Let T be some index set and let $\mathbf{R}^T$ denote the collection of real-valued functions on T. Let $\mathcal{R}^T$ be the smallest σ-field containing the sets of the form $\{\phi \in \mathbf{R}^T : \phi(t) \in B, t \in T, B \in \mathcal{B}_1\}$. Here $\mathcal{B}_k$ denotes the Borel subsets of $\mathbf{R}^k$ for $k \geq 1$. Let $\mathcal{R}_0^T$ denote the collection of sets

$$\{\phi \in \mathbf{R}^T : (\phi(t_1), ..., \phi(t_k)) \in B, t_j \in T, j = 1, ...k, B \in \mathcal{B}_k, k \geq 1\}. \tag{1}$$

P1: Show that $\mathcal{R}_0^T$ is a field.

A *system* $\mathbf{S}_T(\mu)$ gives, for each $\mathbf{t} \in T^k$, $k \geq 1$, a probability measure $\mu_{\mathbf{t}}$ on $(\mathbf{R}^k, \mathcal{B}_k)$. A system $\mathbf{S}_T(\mu)$ is *consistent* if for all $k \geq 1$, $t_j \in T$ and $B_j \in \mathcal{B}_1$, $1 \leq j \leq k$, and permutation π of $(1, ..., k)$,

$$\mu_{(t_1,...,t_k)}(B_1 \times ... \times B_k) = \mu_{(t_{\pi_1},...,t_{\pi_k})}(B_{\pi_1} \times ... \times B_{\pi_k}) \tag{2}$$

and

$$\mu_{(t_1,...,t_k)}(B_1 \times ... \times B_k) = \mu_{(t_1,...,t_k,t_{k+1})}(B_1 \times ... \times B_k \times \mathbf{R}). \tag{3}$$

(2) and (3) are *Kolmogorov's consistency conditions.*

Let $(\Omega, \mathcal{F}, P)$ be a probability space. A T-indexed *stochastic process* on this space is a collection $\{X(t) : t \in T\}$ of random variables. This stochastic process has the system $\mathbf{S}_T(\mu)$ of *finite-dimensional distributions* if for all $k \geq 1$, $t_j \in T$ and $B_j \in \mathcal{B}_1$ for $1 \leq j \leq k$,

$$P((X(t_1), ..., X(t_k)) \in B_1 \times ... \times B_k) = \mu_{(t_1,...,t_k)}(B_1 \times ... \times B_k). \tag{4}$$

Theorem 1 *Kolmogorov's Existence Theorem:*

Version I: If $\mathbf{S}_T(\mu)$ is a consistent system, then there exists a probability measure P on $(\mathbf{R}^T, \mathcal{R}^T)$ such that the stochastic process $\{\omega(t) : t \in T\}$ for $\omega \in \mathbf{R}^T$ has $\mathbf{S}_T(\mu)$ as its system of finite-dimensional distributions.

Version II: If $\mathbf{S}_T(\mu)$ is a consistent system, then there exists on some probability space $(\Omega, \mathcal{F}, P)$ a stochastic process $\{X(t) : t \in T\}$ with $\mathbf{S}_T(\mu)$ as its system of finite-dimensional distributions.

Kolmogorov's Existence Theorem allows us to assume existence of a stochastic process after verifying the consistency of a given system of finite-dimensional distributions. The following example shows us though that the finite-dimensional distributions don't specify the path properties of a stochastic process.

Example: Let $(\Omega, \mathcal{F}, P) = ((0, 1], \mathcal{B}, \lambda)$ where $\mathcal{B}$ are the Borel sub-sets of the (0,1]. Let $T = (0, 1]$ as well. Let $X(t, \omega) = 0$ for all t and ω and let $Y(t, \omega) = \mathbf{1}_t(\omega)$. It is easy to see that the finite-dimensional distributions of both X and Y correspond to an unit mass at the origin in all dimensions. The paths, or realizations, of X, $\{X(t, \omega : t \in T\})$ are continuous; those of Y are not.

Recall that for a measurable space $(\Omega, \mathcal{F})$ and function $X : \Omega \to \mathbf{R}$. $\sigma(X)$, the σ-field generated by X, is the smallest σ-field that contains the sets $\{X^{-1}(B) : B \in \mathcal{B}_1\}$. (It is easy to check that actually $\sigma(X) = \{X^{-1}(B) : B \in \mathcal{B}_1\}$.)

Theorem 2 *Let $(\Omega, \mathcal{F})$ be a measurable space and $\{\phi(t,\omega) : t \in T\}$ be a family of $\mathbf{R}^T$-valued functions on Ω which are $\mathcal{F}$-measurable for each fixed $t \in T$. For $S \subset T$ let $\mathcal{F}_S = \sigma(\{\phi(t,\cdot) : t \in S\})$; that is $\mathcal{F}_S$ is the smallest σ-field that contains each of the sub-σ-fields $\mathcal{F}_t = \sigma(\{\phi(t,\cdot)\})$ for $t \in S$.*

(i) If $A \in \mathcal{F}_T$, $\omega \in A$ and $\phi(t,\omega) = \phi(t,\omega')$ for all $t \in T$, then $\omega' \in A$ as well.

(ii) If $A \in \mathcal{F}_T$, then $A \in \mathcal{F}_C$ for some countable subset C of T.

P2: Take $T = [0,1]$ and show that $\mathcal{C}([0,1])$, the set of continuous functions on [0,1], is not in $\mathcal{R}^T$. Hint: Use Theorem 2 above.

Let (T, d) be a separable metric space. We say $t_n \to t$ if $d(t_n, t) \to 0$. Let D be a countable dense subset of T. A function $\phi \in \mathcal{R}^T$ is *separable* with respect to D, or D-*separable*, if for all $t \in T$ there exists $\{t_n : n \geq 1\} \subset D$ with $t_n \to t$ and $\phi(t_n) \to \phi(t)$. $\{X_t : t \in T\}$ defined on $(\Omega, \mathcal{F}, P)$ is a *separable stochastic process* with respect to D if there exists $N \in \mathcal{F}$ with $P(N) = 0$ and for all $\omega \in N^c$ the realization $X(\cdot,\omega)$ is D-separable.

Theorem 3 *Suppose that $\{X_t : t \in T\}$ is a stochastic process defined on the probability space $(\Omega, \mathcal{F}, P)$. Then there exists defined on the same probability space a separable process $\{\tilde{X}_t : t \in T\}$ with $P(\tilde{X}_t = X_t) = 1$ for each fixed t.*

Rephrased, this theorem tells us that for any stochastic process indexed by elements of a separable metric space we have a separable version.

P3: Verify that $\{X_t : t \in T\}$ and its separable version $\{\tilde{X}_t : t \in T\}$ have the same finite-dimensional distributions.

P4: Construct a consistent system of finite-dimensional distributions on $T = \{0, 1, 2, ...\}$.

P5: Construct a consistent system of finite-dimensional distributions on $T = [0, 1]$.

P6: Construct a consistent system of finite-dimensional distributions on $T = [0, 1]^n$.

Let X be a random variable defined on the probability space $(\Omega, \mathcal{F}, P)$ with $E|X| < \infty$. Let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$. The *conditional expectation of X given $\mathcal{G}$*, $E(X|\mathcal{G})$, is a random variable satisfying the following 2 conditions:

(1) $E(X|\mathcal{G})$ is measurable $\mathcal{G}$ and $E|E(X|\mathcal{G})| < \infty$.

(2) $E(E(X|\mathcal{G})\mathbf{1}_G) = E(X\mathbf{1}_G)$ for all $G \in \mathcal{G}$.

A *version* of the *conditional probability of A given $\mathcal{G}$*, $P(A|\mathcal{G})$, is given by $E(\mathbf{1}_A|\mathcal{G})$.

Note: the existence of conditional expectations and probabilities is an easy consequence of the Radon-Nikodym theorem.

P7*: Let $(\Omega, \mathcal{F}, P) = ((0,1], \mathcal{B}, \lambda)$ where $\mathcal{B}$ denotes the Borel subsets of (0,1] and λ denotes Lebesgue measure. Let $X(\omega) = \omega^2$ and for $n \geq 1$ let

$$\mathcal{G}_n = \sigma(\{(\frac{k-1}{n}, \frac{k}{n}] : 1 \leq k \leq n\}).$$

Find $E(X|\mathcal{G}_n)$.

The theorem that can be used to justify the calculations providing the solution to P7 is as follows.

Theorem 4 *Let $\mathcal{G} = \sigma(\mathcal{P})$ where $\mathcal{P}$ is a π-system and $\Omega = \cup A_k$ for some finite or countable sequence of sets $\{A_k\} \subset \mathcal{P}$. Then a $\mathcal{G}$-measurable function g is a version of $E(X|\mathcal{G})$ if for all $G \in \mathcal{P}$,*

$$E(g\mathbf{1}_G) = E(X\mathbf{1}_G).$$

The following theorem lists many important properties of conditional expectation.

Theorem 5 *If X, Y and $\{X_n : n \geq 1\}$ all have finite expectations, then the following hold a.s. P.*

(a) If $X = a$ a.s. P, then $E(X|\mathcal{G}) = a$.

(b) $E(aX + bY|\mathcal{G}) = aE(X|\mathcal{G}) + bE(Y|\mathcal{G})$.

(c) If $X \leq Y$ a.s. P, then $E(X|\mathcal{G}) \leq E(Y|\mathcal{G})$.

(d) $|E(X|\mathcal{G})| \leq E(|X||\mathcal{G})$.

(e) For a convex function ϕ, $\phi(E(X|\mathcal{G})) \leq E(\phi(X)|\mathcal{G})$.

(f) If $X_n \to X$ a.s. P with $|X_n| \leq Y$ for all n, then $E(X_n|\mathcal{G}) \to E(X|\mathcal{G})$.

The following theorem gives iteration and factorization of conditional expectations for integrable random variables.

Theorem 6 *Suppose throughout that X is integrable. Then the following hold a.s. P.*

(a) If $\mathcal{H} \subset \mathcal{G} \subset \mathcal{F}$ with $\mathcal{H}$ and $\mathcal{G}$ both sub-σ-fields of $\mathcal{F}$, then

$$E(E(X|\mathcal{G})|\mathcal{H}) = E(X|\mathcal{H}).$$

(b) If Y is measurable $\mathcal{G}$ and XY is integrable, then $E(XY|\mathcal{G}) = YE(X|\mathcal{G})$.

P8*: ($\uparrow$ P7) (a) Find expressions for $E(E(X|\mathcal{G}_n)|\mathcal{G}_{n+1})$ and $E(E(X|\mathcal{G}_{n+1})|\mathcal{G}_n)$. Hint: first consider

$$\omega \in (\frac{j-1}{n+1}, \frac{j}{n+1}] \qquad \text{where} \qquad \frac{j-1}{n+1} < \frac{k}{n} \leq \frac{j}{n+1}.$$

Then reverse the roles of n and $n+1$.

(b) Give expressions for $E(E(X|\mathcal{G}_n)|\mathcal{G}_{2n})$ and $E(E(X|\mathcal{G}_{2n})|\mathcal{G}_n)$. (This should be much easier than (a).)

Note: frequently $E(X|Y)$ is written for $E(X|\sigma(Y))$.

Theorem 7 *X is measurable $\sigma(Y)$ if and only if $X = \phi(Y)$ for some measurable function ϕ.*

P9: Give a proof of Theorem 7. Hint: start with a simple random variable X. Then go to an approximating sequence of simple functions.

P10: Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $(X,Y) : \Omega \to \{0,1,2,...\}^2$ be measurable with

$$P((X,Y) \in B) = \sum_{(j,k)\in B} p(j,k).$$

Give expressions for $P(A|X)$ and $E(X|Y)$.

A stochastic process $\{X_t : t \in T\}$ defined on $(\Omega, \mathcal{F}, P)$ has *state space* S if $X_t : \Omega \to S$ for all $t \in T$. A (discrete) *filtration* in $\mathcal{F}$ is a sequence of σ-fields $\{\mathcal{F}_n : n \geq 0\}$ with $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ for all $n \geq 0$. A stochastic process $\{X_n : n \geq 0\}$ is a Markov chain if it has a discrete state space and for the filtration $\{\mathcal{F}_n : n \geq 0\}$ given by $\mathcal{F}_n = \sigma(X_0, X_1, ..., X_n)$

$$P(X_{n+1} = b|\mathcal{F}_n) = P(X_{n+1} = b|X_n).$$

This is known as the *Markov property*.

P11: Discuss the definition of Markov chain as give by DK on pages 267-268. Note: 'adapted' is defined on page 4.

The *lim sup* of a sequence of sets $\{A_n : n \geq 1\}$ is given by

$$\limsup A_n = \bigcap_{n\geq 1} \bigcup_{k\geq n} A_k.$$

This is commonly referred to as "A_n infinitely often" and written $[A_n\ i.o.]$. The *lim inf* of $\{A_n : n \geq 1\}$ is given by

$$\liminf A_n = \bigcup_{n\geq 1} \bigcap_{k\geq n} A_k.$$

Notice that $[A_n\ i.o.]^c = \liminf(A_n^c)$.

P12: (a) Show that $\omega \in [A_n\ i.o.]$ if and only if ω is in infinitely many of the A_n's.

(b) Show that $\limsup_n \mathbf{1}_{A_n} = \mathbf{1}_{[A_n i.o.]}$.

The following simple lemma is one of the most important tools of probability theory. We will use it in looking at the recurrence of events for particular stochastic processes.

Lemma 8 *The First Borel-Cantelli Lemma.*

Let $\{A_n : n \geq 1\}$ *be a sequence of sets. If* $\sum_n P(A_n) < \infty$ *then* $P(A_n \text{ i.o.}) = 0$.

<u>Example 1</u>: Consider simple random walk $\{S_n : n \geq 0\}$ with $S_n = 0$ and steps of height 1 with probability p and height -1 with probability $q = 1 - p$ taken independently at each epoch $n \geq 1$. Letting $A_n = [S_n = 0]$, we see that if $p \neq q$ then

$$\sum_{n \geq 1} P(A_n) = (1 - 4pq)^{-1/2} - 1 < \infty.$$

This tells us that 0 is not a recurrent state for S_n if $p \neq q$. If we take $p = q = 1/2$ we find that $\sum_{n \geq 1} P(A_n) = \infty$. This doesn't directly help us with recurrence though. The A_n's here are clearly not independent and we can't rely on the Second Borel-Cantelli Lemma.

Lemma 9 *The Second Borel-Cantelli Lemma.*

Let $\{A_n : n \geq 1\}$ *be a sequence of independent sets. If* $\sum_n P(A_n) = \infty$ *then* $P(A_n \text{ i.o.}) = 1$.

P13: The calculation in Example 1 relied on the identity

$$\binom{2k}{k} = \binom{-1/2}{k}(-4)^k.$$

Show that this is indeed true.

P14: Suppose that the random walk of Example 1 starts at some given integer level a; that is suppose that $S_0 = a$ with probability 1. Let $N_n(a) = \sum_{k=1}^n \mathbf{1}_{\{a\}} S_k$ and $e_n(a) = E(N_n(a))$. For a given value of p show that $\lim_{n \to \infty} e_n(a)$ is either finite or infinite for all finite a.

The stochastic process $\{X_n : n \geq 0\}$ is a *time-homogeneous* Markov chain if it is a Markov chain whose *transition probabilities* $P(X_{n+1} = b | X_n)$ are the same for all n. Let

$$p(a, b) = P(X_{n+1} = b | X_n = a).$$

It is natural to use the matrix $\mathbf{P}$ of transition probabilities

$$(\mathbf{P})_{a,b} = p(a, b) \quad \text{for} \quad a, b \in S$$

to describe and characterize these transition probabilities. The simple random walk described above is an example of a time-homogeneous Markov chain.

<u>Example 2</u>: Let $\{T_n : n \geq 0\}$ be simple random walk with state space $S = \{0, 1, ..., N\}$ where the fixed level N is a reflecting barrier and level 0 is an absorbing boundary. For $0 \leq k \leq N$, let

$$\alpha_k = P(\cup_{n \geq 0}[T_n = N] | T_0 = k).$$

It is easy to see that $\alpha_0 = 0$, $\alpha_N = 1$ and, using the Markov property, for $1 \leq k \leq N-1$,

$$\alpha_k = p\alpha_{k+1} + q\alpha_{k-1}.$$

In the case $p = q = 1/2$, this gives

$$\alpha_k = \frac{k}{N}.$$

P15: Find the α_k's for $p \neq q$.

By reversing the roles of the absorbing and reflecting boundaries in this example, it is easy to see that for the unrestricted simple symmetric (p=q=1/2) random walk S_n of Example 1,

$$P(\cup_{n\geq 1}[S_1 < N, S_2 < N, ...S_{n-1} < N, S_n = 0] | S_0 = k) = 1 - \frac{k}{N}$$

for $0 < k < N$. Thus

$$\lim_{N\to\infty} P(\cup_{n\geq 1}[S_1 < N, S_2 < N, ...S_{n-1} < N, S_n = 0] | S_0 = k) = 1.$$

This shows that for symmetric random walk, for any initial level k, with probability 1 the walk returns to level 0 before going off to ∞. The walk should then return to 0 infinitely often with probability 1 because after any given return, it will return again with probability 1. These ideas will be made more precise a bit later.

Let $\{X_n : n \geq 0\}$ be a time-homogeneous Markov chain with transition matrix $\mathbf{P}$. If X_0 has distribution given by $P(X_0 = a) = \lambda(a)$ for a probability measure λ on S we say that $\{X_n : n \geq 0\}$ is Markov$(\lambda, \mathbf{P})$. Here are some properties of this stochastic process.

Theorem 10 *If $\{X_n : n \geq 0\}$ is Markov$(\lambda, \mathbf{P})$ then*

a. $P(X_0 = a_0, X_1 = a_1, ..., X_n = a_n) = \lambda(a_0)p(a_0, a_1)...p(a_{n-1}, a_n)$,

b. conditional on the event $[X_m = a]$, $\{X_{n+m} : n \geq 0\}$ *is Markov*$(\delta_a, \mathbf{P})$ *and independent of* $(X_0, ..., X_m)$,

c. $p^{(n)}(a, b) := P(X_{n+m} = b | X_m = a) = (\mathbf{P}^n)_{(a,b)}$,

and, taking λ to represent the vector $(\lambda(a))$,

d. $P(X_n = b) = \sum_{a\in S} \lambda(a)p^{(n)}(a, b) = (\lambda^t \mathbf{P}^n)_b$.

P16: Suppose that $\{X_n : n \geq 0\}$ is Markov$(\lambda, \mathbf{P})$. Verify that the pair λ and $\mathbf{P}$ determine the finite dimensional distributions of $\{X_n : n \geq 0\}$.

P17: (With thanks to Norris.) (a) A flea hops about at random on the vertices of a triangle, with all jumps equally likely. Find the probability that the flea is back at his starting vertex after exactly n jumps.

(b) A second flea hopping about on the triangle is twice as likely to jump clockwise as counter-clockwise. Find the probability that this flea is back at his starting vertex after exactly n jumps.

P18: (With more thanks to Norris.) Let X_0 be a random variable with countable state space S and let $U_1, U_2, ...$ be a sequence of independent r.v.'s, all distributed uniformly on the unit interval. Let the function

$$G : S \times [0,1] \to S$$

and define iteratively

$$X_{n+1} = G(X_n, U_{n+1}).$$

Show that $\{X_n : n \geq 0\}$ is a Markov chain and express the transition matrix $\mathbf{P}$ in terms of the function G. Can any transition matrix $\mathbf{P}$ be expressed in terms of some such G?

P19: (With thanks to Bhattacharya and Waymire.) Finding a transition matrix $\mathbf{P}$. A reservoir has capacity N (integer) units. The daily runoff into the reservoir is assumed to be modeled by independent identically distributed random variables with state space $\{0, 1, 2, ...\}$. For $k \geq 0$, let p_k denote the probability of the runoff entering the reservoir on a given day being exactly k units. The reservoir releases exactly 1 unit of water daily if it is not empty or at capacity. If it is empty it doesn't release anything; if it is at capacity it has to release everything over N units. Let X_n denote the units of water in the reservoir on day n. Find the transition matrix $\mathbf{P}$ for this stochastic process.

If for states a and b of a Markov chain, $p_{a,b}^{(n)} > 0$ for some $n > 0$, then we say that a *leads to* b and write $a \to b$. If both $a \to b$ and $b \to a$, then we say that a and b *communicate*. This equivalence relation can be used to partition S into *communicating classes*. In particular a class C of states is *closed* if for $a \in C$ and $a \to b$ then $b \in C$ as well.

Define the *hitting time* of a set $A \subset S$ via

$$H^A(\omega) = \inf\{n \geq 0 : X_n(\omega) \in A\}.$$

Let

$$h_b^A = P(H^A < \infty | X_0 = b).$$

Theorem 11 *The vector* $\{h_b^A : b \in S\}$ *is the minimal non-negative solution to*

$$h_b^A = \mathbf{1}_A(b) + \sum_{c \in S} p_{b,c} h_c^A \mathbf{1}_{A^c}(c).$$

Let $\mu_b^A = E(H^A | X_0 = b)$.

Theorem 12 *The vector* $\{\mu_b^A : b \in S\}$ *is the minimal non-negative solution to*

$$\mu_b^A = \Big(1 + \sum_{c \in A^c} p_{b,c} \mu_c^A \mathbf{1}_{A^c}(c)\Big) \mathbf{1}_{A^c}(b).$$

P20: Give a proof of Theorem 12.

P21: Calculate $\mu_b^{\{0\}}$ for the restricted random walk $\{T_n : n \geq 0\}$ with a reflecting barrier at level N and absorbing barrier at 0.

Let $\{\mathcal{F}_n : n \geq 0\}$ be a filtration on the probability space $(\Omega, \mathcal{F}, P)$. A random variable $\tau : \Omega \to \{0, 1, ...\} \cup \{\infty\}$ is a *stopping time* if

$$[\tau \leq n] \in \mathcal{F}_n \quad \text{for all} \quad n.$$

For a stopping time τ define

$$\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap [\tau \leq n] \in \mathcal{F}_n : 0 \leq n < \infty\}.$$

P22: Verify that $\mathcal{F}_\tau$ is a σ-field.

Theorem 13 *Strong Markov property. If $\{X_n : n \geq 0\}$ is Markov$(\lambda, \mathbf{P})$ and τ is a stopping time relative to the filtration $\{\mathcal{F}_n : n \geq 0\}$, then conditional on $\tau < \infty$ and $X_\tau = a$, $\{X_{\tau+n} : n \geq 0\}$ is Markov$(\delta_a, \mathbf{P})$ and independent of $(X_0, ..., X_\tau)$.*

Let $\{X_n : n \geq 0\}$ be Markov$(\lambda, \mathbf{P})$. From here on, use the simplifying notation

$$P_a(B) = P(B|X_0 = a).$$

State $a \in S$ is *recurrent* if $P_a([X_n = a]i.o.) = 1$ and *transient* if $P_a([X_n = a]i.o.) = 0$. The *first passage time* to state $a \in S$ is given by

$$T_a = \inf\{n \geq 1 : X_n = a\}.$$

It is understood that $T_a := \infty$ on the set $\cap_{n \geq 1}[X_n \neq a]$. The sequence of kth passage times to state a, $\{T_a^{(k)} : k \geq 0\}$, is given by $T_a^{(0)} = 0$ and for $k \geq 1$,

$$T_a^{(k)} = \inf\{n \geq T_a^{(k-1)} + 1 : X_n = a\}.$$

P23: Show that for each k, $T_a^{(k)}$ is a stopping time.

P24: Show that any fixed non-negative integer m is a stopping time.

The *inter-arrival times* between visits to a, or excursion lengths between visits to a, are defined via

$$S_a^{(k)} = T_a^{(k)} - T_a^{(k-1)}$$

on the set $[T_a^{(k-1)} < \infty]$. If $T_a^{(k-1)} = \infty$ then define $S_a^{(k)} := 0$.

Lemma 14 *For each $k \geq 1$, conditional on $[T_a^{(k)} < \infty]$, the inter-arrival time $S_a^{(k+1)}$ is independent of $\{X_j : 0 \leq j \leq T_a^{(k)}\}$.*

P25: Give a proof of Lemma 14.

Define

$$f_a = P_a(T_a < \infty)$$

and

$$V_a = \sum_{n=0}^{\infty} \mathbf{1}[X_n = a].$$

Lemma 15 *For each $k \geq 0$*

$$P_a(V_a > k) = P_a(T_a^{(k)} < \infty) = f_a^k.$$

P26: Give a proof of Lemma 15. Hint: use induction.

Corollary 16 *$E_a(V_a)$ is finite if $f_a < 1$ and infinite if $f_a = 1$.*

Theorem 17 *Each state in a Markov chain is recurrent or transient. Additionally:*

a. If $f_a = 1$, state a is recurrent and $\sum_{n \geq 0} p^{(n)}(a,a) = \infty$.

b. If $f_a < 1$, state a is transient and $\sum_{n \geq 0} p^{(n)}(a,a) < \infty$.

P27: (With thanks to Norris) Let $\{X_n : n \geq 0\}$ be a Markov chain with state space $\{0, 1, 2, ...\}$ and transition probabilities given by $p(0,1) = 1$, $p(k, k+1) + p(k, k-1) = 1$, and

$$p(k, k+1) = \left(\frac{k+1}{k}\right)^2 p(k, k-1) \quad \text{for} \quad k \geq 1.$$

(This can be thought of as a birth-death chain with guaranteed rebirth.)

(a) Show that if $X_0 = 0$, then the probability that X_n never returns to 0 is $1 - 6/\pi^2$.

(b) Show that $P(X_n \to \infty \text{ as } n \to \infty) = 1$.

(c) Change the problem by taking the transition probabilities to be

$$p(k, k+1) = \left(\frac{k+1}{k}\right)^\alpha p(k, k-1) \quad \text{for} \quad k \geq 1$$

with $\alpha \in (0, \infty)$ being a fixed parameter. Find the value of $P(X_n \to \infty \text{ as } n \to \infty)$.

P28: (Random Fibonacci's) Let $\{Y_n : n \geq 1\}$ be a sequence of i.i.d. symmetric Bernoulli r.v.'s. (That is; $P(Y_n = 1) = P(Y_n = -1) = 1/2$ for each n.) Define the random sequence $\{F_n : n \geq 0\}$ via $F_0 = 0$, $F_1 = 1$ and, for larger n,

$$F_{n+1} = F_n + Y_n F_{n-1}.$$

(a) Is $\{F_n : n \geq 0\}$ a Markov chain?

(b) Show that $\{X_n = (F_{n-1}, F_n) : n \geq 1\}$ is a Markov chain.

(c) Diagram the possible values of the first few steps of this chain.

(d) Show that the probability of hitting (1,1) starting from (1,2) is $(3 - \sqrt{5})/2$.

(e) Show that $\{X_n : n \geq 1\}$ is transient and, with probability 1, $F_n \to \infty$ as $n \to \infty$.

Theorem 18 *States in a communicating class C are either all recurrent or transient.*

A Markov chain is *irreducible* if the state space forms a single communicating class. Often this is stated in terms of the transition matrix $\mathbf{P}$.

Theorem 19 *a. If C is a recurrent class then it is closed.*

b. Every finite closed class is recurrent.

c. If $\mathbf{P}$ is irreducible and recurrent then $f_a = 1$ for all $a \in S$.

P29: Give a proof of Theorem 19.

A measure λ on S is an *invariant* measure for $\mathbf{P}$ if

$$\lambda^t \mathbf{P} = \lambda^t.$$

Lemma 20 *If $\{X_n : n \geq 0\}$ is Markov$(\lambda, \mathbf{P})$ with λ invariant for $\mathbf{P}$, then for any $m \geq 1$, $\{X_{m+n} : n \geq 0\}$ is also Markov$(\lambda, \mathbf{P})$.*

P30: Give a proof of Lemma 20.

Theorem 21 *If S is finite and for some $b \in S$*

$$\lim_{n\to\infty} p_{b,a}^{(n)} = \pi_a \quad \textit{for each} \quad a \in S,$$

then $\pi = (\pi_a)$ is an invariant probability measure for the transition matrix $\mathbf{P}$.

Fix $b \in S$ and let

$$\gamma_a^b = E_b \sum_{n=0}^{T_b - 1} \mathbf{1}[X_n = a].$$

This is the expected number of visits to state a during a sojourn away from state b. Notice that if a return to b occurs with probability 1; that is, if $f_b = 1$; then this could be written as

$$\gamma_a^b = E_b \sum_{n=1}^{T_b} \mathbf{1}[X_n = a].$$

Theorem 22 *If* $\mathbf{P}$ *is irreducible and recurrent, then*

1. $\gamma_b^b = 1$
2. $(\gamma_a^b)^t \mathbf{P} = (\gamma_a^b)^t$, *and*
3. $0 < \gamma_a^b < \infty$ *for all* $a \in S$.

Note: the proof of the theorem uses the strong Markov property via the following calculation

$$\begin{aligned} P_b(X_{n-1} = c, X_n = a, T_b \geq n) &= P_b(X_n = a | X_{n-1} = c, T_b \geq n) P(X_{n-1} = c, T_b \geq n) \\ &= p(c, a) P(X_{n-1} = c, T_b \geq n). \end{aligned}$$

Theorem 23 *If* $\mathbf{P}$ *is irreducible, then* $(\gamma_a^b : a \in S)$ *is the minimal non-negative solution to* $\lambda^t \mathbf{P} = \lambda^t$ *with* $\lambda_b = 1$. *Furthermore, if* $\mathbf{P}$ *is recurrent and* $\lambda_b = 1$, *then* $\lambda_a = \gamma_a^b$ *for all* $a \in S$.

P31: Sketch the proof of the minimality part of Theorem 23.

Let the expected return time to state b be given by

$$m_b = E_b T_b.$$

If $m_b < \infty$, then state b is *positive recurrent*. Notice that positive recurrence implies recurrence. If state b is recurrent with $m_b = \infty$, then b is *null recurrent*. In either case

$$\sum_{a \in S} \gamma_a^b = \sum_{a \in S} E_b \sum_{n=0}^{T_b - 1} \mathbf{1}[X_n = a] = E_b \sum_{n=0}^{T_b - 1} \sum_{a \in S} \mathbf{1}[X_n = a] = E_b T_b = m_b.$$

P32: Show that unrestricted symmetric random walk is null recurrent.

Theorem 24 *If* $\mathbf{P}$ *is irreducible, then the following are equivalent.*

1. *All states* $a \in S$ *are positive recurrent.*
2. *Some state* $b \in S$ *is positive recurrent.*
3. $\mathbf{P}$ *has an invariant probability measure* π *with* $\pi_a = 1/m_a$ *for all* $a \in S$.

P33: Go back to the flea problem (P17) and find the invariant probability measures.

A state $b \in S$ is *aperiodic* if for all n sufficiently large $p^{(n)}(b, b) > 0$.

P34: Show that b is aperiodic if and only if the least common divisor of $\{n : p^{(n)}(b, b) > 0\}$ is 1.

Lemma 25 *If* $\mathbf{P}$ *is irreducible and and state* b *is aperiodic then all states are aperiodic.*

Theorem 26 *If* $\mathbf{P}$ *is irreducible and aperiodic with invariant probability distribution* π *and* $\{X_n : n \geq 0\}$ *is Markov*$(\lambda, \mathbf{P})$ *for some probability distribution* λ *on* S*, then for all states* $b \in S$

$$\lim_{n\to\infty} P(X_n = b) = \pi_b.$$

Notice that taking $\lambda = \delta_a$ gives $\lim_{n\to\infty} p^{(n)}(a, b) = \pi_b$.

P35: (Norris) A particle moves around on the vertices of a cube randomly. At each step it is equally likely to move to each of the three adjacent vertices independently of past positions. Let α denote the initial vertex and ζ the opposite vertex. Find the following:

(a.) $E_\alpha T_\alpha$, the expected number of steps until the particle first returns to α.

(b.) The expected number of visits to ζ before returning to α.

(c.) The expected number of steps until the first visit to ζ.

P36: (Norris) An approach to computation of the invariant distribution. Let S be finite and $\mathbf{P}$ be a corresponding transition matrix. Let the vector j have $j_a = 1$ for all $a \in S$ and the matrix $\mathbf{J}$ have $J_{a,b} = 1$ for all $a, b \in S$. Show that π is invariant for $\mathbf{P}$ if and only if

$$\pi^t(\mathbf{I} - \mathbf{P} + \mathbf{J}) = j^t.$$

Assume further that $\mathbf{P}$ is irreducible and show that $\mathbf{I} - \mathbf{P} + \mathbf{J}$ is invertible.

P37: Throw a fair six-sided die repeatedly. Let X_n denote the sum of the first n throws. Find

$$\lim_{n\to\infty} P(X_n \text{ is a multiple of } 13).$$

P38: Let $\{X_n : n \geq 0\}$ be a birth-death Markov chain with guaranteed rebirth. That is, let $\{X_n : n \geq 0\}$ be a Markov chain with transition matrix $\mathbf{P}$ given by $p(0, 1) = 1$ and, for $k \geq 1$, $p(k, k+1) = p_k$, $p(k, k-1) = q_k$ with $p_k + q_k = 1$ and both $p_k > 0$ and $q_k > 0$. Find examples of particular sequences $\{p_k : k \geq 1\}$ that give respectively, transience, recurrence, and positive recurrence.

Take $\mathbf{P}$ to be an irreducible transition matrix and suppose that $\{X_k : k \geq 0\}$ is Markov$(\lambda, \mathbf{P})$. Let

$$V_b(n) = \sum_{k=0}^{n-1} \mathbf{1}[X_k = b].$$

Recall that $m_b = E_b T_b$ where T_b is the first passage time to state b.

Proposition 1

$$\frac{V_b(n)}{n} \to \frac{1}{m_b} \quad as \quad n \to \infty \quad a.s.P.$$

Theorem 27 *Ergodic Theorem I: If* $\mathbf{P}$ *is positive recurrent with invariant probability distribution* π*, then for any bounded* $f : S \to \mathbf{R}$*,*

$$\frac{1}{n}\sum_{k=0}^{n-1} f(X_k) \to E_\pi f(X_0) \quad \textit{as} \quad n \to \infty \quad \textit{a.s.P.}$$

Here $E_\pi f(X_0) = \sum_{a \in S} f(a)\pi_a$.

Theorem 28 *Ergodic Theorem II: If* $\mathbf{P}$ *is positive recurrent with invariant probability distribution* π*, then for any* $f : S \to \mathbf{R}$ *with*

$$E_\pi |f(X_0)| < \infty,$$

$$\frac{1}{n}\sum_{k=0}^{n-1} f(X_k) \to E_\pi f(X_0) \quad \textit{as} \quad n \to \infty \quad \textit{a.s.P.}$$

P39: (Norris) A professor has N umbrellas. When walking to and from campus in the morning and afternoon she carries an umbrella if it is raining, and goes without if it is not raining. Suppose that it rains on each walk with probability p independently of past weather. Find the long-run proportion of walks on which the professor gets wet?

P40: (Norris) A promoter has persuaded an opera singer to undertake a long performance tour. Having a fine artistic temperament, she is liable to pull out each night with probability 1/2. Once this has happened she will not sing again until the promoter convinces her of his high regard. He does this by sending flowers every day until she returns. Flowers costing x thousand dollars, $0 \le x \le 1$ brings about a reconciliation with probability $\sqrt{x}$. The promoter stands to make \$750 from each concert. How much should he spend on flowers?

Let $\{X_t : t \ge 0\}$ be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, P)$ with discrete state space S. Let $\mathcal{F}_t = \sigma(\{X_s : 0 \le s \le t\})$ for $t \ge 0$. Thus $\{X_t : t \ge 0\}$ is *adapted* to the *filtration* $\{\mathcal{F}_t : t \ge 0\}$ The process $\{X_t : t \ge 0\}$ is a *Markov* process if for all $t, s \ge 0$

$$P(X_{t+s} = b \mid \mathcal{F}_t) = P(X_{t+s} = b \mid \sigma(X_t)).$$

The transition probabilities of the process are

$$p_{a,b}(t) = P(X_t = b \mid X_0 = a) \quad \text{for} \quad t > 0.$$

This gives the matrix of transition probabilities

$$\mathbf{P}(t) = (p_{a,b}(t)) \quad \text{for} \quad t > 0.$$

The Markov process has *homogeneous* transition probabilities if

$$P(X_{t+s} = b \mid X_s = a) = p_{a,b}(t) \quad \text{for all} \quad s, t > 0.$$

The finite dimensional distributions of the process can be expressed in terms of the transition probabilities as follows:

$$P_\pi(X_0 = a_0, X_{t_1} = a_1, ...X_{t_k} = a_k) = \pi(a_0)p_{a_0,a_1}(t_1)p_{a_1,a_2}(t_2 - t_1)\cdots p_{a_{k-1},a_k}(t_k - t_{k-1}).$$

Notice that $\mathbf{P}(t)$ must be known for all $t \geq 0$ to specify the finite dimensional distributions. The *Chapman-Kolmogorov equations* are given by

$$p_{a,b}(t+s) = \sum_{c \in S} p_{a,c}(t) p_{c,b}(s) \quad \text{for} \quad s,t > 0.$$

The finite dimensional distributions must satisfy the Chapman-Kolmogorov equations in order to be consistent. In matrix notation this becomes the *semigroup property*

$$\mathbf{P}(t+s) = \mathbf{P}(t)\mathbf{P}(s) \quad \text{for} \quad s,t > 0.$$

Additionally we will assume that

$$\mathbf{P}(0) = \mathbf{I} \qquad \text{and} \qquad \lim_{t \downarrow 0} \mathbf{P} = \mathbf{I}.$$

That is; the probability that the process instantaneously leaves a state that it has entered is equal to 0. Along with the continuity of $\mathbf{P}$ at $t = 0$ we assume that $\mathbf{P}$ is differentiable at $t = 0$. Set

$$q_{a,b} = \lim_{t \downarrow 0} \frac{p_{a,b}(t) - \delta_{a,b}}{t}$$

and

$$\mathbf{Q} = \Big(q_{a,b} \Big).$$

In matrix notation

$$\mathbf{Q} = \lim_{t \downarrow 0} \frac{\mathbf{P}(t) - \mathbf{I}}{t}.$$

Notice that since $\sum_{b \in S} p_{a,b}(t) = 1$, if summation and differentiation can be exchanged,

$$\sum_{b \in S} q_{a,b} = 0.$$

In particular

$$q_{a,a} = \lim_{t \downarrow 0} \frac{p_{a,a}(t) - 1}{t} \leq 0$$

and, for $a \neq b$,

$$q_{a,b} = \lim_{t \downarrow 0} \frac{p_{a,b}(t)}{t} \geq 0.$$

P41: Compound Poisson process. Let $\{Y_k : k \geq 0\}$ be a sequence of i.i.d. discrete random variables with state space S. Let $\{N_t : t \geq 0\}$ be an independent Poisson process with constant rate $\lambda > 0$. Let

$$X_t = \sum_{k=1}^{N_t} Y_k.$$

Show that $\{X_t : t \geq 0\}$ is a homogeneous Markov process.

P42: Show that a random variable X is an exponential random variable if and only if

$$P(X > s + t | X > s) = P(X > t) \quad \text{for all} \quad s,t > 0.$$

Let $\{X_t : t \geq 0\}$ be a homogeneous Markov process with discrete state space S and transition matrix $P(t)$ satisfying $P(0) = \mathbf{I}$ and $\mathbf{P}'(0) = \mathbf{Q}$. The semi-group property then gives *Kolmogorov's forward equation*

$$\mathbf{P}'(t) = \mathbf{P}(t)\mathbf{Q}$$

and *Kolmogorov's backward equations*

$$\mathbf{P}'(t) = \mathbf{Q}\mathbf{P}(t).$$

Notice that this gives

$$\mathbf{P}^{(k)}(0) = \mathbf{Q}^k$$

for $k \geq 0$ with $\mathbf{Q}^0 = \mathbf{I}$. For finite state space S this immediately gives

$$P(t) = \exp^{t\mathbf{Q}} = \sum_{k\geq 0} \frac{\mathbf{Q}^k t^k}{k!}.$$

For an infinite state space this representation can be problematic. However if $\sup_{a\in S}(-q_{a,a}) < \infty$, then this representation can be justified via inductive bounds on $|(\mathbf{Q}^n)_{a,b}|$.

P43: Fix $0 < \alpha < \beta$ and take

$$\mathbf{Q} = \begin{pmatrix} -\beta & \alpha & \beta - \alpha \\ \alpha & -2\alpha & \alpha \\ \beta - \alpha & \alpha & -\beta \end{pmatrix}.$$

Find the associated transition matrix $\mathbf{P}(t)$ and caculate the equilibrium probability.

Let S be a discrete space. A $\mathbf{Q}$ matrix on S has

$$\sum_{b\in S} q_{a,b} = 0, \qquad q_{a,a} \leq 0, \quad \text{and} \quad q_{a,b} \geq 0 \quad \text{for} \quad a \neq b.$$

The proof of the following theorem relies on the derivation of an iterative integral expression for $\mathbf{P}(t)$.

Theorem 29 *For any $\mathbf{Q}$ matrix, there exists a minimal non-negative solution $\mathbf{P}(t)$ of the backward equation $\mathbf{P}'(t) = \mathbf{Q}\mathbf{P}(t)$ with*

$$\sum_{b\in S} p_{a,b}(t) \leq 1$$

where $p_{a,b}(t) = (\mathbf{P})_{a,b}(t)$. If $\sum_{b\in S} p_{a,b}(t) = 1$ then the solution gives the unique transition matrix associated with $\mathbf{Q}$.

In the proof of Theorem 29 it is seen that the integral equation

$$p_{a,b}(t) = \delta_{a,b} e^{-q_a} + \int_{s=0}^{t} e^{-q_a(t-s)} \sum_{c\neq a} q_{a,c} p_{c,b}(s) ds$$

is equivalent to the backward equations with $\mathbf{P}(0) = \mathbf{I}$.

This gives the following sequence of iterative approximations to $p_{a,b}(t)$. Initiate the sequence by taking

$$p_{a,b}^{(0)}(t) = \delta_{a,b} e^{-q_a t}$$

and then set

$$p_{a,b}^{(n+1)}(t) = \delta_{a,b} e^{-q_a t} + \int_{s=0}^{t} e^{-q_a(t-s)} \sum_{c \neq a} q_{a,c} p_{c,b}^{(n)}(s) ds \quad for \quad n \geq 1.$$

P44: Show that $p_{a,b}^{(n)}(t)$ increases in n for fixed a, b, and t.

Let $\tilde{p}_{a,b}(t) = \lim_{n \to \infty} p_{a,b}^{(n)}(t)$.

P45: Show that $\sum_{b \in S} \tilde{p}_{a,b}(t) \leq 1$.

P46: Show that any other non-negative solution $p_{a,b}(t)$ to the integral equations with $\mathbf{P}(0) = \mathbf{I}$ satisfies

$$\tilde{p}_{a,b}(t) \leq p_{a,b}(t).$$

Proposition 2 *Let* $\tau_0 = \inf\{t > 0 : X_t \neq X_0\}$. *Then* $P_a(\tau_0 > t) = e^{-q_a t}$.

Let $\{\mathcal{F}_t : t \geq 0\}$ be a filtration on the probability space $(\Omega, \mathcal{F}, P)$. A random variable $\tau : \Omega \to [0, \infty]$ is a *stopping time* if

$$[\tau \leq t] \in \mathcal{F}_t \quad \text{for all} \quad t \geq 0.$$

For a stopping time τ define

$$\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap [\tau \leq t] \in \mathcal{F}_t \text{ for all } t \geq 0\}.$$

P47: Verify that $\mathcal{F}_\tau$ is a σ-field and that if the Markov process $\{X_t : t \geq 0\}$ is adapted to the filtration $\{\mathcal{F}_t : t \geq 0\}$ then $\{X_t : 0 \leq t \leq \tau\}$ is measurable $\mathcal{F}_\tau$.

The previous proposition shows that a homogeneous Markov process $\{X_t : t \geq 0\}$ with discrete state space S, transition semi-group $\{\mathbf{P}(t) : t \geq 0\}$ satisfying $\mathbf{P}(0) = \mathbf{I}$, and *infinitesimal generator* $\mathbf{Q} = \mathbf{P}'(0)$ has, at any fixed time t, right continuous sample paths with probability 1.

Proposition 3

$$P(\lim_{s \downarrow 0} X_s = a) = 1.$$

Corollary 30 *For any fixed* $t \geq 0$,

$$P_a(\lim_{s \downarrow t} X_s = X_t) = 1.$$

These processes also enjoy the strong Markov property.

Proposition 4 *Let τ be a stopping time. On the set $[\tau < \infty]$, given $\mathcal{F}_\tau$, $\{X_{t+\tau} : t \geq 0\}$ is a homogeneous Markov process with infinitesimal generator $\mathbf{Q}$, transition semi-group $\{\mathbf{P}(t) : t \geq 0\}$, and initial distribution $\lambda(a) = P(X_\tau = a)$.*

Rephrased, this proposition gives, for example, on the set $[\tau < \infty]$, for any $A \in \mathcal{F}_\tau$

$$P(X_{t+\tau} = b \,\Big|\, A \cap [X_\tau = a]) = p_{a,b}(t).$$

One method of proof involves approximation by discrete time increments. The results of some of the following problems are helpful.

P48: Show that τ_0 as defined in Proposition 2 is a stopping time.

P49: Show that if τ is a stopping time then for any fixed t the random variable $\tau \wedge t = \min\{\tau, t\}$ is also a stopping time.

P50: Show that any fixed time t is a stopping time.

P51: Show that for any two stopping times τ and σ relative to the same filtration $\{\mathcal{F}_t : t \geq 0\}$, $\gamma = \tau \wedge \sigma$ is also a stopping time.

P52: Show that if τ and σ are both stopping times relative to the same filtration $\{\mathcal{F}_t : t \geq 0\}$ and $\tau \leq \sigma$, then $\mathcal{F}_\tau \subset \mathcal{F}_\sigma$.

For a homogeneous Markov process $\{X_t : t \geq 0\}$ with discrete state space S, stochastic transition semi-group $\{\mathbf{P}(t) : t \geq 0\}$ satisfying $\mathbf{P}(0) = \mathbf{I}$, and *infinitesimal generator* $\mathbf{Q} = \mathbf{P}'(0)$ define the sequence of stopping times $\{\tau_k : k \geq 0\}$ and the sequence of holding times $\{T_k : k \geq 0\}$ via

$$T_0 = \tau_0 = \inf\{t > 0 : X_t \neq X_0\},$$

$$\tau_k = \inf\{t > \tau_{k-1} : X_t \neq X_{\tau_{k-1}}\} \quad \text{for} \quad k \geq 1,$$

and

$$T_k = \tau_k - \tau_{k-1} \quad \text{for} \quad k \geq 1.$$

The *jump process* associated with $\{X_t : t \geq 0\}$ is the sequence given by $Y_0 = X_0$ and, for $n \geq 1$,

$$Y_n = X_{\tau_{n-1}}.$$

Proposition 5 *$\{Y_n : n \geq 0\}$ is a Markov chain with transition matrix $\mathbf{P}$ given by*

$$p_{a,b} = \left\{ \begin{array}{ll} 0 & \text{if } b = a \text{ and } q_a \neq 0 \\ \frac{q_{a,b}}{q_a} & \text{if } b \neq a \text{ and } q_a \neq 0 \\ 1 & \text{if } b = a \text{ and } q_a = 0 \\ 0 & \text{if } b \neq a \text{ and } q_a = 0. \end{array} \right\}$$

Proposition 5 is just part of a larger result.

Theorem 31 *(a) If $\{X_t : t \geq 0\}$ is Markov, then the associated jump process $\{Y_n : n \geq 0\}$ is a Markov chain with transition matrix $\mathbf{P}$ as given in Proposition 5. Furthermore, conditional on the Y_n's, the holding times $\{T_n : n \geq\}$ are independent exponential random variables with respective parameters q_{Y_n}.*

(b) If $\{Y_n : n \geq 0\}$ is a Markov chain with transition matrix $\mathbf{P}$ as given in Proposition 5 and, conditional on the Y_n's, $\{T_n : n \geq 0\}$ is a sequence of independent exponential random variables with respective parameters q_{Y_n}, then

$$X_t = \sum_{n \geq 0} Y_n \mathbf{1}[\sum_{0}^{n-1} T_k \leq t < \sum_{0}^{n} T_k]$$

is a Markov process with infintesimal generator $\mathbf{Q}$.

The *explosion time* of the process $\{X_t : t \geq 0\}$ is defined to be

$$\xi = \lim_{n \to \infty} \tau_n = \sum_{k \geq 0} T_k.$$

If $P_a(\xi = \infty) = 1$ for all $a \in S$ then $\{X_t : t \geq 0\}$ is *non-explosive* or *conservative*.

Example: Consider the pure birth process in continuous time with state space $S = \{0, 1, 2, ...\}$. This corresponds to a $\mathbf{Q}$ matrix with $q_{a,a+1} = q_a$. This can be thought of as modeling a population that is increasing over time with the rate of increase depending on the population level. The question that comes to mind is: does the population explode in finite time with positive probability? This corresponds to asking if $P(\xi < \infty) > 0$. In this case, given that the process starts in state $a = 0$, $\{T_k : k \geq 0\}$ is a sequence of independent exponentials with non-random rates $\{q_k : k \geq 0\}$. The following proposition gives exact criteria for explosive behavior.

Proposition 6 *Let $\{T_k : k \geq 0\}$ be a sequence of independent exponentials with rates $\{q_k : k \geq 0\}$ and let $\xi = \sum_{k \geq 0} T_k$. Then*

$$P(\xi < \infty) = 1 \text{ if } \sum_{k \geq 0} 1/q_k < \infty$$

and

$$P(\xi = \infty) = 1 \text{ if } \sum_{k \geq 0} 1/q_k = \infty.$$

Corollary 32 *A pure birth process explodes in finite time if and only if*

$$\sum_{k \geq 0} 1/q_k < \infty.$$

In the general setting the following theorem holds.

Theorem 33 *The Markov process $\{X_t : t \geq 0\}$ with infinitesimal generator $\mathbf{Q}$ and stochastic transition semi-group $\{\mathbf{P}(t) : t \geq 0\}$ is non-explosive if any of the following hold.*

(a) S is finite.

(b) $\sup_{a \in S} q_a < \infty$.

(c) $X_0 = a$ with state a recurrent for the associated jump process $\{Y_n : n \geq 0\}$.

Part (c) of Theorem 33 illustrates the utility in the continuous time setting of understanding recurrence and transience of discrete time chains. This will be seen later as well.

If explosion occurs in finite time for a process $\{X_t : t \geq 0\}$, it is common to adjoin a *coffin state* Δ to S and set

$$X_t = \Delta \text{ if } t \geq \xi.$$

This is called the *minimal* process associated with the infinitesimal generator $\mathbf{Q}$ and stochastic semi-group $\{\mathbf{P}(t) : t \geq 0\}$.

An explosive process can be extended by introducing a probability distribution ψ on the state space S and at time ξ restarting the process X_t in S according to the distribution ψ. Then we have a new *extended* process $\{\tilde{X}_t : t \geq 0\}$ with

$$P(\tilde{X}_\xi = a) = \psi(a) \text{ for } a \in S$$

with local evolution from time $t = \xi$ governed by the infinitesimal generator $\mathbf{Q}$. This extension of the process may again explode, so this construction results in a sequence of inter-explosion times $\{\xi_k : k \geq 0\}$ with $\xi_0 = \xi$. If $\{\tilde{X}_t : t \geq 0\}$ has initial distribution ψ then the ξ_k's are iid. This extension of the process changes the semi-group from $\{\mathbf{P}(t) : t \geq 0\}$ to $\{\tilde{\mathbf{P}}(t) : t \geq 0\}$ with

$$\begin{aligned}
(\tilde{\mathbf{P}}(t))_{a,b} &= P_a(\tilde{X}_t = b) \\
&= P(\tilde{X}_t = b, \xi_0 > t) + \sum_{n \geq 1} P(\tilde{X}_t = b, \sum_0^{n-1} \xi_k \leq t < \sum_0^n \xi_k)
\end{aligned}$$

P53: Consider the pure birth process on $S = \{1, 2, 3, ...\}$ with explosion. Assume that it starts in state 1 with probability 1 and returns to state 1 with probability 1 at explosion. Show that for this example $\tilde{\mathbf{P}}(t)$ satisfies the backward equations but not the forward equations.

The following gives a sketch of the proof of Proposition 4 on page 20. Recall that this is the strong Markov property for continuous time Markov chains. Let τ be a stopping time relative to the canonical filtration $\{\mathcal{F}_t : t \geq 0\}$. Define a decreasing sequence of stopping times via

$$\tau_n = \left\{ \begin{array}{ll} \sum_{k \geq 0} \frac{k}{2^n} \mathbf{1}[\frac{k-1}{2^n} < \tau \leq \frac{k}{2^n}] & \text{if } \tau < \infty \\ \infty & \text{if } \tau = \infty. \end{array} \right\}$$

As defined, $\tau_n \downarrow \tau$ a.s. P as $n \to \infty$. From P52, $\mathcal{F}_\tau \subset \mathcal{F}_{\tau_n}$ for each n. For n and j let $Y_j^{(n)} = X(j/2^n)$. Then $\{Y_j^{(n)} : j \geq 0\}$ is a Markov chain with transition matrix $\mathbf{P}^{(n)} = \mathbf{P}(2^{-n})$. τ_n is a stopping time relative to both the continuous filtration $\{\mathcal{F}_t : t \geq 0\}$ and the discrete filtration $\mathcal{F}^{(n)} = \{\mathcal{F}_{j/2^n} : j \geq 0\}$. Let $\mathcal{G}_{\tau_n}$ be the stopping time σ-field relative to $\mathcal{F}^{(n)}$; that is

$$\mathcal{G}_{\tau_n} = \{A \in \mathcal{F} : A \cap [\tau_n \leq j/2^n] \in \mathcal{F}_{j/2^n} \text{ for all } j \geq 0\}.$$

We then have (please check)

$$\mathcal{F}_\tau \subset \mathcal{F}_{\tau_n} \subset \mathcal{G}_{\tau_n}.$$

If t is fixed and $\frac{j-1}{2^n} < t \leq \frac{j}{2^n}$ then

$$P(X(j/2^n + \tau_n) = b, X(\tau_n) = a) \to P(X(t+\tau) = b, X(\tau) = a) \text{ as } n \to \infty.$$

¿From the strong Markov property for discrete chains, for $A \in \mathcal{F}_\tau$

$$P(X(j/2^n + \tau_n) = b | A \cap [X(\tau_n) = a]) = p_{a,b}(j/2^n) \to p_{a,b}(t) \text{ as } n \to \infty.$$

Additionally, for $A \in \mathcal{F}_\tau$,

$$P(A \cap [X(\tau_n) = a]) \to P(A \cap [X(\tau) = a]) \text{ as } n \to \infty.$$

This gives

$$p_{a,b}(t) = \lim_{n\to\infty} P(X(j/2^n + \tau_n) = b | A \cap [X(\tau_n) = a]) = P(X(t+\tau) = b | A \cap [X(\tau) = a]).$$

This isn't quite complete; it is straightforward to complete the proof by replacing $[X(t+\tau) = b]$ with a set of the form $\cap_{k=1}^m [X(t_k + \tau) = b_k]$.

Let $\{X_t : t \geq 0\}$ be a minimal homogenous Markov process with infinitesimal generator $\mathbf{Q}$. Denote the associated 'jump chain' by $\{Y_n : n \geq 0\}$. For $A \subset S$ let $D^A = \inf\{t \geq 0 : X_t \in A\}$ denote the first entry time to A. Recall that for the discrete time chain $\{Y_n\}$, the first entry time to A is defined as $H^A = \inf\{n \geq 0 : Y_n \in A\}$. Then

$$[H^A < \infty] = [D^A < \infty].$$

Again let

$$h_a^A = P_a(H^A < \infty) = P_a(D^A < \infty).$$

Theorem 34 *The vector $(H_a^A : a \in S)$ is the minimal non-negative solution to the system*

$$\begin{aligned} h_a^A &= 1 \quad \textit{for} \quad a \in A \\ \sum_{b \in S} q_{a,b} h_b^A &= 0 \quad \textit{for} \quad a \notin A. \end{aligned}$$

Define the expected first entry time to A as

$$k_a^A = E_a D^A.$$

Theorem 35 *If $q_a > 0$ for all $a \in S$, then the vector $(k_a^A : a \in S)$ is the minimal non-negative solution to the system*

$$\begin{aligned} k_a^A &= 0 \quad \textit{for} \quad a \in A \\ -\sum_{b \in S} q_{a,b} k_b^A &= 1 \quad \textit{for} \quad a \notin A. \end{aligned}$$

State $a \in S$ is *recurrent* if

$$P_a(\{t > 0 : X_t = a\} \text{ is unbounded}) = 1.$$

P54: Show that state a is recurrent if and only if

$$P_a(\sup\{t > 0 : X_t = a\} = \infty) = 1.$$

State $a \in S$ is *transient* if

$$P_a(\{t > 0 : X_t = a\} \text{ is unbounded}) = 0.$$

P55: Show that state a is transient if and only if

$$P_a(\sup\{t > 0 : X_t = a\} < \infty) = 1.$$

Theorem 36 *(1) If state a is recurrent for $\{Y_n : n \geq 0\}$, then a is recurrent for $\{X_t : t \geq 0\}$.*

(2) If state a is transient for $\{Y_n : n \geq 0\}$, then a is transient for $\{X_t : t \geq 0\}$.

(3) Each state is either recurrent or transient.

(4) Recurrence and transience are class properties.

Define the first passage time to state a as

$$\eta_a = \inf\{t > T_0 : X_t = a\}.$$

State a is recurrent if $P_a(\eta_a < \infty) = 1$. State a is *positive recurrent* if

$$E_a \eta_a < \infty.$$

State a is *null recurrent* if a is recurrent and

$$E_a \eta_a = \infty.$$

Theorem 37 *(1) If $q_a = 0$ or $P_a(\eta_a < \infty) = 1$, then state a is recurrent and $\int_{t=0}^{\infty} p_{a,a}(t)dt = \infty$.*

(2) If $q_a > 0$ or $P_a(\eta_a < \infty) < 1$, then state a is transient and $\int_{t=0}^{\infty} p_{a,a}(t)dt < \infty$.

Transience and recurrence for $\{X_t\}$ can also be determined by looking at a discrete time chain $\{Z_n : n \geq 0\}$ defined by fixing $h > 0$ and setting

$$Z_n = X_{nh} \quad \text{for} \quad n \geq 0.$$

This chain has transition matrix $\mathbf{P}(h)$.

Theorem 38 *(1) If a is recurrent for $\{X_t\}$ it is also recurrent for $\{Z_n\}$.*

(2) If a is transient for $\{X_t\}$ it is also transient for $\{Z_n\}$.

A vector $\lambda = (\lambda_a : a \in S)$ is *invariant* for $\mathbf{Q}$ if

$$\lambda^t \mathbf{Q} = 0.$$

Note that if λ is invariant for $\mathbf{Q}$ then the backward equations give

$$(\lambda^t \mathbf{P}(t))' = \lambda^t \mathbf{P}'(t) = \lambda^t \mathbf{Q}\, \mathbf{P}(t) = 0$$

for all t. Thus $\lambda^t \mathbf{P}(t)$ is constant in t with

$$\lambda^t \mathbf{P}(t) = \lambda^t \mathbf{P}(0) = \lambda^t \mathbf{I} = \lambda^t.$$

That is; if λ is a probability distribution on S and invariant for $\mathbf{Q}$, then λ is an invariant distribution for $\mathbf{P}(t)$ for any t. Let $\mathbf{P_Q}$ denote the transition matrix of the associated discrete time chain $\{Y_n : n \geq 0\}$.

Theorem 39 *The following are equivalent.*

(1) λ *is invariant for* $\mathbf{Q}$.

(2) The vector π *with* $\pi_a = \lambda_a q_a$ *is invariant for* $\mathbf{P_Q}$.

Theorem 40 *If* $\mathbf{Q}$ *is irreducible and recurrent then it has an invariant vector* λ *that is unique up to scalar multiples.*

Theorem 41 *If* $\mathbf{Q}$ *is irreducible the following are equivalent:*

(1) Each state in S *is positive recurrent.*

(2) Some a *in* S *is positive recurrent.*

(3) $\mathbf{Q}$ *is non-explosive and has an invariant probability distribution* λ.

In this case, $\lambda_a = 1/(q_a E_a \eta_a)$.

Notice that if λ and π are invariant probability distributions for $\mathbf{Q}$ and $\mathbf{P_Q}$ respectively, then

$$\lambda_a = \frac{\pi_a/q_a}{\sum_b (\pi_b/q_b)}.$$

P56: Show that if for any $h >$, the transition matrix $\mathbf{P}(h)$ is irreducible and positive recurrent with invariant probability distribution ρ, then $\gamma^t \mathbf{P}(t) \to \rho$ as $t \to \infty$ for any initial probability distribution γ. Argue that $\rho = \lambda$ with λ the invariant probability distribution for $\mathbf{Q}$.

Theorem 42 *The Ergodic Theorem:*

Suppose that S forms a single closed class of positive recurrent states. If $f : S \to \mathbf{R}$ has

$$E_a\left|\int_{s=0}^{t} f(X_s)ds\right| < \infty \quad \textit{for some} \quad a \in S$$

then for all initial distributions of X_0,

$$\lim_{t\to\infty} \frac{1}{t}\int_{s=0}^{t} f(X_s)ds = \frac{1}{E_a\eta_a}E_a\int_{s=0}^{\eta_a} f(X_s)ds \quad \textit{a.s. } P.$$

P57: Show that $\frac{1}{E_a\eta_a}E_a\int_{s=0}^{\eta_a} f(X_s)ds = E_\lambda f(X_0)$ with λ being the invariant probability measure for $\mathbf{Q}$.