Contents

1 Categories

Problem 1-1: The objects of $\mathbf{Rel}$ are sets, and an arrow $A \to B$ is a relation from A to B, that is, a subset $R \subseteq A \times B$. The equality relation $\{\langle a, a \rangle \in A \times A \mid a \in A\}$ is the identity arrow on a set A. Composition in $\mathbf{Rel}$ is to be given by

$$\{\langle a, c \rangle \in A \times C \mid \exists b(\langle a, b \rangle \in R \,\&\, \langle b, c \rangle \in S)\}$$

for $R \subseteq A \times B$ and $S \subseteq B \times C$.

(a) Show that $\mathbf{Rel}$ is a category.
(b) Show also that there is a functor $G : \mathbf{Sets} \to \mathbf{Rel}$ taking objects to themselves and each function $f : A \to B$ to its graph,

$$G(f) = \{\langle a, f(a) \rangle \in A \times B \mid a \in A\}$$

(c) Finally, show that there is a functor $C : \mathbf{Rel}^{\mathrm{op}} \to \mathbf{Rel}$ taking each relation $R \subseteq A \times B$ to its converse $R^c \subseteq B \times A$, where,

$$\langle a, b \rangle \in R^c \Leftrightarrow \langle b, a \rangle \in R$$

Max:

(a) By definition, the composition of two relations is another relation, so composition is closed. It suffices to show the identity and associativity rules.

For identity, let $1_A \subseteq A \times A$ and $R \subseteq A \times B$ for some sets A, B. Then $R \circ 1_A$ is,

$$\{\langle a, b \rangle \in A \times B \mid \exists a'(\langle a, a' \rangle \in 1_A \wedge \langle a', b \rangle \in R)\}$$

Since 1_A is the identity relation, $a = a'$, so the set is equivalent to,

$$\{\langle a, b \rangle \in A \times B \mid \langle a, b \rangle \in R\} = R$$

The case for $1_A \circ R$ follows similarly.

For associativity, let $R \subseteq A \times B$, $S \subseteq B \times C$, and $T \subseteq C \times D$ for sets A, B, C, D. Then,

$$\begin{aligned}
T \circ (S \circ R) &= \{\langle a, d \rangle \in A \times D \mid \exists c(\langle a, c \rangle \in (S \circ R) \wedge \langle c, d \rangle \in T)\} \\
&= \{\langle a, d \rangle \in A \times D \mid \exists c(\exists b(\langle a, b \rangle \in R \wedge \langle b, c \rangle \in S) \wedge \langle c, d \rangle \in T)\} \\
&= \{\langle a, d \rangle \in A \times D \mid \exists c \exists b(\langle a, b \rangle \in R \wedge \langle b, c \rangle \in S \wedge \langle c, d \rangle \in T\} \\
&= \{\langle a, d \rangle \in A \times D \mid \exists b \exists c(\langle a, b \rangle \in R \wedge \langle b, c \rangle \in S \wedge \langle c, d \rangle \in T\} \\
&= \{\langle a, d \rangle \in A \times D \mid \exists b(\langle a, b \rangle \in R \wedge \exists c(\langle b, c \rangle \in S \wedge \langle c, d \rangle \in T))\} \\
&= \{\langle a, d \rangle \in A \times D \mid \exists b(\langle a, b \rangle \in R \wedge \langle b, d \rangle \in T \circ S)\} \\
&= (T \circ S) \circ R
\end{aligned}$$

(b) $G(f)$ is clearly a subset of $G(A) \times G(B) = A \times B$, so it suffices to show identities and composition are preserved.

For identities, let $\iota_A : A \to A$ be the identity function on set A. Then,

$$G(\iota_A) = \{\langle a, \iota_A(a) \rangle \in A \times A \mid a \in A\} = \{\langle a, a \rangle \in A \times A \mid a \in A\} = 1_A$$

For composition, let $f : A \to B$ and $g : B \to C$. Then,

$$
\begin{aligned}
G(g) \circ G(f) &= \{\langle a, c\rangle \in A \times C \mid \exists b(\langle a, b\rangle \in G(f) \wedge \langle b, c\rangle \in G(g))\} \\
&= \{\langle a, c\rangle \in A \times C \mid \exists b(b = f(a) \wedge c = g(b))\} \\
&= \{\langle a, c\rangle \in A \times C \mid \exists b(c = g(f(a)))\} \\
&= \{\langle a, c\rangle \in A \times C \mid c = g(f(a))\} \\
&= \{\langle a, c\rangle \in A \times C \mid \exists b(c = (g \circ f)(a))\} = G(g \circ f)
\end{aligned}
$$

(c) Since $R^c \subseteq C(B) \times C(A) = B \times A$, morphisms are compatible with objects, so it suffices to show identities and composition are preserved.

For identities, let $1_A : A \to A$ be the identity relation on set A. Then,

$$\langle a, a\rangle \in (1_A)^c \iff \langle a, a\rangle \in 1_A$$

So $(1_A)^c = 1_A$.

For composition, let $R \subseteq A \times B$ and $S \subseteq B \times C$. Then,

$$
\begin{aligned}
\langle a, c\rangle \in (S \circ R)^c &\iff \langle c, a\rangle \in S \circ R \\
&\iff \exists b(\langle c, b\rangle \in R \wedge \langle b, a\rangle \in S) \\
&\iff \exists b(\langle a, b\rangle \in S^c \wedge \langle b, c\rangle \in R^c) \\
&\iff \langle a, c\rangle \in R^c \circ S^c
\end{aligned}
$$

So $(S \circ R)^c = R^c \circ S^c$.

$\square$

Problem 1-2: Consider the following isomorphisms of categories and determine which ones hold.

(a) $\mathbf{Rel} \cong \mathbf{Rel}^{\mathrm{op}}$

(b) $\mathbf{Sets} \cong \mathbf{Sets}^{\mathrm{op}}$

(c) For a fixed set X with a power set $P(X)$, as poset categories $P(X) \cong P(X)^{\mathrm{op}}$ (the arrows in $P(X)$ are subset inclusions $A \subseteq B$ for all $A, B \subseteq X$).

Max:

(a) Consider the functor from problem 1-1(c). We can define in the same way a converse functor from $\mathbf{Rel} \to \mathbf{Rel}^{\mathrm{op}}$ taking each relation to its converse, and since converses are involutions, this is a two-sided inverse.

(b) Suppose F is an isomorphism from $\mathbf{Sets} \to \mathbf{Sets}^{\mathrm{op}}$. Then for any sets A, B, we would have that $\mathrm{Hom}_{\mathbf{Sets}}(A, B) \cong \mathrm{Hom}_{\mathbf{Sets}^{\mathrm{op}}}(F(A), F(B)) \cong \mathrm{Hom}_{\mathbf{Sets}}(F(B), F(A))$, i.e., $|A^B| = |F(B)^{F(A)}|$.

Letting B be the null set, this would mean (for non-empty A), that $0 = |A^{\{\}}| = |F(\{\})^{F(A)}|$. Since there is only one null set, there is some non-empty A such that $F(A)$ is also non-empty. But there is always at least one set-function from any set to a non-empty set, which is a contradiction.

(c) We can define a functor $F : P(X) \to P(X)^{\mathrm{op}}$ by mapping sets to their complement in X and reversing inclusions. This is evidently functorial, and bijective on objects (since compliments are involutions) and on morphisms since $A \subseteq B \iff X - B \subseteq X - A$. It follows that F is an isomorphism of categories.

$\square$

Problem 1-3:

(a) Show that in $\mathbf{Sets}$, the isomorphisms are exactly the bijections.

(b) Show that in $\mathbf{Monoids}$, the isomorphisms are exactly the bijective homomorphisms.

(c) Show that in $\mathbf{Posets}$, the isomorphisms are *not* the same as bijective homomorphisms.

Max:

(a) This follows from elementary set theory, that if $f : A \to B$ and $g : B \to A$ are such that $g \circ f$ and $f \circ g$ are identities, then f and g are surjective and injective and thus bijective.

(b) As a $\mathbf{Monoids}$ morphism is a set function that preserves the monoid structure, any isomorphism is a bijection by (a).

Conversely, any bijection $f : M \to N$ between monoids M, N satisfies,

$$f(m \cdot m') = f(m) \cdot f(m')$$

For all $m, m' \in M$.

Now if $n, n' \in N$ so that $n = f(m)$ and $n' = f(m')$ (which we can assume by surjectivity of f), we have,

$$f^{-1}(n \cdot n') = f^{-1}(f(m) \cdot f(m')) = f^{-1}(f(m \cdot m')) = m \cdot m' = f^{-1}(n) \cdot f^{-1}(n')$$

(c) By elementary set theory, any homomorphism which is an isomorphism must have its categorical inverse be the set function inverse. So it suffices to find a bijective homomorphism whose set theoretic inverse is not a homomorphism.

Let $\mathcal{P}(2)$ be the poset of subsets of a 2-element set $2 = \{a, b\}$ under inclusion, and $\mathbf{4}$ be the chain on four elements $1 \leq 2 \leq 3 \leq 4$. Then we can define a function $f : \mathcal{P}(2) \to \mathbf{4}$ like,

$$\begin{aligned} \{\} &\mapsto 1 \\ \{a\} &\mapsto 2 \\ \{b\} &\mapsto 3 \\ \{a, b\} &\mapsto 4 \end{aligned}$$

We have clearly exhibited an order preserving function which is also bijective. However, the inverse map is not a morphism since in particular $2 \leq 3$ but it is not the case that $\{a\} \subseteq \{b\}$.

$\square$

Problem 1-4: Let X be a topological space and preorder the points by *specialization*: $x \leq y$ iff y is contained in every open set that contains x. Show that this is a preorder, and that it is a poset if X is T_0 (for any two distinct points, thre is some open set containing one but not the other). Show that the ordering is trivial if X is T_1 (for any two distinct points, each is contained in an open set not containing the other).

Max: Identity is clear since x is trivially contained in every open set that contains itself.

For transitivity, suppose $x \leq y$ and $y \leq z$. Then if U is open in X, $x \in U \Rightarrow y \in U \Rightarrow z \in U$ so in particular $x \in U \Rightarrow z \in U$ so $x \leq z$. It follows that specialization is at least a preorder.

If X is T_0, then suppose $x \leq y$ and $y \leq x$. If $x \neq y$, we can choose without loss of generality an open U such that $x \in U$ and $y \notin U$, but the latter is a contradiction since $x \leq y$.

Finally if X is T_1, and $x \leq y$, $x \neq y$ creates a contradiction, because we can choose some open U, V so that $x \in U$, $y \in V$ and $U \cap V = \emptyset$, so in particular $x \in U$ and $y \notin U$, but by $x \leq y$, $y \in U$. So the order is trivial - the only pairs x, y for which $x \leq y$ are those where $x = y$. $\square$

Problem 1-5: For any category $\mathbf{C}$, define a functor $U : \mathbf{C}/C \to \mathbf{C}$ from the slice category over an object C that "forgets about C." Find a functor $F : \mathbf{C}/C \to C^{\to}$ to the arrow category such that $\mathbf{dom} \circ F = U$.

Max: For U, define $U(f) = \mathbf{dom}\, f$ on objects. Then considering $g : f \to f'$ as a function $\mathbf{dom}\, f \to \mathbf{dom}\, f'$ defines the action on morphisms.

Since we define identities in the slice categories as identites on domains, identites are preserved. Similarly, since we define the composition in $\mathbf{C}/C$ as the composition of the underlying functions on domains, it follows that U preserves composition.

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "1_a"'] & c \\ a \arrow[ur, "f'"'] & \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} a \arrow[d, "1_a"'] \\ b \end{tikzcd} \qquad\qquad \begin{tikzcd} a \arrow[r, "f"] \arrow[d, "g"'] & c \\ b \arrow[ur, "f'"] \arrow[d, "h"'] & \\ c \arrow[uur, "f''"'] & \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} a \arrow[d, "g"'] \\ b \arrow[d, "h"'] \\ c \end{tikzcd}$$

Now define F on objects to take each morphism $f : a \to c$ to itself, and each morphism $g : f \to f'$ to $\langle g, 1_C \rangle$. Letting $f \xrightarrow{g} f' \xrightarrow{g'} f''$, we have,

$$F(g') \circ F(g) = \langle g', 1_C \rangle \circ \langle g, 1_C \rangle = \langle g' \circ g, 1_C \rangle = F(g' \circ g)$$

Then the composition of functors $\mathbf{dom} \circ F$ has, for objects $f \in \operatorname{ob} \mathbf{C}/C$,

$$(\mathbf{dom} \circ F)(f) = \mathbf{dom}\, F(f) = \mathbf{dom}\, f = U(f)$$

And for morphisms $g : f \to f'$,

$$(\mathbf{dom} \circ F)(g) = \mathbf{dom}\langle g, 1_C \rangle = g = U(g)$$

$\square$

Problem 1-6: Construct the "coslice category" $C/\mathbf{C}$ of a category $\mathbf{C}$ under an object C from the slice category $\mathbf{C}/C$ and the "dual category" operation $-^{\mathrm{op}}$.

Max: Consider the category $(\mathbf{C}^{\mathrm{op}}/C^{\mathrm{op}})^{\mathrm{op}}$. It has as objects maps $f^{\mathrm{op}} : X^{\mathrm{op}} \to C^{\mathrm{op}}$ in $\mathbf{C}^{\mathrm{op}}$, and as morphisms maps g^{op} satisfying,

$$\begin{tikzcd}
X^{\mathrm{op}} \arrow[r, "f^{\mathrm{op}}"] \arrow[d, "h^{\mathrm{op}}"'] & C^{\mathrm{op}} \\
Y^{\mathrm{op}} \arrow[ur, "f'^{\mathrm{op}}"'] &
\end{tikzcd}$$

Which "unravels" (by reversing arrows) to the following equivalent diagram in $\mathbf{C}$,

$$\begin{tikzcd}
X & C \arrow[l, "f"'] \arrow[dl, "f'"] \\
Y \arrow[u, "h"] &
\end{tikzcd}$$

So we find that the category $(\mathbf{C}^{\mathrm{op}}/C^{\mathrm{op}})^{\mathrm{op}}$ is nothing more than the category of maps f with domain C, and morphisms $h : Y \to X$ satisfying the above diagrams. This is precisely what we would expect from a "coslice" category $C/\mathbf{C}$. $\square$

Problem 1-7: Let $2 = \{a, b\}$ be any set with exactly 2 elements a and b. Define a functor $F : \mathbf{Sets}/2 \to \mathbf{Sets} \times \mathbf{Sets}$ with $F(f : X \to 2) = (f^{-1}(a), f^{-1}(b))$. Is this an isomorphism of categories? What about the analagous situation with a one-element set $1 = \{a\}$ instead of 2?

Max: Consider the object $(\{1\}, \{1\})$ in $\mathbf{Sets} \times \mathbf{Sets}$, and suppose it was the image of some function $f : X \to 2$ under F, so $F(f) = (f^{-1}(a), f^{-1}(b)) = (\{1\}, \{1\})$.

$f^{-1}(a) \cap f^{-1}(b) = \emptyset$ by elementary set theory, but $f^{-1}(a) = \{1\}$ and $f^{-1}(b) = \{1\}$, so $f^{-1}(a) \cap f^{-1}(b) = \{1\} \cap \{1\} = \{1\}$. So $\emptyset = \{1\}$, a contradiction. Thus F is not surjective on objects and cannot be an isomorphism of categories.

We don't run into this problem for the one-element set $1 = \{a\}$ - here we can define an inverse functor $G : \mathbf{Sets} \to \mathbf{Sets}\ /1$ that maps each set X to the unique function $X \to 1$ and acts identically on functions. It is straight-forward to show that this is functorial and a two-sided inverse to F defined analogously as with 2. $\square$

Problem 1-8: Any category $\mathbf{C}$ determines a preorder $P(\mathbf{C})$ by defining a binary relation $\leq$ on the objects by

$$A \leq B \text{ if and only if there is an arrow } A \to B$$

Show that P determines a functor from categories to preorders, by defining its effect on functors between categories and and checking the required conditions. Show that P is a (one-sided) inverse to the evident inclusion functor of preorders into categories.

Max: If F is a functor from $\mathbf{C} \to \mathbf{D}$, let $P(F)$ coincide with the action of F on objects.

Then if $C \leq C'$ in $P(\mathbf{C})$, there is some morphism $f : A \to B$ by definition. Then $F(f)$ is a morphism from $F(A) \to F(B)$ in $\mathbf{D}$ by functorality, so $F(A) \leq F(B)$ in $P(\mathbf{D})$. It follows that $P(F)$ is order-preserving, so this mapping of morphisms is well-defined.

The identity functor $\mathbb{1}_{\mathbf{C}}$ gets mapped by P to the identity set-function on $P(\mathbf{C}) = \operatorname{ob} \mathbf{C}$, so identities are carried to identities.

If $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{E}$, then $P(G \cdot F)$ goes to the set-function composition of F and G on objects, which is exactly $P(G) \circ P(F)$, so composition is preserved.

Finally to show that P is a one-sided inverse let $I : \mathbf{Pre} \to \mathbf{Cat}$ be the inclusion functor regarding pre-orders as categories. Consider the category $P(I(X))$: its objects are the objects of $I(X)$, which are the objects of X, and we have,

$$a \underset{P(I(X))}{\leq} b \Longleftrightarrow \operatorname{Hom}_{I(X)}(a, b) \neq \emptyset \Leftrightarrow a \underset{X}{\leq} b$$

So that the two pre-orders, X and $P(I(X))$, are equal.

Finally if f is an order-preserving map $X \to Y$, both I and P preserve f's action on objects by definitions, so $P(I(f)) = f$. $\square$

Problem 1-9: Describe the free categories on the following graphs by determining their objects, arrows, and composition operations.

(a)

$$a \xrightarrow{e} b$$

(b)

$$\begin{tikzcd} a \arrow[r, shift left, "e"] & b \arrow[l, shift left, "f"] \end{tikzcd}$$

(c)

$$\begin{tikzcd} a \arrow[r, "e"] \arrow[dr, "g"'] & b \arrow[d, "f"] \\ & c \end{tikzcd}$$

(d)

$$\begin{tikzcd} a \arrow[r, shift left, "e"] \arrow[dr, "g"'] & b \arrow[l, shift left, "h"] \arrow[d, "f"] & & d \\ & c \end{tikzcd}$$

Max: In each case let denote by G the graph and by $\mathbf{C}(G)$ the free category on G. Since rules for composing paths and the objects of $\mathbf{C}(G)$ are given by definition, we see it sufficing to only characterize the arrows of $\mathbf{C}(G)$.

(a) Clearly a path cannot have more than one edge, since e is not compatible with itself under path composition. It follows that there are exactly three morphisms on $\mathbf{C}(G)$:

- $\mathrm{Hom}_{\mathbf{C}(G)}(a, a) = \{1_a\}$
- $\mathrm{Hom}_{\mathbf{C}(G)}(b, b) = \{1_b\}$
- $\mathrm{Hom}_{\mathbf{C}(G)}(a, b) = \{e\}$

(b) We claim that every path in $\mathbf{C}(G)$ is alternating between e and h. We prove this by induction on path length n.

The case $n = 0$ is immediate.

Now suppose that the claim holds for paths of length n, and consider a path of length $n + 1$, which by induction will be of the form,

$$p \circ A$$

For some edge p and path A alternating in e and h.

If A is empty, the result is immediate since p must be e or h.

If A starts with e, then the only option for p is h and the result follows. The case A starting with h is similar.

Now that we know what all paths look like in $\mathbf{C}(G)$, we can determine all the arrows like so:

- $\mathrm{Hom}_{\mathbf{C}(G)}(a, a) = \{(he)^n\}_{n \geq 0}$
- $\mathrm{Hom}_{\mathbf{C}(G)}(b, b) = \{(eh)^n\}_{n \geq 0}$
- $\mathrm{Hom}_{\mathbf{C}(G)}(a, b) = \{(eh)^n e\}_{n \geq 0}$
- $\mathrm{Hom}_{\mathbf{C}(G)}(b, a) = \{(he)^n h\}_{n \geq 0}$

(c) Suppose there is a path of length 3, pqr. $s(p) = t(q)$ cannot be c as that would leave no options for p. This means $s(q) = t(r)$ cannot be b (this would force $t(q) = c$), which means $s(r)$ cannot be a (this would force $t(r) = b$).

Similarly, $t(r)$ cannot be c as that would leave no options for q, which means $s(r)$ cannot be b either.

Finally $s(r)$ cannot be c since there are no edges of source c.

It follows that there are no paths of length 3, and in fact greater than 3 neither since those paths would have length-3 subpaths.

From this we know that all paths are at most length 2, and a simple check shows that only the following are valid:

- $\mathrm{Hom}_{\mathbf{C}(G)}(a, a) = \{1_a\}$
- $\mathrm{Hom}_{\mathbf{C}(G)}(b, b) = \{1_b\}$
- $\mathrm{Hom}_{\mathbf{C}(G)}(a, b) = \{e\}$
- $\mathrm{Hom}_{\mathbf{C}(G)}(b, c) = \{f\}$
- $\mathrm{Hom}_{\mathbf{C}(G)}(a, c) = \{g, fe\}$
- $\mathrm{Hom}_{\mathbf{C}(G)}(c, c) = \{1_c\}$

(d) We claim that every path in $\mathbf{C}(G)$ is of the form,

$$p \circ A$$

Where p is either an empty path or one of f, g, and A is an alternating sequence of e and h. We prove this by induction on path length n.

The case $n = 0$ is immediate.

Now suppose that the claim holds for paths of length n, and consider a path of length $n + 1$, which by induction will be of the form,

$$p_0 \circ p \circ A$$

For some edge p_0, p as described above and path A alternating in e and h.

If p is empty, then the result is immediate since if p_0 is f or g it is directly of the needed form, and if its either e or h then the entire path is alternating.

If p is f, then p_0 must be an edge from c. This forces p_0 to be empty and the result follows. The case $p = g$ is similar.

Now that we know what all paths look like in $\mathbf{C}(G)$, we can determine all the arrows like so:

- $\mathrm{Hom}_{\mathbf{C}(G)}(a, a) = \{(he)^n\}_{n \geq 0}$
- $\mathrm{Hom}_{\mathbf{C}(G)}(b, b) = \{(eh)^n\}_{n \geq 0}$
- $\mathrm{Hom}_{\mathbf{C}(G)}(a, b) = \{(eh)^n e\}_{n \geq 0}$

- $\mathrm{Hom}_{\mathbf{C}(G)}(b, a) = \{(he)^n h\}_{n \geq 0}$
- $\mathrm{Hom}_{\mathbf{C}(G)}(b, c) = \{f(eh)^n\}_{n \geq 0}$
- $\mathrm{Hom}_{\mathbf{C}(G)}(a, c) = \{f(eh)^n e\}_{n \geq 0} \cup \{g(he)^n\}_{n \geq 0}$
- $\mathrm{Hom}_{\mathbf{C}(G)}(c, c) = \{1_c\}$

And all other hom-sets are empty.

$\square$

Problem 1-10: How many free categories on graphs are there which have exactly six arrows? Draw the graphs that generate these categories.

Max: First of all, the underlying graph cannot contain any cycles, since any cycle is a path that can be composed with itself infinitely, resulting in infinitely many arrows.

Then there cannot be more than 6 nodes + edges since each node admits an identity arrow to itself. This gives us a path to a solution by enumerating all (finitely many) non-isomorphic directed acyclic graphs with $|V| + |E| \leq 6$.

Doing this for all possible number of nodes, we get:

0) 0 nodes generates the empty category, which has no arrows.
1) With 1 node our only option is to have no edges, since we can't have a loop, which generates the singleton category.
2) With 2 nodes our only option is to have four edges from one node to the other:

$$\begin{tikzcd} \star \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \star \end{tikzcd}$$

3) We get three graphs whose free category has 6 arrows:

$$\begin{tikzcd} \star \arrow[d] & \\ \star \arrow[r] & \star \end{tikzcd} \qquad \begin{tikzcd} \star \arrow[d, shift right=2] \arrow[d] \arrow[d, shift left=2] & \\ \star & \star \end{tikzcd} \qquad \begin{tikzcd} \star \arrow[d, shift right=1] \arrow[d, shift left=1] \arrow[dr] & \\ \star & \star \end{tikzcd}$$

4) We have the following four graphs on four nodes:

$$\begin{tikzcd} \star & \star \\ \star \arrow[u] \arrow[r] & \star \end{tikzcd} \qquad \begin{tikzcd} \star \arrow[d] & \star \\ \star & \star \arrow[l] \end{tikzcd} \qquad \begin{tikzcd} \star \arrow[d] & \star \arrow[d] \\ \star & \star \end{tikzcd} \qquad \begin{tikzcd} \star \arrow[d, shift right=1] \arrow[d, shift left=1] & \star \\ \star & \star \end{tikzcd}$$

   To cut back on checking, we can notice first that there can be at most two edges (by the above logic $|V| + |E| \leq 6$). If there are two edges, there are going to be two disconnected graphs, either of order 3 or 2. If 3, the size 3 subgraph has to end up being as a free category with 5 arrows, all of whom we enumerated while working through case (3). If 2, there is only one possibility enumerated last above. The case of one edge yields only 5 morphisms, and no edges yields only 4.

5) There is just one such graph, namely

$$\begin{tikzcd} \star \arrow[d] & \star & \star \\ \star & \star & , \end{tikzcd}$$

   We can conclude this by noticing there can be at most one edge, and any such choice forces the above graph (up to isomorphism). There cannot be no edge since this yields just five morphisms.

6) With 6 nodes we must not select any edges since then we'd have more than 6 arrows. This gives us a single option, the discrete category of 6 elements:

$$\begin{tikzcd}
\star & \star & \star \\
\\
\star & \star & \star
\end{tikzcd}$$

$\square$

Problem 1-11: Show that the free monoid functor

$$M : \mathbf{Sets} \to \mathbf{Mon}$$

exists, in two different ways:

(a) Assume the particular choice $M(X) = X^*$ and define its effect

$$M(f) : M(A) \to M(B)$$

on a function $f : A \to B$ to be

$$M(f)(a_1 ... a_k) = f(a_1) ... f(a_k), \quad a_1, ... a_k \in A$$

(b) Assume only the UMP of the free monoid and use it to determine M on functions, showing the result to be a functor.

Reflect on how these two approaches are related.

Max:

(a) Let $f : A \to B$ and $g : B \to C$. Then,

$$M(g \circ f)(a_1 ... a_k) = g(f(a_1)) ... g(f(a_n)) = M(g)(f(a_1) ... f(a_n)) = M(g)(M(f)(a_1 ... a_n))$$

And, for the identity function $\iota_A : A \to A$,

$$M(\iota_A)(a_1 ... a_n) = \iota_A(a_1) ... \iota_A(a_n) = a_1 ... a_n$$

So M as defined is a functor.

(b) Let $A, B \in \mathrm{ob}(\mathbf{Sets})$ and $i_A : A \to |M(A)|$ and $i_B : B \to |M(B)|$ be the canonical maps from the UMP for A, B respectively. Then for any set function $f : A \to B$, we can define $M(f) \in \mathrm{Hom}_{\mathbf{Mon}}(M(A), M(B))$ as the unique monoid morphism making the following square commute by UMP:

$$\begin{tikzcd}
{|M(A)|} \arrow[r, "{|M(f)|}"] & {|M(B)|} \\
A \arrow[u, "i_A"] \arrow[r, "f"] & B \arrow[u, "i_B"']
\end{tikzcd}$$

If ι_A is the identity function $A \to A$, then the identity monoid morphism $I_A : M(A) \to M(A)$ coincides with the set identity function $|M(A)| \to |M(A)|$. Then the diagram below clearly commutes:

$$\begin{tikzcd}
{|M(A)|} \arrow[r, "{|I_A| = \iota_{M(A)}}"] & {|M(A)|} \\
A \arrow[u, "i_A"] \arrow[r, "\iota_A"] & A \arrow[u, "i_A"']
\end{tikzcd}$$

So $I_A = M(\iota_A)$ by uniqueness of the UMP.

Finally for composition, suppose we have sets A, B, C and $f : A \to B$, $g : B \to C$, then, we have:

$$\begin{tikzcd}
{|M(A)|} \arrow[r, "|M(f)|"] \arrow[rr, bend left, "|M(g)| \circ |M(f)| = |M(g) \circ M(f)|"] & {|M(B)|} \arrow[r, "|M(g)|"] & {|M(C)|} \\
A \arrow[u, "i_A"'] \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & B \arrow[u, "i_B"'] \arrow[r, "g"] & C \arrow[u, "i_C"']
\end{tikzcd}$$

Since the inner two squares commute, so does the outer square. It follows by uniqueness of UMP that $M(g) \circ M(f) = M(g \circ f)$.

□

Problem 1-12: Verify the UMP for free categories on graphs, defined as above with arrows being sequences of edges. Specifically, let $\mathbf{C}(G)$ be the free category on the graph G, so defined, and $i : G \to U(\mathbf{C}(G))$ the graph homomorphism taking vertices and edges to themselves, regarded as objects and arrows in $\mathbf{C}(G)$. Show that for any category $\mathbf{D}$ and graph homomorphism $h : G \to U(\mathbf{D})$, there is a unique functor

$$\overline{h} : \mathbf{C}(G) \to \mathbf{D}$$

with

$$U\left(\overline{h}\right) \circ i = h,$$

where $U : \mathbf{Cat} \to \mathbf{Graph}$ is the underlying graph functor.

Max: The condition on $\overline{h}$ mandates that $\overline{h}$ takes each node a, regarded as an element of $\mathbf{C}(G)$, to the object $h(a)$ in $\mathbf{D}$, and that it takes each edge e, regarded as a morphism of $\mathbf{C}(G)$, to the edge representing the morphism $h(e)$ in $U(\mathbf{D})$.

That this uniquely pins down $\overline{h}$ follows from the fact that it is a graph homomorphism, since we know it on objects and by definition any morphism p in $\mathbf{C}(G)$ can be written as some finite sequence of edges,

$$p = e_1 \circ ... \circ e_n$$

So that $\overline{h}(p)$ is equal to $\overline{h}(e_1) \circ ... \circ \overline{h}(e_n)$ by the fact that $\overline{h}$ is a functor.

Now since each non-identity morphism p in $\mathbf{C}(G)$ is uniquely expressible as some sequence of edges, it is no issue to define $\overline{h}(p)$ *as* $h(e_1) \circ ... \circ h(e_n)$, where we send the empty sequence to the identity morphism on $h(a)$, thus preserving identities, and we send objects to their image in h.

Then for any non-empty composition-compatible paths $p = e_1...e_n$ and $q = s_1...s_m$, we have,

$$\begin{aligned}
\overline{h}(p \circ q) = \overline{h}(e_1...e_n \circ s_1...s_m) &= \overline{h}(e_1...e_n s_1...s_m) \\
&= h(e_1) \circ ... \circ h(e_n) \circ h(s_1) \circ ... \circ h(s_m) \\
&= \overline{h}(e_1...e_n) \circ \overline{h}(s_1...s_m) \\
&= \overline{h}(p) \circ \overline{h}(q)
\end{aligned}$$

For empty paths then the preservation of composition is immediate. □

Problem 1-13: Use the Cayley representation to show that every small category is isomorphic to a "concrete" one, that is, one in which the objects are sets and the arrows are functions between them.

Max: Define a functor $\overline{(-)} : \mathbf{C} \to \overline{\mathbf{C}}$ on objects as sending $C \to \overline{C}$ and on morphisms $g : C \to D$ to $\overline{g} : \overline{C} \to \overline{D}$. It suffices to show that $\overline{(-)}$ is a functor, is bijective on objects, and is bijective on Hom-sets.

For preservation of identity, $\overline{1_A}$ acts by post-composition with the identity, and therefore is the set-functional identity $\overline{A} \to \overline{A}$; since composition occurs in $\overline{\mathbf{C}}$ as set-function composition, $\overline{1_A}$ is the identity in $\overline{\mathbf{C}}$.

For functoral composition, if $f : A \to B$ and $g : B \to C$ in $\mathbf{C}$, and if $h \in \overline{A}$, then,

$$\left(\overline{g} \circ \overline{f}\right)(h) = \overline{g}\left(\overline{f}(h)\right) = \overline{g}(f \circ h) = g \circ (f \circ h) = (g \circ f) \circ h = \left(\overline{g \circ f}\right)(h)$$

So that $\overline{g} \circ \overline{f} = \overline{g \circ f}$, i.e. $\overline{(-)}$ preserves composition.

For bijectivity on objects, surjectivity is clear by definition since the objects are precisely the elements $\overline{C}$ for $C \in \text{ob}(\mathbf{C})$. Injectivity is also clear since we can always recover the original object C uniquely from the identity $1_C \in \overline{C}$, which could never be in any other $\overline{D}$ for $D \neq C$.

For bijectivity on morphisms, surjectivity is again clear since every morphism is defined as $\overline{g}$ for some $g \in \text{mor}(\mathbf{C})$. Now if $g : A \to B$ for objects A, B in $\mathbf{C}$, $\overline{g}$ acts by post-composition with g from $\overline{A} \to \overline{B}$. It follows that the original g can be recovered uniquely as $\overline{g}(1_A)$, so we also have injectivity. $\square$

Problem 1-14: The notion of a category can also be defined with just one sort (arrows) . rather than two (arrows and objects); the domains and codomains are taken to be certain *arrows* that act as units under composition, which is partially defined. Read about this definition in section I.1 of Mac Lane's *Categories for the Working Mathematician*, and do the exercise mentioned there, showing that it is equivalent to the usual definition.

Max:

- ($\Longrightarrow$) Suppose we have an axiomatization of $\mathbf{C}$ as objects and arrows. Then consider just the collection of arrows, with composition defined as in $\mathbf{C}$. We show each axiom is satisfied.

  (i) If $(k \circ g)$ and f are composable then it is not hard to see that g is composable with f and k is composable with $g \circ f$. Then by associativity $(k \circ g) \circ f = k \circ (g \circ f)$.

  (ii) If $k \circ g$ and $g \circ f$ are valid compositions then by considering the constraints on domains/codomains, we find that $k \circ g \circ f$ is valid.

  (iii) For any arrow $g : A \to B$ we have $g \circ 1_A = g$ and $1_B \circ g = g$, so there exist such arrows.
- ($\Longleftarrow$) With the collection of arrows as A, and composability of g, f as gf when allowed, we will define a category $\mathbf{C}$ as follows:
  - For objects, the collection of identity arrows u so that $uf = f$ and $gu = u$ when such compositions are defined.
  - For morphisms from $u \to u'$, the collection of arrows f so that uf is defined and fu' is also defined (and necessarily both equal to f).
  - For composition, $g \circ f = gf$

  First of all, composition is well-defined since if uf, fu', $u'g$, and gu'' are defined for identity arrows u, u', u'', then by axiom (ii) $fu'g$ is defined and equal to $(fu')g = fg$ since $fu' = f$.

  Now by axiom (iii), any identity u has uu' defined for some other identity u'. But then by the definition of identities,

  $$u = uu' = u'$$

  It follows that $uu = u$ so that u is a morphism in $\mathbf{C}$ from $u \to u$ with $u \circ f = f$ and $g \circ u = g$ for any g, f where such equations are defined. It follows that each object u has a two-sided identity in $\mathbf{C}$.

  Associativity follows directly from axiom (i).

□