Optimal Shuffle Distance

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This problem pertains to a deck of cards, but more abstractly to an ordered list of distinguishable elements and a metric over the space of possible permutations.

Define a deck D of size n to be an ordered list of n cards. Define a shuffle ψ to be any action that takes a deck D to another deck $\hat{D} \equiv \psi(D)$ that contains all of the same elements of D, but not necessarily in the same order. That is:

$$\psi : \mathbb{D}_n \to \mathbb{D}_n \ni c \in D \iff c \in \hat{D} \quad \forall c \in D$$

where $\mathbb{D}_n$ is the space of all possible decks of size n.

First Problem:

Define a metric $< \cdot, \cdot >: \mathbb{D}_n^2 \to \mathbb{R}$ that measures the sum of the absolute distances each element "traveled" in the shuffle. More formally:

$$< D, \hat{D} > \equiv \sum_{k=1}^{n} |I(c_k) - I(\hat{c_k})|$$

where $\hat{c_k}$ is c_k in the shuffled deck, and $I(c_k)$ gives the ordered index of card k.

1. *Given n, find ψ that maximizes $< D, \hat{D} >$.*

Second Problem:

Define a metric $< \cdot, \cdot >: \mathbb{D}_n^2 \to \mathbb{R}$ that measures the sum of the absolute distances each element "traveled" in the shuffle *from its neighbors*. If the original cards were on the outside of the deck, then just measure the distance it traveled from its one adjacent neighbor. More formally:

$$\begin{aligned} < D, \hat{D} > \equiv \\ & |I(c_1) - I(\hat{c_2})| \\ & + \Big(\sum_{k=2}^{n-1} |I(c_k) - I(\widehat{c_{k-1}})| + |I(c_k) - I(\widehat{c_{k+1}})| \Big) \\ & + |I(c_{n-1}) - I(\hat{c_n})| \end{aligned}$$

where $\hat{c_k}$ is c_k in the shuffled deck, and $I(c_k)$ gives the ordered index of card k.

2. *Given n, find ψ that maximizes $< D, \hat{D} >$.*