
$x(t) = e^{-at}u(t), \quad Re(a) > 0$	$X(j\Omega) = \frac{1}{a+j\Omega}$
$x(t) = te^{-at}u(t), \quad Re(a) > 0$	$X(j\Omega) = \frac{1}{(a+j\Omega)^2}$
$x(t) = e^{-a\lvert t\rvert}, \quad a > 0$	$X(j\Omega) = \frac{2a}{a^2+\Omega^2}$
$x(t) = \frac{1}{\sqrt{2\pi}}e^{-t^2/2}$	$X(j\Omega) = e^{-\Omega^2/2}$
$x(t) = \cos(\Omega_0 t)$	$X(j\Omega) = \pi\delta(\Omega - \Omega_0) + \pi\delta(\Omega + \Omega_0)$
$x(t) = \sin(\Omega_0 t)$	$X(j\Omega) = \frac{\pi}{j}\delta(\Omega - \Omega_0) - \frac{\pi}{j}\delta(\Omega + \Omega_0)$
$x(t) = \sum_{n=-\infty}^{\infty}\delta(t - nT_s)$	$X(j\Omega) = \frac{2\pi}{T_s}\sum_{k=-\infty}^{\infty}\delta\left(\Omega - k\frac{2\pi}{T_s}\right)$

$$x'(t) \longleftrightarrow j\Omega X(j\Omega) \qquad x_1 * x_2(t) \longleftrightarrow 2\pi X_1(j\Omega)X_2(j\Omega)$$

$$\int_0^t x(s)\,ds \longleftrightarrow X(j\Omega)\left[\frac{1}{j\Omega} + \pi\delta(\Omega)\right] \qquad x(at) \quad \longleftrightarrow \quad \frac{1}{|a|}X(j\Omega/a)$$

10.5 Basic Laplace Transforms Pairs

Signal	Transform	Region of Convergence
$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\,ds$	$X(s) = \int_{\mathbb{R}} x(t)e^{-st}\,dt$	
$x(t) = u(t)$	$X(s) = \frac{1}{s}$	$Re(s) > 0$
$x(t) = tu(t)$	$X(s) = \frac{1}{s^2}$	$Re(s) > 0$
$x(t) = \delta(t-\tau), \quad \tau \geq 0$	$X(s) = e^{-s\tau}$	$s \in \mathbb{R}$
$x(t) = e^{-at}u(t)$	$X(s) = \frac{1}{s+a}$	$Re(s) > -a$
$x(t) = te^{-at}u(t)$	$X(s) = \frac{1}{(s+a)^2}$	$Re(s) > -a$
$x(t) = \cos(\Omega_1 t)u(t)$	$X(s) = \frac{s}{s^2+\Omega_1^2}$	$Re(s) > 0$
$x(t) = \sin(\Omega_1 t)u(t)$	$X(s) = \frac{\Omega_1}{s^2+\Omega_1^2}$	$Re(s) > 0$
$x(t) = e^{-at}\cos(\Omega_1 t)u(t)$	$X(s) = \frac{s+a}{(s+a)^2+\Omega_1^2}$	$Re(s) > -a$
$x(t) = e^{-at}\sin(\Omega_1 t)u(t)$	$X(s) = \frac{\Omega_1}{(s+a)^2+\Omega_1^2}$	$Re(s) > -a$
$x(t) = \delta(t-\tau), \quad \tau < 0$	$X(s) = e^{-s\tau}$	$s \in \mathbb{R}$
$x(t) = -u(-t)$	$X(s) = \frac{1}{s}$	$Re(s) < 0$
$x(t) = -tu(-t)$	$X(s) = \frac{1}{s^2}$	$Re(s) < 0$
$x(t) = -e^{-at}u(-t)$	$X(s) = \frac{1}{s+a}$	$Re(s) < -a$

10.6 Basic Z-Transforms Pairs

Signal	Transform	Region of Convergence
$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1}\, dz$	$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$	
$x[n] = \delta[n]$	$X(z) = 1$	$z \in \mathbb{C}$
$x[n] = u[n]$	$X(z) = \frac{1}{1-z^{-1}}$	$\|z\| > 1$
$x[n] = \alpha^n u[n]$	$X(z) = \frac{1}{1-\alpha z^{-1}}$	$\|z\| > \|\alpha\|$
$x[n] = n\alpha^n u[n]$	$X(z) = \frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\|z\| > \|\alpha\|$
$x[n] = \cos(\omega_1 n) u[n]$	$X(z) = \frac{1-z^{-1}\cos\omega_1}{1-z^{-1}2\cos\omega_1+z^{-2}}$	$\|z\| > 1$
$x[n] = \sin(\omega_1 n) u[n]$	$X(z) = \frac{z^{-1}\sin\omega_1}{1-z^{-1}2\cos\omega_1+z^{-2}}$	$\|z\| > 1$
$x[n] = r^n \cos(\omega_1 n) u[n]$	$X(z) = \frac{1-z^{-1}r\cos\omega_1}{1-z^{-1}2r\cos\omega_1+r^2 z^{-2}}$	$\|z\| > r$
$x[n] = r^n \sin(\omega_1 n) u[n]$	$X(z) = \frac{z^{-1}r\cos\omega_1}{1-z^{-1}2r\cos\omega_1+r^2 z^{-2}}$	$\|z\| > r$
$x[n] = u[-n-1]$	$X(z) = \frac{1}{1-z^{-1}}$	$\|z\| < 1$
$x[n] = -\alpha^n u[-n-1]$	$X(z) = \frac{1}{1-\alpha z^{-1}}$	$\|z\| < \|\alpha\|$
$x[n] = -n\alpha^n u[-n-1]$	$X(z) = \frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\|z\| < \|\alpha\|$

$$nx[n] \longleftrightarrow -zX'(z)$$