Useful Formulas

Basic Trigonometric Formulas:

$$\begin{aligned}
\sin(x+y) &= \sin(x)\cos(y)+\cos(x)\sin(y)\\
\sin(x-y) &= \sin(x)\cos(y)-\cos(x)\sin(y)\\
\\
\cos(x+y) &= \cos(x)\cos(y)-\sin(x)\sin(y)\\
\cos(x-y) &= \cos(x)\cos(y)+\sin(x)\sin(y)\\
\\
\tan(x+y) &= \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}\\
\tan(x-y) &= \frac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}\\
\\
\sin(2x) &= 2\sin(x)\cos(x)\\
\cos(2x) &= \cos^2(x)-\sin^2(x) = 2\cos^2(x)-1 = 1-2\sin^2(x)\\
\tan(2x) &= \frac{2\tan(x)}{1-\tan^2(x)}\\
\\
\sin^2(x) &= \frac{1-\cos(2x)}{2}\\
\cos^2(x) &= \frac{1+\cos(2x)}{2}\\
\cos(x/2) &= \sqrt{(1+\cos(x))/2}
\end{aligned}$$

$$\begin{aligned}
\cos(x+\pi/2) &= -\sin(x) \\
\sin(x+\pi/2) &= \cos(x)
\end{aligned}$$

$$\begin{aligned}
\cos(x-\pi/2) &= \sin(x) \\
\sin(x-\pi/2) &= -\cos(x)
\end{aligned}$$

$$\begin{aligned}
\cos(x+\pi) &= -\cos(x) \\
\sin(x+\pi) &= -\sin(x)
\end{aligned}$$

$$\begin{aligned}
\tan(x-\pi/2) &= \frac{\sin(x-\pi/2)}{\cos(x-\pi/2)} = \frac{-\cos(x)}{\sin(x)} = -\cot(x) \\
\tan(x+\pi/2) &= \frac{\sin(x+\pi/2)}{\cos(x+\pi/2)} = \frac{\cos(x)}{-\sin(x)} = -\cot(x) \\
\tan(\pi/2-x) &= \cot(x)
\end{aligned}$$

Composition of Fourier and Linear Transforms: Let T be a linear transform and $S = (T*)^{-1}$. Then

$$\widehat{(f \circ T)} = |\det T|^{-1} \widehat{f} \circ S$$

and if T is a rotation

$$\widehat{(f \circ T)} = \widehat{f} \circ T.$$

John's Equation: Let $x, y, u, v \in \mathbb{R}^n$ and $f \in C^2(\mathbb{R}^n)$. Define

$$\begin{aligned}
p(x,y) &\equiv \int_{\mathbb{R}} f(x + t(x-y))\, dt \\
g(u,v) &\equiv \int_{\mathbb{R}} f(u + tv)\, dt.
\end{aligned}$$

Then p and g are ultra-hyperbolic and satisfy John's equation

$$\begin{aligned}
\frac{\partial^2 p}{\partial x_i \partial y_j} - \frac{\partial^2 p}{\partial y_i \partial x_j} &= 0 \\
\frac{\partial^2 g}{\partial u_i \partial v_j} - \frac{\partial^2 g}{\partial v_i \partial u_j} &= 0.
\end{aligned}$$

Mellin Tranform: The Mellin Transform is defined on $(0, \infty)$ by

$$Mf(s) = \int_0^\infty f(x) x^{s-1}\, dx.$$

It has the following properties:

$$
\begin{aligned}
Mf'(s) &= (1-s)Mf(s-1) \\
M(x^p f)(s) &= Mf(s+p) \\
M(x^p f^{(p)})(s) &= (-1)^p \frac{\Gamma(s+p)}{\Gamma(s)} Mf(s), \quad s > 0 \\
M(f * g)(s) &= MfMg, \quad f * g(s) = \int_0^\infty f(r) g(s/r) \frac{dr}{r}.
\end{aligned}
$$

CHIRP-Z Algorithm: Let $f[n] = f_c(nT)$ for $n \in \mathbb{Z}$ and $f_c(t) = 0$ for $t < 0$ and $t > NT$. The DFT of $f[n]$ enables us to calculate samples of the FT of f_c at $\frac{2\pi k}{NT}$. The CHIRP-Z algorithm provides as efficient way to calculate the frequency samples at $2\pi k D$ for $D \in \mathbb{R}$. Note that

$$
\begin{aligned}
F(e^{j\omega})\big|_{\omega = 2\pi kTD} &= F(e^{j2\pi kTD}) \\
&= \frac{1}{T} \sum_l F_c\left(j\left(2\pi kD - \frac{2\pi l}{T}\right)\right),
\end{aligned}
$$

where T = spatial sampling rate and D = (linear) frequency sampling rate. This can be calculated by

$$
\begin{aligned}
F(e^{j\omega})\big|_{\omega = 2\pi kTD} &= \sum_n f[n] e^{-2\pi jknTD} \\
&= \sum_n f[n] e^{-2\pi jTD[k^2 + n^2 - (n-k)^2]/2} \\
&= e^{-2\pi jk^2 TD/2} \sum_n f[n] e^{-2\pi jn^2 TD/2} e^{2\pi j(n-k)^2 TD/2}.
\end{aligned}
$$

where we have used that $kn = \frac{1}{2}[k^2 + n^2 - (n-k)^2]$. If we let

$$
\begin{aligned}
g[n] &\equiv f[n] e^{-2\pi jn^2 TD/2} \\
h[n] &\equiv e^{-2\pi jn^2 TD/2}
\end{aligned}
$$

then the above can be written as

$$
F(e^{j\omega})\big|_{\omega = 2\pi kTD} = h[k](g[k] * h^*[k]).
$$

The algorithm proceeds as follows: $M \leq \text{floor}\left(\frac{1}{DT}\right)$, $L \geq M + N - 1$,

$$
F[k] = h[k]\, \text{IDFT}_L \left(\text{DFT}_L(g) \text{DFT}_L(h)\right)[k]
$$