Stochastic Processes; Notes and Problems

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A standard Brownian motion, or Wiener process, is a stochastic process $\{B_t : t \geq 0\}$ with state space $(\mathbf{R}, \mathcal{B})$ defined on some probability space $(\Omega, \mathcal{F}, P)$ satisfying the following 3 conditions:

(1) The process starts at 0; $P(B_0 = 0) = 1$.

(2) The increments of the process are independent; for any $n \geq 1$ and $0 = t_0 < t_1 < ... < t_n$,

$$P(\cap_{k=1}^{n}[B_{t_k} - B_{t_{k-1}} \in A_k]) = \prod_{k=1}^{n} P(B_{t_k} - B_{t_{k-1}} \in A_k)$$

for $A_k \in \mathcal{B}$, $1 \leq k \leq n$.

(3) Each increment $B_t - B_s$ is normally distributed with mean 0 and variance $|t - s|$; that is it has density

$$f(x) = (2\pi|t-s|)^{-1/2}\, e^{-x^2/2|t-s|}.$$

Theorem 1 *Brownian motion exists on some probability space.*

E1: Give a proof of Theorem 1. (One approach is via Kolmogorov's consistency conditions.)

Theorem 2 *There exists a version of* $\{B_t : t \geq 0\}$ *that is a.s. continuous.*

Stochastic processes $\{X_t : t \in T\}$ and $\{\widetilde{X}_t : t \in T\}$ defined on $(\Omega, \mathcal{F}, P)$ and $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$ respectively with common state space $(E, \mathcal{E})$ are *equivalent* if they have the same finite-dimensional distributions. They are then termed *versions* of each other. Here E is a topological space and $\mathcal{E}$ is the Borel σ-field on E. Two versions X and $\widetilde{X}$ of a stochastic process defined on a common probability space are *modifications* of each other if

$$P(X_t = \widetilde{X}_t) = 1 \text{ for each } t \in T.$$

The versions are *indistinguishable* if there exists $\Lambda \in \mathcal{F}$ with $P(\Lambda) = 1$ and

$$X_t(\omega) = \widetilde{X}_t(\omega) \text{ for all } t \in T \text{ and } \omega \in \Lambda.$$

The stochastic process $\{X_t : t \in \mathbf{R}\}$ with state space $(E, \mathcal{E})$ is *a.s. continuous* if there exists $\Lambda \in \mathcal{F}$ with $P(\Lambda) = 1$ and $f_\omega(t) = X_t(\omega)$ a continuous function of t for all $\omega \in \Lambda$.

Theorem 3 *Kolmogorov's continuity criterion. If the stochastic process* $\{X_t : t \in \mathbf{R}\}$ *has state space* $(\mathbf{R}, \mathcal{B})$ *and there exists* $\alpha, \beta, \kappa > 0$ *such that*

$$E(|X_{t+h} - X_t|^\alpha) \leq \kappa h^{1+\beta} \textit{ for all } t \in \mathbf{R} \textit{ and } h > 0,$$

then there exists a modification of $\{X_t\}$ *which is a.s. continuous.*

E2: Suppose that $\{X_t : t \in [0,1]\}$ satisfies Kolmogorov's continuity condition, let D denote the dyadic numbers in [0,1], and assume that $\{X_t : t \in D\}$ is a.s. continuous. Define

$$\widetilde{X}_t = \lim s \to t : s \in DX_s$$

and show that $\{(\widetilde{X}_t : t \in [0,1]\}$ is a version of $\{X_t : t \in [0,1]\}$.

The following simple lemma is key to the proof of Kolmogorov's continuity theorem.

Lemma 4 *If Δ is a finite subset of $T \times T$, then*

$$E(\sup_{(s,t)\in\Delta} |X_s - X_t|^\alpha) \le |\Delta| \sup_{(s,t)\in\Delta} E|X_s - X_t|^\alpha.$$

If $\alpha < 1$, then the following simple real variables lemma is useful.

Lemma 5 *If $\alpha \in (0,1)$ and both $a \ge 0$ and $b \ge 0$, then*

$$(a+b)^\alpha \le a^\alpha + b^\alpha.$$

E3: Provide a proof of Lemma 5.

Proposition 1 *Let* $\mathbf{H}$ *be a real separable Hilbert space. There exists a probability space* $(\Omega, \mathcal{F}, P)$ *and a family of random variables* $\{X_h : h \in \mathbf{H}\}$ *such that the following hold.*

(i.) The map $h \to X_h$ *is linear.*

(ii.) X_h *is a normal random variable with mean 0 and variance* $||h||^2_{\mathbf{H}}$.

Note: the process $\{X_h : h \in \mathbf{H}\}$ is generally referred to as a *Gaussian process indexed by* $\mathbf{H}$.

Example 1: Taking $\mathbf{H} = \mathbf{L}_2([0,\infty), \mathcal{B}_{[0,\infty)}, \lambda)$ and letting $h_t = \mathbf{1}[0,t]$ for $t \geq 0$ gives

$$\{B_t : t \geq 0\} =^d \{X_{h_t} : t \geq 0\}$$

where $\{B_t : t \geq 0\}$ is a standard Brownian motion.

E4: Show that $\{B_t : t \geq 0\} =^d \{X_{h_t} : t \geq 0\}$.

Example 2: *Brownian sheet* is a mean 0 Gaussian process indexed by $t \in [0,\infty)^d$, $d \geq 2$ with the finite dimensional distributions specified by

$$E(B_s B_t) = \prod_{i=1}^{d} (s_i \wedge t_i).$$

This corresponds to taking $h_t = \prod_{i=1}^d \mathbf{1}[0, t_i]$

Example 3: *Levy's Brownian motion* is a mean 0 Gaussian process indexed by $t \in \mathbf{R}^d$, $d \geq 2$ with the finite dimensional distributions specified by the following.

(i.) $P(Z_0 = 0) = 1$.

(ii.) $E(Z_t - Z_s)^2 = |t - s|$.

This corresponds to considering the Hilbert space $\mathbf{H} = \mathbf{L}_2(\mathbf{R}^d, \mathcal{B}_d, \mu)$ where the measure μ has density $f_d(x) = |x|^{1-d}$ with respect to Lebesgue measure and taking $h_t = c_d \mathbf{1}[A_t]$ where

$$A_t = \left\{ s \in \mathbf{R}^d : |s - \frac{t}{2}| \leq \frac{|t|}{2} \right\}$$

is the ball of radius $|t|/2$ centered at $t/2$ and c_d is a constant depending on the dimension d.

E5: Find an expression for the covariance of Levy's Brownian motion and verify that the h_t defined in Example 3 gives a Gaussian process with the specified finite dimensional distributions.

Example 4: *Fractional Brownian motion of order* $\beta \in (0, 1)$ is a mean 0 Gaussian process indexed by $t \in \mathbf{R}$ with finite dimensional distributions specified by the following.

(i.) $P(B_0 = 0) = 1$.

(ii.) $E(B_t - B_s)^2 = |t - s|^{2\beta}$.

This process can be shown to exist by taking

$$h_t(x) = c_\beta(|t - x|^{\beta - 1/2} sign(t - x) + |x|^{\beta - 1/2} sign(x)).$$

E6: Find an expression for the covariance of fractional Brownian motion and verify that the h_t defined in Example 4 gives a Gaussian process with the specified finite dimensional distributions.

Note: fractional Brownian motion also exists in higher dimensions. The structure of h_t there is somewhat more complicated.

Theorem 6 *A generalization of Kolmolgorov's Continuity Theorem.*

Let $\{X_t : t \in [0,1]^d\}$ be a real-valued stochastic process with

$$E(|X_s - X_t|^\alpha) \le \kappa |s-t|^{d+\beta} \text{ for all } s,t \in [0,1]^d$$

for some $\alpha, \beta, \kappa > 0$. Then there exists a version $\{\widetilde{X}_t\}$ of $\{X_t\}$ which is a.s. continuous. Furthermore

$$E \sup_{s \ne t} \left(\frac{|\widetilde{X}_s - \widetilde{X}_t|}{|t-s|^\gamma} \right)^\alpha < \infty \tag{1}$$

for all $\gamma \in (0, \beta/\alpha)$. In particular, the paths of $\widetilde{X}$ are Holder continuous of order γ for all $\gamma \in (0, \beta/\alpha)$.

E7: In class we showed the following:

$$E \sup_{s,t \in D : |s-t| < 2^{-N}} |X_s - X_t|^\alpha \ \le \ C\, 2^{-N\beta}$$

where $C = C(\alpha, d) \in (0, \infty)$. Use this to deduce the conclusion of the theorem. Note: a function $f : \mathbf{R}^d \to \mathbf{R}$ is Holder continuous of order (or exponent) $\gamma > 0$ if

$$\sup_{x \ne y} \frac{|f(x) - f(y)|}{|x-y|^\gamma} \ < \ \infty.$$

The theorem above specifies that the increments of the stochastic process X, which has index space $\mathbf{R}^d$, be appropriately small in L_α. The stronger assumption that the process X is Gaussian permits weaker assumptions on the indexing space. Restated, Gaussian processes are continuous on quite large and rich indexing spaces.

Theorem 7 *(Dudley) Metric entropy and continuity of Gaussian processes*

Let $\{Z_h : h \in \mathbf{H}_0\}$ be a mean 0 Gaussian process indexed by $\mathbf{H}_0 \subset \mathbf{H}$ where $\mathbf{H}$ is a real separable Hilbert space. Let

$$N(\delta) = \min\{N : \text{there exists } h_1, ..., h_N \in \mathbf{H} \text{ with } \min_{1 \le n \le N} ||h - h_n||_{\mathbf{H}} < \delta \text{ for all } h \in \mathbf{H}_0\}.$$

If

$$\int_0^1 \sqrt{\ln(N(\delta))}\, d\delta < \infty$$

then there exists an a.s. continuous modification of $\{Z_h : h \in \mathbf{H}_0\}$.

E8: Figure out what $N(\delta)$ is for $\{B_t : 0 \le t \le 1\}$, Brownian motion restricted to the unit interval. What about Brownian sheet on a unit rectangle or Levy's Brownian motion on the unit disc centered at the origin? (Take an arbitrary dimension d.)

E9: Make up a new Gaussian process indexed by a subset of a real separable Hilbert space that has an a.s. continuous modification. Verify the continuity.

Let $\{X_n : n \geq 0\}$ be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, P)$ with $\{X_n\}$ *adapted* to a filtration $\{\mathcal{F}_n\}$; that is X_n is measurable $\{\mathcal{F}_n\}$ for each n. If each X_n is integrable and for each $n \geq 1$

$$E(X_n|\mathcal{F}_{n-1}) = X_{n-1} \qquad a.s.\ P$$

then $\{X_n : n \geq 0\}$ is a *martingale sequence relative to* $\{\mathcal{F}_n\}$. (Recall that a discrete filtration in $\mathcal{F}$ is a sequence of σ-fields $\{\mathcal{F}_n : n \geq 0\}$ with $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ for all $n \geq 0$.)

Example 1: Let $\{B_t : t \geq 0\}$ be standard Brownian motion, and take $\{s_n : n \geq 0\}$ be an increasing sequence with $s_0 = 0$. For each n let

$$\mathcal{F}_n = \sigma(\{B_{s_k} : 0 \leq k \leq n\}).$$

Then $\{B_{s_n} : n \geq 0\}$ is a martingale sequence relative to $\mathcal{F}_n$. If $s_n \to s < \infty$, then $B_{s_n} \to B_s$ a.s. P.

Example 2: Let $(\Omega, \mathcal{F}, P) = ((0,1], \mathcal{B}, \lambda)$ and set $U(\omega) = \omega$. Let

$$\mathcal{F}_n = \sigma(\{((k-1)2^{-n}, k2^{-n}] : 1 \leq k \leq 2^n\})$$

and

$$U_n = E(U|\mathcal{F}_n).$$

Then $\{U_n\}$ is a martingale sequence relative to $\{\mathcal{F}_n\}$ with $U_n \to U$ a.s. P as $n \to \infty$.

Example 3: Let $\{Y_k : k \geq 1\}$ be a sequence of independent and integrable mean 0 r.v.'s. Take $S_0 = 0$ and for $n \geq 1$, let

$$S_n = \sum_{k=1}^{n} Y_k$$

and

$$\mathcal{F}_n = \sigma(\{Y_k : 1 \leq k \leq n\}).$$

Then $\{S_n\}$ is a martingale sequence relative to $\{\mathcal{F}_n\}$. If the Y_k's are also in L_2 with $\sum_{k\geq 1} Var(Y_k) < \infty$, then S_n converges a.s. P to some finite r.v. as $n \to \infty$.

The following gives a link between Markov chains and martingale sequences. Assume that the sequence $\{X_n : n \geq 0\}$ has discrete state space $\mathbf{S}$ and take the associated filtration $\{\mathcal{F}_n\}$ to be the canonical one; that is let $\mathcal{F}_n = \sigma(X_0, ..., X_n)$.

Theorem 8 *The following are equivalent.*

(i.) $\{X_n : n \geq 0\}$ *is a time-homogeneous Markov chain with transition matrix* $\mathbf{P}$.

(ii.) For all bounded $f : \mathbf{S} \to \mathbf{R}$, $\{M_n^f : n \geq 0\}$ *is a martingale relative to* $\{\mathcal{F}_n\}$ *where*

$$M_n^f = f(X_n) - f(X_0) - \sum_{k=0}^{n-1} ((\mathbf{P} - I)f)(X_k).$$

Here for a matrix $A = (a_{kj})$, the vector (Af) is given by

$$(Af)(k) = \sum_{j \in \mathbf{S}} a_{kj} f(j).$$

Let $\{X_n : n \geq 0\}$ be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, P)$ with $\{X_n\}$ adapted to a filtration $\{\mathcal{F}_n\}$. If each X_n is integrable and for each $n \geq 1$

$$E(X_n | \mathcal{F}_{n-1}) \geq X_{n-1} \qquad a.s.\ P$$

then $\{X_n : n \geq 0\}$ is a *sub-martingale sequence relative to* $\{\mathcal{F}_n\}$.

Example 2 continued: Let the V_n denote the "lower" $\mathcal{F}_n$-measurable approximation to U, so

$$V_n = U_n - 2^{-n-1}.$$

Then

$$E(V_{n+1} | \mathcal{F}_n) = V_n + 2^{-n-2} \geq V_n,$$

and we see that $\{V_n\}$ is a sub-martingale sequence.

Example 3 continued: Let

$$X_n = |S_n|.$$

Then

$$E(X_{n+1} | \mathcal{F}_n) = E(|S_{n+1}| | \mathcal{F}_n) \geq |E(S_{n+1} | \mathcal{F}_n)| = |S_n| = X_n,$$

giving a sub-martingale sequence.

Indeed, this can be generalized as follows.

Proposition 2 *If* $\{Y_n : n \geq 1\}$ *is a martingale sequence relative to the filtration* $\{\mathcal{F}_n : n \geq 1\}$ *and* φ *is convex with* $X_n := \varphi(Y_n)$ *integrable for each* $n \geq 1$*, then* $\{X_n : n \geq 1\}$ *is a sub-martingale relative to* $\{\mathcal{F}_n : n \geq 1\}$*.*

Theorem 9 *The Optional Stopping Theorem, Version I*

Suppose that $\{X_n\}$ *is a martingale sequence adapted to a filtration* $\{\mathcal{F}_n\}$*,* τ *is a stopping time relative to the same filtration, and either*

(i.) $P(\tau \leq n) = 1$ *for some fixed* $n < \infty$

or

(ii.) $P(\tau < \infty) = 1$ *and for each* $n \geq 1$*,* $[\tau \leq n] \subset [|X_n| \leq C]$ *for some* $C < \infty$*.*

Then

$$EX_\tau = EX_0.$$

Theorem 10 *The Optional Stopping Theorem, Version II.*

If $\{X_n : n \geq 0\}$ *is a sub-martingale sequence and* τ_1*,* τ_2 *are stopping times with* $1 \leq \tau_1 \leq \tau_2 \leq n$ *for some* $n < \infty$*, then*

$$(X_{\tau_1}, X_{\tau_2})$$

forms a sub-martingale sequence; that is

$$E(X_{\tau_2}|\mathcal{F}_{\tau_1}) \geq X_{\tau_1}$$

a.s. P. In particular

$$E(X_{\tau_2}) \geq E(X_{\tau_1}).$$

Corollary 11 *If* $\{X_n : n \geq 0\}$ *is a martingale sequence and* τ_1*,* τ_2 *are stopping times with* $1 \leq \tau_1 \leq \tau_2 \leq n$ *for some* $n < \infty$*, then*

$$(X_{\tau_1}, X_{\tau_2})$$

forms a martingale sequence; that is

$$E(X_{\tau_2}|\mathcal{F}_{\tau_1}) = X_{\tau_1}.$$

a.s. P. In particular

$$E(X_{\tau_2}) = E(X_{\tau_1}).$$

Let $\{X_n : n \geq 0\}$ be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, P)$ with $\{X_n\}$ adapted to a filtration $\{\mathcal{F}_n\}$. If each X_n is integrable and for each $n \geq 1$

$$E(X_n|\mathcal{F}_{n-1}) \leq X_{n-1} \qquad a.s.\ P$$

then $\{X_n : n \geq 0\}$ is a *super-martingale sequence relative to* $\{\mathcal{F}_n\}$.

Example: Notice that if $\{X_n\}$ is a sub-martingale, then $\{-X_n\}$ is a super-martingale.

E10: (Billingsley) Suppose that $\{\Delta_k : k \geq 0\}$ are independent mean 0 r.v.'s. Let $X_1 = \Delta_1$ and $X_{n+1} = X_n + \Delta_{n+1} f_n(X_1, ..., X_n)$ for $n \geq 1$. Suppose that the X_n's are integrable. Show that the X_n's form a martingale sequence.

E11: (Billingsley) Suppose that $\{X_n : n \geq 1\}$ is a mean 0 martingale sequence. Show that for $n, k \geq 1$,

$$E(X_{n+k} - X_n)^2 = \sum_{j=1}^{k} E(X_{n+j} - X_{n+j-1})^2.$$

This is the equivalent of showing that the variance of the sum is the sum of the variances for a sum of independent r.v.'s. Show that if

$$\sum_{n \geq 1} E(X_{n+1} - X_n)^2 < \infty$$

then X_n converges a.s. P.

E12: (Billingsley) Show that if $\{X_n : n \geq 1\}$ is a sub-martingale sequence adapted to $\{\mathcal{F}_n : n \geq 0\}$, then it can be decomposed into a $\{\mathcal{F}_n\}$ adapted martingale and an increasing predictable process. That is, show that

$$X_n = Y_n + Z_n$$

where $\{Y_n : n \geq 1\}$ is a martingale sequence and $0 = Z_1 \leq Z_2 \leq Z_3 \leq ...$ with Z_{n+1} measurable $\mathcal{F}_n$ for each $n \geq 1$.

E13: (Billingsley) Let $\{X_n : n \geq 1\}$ be a martingale and assume that $|X_1|$ and $|X_n - X_{n-1}|$ are uniformly bounded by a constant. Let τ be a stopping time with $E\tau < \infty$. Show that X_τ is integrable with $EX_\tau = EX_1$.

E14: (BMP) Let $\{X_n : n \geq 1\}$ be a sub-martingale sequence with EX_n constant in n. Show that $\{X_n : n \geq 1\}$ is a martingale sequence.

E15: (Billingsley) Show that $\{X_n : n \geq 1\}$ is a martingale sequence relative to $\{\mathcal{F}_n : n \geq 0\}$ if and only if for all n and stopping times τ with $\tau \leq n$,

$$E(X_n | \mathcal{F}_\tau) = X_\tau.$$

E16: (BMP) Suppose that $\{X_n : n \geq 1\}$ and $\{Y_n : n \geq 1\}$ are two sub-martingale sequences relative to a common filtration $\{\mathcal{F}_n : n \geq 0\}$. Let τ be a stopping time with $X_\tau \geq Y_\tau$ on the set $[\tau < \infty]$. Let

$$Z_n = Y_n \mathbf{1}[n < \tau] + X_n \mathbf{1}[n \geq \tau].$$

Show that $\{Z_n : n \geq 0\}$ is a sub-martingale sequence.

The following maximal inequality for sub-martingales can be regarded as a generalization of Kolmogorov's maximal inequality for sums of independent random variables.

Theorem 12 *Doob's Inequality*

If $\{X_n : n \geq 1\}$ *is a sub-martingale sequence, then for any* $\alpha > 0$

$$P(\max_{1 \leq k \leq n} X_k \geq \alpha) \leq \frac{E|X_n|}{\alpha}.$$

Here is a simple "martingale convergence theorem". Note that it covers sub-martingales as well.

Theorem 13 *Let* $\{X_n : n \geq 1\}$ *be a sub-martingale sequence. If* $\sup_n E|X_n| \leq K < \infty$*, then* $X_n \to X$ *a.s.* P*, where* $E|X| < \infty$.

The proof of this theorem relies on the upcrossing lemma.

Here the random variable $N_n(\alpha, \beta)$ is the number of upcrossings of the interval $[\alpha, \beta]$ by the process $\{X_1, ..., X_n\}$. An upcrossing occurs if $X_k \leq \alpha$, and for some $j \geq 1$, $X_{k+j} \geq \beta$. If no upcrossings occur, $N_n(\alpha, \beta) = 0$. If at least 1 upcrossing occurs, $N_n(\alpha, \beta)$ is defined as follows. Let $\tau_0 = 0$ and then define $\tau_1, \tau_2, ...\tau_n$ iteratively for $j \geq 1$ via

$$\tau_{2j-1} = \min\{j : \tau_{2j-2} < k \leq n, X_k \leq \alpha\}$$

and

$$\tau_{2j} = \min\{j : \tau_{2j-1} < k \leq n, X_k \geq \beta\}.$$

If the described event does not occur, then the τ_k's are taken to be n. Finally then, if $N_n(\alpha, \beta) > 0$, it is given by

$$N_n(\alpha, \beta) = \max\{j : 1 \leq j \leq n/2, X_{\tau_{2j-1}} \leq \alpha < \beta \leq X_{\tau_{2j}}\}.$$

Lemma 14 *Upcrossing Lemma*

If $\{X_n : n \geq 1\}$ *is a sub-martingale sequence then*

$$EN_n(\alpha, \beta) \leq \frac{E|X_n| + |\alpha|}{\beta - \alpha}.$$

The concept of uniform integrability is often useful when using martingale sequences. A collection of r.v.'s $\{Y_\alpha : \alpha \in A\}$ is *uniformly integrable* if

$$\sup_{\alpha \in A} E|Y_\alpha| \mathbf{1}[|Y_\alpha| > M] \to 0 \quad \text{as} \quad M \to \infty.$$

The idea behind uniform integrability is somewhat akin to tightness.

E17: Suppose that the sequence $\{X_n : n \geq 1\}$ is uniformly integrable. Show that $\sup_n E|X_n|$ is finite.

E18: Suppose that the sequences $\{X_n : n \geq 1\}$ and $\{Y_n : n \geq 1\}$ are both uniformly integrable. Let $Z_n = X_n + Y_n$. Show that $\{Z_n : n \geq 1\}$ is uniformly integrable as well. Hint: $|x+y| \leq 2\, max(|x|, |y|)$.

Theorem 15 *If $\{X_n : n \geq 1\}$ is uniformly integrable and $X_n \to X$ a.s. P, then X is integrable and both $EX_n \to EX$ and $X_n \to X$ in L_1.*

The following theorem provides some classes of uniformly integrable r.v.'s.

Theorem 16 *Let X be an integrable r.v. on the probability space $(\Omega, \mathcal{F}, P)$ and let $\{\mathcal{F}_\alpha : \alpha \in A\}$ be a collection of sub-σ-fields of $\mathcal{F}$. Then $\{E(X|\mathcal{F}_\alpha) : \alpha \in A\}$ is uniformly integrable.*

Theorem 17 *If X is an integrable r.v. and $\{\mathcal{F}_n : n \geq 1\}$ is a filtration, then*

$$E(X|\mathcal{F}_n) \to E(X|\mathcal{F}_\infty)$$

a.s. P where $\mathcal{F}_\infty = \sigma(\cup_{n\geq 1}\mathcal{F}_n)$.

E19: In the proof of Theorem 17 above, Theorem 16, E17, and the Martingale Convergence Theorem (Theorem 13) together tell us that X_n converges a.s. to an integrable limiting r.v.. Explain why the limiting r.v. is measurable $\mathcal{F}_\infty$.

Corollary 18 *If $A \in \mathcal{F}_\infty$, then $P(A|\mathcal{F}_n) \to \mathbf{1}_A$ a.s. P.*

This corollary provides, for example, a proof of Kolmogorov's 0-1 law.

Theorem 19 *If $X_n \to X$ in probability, then the following are equivalent.*

(i.) $\{X_n : n \geq 1\}$ *is uniformly integrable.*

(ii.) $X_n \to X$ *in* L_1 *with* $E|X| < \infty$.

(iii.) $E|X_n| \to E|X|$ *with* $E|X| < \infty$.

For sub-martingale sequences, this leads to the following.

Theorem 20 *If $\{X_n : n \geq 1\}$ is a sub-martingale sequence, then the following are equivalent.*

(i.) $\{X_n : n \geq 1\}$ *is uniformly integrable.*

(ii.) $X_n \to X$ *a.s.* P *and* $X_n \to X$ *in* L_1 *with* $E|X| < \infty$.

(iii.) $E|X_n| \to E|X|$ *in* L_1 *with* $E|X| < \infty$.

Compare the following lemma to Theorem 17.

Lemma 21 *If $\{X_n : n \geq 1\}$ is a martingale sequence with $X_n \to X$ in L_1 where $E|X| < \infty$, then $X_n = E(X|\mathcal{F}_n)$ for each $n \geq 1$.*

E20: Prove the following:

Theorem 22 *If $\{X_n : n \geq 1\}$ is a martingale sequence, then the following are equivalent.*

(i.) $\{X_n : n \geq 1\}$ is uniformly integrable.

(ii.) $X_n \to X$ a.s. P and $X_n \to X$ in L_1 with $E|X| < \infty$.

(iii.) $E|X_n| \to E|X|$ in L_1 with $E|X| < \infty$.

(iv.) There exists an integrable r.v. X with $X_n = E(X|\mathcal{F}_n)$ for each $n \geq 1$.

E21: (Durrett) Let $\{X_n : n \geq 1\}$ be a sequence of r.v.'s taking values in the interval [0,1] that is adapted to the filtration $\{\mathcal{F}_n : n \geq 1\}$. Fix α, $\beta > 0$ with $\alpha + \beta = 1$ and suppose that

$$P(X_{n+1} = \alpha + \beta X_n|\mathcal{F}_n) = X_n \quad \text{and} \quad P(X_{n+1} = \beta X_n|\mathcal{F}_n) = 1 - X_n.$$

Show that

$$P(\lim_n X_n \in \{0\} \cup \{1\}) = 1$$

and, on the set $[X_0 = \theta]$,

$$P(\lim_n X_n = 1) = \theta.$$

A *reversed* (or backward) martingale can be thought of as a martingale indexed by $\{..., -3, -2, -1\}$ relative to the increasing filtration $\{\mathcal{F}_{-n} : n \geq 1\}$. More precisely, $\{X_{-n} : n \geq 1\}$ is a reversed martingale relative to $\{\mathcal{F}_{-n} : n \geq 1\}$ if

- $\mathcal{F}_{-n-1} \subset \mathcal{F}_{-n}$
- X_{-n} is measurable $\mathcal{F}_{-n}$
- $E|X_{-n}| < \infty$ for all $n \geq 1$
- $E(X_{-n}|\mathcal{F}_{-n-1}) = X_{-n-1}$ a.s. P.

Theorem 23 *If $\{X_{-n} : n \geq 1\}$ is a reversed martingale then $\lim_{n\to\infty} X_{-n} = X$ exists a.s. P and is integrable with $EX = EX_{-n}$ for all $n \geq 1$.*

Theorem 24 *Let $\{\mathcal{F}_n : n \geq 1\}$ be a sequence of decreasing σ-fields with $\mathcal{G} = \cap_{n\geq 1}\mathcal{F}_n$. If Z is an integrable r.v., then*

$$E(Z|\mathcal{F}_n) \to E(Z|\mathcal{G}) \quad a.s.\ P.$$

The simple idea of reversed martingales is surprisingly powerful when applied to some classical problems in probability. For example, Theorem 23 can be used together with Kolmogorov's 0-1 Law to give a short proof of the Strong Law of Large Numbers for i.i.d. integrable r.v.'s.

The following maximal inequalities follow from Doob's inequality.

Theorem 25 *If $\{X_n : n \geq 1\}$ is a submartingale with $E(X_n^+)^p < \infty$ for each $n \geq 1$ and some $p > 1$, then*

$$E(\max_{k \leq n} X_k^+)^p \leq (\frac{p}{p-1})^p E(X_n^+)^p.$$

This leads to the following convergence theorem.

Theorem 26 *If $\{X_n : n \geq 1\}$ is a martingale with $\sup_n E|X_n|^p < \infty$, then $X_n \to X$ a.s. P and $E|X_n - X|^p \to 0$.*

Let $\{\mathcal{F}_n : n \geq 0\}$ be a filtration. The sequence $\{A_n : n \geq 1\}$ is *predictable* with respect to $\{\mathcal{F}_n : n \geq 0\}$ if for each n A_n is measurable $\mathcal{F}_{n-1}$.

Theorem 27 *Doob's Decomposition. Any submartingale $\{X_n : n \geq 0\}$ can be decomposed uniquely as*

$$X_n = M_n + A_n$$

where $\{M_n\}$ is a martingale sequence and $\{A_n\}$ is a nondecreasing predictable sequence with $A_0 = 0$.

E22: (Durrett) Let $\{Y_n : n \geq 1\}$ be a sequence of i.i.d. mean 0 r.v.'s. Fix $k \geq 2$ and assume that $E|Y_n|^k < \infty$. Show that

$$T_n = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} Y_{i_1} \cdots Y_{i_k}$$

is a martingale.

E23: (Durrett) Let $\{X_n : n \geq 1\}$ and $\{Y_n : n \geq 1\}$ both be submartingales relative to the same filtration. Show that $Z_n = X_n \vee Y_n$ is also a submartingale.

E24: (Durrett) Let $\{X_n : n \geq 0\}$ be a martingale with $X_0 = 0$ a.s. P and $EX_n^2 < \infty$ for all n. Show that for all $a > 0$

$$P(\max_{k \leq n} X_k \geq a) \leq \frac{EX_n^2}{EX_n^2 + a^2}.$$

Hint: $(X_n + \lambda)^2$ is a submartingale. Consider optimizing over λ.

E25: Let $\{X_n : n \geq 0\}$ be a martingale and take $\Delta_n = X_n - X_{n-1}$. Show that if a sequence $b_n \uparrow \infty$ and

$$\sum_{n \geq 1} \frac{E\Delta_n^2}{b_n^2} < \infty$$

then

$$\frac{X_n}{b_n} \to 0 \quad a.s. \quad P.$$

A particularly important example of Doob's Decomposition is the following. Let $\{Y_n\}$ be a martingale. Then the submartingale $\{Y_n^2\}$ is a submartingale which has the decomposition

$$Y_n^2 = M_n + A_n$$

where

$$A_n = \sum_{k=1}^{n} (E(Y_k^2 | \mathcal{F}_{k-1}) - Y_{k-1}^2).$$

Notice that $EA_n = EY_n^2$. Let $A_\infty = \lim_{n\to\infty} A_n$.

Theorem 28 *For $\omega \in [A_\infty < \infty]$, $\lim_{n\to\infty} Y_n(\omega)$ exists except for perhaps ω in a subset of probability 0.*

The proof relies on the following simple lemma.

Lemma 29 *If $\{X_n : n \geq 0\}$ is a submartingale and τ is a stopping time, then $Y_n = X_{\tau \wedge n}$ is a submartingale sequence.*

E26: Give a proof of Lemma 29.

Doob's decomposition and Theorem 28 above lead to the following strong limit result.

Theorem 30 *If $f : [0, \infty) \to [1, \infty)$ is an increasing function with*

$$\int_0^\infty (f(t))^{-2} dt < \infty$$

then

$$\frac{Y_n}{f(A_n)} \to 0 \quad a.s. \quad \text{on the set} \quad [A_\infty = \infty].$$

Theorem 31 *A Martingale Central Limit Theorem. Let $\{M_n : n \geq 1\}$ be a mean 0 martingale sequence and take*

$$V_n^2 = \sum_{k=1}^{n} E((M_k - M_{k-1})^2 | \mathcal{F}_{k-1}).$$

Suppose that

$$\sup_n |M_n - M_{n-1}| \leq K < \infty$$

and

$$\lim_n V_n^2 = \infty.$$

Then for $\tau_t = \min\{n : V_n^2 \geq t\}$,

$$\frac{M_{\tau_t}}{\sqrt{t}} \to^d \ Z$$

where Z is a standard normal r.v.

E27: As in E22, let $\{Y_n : n \geq 1\}$ be a sequence of i.i.d. mean 0 r.v.'s. Fix $k \geq 2$ and assume that $E|Y_n|^k < \infty$. Sketch a strong law and central limit theorem for the martingale

$$T_n = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} Y_{i_1} \cdots Y_{i_k}.$$

Note: you can impose some more conditions on the Y_n's if needed. This class of random variables is related to *U-statistics*.

We have been looking at discretely indexed martingales to date. We now go on to martingales indexed by a continuous 'time' parameter.

Let $\{\mathcal{F}_t : t \geq 0\}$ be a filtration; that is assume that each $\mathcal{F}_t$ is a σ-field and $\mathcal{F}_s \subset \mathcal{F}_t$ for $s < t$. $\{X_t : t \geq 0\}$ is an integrable *sub-martingale* with respect to $\{\mathcal{F}_t : t \geq 0\}$ if

- $\{X_t : t \geq 0\}$ is adapted to $\{\mathcal{F}_t : t \geq 0\}$; that is X_t is measurable $\mathcal{F}_t$ for each t.
- $E|X_t| < \infty$ for each t.
- $E(X_t|\mathcal{F}_s) \geq X_s$ a.s. P for all $0 \leq s < t < \infty$.

If $\{-X_t : t \geq 0\}$ is an integrable *sub-martingale* with respect to $\{\mathcal{F}_t : t \geq 0\}$, then $\{X_t : t \geq 0\}$ is an integrable *super-martingale* with respect to $\{\mathcal{F}_t : t \geq 0\}$. If both $\{X_t : t \geq 0\}$ and $\{-X_t : t \geq 0\}$ are integrable sub-martingales, then $\{X_t : t \geq 0\}$ is an integrable *martingale*; that is

- $E(X_t|\mathcal{F}_s) = X_s$ a.s. P for all $0 \leq s < t < \infty$.

Note: a process $\{X_t : t \geq 0\}$ has a canonical filtration $\{\mathcal{F}_t : t \geq 0\}$ associated with it that is given by $\mathcal{F}_t = \sigma(\{X_s : 0 \leq s \leq t\})$.

The most important family of examples of martingales with a continuous time index are described in the following.

Proposition 3 *Let $\{B_t : t \geq 0\}$ be standard Brownian motion. The following processes are each martingales relative to the filtration $\{\mathcal{F}_t : t \geq 0\}$.*

(1) $\{B_t : t \geq 0\}$.

(2) $\{B_t^2 - t : t \geq 0\}$.

(3) $\{\exp(\alpha B_t - \alpha^2 t/2) : t \geq 0\}$ *for any fixed* $\alpha \in \mathbf{R}$.

E28: (Revuz and Yor) Choose some function $f \in L_2([0,\infty), \lambda)$ where λ is Lebesgue measure. Let $\{Z_t : t \geq 0\}$ be the associated mean 0 Gaussian process defined via

$$Z_t = X(f \cdot \mathbf{1}[0,t]).$$

Show that the following are martingales:

(a) $\{Z_t : t \geq 0\}$

(b) $\{Z_t^2 - \int_{[0,t]} f^2(s)ds : t \geq 0\}$

(c) $\{\exp(\alpha Z_t - \alpha^2 \int_{[0,t]} f^2(s)ds/2) : t \geq 0\}$ for any fixed $\alpha \in \mathbf{R}$.

The following inequalities for submartingales with a continuous time index are completely analogous to those seen in the discrete time setting.

Theorem 32 *Doob's L_p inequality. If $\{X_t : a \leq t \leq b\}$ is right-continuous and a martingale or positive submartingale, then for any $p \geq 1$ and $\lambda > 0$*

$$P(\sup_{a \leq t \leq b} |X_t| \geq \lambda) \leq \lambda^{-p} \sup_{a \leq t \leq b} E|X_t|^p = \lambda^{-p} E|X_b|^p$$

and for $p > 1$

$$E^{1/p}(\sup_{a \leq t \leq b} |X_t|)^p \leq \frac{p}{p-1} \sup_{a \leq t \leq b} E^{1/p}|X_t|^p = \frac{p}{p-1} E^{1/p}|X_b|^p.$$

The following proposition gives an example of the application of Doob's inequality in the continuous time setting.

Proposition 4 *For any $\lambda > 0$ and $t > 0$*

$$P(\sup_{0 \leq s \leq t} B_s \geq \lambda t) \leq e^{-\lambda^2 t/2}.$$

The following extends the earlier optional stopping theorems.

Theorem 33 *The Optional Stopping Theorem, Version III. If $\{X_n\}$ is adapted to a filtration $\{\mathcal{F}_n\}$ it is a martingale relative to $\{\mathcal{F}_n\}$ if and only if*

$$EX_{\tau_1} = EX_{\tau_2}$$

for each pair of stopping times τ_1 and τ_2 satisfying

$$\tau_1 \leq \tau_1 \leq M$$

for some $M < \infty$.

In this context, the following version of the Upcrossing Lemma will be useful.

Lemma 34 *Upcrossing Lemma, Version II. Suppose that $T \subset [0, \infty)$ is countable. If $\{X_t : t \in T\}$ is a submartingale and $N_T(a, b)$ is the number of upcrossings of $[a, b]$ by $\{X_t : t \in T\}$ then*

$$EN_T(a, b) \leq \sup_{t \in T} \frac{E(X_t - a)^+}{b - a}.$$

E29: Give a proof of this second version of the Upcrossing Lemma.

Theorem 35 *Regularization Theorem: If $\{X_t : t \in T\}$ is a submartingale, then there exists $\Lambda \subset \Omega$ with $P(\Lambda) = 1$ and for each $\omega \in \Lambda$ and $t > 0$*

$$\lim_{\{r \uparrow t : r \in \mathbf{Q}\}} X_r(\omega) \quad \textit{exists}$$

and for each $\omega \in \Lambda$ and $t \geq 0$

$$\lim_{\{r \downarrow t : r \in \mathbf{Q}\}} X_r(\omega) \quad \textit{exists.}$$

We use these limits to define two new processes:

$$X_{t^-} := \limsup_{\{r \uparrow t : r \in \mathbf{Q}\}} X_r(\omega) \mathbf{1}(\Lambda)$$

and

$$X_{t^+} := \limsup_{\{r \downarrow t : r \in \mathbf{Q}\}} X_r(\omega) \mathbf{1}(\Lambda).$$

To go with these processes we need new filtrations:

$$\mathcal{F}_{t^-} := \sigma(\cup_{s<t} \mathcal{F}_s)$$

where $\mathcal{F}_{0^-} := \mathcal{F}_0$ and

$$\mathcal{F}_{t^+} := \cap_{s>t} \mathcal{F}_s.$$

Extending our limit theory for reversed martingales to cover sub-martingales as well will be helpful in working with the above processes.

Theorem 36 *(1) If $\{X_{-n} : n \geq 0\}$ is a submartingale, then $\lim_{n \to \infty} X_{-n} := X$ exists a.s. P. (2) In addition, if $\sup_n E|X_{-n}| < \infty$, then $\{X_{-n} : n \geq 0\}$ is uniformly integrable and $X_{-n} \to X$ in L_1 with $X \leq E(X_{-n}|\mathcal{F}_\infty)$ for all $n \geq 0$ where*

$$\mathcal{F}_\infty = \cap_{n \geq 0} \mathcal{F}_{-n}.$$

E30: Give a proof of (1) in the theorem above.

Corollary 37 *Let $\{X_n : n \geq 0\}$ and X be random variables with $X_n \to X$ a.s. P where $\sup |X_n| \leq Y$ with $EY < \infty$. If $\{\mathcal{F}_n : n \geq 0\}$ is a sequence of increasing σ-fields, then $E(X_n|\mathcal{F}_n) \to E(X|\mathcal{F})$ a.s. P where $\mathcal{F} = \sigma(\cup \mathcal{F}_n)$. If $\{\mathcal{F}_n : n \geq 0\}$ is a sequence of decreasing σ-fields, then $E(X_n|\mathcal{F}_n) \to E(X|\mathcal{F})$ a. s. P where $\mathcal{F} = \sigma(\cap \mathcal{F}_n)$.*

Proposition 5 *Let* $\{X_t : t \in T\}$ *be a submartingale adapted to the filtration* $\{\mathcal{F}_t : t \geq 0\}$*. Suppose that* $E|X_t| < \infty$ *for each* $t \geq 0$*. Then* $E|X_{t+}| < \infty$ *for each* $t \geq 0$ *as well and*

$$X_t \leq E(X_{t+}|\mathcal{F}_t).$$

If EX_t *is a right-continuous function of* t*, then*

$$X_t = E(X_{t+}|\mathcal{F}_t).$$

Finally, $\{X_{t+} : t \in T\}$ *is a submartingale adapted to the filtration* $\{\mathcal{F}_{t+} : t \geq 0\}$ *and* $\{X_{t+} : t \in T\}$ *is a martingale if* $\{X_t : t \geq 0\}$ *is a martingale.*

Proposition 6 *Let* $\{X_t : t \in T\}$ *be a submartingale adapted to the filtration* $\{\mathcal{F}_t : t \geq 0\}$*. Suppose that* $E|X_t| < \infty$ *for each* $t \geq 0$*. Then* $E|X_{t-}| < \infty$ *for each* $t \geq 0$ *as well and*

$$X_{t-} \leq E(X_t|\mathcal{F}_{t-}).$$

If EX_t *is a right-continuous function of* t*, then*

$$X_{t-} = E(X_t|\mathcal{F}_{t-}).$$

Finally, $\{X_{t-} : t \in T\}$ *is a submartingale adapted to the filtration* $\{\mathcal{F}_{t-} : t \geq 0\}$ *and* $\{X_{t-} : t \in T\}$ *is a martingale if* $\{X_t : t \geq 0\}$ *is a martingale.*

E31: Give a proof of Proposition 6.

The previous development leads to the following.

Theorem 38 *If $\{X_t : t \geq 0\}$ is a right-continuous submartingale relative to the filtration $\{\mathcal{F}_t : t \geq 0\}$ then the following hold.*

(1) $\{X_t : t \geq 0\}$ is a submartingale relative to both $\{\mathcal{F}_{t^+} : t \geq 0\}$ and the completion of $\{\mathcal{F}_{t^+} : t \geq 0\}$.

(2) Almost every path $\{X_t(\omega) : t \geq 0\}$ is cadlag.

E32: Give a proof of Theorem 38.

Theorem 39 *Let $\{X_t : t \geq 0\}$ be a submartingale relative to a right-continuous and complete filtration $\{\mathcal{F}_t : t \geq 0\}$. If EX_t is right-continuous as a function of t, then $\{X_t : t \geq 0\}$ has a cadlag modification which is a submartingale relative to $\{\mathcal{F}_t : t \geq 0\}$.*

This means that we can routinely assume that we are working with submartingales with right-continuous paths a.s.. The following demonstrates the utility of this simplification.

Lemma 40 *Upcrossing Lemma, Version III. If $\{X_t : t \geq 0\}$ is a right-continuous submartingale then for $a < b$,*

$$EN_{[0,\infty)}(a, b) \leq \frac{\sup_{t \geq 0} E(X_t - a)^+}{b - a}.$$

E33: Sketch the proof of the Upcrossing lemma, version III.

This gives the following.

Theorem 41 *If $\{X_t : t \geq 0\}$ is a right-continuous submartingale with $\sup_t EX_t^+ < \infty$, then $\lim_{t\to\infty} X_t$ exists a.s. P.*

Corollary 42 *If $\{Y_t : t \geq 0\}$ is a non-negative right-continuous supermartingale then $\lim_{t\to\infty} Y_t$ exists a.s. P.*

We now turn our attention to optional stopping theory in the setting of continuous time. Throughout take

$$\mathcal{F}_\infty = \sigma(\cup_{t\geq 0}\mathcal{F}_t)$$

and assume that $\{X_t : t \geq 0\}$ is cadlag a.s. P. The following parallels an earlier theorem with a discrete time index.

Theorem 43 *If $\{X_t : t \geq 0\}$ is a martingale, the following are equivalent.*

(1) X_t converges in L_1 to a limiting r.v. X as $t \to \infty$.

(2) There exists an integrable r.v. X_∞ such that $X_t = E(X_\infty|\mathcal{F}_t)$.

(3) $\{X_t : t \geq 0\}$ is uniformly integrable.

If the above hold, then $X_t \to X_\infty$ a.s. P. If also $\sup_{t\geq 0} E|X_t|^p < \infty$, then $X_t \to X_\infty$ in L_p and the L_p versions of (1)-(3) hold as well.

Theorem 44 *Optional stopping in continuous time.*

(1) If $\{X_t : t \geq 0\}$ is a martingale and S and T are stopping times with $S \leq T \leq m < \infty$, then

$$X_S = E(X_T|\mathcal{F}_S) \quad a.s.\ P.$$

(2) If $\{X_t : t \geq 0\}$ is an uniformly integrable martingale then $\{X_S : S$ is a stopping time$\}$ is an uniformly integrable collection of r.v.'s and if $S \leq T$,

$$X_S = E(X_T|\mathcal{F}_S) = E(X_\infty|\mathcal{F}_S).$$

Proposition 7 *A cadlag adapted process $\{X_t : t \geq 0\}$ is a martingale if and only if for all bounded stopping times T, X_T is integrable and $EX_T = EX_0$.*

Example: Consider standard Brownian motion $\{B_t : t \geq 0\}$ with $B_0 = 0$ and for fixed $a \neq 0$ let

$$T_a = \inf\{t > 0 : B_t = a\}.$$

We know from previous work that Brownian motion is a continuous martingale. In particular $B_{T_a} = a$. Then

$$a = EB_{T_a} \neq EB_0 = 0.$$

We can conclude that the stopping time T_a is not bounded.

Here are some more applications of the optional stopping theory to first passage times for standard Brownian motion. Recall the family of mean 1 exponential martingales associated with Brownian motion $\{B_t : t \geq 0\}$

$$M_t^\alpha = e^{\alpha B_t - \alpha^2 t/2}$$

for $\alpha \in \mathbf{R}$. We again define the first passage time of B_t to level a as

$$T_a = \inf\{t > 0 : B_t = a\}.$$

From the a.s. continuity of Brownian motion,

$$M_{T_a}^\alpha = e^{\alpha a - \alpha^2 T_a/2}.$$

This non-negative r.v. is bounded above by $e^{\alpha a}$. Indeed, if $\alpha > 0$ then for any $t \leq T_a$, $M_t^\alpha \leq e^{\alpha a}$. This guarantees that the stopped martingale $M_{t\wedge T_a}^\alpha$ is uniformly integrable. Then the optional stopping theorem gives

$$1 = EM_0^\alpha = EM_{t\wedge T_a}^\alpha$$

for all t finite. Letting $t \to \infty$ and using the dominated convergence theorem gives

$$1 = EM_{T_a}^\alpha = Ee^{\alpha a - \alpha^2 T_a/2}.$$

Setting $\lambda = \alpha^2/2$ gives

$$Ee^{-\lambda T_a} = e^{-a\sqrt{2\lambda}}$$

for $\lambda > 0$. This can be used in turn to show that

$$P(T_a \leq t) \leq e^{-a^2/2t}.$$

In particular, letting $t \to \infty$ shows that T_a is finite a.s..

In the following consider the shifted Brownian motion $x+B_t$ where $x \in \mathbf{R}$. Denote the corresponding probability measure as P_x and expectation as E_x.

Proposition 8 *If $a < x < b$, then*

$$P_x(T_a < T_b) + \frac{b-x}{b-a}.$$

The proof of the above proposition is an easy consequence of applying an optional stopping theorem in connection with the stopped martingale $B_{t\wedge T_a \wedge T_b}$.

Similarly, using the stopped martingale $B_{t\wedge T_a\wedge T_b}^2 - t \wedge T_a \wedge T_b$ gives

$$E_x T_a \wedge T_b = (b+a)x - ab.$$

E34: Show that the rescaled process $\{a^2 B_{t/a} : t \geq 0\}$ is also a Brownian motion.

E35: For $b > 0$, let
$$\sigma_b = \inf\{t : B_t < t - b\}.$$
Show that σ_b is finite a.s. for b finite, but $\lim_{b\to\infty} \sigma_b = \infty$ a.s.. Hint: use the following variant on the exponential martingale for Brownian motion:
$$e^{-(\sqrt{1+2\lambda}-1)(B_t - t) - \lambda t}$$
where $\lambda > 0$.

E36: Let $\{M_t : t \geq 0\}$ be a positive continuous martingale converging to 0 a.s. as $t \to \infty$. Let $M* = \sup_{t\geq 0} M_t$. . Show that, for any $x > 0$,
$$P(M* \geq x \,|\mathcal{F}_0) = 1 \wedge (M_0/x).$$
Take X to be a positive $\mathcal{F}_0$ measurable r.v.. Show that
$$P(M* \geq X \,|\mathcal{F}_0) = 1 \wedge (M_0/X).$$
Now show that M_0 is the largest $\mathcal{F}_0$ measurable r.v. smaller than $M*$ and that the distribution of $M*$ is the same as that of M_0/U where U is independent of M_0 and uniformly distributed on the unit interval.

In the following let (S,d) denote a separable locally compact metric space, and $\mathcal{S}$ the Borel σ-field on S. Let μ be a σ-finite measure on $(S,\mathcal{S})$. A family of non-negative functions $\{p_t : t \geq 0\}$ with each $p_t : S \times S \to [0,\infty)$ being jointly measurable in t, x, and y, is a family of *transition densities with respect to* μ if both:

(1) For all $t \geq 0$ and $x \in S$, $\int_S p_t(x,y)d\mu(y) = 1$.

(2) For all $s,t \geq 0$ and $x,y \in S$,

$$p_{t+s}(x,y) = \int_S p_t(x,z)p_s(z,y)d\mu(z).$$

Example: Take $S = \mathbf{R}$ and set $p_t(x,y) = (2\pi t)^{-1/2}e^{-(x-y)^2/2t}$.

The family of *transition operators* $\{\mathcal{T}_t : t \geq 0\}$ associated with $\{p_t : t \geq 0\}$ is given by

$$\mathcal{T}_t f(x) = \int_S p_t(x,y)f(y)d\mu(y).$$

Notice that $\mathcal{T}_t$ maps functions to functions. We will start by assuming that we are working with uniformly bounded functions; i.e. we assume above that $f \in L_\infty(S)$.

A family of operators $\{\mathcal{T}_t : t \geq 0\}$ is a *Markov semigroup* if it satisfies the following four conditions.

(1) For each $t \geq 0$, $\mathcal{T}_t$ is a bounded linear operator from $L_\infty(S)$ to $L_\infty(S)$.

(2) For each $f \in L_\infty(S)$, $\mathcal{T}_0 f = f$ a.e. with respect to μ.

(3) For all $s,t \geq 0$, $\mathcal{T}_{t+s}f = \mathcal{T}_t\mathcal{T}_s f$.

(4) If $f \in L_\infty(S)$ has $f(x) \geq 0$ for all $x \in S$, then for all $t \geq 0$ and $x \in S$, $\mathcal{T}_t f(x) \geq 0$ as well.

Lemma 45 *The family of transition operators associated with a family of transition densities is a Markov semigroup.*

A Markov semigroup $\{\mathcal{T}_t : t \geq 0\}$ is a *Feller semigroup* if the following two conditions also hold.

(1) For each $t \geq 0$ and $f \in C_0(S)$, $\mathcal{T}_t f$ is also in $C_0(S)$.

(2) For all $f \in C_0(S)$, $\lim_{t\downarrow 0}\sup_{x\in S}|\mathcal{T}_t f(x) - f(x)| = 0$.

Note: $C_0(S)$ is defined to be the family of continuous real-valued functions on S which are uniformly small outside a compact set; that is $f \in C_0(S)$ if for any $\epsilon > 0$ there exists compact $K_\epsilon \subset S$ with $\sup_{x\in K_\epsilon^c}|f(x)| < \epsilon$.

Lemma 46 *If (1) above holds, then (2) is equivalent to*

(2') For all $f \in C_0(S)$, $\lim_{t \downarrow 0} \sup_{s \geq 0} \sup_{x \in S} |\mathcal{T}_{s+t} f(x) - \mathcal{T}_s f(x)| = 0$.

The following proposition gives a mechanism for checking to see if a given transition density results in a Feller semigroup. Note: this is a sufficient condition; not a necessary one.

Proposition 9 *If for all $\delta > 0$, the family of transition densities $\{p_t : t \geq 0\}$ with respect to μ satisfies*

$$\lim_{t \to 0} \sup_{x \in S} \int_{y: d(x,y) > \delta} p_t(x, y) d\mu(y) = 0,$$

then the corresponding family of transition operators is a Feller semigroup.

Example: It is easy to use this proposition to check that the family of transition operators corresponding to Brownian motion is a Feller semi-group.

We say that $\{X_t : t \geq 0\}$ is a *Markov process with initial measure ν, filtration $\{\mathcal{F}_t : t \geq 0\}$ and transition operators $\{\mathcal{T}_t : t \geq 0\}$* if there exists $(\Omega, \mathcal{F}, P)$ such that $\{X_t : t \geq 0\}$ is adapted to $\{\mathcal{F}_t : t \geq 0\}$, $P_\nu(X_0 \in A) = \nu(A)$, and for all $s, t \geq 0$ and $f \in C_0(S)$

$$E_\nu(f(X_{t+s}) | \mathcal{F}_s) \; = \; \mathcal{T}_t f(X_s) \quad P_\nu \;\; a.s..$$

We say simply that $\{X_t : t \geq 0\}$ is a Markov process if it is a Markov process for all initial measures given by δ_x for $x \in S$.

Here's a version of the Chapman-Kolmogorov equations in this setting.

Lemma 47 *For a Markov process $\{X_t : t \geq 0\}$ with initial measure ν and Markov semigroup $\{\mathcal{T}_t : t \geq 0\}$ corresponding to transition densities $\{p_t : t \geq 0\}$ relative to μ,*

$$E_\nu \prod_{k=1}^n \phi(X_{t_k}) = \int_S \cdots \int_S \prod_{k=1}^n \phi(x_k) p_{t_k - t_{k-1}}(x_{k-1}, x_k) \prod_{k=1}^n d\mu(x_k) d\nu(x_0) \quad \textit{for} \quad 0 = t_0 < t_1 < ... < t_n.$$

Complete and augment the filtration as follows. Let $\mathcal{F}_t^\mu$ denote the completion of $\mathcal{F}_t$ with respect to a fixed probability measure μ. Let

$$\bar{\mathcal{F}}_t = \cap_\mu \mathcal{F}_t^\mu$$

where the intersection is over all probability measures μ on $(S, \mathcal{F}_\infty)$. Augment the filtration by setting

$$\mathcal{F}_t^* = \cap_{s>t} \bar{\mathcal{F}}_t.$$

Lemma 48 *If $\{X_t : t \geq 0\}$ is Markov with transition operators $\{\mathcal{T}_t : t \geq 0\}$ and filtration $\{\mathcal{F}_t : t \geq 0\}$ then it is also Markov with transition operators $\{\mathcal{T}_t : t \geq 0\}$ relative to the complete filtration $\{\bar{\mathcal{F}}_t : t \geq 0\}$.*

Recall that we started with a locally compact metric space (S, d). If it isn't compact, adjoin a *cemetery state* Δ to compactify it. Let $S_\Delta = S \cup \{\Delta\}$. Extend a function f from S to S_Δ by defining $f(\Delta) = 0$.

Theorem 49 *Let $\{\mathcal{T}_t : t \geq 0\}$ be a Feller semigroup and let $\{X_t : t \geq 0\}$ be the corresponding Markov process. $\{X_t : t \geq 0\}$ has a right-continuous S_Δ-valued modification that is itself Feller with transition semigroup $\{\mathcal{T}_t : t \geq 0\}$.*

Theorem 50 *Suppose that $\{X_t : t \geq 0\}$ is a right-continuous S_Δ-valued Feller process adapted to the filtration $\{\mathcal{F}_t : t \geq 0\}$. Then it is also a Feller process relative to the complete augmented filtration $\{\mathcal{F}_t^* : t \geq 0\}$.*

A Markov process $\{X_t : t \geq 0\}$ is also a *strong Markov process* if, for all $\{\mathcal{F}_t : t \geq 0\}$ stopping times T,

(1) X_T is measurable $(\Omega, \mathcal{F})$ to $(S_\Delta, \mathcal{F}_\infty)$ and

(2) for all $f \in L_\infty(S)$, all $t \geq 0$ and $x \in S$,

$$E_x(f(X_{t+T}) | \mathcal{F}_T) \mathbf{1}_{[T<\infty]} = E_{X_T} f(X_t).$$

Theorem 51 *Any right-continuous Feller process is also a strong Markov process.*

Example: Brownian motion is a continuous Feller process. Thus it enjoys the strong Markov property.

We now will define the generator of a Markov process, and use it to bring together Markov processes and martingales. The *resolvent* of the family of transition operators $\{\mathcal{T}_t : t \geq 0\}$ is

$$\mathcal{R}_\lambda f(x) = \int_0^\infty e^{-\lambda t} \mathcal{T}_t f(x) dt.$$

This has many interesting properties that we won't explore. Define the generator of $\{\mathcal{T}_t : t \geq 0\}$ as

$$\mathcal{A}f(x) = \lambda f(x) - \mathcal{R}_\lambda^{-1} f(x)$$

for $x \in S$ and $f \in \mathcal{D}(A)$ where

$$\mathcal{D}(A) = \{f : f = \mathcal{R}_\lambda g \text{ for some } g \in C_0(S)\}.$$

Oddly enough, $\mathcal{A}f(x)$ does not depend on the variable λ.

Theorem 52 *If $\{\mathcal{T}_t : t \geq 0\}$ is a Feller semigroup, then*

$$\mathcal{D}(A) = \{g \in C_0(S) : \lim_{t \downarrow 0} \frac{\mathcal{T}_t g - g}{t} \text{ exists in } C_0(S)\}$$

and, if $g \in \mathcal{D}(A)$, then

$$\mathcal{A}g = \lim_{t \downarrow 0} \frac{\mathcal{T}_t g - g}{t}.$$

In other words, we can regard the generator acting pointwise as satisfying

$$\lim_{t \downarrow 0} \sup_{x \in S} \left| \mathcal{A}g(x) - \frac{\mathcal{T}_t g(x) - g(x)}{t} \right| = 0.$$

Theorem 53 *The Martingale Problem. If $\{X_t : t \geq 0\}$ is a Feller process relative to a complete augmented filtration, then for any $f \in \mathcal{D}(A)$ and initial measure P_x, $x \in S$,*

$$M_t^f = f(X_t) - f(X_0) - \int_0^t \mathcal{A}f(X_s) ds$$

is a mean 0 martingale relative to the same filtration.

Theorem 54 *(A Converse.) Take $F \subset \mathcal{D}(A)$ with $\mathcal{A}f \in C_0(S)$ for all $f \in F$. If there exists a linear operator $\mathcal{B} : F \to C_0(S)$ with*

$$N_t^f = f(X_t) - f(X_0) - \int_0^t \mathcal{B}f(X_s) ds$$

being a mean 0 martingale P_x a.s. for all $x \in S$ and $f \in F$, then $\mathcal{B}f = \mathcal{A}f$ for all $f \in F$.

Example: It is not too hard to show that for Brownian motion the generator $\mathcal{A}$ satisfies

$$\mathcal{A}f = \frac{f''}{2}.$$

The above theorem then gives a multitude more martingales built from Brownian motion. One is

$$B_t^4 - B_0^4 - \int_0^t 6B_s^2 ds.$$

You might notice that this formulation could be used to find an inductive form for the moments of Gaussian r.v's.

The following theorem is presented to give you an indication of the rich connections between Brownian motion and martingales.

Theorem 55 *The simple version.*

Most continuous martingales are time-changes of Brownian motion.

Here's a little more precision.

Theorem 56 *Let $\{M_t : t \geq 0\}$ be a continuous martingale relative to the filtration $\{\mathcal{F}_t : t \geq 0\}$ with $EM_t^2 \uparrow \infty$ as $t \uparrow \infty$. Then there exists a collection of stopping times $\{T_s : s \geq 0\}$, increasing in s, such that*

$$\{M_{T_s} : s \geq 0\}$$

is a Brownian motion.

Annotated Bibliography

Markov Chains, by Norris. This is a relatively inexpensive paperback published by Cambridge University Press. It contains a well-written complete presentation of Markov chains with a discrete state space indexed by both discrete and continuous time. It has short sections on martingales and potential theory (for example) that give a brief introduction. It is mathematically very solid, although it does not require measure theory.

Martingale Limit Theory and its Application, by Hall and Heyde. This is a classic book on discrete time index martingales. It has all sorts of interesting results, including a nice treatment of exchangeability. It is much more technical than Norris. This has a complete (and quite complicated) presentation of the central limit theory for martingales with a discrete time index.

Continuous Martingales and Brownian Motion, by Revuz and Yor. This is a very nice book. It starts with a treatment of Gaussian processes using Hilbert spaces and goes on to martingales, Markov processes, stochastic integration, local time, Girsanov's theorem and many other related topics. Highly recommended.

Multiparameter Processes, by Khoshnevisan. There is a nice treatment of Markov processes in this book. Many other topics are covered as well. If you have an interest in multiparameter processes, this is the book for you.

Stochastic Processes with Applications, by Bhattacharya and Waymire. This book has been out of print for several years. It is significantly more applied (and less theoretical) than the previous three books. Has a slew of nice examples.

Probability: Theory and Examples, by Durrett. This contains a nice outline of martingale limit theory in the discrete time setting. Be careful though, there are some omissions and typographical errors.

Probability and Measure, by Billingsley. A solid reference on measure-theoretic probability. Good discussions. Highly recommended.