

# 1 Categories

**Problem 1-1:** The objects of **Rel** are sets, and an arrow  $A \rightarrow B$  is a relation from  $A$  to  $B$ , that is, a subset  $R \subseteq A \times B$ . The equality relation  $\{\langle a, a \rangle \in A \times A \mid a \in A\}$  is the identity arrow on a set  $A$ . Composition in **Rel** is to be given by

$$\{\langle a, c \rangle \in A \times C \mid \exists b(\langle a, b \rangle \in R \ \& \ \langle b, c \rangle \in S)\}$$

for  $R \subseteq A \times B$  and  $S \subseteq B \times C$ .

- (a) Show that **Rel** is a category.
- (b) Show also that there is a functor  $G : \mathbf{Sets} \rightarrow \mathbf{Rel}$  taking objects to themselves and each function  $f : A \rightarrow B$  to its graph,

$$G(f) = \{\langle a, f(a) \rangle \in A \times B \mid a \in A\}$$

- (c) Finally, show that there is a functor  $C : \mathbf{Rel}^{\text{op}} \rightarrow \mathbf{Rel}$  taking each relation  $R \subseteq A \times B$  to its converse  $R^c \subseteq B \times A$ , where,

$$\langle a, b \rangle \in R^c \Leftrightarrow \langle b, a \rangle \in R$$

Max:

- (a) By definition, the composition of two relations is another relation, so composition is closed. It suffices to show the identity and associativity rules.

For identity, let  $1_A \subseteq A \times A$  and  $R \subseteq A \times B$  for some sets  $A, B$ . Then  $R \circ 1_A$  is,

$$\{\langle a, b \rangle \in A \times B \mid \exists a'(\langle a, a' \rangle \in 1_A \wedge \langle a', b \rangle \in R)\}$$

Since  $1_A$  is the identity relation,  $a = a'$ , so the set is equivalent to,

$$\{\langle a, b \rangle \in A \times B \mid \langle a, b \rangle \in R\} = R$$

The case for  $1_A \circ R$  follows similarly.

For associativity, let  $R \subseteq A \times B$ ,  $S \subseteq B \times C$ , and  $T \subseteq C \times D$  for sets  $A, B, C, D$ . Then,

$$\begin{aligned} T \circ (S \circ R) &= \{\langle a, d \rangle \in A \times D \mid \exists c(\langle a, c \rangle \in (S \circ R) \wedge \langle c, d \rangle \in T)\} \\ &= \{\langle a, d \rangle \in A \times D \mid \exists c(\exists b(\langle a, b \rangle \in R \wedge \langle b, c \rangle \in S) \wedge \langle c, d \rangle \in T)\} \\ &= \{\langle a, d \rangle \in A \times D \mid \exists c \exists b(\langle a, b \rangle \in R \wedge \langle b, c \rangle \in S \wedge \langle c, d \rangle \in T)\} \\ &= \{\langle a, d \rangle \in A \times D \mid \exists b \exists c(\langle a, b \rangle \in R \wedge \langle b, c \rangle \in S \wedge \langle c, d \rangle \in T)\} \\ &= \{\langle a, d \rangle \in A \times D \mid \exists b(\langle a, b \rangle \in R \wedge \exists c(\langle b, c \rangle \in S \wedge \langle c, d \rangle \in T))\} \\ &= \{\langle a, d \rangle \in A \times D \mid \exists b(\langle a, b \rangle \in R \wedge \langle b, d \rangle \in T \circ S)\} = (T \circ S) \circ R \end{aligned}$$

- (b)  $G(f)$  is clearly a subset of  $G(A) \times G(B) = A \times B$ , so it suffices to show identities and composition are preserved.

For identities, let  $\iota_A : A \rightarrow A$  be the identity function on set  $A$ . Then,

$$G(\iota_A) = \{\langle a, \iota_A(a) \rangle \in A \times A \mid a \in A\} = \{\langle a, a \rangle \in A \times A \mid a \in A\} = 1_A$$

For composition, let  $f : A \rightarrow B$  and  $g : B \rightarrow C$ . Then,

$$\begin{aligned}
G(g) \circ G(f) &= \{ \langle a, c \rangle \in A \times C \mid \exists b (\langle a, b \rangle \in G(f) \wedge \langle b, c \rangle \in G(g)) \} \\
&= \{ \langle a, c \rangle \in A \times C \mid \exists b (b = f(a) \wedge c = g(b)) \} \\
&= \{ \langle a, c \rangle \in A \times C \mid \exists b (c = g(f(a))) \} \\
&= \{ \langle a, c \rangle \in A \times C \mid c = g(f(a)) \} \\
&= \{ \langle a, c \rangle \in A \times C \mid \exists b (c = (g \circ f)(a)) \} = G(g \circ f)
\end{aligned}$$

(c) Since  $R^c \subseteq C(B) \times C(A) = B \times A$ , morphisms are compatible with objects, so it suffices to show identities and composition are preserved.

For identities, let  $1_A : A \rightarrow A$  be the identity relation on set  $A$ . Then,

$$\langle a, a \rangle \in (1_A)^c \iff \langle a, a \rangle \in 1_A$$

So  $(1_A)^c = 1_A$ .

For composition, let  $R \subseteq A \times B$  and  $S \subseteq B \times C$ . Then,

$$\begin{aligned}
\langle a, c \rangle \in (S \circ R)^c &\iff \langle c, a \rangle \in S \circ R \\
&\iff \exists b (\langle c, b \rangle \in R \wedge \langle b, a \rangle \in S) \\
&\iff \exists b (\langle a, b \rangle \in S^c \wedge \langle b, c \rangle \in R^c) \\
&\iff \langle a, c \rangle \in R^c \circ S^c
\end{aligned}$$

So  $(S \circ R)^c = R^c \circ S^c$ .

□

**Problem 1-2:** Consider the following isomorphisms of categories and determine which ones hold.

- (a)  $\mathbf{Rel} \cong \mathbf{Rel}^{\text{op}}$
- (b)  $\mathbf{Sets} \cong \mathbf{Sets}^{\text{op}}$
- (c) For a fixed set  $X$  with a power set  $P(X)$ , as poset categories  $P(X) \cong P(X)^{\text{op}}$  (the arrows in  $P(X)$  are subset inclusions  $A \subseteq B$  for all  $A, B \subseteq X$ ).

*Max:*

- (a) Consider the functor from problem 1-1(c). It is clear from the symmetry that  $C$
- (b)
- (c)

□

**Problem 1-3:**

- (a) Show that in **Sets**, the isomorphisms are exactly the bijections.
- (b) Show that in **Monoids**, the isomorphisms are exactly the bijective homomorphisms.
- (c) Show that in **Posets**, the isomorphisms are *not* the same as bijective homomorphisms.

Max:

- (a) This follows from elementary set theory, that if  $f : A \rightarrow B$  and  $g : B \rightarrow A$  are such that  $g \circ f$  and  $f \circ g$  are identities, then  $f$  and  $g$  are surjective and injective and thus bijective.
- (b) As a **Monoids** morphism is a set function that preserves the monoid structure, any isomorphism is a bijection by (a).

Conversely, any bijection  $f : M \rightarrow N$  between monoids  $M, N$  satisfies,

$$f(m \cdot m') = f(m) \cdot f(m')$$

For all  $m, m' \in M$ .

Now if  $n, n' \in N$  so that  $n = f(m)$  and  $n' = f(m')$  (which we can assume by surjectivity of  $f$ ), we have,

$$f^{-1}(n \cdot n') = f^{-1}(f(m) \cdot f(m')) = f^{-1}(f(m \cdot m')) = m \cdot m' = f^{-1}(n) \cdot f^{-1}(n')$$

- (c) By elementary set theory, any homomorphism which is an isomorphism must have its categorical inverse be the set function inverse. So it suffices to find a bijective homomorphism whose set theoretic inverse is not a homomorphism.

Let  $\mathcal{P}(2)$  be the poset of subsets of a 2-element set  $2 = \{a, b\}$  under inclusion, and  $4$  be the chain on four elements  $1 \leq 2 \leq 3 \leq 4$ . Then we can define a function  $f : \mathcal{P}(2) \rightarrow 4$  like,

$$\begin{aligned}\{\} &\mapsto 1 \\ \{a\} &\mapsto 2 \\ \{b\} &\mapsto 3 \\ \{a, b\} &\mapsto 4\end{aligned}$$

We have clearly exhibited an order preserving function which is also bijective. However, the inverse map is not bijective since in particular  $2 \leq 3$  but it is not the case that  $\{a\} \subseteq \{b\}$ .

□

**Problem 1-4:** Let  $X$  be a topological space and preorder the points by *specialization*:  $x \leq y$  iff  $y$  is contained in every open set that contains  $x$ . Show that this is a preorder, and that it is a poset if  $X$  is  $T_0$  (for any two distinct points, there is some open set containing one but not the other). Show that the ordering is trivial if  $X$  is  $T_1$  (for any two distinct points, each is contained in an open set not containing the other).

*Max:* Identity is clear since  $x$  is trivially contained in every open set that contains itself.

For transitivity, suppose  $x \leq y$  and  $y \leq z$ . Then if  $U$  is open in  $X$ ,  $x \in U \Rightarrow y \in U \Rightarrow z \in U$  so in particular  $x \in U \Rightarrow z \in U$  so  $x \leq z$ . It follows that specialization is at least a preorder.

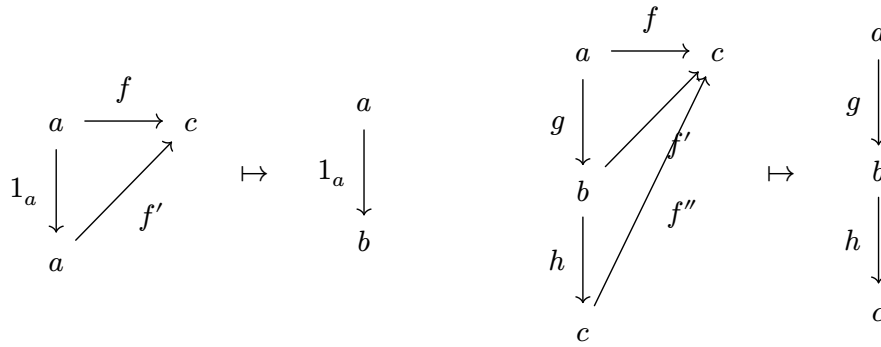
If  $X$  is  $T_0$ , then suppose  $x \leq y$  and  $y \leq x$ . If  $x \neq y$ , we can choose without loss of generality an open  $U$  such that  $x \in U$  and  $y \notin U$ , but the latter is a contradiction since  $x \leq y$ .

Finally if  $X$  is  $T_1$ , and  $x \leq y$ ,  $x \neq y$  creates a contradiction, because we can choose some open  $U, V$  so that  $x \in U$ ,  $y \in V$  and  $U \cap V = \emptyset$ , so in particular  $x \in U$  and  $y \notin U$ , but by  $x \leq y$ ,  $y \in U$ . So the order is trivial - the only pairs  $x, y$  for which  $x \leq y$  are those where  $x = y$ .  $\square$

**Problem 1-5:** For any category  $\mathbf{C}$ , define a functor  $U : \mathbf{C}/C \rightarrow \mathbf{C}$  from the slice category over an object  $C$  that “forgets about  $C$ .” Find a functor  $F : \mathbf{C}/C \rightarrow \mathbf{C}^{\rightarrow}$  to the arrow category such that  $\mathbf{dom} \circ F = U$ .

*Max:* For  $U$ , define  $U(f) = \mathbf{dom} f$  on objects. Then considering  $g : f \rightarrow f'$  as a function  $\mathbf{dom} f \rightarrow \mathbf{dom} f'$  defines the action on morphisms.

Since we define identities in the slice categories as identities on domains, identities are preserved. Similarly, since we define the composition in  $\mathbf{C}/C$  as the composition of the underlying functions on domains, it follows that  $U$  preserves composition.



Now define  $F$  on objects to take each morphism  $f : a \rightarrow c$  to itself, and each morphism  $g : f \rightarrow f'$  to  $\langle g, 1_C \rangle$ . Letting  $f \xrightarrow{g} f' \xrightarrow{g'} f''$ , we have,

$$F(g') \circ F(g) = \langle g', 1_C \rangle \circ \langle g, 1_C \rangle = \langle g' \circ g, 1_C \rangle = F(g' \circ g)$$

Then the composition of functors  $\mathbf{dom} \circ F$  has, for objects  $f \in \mathbf{ob} \mathbf{C}/C$ ,

$$(\mathbf{dom} \circ F)(f) = \mathbf{dom} F(f) = \mathbf{dom} f = U(f)$$

And for morphisms  $g : f \rightarrow f'$ ,

$$(\mathbf{dom} \circ F)(g) = \mathbf{dom} \langle g, 1_C \rangle = g = U(g)$$

□

**Problem 1-6:** Construct the “coslice category”  $C/\mathbf{C}$  of a category  $\mathbf{C}$  under an object  $C$  from the slice category  $\mathbf{C}/C$  and the “dual category” operation  $-^{\text{op}}$ .

**Problem 1-7:** Let  $2 = \{a, b\}$  be any set with exactly 2 elements  $a$  and  $b$ . Define a functor  $F : \mathbf{Sets}/2 \rightarrow \mathbf{Sets} \times \mathbf{Sets}$  with  $F(f : X \rightarrow 2) = (f^{-1}(a), f^{-1}(b))$ . Is this an isomorphism of categories? What about the analagous situation with a one-element set  $1 = \{a\}$  instead of  $2$ ?



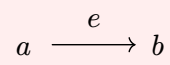
**Problem 1-8:** Any category  $\mathbf{C}$  determines a preorder  $P(\mathbf{C})$  by defining a binary relation  $\leq$  on the objects by

$$A \leq B \text{ if and only if there is an arrow } A \rightarrow B$$

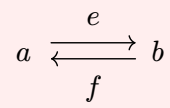
Show that  $P$  determines a functor from categories to preorders, by defining its effect on functors between categories and checking the required conditions. Show that  $P$  is a (one-sided) inverse to the evident inclusion functor of preorders into categories.

**Problem 1-9:** Describe the free categories on the following graphs by determining their objects, arrows, and composition operations.

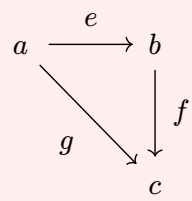
(a)



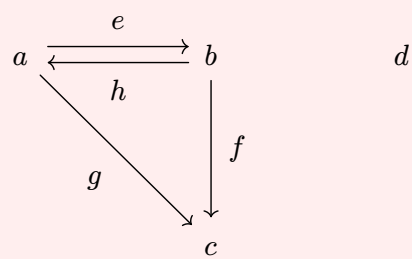
(b)



(c)



(d)



**Problem 1-10:** How many free categories on graphs are there which have exactly six arrows? Draw the graphs that generate these categories.

**Problem 1-11:** Show that the free monoid functor

$$M : \mathbf{Sets} \rightarrow \mathbf{Mon}$$

exists, in two different ways:

- (a) Assume the particular choice  $M(X) = X^*$  and define its effect

$$M(f) : M(A) \rightarrow M(B)$$

on a function  $f : A \rightarrow B$  to be

$$M(f)(a_1 \dots a_k) = f(a_1) \dots f(a_k), \quad a_1, \dots, a_k \in A$$

- (b) Assume only the UMP of the free monoid and use it to determine  $M$  on functions, showing the result to be a functor.

Reflect on how these two approaches are related.

**Problem 1-12:** Verify the UMP for free categories on graphs, defined as above with arrows being sequences of edges. Specifically, let  $\mathbf{C}(G)$  be the free category on the graph  $G$ , so defined, and  $i : G \rightarrow U(\mathbf{C}(G))$  the graph homomorphism taking vertices and edges to themselves, regarded as objects and arrows in  $\mathbf{C}(G)$ . Show that for any category  $\mathbf{D}$  and graph homomorphism  $f : G \rightarrow U(\mathbf{D})$ , there is a unique functor

$$\bar{h} : \mathbf{C}(G) \rightarrow \mathbf{D}$$

with

$$U(\bar{h}) \circ i = f,$$

where  $U : \mathbf{Cat} \rightarrow \mathbf{Graph}$  is the underlying graph functor.

**Problem 1-13:** Use the Cayley representation to show that every small category is isomorphic to a “concrete” one, that is, one in which the objects are sets and the arrows are functions between them.

**Problem 1-14:** The notion of a category can also be defined with just one sort (arrows) . rather than two (arrows and objects); the domains and codomains are taken to be certain *arrows* that act as units under composition, which is partially defined. Read about this definition in section I.1 of Mac Lane's *Categories for the Working Mathematician*, and do the exercise mentioned there, showing that it is equivalent to the usual definition.

