Eco 673 Lecture 6, 2017-01-19

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Rounding Error

$p \to$ # digits

$\beta \to$ base (2 in computer language)

$\varepsilon \to \beta^{-p+1}/2$

In R:

$p \to 53$ $\varepsilon = 2^{-53}/2.$

$\beta \to 2$

ex. $p=3$, $\beta=10$

$\varepsilon = 10^{-2}/2 = 0.05$ (machine epsilon for this computer).

Now

$$\begin{array}{r} 1.00 \times 10^0 \\ +\ 0.005 \times 10^0 \\ \hline 1.00 \end{array}$$

dropped.

3 digits

floating point.

for this computer, $\pi = 3.14$.

Error: 0.0016

Error: 0.0016

Ulps (Units of the last place). $= \beta^{-p+1} = 10^{-2} = 0.01 = 1$ Ulps

$0.0016 = 0.16$ Ulps.

absolute error.

Next, $2\pi = 6.28$ (6.2832) True.

error: $0.0032 = 0.32$ Ulps.

Relative Error: $\frac{\text{Error}}{\text{True}}$

for π, $\frac{0.0016}{3.1416} = 0.0005$

2π, $\frac{0.0032}{6.2832} = 0.0005$.

We want to report relative error in term of ε.

$\varepsilon = 0.005$, relative error $= 0.1\,\varepsilon$.

ex.2

$$\begin{array}{r} 2.15 \times 10^{12} \\ +\ 1.25 \times 10^{-5} \\ \hline \end{array} \longrightarrow \begin{array}{r} 2.15 \times 10^{12} \\ +\ 0.00\cdots \times 10^{12} \\ \hline 2.15 \times 10^{12} \end{array}$$

relative error $= \frac{1.25 \times 10^{-5}}{2.15 \times 10^{12} + 1.25 \times 10^{-5}}$

$=$ very small.

not contaiminate.

Now, $10.1 - 9.93$

$$\begin{array}{r} 1.01 \times 10^1 \\ - \; 0.99 \times 10^1 \\ \hline 0.02 \times 10^1 = 0.2 \end{array}$$

true value $= 0.17$, error $= 0.2 - 0.17 = 0.03$

relative error $= \frac{0.03}{0.17} = 0.1764706/\varepsilon = 35.2\varepsilon$.

catastraphic cancellation

when $x-y$ and $x \approx y$.

It get worse, $f(x)-f(y)$.

operation increasing rounding error.

$\sum_{i=1}^{\text{large}} (x_i - y_i) \Rightarrow$ error accumulate thorough operations.

ex. $x^2 - y^2 = (x+y)(x-y)$

square operation increase error.

more precise.

$\rightarrow$ badly condition (ill condition).

Conditioning:

$x \to$ given.

$\hat{x} \to$ observed

saved by computer.

$f(x)$, $f(\hat{x})$

$$\text{Condition} = \frac{|f(\hat{x}) - f(x)| / |f(x)|}{|\hat{x} - x| / x}$$

elasticity of error.

Use Taylor:

$$f(x) = f(\hat{x}) + f'(x)(x - \hat{x})$$

$$\text{Cond} = \frac{|f'(x)(x-\hat{x})| / f(x)}{|x - \hat{x}| / |x|} = \left| \frac{x f'(x)}{f(x)} \right|$$

suppose $f(x) = x - y$

$$\text{Cond} = \left| \frac{x \cdot 1}{x - y} \right| = \frac{|x|}{|x - y|} \to \text{very high if } x \approx y$$

Bad Condition

$\Rightarrow$ High Rounding Error.

Consider this one,

$$\left. \begin{aligned} x + \alpha y &= 1 \\ \alpha y + y &= 0 \end{aligned} \right\} \Rightarrow x = \frac{1}{1 - \alpha^2}$$

α could effect x if not properly saved.

$$f(x) = \frac{1}{1 - x^2}$$

$$f'(x) = \frac{2x}{(1-x^2)^2}$$

$$\text{Cond} = \left| \frac{x \frac{2x}{(1-x^2)}}{\frac{1}{1-x^2}} \right| = \left| \frac{2x^2}{1-x^2} \right|$$

very bad if $x \approx 1$, or $x \approx -1$.

will make $x + y = 1$

$x + y = 0$

matrix no longer invertable.

If $|A| \approx 0 \Rightarrow$ Bad Condition.

small change $\Rightarrow$ big error.

In econometrics, this is multicolinearity.

Solutions: $\Rightarrow$ Selecting the right method.

machine epsilon $= 10^{-16}$.

ex. $\begin{pmatrix} \varepsilon & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$x_1 = \frac{2-3}{\quad} \approx 1$

$$x_1 = \frac{2-3}{2\varepsilon-1} \approx 1$$

$$x_2 = \frac{3\varepsilon-1}{2\varepsilon-1} = 1$$

$x_1 = x_2 = 1.$

Now, lets solve,

1) x_1 first

$$L1 = -\tfrac{1}{2}L2 + L1$$

$$-\tfrac{1}{2}+\varepsilon. \quad \begin{pmatrix} -\frac{1}{2} & 0 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -0.5 \\ 3 \end{pmatrix} \longrightarrow x_1 = 1,\ x_2 = 1.$$

2) x_2 first

$$L_2 = L_2 - \tfrac{1}{\varepsilon}L.$$

$$\begin{pmatrix} \varepsilon & 1 \\ 0 & 2-\frac{1}{\varepsilon} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3-\frac{1}{\varepsilon} \end{pmatrix}$$

$$x_2 = \frac{3-\frac{1}{\varepsilon}}{2-\frac{1}{\varepsilon}} = 1$$

both number really big.

$$\varepsilon x_1 + x_2 = 1$$

$\Sigma x_1 = 0.$

gives $\underline{x_1 = 0}$! completed wrong.

(b/c we tried to eliminate ε, which is close to 0).

$$\frac{\sum_{i=1}^{n} x_i}{\text{sum}(x)}$$

loop: $S \leftarrow 0$

for (i in 1:(n))

$S \leftarrow S + x[i]$.

Algorithm 2.1.

$C \leftarrow 0$

$S \leftarrow x[1]$

for (i in 2:n)

$y \leftarrow x[i] - C$

$T \leftarrow S + y$

$C \leftarrow (T - S) + y$

$S \leftarrow T$

};

x <- rep(.1, 10000)

sum(x) - 1000

use 2.1, you can reduce error to zero.

Difficult: $\sum_{i=1}^{n} a_i x^i$

$n \circledast,\ n \oplus,\ n$ ⓧ

Solution: rewrite in $S = a_0 + x(a_1 + x(a_2 + x a_4)$.

$S = a_n$

$i = (n-1):1$

$S \to A[i] + S * X$

Logical Issue.

if (x == value)

floating point.

use all.equal((.1+.1+.1), .3, tol=1e-16) to replace (==)

But, not always TRUE.

... FALSE.

return "string" when TRUE.

use <u>isTRUE</u> to solve this!

<u>Linear Algebra</u>

Solve $Ax=b$

$$\Rightarrow x = \underset{\sim}{A^{-1}} b$$

$\quad \hookrightarrow \frac{1}{\underset{\sim}{|A|}}\,\text{cofactor}(A).$

$\quad\quad \hookrightarrow a_{11}|A_{-11}|(-1)^{1+1} + a_{21}|A_{-21}|(-1)^{1+2} + \cdots$

<u>Order:</u> $|0| = O(n^2)$, $A_{n\times n} \to O(n^2)$.

order $O(n!)$

Instead, we <u>solving</u>

$Ax=b$ by substitution.

$$\downarrow$$

$$\tilde{A}x = \tilde{b}$$

$$\downarrow$$

Forward solve. $\downarrow$ $\begin{pmatrix} \circ & & 0 \\ \circ\ \circ & \ddots & \\ \circ\ \circ\ \circ & & \end{pmatrix} x = \tilde{b}$ ← <u>stable</u>

or

Backward solve $\uparrow$ $\begin{pmatrix} & \ddots & \\ & 0 & \end{pmatrix}$ ← <u>not stable</u>

Back-ward Solve | $\begin{pmatrix} 0 & \diagdown \end{pmatrix}$

in R:

$x \leftarrow$ solve(A, b) $\Rightarrow$ vector

or $x \leftarrow$ solve(A) %*% b $\Rightarrow$ matrix.

Instead, $AA^{-1} = I$.

Solving N system of equations.

$$\begin{cases} Ax = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = A^{-1}_{\cdot 1} \\ Ax = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} = A^{-1}_{\cdot 2} \end{cases}$$

ex.

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} \\ \tilde{a}_{21} & \tilde{a}_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A \begin{pmatrix} \tilde{a}_{11} \\ \tilde{a}_{12} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad A \begin{pmatrix} \tilde{a}_{12} \\ \tilde{a}_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

1) LU Decomposition

LU Decomposition

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 1 & 1 & 2 \end{pmatrix}$$

$$\underbrace{A = \underset{\text{lower tri}}{L}\,\underset{\text{upper tri.}}{U}}_{O(n^3/3)}$$

$L\underbrace{Ux}_{z} = b$

↓

$Lz = b \rightarrow z$ by forward solve

forwardsolve(A,b)

↓

$Ux = z \rightarrow x$ by backward solve.

backsolve(A, b)

$O(n^2)$

ex. fixed A ⇒ 1000's of bs

⇒ L, U once.

⇒ 1000's back to forward solve.

Package(Matrix).