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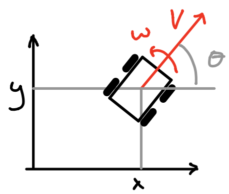
AA 274 - HW 1

Collaborators: Michael Anderson

All work included herein is my own



$$\dot{x}(t) = V\cos(\theta(t)) \qquad \dot{y}(t) = V\sin(\theta(t)) \qquad \dot{\theta}(t) = \omega(t)$$

$$\dot{X}(t) = \begin{bmatrix} V\cos(\theta(t)) \\ V\sin(\theta(t)) \\ \omega(t) \end{bmatrix} = \begin{bmatrix} \cos(\theta(t)) & 0 \\ \sin(\theta(t)) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V \\ \omega \end{bmatrix}$$

$$|V(t)| \leq 0.5 \ m/s \quad \text{and} \quad |\omega(t)| \leq 1.0 \ rad/s \qquad u(t) = \begin{bmatrix} V(t) \\ \omega(t) \end{bmatrix}$$

Problem 1.

Drive from one waypoint to another, minimize time and effort.

i.e. minimize: $J = \int_{t_0}^{t_f} \underbrace{[\lambda + V(t)^2 + \omega(t)^2]}_{g(x,u,t)} dt$

$x(0) = 0, \ y(0) = 0, \ \theta(0) = -\pi/2$

$x(t_f) = 5, \ y(t_f) = 5, \ \theta(t_f) = -\pi/2$

$$X_0 = \begin{bmatrix} 0 \\ 0 \\ -\frac{\pi}{2} \end{bmatrix} \qquad X_{t_f} = \begin{bmatrix} 5 \\ 5 \\ -\frac{\pi}{2} \end{bmatrix}$$

i. Derive the Hamiltonian and conditions of optimality:

Hamiltonian: $H = g(x(t), u(t), t) + p^T(t)[a(x(t), u(t), t)]$

$p(t) = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$ (co-state vector)

$$g(x(t), u(t), t) = [\lambda + V(t)^2 + \omega(t)^2] = \lambda + \begin{bmatrix} V \\ \omega \end{bmatrix}^T \begin{bmatrix} V \\ \omega \end{bmatrix}$$

$$[a(x(t), u(t), t)] = \dot{X}(t) = \begin{bmatrix} V\cos(\theta(t)) \\ V\sin(\theta(t)) \\ \omega(t) \end{bmatrix} = \begin{bmatrix} \cos(\theta(t)) & 0 \\ \sin(\theta(t)) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V \\ \omega \end{bmatrix}$$

Hamiltonian:

$$\boxed{H = g(x(t), u(t), t) + p^T(t)[a(x(t), u(t), t)] = \lambda + \begin{bmatrix} V \\ \omega \end{bmatrix}^T \begin{bmatrix} V \\ \omega \end{bmatrix} + p^T(t) \begin{bmatrix} \cos(\theta(t)) & 0 \\ \sin(\theta(t)) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V \\ \omega \end{bmatrix}}$$

$$H = \lambda + V(t)^2 + \omega(t)^2 + p_1 V\cos(\theta(t)) + p_2 V\sin(\theta(t)) + p_3\omega(t)$$

conditions of optimality:

1. $$\dot{x}^*(t) = \begin{bmatrix} \dot{x}^*(t) \\ \dot{y}^*(t) \\ \dot{\theta}^*(t) \end{bmatrix} = \frac{\partial H}{\partial p}(x^*(t), u^*(t), p^*(t), t) = \begin{bmatrix} V^*\cos(\theta^*(t)) \\ V^*\sin(\theta^*(t)) \\ \omega^*(t) \end{bmatrix}$$

2. $$\dot{p}^*(t) = \begin{bmatrix} \dot{p}_1^* \\ \dot{p}_2^* \\ \dot{p}_3^* \end{bmatrix} = -\frac{\partial H}{\partial x}(x^*, u^*, p^*, t) = -\begin{bmatrix} 0 \\ 0 \\ -p_1(t) V^*\sin(\theta^*) + p_2(t) V^*\cos(\theta^*) \end{bmatrix}$$

3. $$0 = \frac{\partial H}{\partial u}(x^*(t), u^*(t), p^*(t), t) = \begin{bmatrix} 2V^* \\ 2\omega^* \end{bmatrix} + \begin{bmatrix} p_1^*(t)\cos(\theta^*(t)) + p_2^*(t)\sin(\theta^*(t)) \\ p_3^*(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2V^* + p_1^*(t)\cos(\theta^*(t)) + p_2^*(t)\sin(\theta^*(t)) = 0 \Rightarrow V^*(t) = \frac{1}{2}\left(-p_1^*(t)\cos(\theta^*(t)) - p_2^*(t)\sin(\theta^*(t))\right)$$

$$2\omega^* + p_3^*(t) = 0 \Rightarrow \omega^* = -\frac{1}{2} p_3^*(t)$$

$$\boxed{\begin{aligned}
\dot{x}^*(t) &= V^*\cos(\theta^*(t)) \\
\dot{y}^*(t) &= V^*\sin(\theta^*(t)) \\
\dot{\theta}^*(t) &= \omega^*(t) \\
\dot{p}_1^* &= 0 \\
\dot{p}_2^* &= 0 \\
\dot{p}_3^* &= p_1(t) V^*\sin(\theta^*) - p_2(t) V^*\cos(\theta^*) \\
V^*(t) &= -\frac{1}{2}\left(p_1^*(t)\cos(\theta^*(t)) + p_2^*(t)\sin(\theta^*(t))\right) \\
\omega^* &= -\frac{1}{2} p_3^*(t) \\
r' &= 0
\end{aligned}}$$

Formulate problem as a 2P-BVP:

Boundary conditions:

1. $$\boxed{x^*(t_0) = \begin{bmatrix} x^*(t_0) \\ y^*(t_0) \\ \theta^*(t_0) \end{bmatrix} = X_0 = \begin{bmatrix} 0 \\ 0 \\ -\frac{\pi}{2} \end{bmatrix}}$$

2. $$\boxed{x^*(t_f) = \begin{bmatrix} x^*(t_f) \\ y^*(t_f) \\ \theta^*(t_f) \end{bmatrix} = X_{t_f} = \begin{bmatrix} 5 \\ 5 \\ -\frac{\pi}{2} \end{bmatrix}}$$

3. $$\left[\frac{\partial h}{\partial x}(x^*(t_f), t_f) - p^*(t_f)\right]^T \underbrace{\delta x_f}_{=0} + \underbrace{\left[H(x^*(t_f), u^*(t_f), p^*(t_f), t_f) + \frac{\partial h}{\partial t}(x^*(t_f), t_f)\right]}_{=0} \delta t_f = 0$$

$$H(x^*(t_f), u^*(t_f), p^*(t_f), t_f) + \cancel{\frac{\partial h}{\partial t}(x^*(t_f), t_f)} = 0$$

$$H(x^*(t_f), u^*(t_f), p^*(t_f), t_f) = 0 \qquad \lambda + \begin{bmatrix} V^* \\ \omega^* \end{bmatrix}^T \begin{bmatrix} V^* \\ \omega^* \end{bmatrix} + p^{*T}(t) \begin{bmatrix} \cos(\theta^*(t)) & 0 \\ \sin(\theta^*(t)) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V^* \\ \omega^* \end{bmatrix} = 0$$

$$\boxed{\lambda + V^*(t_f)^2 + \omega^*(t_f)^2 + p_1^*(t_f) V^* \cos(\theta^*(t_f)) + p_2^*(t_f) V^* \sin(\theta^*(t_f)) + p_3^*(t_f)\omega^*(t_f) = 0}$$

Dummy variable

$t_f = r, \quad r' = 0$

unknowns

$x, y, \theta, p_1, p_2, p_3, r, \lambda$

$\tau = t/t_f \qquad \frac{d}{d\tau}(\cdot) = t_f \frac{d}{dt}(\cdot) = r\frac{d}{dt}(\cdot)$

$$V^*(\tau) = -\frac{1}{2}\left(p_1^*(\tau)\cos(\theta^*(\tau)) + p_2^*(\tau)\sin(\theta^*(\tau))\right)$$

$$\omega^*(\tau) = -\frac{1}{2} p_3^*(\tau)$$

$$\text{ODEs}\begin{cases} \dot{x}^*(\tau) = r V^* \cos(\theta^*(\tau)) \\ \dot{y}^*(\tau) = r V^* \sin(\theta^*(\tau)) \\ \dot{\theta}^*(\tau) = r\omega^*(\tau) \\ \dot{p}_1^*(\tau) = 0 \\ \dot{p}_2^*(\tau) = 0 \\ \dot{p}_3^*(\tau) = r\, p_1(\tau) V^*(\tau)\sin(\theta^*(\tau)) - r\, p_2(\tau) V^*(\tau)\cos(\theta^*(\tau)) \\ \dot{r}(\tau) = 0 \end{cases}$$

$$\text{BCs}\begin{cases} x^*(0) = 0 \\ y^*(0) = 0 \\ \theta^*(0) + \frac{\pi}{2} = 0 \\ x^*(1) - 5 = 0 \\ y^*(1) - 5 = 0 \\ \theta^*(1) + \frac{\pi}{2} = 0 \\ \lambda + V^*(t_f)^2 + \omega^*(t_f)^2 + p_1^*(t_f) V^* \cos(\theta^*(t_f)) + p_2^*(t_f) V^* \sin(\theta^*(t_f)) + p_3^*(t_f)\omega^*(t_f) = 0 \end{cases}$$

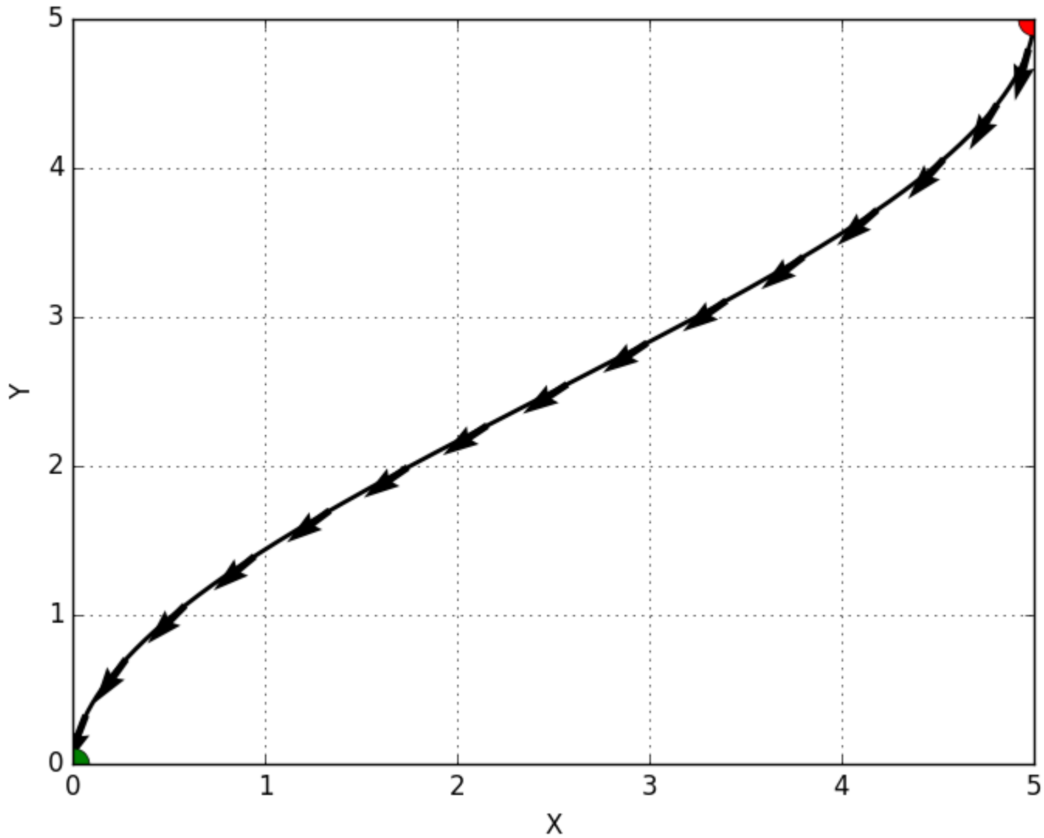
ii. Edit traj_opt.py to solve 2P-BVP with largest possible λ (larger $\lambda \Rightarrow$ greater penalty on time $\Rightarrow$ faster performance) ☑

iii. Extract trajectories ☑

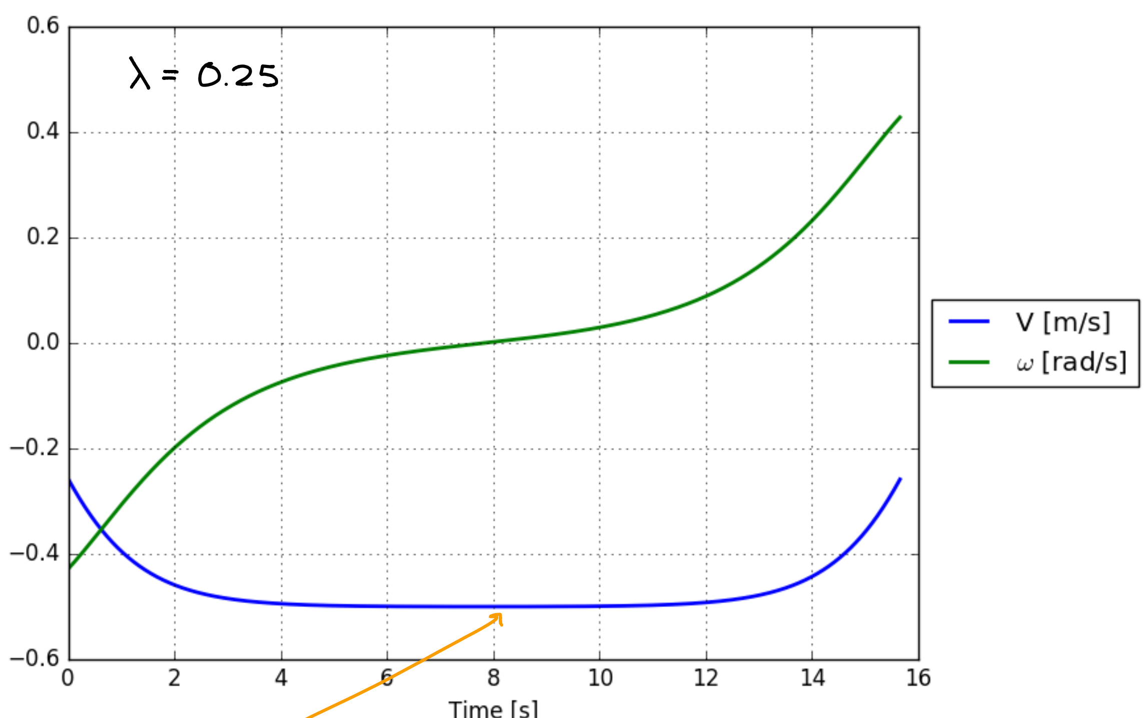
iv. plot (x, y) and $(V(t), \omega(t))$ ☑

v. save the data ☑

Trajectory



Control Histories



Right on the line!

Problem 2 – Dynamically extended robot model:

$\dot{x}(t) = V\cos(\theta(t))$
$\dot{y}(t) = V\sin(\theta(t))$
$\dot{V}(t) = a(t)$
$\dot{\theta}(t) = \omega(t)$

$$\begin{bmatrix} \ddot{x}(t) \\ \ddot{y}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(\theta) & -V\sin(\theta) \\ \sin(\theta) & V\cos(\theta) \end{bmatrix}}_{J} \begin{bmatrix} a \\ \omega \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$\det(J) = V$, for $V > 0$, invertible

polynomial basis expansion:

$$x(t) = \sum_{i=1}^{n} x_i \psi_i(t), \quad y(t) = \sum_{i=1}^{n} y_i \psi_i(t),$$

ψ_i, $i = 1, \dots, n$ are basis functions

x_i, y_i are coefficients to be designed

(i) Take the basis functions $\psi_1(t) = 1$, $\psi(t) = t$, $\psi(t) = t^2$, $\psi(t) = t^3$

write a set of linear expressions in x_i, y_i, $i = 1, \dots, n$ to express the following initial conditions:

$x(0) = 0$
$y(0) = 0$
$V(0) = 0.5$
$\theta(0) = -\frac{\pi}{2}$

$x(t_f) = 0$
$y(t_f) = 0$
$V(t_f) = 0.5$
$\theta(t_f) = -\frac{\pi}{2}$

$t_f = 15$

$V(0) = 0.5 \Rightarrow \dot{x} = V^0 \cos(\theta^0)$

$x(0)$: $\quad x_1 + x_2 t + x_3 t^2 + x_4 t^3 = 0$
$x(t_f)$: $\quad x_1 + x_2 t_f + x_3 t_f^2 + x_4 t_f^3 = 5$
$\dot{x}(0)$: $\quad 0 + x_2 + 2x_3 t_0 + 3x_4 t_0^2 = V^0 \cos(\theta^0)$
$\dot{x}(t_f)$: $\quad 0 + x_2 + 2x_3 t_0 + 3x_4 t_0^2 = V^f \cos(\theta^f)$
$y(0)$: $\quad y_1 + y_2 t + y_3 t^2 + y_4 t^3 = 0$
$y(t_f)$: $\quad y_1 + y_2 t_f + y_3 t_f^2 + y_4 t_f^3 = 5$
$\dot{y}(0)$: $\quad 0 + y_2 + 2y_3 t_0 + 3y_4 t_0^2 = V^0 \sin(\theta^0)$
$\dot{y}(t_f)$: $\quad 0 + y_2 + 2y_3 t_0 + 3y_4 t_0^2 = V^f \sin(\theta^f)$

$$\boxed{\begin{bmatrix} \psi_1^0 & \psi_2^0 & \psi_3^0 & \psi_4^0 & 0 & 0 & 0 & 0 \\ \psi_1^{t_f} & \psi_2^{t_f} & \psi_3^{t_f} & \psi_4^{t_f} & 0 & 0 & 0 & 0 \\ 0 & \psi_1^0 & 2\psi_2^0 & 3\psi_3^0 & 0 & 0 & 0 & 0 \\ 0 & \psi_1^{t_f} & 2\psi_2^{t_f} & 3\psi_3^{t_f} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \psi_1^0 & \psi_2^0 & \psi_3^0 & \psi_4^0 \\ 0 & 0 & 0 & 0 & \psi_1^{t_f} & \psi_2^{t_f} & \psi_3^{t_f} & \psi_4^{t_f} \\ 0 & 0 & 0 & 0 & 0 & \psi_1^0 & 2\psi_2^0 & 3\psi_3^0 \\ 0 & 0 & 0 & 0 & 0 & \psi_1^{t_f} & 2\psi_2^{t_f} & 3\psi_3^{t_f} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ V^0\cos(\theta^0) \\ V^f\cos(\theta^f) \\ 0 \\ 5 \\ V^0\sin(\theta^0) \\ V^f\sin(\theta^f) \end{bmatrix}}$$

ii. Edit traj_df.py to solve the system of equations ☑

iii. rescale the velocity trajectory while keeping the geometric aspects of the trajectory the same

path parameter: s (arc length) $\longrightarrow$ $V(t) = \underbrace{V(s(t))}_{\text{pseudo-velocities w.r.t } s}\underbrace{\dot{s}(t)}_{\text{timing law}}$, $\omega(t) = \underbrace{\omega(s(t))}_{\text{pseudo-velocities w.r.t } s}\underbrace{\dot{s}(t)}_{\text{timing law}}$

$V(s) = \dfrac{V(t)}{\dot{s}(t)}$ $\qquad$ $\omega(s) = \dfrac{\omega(t)}{\dot{s}(t)}$

To scale V and ω, increase Δt where necessary:

For each step i in time history:
$\quad dt_i = 0.005$
$\quad$ if $V_i >= V_{max}$:
$\qquad dt_i = dt_i \times \dfrac{V_i}{V_{max}}$
$\quad$ if $\omega_i >= \omega_{max}$:
$\qquad dt_i = dt_i \times \dfrac{\omega_i}{\omega_{max}}$
$\quad \tilde{\dot{s}}(t) = \dfrac{ds_i}{dt_i}$

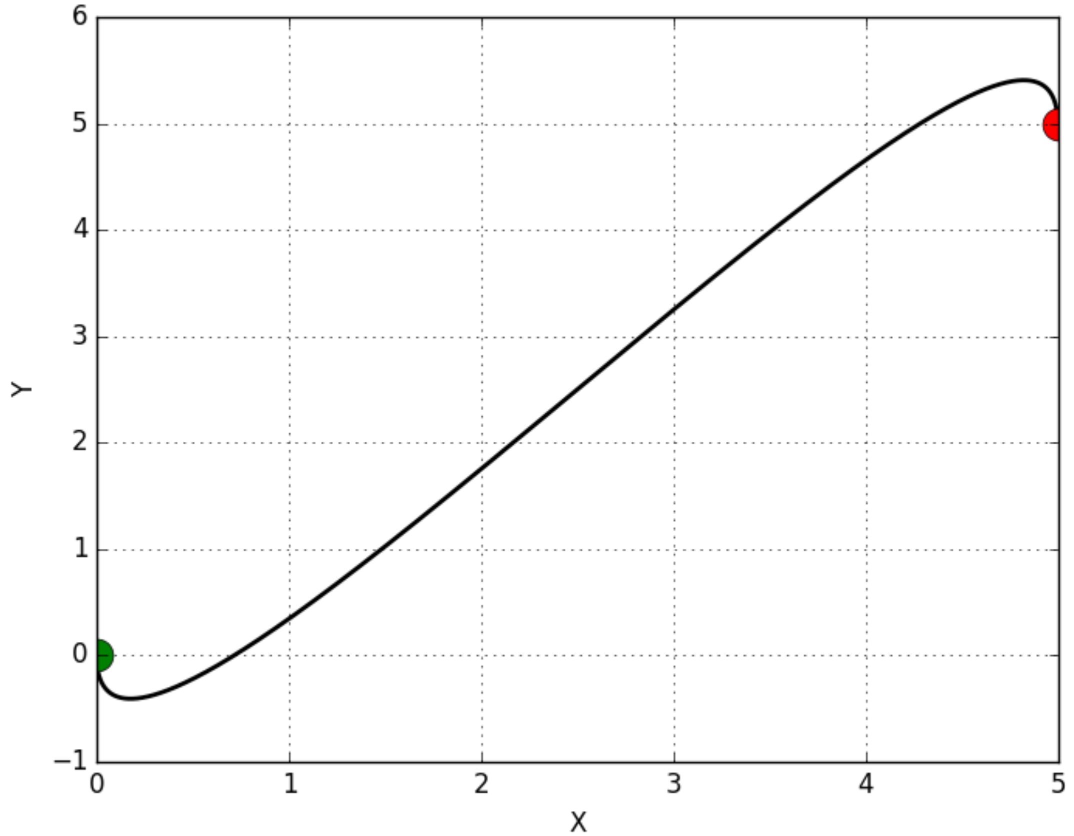
$\tilde{V}(t) = \tilde{\dot{s}}(t) V(s) = \tilde{\dot{s}}(t)$
$\tilde{\omega}(t) = \tilde{\dot{s}}(t)\omega(s)$
$\tilde{\theta}(t) = \int_0^{t_f} \tilde{\omega}(t)\,dt + \theta(0)$
$\tilde{\dot{x}}(t) = \tilde{V}(t)\cos(\tilde{\theta}(t))$
$\tilde{\dot{y}}(t) = \tilde{V}(t)\sin(\tilde{\theta}(t))$
$\tilde{x}(t) = \int \tilde{\dot{x}}(t)\,dt$
$\tilde{y}(t) = \int \tilde{\dot{y}}(t)\,dt$

$\tilde{\ddot{x}}(t) = \tilde{\dot{V}}\cos\theta - \tilde{\omega}V\sin\theta$
$\tilde{\ddot{y}}(t) = \tilde{\dot{V}}\sin\theta + \tilde{\omega}V\cos\theta$

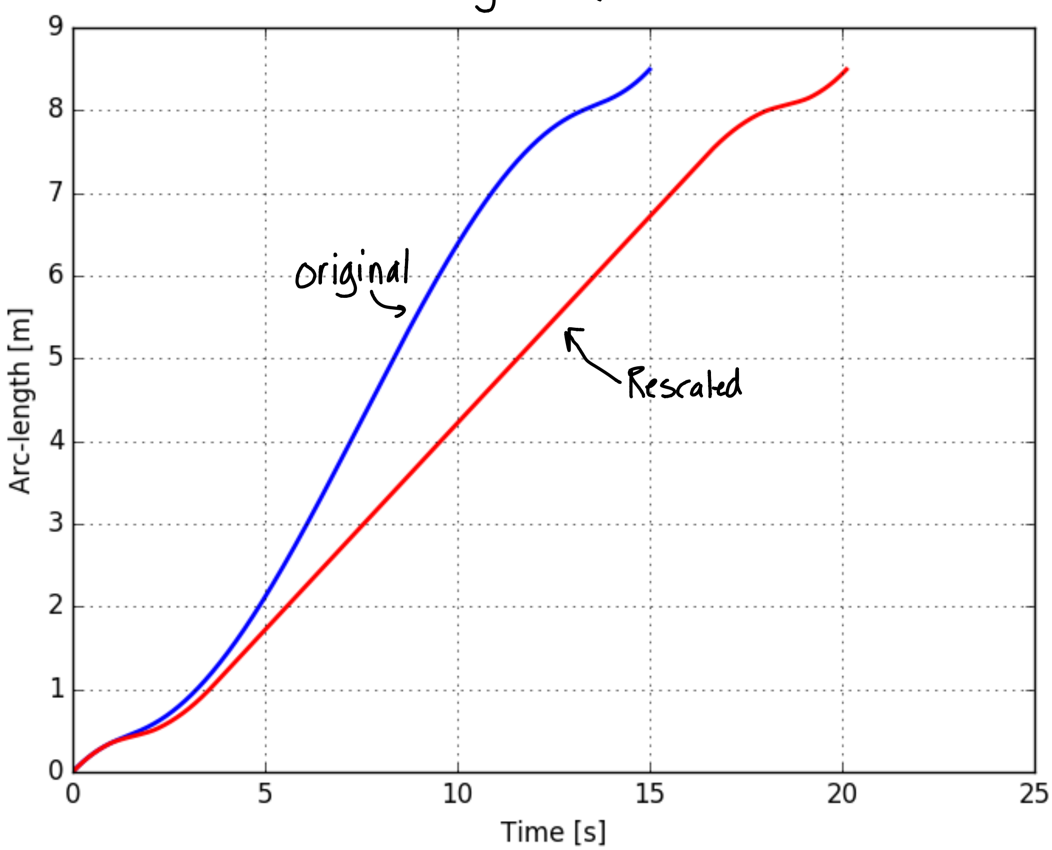
iv. Compute and plot the scaled trajectory ☑

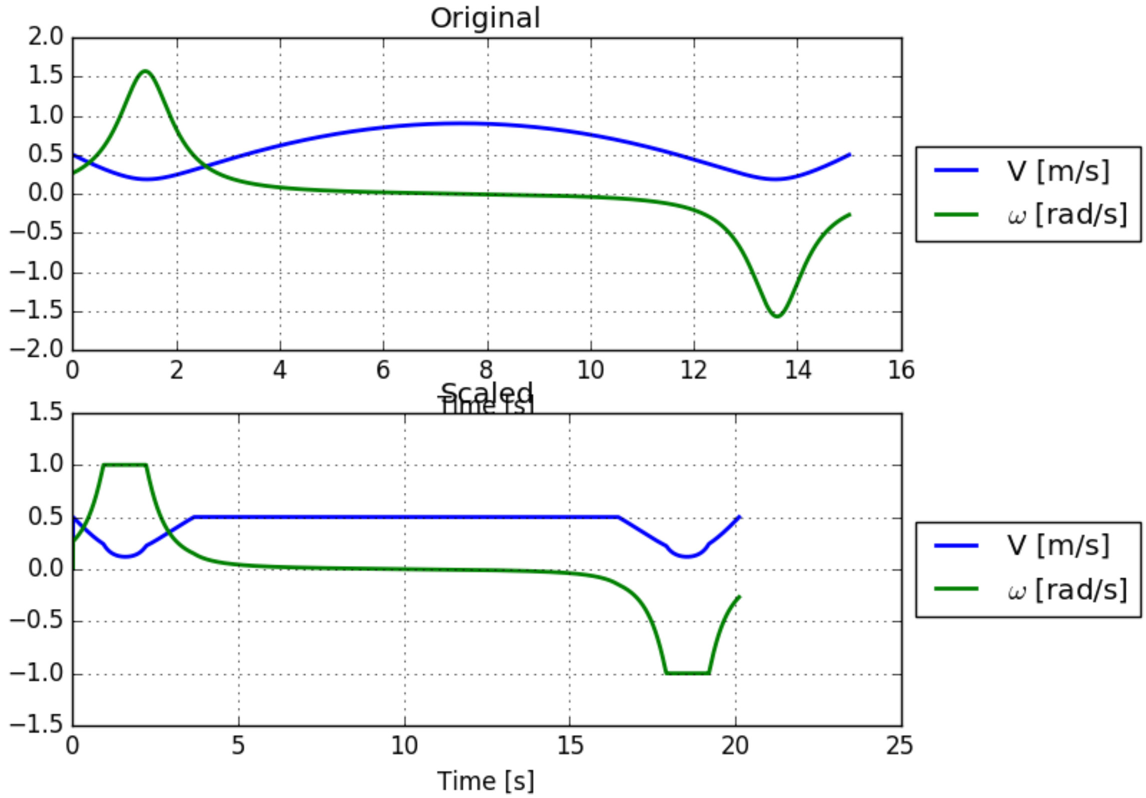
<u>Validate</u>: confirm that the disturbance-free trajectory arrives while the perturbed trajectory drifts

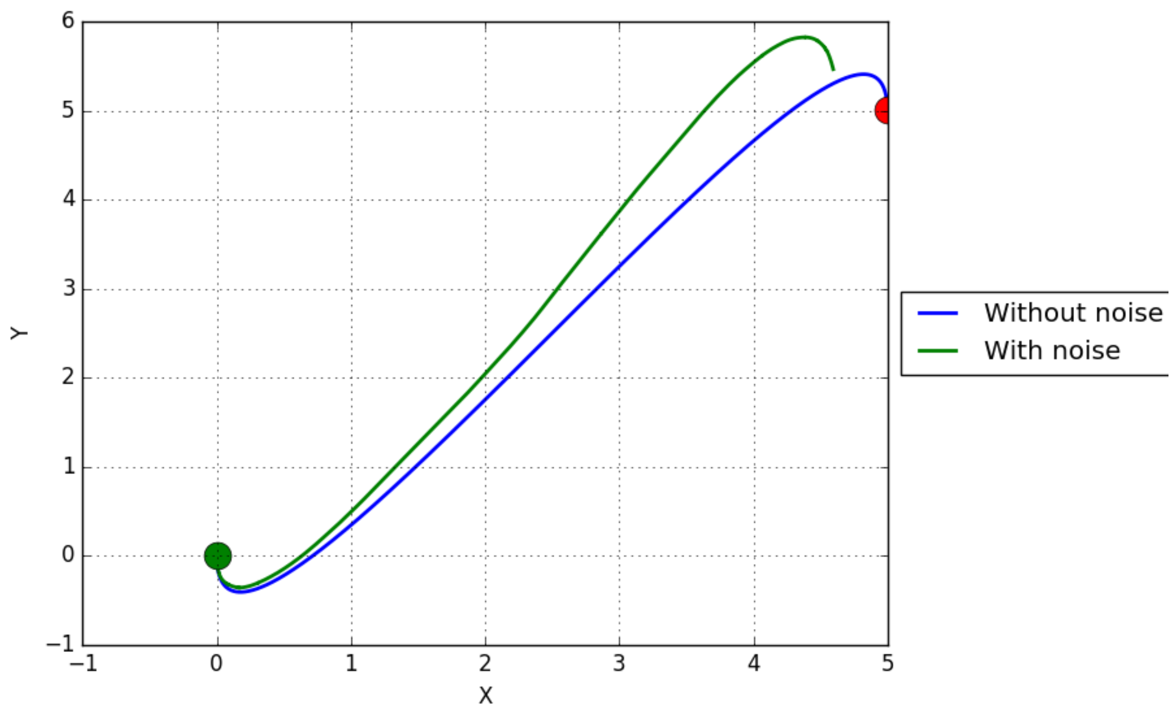
Rescaled trajectory (identical to original)

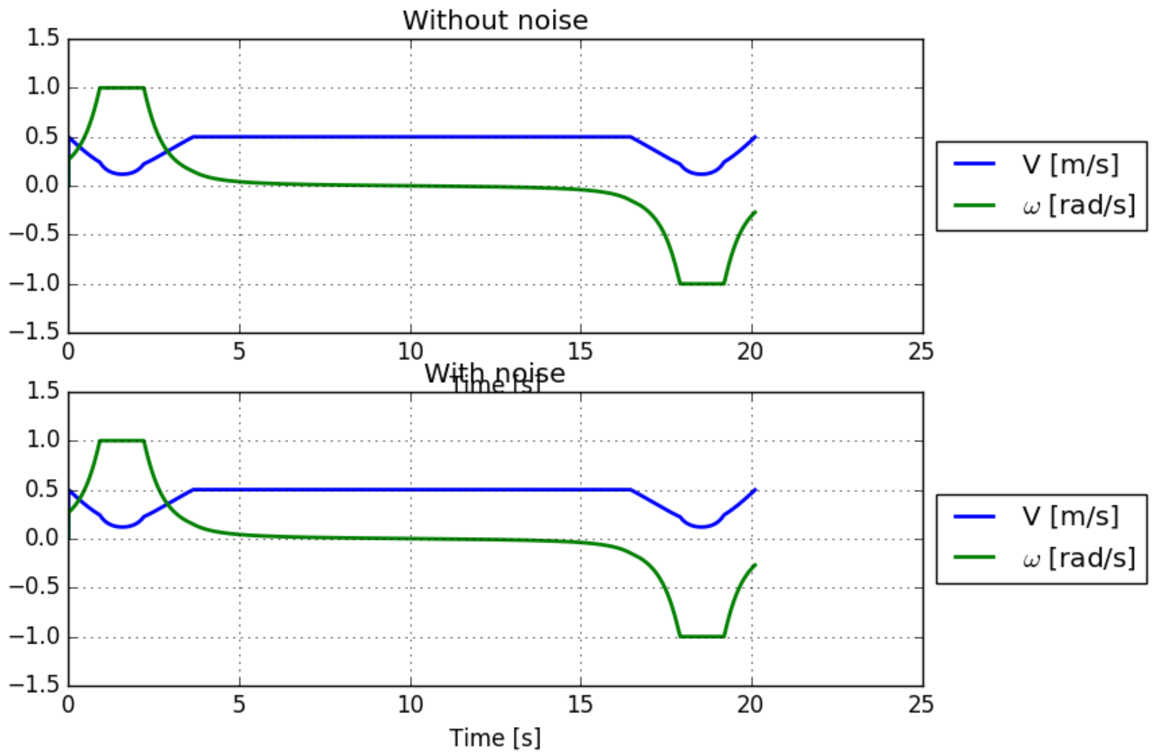


arc lengths s(t)




Original
V [m/s]
ω [rad/s]
Time [s]
Scaled
V [m/s]
ω [rad/s]
Time [s]

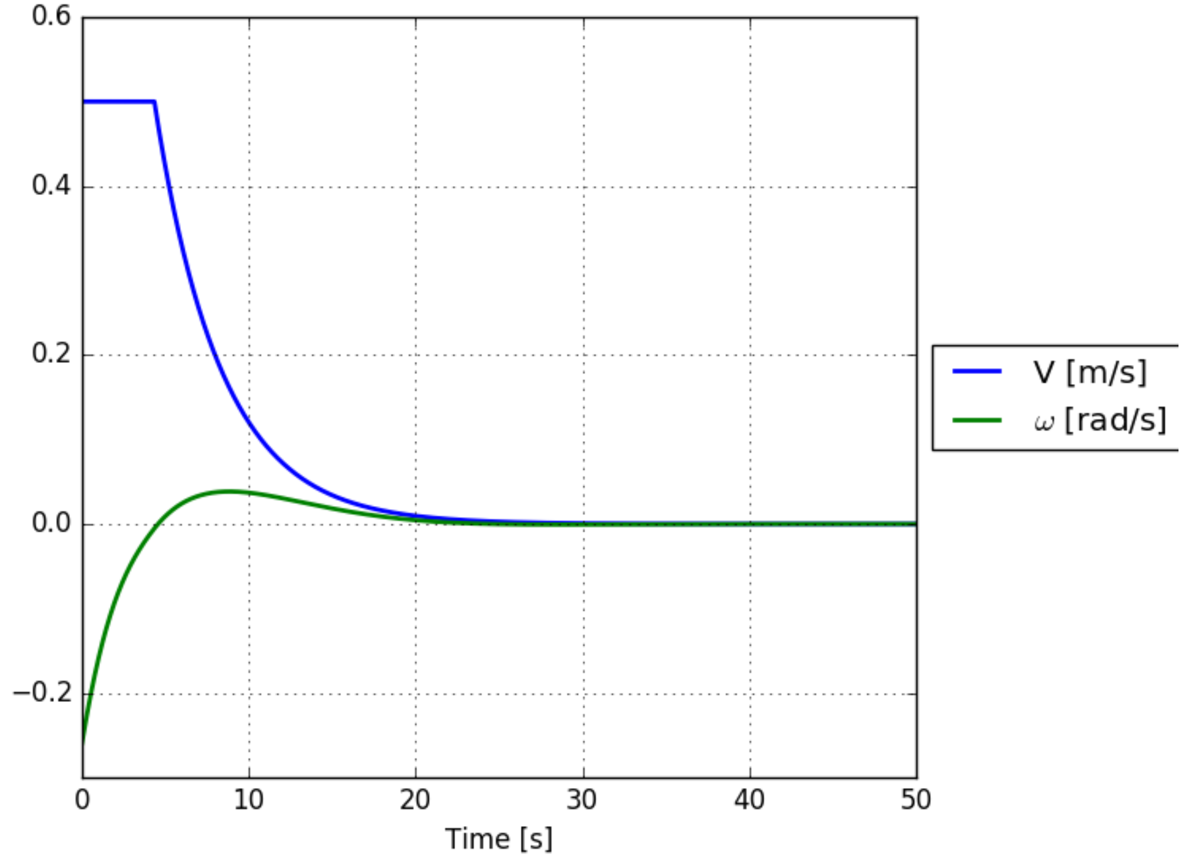

Without noise
With noise
X
Y

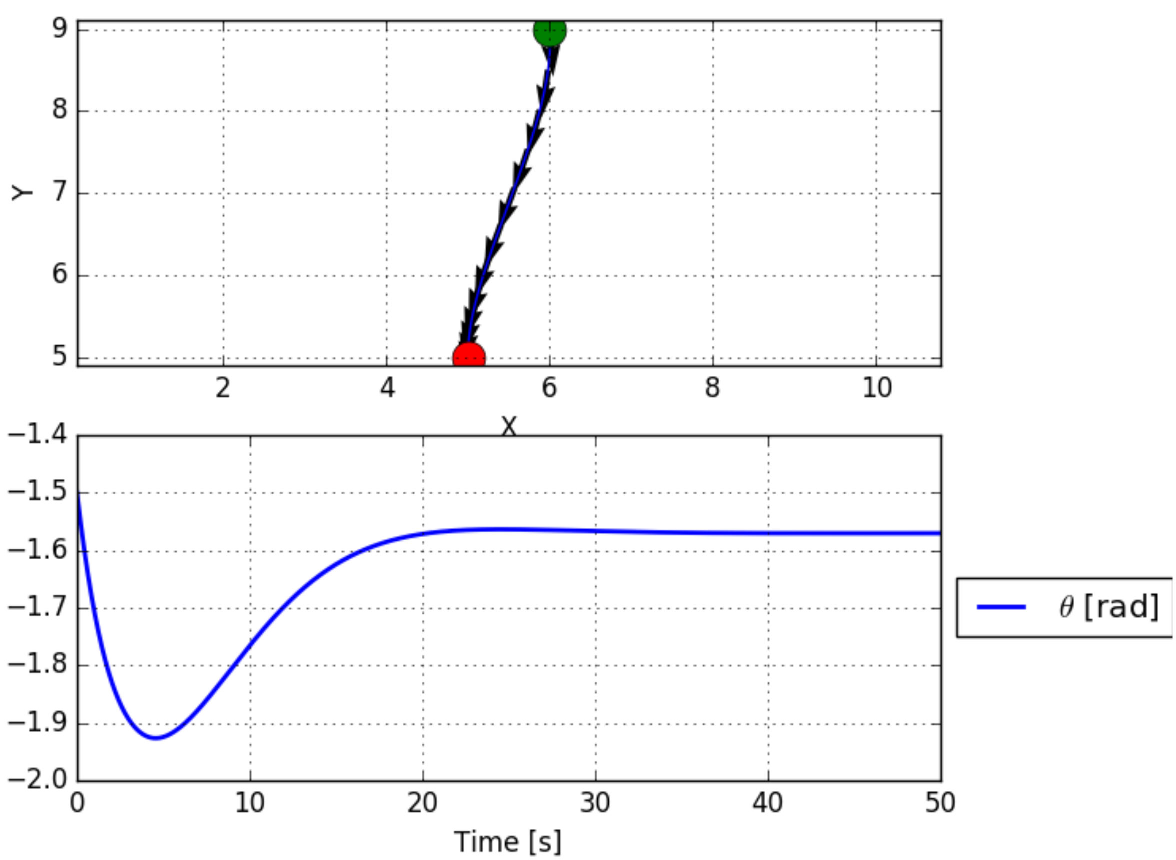


Problem 3 - closed loop control

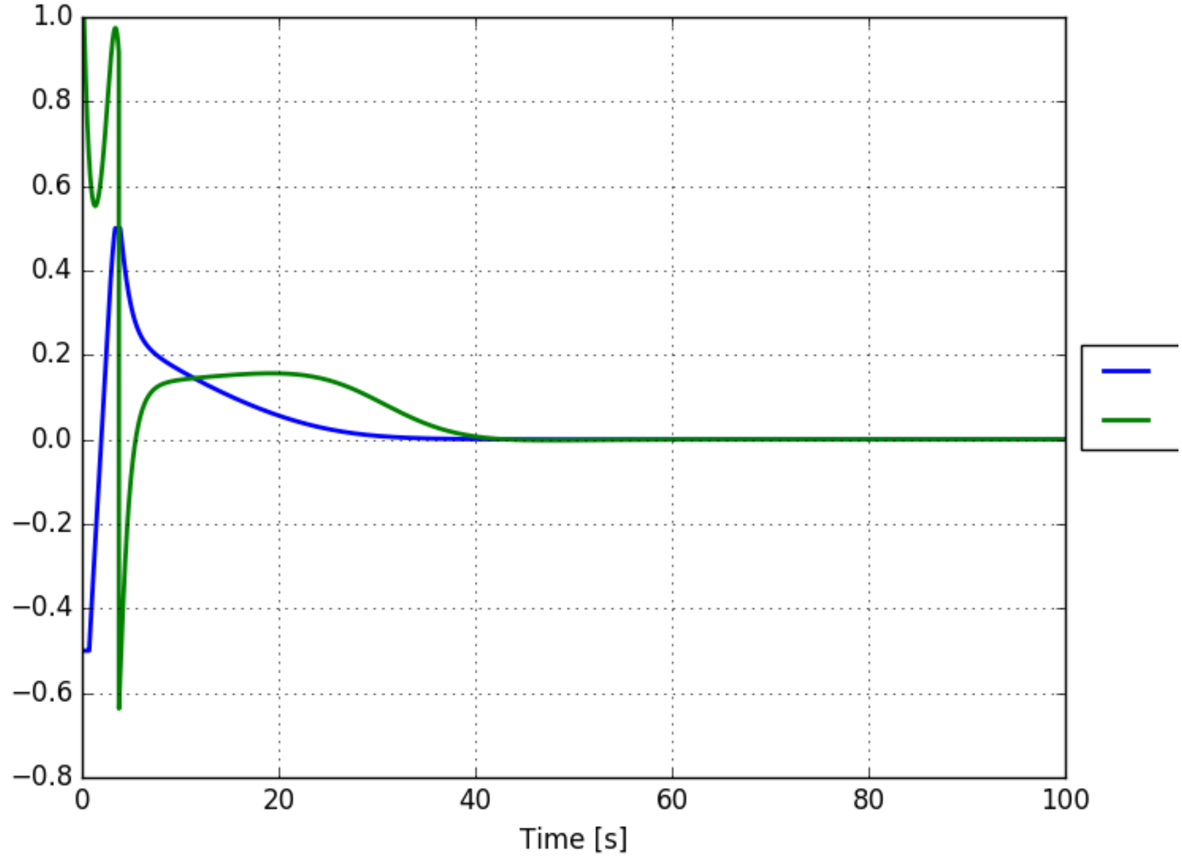
Task: edit control function ctrl_pose and program the given control law ☑

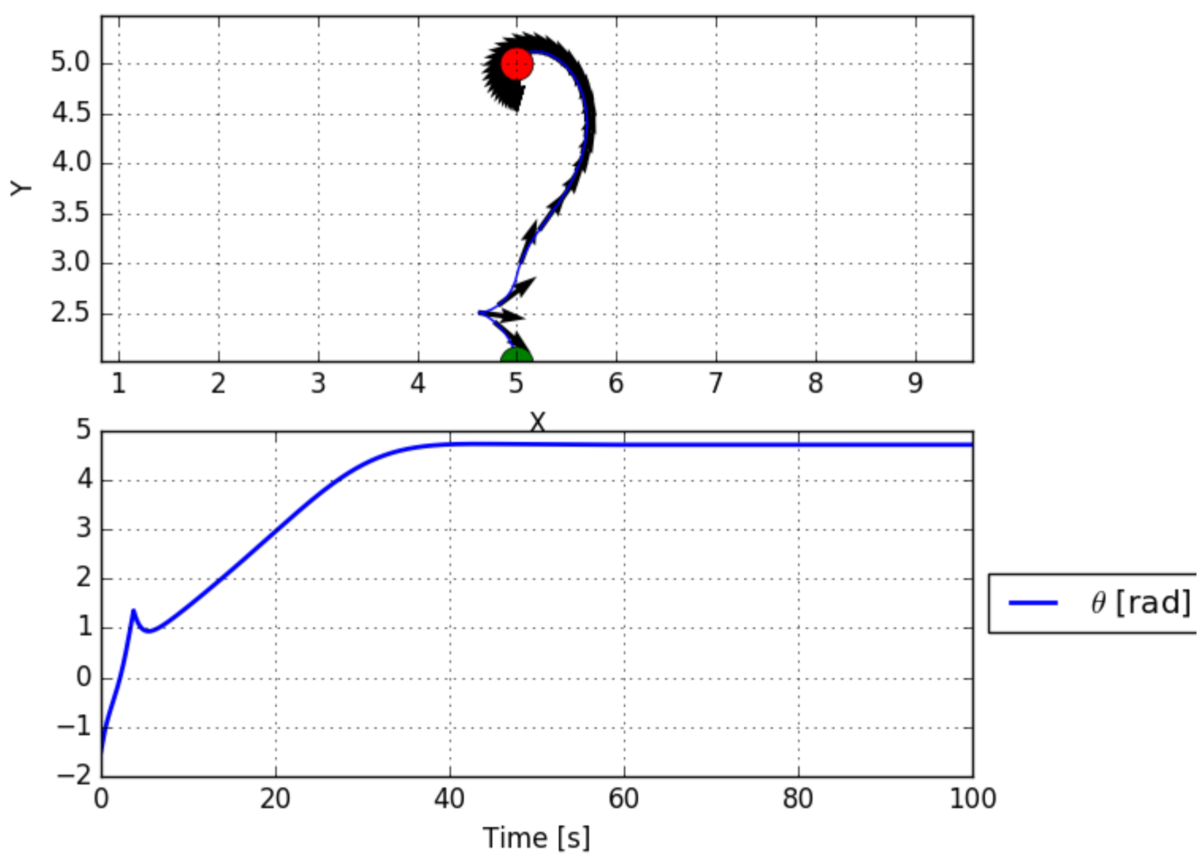
Problem 3 - forward parking



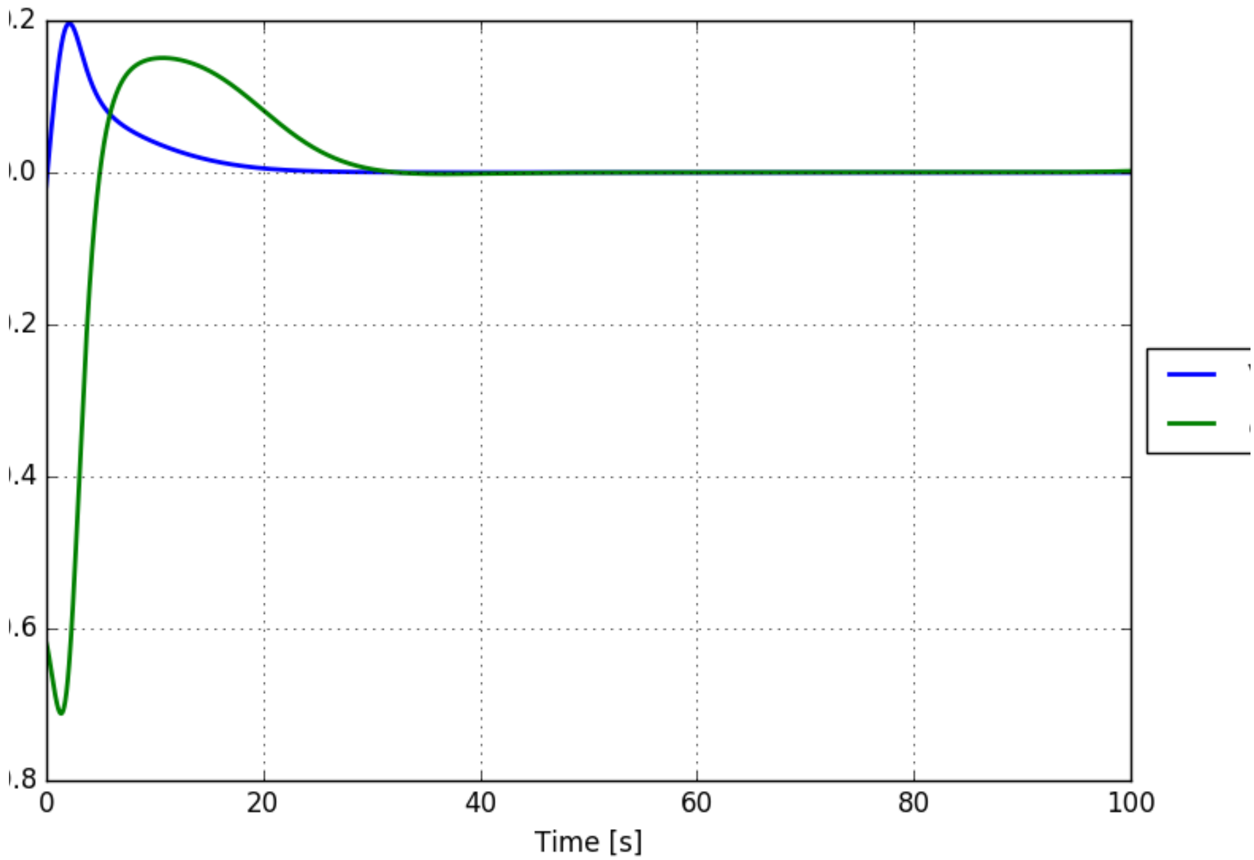


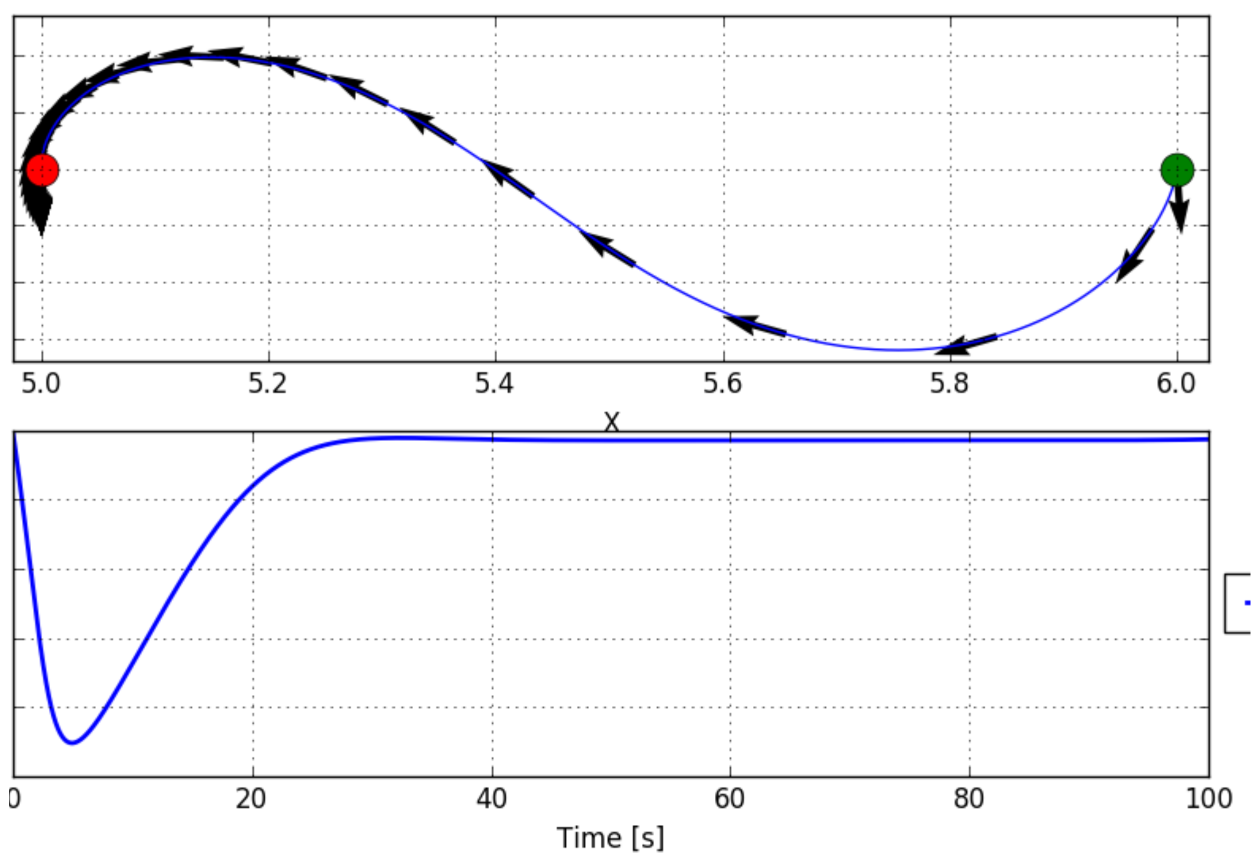
Problem 3 - backward parking





Problem 3 - parallel parking





Problem 4:

i. Write down the form of the dynamic compensator as a set of differential algebraic equations with input (u_1, u_2), output (V, ω). Clearly define any internal states you might need.

$$u_1 = \ddot{x}_d + \underline{k_{px}}(x_d - x) + \underline{k_{dx}}(\dot{x}_d - \dot{x})$$

$$u_2 = \ddot{y}_d + \underline{k_{py}}(y_d - y) + \underline{k_{dy}}(\dot{y}_d - \dot{y})$$

control gains

input: η

$$\eta = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \dot{\eta} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \cos(\theta) & 0 \\ \sin(\theta) & 0 \end{bmatrix} \begin{bmatrix} V \\ \omega \end{bmatrix}$$

$$V = \xi, \quad \dot{\xi} = a \Rightarrow \dot{\eta} = \xi \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

new input: $\ddot{\eta}$

$$\ddot{\eta} = \dot{\xi}\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + \xi\dot{\theta}\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\xi\sin\theta \\ \sin\theta & \xi\cos\theta \end{bmatrix}\begin{bmatrix} a \\ \omega \end{bmatrix}$$

$$\begin{bmatrix} a \\ \omega \end{bmatrix} = \begin{bmatrix} \cos\theta & -\xi\sin\theta \\ \sin\theta & \xi\cos\theta \end{bmatrix}^{-1} \underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}_{\ddot{\eta} = \begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = u}$$

$$\begin{bmatrix} \cos\theta & -\xi\sin\theta \\ \sin\theta & \xi\cos\theta \end{bmatrix}^{-1} = \begin{bmatrix} \xi\cos\theta & \xi\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\frac{1}{\xi}$$

$$\begin{bmatrix} a \\ \omega \end{bmatrix} = \begin{bmatrix} \xi\cos\theta & \xi\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\frac{1}{\xi} \qquad a = \dot{\xi} = (\xi u_1\cos\theta + \xi u_2\sin\theta)\frac{1}{\xi}$$

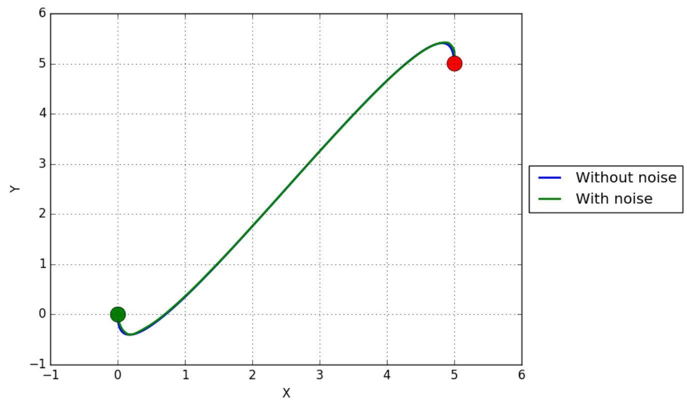
$$\boxed{\begin{aligned} \dot{\xi} &= u_1\cos\theta + u_2\sin\theta \\ V &= \xi \\ \omega &= \frac{u_1\cos\theta - u_2\sin\theta}{\xi} \end{aligned}}$$

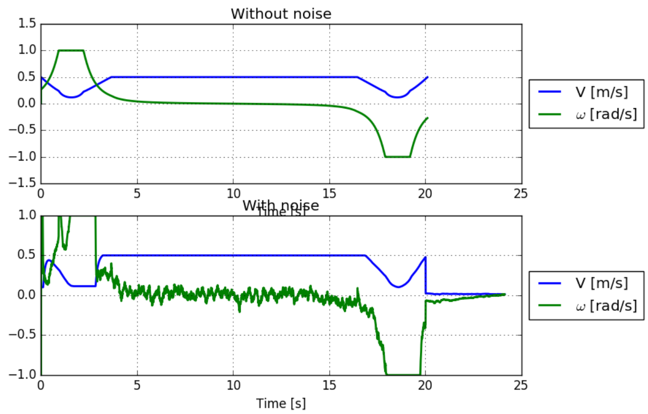
ii. Edit ctrl_traj ☑

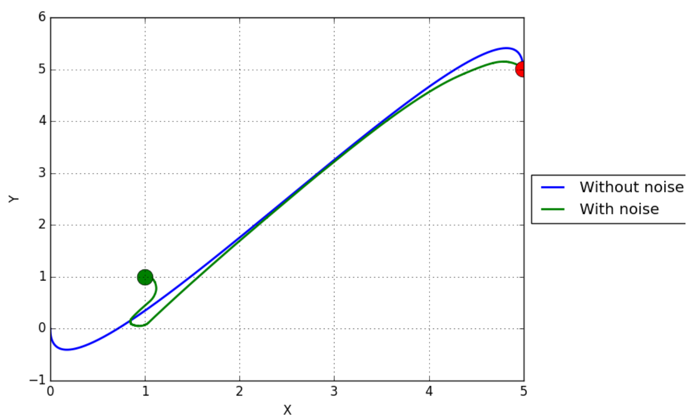
iii. modify ctrl_traj to switch to ctrl_pose when ρ is small enough ☑

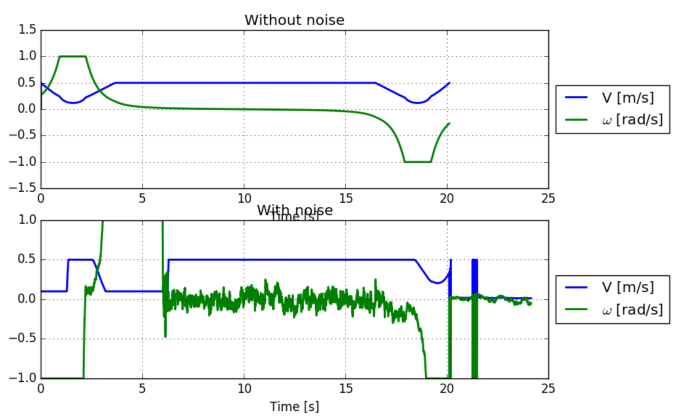
Validate: confirm that the disturbance-free trajectory and the perturbed trajectory both arrive ☑

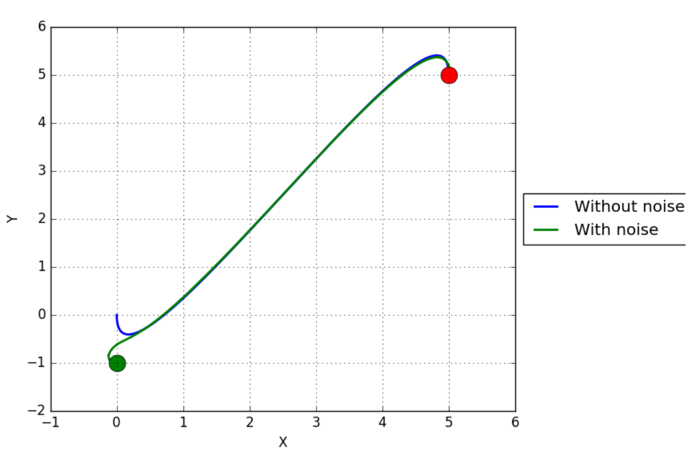
Various initial conditions for Problem 4

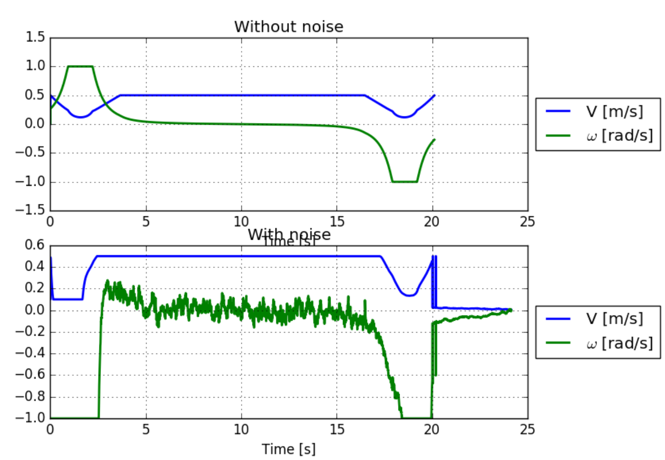












Problem 5 - ROS stuff

☑