HW5

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Contents

We all contributed equally for this homework.

Question 0

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Question 1

- **(a)**

$$
\begin{aligned}
\frac{\partial RSS(a,b)}{\partial a} &= \sum_{i=1}^{n} \frac{\partial}{\partial a}(y_i - a - bx_i)^2 \\
&= \sum_{i=1}^{n} -2(y_i - a - bx_i) \\
&= -2\sum_{i=1}^{n} y_i + 2a\sum_{i=1}^{n} 1 + 2b\sum_{i=1}^{n} x_i
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial RSS(a,b)}{\partial b} &= \sum_{i=1}^{n} \frac{\partial}{\partial b}(y_i - a - bx_i)^2 \\
&= \sum_{i=1}^{n} -2x_i(y_i - a - bx_i) \\
&= -2\sum_{i=1}^{n} x_i y_i + 2a\sum_{i=1}^{n} x_i + 2b\sum_{i=1}^{n} x_i^2
\end{aligned}
$$

- **(b)**

  The estimators, $\hat{\beta}_0, \hat{\beta}_1$, can be found by setting the above partial derivatives equal to 0 to minimize them:

$$
\begin{aligned}
\frac{\partial RSS(\hat{\beta}_0,\hat{\beta}_1)}{\partial \hat{\beta}_0} &= -2\sum_{i=1}^{n} y_i + 2\hat{\beta}_0\sum_{i=1}^{n} 1 + 2\hat{\beta}_1\sum_{i=1}^{n} x_i \\
0 &= -2\sum_{i=1}^{n} y_i + 2\hat{\beta}_0\sum_{i=1}^{n} 1 + 2\hat{\beta}_1\sum_{i=1}^{n} x_i \\
2\sum_{i=1}^{n} y_i &= 2\hat{\beta}_0 n + 2\hat{\beta}_1\sum_{i=1}^{n} x_i \\
2\sum_{i=1}^{n} \frac{n}{n} y_i &= 2\hat{\beta}_0 n\frac{n}{n} + 2\hat{\beta}_1\sum_{i=1}^{n} \frac{n}{n} x_i \\
2n\bar{y} &= 2n\hat{\beta}_0 + 2\hat{\beta}_1 n\bar{x} \\
\bar{y} &= \hat{\beta}_0 + \hat{\beta}_1\bar{x}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial RSS(\hat{\beta}_0, \hat{\beta}_1)}{\partial \hat{\beta}_1} &= \sum_{i=1}^{n} -2x_i(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) \\
0 &= \sum_{i=1}^{n} -2x_i(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) \\
&= \sum_{i=1}^{n} x_i(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) \\
\text{Subsitute: } \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} \\
&= \sum_{i=1}^{n} x_i(y_i - \bar{y} + \hat{\beta}_1 \bar{x} - \hat{\beta}_1 x_i) \\
&= \sum_{i=1}^{n} x_i(y_i - \bar{y}) - \sum_{i=1}^{n} \hat{\beta}_1 x_i(x_i - \bar{x}) \\
\sum_{i=1}^{n} \hat{\beta}_1 x_i(x_i - \bar{x}) &= \sum_{i=1}^{n} x_i(y_i - \bar{y}) \\
\hat{\beta}_1 &= \frac{\sum_{i=1}^{n} x_i(y_i - \bar{y})}{\sum_{i=1}^{n} x_i(x_i - \bar{x})} \\
\hat{\beta}_1 &= \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}
\end{aligned}
$$

So our estimators are:

$$
\begin{aligned}
\hat{\beta}_1 &= \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2} \\
&= \frac{\mathrm{Cov}(x, y)}{\mathrm{Var}(x)} \\
\hat{\beta}_0 &= \bar{y} - \hat{\beta}_1 \bar{x}
\end{aligned}
$$

- **(c)**

$$
\begin{aligned}
\hat{\beta}_1 &= \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2} \\
&= \frac{\sum_{i=1}^{n}(x_i - \bar{x})(\beta_1(x_i - \bar{x}) + (\epsilon_i - \bar{\epsilon}))}{\sum_{i=1}^{n}(x_i - \bar{x})^2} \\
&= \frac{\sum_{i=1}^{n}(x_i - \bar{x})(\beta_1(x_i - \bar{x}))}{\sum_{i=1}^{n}(x_i - \bar{x})^2} \ + \ \frac{\sum_{i=1}^{n}(x_i - \bar{x})(\epsilon_i - \bar{\epsilon})}{\sum_{i=1}^{n}(x_i - \bar{x})^2} \\
&= \frac{\beta_1 \sum_{i=1}^{n}(x_i - \bar{x})^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2} \ + \ \frac{\sum_{i=1}^{n}(x_i - \bar{x})\epsilon_i}{\sum_{i=1}^{n}(x_i - \bar{x})^2} \ + \ 0 \\
&= \beta_1 + \frac{\sum_{i=1}^{n}(x_i - \bar{x})\epsilon_i}{\sum_{i=1}^{n}(x_i - \bar{x})^2} \\
\mathrm{E}(\hat{\beta}_1) &= \beta_1 + \mathrm{E}\left(\frac{\sum_{i=1}^{n}(x_i - \bar{x})\epsilon_i}{\sum_{i=1}^{n}(x_i - \bar{x})^2}\right) \\
&= \beta_1
\end{aligned}
$$

Therefore $\mathrm{E}(\hat{\beta}_1) = \beta_1$ and is unbiased.

- **(d)**

$$
\begin{aligned}
\mathrm{SE}(\hat{\beta}_1)^2 &= \frac{\sigma^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2} \\
&= \frac{\sigma^2}{(n-1)\mathrm{Var}(x)} \\
\sqrt{\mathrm{SE}(\hat{\beta}_1)^2} &= \sqrt{\frac{\sigma^2}{(n-1)\mathrm{Var}(x)}} \\
&= \frac{\sigma}{\sqrt{(n-1)\mathrm{Var}(x)}} \\
&= \frac{\sigma}{\sqrt{n-1}\cdot \mathrm{sd}(x)}
\end{aligned}
$$

Question 2

```
set.seed(2)
beta0 <- 3
beta1 <- 1
x <- runif(100) - 0.5
eps <- rnorm(100)
y <- beta0 + beta1 * x + eps
```

- **(a)**

```
cov_xy <- cov(x,y)  # covariance of x, y
var_x <- var(x)     # variance of x, y
beta_1 <- cov_xy / var_x; beta_1  # beta_1 estimator
```

```
## [1] 0.9485236
```

- **(b)**

```
m <- 3               # new seed choice

set.seed(m)          # set new seed to m
beta0 <- 3
beta1 <- 1
x <- runif(100) - 0.5
eps <- rnorm(100)
y <- beta0 + beta1 * x + eps

cov_xy <- cov(x,y)  # covariance of x, y
var_x <- var(x)     # variance of x, y
beta_1 <- cov_xy / var_x; beta_1  # new beta_1 estimator
```

```
## [1] 1.020513
```

- **(c)**

```
B <- 500  # B = 500 for iteration
beta1.vec <- rep(0,B) # beta vector filled with 0s, of size B

for (j in 1:B) {
  set.seed(m + j) # set new seed to m + j
  beta0 <- 3
```

```
  beta1 <- 1
  x <- runif(100) - 0.5
  eps <- rnorm(100)
  y <- beta0 + beta1 * x + eps

  cov_xy <- cov(x,y)  # covariance of x, y
  var_x <- var(x)     # variance of x, y
  beta_1 <- cov_xy / var_x;  # new beta_1 estimator
  beta1.vec[j] <- beta_1
}

head(beta1.vec) # show first few indexes of beta vector
```

```
## [1] 1.0850856 1.1529613 0.9171344 0.8368420 0.7559993 1.6825957
```

- **(d)**

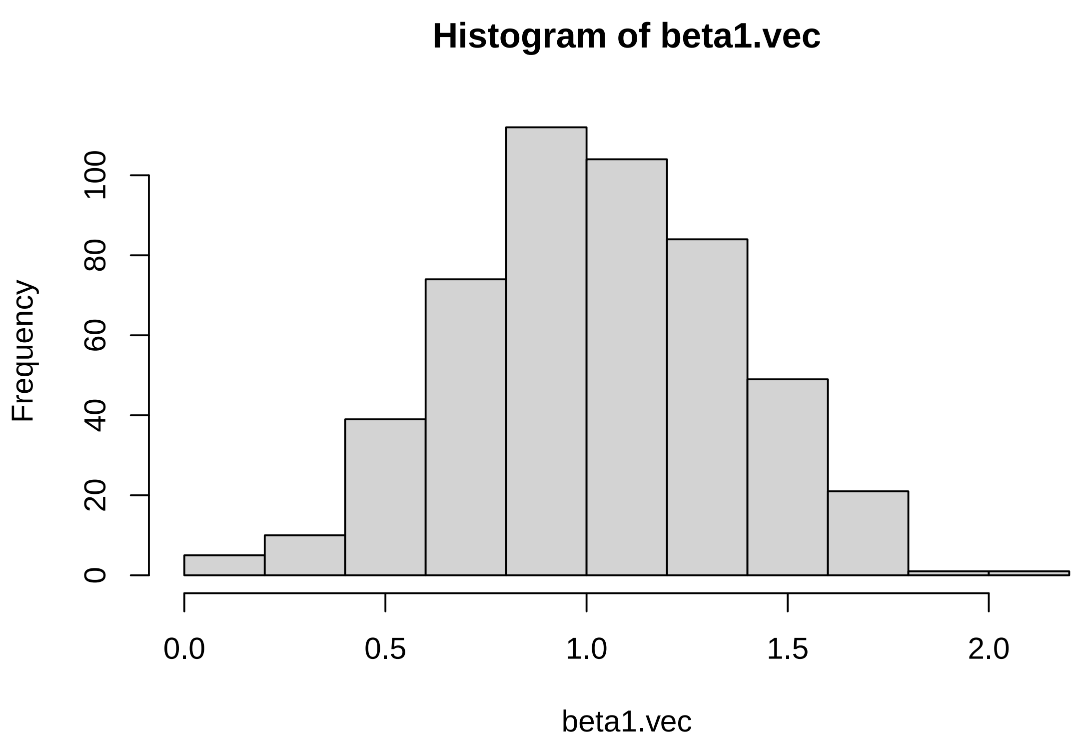
```
mean(beta1.vec)
```

```
## [1] 1.027777
```

Here our sample mean estimator $\hat{\beta}_1 \approx 1.0278$ and our true $\beta_1 = 1$, therefor our estimator is very close to our true coefficient. This makes sense due to $\hat{\beta}_1$ being an unbiased estimator of β_1 as proved above.

- **(e)**

```
hist(beta1.vec)
```



We can see here $\hat{\beta}_1$ is normally distributed with its mean centered at $\beta_1 = 1$ as we would expect.