Series on Practical Mathematics

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Quantitative Methods for the Analysis of Biathlon Performance

$K\Gamma_{\partial_{\mathcal{L}}^{\infty}}$ Gschwend & Co.

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Preface

> *When you can measure what you are speaking about, and express it in numbers, you know something about it; but when you cannot measure it, when you cannot express it in numbers, your knowledge is of a meagre and unsatisfactory kind...*
>
> — William Thomson, Lord Kelvin

Consider this: As of 2024, the United States has only won a single gold medal at the Biathlon World Championships (BWCH) and the Winter Olympics (Lowell Bailey at the 2017 BWCH in the men's individual). In no other shooting sport has the United States lagged so far behind.

The impetus for this report is the observation that careful and systematic study of a process, and quantification of results, is the best way to improve at that endeavor.

At the beginning of this report, I laid out my hope that a systematic, quantitative approach to the analysis of biathlon performance would be helpful to biathletes and coaches. Naturally, I think the general methodology is powerful and can help to distinguish good and bad practices, evaluate progress, and improve the training rigor of biathlon in the United States.

- Learning is an evolutionary process, with many fits and starts but eventually settling on something that is the truth
- Engineers and mathematicians are naturally understanding of this concept, because the sort of person that is directed down that path tends to be highly systematic
- Bill James and sabermetricians (also Elo ratings in chess), there is a very long and distinguished history of applying quantitative methods to the analysis of sport. Not all methods are statistical, as there is a great deal of research on biomechanics - although that is not the focus of this report
- When I was an undergraduate at Stanford University, the demand for generalization in mathematics courses was everywhere

- I love writing documents in LaTeX!
- The breadth of analysis will be wrapped up with recordkeeping procedures, available technology (e.g., heart rate monitors, accelerometers, motion capture), and other considerations.
- I hope that this report will improve the quality of biathlon training in the United States

Data used in these analyses is taken from historical data compiled by the Colorado Biathlon Club and the International Biathlon Union (IBU); IBU data can be accessed through the Datacenter.

I've also included select computer programming code for MATLAB, Python, and R. Similar analyses can be completed in Microsoft Excel with Solver and the Data Analysis add-in.

Several sections are both difficult and unnecessary, but are included for the curious reader. You may identify these sections when you see  , because they may remind you of the pain of doing five penalty loops.

While the information in this report should be helpful to biathletes and coaches, it is neither sponsored nor endorsed by the United States Biathlon Association (USBA). Reach out to your local club or the USBA for more information, particularly in the realm of training manuals and protocols.

- **List sections that are most useful for biathletes and coaches, followed by sections that are optional (give a roadmap, like what you'd get in an academic book)**
- **Further discuss the motivation for writing this, in particular drawing on the experiences of the NFL, NBA, and MLB with sabermetric-like analysis (the idea being that quantitative analysis can help to improve aspects of performance). In particular, the NFL and NBA (and high schools) hold combines (so does the NHL) to gather data when making decisions about offering scholarships and selecting players. The NFL Scouting Combine includes physical measurements, athletic tests (vertical jump, 40-yard dash), position drills, drug testing, a Wonderlic exam, and interviews.**

Kyle A. Gschwend
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Chapter 1

Overview of Biathlon

1.1 History

Biathlon is a winter sport combining cross-country skiing and rifle marksmanship. It presents a formidable challenge, because it tests an athlete's physical endurance and fine motor skill.

1.1.1 Military Origins

The origins of biathlon can be traced back to ancient Scandinavia, where hunters would ski long distances through snow-covered landscapes while carrying weapons for hunting and self-defense. However, the primary impetus for the modern development of the sport lies in its utility as training for ski warfare. There are historical records of ski-mounted soldiers being used in Norway and Sweden as early as the 18th century. The first known biathlon competition taking place in Norway in 1767 when Norwegian soldiers competed in a ski race with shooting components; in Norway, biathlon is called *skiskyting* (*ski shooting*). By the late 19th century, the Norwegian military organized competitions known as military patrols, which combined cross-country skiing with rifle shooting.

The importance of ski warfare was clearly shown in the two World Wars and the Winter War. World War I was the first major conflict to see widespread use of skiing in warfare, particularly in fighting in the Alps between Austria-Hungary and Italy. Ski-mounted troops played crucial roles in reconnaissance, transportation, and combat. The Winter War between Finland and the Soviet Union in 1939-1940 showcased the strategic importance of skiing and marksmanship in cold-weather warfare. Finnish ski troops employed hit-and-run tactics and utilized their superior skiing skills to outmaneuver and harass larger Soviet forces in rural, heavily forested terrain. Their success in defending Finland against the Soviet invasion highlighted the effectiveness of ski warfare and further emphasized the importance of marksmanship while skiing. World War

II saw widespread use of ski troops and cold-weather warfare tactics. The German Gebirgsjäger (mountain troops) and Soviet ski divisions, played important roles in various military campaigns on the Eastern Front.

These experiences showcased the value of military training in cross-country skiing and marksmanship, which continues to this day. For instance, the United States Armed Forces currently train soldiers, sailors, and Marines in ski warfare and hold biathlon races to encourage the development of skill in both domains. For a history of biathlon in the United States, please see [12].

1.1.2 The Modern Sport

Biathlon gained popularity as a competitive sport in the early 20th century, with the first known biathlon competition for civilians held in Norway in 1912. Military patrol, the predecessor of biathlon was included in the official program of the Winter Olympics at the 1924 Chamonix Games, and as a demonstration event at the 1928, 1936, and 1948 games. However, after World War II enthusiasm for including military patrol on the Olympic program waned because of its explicit military structure. It wasn't until the 1960 Squaw Valley Games that the civilian variant of biathlon became an official Olympic sport.

In the latter half of the 20th century, biathlon underwent significant technological and organizational advancements, including improvements in ski and rifle technology, the standardization of competition formats, and the establishment of the International Biathlon Union (IBU) in 1993 to govern the sport globally. Centerfire cartridges were used for most of biathlon's early history, before the shooting distance was changed to 50m in 1978 and rimfire cartridges became standard. Biathlon continues to be a popular winter sport, usually the second most popular in Europe, with competitions ranging from local and regional events to World Cup races and the Winter Olympics.

1.2 The Biathlon Rifle

1.2.1 Handling a Firearm

Before handling a biathlon rifle, or for that matter any firearm, it is necessary to be of clear and sound mind. Firearms are dangerous instruments, and errant, negligent handling can lead to injury or death for you or those around you.

Following to the hilt three safety rules will ensure that no accidents occur:

1. **ALWAYS POINT THE MUZZLE IN A SAFE DIRECTION**.
2. **KEEP THE ACTION OPEN AND RIFLE UNLOADED UNTIL READY TO SHOOT**.
3. **KEEP YOUR FINGER OFF THE TRIGGER UNTIL READY TO SHOOT**.

If you cannot follow these rules, do not participate in biathlon and do not handle any firearm. For further information on biathlon safety, please consult the USBA Rifle Safety Certification Program (the Red Book). Clinics are regularly held.

1.2.2 The Anschütz 1827F

Biathlon rifles are bolt-action rifles which must be carried on the athlete's back during a race. They must also perform well in cold weather and cannot be fitted with an optical sight. The standard for international competition is the **Anschütz 1827F**.

Figure 1.1: The Anschütz 1827F biathlon rifle. It is the world standard, making up at least 95% of rifles used in international competition.

Its signature feature is the Fortner bolt (a straight-pull mechanism) that can reduce the interval between shots by 1-3 seconds.

1.2.3 Biathlon Rifle Vendors in the United States

- **Altius Handcrafted Firearms** (Marc Sheppard)
  - Altius is the only Anschütz factory-authorized service center in the United States.
- **Lost Nation R&D** (Ethan Dreissigacker)
  - Lost Nation R&D offers three firearms as the base for its biathlon rifle: the Savage MKII-FV; the CZ 457; and the Anschütz 1827F.

– The signature accessory is the Versatile Biathlon Stock (VBS), which is very easy to adjust and is a good option for a club rifle.

Lapua and **SK** ammunition are preferred when shooting biathlon rifles, since they tend to perform consistently in cold weather. Boxes of each brand can be bought at **Altius**, **Champion's Choice**, or **MidwayUSA**.

1.3 Race Formats

Biathlon features a variety of race formats that test athletes' skiing speed, endurance, and shooting accuracy in different ways.

1.3.1 Individual

The individual race is the classic biathlon format, featuring a challenging course with a combination of skiing segments and shooting stages. Athletes start at intervals, usually 30 seconds apart, and compete against the clock. In each shooting stage, athletes must shoot at targets from both the prone and standing positions, with penalty loops added for missed targets.

1.3.2 Sprint

The sprint race is a shorter and faster-paced version of the individual race. Athletes ski a shorter course with fewer shooting stages compared to the individual race. The format typically consists of two shooting stages (one prone and one standing) interspersed with skiing segments. The sprint race emphasizes skiing speed and shooting accuracy under time pressure.

1.3.3 Pursuit

The pursuit race follows the sprint race and is based on the athletes' finishing positions in the sprint. Athletes start the pursuit race with staggered start times determined by their sprint race results, with the winner of the sprint starting first and other athletes starting behind them at intervals based on their time deficits. The pursuit format adds an exciting element of head-to-head competition as athletes chase down their rivals on the course.

1.3.4 Mass Start

The mass start race brings all athletes together on the course at the same time, creating a dynamic and intense competition. Athletes start in rows or waves and compete head-to-head over a longer course with multiple shooting stages. The mass start format requires athletes to navigate crowded courses and maintain focus amidst the chaos of simultaneous skiing and shooting.

1.3.5 Relays

The relay race is a team event in which teams of four athletes each compete in succession, with each team member completing a designated portion of the course before handing off to the next teammate. The relay format features a combination of skiing and shooting segments, with each athlete completing two shooting stages (one prone and one standing). The relay race emphasizes teamwork, strategy, and consistent shooting performance across all team members.

Mixed Relays

The mixed relay follows the same format as the relay race but features teams composed of two men and two women. Each team member completes one leg of the race, with the order of legs alternating between male and female athletes. The mixed relay adds an element of gender balance and teamwork, highlighting the versatility and cooperation among biathletes.

1.4 Select Biathlon Venues in the Contiguous United States

Name	Location	Elevation (ft)
Auburn Ski Club	Soda Springs, CA	7,263
Snow Mountain Ranch	Tabernash, CO	8,755
Fort Kent Outdoor Center	Fort Kent, ME	913
Nordic Heritage Center	Presque Isle, ME	874
Crosscut Mountain Sports Center	Bozeman, MT	6,254
West Yellowstone SEF	West Yellowstone, MT	6,679
Saratoga Biathlon Club	Hadley, NY	829
Mt. Van Hoevenberg	Lake Placid, NY	2,019
Soldier Hollow	Midway, UT	5,562
Craftsbury Outdoor Center	Craftsbury Common, VT	1,181
Ethan Allen Firing Range	Jericho, VT	670
Stevens Pass Nordic Center	Leavenworth, WA	3,013
Ariens Nordic Center	Brillion, WI	817
Casper Mountain	Casper, WY	7,976

Table 1.1: Biathlon Venues in the United States.

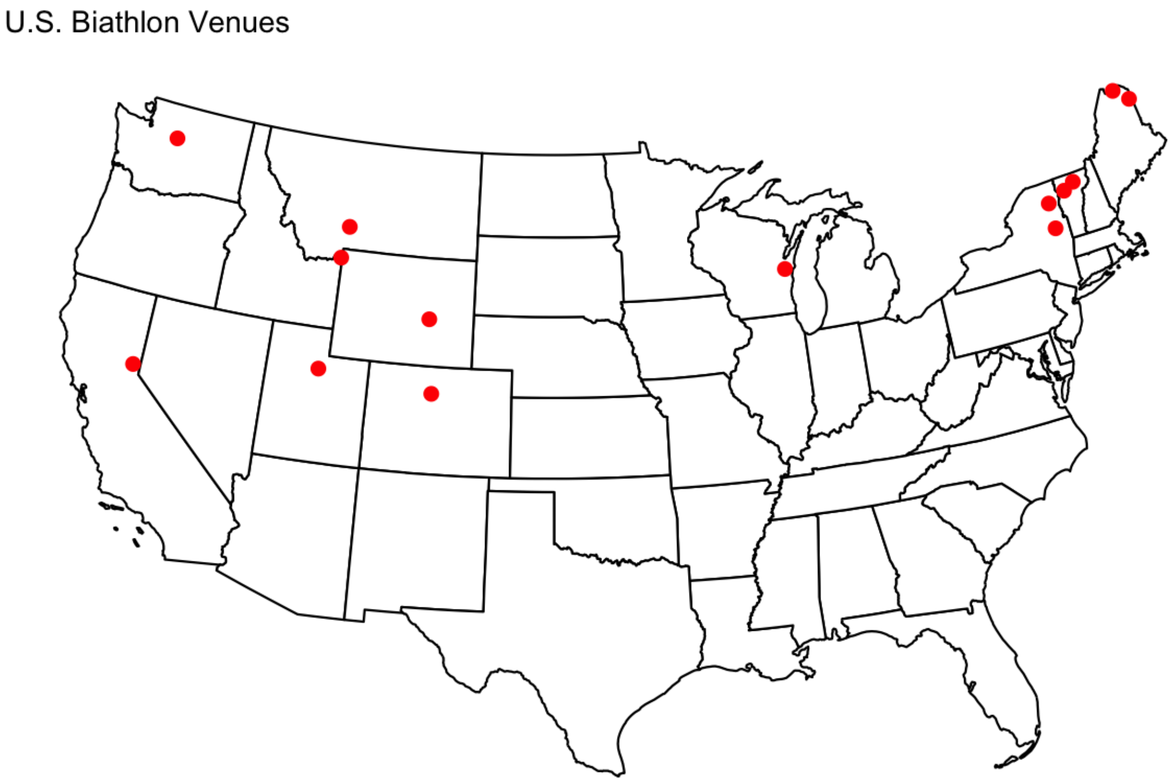

Figure 1.2: The map locations of the biathlon venues from Table 1.1.

Chapter 2

Computer Programming

Microsoft Excel, MATLAB, Python, and R. The primary use is statistical computing.

MATLAB, Python, and R are three powerful programming environments widely used in data analysis, including the specialized domain of sports performance analysis like biathlon. Each offers unique tools and libraries/frameworks suited for handling various aspects of biathlon performance data, from preprocessing to in-depth statistical analysis and visualization.

MATLAB offers a tightly integrated environment for analysis and simulation, Python provides versatility and a rich set of libraries for data manipulation, machine learning, and visualization, while R stands out for its statistical analysis capabilities and specialized packages tailored for complex data analysis. The choice among them can depend on the specific needs of the project, the analyst's familiarity with the language, and the complexity of the data being analyzed.

While Excel and SQL are powerful tools, I won't go into much detail on them because in terms of computational power and the ease of use for more complex tools, MATLAB, Python, and R are far superior. However, as a storage for basic data Excel and SQL are superior; although, for most data purposes in biathlon, SQL is overkill.

2.1 Microsoft Excel

Data Organization: Excel serves as a versatile tool for biathlon coaches to organize various types of data related to athletes, competitions, training sessions, and performance metrics. Coaches can create separate worksheets or tabs within a workbook to manage different aspects of their coaching program, ensuring that data is structured and easily accessible.

Data Analysis: Excel provides powerful features for analyzing data, including functions, formulas, and pivot tables. Biathlon coaches can use Excel to calculate performance metrics such as shooting accuracy, ski times, and overall race results. They can also perform statistical analysis to identify trends, patterns, and correlations in athlete performance, helping them make informed coaching decisions.

Visualization: Visualizing data is essential for communicating insights effectively. Excel offers a wide range of charting tools, including bar graphs, line charts, and scatter plots, allowing coaches to create visually appealing representations of their data. Whether it's tracking athlete progress over time or comparing performance across different competitions, Excel's visualization capabilities help coaches convey information clearly to athletes, team managers, and other stakeholders.

Custom Reporting: Biathlon coaches often need to generate custom reports to summarize key information and track progress. Excel enables coaches to create tailored reports by organizing data into tables and charts and formatting them to meet specific requirements. Whether it's a summary of training sessions, a breakdown of competition results, or an analysis of equipment usage, Excel provides flexibility in designing reports to support decision-making and communication.

Data Integration: Excel facilitates the integration of data from different sources, enabling coaches to consolidate information from various sources into a single workbook. Coaches can import data from external sources such as race results files, training logs, and athlete performance tracking systems, and then manipulate and analyze the data using Excel's tools and functions. This integration capability streamlines the process of managing and analyzing diverse data sources.

Collaboration and Sharing: Excel supports collaboration among coaches, athletes, and other stakeholders by allowing multiple users to work on the same workbook simultaneously. Coaches can share workbooks via cloud storage services or email, enabling real-time collaboration and feedback. Excel's commenting and review features facilitate communication and collaboration, helping coaches and athletes collaborate effectively regardless of their location.

Training Planning and Tracking: Excel serves as a valuable tool for planning and tracking training programs. Coaches can create training schedules, record session details, and track athlete progress over time using Excel's spreadsheet functionality. By organizing training data in Excel, coaches can monitor adherence to the training plan, track improvements in performance, and adjust training programs as needed to optimize athlete development.

2.2 MATLAB

- **Data Import and Preprocessing**: MATLAB provides robust functions for importing data from different file types. Its Signal Processing Toolbox can be particularly useful for filtering noise from biathlon timing and biometric data.

- **Analyis and Visualization**: With its rich set of built-in functions for statistical analysis and machine learning, MATLAB simplifies the process of performing complex analyses, such as determining the impact of environmental conditions on shooting accuracy. MATLAB's plotting functions allow for the creation of high-quality graphs and visualizations to represent athletes' performance trends and comparisons effectively.

- **Simulation**: The Simulink environment within MATLAB is excellent for simulating biathlon races under different conditions, helping to strategize race pacing and energy expenditure.

MATLAB is developed by MathWorks and, unlike Python and R, is not free.

2.3 Python

- **Versatility and Libraries**: Python's versatility and the extensive libraries available for data analysis (pandas, NumPy) and machine learning (scikit-learn, TensorFlow, PyTorch) make it a powerful tool for biathlon performance analysis. Its libraries for statistical analysis (statsmodels) can model biathlon shooting accuracy or predict performance outcomes.

- **Visualization**: Python offers various libraries for data visualization (matplotlib, seaborn, Plotly), which can be used to generate insightful plots, heat maps, and interactive charts for exploring biathlon performance data.

2.4 R

R is a dedicated statistical computing environment.

- **Statistical Analysis and Modeling**: R is particularly strong in statistical analysis and modeling. It's well-suited for handling the complexities of biathlon data, such as calculating shooting percentages, time differences, and correlating various performance metrics with outcomes.

- **Packages**: Specific packages like lme4 for linear mixed-effects models can analyze time series data from biathlon races, taking into account the repeated measures from the same athlete across different events. The ggplot2 package allows for sophisticated data visualization, making it easier to communicate findings.

- **Environment for Statistical Computing**: R is designed around statistical computing, offering a more direct approach for specific statistical methods crucial in sports science research. This makes it especially useful for academically rigorous analysis of biathlon performance.

Definitely use **RStudio**.

2.5 SQL

SQL, or Structured Query Language, is a critical tool for biathlon coaches in managing and analyzing data related to their athletes, competitions, and training programs. Here's a more detailed summary tailored to biathlon coaches:

Data Management: Biathlon coaches deal with a vast amount of data, including athlete profiles, training schedules, competition results, and equipment inventory. SQL enables coaches to create and maintain a structured database to organize this information efficiently. They can define tables for athletes, competitions, training sessions, shooting performance, and more, ensuring data consistency and integrity.

Data Analysis: SQL provides powerful capabilities for querying and analyzing data. Coaches can use SQL queries to extract valuable insights from their database. For example, they can calculate performance metrics such as shooting accuracy, ski times, and overall race results. By filtering and aggregating data, coaches can identify patterns, trends, and areas for improvement in athlete performance.

Custom Reporting: Biathlon coaches often need to generate custom reports for various stakeholders, including athletes, team managers, and sponsors. SQL allows coaches to create tailored reports by retrieving specific data from the database. Whether it's a summary of training progress, a breakdown of competition results, or an analysis of equipment usage, SQL queries can produce detailed reports to support decision-making and communication.

Performance Tracking: Tracking athlete performance over time is essential for monitoring progress and making informed coaching decisions. SQL enables coaches to store historical data and track changes in performance metrics longitudinally. By comparing performance data across training seasons and competition seasons, coaches can evaluate the effectiveness of training programs and adjust strategies accordingly.

Data Integration: In addition to managing internal data, biathlon coaches may need to integrate external data sources into their analysis. SQL supports data integration by facilitating the import and export of data between different systems and formats. Whether it's importing race results from external competitions or syncing training data from wearable devices, SQL queries can streamline the process of integrating diverse data sources.

Data Security and Privacy: Protecting sensitive data is paramount for biathlon coaches, especially when dealing with athlete health information and performance data. SQL provides robust security features to control access to the database and encrypt sensitive data. Coaches can implement user authentication, authorization rules, and data encryption to safeguard confidential information and comply with privacy regulations.

Chapter 3

Physics

See this: **link**.
And this: **link**.

3.1 Overview of Differential Equations

3.2 Snow Physics

Snow physics plays a crucial role in understanding and optimizing cross-country skiing performance. The interaction between ski bases and snow involves complex dynamics influenced by snow crystal structure, temperature, humidity, and the mechanical properties of snow. These interactions determine the glide and grip of skis, which are critical for efficiency and speed in cross-country skiing.

Snow undergoes metamorphism after it falls, changing its crystal structure over time due to temperature gradients and sublimation. Fresh snow, characterized by sharp, angular crystals, behaves differently under ski bases compared to older, rounded, or sintered snow crystals. Understanding these transformations is key to predicting snow behavior and selecting the appropriate ski wax or structure.

The glide phase in cross-country skiing is significantly affected by the friction between the ski base and the snow, which can be categorized into:

- **Dry Friction:** Occurs in the absence of a water layer between the ski and the snow. It is influenced by the roughness of the ski base and the hardness of the snow crystals.
- **Wet Friction:** Arises when a thin film of water is present, usually resulting from the melting of snow due to the pressure exerted by the ski. The right wax can help reduce wet friction by repelling water.
- **Viscous Friction:** Related to the shearing of the thin water layer. The effectiveness of ski waxes in reducing viscous friction depends on their ability to minimize water adhesion.

The temperature and state of the snow surface vary considerably, affecting ski performance. Waxes are designed to operate optimally within specific temperature ranges, aiming to minimize friction and are chosen based on the air and snow temperature, snow crystal type, and anticipated snow transformation during a race.

For classic cross-country skiing, the grip of the ski on the snow during the kick phase is crucial for effective propulsion. The grip is influenced by the mechanical interlocking between the ski base and the snow, which can be enhanced through textured ski bases or grip waxes. These are selected based on snow conditions to maximize grip while minimizing friction during the glide phase.

Understanding snow physics allows cross-country skiers and their support teams to make informed decisions about wax selection, ski preparation, and technique on race day, which includes:

- **Wax Selection and Ski Preparation:** Choosing waxes and structuring the ski base to match specific snow conditions, optimizing glide and grip.
- **Technique Adaptation:** Adjusting technique based on snow conditions to optimize grip and glide.
- **Equipment Choices:** The design and material composition of skis, poles, and wax evolve based on snow physics research, aiming to find the best combinations for speed and efficiency.

3.3 Atmospheric Physics & Forecasting

See Mark Jacobson's book "Fundamentals of Atmospheric Modeling" as a reference.

3.3.1 Wind Speed Models

Rayleigh and Weibull distributions for the wind speed distribution, which can be found for different courses.

3.4 Dimensionless Numbers

Derived from fluid dynamics

3.4.1 Drag Coefficient

$$C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A} \tag{3.1}$$

3.4.2 Strouhal Number

The Strouhal number characterizes the ratio of oscillatory forces to inertial forces in fluid flow. In biathlon, a modified Strouhal number could potentially quantify the oscillatory dynamics of skiing movements, such as pole plant frequency or shooting rhythm, relative to skiing speed and other factors.

$$\mathrm{St} = \frac{fL}{V} \tag{3.2}$$

Where:

- f is the frequency of oscillation
- L is the characteristic length scale
- V is the velocity

3.5 Biomechanics

See this resource: **link**.

3.6 Calculus of Variations

3.6.1 Shortest Path Problem

Minimum Path w/ Constraints

Can be used to identify an optimal path on a ski trail, subject to a smoothness constraint and accounting for the curvature; also subject to a momentum constraint (is this angular momentum)?

This can be solved by a dense network model.

Chapter 4

Probability & Statistics

If you torture the data long enough, it will confess.

— Ronald Coase

Statistics: The only science that enables different experts using the same figures to draw different conclusions.

— Evan Esar

There are three kinds of lies: lies, damned lies, and statistics.

— Mark Twain

After reading these quotes, you could be forgiven for concluding that statistical analysis is a dodgy subject, something practiced by quantitative loan sharks and charlatans. I won't deny that statistical methods are more prone to sophistry than those of pure mathematics, but throwing away the entire enterprise is hardly sensible. The aim of the chapter is to provide an overview of methods and their application to biathlon performance.

4.1 Overview

Probability and statistics are branches of mathematics that are essential for understanding and analyzing uncertainty and variability in data. They play a crucial role in various fields such as science, engineering, economics, finance, social sciences, and many others. The primary purposes of probability and statistics are as follows:

1. **Quantifying Uncertainty**: Probability theory provides a framework for quantifying uncertainty and randomness in events or outcomes. It allows us to assign numerical values, called probabilities, to

different possible outcomes of an experiment or process, enabling us to make informed decisions in the face of uncertainty.

2. **Making Predictions and Inferences**: Statistics involves the collection, analysis, interpretation, and presentation of data. By using statistical methods, we can make predictions about future events or outcomes based on past data, as well as draw conclusions and make inferences about populations based on sample data.

3. **Testing Hypotheses**: Statistical hypothesis testing is a fundamental tool for making decisions and drawing conclusions in scientific research and experimentation. It allows researchers to test hypotheses about the relationships between variables, the effects of treatments or interventions, and the validity of theories or models.

4. **Modeling and Analysis**: Probability and statistics are used to develop mathematical models that describe the behavior of complex systems and phenomena. These models can be used to analyze data, identify patterns and trends, and make predictions about future behavior.

5. **Data-driven Decision Making**: In today's data-rich world, probability and statistics play a vital role in data-driven decision-making. By analyzing large volumes of data, extracting meaningful insights, and identifying patterns and trends, organizations can make informed decisions, optimize processes, and drive innovation and growth.

Overall, probability and statistics provide essential tools and techniques for understanding, analyzing, and interpreting data, enabling us to make better decisions, solve problems, and gain insights into the world around us.

4.1.1 Measurement of Effect Sizes

Measurement theory in statistics is concerned with the connection between empirical observations and the theoretical properties they represent. It examines the nature of measurements, the scales on which they are expressed, and the various errors associated with them. Measurement theory deals with the validity, reliability, and accuracy of data. For example, it's crucial in developing and validating tests and scales in psychological and educational testing, ensuring that they accurately measure the constructs they are intended to.

Validity is about whether the measurement method actually measures what it purports to measure; reliability refers to the consistency of a set of measurements or measuring instrument, often judged by the consistency of scores obtained by the same individuals when retested; and accuracy pertains to how close measurements are to the true or accepted value.

Effect sizes, on the other hand, are a key output of statistical analyses. They describe the strength of a phenomenon, providing a scale of impact or difference that is independent of sample size, which is particularly useful in meta-analysis. Commonly used effect size measures include Cohen's d, Pearson's r, and odds ratios. They help determine the practical significance of research findings, as opposed to statistical significance, which does not always indicate a finding is meaningful in a real-world context.

- **Cohen's d** measures the difference between two means expressed in standard deviation units and is commonly used to quantify the difference between two groups.
- **Pearson's r** provides a measure of the strength of linear association between two continuous variables. (also talk about the Spearman rank correlation)
- **Odds ratios** are used in the context of binary data to compare the odds of an event occurring in one group to the odds of it occurring in another group.

Statistical Power

Statistical power is a critical concept in experimental design and hypothesis testing. It refers to the probability that a statistical test will correctly reject the null hypothesis when the alternative hypothesis is true. In simpler terms, it measures the likelihood that a study will detect a true effect if it exists.

Statistical power is crucial because it determines the likelihood of detecting real effects. Studies with low power are more likely to miss true effects, leading to false-negative results (i.e., failing to reject the null hypothesis when the alternative hypothesis is true). Conversely, studies with high power are more likely to detect true effects accurately.

Several factors influence statistical power, including the sample size, effect size, significance level (alpha), and variability in the data. Generally, larger sample sizes, larger effect sizes, and lower variability increase statistical power. Additionally, a less stringent significance level (e.g., using a higher alpha level) can increase power but also increase the risk of Type I errors (false positives).

4.1.2 Correlation Coefficients

$$\rho(X, y) = \frac{\mathrm{Cov}(X, Y)}{\sigma_X \sigma_Y}$$

$$\rho_{\mathrm{Spear}} = 1 - \frac{6 \sum_i d_i^2}{n\,(n^2 - 1)}, \quad d_i = \mathrm{R}(X_i) - \mathrm{R}(Y_i)$$

A Brief Rant on *Correlation Does Not Imply Causation*

That's generally true, but also besides the point.

This nostrum is the sort of thing you hear from someone, typically a well-heeled college graduate, often with post-nominals conferred by a member of the Ivy League, who has never bothered to think about causality because their connoisseurship of thought-terminating clichés has addled their brain.[1]

See the Bradford Hill criteria, originally developed for use in epidemiology but which have broad applicability for understanding causality.

Ah, the age-old adage: "Correlation does not imply causation." It's like the buzzkill at a party full of interesting insights and provocative ideas. Sure, it's technically true, but its overuse can sometimes feel like a cop-out—a way to shut down a conversation rather than engage in meaningful exploration.

Yes, correlation is not the same as causation. Just because two variables are associated with each other doesn't mean that one causes the other. There could be other factors at play, lurking in the shadows, waiting to be uncovered by a diligent researcher armed with statistical tools and a curious mind.

But here's the thing: acknowledging that correlation doesn't prove causation shouldn't stop us from digging deeper. It shouldn't be an excuse to throw up our hands and say, "Well, we can't know for sure, so let's move on." Instead, it should be a call to action—a challenge to investigate further, to consider alternative explanations, and to explore the underlying mechanisms driving the observed patterns.

Sometimes, yes, correlation is just correlation. Maybe it's a coincidence, a fluke, a statistical artifact. But other times, there's a story waiting to be uncovered—a narrative of cause and effect, of action and consequence, waiting for someone brave enough to venture into the unknown and unravel its mysteries.

So, yes, let's remember that correlation does not imply causation. But let's also remember that it's not the end of the story—it's just the beginning. Let's use it as a springboard for curiosity, for inquiry, for discovery. Because who knows what wonders lie beyond those seemingly innocuous correlations, waiting to be uncovered by intrepid explorers of the statistical frontier?

4.1.3 Probability Distributions

Only include those that may be of use in biathlon. (Weibull-type distributions could be useful for simulating wind speeds and the time between shot intervals - including the generalized gamma distribution). The Bernoulli/Binomial distribution; the Gaussian distribution; the GEV distribution; some other stuff.

Parameter Estimation

Method of moments and maximum likelihood

4.1.4 The Binomial Distribution

The binomial distribution is a discrete probability distribution that models the number of successes in a fixed number of independent Bernoulli trials, where each trial has two possible outcomes: success or failure.

[1] The author is a graduate of Stanford and Columbia, occasionally suffering from this common malady. Symptoms include appeals to authority, pretentiousness, and dismissing Cornell as a mere public school.

1. **Definition**: The binomial distribution describes the probability of obtaining a specific number of successes k in n independent trials, each with a probability p of success and $1-p$ of failure. The random variable X follows a binomial distribution, denoted as $X \sim \text{Bin}(n,p)$.

2. **Probability Mass Function (PMF)**: The probability mass function of the binomial distribution gives the probability of obtaining exactly k successes in n trials. It is given by:

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

where:

- $\binom{n}{k}$ is the binomial coefficient,
- p is the probability of success in each trial,
- $(1-p)$ is the probability of failure in each trial.

3. **Properties**:

- **Mean and Variance**: The mean (expected value) of a binomial distribution is $\mu = np$, and the variance is $\sigma^2 = np(1-p)$.
- **Shape**: The shape of the binomial distribution depends on the values of n and p. As n increases, the distribution becomes more symmetric and approaches a bell-shaped curve resembling the normal distribution, especially when p is close to 0.5.
- **Sum of Independent Binomial Variables**: If $X_1, X_2, \ldots, X_k$ are independent random variables following a binomial distribution with parameters (n,p), then their sum $Y = X_1 + X_2 + \ldots + X_k$ also follows a binomial distribution with parameters (kn, p).

4. **Applications**:

- **Coin Flips**: The binomial distribution can model the number of heads obtained in a series of coin flips.
- **Product Quality Control**: It can be used to assess the likelihood of defective items in a production batch.
- **Biological Experiments**: It is used to analyze the outcomes of biological experiments, such as the number of successful drug trials out of a fixed number of experiments.
- **Survey Sampling**: It is employed in survey sampling to estimate the proportion of a population with a specific characteristic.

4.1.5 The Normal Distribution

1. **Definition**: The normal distribution, also known as the Gaussian distribution, is a continuous probability distribution that is symmetric about its mean. It is characterized by two parameters: the mean μ and the standard deviation σ. The random variable X follows a normal distribution, denoted as $X \sim \mathcal{N}(\mu, \sigma^2)$.

2. **Probability Density Function (PDF)**: The probability density function of the normal distribution is given by:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

   where:

   - μ is the mean of the distribution,
   - σ is the standard deviation,
   - π is the mathematical constant pi (approximately 3.14159),
   - e is the mathematical constant Euler's number (approximately 2.71828).

3. **Properties**:

   - **Symmetry**: The normal distribution is symmetric about its mean μ, with the highest point of the distribution occurring at $x = \mu$.
   - **Central Limit Theorem**: The normal distribution arises naturally as the limiting distribution of the sample mean of a large number of independent and identically distributed random variables. This is known as the Central Limit Theorem, which states that the distribution of the sample mean approaches a normal distribution as the sample size increases, regardless of the original distribution of the random variables.
   - **68-95-99.7 Rule**: Approximately 68

4. **Applications**:

   - **Natural Phenomena**: Many natural phenomena, such as human heights, IQ scores, and errors in measurements, follow approximately normal distributions.
   - **Statistical Inference**: The normal distribution is widely used in statistical inference, including hypothesis testing, confidence intervals, and linear regression.
   - **Financial Modeling**: It is employed in finance to model stock prices, interest rates, and other financial variables.

- **Quality Control**: The normal distribution is used in quality control to analyze process variability and set tolerance limits for product specifications.

Jarque-Bera test

1. **Definition**: The Jarque-Bera test is a statistical test used to assess whether the data in a sample comes from a normally distributed population. It is based on the skewness and kurtosis of the sample data.

2. **Test Statistic**: The Jarque-Bera test statistic is calculated as:

$$JB = \frac{n}{6} \left(S^2 + \frac{1}{4}(K - 3)^2 \right)$$

   where:

   - n is the sample size,
   - S is the sample skewness,
   - K is the sample excess kurtosis (kurtosis minus 3).

3. **Null Hypothesis**: The null hypothesis of the Jarque-Bera test is that the data in the sample is normally distributed.

4. **Decision Rule**:

   - If the Jarque-Bera test statistic JB is greater than a critical value from the chi-square distribution with 2 degrees of freedom at a chosen significance level (e.g., 5
   - If JB is less than the critical value, then the null hypothesis cannot be rejected, suggesting that the data may be normally distributed.

5. **Applications**:

   - The Jarque-Bera test is commonly used in finance, econometrics, and other fields to check the assumption of normality in data before applying certain statistical methods that require normally distributed data, such as parametric tests and regression analysis.
   - It is also used in algorithmic trading to detect departures from normality in financial time series data, which may affect the performance of trading strategies.

Example: BMI of Professional Biathletes

4.1.6 Order Statistics

Order statistics are a fundamental concept in statistics that involve arranging a set of observations in ascending or descending order and analyzing the statistical properties of these ordered values.

1. **Definition**: Given a sample of n observations $X_1, X_2, \ldots, X_n$, the order statistics are the n random variables obtained by arranging the observations in ascending order. In other words, $X_{(1)}$ is the smallest observation, $X_{(2)}$ is the second smallest, and so on, up to $X_{(n)}$, which is the largest observation.

2. **Notation**: Order statistics are typically denoted using parentheses, with $X_{(k)}$ representing the k-th order statistic. The subscript k ranges from 1 to n, corresponding to the k-th smallest observation in the sample.

3. **Properties**:

   - **Minimum and Maximum**: The first order statistic $X_{(1)}$ represents the minimum value in the sample, while the n-th order statistic $X_{(n)}$ represents the maximum value.
   - **Range**: The range of the sample is defined as the difference between the maximum and minimum values: $X_{(n)} - X_{(1)}$.
   - **Percentiles**: Order statistics are used to calculate percentiles and quantiles of a distribution. For example, the median of the sample is the value of $X_{(n/2)}$ if n is even, or the average of $X_{(n/2)}$ and $X_{(n/2+1)}$ if n is odd.
   - **Distribution**: The joint distribution of order statistics can be derived for specific distributions. For example, if the X_i's are independent and identically distributed (i.i.d.) random variables from a continuous distribution, the distribution of the order statistics can be expressed in terms of the cumulative distribution function (CDF) of the original distribution.

4. **Applications**:

   - **Reliability Analysis**: Order statistics are used in reliability analysis to estimate parameters such as the mean time to failure and the reliability of systems.
   - **Extreme Value Theory**: Order statistics are central to extreme value theory, which deals with the statistical behavior of extreme events, such as floods, earthquakes, and financial market crashes.
   - **Nonparametric Statistics**: Order statistics are often used in nonparametric statistics to construct statistical tests and confidence intervals without assuming specific distributions for the data.

Extreme Value Theory  ●●●●●

By the Fisher-Tippett-Gnedenko theorem, the distribution of an order statistic will have the probability distribution function:

$$\boxed{f(x;\mu,\sigma,\xi)=\frac{1}{\sigma}\left[1+\xi\left(\frac{x-\mu}{\sigma}\right)\right]^{-1/\xi-1}\exp\left\{-\left[1+\xi\left(\frac{x-\mu}{\sigma}\right)\right]^{-1/\xi}\right\}} \tag{4.1}$$

This limiting distribution is called the **generalized extreme value (GEV) distribution**. A second extreme value theorem, known as the Pickands-Balkema-de Haan theorem, deals with the tail distribution of a random variable; specifically that the generalized Pareto distribution is an excellent descriptor of the tail for a wide range of distributions. While an interesting topic, this will not come into play in the analysis of biathlon performance.

4.1.7 Monte Carlo Methods

Monte Carlo methods are a class of computational techniques used to approximate solutions to complex mathematical problems through random sampling. They are widely used in various fields, including physics, engineering, finance, and statistics, to solve problems that may be analytically intractable or computationally expensive. Here's a breakdown of Monte Carlo methods:

1. **Definition**: Monte Carlo methods rely on repeated random sampling to obtain numerical results. The basic idea is to simulate the behavior of a system or process by generating random samples from the input parameter space and then averaging the results to obtain an estimate of the desired quantity.

2. **Steps**:

   (a) **Random Sampling**: Random samples are generated from the input parameter space according to a specified probability distribution. These samples are typically drawn from uniform, normal, or other distributions depending on the problem.

   (b) **Simulation**: The system or process of interest is simulated using the random samples as inputs. This may involve solving equations, running simulations, or performing other computational tasks.

   (c) **Analysis**: The results of the simulations are analyzed to obtain estimates of the desired quantities, such as means, variances, probabilities, or optimal decisions.

3. **Types of Monte Carlo Methods**:

   - **Monte Carlo Integration**: Monte Carlo integration is used to approximate the value of a multidimensional integral by averaging the function over random samples from the input space.

- **Monte Carlo Simulation**: Monte Carlo simulation is used to model and analyze the behavior of complex systems or processes by generating random samples from the input parameter space and observing the resulting outputs.
- **Monte Carlo Optimization**: Monte Carlo optimization is used to find optimal solutions to optimization problems by randomly sampling the solution space and evaluating the objective function.

4. **Applications**:

- **Finance**: Monte Carlo methods are used in option pricing, risk analysis, and portfolio optimization in finance.
- **Physics**: They are used in computational physics to simulate particle interactions, nuclear reactions, and other physical phenomena.
- **Engineering**: Monte Carlo methods are used in reliability analysis, design optimization, and uncertainty quantification in engineering.
- **Statistics**: They are used in Bayesian inference, Markov chain Monte Carlo (MCMC) methods, and statistical hypothesis testing.

Inverse Transform Sampling

A basic Monte Carlo method, but very powerful

Inverse transform sampling, also known as the inverse probability integral transform, is a method used to generate random samples from any probability distribution, provided its cumulative distribution function (CDF) is known. The technique leverages the fact that uniformly distributed random variables can be transformed into random variables of any desired distribution.

The method involves the following steps:

1. Identify the target probability distribution and its CDF, $F_X(x)$, where $F_X(x) = P(X \leq x)$ for a random variable X.
2. Generate a random sample U from a uniform distribution over the interval $[0, 1]$.
3. Use the inverse of the CDF, $F_X^{-1}(u)$, to transform the uniformly distributed sample U into a sample X from the desired distribution: $X = F_X^{-1}(U)$.

The fundamental principle behind inverse transform sampling is that if U is a uniformly distributed random variable on $[0, 1]$, then the random variable $X = F_X^{-1}(U)$ will have $F_X(x)$ as its CDF. This is due to the properties of uniform distributions and the nature of the CDF, which guarantees that the probability is uniformly spread over the range of possible outcomes.

Inverse transform sampling is widely used in simulations where specific probability distributions are required. It is particularly useful because it allows for the generation of random variables with a specified distribution from a source of uniform random numbers, which are readily available in most programming environments.

While versatile, the method requires that the inverse of the CDF, F_X^{-1}, can be easily computed. For distributions where the inverse CDF is not analytically tractable or is difficult to compute, alternative sampling methods may be more efficient.

Insert of table of common distributions and their quantile functions, along with commands in the major programming languages.

The Weibull distribution:

$$F = 1 - e^{-(x/\lambda)^k} \implies Q(p) = \lambda\left[-\log(1-p)\right]^{1/k}$$

The generalized extreme value distribution:

$$F = e^{-t(x)}, \quad t(x) = \left[1 + \xi\left(\frac{x-\mu}{\sigma}\right)\right]^{-1/\xi}$$

$$\implies Q(p) = \mu + \frac{\sigma}{\xi}\left[\frac{1}{(-\log p)^{\xi}} - 1\right]$$

Rejection Sampling

Can be used to draw random variables from a more complicated distribution (one without an analytical CDF), such as when the Rice distribution is the target distribution and the Rayleigh distribution is the proposal distribution. In general, we want:

$$M f_{\text{prop}} \geq f_{\text{tar}} \implies M \geq \frac{f_{\text{tar}}}{f_{\text{prop}}}$$

For the case where the Rice distribution is the target and the Rayleigh distribution is the proposal, the ratio is:

$$\frac{f_{\text{tar}}}{f_{\text{prop}}} = e^{-\nu^2/2\sigma^2} I_0\left(\frac{r\nu}{\sigma^2}\right)$$

4.2 Regression Analysis

Regression analysis is a statistical method used to study the relationship between one or more independent variables and a dependent variable. It is widely used in various fields, including economics, finance, psychology, and biology, to model and predict the behavior of complex systems. Here's a breakdown of regression analysis:

1. **Definition**: Regression analysis involves fitting a mathematical model to observed data in order to estimate the relationship between variables. The model typically takes the form of a linear equation, although other functional forms such as polynomial, exponential, and logarithmic can also be used depending on the nature of the data.

2. **Types of Regression Models**:

   - **Simple Linear Regression**: In simple linear regression, there is only one independent variable and one dependent variable, and the relationship between them is assumed to be linear.
   - **Multiple Linear Regression**: Multiple linear regression extends simple linear regression to cases where there are multiple independent variables. The model takes the form of a linear equation with multiple coefficients.
   - **Nonlinear Regression**: Nonlinear regression is used when the relationship between variables cannot be adequately captured by a linear model. It allows for more flexible functional forms, such as polynomial or exponential equations.

3. **Model Estimation**: The parameters of the regression model, including the coefficients and intercept, are estimated using statistical techniques such as ordinary least squares (OLS) or maximum likelihood estimation (MLE). These techniques aim to minimize the difference between the observed data and the predicted values from the model.

4. **Model Assessment**:

   - **Goodness-of-Fit**: Measures such as the coefficient of determination (R^2) and the adjusted R^2 are used to assess how well the regression model fits the data. A higher R^2 indicates a better fit.
   - **Residual Analysis**: Residuals, which are the differences between observed and predicted values, are analyzed to check for patterns or outliers that may indicate problems with the model.
   - **Hypothesis Testing**: Statistical tests, such as t-tests and F-tests, are used to assess the significance of individual coefficients and overall model fit.

5. **Applications**:

   - Regression analysis is used for prediction and forecasting, such as predicting sales based on advertising expenditure or forecasting stock prices based on past performance.
   - It is used in scientific research to study the relationship between variables, such as the effect of temperature on plant growth or the relationship between income and education level.
   - Regression analysis is also used in policy evaluation and decision-making, such as assessing the impact of public health interventions or determining the factors influencing voting behavior.

4.2.1 Ordinary Least Squares

Ordinary Least Squares (OLS) regression is a method used to estimate the parameters of a linear regression model by minimizing the sum of the squared differences between the observed and predicted values of the dependent variable. It is one of the most common techniques for fitting a linear model to data. Here's a breakdown of OLS regression:

1. **Definition**: OLS regression aims to find the best-fitting line through a set of data points by minimizing the sum of the squared vertical distances between the observed and predicted values. The linear regression model takes the form:

   $$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \ldots + \beta_n X_n + \varepsilon$$

   where:

   - Y is the dependent variable,
   - $X_1, X_2, \ldots, X_n$ are the independent variables,
   - $\beta_0, \beta_1, \ldots, \beta_n$ are the regression coefficients,
   - ε is the error term representing the difference between the observed and predicted values.

2. **Minimization of Residuals**: OLS regression minimizes the sum of the squared residuals (also known as errors or residuals) between the observed and predicted values of the dependent variable. The residuals are calculated as the differences between the observed Y values and the predicted Y values from the regression model.

3. **Estimation of Coefficients**: The regression coefficients $\beta_0, \beta_1, \ldots, \beta_n$ are estimated by solving the normal equations, which are a set of equations derived from the first-order conditions of the minimization problem. The coefficients minimize the sum of squared residuals, making the regression line the best linear fit to the data.

4. **Properties**:

   - OLS regression coefficients have desirable properties such as unbiasedness, efficiency, and consistency under certain assumptions about the data and the error term.
   - The fitted values obtained from OLS regression are the best linear unbiased estimators (BLUE) of the true values of the dependent variable, given the assumptions of the model.

5. **Applications**:

   - OLS regression is used in various fields, including economics, finance, social sciences, and engineering, to analyze the relationship between variables, make predictions, and test hypotheses.

- It is commonly used for modeling and forecasting, hypothesis testing, causal inference, and policy analysis.

$$y = X\beta \implies \widehat{\beta} = (X^T X)^{-1} X^T y$$

4.2.2 Binomial Regression

Binomial regression is a technique for predicting the number of hits on target in biathlon, while accounting for other independent variables – these could include experience, wind speed, heart rate, visibility, and position (i.e., prone or standing). (It is a general version of logistic regression)

Suppose an athlete takes n shots and has k hits in a set, with a hit probability of p. Then the probability of having k hits is:

$$\mathbb{P}(Y = k) = \binom{n}{k} p^k (1-p)^{n-k} \tag{4.2}$$

$$\ell \propto -\sum_i k_i \log\left(\frac{p_i}{1-p_i}\right) - n\sum_i \log(1-p_i), \quad p = \frac{1}{1+e^{-\beta^T x}}$$

$$\implies \ell \propto -\sum_i k_i \beta^T x_i + n\sum_i \log\left(1 + e^{\beta^T x_i}\right)$$

$$\nabla_\beta \ell = -X^T\left(k - \frac{n}{1+e^{-X\beta}}\right), \quad \nabla^2_\beta \ell = X^T W X$$

With $W = \text{diag}[np(1-p)]$, which tells us that:

$$\beta_{j+1} = \beta_j + (X^T W_j X)^{-1} X^T (k - np_j)$$

4.2.3 Mixed Effects Regression

Mixed effects regression, also known as hierarchical or multilevel regression, accounts for hierarchical data structures, such as repeated measurements within individuals or clusters (e.g., athletes, races). In biathlon, mixed effects regression could be used to analyze longitudinal data from individual athletes or to account for clustering effects within races or competitions.

$$y = X\beta + Z\lambda + \epsilon$$

4.2.4 Time Series Analysis

Time series regression is used when the outcome variable is measured over time. In biathlon, time series regression could be used to analyze trends and patterns in race times, shooting accuracy, and other performance metrics over multiple seasons or competitions.

4.2.5 Ordinal Logistic Regression ●●●●●

Ordinal logistic regression, also known as ordered logit regression, is used when the dependent variable is ordinal, meaning that it has a clear ordering or ranking but the intervals between the rankings are not necessarily equal. This method is commonly applied in situations where you want to predict an ordinal outcome based on one or more independent variables.

The core of ordinal logistic regression is the proportional odds model, which can be represented as follows. Suppose Y is the ordinal dependent variable with J ordered categories. The model can be written in terms of cumulative odds up to the jth category:

$$\log \frac{\mathbb{P}(Y \leq j|x)}{\mathbb{P}(Y > j|x)} = \alpha_j - \beta^T x$$

Proportional odds is the key assumption of ordinal logistic regression is that the relationship between the cumulative log odds of being in a lower ordered category and the independent variables is the same for each threshold. This is also known as the parallel lines assumption.

The independence of irrelevant alternatives means that the choice of one category over another does not depend on the presence or absence of other alternatives.

Example: Picking a Wax Based on Weather Conditions

The goal is to pick the wax with the best handling on the course, given the weather conditions.

4.2.6 Regularization ●●●●●

The Bayesian information criterion (BIC)

Stepwise Regression

The LASSO

The Least Absolute Shrinkage and Selection Operator (LASSO) is a method used in regression analysis that combines variable selection and regularization to enhance the prediction accuracy and interpretability of the resulting statistical model. Developed by Robert Tibshirani in 1996, LASSO is especially beneficial in scenarios with a large number of predictors or when there is a high degree of multicollinearity among the predictors.

LASSO introduces several key features into the regression analysis:

- **Regularization:** LASSO applies a penalty to the coefficients of the regression variables, equivalent to the absolute value of the coefficients (the L1-norm). This process is controlled by a regularization parameter, λ, which dictates the extent of coefficient shrinkage towards zero.
- **Variable Selection:** The LASSO method can zero out some coefficients, effectively performing variable selection by excluding non-contributing variables from the model.
- **Prevention of Overfitting:** Through its regularization aspect, LASSO helps prevent the model from overfitting, potentially improving its generalizability to unseen data.

The objective function for LASSO regression can be expressed as follows:

$$\text{Minimize}\left\{\frac{1}{2n}\sum_{i=1}^{n}(y_i-\beta_0-\sum_{j=1}^{p}\beta_j x_{ij})^2+\lambda\sum_{j=1}^{p}|\beta_j|\right\}$$

where y_i is the response variable for the i^{th} observation, x_{ij} is the j^{th} predictor for the i^{th} observation, β_0 is the intercept, β_j are the coefficients for the predictors, n is the number of observations, p is the number of predictors, and λ is the regularization parameter.

LASSO's methodology is particularly valuable in various contexts:

- **High-Dimensional Data:** LASSO is effective in analyzing datasets where the number of features surpasses the number of observations, such as in genomics.
- **Feature Selection:** It is used in fields where identifying the most relevant predictors is critical, such as in finance for risk modeling or marketing for customer segmentation.
- **Multicollinearity:** LASSO can handle scenarios with highly correlated predictors, providing a more stable model compared to traditional regression methods.

When employing LASSO, several factors must be taken into account:

- The selection of λ is crucial, as it significantly impacts the model's complexity. Cross-validation is typically employed to find an optimal value.
- LASSO may not perform well when all predictors have equal importance or when the goal is solely prediction accuracy rather than interpretability or variable selection.
- In cases with multiple highly correlated predictors of equal relevance, LASSO may select one and ignore others, which might not be desirable depending on the context.

4.3 Design of Experiments

The statistical design of experiments (DoE) is a structured, organized method for determining the relationship between factors affecting a process and the output of that process. It is used to find cause-and-effect relationships and to optimize the performance of a process. In the context of biathlon, DoE can be applied to various aspects such as athlete training, equipment selection, and technique optimization.

4.3.1 Randomization

Randomization is a fundamental principle in the design of experiments (DoE), serving as a crucial method for ensuring that the experiment's outcomes are statistically valid and free from bias. The core purpose of randomization is to minimize the impact of confounding variables—factors other than the ones being studied—that could influence the results. By randomly assigning subjects, treatments, or experimental units to different groups, researchers can ensure that each group is statistically equivalent at the start of the experiment. This process helps in achieving unbiased estimates of treatment effects and enhances the reliability of the conclusions drawn from the experiment.

There are three key aspects of randomization:

- **Elimination of Bias**: Randomization helps in eliminating selection bias and assignment bias, ensuring that the experiment's results are not skewed by external factors.
- **Creating Comparable Groups**: It ensures that the treatment and control groups are comparable with respect to known and unknown factors, which might influence the outcome, thus attributing observed differences in outcomes directly to the treatment effect.
- **Foundation for Statistical Analysis**: Randomization provides a solid foundation for the statistical methods used to analyze the data. Since the assignment of treatments is random, the statistical analysis can make valid inferences about the probability of observing the results under the null hypothesis.

Implementing randomization requires three steps:

1. **Simple Randomization**: This involves using a random process, like flipping a coin or drawing numbers, to assign subjects to groups. Each subject has an equal chance of being assigned to any group.
2. **Stratified Randomization**: In this method, subjects are grouped into strata based on certain characteristics (e.g., age, gender), and then within each stratum, subjects are randomly assigned to treatment groups. This ensures that each group is balanced in terms of those characteristics.

3. **Block Randomization**: This method is used to ensure that each treatment group has an equal number of subjects by dividing them into blocks of a certain size that is a multiple of the number of treatments, and then randomly assigning treatments within each block.

In the context of biathlon or any sports science research, randomization can significantly enhance the quality and reliability of experimental results. For instance, when testing the effect of different training regimens on athlete performance, randomization ensures that differences in performance outcomes can be attributed with greater confidence to the training regimen itself, rather than to other confounding variables such as pre-existing fitness levels, nutritional status, or equipment used. Similarly, in equipment testing, randomization helps in isolating the effect of equipment changes from environmental conditions or athlete's physical condition on the day of testing.

4.3.2 Replication

Replication involves repeating the experiment multiple times to ensure that the results are consistent and reliable. This helps in quantifying the variation in the experiment and increases the precision of the results. Replication is crucial for validating the findings of an experiment and for understanding the natural variability in the data. Replication allows researchers to distinguish between real effects caused by the treatments or factors under investigation and random noise or fluctuations inherent in any experimental process.

The key aspects of replication are:

- **Statistical Precision**: Replication increases the precision of the experiment's results by providing a more accurate estimate of the treatment effects. With more data points collected through replication, statistical analyses have a stronger basis for detecting true effects.
- **Estimation of Variability**: It enables researchers to estimate the natural variability or error in the experiment. Understanding this variability is essential for assessing the significance of the experimental results.
- **Increased Reliability**: Replicating experiments and finding consistent results across replicates enhances the reliability and credibility of the findings. It demonstrates that the observed effects are not due to chance or specific to a particular set of experimental conditions.

There are also several types of replication:

1. **True Replication**: This involves repeating the entire experiment from the beginning, including the randomization of subjects or units to treatments. True replication is the most robust form of replication, as it includes all sources of variability.

2. **Technical Replication**: In this form, certain steps of the experiment are repeated, such as the measurement or observation phase, without redoing the entire experiment. This can help in assessing the reliability of specific experimental procedures or measurements.

3. **Biological Replication**: Especially relevant in biological and medical research, this type of replication involves using different biological samples (e.g., different individuals, cell batches, or animals) in each replicate to account for biological variability.

In the context of biathlon, replication can be applied in various ways to enhance the quality of experimental findings:

- **Training Regimen Evaluation**: By replicating the study across multiple athletes or over different training cycles, researchers can confidently identify the effects of specific training regimens on performance, taking into account individual variability and seasonal conditions.

- **Equipment Testing**: Replicating equipment tests under various conditions (e.g., different snow types, temperatures) helps in reliably assessing the performance benefits of new skis, rifles, or other gear.

- **Technique Optimization**: Replicating technique analysis sessions with multiple athletes or under different fatigue levels can provide insights into the most efficient techniques that are robust across different conditions.

4.3.3 Blocking

Blocking is a technique used in the design of experiments (DoE) to account for variability among experimental units that cannot be controlled or eliminated. By organizing these units into blocks or groups that are similar in some respect that is expected to affect the response to the treatments, researchers can more accurately isolate and measure the effect of the treatments themselves. Blocking thus helps in reducing the impact of confounding variables, making the comparison between treatment groups more precise and reliable.

The key aspects of blocking are:

- **Control for Variability**: Blocking controls for variation among experimental units by grouping similar units together. This means that any variation within a block is likely due to the treatment rather than differences between the units.

- **Increase in Precision**: By controlling for external variability, blocking increases the precision of the experiment. It allows for a clearer comparison of treatment effects within homogeneous groups.

- **Efficient Use of Resources**: Blocking can lead to more efficient experiments by ensuring that comparisons are made within groups of units that are similar in ways that might affect the outcome of the experiment. This can sometimes reduce the number of experimental units needed to detect a treatment effect.

There are several steps need to implement blocking:

1. **Identification of Blocking Factors**: The first step is to identify variables that are likely to introduce variability into the experiment but are not of primary interest. Common blocking factors include age, gender, location, and time of day.

2. **Formation of Blocks**: Once blocking factors are identified, experimental units are grouped into blocks based on these factors, ensuring that each block is as homogeneous as possible with respect to the blocking factor.

3. **Random Assignment within Blocks**: Treatments are then randomly assigned to experimental units within each block. This maintains the benefits of randomization while controlling for the blocking factor.

In biathlon, blocking can be applied to various experimental setups to improve the accuracy and reliability of findings:

- **Training Studies**: When evaluating the effectiveness of different training programs, athletes could be blocked by factors such as age, gender, or previous performance levels to control for their effects on the outcomes of interest.

- **Equipment Tests**: When testing new equipment, such as skis or rifles, conditions like snow type or temperature could serve as blocking factors. By conducting tests within these blocks, researchers can more accurately assess the performance of the equipment under specific conditions.

- **Technique Optimization**: Techniques could be evaluated within blocks defined by athlete fatigue levels or environmental conditions, allowing for a more precise understanding of how different techniques perform under varied conditions.

4.3.4 Applications in Biathlon

Performance Improvement

DoE can be used to systematically test different training regimens, nutrition plans, and recovery strategies to find the most effective combination for improving athlete performance.

Equipment Optimization

By applying DoE principles, biathletes and their teams can evaluate the effects of different equipment choices (e.g., skis, poles, rifles) under varying conditions to select the best equipment for different races and conditions.

Technique Optimization

DoE can also be applied to refine shooting and skiing techniques. By systematically varying technique elements and measuring their impact on performance and accuracy, athletes can find the most efficient techniques for both skiing and shooting.

4.3.5 Response Surface Methodology

Use this technique from aircraft design.

4.3.6 Conclusion

The statistical design of experiments is a powerful tool for optimizing performance in biathlon. By applying principles such as randomization, replication, and blocking, biathletes and their coaches can make data-driven decisions to enhance training, equipment selection, and technique. This systematic approach leads to more effective training programs, better equipment choices, and improved competition results.

4.4 Avoiding Statistical Malpractice

Have I convinced you that statistical methods are useful and not solely the province of liars and cheats? Before sending an email explaining what a poor job I've done, please re-read the chapter twice over; if you are still not convinced, then by all means send an email explaining how incompetent I am and that my alma maters should revoke the degrees I've earned.

Anyway, now that we are agreed that statistics can provide value for characterizing biathlon performance, we should acknowledge the partial truth of the quotes at the beginning of this chapter. Statistics does have an unfortunate reputation, some of which is deserved; most questionable academic research that has been under scrutiny in the recent **replication crisis** relied on statistical methods to make their conclusions.

One obvious reason for the proliferation of questionable research practices and dodgy conclusions are the incentives, financial and reputational, that are faced by the practitioners. Consider the following examples:

- **Tobacco Industry**: *Smoking doesn't cause lung cancer.*
- **Pharmaceuticals**: *This drug we've developed, over a time period and with outcomes of our choosing, is highly efficacious.*
- **Investments**: *Our asset allocation strategy, over a time period and with outcomes of our choosing, generates positive alpha.*
- **Academics**: *I've discovered a surprising new result, which went unnoticed by my peers, that is going to upend the field.*

Except for the smoking example, each of these claims may be true. However, skepticism is warranted because upon further inspection many of these claims turn out to be exaggerated or non-existent.

So how can you avoid this fate? Follow several guidelines: honesty, transparency, etc.

Chapter 5

Shooting Performance

Aim for the flattop!

— Ray Stantz, *Ghostbusters*

5.1 Overview of Shooting Performance

Shooting performance in biathlon, a sport combining cross-country skiing and rifle shooting, is influenced by a variety of determinants. Each plays a critical role in the overall performance of an athlete during a biathlon event. Below is a summary of the key determinants grouped into a single comprehensive list:

- **Physical Factors**
  - Endurance and Fitness: The ability to maintain high performance in skiing without excessively elevating heart rate and fatigue, which can affect shooting accuracy.
  - Stability and Muscle Control: Physical strength, especially in the core and upper body, aids in stabilizing the rifle during shooting.
  - Breath Control: The ability to control breathing is essential for stabilizing the shot during the moment of firing.
- **Psychological Factors**
  - Focus and Concentration: Maintaining concentration on shooting despite the physical exertion from skiing and external pressures.

- Stress and Anxiety Management: Effective handling of competition pressure and anxiety, which can impact shooting steadiness and accuracy.
- Mental Endurance: The capacity to stay mentally strong and focused throughout the race, managing fluctuations in performance and conditions.

- **Technical Factors**
  - Shooting Technique: Proper positioning, rifle handling, and aiming techniques that ensure accuracy.
  - Transition Efficiency: The ability to quickly and efficiently transition from skiing to shooting and vice versa, minimizing time lost.
  - Shot Timing: Choosing the optimal moment for taking shots, considering factors like breathing and heart rate.

- **Environmental Factors**
  - Weather Conditions: Wind, temperature, and visibility can significantly affect shooting accuracy and require adjustments in aiming and technique.
  - Altitude: Higher altitudes can influence an athlete's physiological state, potentially affecting shooting performance.

Each of these determinants requires careful attention and training to optimize shooting performance in biathlon. Athletes and coaches focus on a holistic training approach that includes physical conditioning, technical skill development, psychological strategies, and adaptation to environmental conditions. Balancing these factors effectively is essential for success in biathlon competitions, where even minor improvements in shooting performance can lead to significant advantages.

5.2 Quantifying Shot Spread in Range Training

5.2.1 Derivation of the Rayleigh Distribution

$$f(r;\sigma) = \frac{r}{\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right) \tag{5.1}$$

$$f(x,y) = \frac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/2\sigma^2}$$

Using the substitutions $x = r\cos\theta$ and $y = r\sin\theta$, the PDF becomes:

$$f(r,\theta) = \frac{r}{2\pi\sigma^2} e^{-r^2/2\sigma^2}$$

Integrating over the angle θ, we see that:

$$f(r) = \int_0^{2\pi} f(r,\theta)\, d\theta = \frac{r}{\sigma^2} e^{-r^2/2\sigma^2}$$

which is simply the Rayleigh distribution.

Relation to the Circular Error Probable

A benchmark figure can also be used for the CEP, such that σ is equal to some benchmark value σ_B:

$$\sigma_B = r\sqrt{-\frac{1}{2\log(1-p_B)}}$$

If we set $p_B = 0.8$ in the prone position, with $r = 22.5$ mm, then $\sigma_B = 12.54$ mm; for the standing position, with $r = 57.5$ mm, $\sigma_B = 32.05$ mm. Progressed can then be tracked relative to the benchmark value.

Accounting for Sighting Error (Rice Distribution)

$$f(r;\sigma,\nu) = \frac{r}{\sigma^2} \exp\left(-\frac{r^2+\nu^2}{2\sigma^2}\right) I_0\left(\frac{r\nu}{\sigma^2}\right)$$

The CDF of the Rice distribution is:

$$F(r) = 1 - Q_1\left(\frac{\nu}{\sigma}, \frac{r}{\sigma}\right)$$

Where Q_1 is the Marcum Q-function, defined by:

$$Q_\nu(a,b) = \frac{1}{a^{\nu-1}} \int_b^\infty x^\nu \exp\left(-\frac{x^2+a^2}{2}\right) I_{\nu-1}(ax)\, dx$$

Therefore, the full CDF reads:

$$F(r) = 1 - \int_{r/\sigma}^\infty x \exp\left(-\frac{x^2\sigma^2+\nu^2}{2\sigma^2}\right) I_0\left(\frac{x\nu}{\sigma}\right)\, dx$$

5.2.2 Maximum Distance Between Two Points

Another quantity we may wish to compute aside from the spread is the maximum distance between two points – I've found this beneficial when examining shooting results from non-biathlon ranges. This standard distance metric is the Euclidean distance:

$$d(x,y) = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$$

5.2.3 Gauss-Laguerre Quadrature

Gauss-Laguerre quadrature is a particular form of Gauss quadrature that is designed to compute integrals of the form:

$$\int_0^\infty e^{-x} f(x)\, dx \approx \sum_{i=1}^{n} w_i f(x_i)$$

The Golub-Welsch algorithm is a numerical method for computing the eigenvalues and eigenvectors of a symmetric tridiagonal matrix.[3] It can be used in conjunction with Gauss-Laguerre quadrature to efficiently compute the nodes and weights for numerical integration over the semi-infinite interval $[0, \infty)$ with respect to the Laguerre weight function e^{-x}.

1. **Generate the Tridiagonal Matrix**: Start by constructing the symmetric tridiagonal matrix T whose eigenvalues and eigenvectors correspond to the nodes and weights of the Gauss-Laguerre quadrature rule. The elements of T can be computed using recurrence relations for Laguerre polynomials.

2. **Apply the Golub-Welsch Algorithm**: Use the Golub-Welsch algorithm to compute the eigenvalues and eigenvectors of T. This involves iteratively reducing the matrix to a diagonal form by orthogonal similarity transformations.

3. **Extract the Nodes and Weights**: Once the eigenvalues and eigenvectors of T are obtained, the nodes and weights of the Gauss-Laguerre quadrature rule can be extracted from the eigenvalues and the first row of the matrix of eigenvectors, respectively.

4. **Normalization**: Normalize the weights to ensure that they sum to 1, if necessary.

5. **Integration**: Use the computed nodes and weights to perform numerical integration over the desired interval $[0, \infty)$ with respect to the Laguerre weight function e^{-x}.

Implementing this procedure in a numerical computing environment such as Python with libraries like NumPy or SciPy allows for efficient computation of Gauss-Laguerre quadrature nodes and weights for various integration tasks.

5.3 Spread Reduction

One maxim of consistent training is that you will improve over time, but that all training suffers from diminishing returns. However, it is worthwhile projecting how much effort is required to attain a certain degree of mastery.

It is widely known that the shot distribution from the bullseye follows a Rayleigh distribution (5.1). In this instance, there is only a single parameter of interest: σ, taken as the spread. Given this, the most appropriate measure of performance improvement is reductions in the spread over time (time being a proxy for experience). A simple model, which seems to accurately reflect improvements, is the equation:[1]

$$\frac{d\sigma}{dt} = k(\sigma_\infty - \sigma) \implies \sigma(t) = \sigma_\infty + (\sigma_0 - \sigma_\infty)e^{-kt}$$

Aside from being an apparently good descriptor of reality, the equation has an initial spread, a spread limit, and exhibits diminishing returns. For a set of parameters, it is also possible to calculate the expected time for reaching a certain spread:

$$t = -\frac{1}{k} \log \left[\frac{\sigma(t) - \sigma_\infty}{\sigma_0 - \sigma_\infty} \right]$$

5.3.1 Learning Curves

For general learning curves, it is helpful to use the generalized logistic function:

$$Y(t) = A + \frac{K - A}{(C + Qe^{-Bt})^{\nu}}$$

Coaches can also use the learning curve in the spread reduction to estimate:

1. The maximum shooting ability of biathletes
2. How long it will take for them to meet that threshold

Qualitative understanding of learning curves involves observing the general trend of how performance improves over time or with experience. Initially, when individuals start learning a new task or skill, their performance may be poor or inconsistent as they are unfamiliar with the task's requirements or techniques. However, with practice and experience, their performance typically improves, and they become more proficient at the task.

Learning curves often exhibit characteristic shapes, such as rapid improvement at the beginning followed by a more gradual increase as learning progresses. Plateaus may occur where performance levels off temporarily before further improvement is observed. These plateaus may indicate a need for additional practice or a different approach to learning.

Quantitative understanding of learning curves involves analyzing numerical data to measure the rate of learning and predict future performance. In this context, learning curves typically plot performance

[1] This is the deterministic version of the Ornstein-Uhlenbeck process:

$$d\sigma = k(\sigma_\infty - \sigma)dt + \beta dW \implies \sigma(t) = \sigma_\infty + (\sigma_0 - \sigma_\infty)e^{-kt} + \beta \int_0^t e^{-k(t-\tau)} dW_\tau$$

The spread is then normally-distributed over time:

$$\sigma(t) \sim \mathcal{N}\left[\sigma_\infty + (\sigma_0 - \sigma_\infty)e^{-kt}, \frac{\beta^2}{2k}\left(1 - e^{-2kt}\right)\right]$$

metrics, such as accuracy, speed, or error rates, against the number of trials, practice sessions, or time spent learning.

Key aspects of quantitative learning curves include:

1. **Initial Performance**: The starting point of the learning curve, representing the baseline performance level before learning begins.

2. **Learning Rate**: The slope of the learning curve, indicating how quickly performance improves over time or with experience. A steeper slope suggests faster learning, while a shallower slope indicates slower learning.

3. **Asymptote**: The maximum or optimal performance level that can be achieved with sufficient learning or practice. As learning progresses, performance approaches this asymptotic level, but may not necessarily reach it.

4. **Variability**: Learning curves may exhibit variability due to factors such as individual differences in learning ability, task complexity, or variations in practice conditions. Understanding and accounting for this variability is important for accurately interpreting learning curve data.

Learning curves have broad applications across various domains, including education, training, skill acquisition, and performance optimization. They provide valuable insights into the learning process, allowing educators, trainers, and learners themselves to:

- Identify areas of weakness or difficulty that may require additional attention or support.

- Optimize learning strategies and instructional methods to maximize learning efficiency and effectiveness.

- Set realistic goals and expectations for learning outcomes based on observed learning rates and performance trajectories.

5.4 Overall Performance Metrics

To create an effective shooting performance metric for biathlon, we must consider the trade-offs between shooting speed and shooting accuracy; in particular, a lack of accuracy will lead to penalty loops. A total time gain or loss will depend on a biathlete's relative quickness in shooting and a biathlete's relative accuracy. Let's consider the following variables:

- **Shooting time**: τ_{sh}, the time taken by the biathlete to complete the shooting stage.

- **Penalty loop time**: τ_{PL}, the time taken to complete one penalty loop.

- **Number of penalties**: p, the number of penalty loops completed by the biathlete during the shooting stage.

Suppose we're comparing a biathlete's performance against the average biathlete. For some performance metric $f(\tau_{sh}, \tau_{PL}, p)$, we should expect the performance metric to decrease as each variable increases. Let Δt be the time gain/loss from the shooting performance. Then, expected time gain/loss from shooting, conditional on the variables $V \equiv \mathbb{E}\left(\Delta t | \tau_{sh}, \tau_{PL}, p\right)$:

$$\mathbb{E}\left(\Delta t | \tau_{sh}, \tau_{PL}, p\right) = \left[\mathbb{E}(\tau_{sh}) - \tau_{sh}\right] + \left[\mathbb{E}(p)\mathbb{E}(\tau_{PL}) - p\tau_{PL}\right] \tag{5.2}$$

If we assume that for an individual biathlete $\tau_{PL} \approx \mathbb{E}(\tau_{PL})$, it follows that:

$$\mathbb{E}\left(\Delta t | \tau_{sh}, \tau_{PL}, p\right) = \left[\mathbb{E}(\tau_{sh}) - \tau_{sh}\right] + \left[\mathbb{E}(p) - p\right]\mathbb{E}(\tau_{PL})$$

Define V^* to be the expected time gain/loss for the minimum shooting time $\tau_{sh}^{\min}$ and with no penalties, which is:

$$V^* \equiv \mathbb{E}\left(\Delta t | \tau_{sh}^{\min}, \tau_{PL}, 0\right) = \left[\mathbb{E}(\tau_{sh}) - \tau_{sh}^{\min}\right] + \mathbb{E}(p)\mathbb{E}(\tau_{PL})$$

Then, we may calculate the Biathlon Adjusted Shooting Performance (BASP) to be:

$$\boxed{\text{BASP} = \frac{V}{V^*} = \frac{\left[\mathbb{E}(\tau_{sh}) - \tau_{sh}\right] + \left[\mathbb{E}(p)\mathbb{E}(\tau_{PL}) - p\tau_{PL}\right]}{\left[\mathbb{E}(\tau_{sh}) - \tau_{sh}^{\min}\right] + \mathbb{E}(p)\mathbb{E}(\tau_{PL})} \in (-\infty, 1]} \tag{5.3}$$

If the number of penalties and penalty time are independent, then $\mathbb{E}(\text{BASP}) = 0$.

BASP quantifies a biathlete's performance during the shooting stage relative to the average performance of competitors. It accounts for both shooting time and penalties incurred, reflecting the time gain or loss compared to the average shooting performance. By using the BASP metric, coaches, athletes, and analysts can evaluate shooting performance in biathlon races and identify areas for improvement, such as marksmanship, shooting speed, or penalty avoidance. Additionally, BASP can be used for benchmarking performance across different races, athletes, and shooting conditions.

To the point about benchmarking, the standards used for evaluating biathletes is should be relative to their peer group – not some absolute standard, although as the level of competition rises BASP can approximate an absolute standard. Based on an analysis of Colorado Biathlon Club and IBU World Cup data, the values of the different parameters when looking for a general, not race-specific, benchmark are as follows:

Level	$\mathbb{E}(\tau_{sh})$	$\mathbb{E}(p)$	$\mathbb{E}(\tau_{PL})$	$\tau_{sh}^{\min}$
Amateur	60 sec	2.5	33 sec	20 sec
Professional	30 sec	1	21 sec	12 sec

Table 5.1: Shooting Performance Metrics for Different Levels of Biathletes.

To compare results, let's suppose that $\tau_{PL} \approx \mathbb{E}(\tau_{PL})$ and set $p = 2$ and $\tau_s h = 45$ seconds. Then we may find that BASP equals 9/35 for the amateur and -12/13 for the professional. With the same shooting statistics, the amateur enjoys a slight time gain from their performance, while the professional has a substantial time loss. Applying BASP to IBU World Cup data, there is a very strong Spearman (rank) correlation between a biathlete's BASP score and their final rank in a competition.

5.5 Psychological Factors

Personality tests and psychology

The Big Five personality tests measure five major dimensions of human personality: Openness, Conscientiousness, Extraversion, Agreeableness, and Neuroticism. These tests can be used to predict success in various domains, including sports. Specifically, for biathlon shooting, certain personality traits may correlate with higher performance levels. The methodologies of factor analysis and item response theory (IRT) play crucial roles in this predictive process.

5.5.1 Factor Analysis  ●●●●●

Factor analysis is a statistical method used to identify underlying variables, or factors, that explain the pattern of correlations among multiple observed variables. In the context of personality testing, it helps in identifying which specific traits contribute most significantly to success in biathlon shooting. For example, high conscientiousness might correlate with the discipline required for shooting accuracy, while low neuroticism could be associated with better stress management during competitions.

- **Identifying Relevant Traits:** Factor analysis can reveal which Big Five traits are most strongly associated with shooting performance in biathlon.
  - You can also develop a network model of how the factors relate, like what I created for sex ability
- **Developing Targeted Interventions:** Understanding these correlations allows coaches to develop targeted training programs that not only focus on physical and technical skills but also on enhancing beneficial personality traits. (e.g., Cognitive Behavioral Therapy for certain types of neuroticism or negative thoughts)

$$y = \mu + \Lambda f + \epsilon$$

$$\mathbb{E}(y|f) = \mu + \Lambda f, \quad \mathrm{Var}(y|f) = \Psi$$

These can be computed using the expectation-maximization (EM) algorithm. See **here**.

5.5.2 Item Response Theory ●●●●●

Item response theory (IRT) is used to analyze the properties of test items, considering the trait levels of respondents. It provides a framework for understanding how individual personality traits influence the likelihood of specific outcomes, such as successful shooting performance in biathlon.

- **Personalized Assessment:** IRT allows for more personalized assessments by estimating the probability that a person with a certain trait level will respond in a specific way to an item related to shooting performance.
- **Predicting Performance:** By applying IRT, psychologists and coaches can more accurately predict how an athlete's personality profile influences their potential for success in shooting disciplines of biathlon.

By integrating the insights gained from factor analysis and IRT with conventional training methods, coaches can identify athletes who possess or have the potential to develop the psychological profile conducive to success in biathlon shooting. This holistic approach to athlete development emphasizes the importance of psychological factors alongside physical and technical training.

The two-parameter IRT model allows an item to vary based on its difficult b_i and discrimination a_i between people of different abilities θ_u.

$$\mathbb{P}(X_{ui} = 1|\theta_u) = \frac{1}{1 + e^{-a_i(\theta_u - b_i)}}$$

Example: Creating a Fitness Testing Battery

This could be a series of tests for shooting and skiing, particularly for screening. In conjunction with ordinal logistic regression.

5.6 Shot Trajectory Analysis ●●●●●

To include:

1. Governing differential equations
2. Bullet drop
3. Temperature effects
4. Parallax

I have not included the topic of differential equations, because compared to probabilistic and statistical methods differential equations are not terribly useful in biathlon (aside from assessing the stiffness of ski poles). Nonetheless, a historically important application of differential equations is in projectile modeling - historically for artillery and naval gunnery.

Modeling the trajectory of a projectile, especially in a context that includes not only basic gravitational forces but also drag and the Coriolis effect, requires a set of coupled differential equations derived from Newton's second law of motion, $F = ma$, where F is the force applied to the object, m is the mass of the object, and a is its acceleration. When considering the factors you've mentioned, the equations can become quite complex due to the nonlinear nature of drag and the rotating reference frame effects introduced by the Coriolis force. Here is a general framework for these equations:

1. **Basic Forces on a Projectile**: - **Gravity**: Acts downward, typically considered as a constant g near the Earth's surface, with a value of approximately $9.81\,\mathrm{m/s}^2$. - **Drag Force**: Depends on the projectile's velocity, the air's density, the cross-sectional area of the projectile perpendicular to its velocity, and a drag coefficient. It acts in the direction opposite to the projectile's motion. The drag force can be modeled as $F_{\text{drag}} = \frac{1}{2}\rho v^2 C_d A$, where ρ is the air density, v is the velocity of the projectile, C_d is the drag coefficient, and A is the cross-sectional area. - **Coriolis Force**: Arises due to the Earth's rotation and affects the projectile's trajectory on a rotating Earth. It can be represented as $F_{\text{Coriolis}} = 2m\mathbf{v} \times \mathbf{\Omega}$, where $\mathbf{v}$ is the velocity of the projectile relative to the Earth's surface, and $\mathbf{\Omega}$ is the Earth's angular velocity vector.

2. **Differential Equations**: The motion of the projectile can be described by a set of differential equations that account for the forces mentioned. Let $\mathbf{r} = (x, y, z)$ be the position vector of the projectile, and $\mathbf{v} = (v_x, v_y, v_z)$ be its velocity vector. The acceleration vector $\mathbf{a}$ is the derivative of the velocity vector with respect to time, $\mathbf{a} = \frac{d\mathbf{v}}{dt}$, and can be expressed as the sum of the accelerations due to gravity, drag, and the Coriolis effect, divided by the mass m.

The governing differential equations, incorporating Newton's second law, are:

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}$$
$$\frac{d\mathbf{v}}{dt} = -g\widehat{\mathbf{z}} - \frac{1}{m}\frac{1}{2}\rho v^2 C_d A \frac{\mathbf{v}}{v} + 2(\mathbf{v} \times \mathbf{\Omega})$$

Where: - $\widehat{\mathbf{z}}$ is the unit vector in the vertical direction. - The term $-g\widehat{\mathbf{z}}$ represents the acceleration due to gravity. - The term $-\frac{1}{m}\frac{1}{2}\rho v^2 C_d A \frac{\mathbf{v}}{v}$ represents the acceleration due to drag, with the direction of drag being opposite to the velocity vector $\mathbf{v}$. - The term $2(\mathbf{v} \times \mathbf{\Omega})$ represents the Coriolis acceleration.

These equations must typically be solved numerically due to their complexity and the nonlinear dependency on velocity for the drag term. Additionally, the exact forms of the equations can vary based on the assumptions made regarding the Earth's shape (spherical or ellipsoidal), the variation of gravitational acceleration with altitude, and the specifics of the projectile's shape and size.

Parallax is a phenomenon where the position or direction of an object appears to differ when viewed from different positions. It's essentially an effect of perspective, highlighting how the relative position of an object changes against a distant background due to a shift in the observer's viewpoint. This concept is foundational in various scientific and engineering fields, including astronomy, where it's used to measure the distances of stars by observing them from different points in Earth's orbit, and in stereoscopy for creating three-dimensional images by simulating parallax.

In the context of shooting sports, parallax becomes relevant in the use of telescopic sights, commonly known as scopes. A scope's reticle (the crosshair or aiming point) must ideally appear to lie on the same optical plane as the target to ensure precise aiming. If the reticle and the target are not on the same focal plane within the scope, a parallax error occurs. This means that if the shooter's eye is not perfectly aligned with the scope's optical axis (a common occurrence due to natural head and hand movements), the reticle appears to move against the target, leading to potential aiming errors.

High-quality scopes often include a parallax adjustment feature, allowing the shooter to minimize or eliminate parallax error by adjusting the scope's focus to match the target's distance. This adjustment aligns the optical plane of the reticle with that of the target, ensuring that the reticle's position appears stationary relative to the target, regardless of minor shifts in eye position. This is particularly important for long-range shooting, where even minute aiming errors can lead to significant misses. The adjustment can be a dial or knob marked with distances on the scope, enabling shooters to fine-tune their sights based on the approximate distance to the target.

Understanding and managing parallax is crucial for achieving high accuracy in shooting sports. It requires a blend of technical knowledge of the optical principles at play and practical skills in adjusting and aligning telescopic sights under varying conditions.

Peep sights, also known as aperture sights, used in shooting sports like biathlon, offer a different approach to minimizing parallax errors compared to telescopic sights. Unlike scopes that may require manual adjustment to align the reticle and target on the same optical plane, peep sights inherently reduce parallax through their design and the way they are used.

A peep sight system typically consists of two main components: a rear aperture or "peep" through which the shooter looks, and a front sight that may be a post, bead, or ring. The shooter aligns the front sight with the target and centers it within the circular aperture of the rear sight.

The principle behind minimizing parallax in peep sights lies in the forced alignment of the eye with the sight due to the small aperture of the peep. When the shooter's eye looks through the small rear aperture, it naturally centers the eye in line with the sight's optical axis. This forced alignment ensures that the line of sight is consistent and directly along the center of the peep and the front sight, making the aiming point more stable and less susceptible to parallax error. Essentially, the small aperture acts as a guide, focusing the eye's line of sight directly towards the target through the center of the front sight.

This configuration exploits the human eye's ability to automatically center objects within a circular frame and its tendency to focus on the most illuminated and distinct elements of the sight picture, which in this case, is the front sight and the target. Since the eye is naturally drawn to center the front sight within the peep sight's aperture, any potential parallax error is minimized because the line of sight remains constant relative to the shooter's eye position.

In summary, peep sights use the natural optical and physiological characteristics of the human eye to ensure consistent aim and minimize parallax error, making them highly effective for precision shooting in sports like biathlon, even without the complex adjustments required for telescopic sights.

Introduce a term for a constant wind to explain the effect on bullet trajectory.

Chapter 6

Ski Performance

It's a lot easier to teach a skier to shoot than a shooter to ski.

— Ancient Biathlon Wisdom

Biathlon could be reasonably described as a ski race with a shooting component. Yes, there will be more variation in outcomes than cross-country skiing, but skiing ability is the primary determinant of differential performance outcomes for professional biathletes - and even more so amongst amateur athletes.

6.1 Overview of Ski Performance

Cross-country skiing performance in biathlon is pivotal for overall success in the sport. This performance is influenced by a comprehensive array of determinants, categorized into physical, technical, tactical, and environmental factors. Here's a detailed overview of these determinants:

- **Physical Factors**
  - **Endurance:** The athlete's ability to sustain high-intensity efforts throughout the race, which is foundational for success.
  - **Strength:** Particularly upper body and core strength, which are vital for effective poling and maintaining technique under fatigue.
  - **Power:** The capacity to generate force quickly, crucial for sprints and uphill sections.
  - **Aerobic and Anaerobic Capacity:** High levels of both are essential for endurance and the ability to recover quickly after intense efforts.

- **Technical Factors**
  - **Ski Technique Efficiency:** Mastery of various skiing techniques to maintain speed with minimal energy expenditure.
  - **Equipment Management:** Optimizing ski selection, waxing, and tuning to match race conditions.
  - **Transitions:** Efficiency in changing techniques or gears to match terrain changes seamlessly.
- **Tactical Factors**
  - **Pacing Strategy:** Knowing when to conserve energy and when to exert maximum effort.
  - **Drafting:** Using competitors to shield from wind and conserve energy.
  - **Route Choice:** Making strategic decisions on course navigation to minimize distance or take advantage of terrain.
- **Environmental Factors**
  - **Weather Conditions:** Adapting to changes in weather, such as temperature and wind, which can affect skiing conditions and physical exertion.
  - **Altitude:** The ability to perform at different altitudes, with higher altitudes requiring specific adaptation due to reduced oxygen availability.

6.2 Performance Attribution

As in financial analysis, we can estimate which of these factors is most important in determining the relative rank of these guys.

6.3 Ski Speed Modeling

It may seem simplistic, but it is worth breaking down how much time is spent in biathlon on different parts of the course. The three fundamental components are time skiing the prescribed course, time skiing penalty loops (or time docked for misses in an individual competition), and time spent shooting at the targets.

6.3.1 Differential Performance

Month

Weather Conditions

6.4 Finishing Time Distributions

6.4.1 Minimum Finishing Time Distribution for Men's Sprint Races

6.4.2 Computer Programming

MATLAB Code

```
% Fit GEV distribution to data
params = gevfit(data);

% Parameters of GEV distribution (shape, location, scale)
shape = params(1);
location = params(2);
scale = params(3);

% Generate random samples from GEV distribution
num_samples = 1000;
samples = gevrnd(shape, scale, location, num_samples, 1);

% Plot histogram of generated samples
histogram(samples, 'Normalization', 'pdf');
hold on;

% Plot PDF of fitted GEV distribution
x = linspace(min(samples), max(samples), 100);
pdf_estimate = gevpdf(x, shape, scale, location);
plot(x, pdf_estimate, 'LineWidth', 2);

legend('Generated Samples', 'Fitted GEV PDF');
xlabel('Value');
ylabel('Probability Density');
title('Fitted Generalized Extreme Value Distribution');
```

Python Code

```python
import numpy as np
from scipy.stats import genextreme
import matplotlib.pyplot as plt

# Fit GEV distribution to data
params = genextreme.fit(data)

# Parameters of GEV distribution (shape, location, scale)
shape = params[0]
location = params[1]
scale = params[2]

# Generate random samples from GEV distribution
num_samples = 1000
samples = genextreme.rvs(shape, loc=location, scale=scale, size=num_samples)

# Plot histogram of generated samples
plt.hist(samples, bins=30, density=True, alpha=0.6, color='g')

# Plot PDF of fitted GEV distribution
x = np.linspace(min(samples), max(samples), 100)
pdf_estimate = genextreme.pdf(x, shape, loc=location, scale=scale)
plt.plot(x, pdf_estimate, 'k-', linewidth=2)

plt.xlabel('Value')
plt.ylabel('Probability Density')
plt.title('Fitted Generalized Extreme Value Distribution')
plt.legend(['Fitted GEV PDF', 'Generated Samples'])
plt.show()
```

R Code

6.5 Offseason Training

6.5.1 Cooper Test

The Cooper test, also known as the Cooper 12-minute run test, is a simple running test used to assess aerobic endurance and cardiovascular fitness.[2] It was developed by Kenneth H. Cooper, a physician in the United States Air Force, in the 1960s as a means of measuring the fitness level of military personnel.

1. **Procedure**: In the Cooper test, participants are instructed to run as far as possible within 12 minutes on a flat track or a measured course. The distance covered in meters or laps is recorded at the end of the test.

2. **Scoring**: The participant's performance is evaluated based on the distance covered within the 12-minute time frame. There are typically standard tables or formulas available to convert the distance covered into an estimate of aerobic fitness level, usually expressed as VO_2 max (maximal oxygen consumption), which is a measure of the body's ability to utilize oxygen during exercise.

3. **Interpretation**: The results of the Cooper test can be interpreted based on established norms or standards for different age and gender groups. Higher distances covered within the 12-minute period indicate better aerobic endurance and cardiovascular fitness.

4. **Applications**: The Cooper test is commonly used in physical education programs, sports training, and military fitness assessments to evaluate aerobic fitness and track changes in fitness over time. It provides a simple and practical measure of cardiovascular endurance that requires minimal equipment and can be easily administered in various settings.

5. **Considerations**: While the Cooper test is a useful tool for assessing aerobic fitness, it may not be suitable for individuals with certain medical conditions or physical limitations. It's important for participants to warm up properly before the test and to avoid overexertion during the running period to prevent injury.

6.5.2 Concept2 Machines

Rower and SkiErg

Chapter 7

Summary

> *Data is not information, information is not knowledge, knowledge is not understanding, understanding is not wisdom.*
>
> — Clifford Stoll

At the beginning of this report, I laid out my hope that a systematic, quantitative approach to the analysis of biathlon performance would be helpful to biathletes and coaches. Naturally, I think the general methodology is powerful and can help to distinguish good and bad practices, evaluate progress, and improve the training rigor of biathlon in the United States.

This quote complements the idea of quantifying data and emphasizes the importance of moving beyond raw data to gain true understanding and wisdom from it, echoing the sentiment of Lord Kelvin's quote about the significance of measurement and numerical expression in acquiring knowledge.

The summary should underscore the necessity of moving from mere data collection to deriving actionable insights, reflecting on Clifford Stoll's sentiment that data alone does not equate to wisdom. Highlight the crucial steps beyond data quantification, emphasizing the transformation of data into information, then knowledge, understanding, and ultimately wisdom. This progression underscores the essence of applying quantitative methods in biathlon performance analysis, stressing the importance of interpreting data to make informed decisions, improve strategies, and enhance performance comprehensively.

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