The Essential Support of the Fourier Transform of the Parallel-Beam Radon Transform

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1 The Continuous Domain

Definition: Define the angular and linear Fourier transforms by

$$\begin{aligned}\widehat{h}(\omega) &\equiv \int_{\mathbb{R}} h(x)e^{-i\omega x}\,dx \\ \widetilde{h}(\sigma) &\equiv \int_{\mathbb{R}} h(x)e^{-2\pi i\sigma x}\,dx.\end{aligned}$$

Definition: Define the Radon transform by

$$Rf(\varphi, s) \equiv \int_{\mathbf{x}\cdot\theta=s} f(\mathbf{x})\,d\mathbf{x} = \int_{\mathbb{R}} f(s\cos\varphi + t\sin\varphi, -s\sin\varphi + t\cos\varphi)\,dt.$$

Lemma 1. *Let $J_k(\cdot)$ be the Bessel function of the first kind with order k. Then $J_k(\cdot)$ satisfies*

$$|J_k(\nu k)| \leq e^{-(|k|/3)(1-\nu^2)^{3/2}}$$

for $0 < \nu < 1$.

Theorem 1. *Suppose that $f \in C_0^\infty(\Omega_R)$, where $\Omega_R \equiv \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < R^2\}$. For $b > 0$ and $0 < \nu < 1$, let $K(\nu, b) = \{(k,\omega) \in \mathbb{Z}\times\mathbb{R} : |\omega| < b, |k| < \frac{1}{\nu}\max(R|\omega|, (1-\nu)b\}$. Then*

$$\int_{(\mathbb{R}\times\mathbb{Z})\setminus K} |\widehat{Rf}(\zeta)|\,d\zeta \leq \frac{8}{\pi^2\nu}\int_{|\xi|>b} |\widehat{f}(\xi)|d\xi + \|f\|_{L^1}\eta(\nu, b),$$

where $\eta(\nu, b)$ decreases exponentially with b, satisfying the estimate

$$0 < \eta(\nu, b) \leq C(\nu)e^{-\lambda(\nu)b}$$

with constants $C(\nu), \lambda(\nu) > 0$.

The main idea of the theorem is as follows. We have

$$\begin{aligned}
\widehat{Rf}(k,\omega) &= \frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{R}} Rf(\varphi,s)e^{-i(k\varphi+\omega s)}\,ds\,d\varphi \\
&= \frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} f(s\cos\varphi+t\sin\varphi,-s\sin\varphi+t\cos\varphi)e^{-i(k\varphi+\omega s)}\,dt\,ds\,d\varphi \\
&= \frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{R}^2} f(\mathbf{x})e^{-i\omega<\mathbf{x},\theta>-ik\varphi)}\,d\mathbf{x}\,d\varphi \\
&= \frac{1}{2\pi}\int_0^{2\pi}\int_{\Omega_R} f(\mathbf{x})e^{-i\omega|\mathbf{x}|\cos(\varphi-\psi)-ik\varphi}\,d\mathbf{x}\,d\varphi \\
&= \frac{1}{2\pi}\int_{\Omega_R} f(\mathbf{x})e^{-ik\psi}\int_0^{2\pi} e^{-i|\mathbf{x}|\omega\cos(\varphi-\psi)-i(\varphi-\psi)k}\,d\varphi\,d\mathbf{x} \\
&= i^k\int_{\Omega_R} f(\mathbf{x})e^{-ik\psi}J_k(-\omega|\mathbf{x}|)\,d\mathbf{x},
\end{aligned}$$

where we have used that substitution $\mathbf{x} = \begin{bmatrix} s\cos\varphi+t\sin\varphi \\ -s\sin\varphi+t\cos\varphi\end{bmatrix}$ and $\mathbf{x} = |x|\begin{bmatrix}\cos\psi \\ \sin\psi\end{bmatrix}$. Note that from the above lemma $J_k(-\omega|\mathbf{x}|)$ decays exponentially as $k\to\infty$ for $\nu|\omega\mathbf{x}|\le|k|$. For this to hold for all $\mathbf{x}\in\Omega_R$ we must have $\nu R|\omega|\le|k|$. In practice b is the Nyquist frequency from sampling s. Also note that $K(\nu_2,b)\subset K(\nu_1,b)$ for $0<\nu_1<\nu_2<1$.

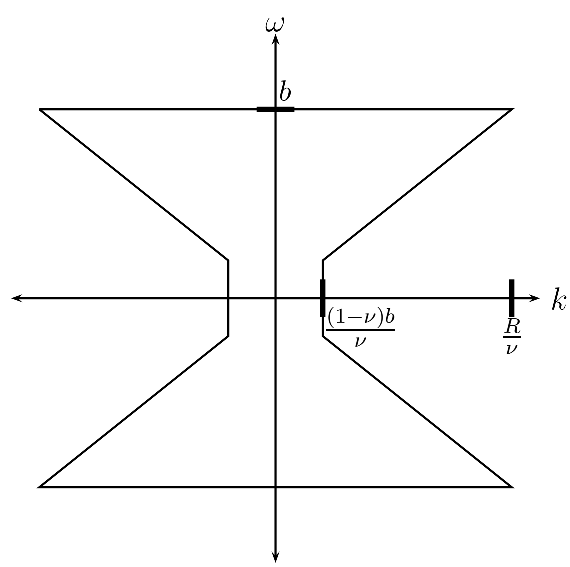

Figure 1: The set $K(\nu,b)$ for $\nu=\frac{4}{5}$.

Now we make sense of the set $K(\nu,b)$ as it applies to the Fast Fourier Transform.

2 Sampling the Radon Tranform

First we define some parameters:

N_φ	number of φ samples (even)
N_s	number of s samples (even)
$T_\varphi = \frac{2\pi}{N_\varphi}$	sampling distance in φ
$T_s = \frac{2R}{N_s}$	sampling distance in s
$\sigma = 2\pi\omega$	linear frequency
$N_\sigma \geq N_s$	number of samples of σ
$N_k = N_\varphi$	number of samples of k
$b = \frac{1}{2T_s}$	linear Nyquist frequency in the s variable

For convenience, let

$$g(t,s) \equiv Rf(2\pi t, s).$$

Then

$$\begin{aligned}\widehat{g}(\tau,\omega) &= \frac{1}{2\pi}\widehat{Rf}\left(\frac{\tau}{2\pi},\omega\right) \\ &= \frac{i^k}{2\pi}\int_{\Omega_R} f(\mathbf{x})e^{-i\tau\psi/(2\pi)}J_k(-\omega|\mathbf{x}|)\,d\mathbf{x}\end{aligned}$$

and

$$\widetilde{g}(\rho,\sigma) = \widehat{g}(2\pi\rho, 2\pi\sigma),$$

so

$$\begin{aligned}\widetilde{g}(\rho,\sigma) &= \int_{\mathbb{R}}\int_0^1 g(t,s)e^{-2\pi i(t\rho+s\sigma)}\,dt\,ds \\ &= \frac{i^k}{2\pi}\int_{\Omega_R} f(\mathbf{x})e^{-i\rho\psi}J_k(-2\pi\sigma|\mathbf{x}|)\,d\mathbf{x}.\end{aligned}$$

Thus if $\widehat{f}(\xi)$ is essentially supported on the set $\{\xi \in \mathbb{R}^2 : |\xi| \leq b\}$, then the essential support of $\widetilde{g}(\rho,\sigma)$ is contained in the set

$$K(\nu,b) \equiv \left\{(\rho,\sigma) \in \mathbb{Z}\times\mathbb{R} : |\sigma| \leq b, |\rho| \leq \frac{1}{\nu}\max(R|\sigma|,(1-\nu)b)\right\}$$

Suppose that $G[k,l]$ is the FFT of a sampling of $g(t,s)$. Then

$$G[k,l] = \widetilde{g}\left(k, \frac{l}{T_s N_\sigma}\right)$$

for $k = -\frac{N_\varphi}{2},\ldots,\frac{N_\varphi}{2}-1$ and $l = -\frac{N_\sigma}{2},\ldots,\frac{N_\sigma}{2}-1$. Thus the essential support of $G[k,l]$ is given by

$$\begin{aligned}K_d(\nu,b) &\equiv \left\{(k,l)\in\mathbb{Z}^2 : |l| \leq \frac{N_\sigma}{2}, |k| \leq \frac{1}{\nu}\max\left(\frac{R}{T_s N_\sigma}|l|, \frac{1-\nu}{2T_s}\right)\right\} \\ &= \left\{(k,l)\in\mathbb{Z}^2 : |l| \leq \frac{N_\sigma}{2}, |k| \leq \frac{1}{\nu}\max\left(\frac{N_s}{2N_\sigma}|l|, \frac{1-\nu}{2T_s}\right)\right\}\end{aligned}$$