Assignment 8

u - parents of X
Y - children of X
Z - parents of X(not X)
blanket is union(x, y, z)

Written Problem (4 pts)

This problem is designed to give you practice thinking about and working with the chain rule and conditional independence.

In illustrating the Markov blanket of a variable X, the slides and textbook (figure 13.4(b)) use subscripted variables U to represent the parents of X, variables Y to represent the children of X, and variables Z to represent the parents, other than X itself, of the Y variables. The Markov blanket, then, is the union of the U, Y, and Z variables. The Gibbs Sampling slides from Week 7 as well as the textbook (equation 13.10) claim that for any Bayes net,

$P(X_i \mid x_1,..,x_{i-1},x_{i+1},..,x_n)$

$= P(X_i \mid markov\_blanket(X_i))$

$= \alpha\ P(X_i \mid parents(X_i))\ \prod_j P(y_j \mid parents(Y_j))$

where the product is over the variable assignments in Y, that is, over the children of X_i with values assigned to them as specified by the corresponding x_k values in the initial conditional probability. As usual, α is a normalization constant.

Given this, let us use **u** to represent the vector of assignments x_k to the U variables for X_i, that is, to represent $parents(X_i)$; use **y** to represent the assignments to the Y variables; and use **z** to represent the assignments to the Z variables. For each child variable Y_j in Y, $parents(Y_j)$ consists of X_i along with all of the assignments in **z** to other parents of Y_j. Notice that the vectors **u**, **y**, and **z** have no x_k's in common and collectively represent all of variables in the Markov blanket of X_i. In short, we have

$P(X_i \mid markov\_blanket(X_i)) = P(X_i \mid \mathbf{u}, \mathbf{y}, \mathbf{z})$

Also notice that since all of the variables except X_i have fixed values, any probabilities that do not involve X_i, such as $P(\mathbf{u})$ or $P(\mathbf{y} \mid \mathbf{z})$, are constants.

Your task is to show that

$P(X_i \mid \mathbf{u}, \mathbf{y}, \mathbf{z}) = \ldots =$

$= c_1\, P(\boldsymbol{u})\, P(X_i, \boldsymbol{z} \mid \boldsymbol{u})\, P(\mathbf{y} \mid X_i, \boldsymbol{z}, \boldsymbol{u})$

$= \ldots = \alpha\ P(X_i \mid parents(X_i))\ \prod_j P(y_j \mid parents(Y_j))$

where c_1 is a constant. In particular, replace each of the ellipses with one or more expressions so that, in the end, you have a sequence of equalities leading from $P(X_i \mid \mathbf{u}, \mathbf{y}, \mathbf{z})$ to the final expression. For each equality, you should specify what you are using to justify that step: the product rule, the chain rule, or conditional independence (as it applies to variables in Bayes nets). If in a step a constant is needed to achieve equality, change the leading constant variable in the new expression (c_1 might become c_2, for instance). In the end the constant becomes α. Hint: You can work forward and/or backwards from any one expression toward another. It shouldn't take many steps to fill in either ellipsis.

Rules applied to previous step to get current step

Rules

$P(X_i \mid u, z, y)\, P(u, z, y)$ start

PR $P(X_i, u, z, y)$

CR $P(X_i, u, z)\, P(y \mid X_i, z, u)$

PR $c_1 P(u)\, P(X_i, z \mid u)\, P(y \mid X_i, z, u)$ middle

PR $P(X_i, z, u)\, P(y \mid X_i, z, u)$

CR $P(y, X_i, z, u)$

Indep $P(Y)\ P(X_i, z, u)$

CR $P(X_i \mid z, u)\ P(z, u)\ P(Y)$

(z, u) all parents

Rename $P(X_i \mid Parents(X_i))\, P(z, u)\, P(Y)$

PR $\alpha P(X_i \mid Parents(X_i))\, P(y_1, y_2, \ldots y_n)$

CR $\alpha P(X_i \mid Parents(X_i))\, \prod_j P(y_j \mid Parents(Y_j))$ end