1 Categories

Problem 1-1: The objects of $\mathbf{Rel}$ are sets, and an arrow $A \to B$ is a relation from A to B, that is, a subset $R \subseteq A \times B$. The equality relation $\{\langle a, a\rangle \in A \times A \mid a \in A\}$ is the identity arrow on a set A. Composition in $\mathbf{Rel}$ is to be given by

$$\{\langle a, c\rangle \in A \times C \mid \exists b(\langle a, b\rangle \in R \,\&\, \langle b, c\rangle \in S)\}$$

for $R \subseteq A \times B$ and $S \subseteq B \times C$.

(a) Show that $\mathbf{Rel}$ is a category.

(b) Show also that there is a functor $G : \mathbf{Sets} \to \mathbf{Rel}$ taking objects to themselves and each function $f : A \to B$ to its graph,

$$G(f) = \{\langle a, f(a)\rangle \in A \times B \mid a \in A\}$$

(c) Finally, show that there is a functor $C : \mathbf{Rel}^{\mathrm{op}} \to \mathbf{Rel}$ taking each relation $R \subseteq A \times B$ to its converse $R^c \subseteq B \times A$, where,

$$\langle a, b\rangle \in R^c \Leftrightarrow \langle b, a\rangle \in R$$

Max:

(a) By definition, the composition of two relations is another relation, so composition is closed. It suffices to show the identity and associativity rules.

For identity, let $1_A \subseteq A \times A$ and $R \subseteq A \times B$ for some sets A, B. Then $R \circ 1_A$ is,

$$\{\langle a, b\rangle \in A \times B \mid \exists a'(\langle a, a'\rangle \in 1_A \land \langle a', b\rangle \in R)\}$$

Since 1_A is the identity relation, $a = a'$, so the set is equivalent to,

$$\{\langle a, b\rangle \in A \times B \mid \langle a, b\rangle \in R\} = R$$

The case for $1_A \circ R$ follows similarly.

For associativity, let $R \subseteq A \times B$, $S \subseteq B \times C$, and $T \subseteq C \times D$ for sets A, B, C, D. Then,

$$\begin{aligned}
T \circ (S \circ R) &= \{\langle a, d\rangle \in A \times D \mid \exists c(\langle a, c\rangle \in (S \circ R) \land \langle c, d\rangle \in T)\} \\
&= \{\langle a, d\rangle \in A \times D \mid \exists c(\exists b(\langle a, b\rangle \in R \land \langle b, c\rangle \in S) \land \langle c, d\rangle \in T)\} \\
&= \{\langle a, d\rangle \in A \times D \mid \exists c \exists b(\langle a, b\rangle \in R \land \langle b, c\rangle \in S \land \langle c, d\rangle \in T\} \\
&= \{\langle a, d\rangle \in A \times D \mid \exists b \exists c(\langle a, b\rangle \in R \land \langle b, c\rangle \in S \land \langle c, d\rangle \in T\} \\
&= \{\langle a, d\rangle \in A \times D \mid \exists b(\langle a, b\rangle \in R \land \exists c(\langle b, c\rangle \in S \land \langle c, d\rangle \in T))\} \\
&= \{\langle a, d\rangle \in A \times D \mid \exists b(\langle a, b\rangle \in R \land \langle b, d\rangle \in T \circ S)\} = (T \circ S) \circ R
\end{aligned}$$

(b) $G(f)$ is clearly a subset of $G(A) \times G(B) = A \times B$, so it suffices to show identities and composition are preserved.

For identities, let $\iota_A : A \to A$ be the identity function on set A. Then,

$$G(\iota_A) = \{\langle a, \iota_A(a)\rangle \in A \times A \mid a \in A\} = \{\langle a, a\rangle \in A \times A \mid a \in A\} = 1_A$$

For composition, let $f : A \to B$ and $g : B \to C$. Then,

$$
\begin{aligned}
G(g) \circ G(f) &= \{\langle a, c \rangle \in A \times C \mid \exists b(\langle a, b \rangle \in G(f) \land \langle b, c \rangle \in G(g))\} \\
&= \{\langle a, c \rangle \in A \times C \mid \exists b(b = f(a) \land c = g(b))\} \\
&= \{\langle a, c \rangle \in A \times C \mid \exists b(c = g(f(a)))\} \\
&= \{\langle a, c \rangle \in A \times C \mid c = g(f(a))\} \\
&= \{\langle a, c \rangle \in A \times C \mid \exists b(c = (g \circ f)(a))\} = G(g \circ f)
\end{aligned}
$$

(c) Since $R^c \subseteq C(B) \times C(A) = B \times A$, morphisms are compatible with objects, so it suffices to show identities and composition are preserved.

For identities, let $1_A : A \to A$ be the identity relation on set A. Then,

$$\langle a, a \rangle \in (1_A)^c \iff \langle a, a \rangle \in 1_A$$

So $(1_A)^c = 1_A$.

For composition, let $R \subseteq A \times B$ and $S \subseteq B \times C$. Then,

$$
\begin{aligned}
\langle a, c \rangle \in (S \circ R)^c &\iff \langle c, a \rangle \in S \circ R \\
&\iff \exists b(\langle c, b \rangle \in R \land \langle b, a \rangle \in S) \\
&\iff \exists b(\langle a, b \rangle \in S^c \land \langle b, c \rangle \in R^c) \\
&\iff \langle a, c \rangle \in R^c \circ S^c
\end{aligned}
$$

So $(S \circ R)^c = R^c \circ S^c$.

$\square$

Problem 1-2: Consider the following isomorphisms of categories and determine which ones hold.

(a) $\mathbf{Rel} \cong \mathbf{Rel}^{\mathrm{op}}$

(b) $\mathbf{Sets} \cong \mathbf{Sets}^{\mathrm{op}}$

(c) For a fixed set X with a power set $P(X)$, as poset categories $P(X) \cong P(X)^{\mathrm{op}}$ (the arrows in $P(X)$ are subset inclusions $A \subseteq B$ for all $A, B \subseteq X$).

Max:

(a) Consider the functor from problem 1-1(c). It is clear from the symmetry that C

(b)

(c)

$\square$

Problem 1-3:

(a) Show that in **Sets**, the isomorphisms are exactly the bijections.
(b) Show that in **Monoids**, the isomorphisms are exactly the bijective homomorphisms.
(c) Show that in **Posets**, the isomorphisms are *not* the same as bijective homomorphisms.

Max:

(a) This follows from elementary set theory, that if $f : A \to B$ and $g : B \to A$ are such that $g \circ f$ and $f \circ g$ are identities, then f and g are surjective and injective and thus bijective.

(b) As a **Monoids** morphism is a set function that preserves the monoid structure, any isomorphism is a bijection by (a).

Conversely, any bijection $f : M \to N$ between monoids M, N satisfies,

$$f(m \cdot m') = f(m) \cdot f(m')$$

For all $m, m' \in M$.

Now if $n, n' \in N$ so that $n = f(m)$ and $n' = f(m')$ (which we can assume by surjectivity of f), we have,

$$f^{-1}(n \cdot n') = f^{-1}(f(m) \cdot f(m')) = f^{-1}(f(m \cdot m')) = m \cdot m' = f^{-1}(n) \cdot f^{-1}(n')$$

(c) By elementary set theory, any homomorphism which is an isomorphism must have its categorical inverse be the set function inverse. So it suffices to find a bijective homomorphism whose set theoretic inverse is not a homomorphism.

Let $\mathcal{P}(2)$ be the poset of subsets of a 2-element set $2 = \{a, b\}$ under inclusion, and $\mathbf{4}$ be the chain on four elements $1 \leq 2 \leq 3 \leq 4$. Then we can define a function $f : \mathcal{P}(2) \to \mathbf{4}$ like,

$$\begin{aligned} \{\} &\mapsto 1 \\ \{a\} &\mapsto 2 \\ \{b\} &\mapsto 3 \\ \{a, b\} &\mapsto 4 \end{aligned}$$

We have clearly exhibited an order preserving function which is also bijective. However, the inverse map is not bijective since in particular $2 \leq 3$ but it is not the case that $\{a\} \subseteq \{b\}$.

$\square$

Problem 1-4: Let X be a topological space and preorder the points by *specialization*: $x \leq y$ iff y is contained in every open set that contains x. Show that this is a preorder, and that it is a poset if X is T_0 (for any two distinct points, thre is some open set containing one but not the other). Show that the ordering is trivial if X is T_1 (for any two distinct points, each is contained in an open set not containing the other).

Max: Identity is clear since x is trivially contained in every open set that contains itself.

For transitivity, suppose $x \leq y$ and $y \leq z$. Then if U is open in X, $x \in U \Rightarrow y \in U \Rightarrow z \in U$ so in particular $x \in U \Rightarrow z \in U$ so $x \leq z$. It follows that specialization is at least a preorder.

If X is T_0, then suppose $x \leq y$ and $y \leq x$. If $x \neq y$, we can choose without loss of generality an open U such that $x \in U$ and $y \notin U$, but the latter is a contradiction since $x \leq y$.

Finally if X is T_1, and $x \leq y$, $x \neq y$ creates a contradiction, because we can choose some open U, V so that $x \in U$, $y \in V$ and $U \cap V = \emptyset$, so in particular $x \in U$ and $y \notin U$, but by $x \leq y$, $y \in U$. So the order is trivial - the only pairs x, y for which $x \leq y$ are those where $x = y$. $\square$

Problem 1-5: For any category $\mathbf{C}$, define a functor $U : \mathbf{C}/C \to \mathbf{C}$ from the slice category over an object C that "forgets about C." Find a functor $F : \mathbf{C}/C \to C^{\to}$ to the arrow category such that $\mathbf{dom} \circ F = U$.

Max: For U, define $U(f) = \mathbf{dom}\, f$ on objects. Then considering $g : f \to f'$ as a function $\mathbf{dom}\, f \to \mathbf{dom}\, f'$ defines the action on morphisms.

Since we define identities in the slice categories as identites on domains, identites are preserved. Similarly, since we define the composition in $\mathbf{C}/C$ as the composition of the underlying functions on domains, it follows that U preserves composition.

$$
\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, "1_a"'] & c \\
a \arrow[ur, "f'"'] &
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
a \arrow[d, "1_a"'] \\
b
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, "g"'] & c \\
b \arrow[ur, "f'"] \arrow[d, "h"'] & \\
c \arrow[uur, "f''"'] &
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
a \arrow[d, "g"'] \\
b \arrow[d, "h"'] \\
c
\end{tikzcd}
$$

Now define F on objects to take each morphism $f : a \to c$ to itself, and each morphism $g : f \to f'$ to $\langle g, 1_C \rangle$. Letting $f \overset{g}{\to} f' \overset{g'}{\to} f''$, we have,

$$F(g') \circ F(g) = \langle g', 1_C \rangle \circ \langle g, 1_C \rangle = \langle g' \circ g, 1_C \rangle = F(g' \circ g)$$

Then the composition of functors $\mathbf{dom} \circ F$ has, for objects $f \in \operatorname{ob} \mathbf{C}/C$,

$$(\mathbf{dom} \circ F)(f) = \mathbf{dom}\, F(f) = \mathbf{dom}\, f = U(f)$$

And for morphisms $g : f \to f'$,

$$(\mathbf{dom} \circ F)(g) = \mathbf{dom}\langle g, 1_C \rangle = g = U(g)$$

$\square$

Problem 1-6: Construct the "coslice category" $C/\mathbf{C}$ of a category $\mathbf{C}$ under an object C from the slice category $\mathbf{C}/C$ and the "dual category" operation $-^{\mathrm{op}}$.

Problem 1-7: Let $2 = \{a, b\}$ be any set with exactly 2 elements a and b. Define a functor $F : \mathbf{Sets}/2 \to \mathbf{Sets} \times \mathbf{Sets}$ with $F(f : X \to 2) = \left(f^{-1}(a), f^{-1}(b)\right)$. Is this an isomorphism of categories? What about the analagous situation with a one-element set $1 = \{a\}$ instead of 2?

Problem 1-8: Any category $\mathbf{C}$ determines a preorder $P(\mathbf{C})$ by defining a binary relation $\leq$ on the objects by

$$A \leq B \text{ if and only if there is an arrow } A \to B$$

Show that P determines a functor from categories to preorders, by defining its effect on functors between categories and and checking the required conditions. Show that P is a (one-sided) inverse to the evident inclusion functor of preorders into categories.

Problem 1-9: Describe the free categories on the following graphs by determining their objects, arrows, and composition operations.

(a)

$$\begin{tikzcd} a \arrow[r, "e"] & b \end{tikzcd}$$

(b)

$$\begin{tikzcd} a \arrow[r, shift left, "e"] & b \arrow[l, shift left, "f"] \end{tikzcd}$$

(c)

$$\begin{tikzcd} a \arrow[r, "e"] \arrow[dr, "g"'] & b \arrow[d, "f"] \\ & c \end{tikzcd}$$

(d)

$$\begin{tikzcd} a \arrow[r, shift left, "e"] \arrow[dr, "g"'] & b \arrow[l, shift left, "h"] \arrow[d, "f"] & d \\ & c & \end{tikzcd}$$

Problem 1-10: How many free categories on graphs are there which have exactly six arrows? Draw the graphs that generate these categories.

Problem 1-11: Show that the free monoid functor

$$M : \mathbf{Sets} \to \mathbf{Mon}$$

exists, in two different ways:

(a) Assume the particular choice $M(X) = X^*$ and define its effect

$$M(f) : M(A) \to M(B)$$

on a function $f : A \to B$ to be

$$M(f)(a_1 ... a_k) = f(a_1) ... f(a_k), \quad a_1, ... a_k \in A$$

(b) Assume only the UMP of the free monoid and use it to determine M on functions, showing the result to be a functor.

Reflect on how these two approaches are related.

Problem 1-12: Verify the UMP for free categories on graphs, defined as above with arrows being sequences of edges. Specifically, let $\mathbf{C}(G)$ be the free category on the graph G, so defined, and $i : G \to U(\mathbf{C}(G))$ the graph homomorphism taking vertices and edges to themselves, regarded as objects and arrows in $\mathbf{C}(G)$. Show that for any category $\mathbf{D}$ and graph homomorphism $f : G \to U(\mathbf{D})$, there is a unique functor

$$\overline{h} : \mathbf{C}(G) \to \mathbf{D}$$

with

$$U\left(\overline{h}\right) \circ i = h,$$

where $U : \mathbf{Cat} \to \mathbf{Graph}$ is the underlying graph functor.

Problem 1-13: Use the Cayley representation to show that every small category is isomorphic to a "concrete" one, that is, one in which the objects are sets and the arrows are functions between them.

Problem 1-14: The notion of a category can also be defined with just one sort (arrows) . rather than two (arrows and objects); the domains and codomains are taken to be certain *arrows* that act as units under composition, which is partially defined. Read about this definition in section I.1 of Mac Lane's *Categories for the Working Mathematician*, and do the exercise mentioned there, showing that it is equivalent to the usual definition.