Math 254 Vector Calculus I - Theorems and Definitions

Conventions: We will denote vectors in bold print and O shall denote the origin.

Distance Formula in Three Dimensions: Let $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ be two points in three space. The distance between P_1 and P_2 is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Equation of a Sphere: An equation of a sphere with center $C(h, k, l)$ and radius r is

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2.$$

Rectangular Coordinates: This is the "normal" coordinate system. We shall denote these by (x, y, z).

Cylindrical Coordinates: The cylindrical coordinates are (r, θ, z), where $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$. Note that $r^2 = x^2 + y^2$ and $\tan\theta = \frac{y}{x}$.

Spherical Coordinates: The spherical coordinates are (ρ, θ, ϕ), where $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi$. Note that $\rho^2 = x^2 + y^2 + z^2$ and $0 \leq \phi \leq \pi$.

Scalar: A real number.

Vectors: A vector provides information of magnitude and direction. A two-dimensional vector is described by an ordered pair $\mathbf{a} =< a_1, a_2 >$ and a three dimensional vector is described by an ordered pair $\mathbf{a} =< a_1, a_2, a_3 >$. The numbers a_1, a_2, a_3 are real numbers and are called the components of $\mathbf{a}$.

Vector Formed by Two Points: Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, we have

$$\overrightarrow{AB} =< x_2 - x_1, y_2 - y_1, z_2 - z_1 > .$$

Position Vector: A position vector is a vector formed by the origin and some other point.

Length or Magnitude of a Vector: The length of the vector $\mathbf{a} =< a_1, a_2, a_3 >$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

Vector Addition: Let $\mathbf{a} =< a_1, a_2, a_3 >$ and $\mathbf{b} =< b_1, b_2, b_3 >$, then

$$\mathbf{a} + \mathbf{b} =< a_1 + b_1, a_2 + b_2, a_3 + b_3 > .$$

Scalar Multiplication: If c is a scalar and $\mathbf{a} =< a_1, a_2, a_3 >$, then the vector $c\mathbf{a}$ is defined by $c\mathbf{a} =< ca_1, ca_2, ca_3 >$.

Properties of Vectors: If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors and c and d are scalars, then

1) $\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$	2) $\mathbf{a}+(\mathbf{b}+\mathbf{c})=(\mathbf{a}+\mathbf{b})+\mathbf{c}$
3) $\mathbf{a}+\mathbf{0}=\mathbf{a}$	4) $\mathbf{a}+(-\mathbf{a})=\mathbf{0}$
5) $c(\mathbf{a}+\mathbf{b})=c\mathbf{a}+c\mathbf{b}$	6) $(c+d)\mathbf{a}=c\mathbf{a}+d\mathbf{a}$
7) $(cd)\mathbf{a}=c(d\mathbf{a})$	8) $1\mathbf{a}=\mathbf{a}$

Standard Basis: Let $\mathbf{i}=<1,0,0>$, $\mathbf{j}=<0,1,0>$, and $\mathbf{k}=<0,0,1>$.

Dot product: Let $\mathbf{a}=<a_1,a_2,a_3>$ and $\mathbf{b}=<b_1,b_2,b_3>$. Then the dot product of $\mathbf{a}$ and $\mathbf{b}$ is the scalar given by $\mathbf{a}\cdot\mathbf{b}=a_1b_1+a_2b_2+a_3b_3$.

Properties of the Dot Product: If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors and c is a scalar, then

1) $\mathbf{a}\cdot\mathbf{a}=\lvert a\rvert^2$	2) $\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}$
3) $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})=\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}$	4) $(c\mathbf{a})\cdot\mathbf{b}=c(\mathbf{a}\cdot\mathbf{b})=\mathbf{a}\cdot(c\mathbf{b})$
5) $\mathbf{0}\cdot\mathbf{a}=0$.	

Theorem 13.3.3: If θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, then $\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}||\mathbf{b}|\cos\theta$.

Theorem 13.3.7: The vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal (perpendicular) if and only if $\mathbf{a}\cdot\mathbf{b}=0$.

Projections: The vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is

$$\mathrm{proj}_{\mathbf{a}}\mathbf{b}=\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}.$$

Cross Product: Let $\mathbf{a}=<a_1,a_2,a_3>$ and $\mathbf{b}=<b_1,b_2,b_3>$. Then the cross product of $\mathbf{a}$ and $\mathbf{b}$ is the vector

$$\mathbf{a}\times\mathbf{b}=<a_2b_3-a_3b_2,a_3b_1-a_1b_3,a_1b_2-a_2b_1>.$$

We may also formulate the cross product as the determinate of the matrix

$$\begin{bmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\a_1&a_2&a_3\\b_1&b_2&b_3\end{bmatrix}.$$

Theorem 13.4.5: The vector $\mathbf{a}\times\mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

Theorem 13.4.6: If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ (so $0\le\theta\le\pi$), then $|\mathbf{a}\times\mathbf{b}|=|\mathbf{a}||\mathbf{b}|\sin\theta$.

Corollary 13.4.7: Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if $\mathbf{a}\times\mathbf{b}=\mathbf{0}$.

Properties of the Cross Product: Let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be vectors and c be a scalar. Then

1) $\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}$
2) $(c\mathbf{a})\times\mathbf{b}=c(\mathbf{a}\times\mathbf{b})=\mathbf{a}\times(c\mathbf{b})$
3) $\mathbf{a}\times(\mathbf{b}+\mathbf{c})=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$
4) $(\mathbf{a}+\mathbf{b})\times\mathbf{c}=\mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c}$
5) $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$
6)$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$

Geometric Properties of the Cross Product: The area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$ is $|\mathbf{a} \times \mathbf{b}|$. The volume of the parallelepiped determined by the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$.

Vector Equation of a Line: Let L be a line in three space and $P(x, y, z)$ be an arbitrary point on L. Let $P_0(x_0, y_0, z_0)$ and $P_1(x_1, y_1, z_1)$ be any (fixed) points on L. Let $\mathbf{r}_0 = \overrightarrow{OP_0}$, $\mathbf{v} = \overrightarrow{P_0P_1}$, and $\mathbf{r} = \overrightarrow{OP}$. Then there is a scalar t such that $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$.

Parametric Equations: Let L, $\mathbf{v} =< a, b, c >$, $\mathbf{r} =< x, y, z >$, and $\mathbf{r}_0 =< x_0, y_0, z_0 >$ be as above. Then the three scalar equations

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct$$

are the parametric equations of L. We can also write this in vector notation $< x, y, z >=< x_0 + ta, y_0 + tb, z_0 + tc >$.

Symmetric Equations: The symmetric equations of L (as above) are

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

Skew Lines: Two lines L_1 and L_2 in three space are skew lines if they are not parallel and do not intersect.

Vector Equation of a Plane: Let S be a plane in three space. Let $P_0(x_0, y_0, z_0)$ be a (fixed) point on the plane and $\mathbf{n}$ be a vector orthogonal to S. Let $P(x, y, z)$ be an arbitrary point in the plane, and let $\mathbf{r}_0$ and $\mathbf{r}$ be the position vectors of P_0 and P, respectively. Then $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$.

Normal Vector: The orthogonal vector $\mathbf{n}$ above is called a normal vector to the plane.

Determination of a Normal Vector: Let P_0, P_1, and P_2 be distinct points on a plane. Let $\mathbf{a} = \overrightarrow{P_0P_1}$ and $\mathbf{b} = \overrightarrow{P_0P_2}$. Then $\mathbf{n} = \mathbf{a} \times \mathbf{b}$.

Scalar Equation of a Plane: Let $P_0(x_0, y_0, z_0)$ be a point on the plane S and $\mathbf{n} =< a, b, c >$ be a normal vector to S. Then an equation of the plane S is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Level Curves: The level curves of a function f of two variables are the curves with equations $f(x, y) = k$ where k is a constant (in the range of f). An example of a graph of the level curves is a topographic map.

Limits: Lets f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say that the limit of $f(x, y)$ as (x, y) approaches (a, b) is L and we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that $|f(x, y) - L| < \varepsilon$ whenever $(x, y) \in D$ and $0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta$.

Failure Test for Limits: If $f(x,y) \to L_1$ as $(x,y) \to (a,b)$ along a path C_1 and $f(x,y) \to L_2$ as $(x,y) \to (a,b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist. Note that if $L_1 = L_2$ this would not guarantee that the limit exists. We need limits along *every* path to be equal to prove that the limit exists.

Continuity: A function f of two variables is called continuous at (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).$$

We say f is continuous on D if f is continuous at every point (a,b) in D.

Partial Derivatives: If f is a function of two variables, its partial derivatives are the functions f_x and f_y, defined by

$$\begin{aligned} \frac{\partial f}{\partial x}(x,y) &= f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}, \\ \frac{\partial f}{\partial y}(x,y) &= f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}. \end{aligned}$$

For instance, to find f_x, regard y as a constant and differentiate $f(x,y)$ with respect to x.

Clairaut's Theorem: Suppose f is defined on a disk D that contains the point (a,b). If the functions f_{xy} and f_{yx} are both continuous on D, then $f_{xy}(a,b) = f_{yx}(a,b)$.

Tangent Planes: Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x,y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0).$$

Theorem 8: If the partial derivatives f_x and f_y exist near (a,b) and are continuous at (a,b), then f is differentiable at (a,b).

Differentials: Let $z = f(x,y)$. Then $dz = f_x(x,y)dx + f_y(x,y)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$.

Chain Rule: Suppose that $z = f(x,y)$ is a differentiable function of x and y, where $x = g(s,t)$ and $y = h(s,t)$ are differentiable functions of s and t. Then

$$\begin{aligned} \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}, \\ \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}. \end{aligned}$$

Implicit Differentiation: If $F(x,y) = 0$, then $dy/dx = -F_x/F_y$.

Directional Derivative: The directional derivative of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} =< a, b >$ is

$$D_{\mathbf{u}} f(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0,y_0)}{h}$$

if the limit exists.

Gradient Vector: If f is a function of two variables x and y, then the gradient of f is the vector function ∇f defined by

$$\nabla f(x,y) =< f_x(x,y), f_y(x,y) > .$$

The gradient vector evaluated at a certain point gives that direction and magnitude of steepest ascent.

How to Calculate a Directional Derivative (Theorem 3): If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} =< a, b >$ and

$$D_{\mathbf{u}}f(x,y) = \nabla f(x,y) \cdot \mathbf{u} = f_x(x,y)a + f_y(x,y)b.$$

Theorem 15: Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}}f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

Local Extreme Points: A function of two variables has a local maximum at (a, b) if $f(x,y) \leq f(a,b)$ when (x,y) is near (a,b). The number $f(a,b)$ is called a local maximum value. If $f(x,y) \geq f(a,b)$ when (x,y) is near (a,b), then $f(a,b)$ is a local minimum value.

Absolute Extreme Points: If the inequalities above hold for all points (x,y) in the domain of f, then f has an absolute maximum (or absolute minimum) at (a,b).

Theorem 2: If f has a local max or min at (a,b) and the first-order partial derivatives of f exist there, then $f_x(a,b) = 0$ and $f_y(a,b) = 0$. In this case we call (a,b) a critical point of f.

Second Derivative Test: Suppose the second partial derivatives of f are continuous on a disk with center (a,b), and suppose that $f_x(a,b) = 0$ and $f_y(a,b) = 0$. Let $D = D(a,b) = f_{xx}(a,b)f_{yy}(x,y) - [f_{xy}(a,b)]^2$. If
(a) If $D(a,b) > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local min.
(b) If $D(a,b) > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local max.
(c) If $D(a,b) < 0$, then $f(a,b)$ is a saddle point.

Extreme Value Theorem: If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute max value $f(x_1, y_1)$ and an absolute min value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

Method of Lagrange Multipliers: To find the maximum and minimum values of $f(x,y,z)$ subject to the constraint $g(x,y,z) = k$:
(a) Find all values of x, y, z,and λ such that

$$\nabla f(x,y,z) = \lambda \nabla g(z,y,z) \qquad \text{and} \qquad g(x,y,z) = k.$$

(b) Evaluate f at all the points (x,y,z) that result from step (a). The largest of these values is the maximum value of f; the smallest is the minimum value of f.

Double Integrals: The double integral of f over the rectangle R is

$$\int_R\int f(x,y)dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

if the limit exists.

Fubini's Theorem: If f is continuous on the rectangle $R = \{(x,y) |\, a \le x \le b,\, c \le y \le d\}$, then

$$\int_R\int f(x,y)dA = \int_a^b\int_c^d f(x,y)dy\,dx = \int_c^d\int_a^b f(x,y)dx\,dy.$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integral exists.

Theorem 11: Let $A(D) =$ area of a region D. If $m \le f(x,y) \le M$ for all (x,y) in D, then

$$mA(D) \le \int_D\int f(x,y)\,dA \le MA(D).$$

Change to Polar Coordinates in a Double Integral: If f is continuous on a *polar* rectangle (i.e. a piece of a circle) R given by $0 \le a \le r \le b$, $\alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$, then

$$\int_R\int f(x,y)\,dA = \int_\alpha^\beta\int_a^b f(r\cos\theta, r\sin\theta)r\,dr\,d\theta.$$

Change to Sherical Coordinates in a Triple Integral: If f is continuous on a spherical regoin S, bounded by $0 \le a \le \rho \le b$, $0 \le \alpha \le \theta \le \beta \le 2\pi$, and $0 \le \mu \le \varphi \le \nu$ (see page one for definition of spherical coordinates), then

$$\int\int_S\int f(x,y,z)\,dV = \int_a^b\int_\alpha^\beta\int_\mu^\nu f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi.$$

Position Vector: The path of an object can be described as $\mathbf{r}(t) =< f(t), g(t) >$. Then the velocity of the object is $\mathbf{r}'(t) = \mathbf{v}(t)$ and its acceleration is given by $\mathbf{r}''(t) = \mathbf{v}'(t) = \mathbf{a}(t)$.

Arc Length: Let $s(t) =$ the length of the path traveled by a position vector $\mathbf{r}(t)$ from 0 to t. Then $\frac{ds}{dt}$ is the velocity of the particle. We have that

$$s = \int_0^t |r'(u)|\,du \quad \text{and thus} \quad \frac{ds}{dt} = |\mathbf{r}'(t)|$$

Unit Tangent Vector: Let $\mathbf{r}(t)$ be a position vector. Then the unit tangent vector of $\mathbf{r}(t)$ is

$$\mathbf{T} = \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}(t)|} = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}.$$

Curvature Vector: Let $\mathbf{r}(t)$ be a position vector and $s(t)$ be the length traveled by $\mathbf{r}(t)$. Then the curvature vector of $\mathbf{r}(t)$ is

$$\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}}{dt}\frac{dt}{ds} = \frac{\mathbf{T}'(t)}{|\mathbf{v}(t)|}.$$

Define the curvature of $\mathbf{r}(t)$ as the magnitude of the curvature vector, i.e.

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{v}(t)|}.$$

Theorem A: Let $\mathbf{r}(t) =< f(t), g(t) >$, $x = f(t)$, and $y = g(t)$. Then

$$\kappa = \frac{|x'y'' - y'x''|}{((x')^2 + (y')^2)^{3/2}}.$$

Unit Normal Vector: The principal unit normal vector $\mathbf{N}$ is given by

$$\mathbf{N} = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds}.$$

Then $\mathbf{T} \cdot \mathbf{N} = 0$.

Components of Acceleration: We can express the acceleration vector $\mathbf{a}(t)$ in terms of $\mathbf{T}$ and $\mathbf{N}$. We have that

$$\mathbf{a} = \frac{d^2s}{dt^2}\mathbf{T} + \left(\frac{ds}{dt}\right)^2 \kappa\mathbf{N} = a_T\mathbf{T} + a_N\mathbf{N}$$

which can also be written as

$$a_T = \mathbf{T} \cdot \mathbf{a} = \frac{\mathbf{r}' \cdot \mathbf{r}''}{|\mathbf{r}'|} \quad \text{and} \quad a_N = |\mathbf{T} \times \mathbf{a}| = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|}.$$

Binormal Vector: The unit binormal vector is defined as $\mathbf{B} = \mathbf{T} \times \mathbf{N}$. Clearly $\mathbf{B}$ is perpendicular to both $\mathbf{T}$ and $\mathbf{N}$.