1 Overview

I view myself as an applied analyst. My research follows certain philosophy of applied mathematics: formulation, solution, interpretation, with special emphasis on the qualitative analysis of partial differential equations (PDEs). The PDE models that I have been working on are motivated by various phenomena arising in biosciences, fluid dynamics, and cross-disciplines, which are rich and active areas for mathematical investigations. After formulating the appropriate (initial and/or boundary value) problems, I utilize analytical approaches to study the qualitative properties of the models, such as existence, uniqueness, regularity and large-time behavior of general and special solutions, stability of steady states, singular limits, controlled dynamics, finite-time singularities *et al*, then apply the theoretical results to interpret the phenomena described by the models. Most of my current research projects are extendable to incorporate more physically relevant problems, whose resolutions will rely on adaptation and renovation of classic approaches in analysis and likely engage young talents.

2 Chemotaxis with Logarithmic Sensitivity

The first area that I have been researching is mathematical biology, especially chemotaxis. Chemotaxis is a fundamental process in modern cell biology, biochemistry, and clinical pathology, through which the movement of an organism or entity is collectively directed by random diffusion and the concentration gradient of a chemical stimulus in the local environment. It is an indispensable mechanism for interpreting various biological processes, such as bacterial foraging, immune response, embryonic development, tissue homeostasis, blood vessel formation, fish pigmentation patterning, tumor angiogenesis, primitive streak formation, slime mould formation, wound healing, just to name a few. Mathematical description of chemotaxis using continuum PDEs was initiated in the 1950s by Patlak and regained popularity in the 1970s through the pioneering work of Keller and Segel. The Patlak-Keller-Segel model is a system of reaction-diffusion-advection equations, which can be formulated in the general form as

$$\begin{aligned} &\partial_t u = D\Delta u - \chi\nabla\cdot[u\nabla\Phi(c)] + f(u),\\ &\partial_t c = \varepsilon\Delta c + g(u,c), \end{aligned} \tag{2.1}$$

where the function $\Phi(c)$, called chemotactic sensitivity, is the primary feature of the model, whose spatial gradient depicts the mechanistic feature of chemotactic movement.

My research has been focusing on one of the Patlak-Keller-Segel type models, which incorporates the logarithmic sensitivity, logistic growth, and polynomial

reaction. In such a functional setting, system (2.1) reads as

$$\begin{aligned} \partial_t u &= D\Delta u - \chi\nabla\cdot[u\nabla(\ln c)] + ru(K-u), \\ \partial_t c &= \varepsilon\Delta c + \mu u^\gamma c - \sigma c, \end{aligned} \tag{2.2}$$

which appeared in the late 1990s as part of the effort to understand the chemotactic movement of biological species that deposit non-diffusive ($\varepsilon = 0$) or slowly-moving ($\varepsilon \ll 1$) chemical signals for succeeding passages in the local environment, such as myxobacteria. Later, the model found applications in cancer biology, especially the onset of tumor angiogenesis.

Model (2.2) has been proven to be capable to describe some of the fundamental phenomena in chemotaxis. This is supported by the explicit and numerical solutions constructed in [8], which mimicked the finite-time aggregation and large-time homogenization of cellular density in different chemotactic processes. Such evidences, together with the biological background and synergy of random diffusion, chemotactic movement and nonlinear reaction, presents an intriguing problem for rigorous mathematical investigation.

My contribution in this area is the rigorous mathematical description of the qualitative properties of solutions to model (2.2). In a series of recent joint works [4, 5, 6, 10, 11, 12, 17, 23, 27, 30, 31, 32, 33, 34, 36], the following topics were studied for the one-dimensional model subject to various initial and/or boundary conditions:

- global asymptotic stability of constant steady states,
- explicit decaying rates of large-amplitude perturbations,
- zero chemical diffusivity limit, i.e., $\varepsilon \to 0$,
- control of large-amplitude solutions by dynamic boundary data.

The first two topics are concerned with the global well-posedness and large-time behavior of large-data classical solutions to the model. Our research demonstrated that large-time homogenization of the cellular density (modeled by $u(x,t)$) is a generic phenomenon in two distinct chemotactic processes: chemoattraction with consumption, and chemorepulsion with production of the chemical stimulus. This is the first time that such a phenomenon is demonstrated in a rigorous mathematical sense. In particular, we fully generalized the preliminary results of [8], regarding the large-time homogenization of the explicit and numerical solutions constructed therein.

The third and fourth topics deal with the question of how boundary conditions affect the qualitative behavior of the solutions. Our investigation of the zero chemical diffusivity limit was initiated in [27] and [11], where the consistency between the chemically diffusive and non-diffusive models, and the formation of boundary layer solutions were discovered, respectively. This is the first

time that the discussion of "vanishing viscosity limit" was brought into the investigation of continuum PDE models of chemotaxis. In [4, 6, 17, 32, 33, **?**, 36], we examined various types of biologically relevant boundary conditions, under which the regular or singular nature of the limit was thoroughly studied. On the other hand, we introduced the boundary control problem in [23], where the first generation of growth/decaying conditions on the dynamic boundary data was identified, under which large-data classical solutions to the model stabilize in the long run. The result was later upgraded in [4] to fit into more physically/biologically relevant situations.

In multi-dimensional spaces, a direct scaling argument shows that the global existence of large-data classical solutions to the model is energy critical in 2D and super critical in 3D. This brings a significant challenge to mathematical analysis, due to the "compressible" nature of the underlying system of equations after applying the Cole-Hopf transformation. Up to date, the problem with general initial data without any smallness restriction is still widely open, whose ultimate resolution may have an impact on the development of analytic techniques in applied PDEs. In the recent joint works [15, 24, 26], we provided definite answers to the question under certain smallness restrictions on the initial data, which are considered "minimal", in the sense that they are less stringent than those in previously reported results.

3 Geophysical Fluid Dynamics

The second area in which I have made contributions is mathematical fluid dynamics. The 2D incompressible Navier-Stokes-Boussinesq equations:

$$\begin{aligned} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P &= \nu \Delta \mathbf{u} + \theta \mathbf{e}_2, \quad \mathbf{x} \in \mathbb{R}^2, \quad t > 0, \\ \partial_t \theta + \mathbf{u} \cdot \nabla \theta &= \kappa \Delta \theta, \\ \nabla \cdot \mathbf{u} &= 0, \end{aligned} \tag{3.1}$$

appear frequently in the modeling of astrophysical, oceanic and atmospheric turbulence, and geophysical fluid flows, where rotation and stratification play a dominant role. It also plays an important role in the study of Rayleigh-Bernard convection. The model can be derived from the Navier-Stokes equations for heat-conducting fluid flows by applying the Boussinesq approximation, whose essence is to ignore the influence of density variations except where they appear in terms multiplied by the gravitational constant. Mathematically, this two-dimensional model shares striking similarities with classic three-dimensional models in mathematical fluid dynamics, such as the Navier-Stokes or Euler equations. One such analogy is the vortex stretching effect, which has been recognized as the major obstruction for solving the Millennium problem concerning the global existence and uniqueness of the 3D Navier-Stokes equations.

In real world applications, flows often move in bounded physical domains with constraints from boundaries, which makes the study of nonlinear PDE models on bounded domains challenging. My first contribution is the establishment of the global existence and uniqueness of classical solutions to (3.1) with either $\kappa = 0$ [7] or $\nu = 0$ [35] in bounded domains with physical boundaries. These partially dissipative versions of the model arise naturally in the study of geophysical turbulence. The results appear to be the first regarding the model in physical bounded domains. A novel spatial-temporal energy method was developed during the course of investigation, which can be adapted to a family of nonlinear PDE models.

My second contribution is the rigorous mathematical description of the global stability of the hydrostatic equilibrium associated with (3.1) when thermal diffusion is insignificant (i.e., $\kappa \approx 0$). The problem was proposed by the Scottish applied mathematician Keith Moffat as his Third 21st Century Problem [19]. Due to the lack of thermal diffusion, the stability analysis is a significant mathematical challenge. In a recent joint work [2], by inventing novel analytic techniques, we proved that the velocity field of the semi-dissipative system is globally asymptotically stable, and the flow described by the model stratifies in the vertical direction when time getting large. Our result particularly demonstrated the effectiveness of the semi-dissipative system as a model for stratification-dominant situations in geophysics. More recently, in a joint work [25], by employing delicate spectral analysis, we identified the explicit decaying rates of the solution of the linearized system around the hydrostatic equilibrium on spatially periodic domains. Our results provided so far the most complete description about the large-time dynamics of the semi-dissipative model. The analytic techniques developed in our studies are expected to be adaptive to other PDE models with partial dissipative structure.

4 Porous Medium Flow

The third research area that I have been actively exploring is hyperbolic balance laws. As an important class of nonlinear PDEs, hyperbolic balance laws arise in various branches of applied sciences, describing the motion of fluids, gas and waves. One of the important models in this area is the compressible Euler equations with frictional damping:

$$\begin{aligned} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) &= 0, \\ \partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla P &= -\alpha \rho \mathbf{u} + \rho \mathbf{f}, \\ \partial_t (\rho E) + \nabla \cdot [\mathbf{u} (\rho E + P)] &= -\alpha \rho |\mathbf{u}|^2 + \rho \mathbf{f} \cdot \mathbf{u}. \end{aligned} \tag{4.1}$$

This model is closely connected with the porous medium equation or generalized Smoluchowski equation. There are a variety of applications of the model in

engineering and natural sciences, such as petroleum engineering, underground filtration, supernova explosion, solar climate modeling and weather forecasting. Because of its physical background and mathematical challenges (strong nonlinearities, weak dissipation *et al*), rigorous study of the model has been one of the focal points in applied analysis for decades.

My first contribution in this area is the rigorous mathematical description of the qualitative properties of the isentropic version of (4.1) with $\mathbf{f} = \mathbf{0}$ in physical domains. In the joint works [20, 21], we studied the global dynamics of one-dimensional entropy weak solutions and multi-dimensional classical solutions to initial-boundary value problems of the model subject to various initial and boundary conditions. In particular, we demonstrated that the model is asymptotically equivalent to the porous medium equation, as time goes to infinity, in the respective solution regimes. The analytic techniques developed in these works have generated positive results in the qualitative analysis of continuum PDE models arising in other scientific areas, such as physiology and material science. For example, in the joint works [13, 14], we established similar results as those in [20, 21], concerning a quasi-linear hyperbolic PDE system modeling human cardiovascular flow. Our analytical results provided useful information for the understanding of core issues in physiology associated with ischemia. Recently, in a joint work [1], we proved the exponential stability of small-data classical solutions to the free boundary problem of a nonlinear hyperbolic PDE model for swelling gels. In particular, we gave an accurate quantification of the explicit decaying rates of the solutions towards equilibrium states, which is expected to help with engineering calculations and design of swelling gels.

Recently, in a joint work [29], we further studied the isentropic version of (4.1) driven by a potential force, i.e., $\mathbf{f} = \nabla\Phi$. By assuming the pressure follows the γ-law and Φ converges to certain non-constant steady state, we showed that under the no-normal-flow boundary condition, the velocity field will vanish and the density converges to a non-constant profile, determined by its initial datum and the potential steady state, in the regime of small-amplitude classical solutions. As a special case when $\mathbf{f} = g\mathbf{e}_n$, i.e., the gravitational force, our result indicates that the flow modeled by the equations will slow down and stratify in the vertical direction as time getting large, and the stratified profile of the density function can be calculated explicitly. In the same work, we studied the full system (4.1) on bounded domains subject to the no-normal-flow boundary condition. By combining energy methods and a Lagrangian approach, we successfully identified the final profiles of the entropy and density functions, which was not recognized before, and proved their asymptotic stability. Our result gives so far the most complete description of the large-time behavior of the non-isentropic system on physical domains in the regime of small-amplitude classical solutions.

5 Binary Fluid Flow

Binary (two-phase) fluid flows occur naturally in areas such as large-scale power systems, pump cavitation, climate systems, and groundwater flow. The Cahn-Hilliard equation is a classic continuum PDE model in mathematical physics, which describes the process of phase separation. It has been utilized in miscellaneous situations where it is coupled with hydrodynamics equations to describe complex phenomena associated with phase separation in binary fluid flows.

My first contribution in this area is the rigorous mathematical analysis of the Cahn-Hilliard-Hele-Shaw system, which consists of the coupling of the Cahn-Hilliard equation with an incompressible flow through Darcy's law:

$$\begin{aligned} \partial_t \phi + \mathbf{u} \cdot \nabla \phi &= M \Delta \mu, \\ \mu &= -\varepsilon \Delta \phi + \varepsilon^{-1}(\phi^3 - \phi), \\ \eta \mathbf{u} &= -\nabla P + \gamma \mu \nabla \phi, \\ \nabla \cdot \mathbf{u} &= 0. \end{aligned} \tag{5.1}$$

This model appeared in various contexts involving spinodal decomposition, such as Hele-Shaw flow and growth of solid tumor. In a joint work [16], we proved various well-posedness results for the model (5.1) on spatially periodic domains in two- and three-dimensional space, which corroborated previous numerical simulations. Moreover, we studied the large-time behavior of the solutions and showed that 1) constant steady states are metastable when they are outside the chemical spinodal, and 2) making the thickness of diffusive interface large leads to constant phase states. Our theoretical results showed consistency with numerical and experimental observations. They are among the first generation of rigorous mathematical results for the model since its initiation.

My second contribution in this area is related to the incompressible Cahn-Hilliard-Brinkman system:

$$\begin{aligned} \partial_t \phi + \mathbf{u} \cdot \nabla \phi &= M \Delta \mu, \\ \mu &= -\varepsilon \Delta \phi + \varepsilon^{-1}(\phi^3 - \phi), \\ \eta \mathbf{u} &= -\nabla P + \nu \Delta \mathbf{u} + \gamma \mu \nabla \phi, \\ \nabla \cdot \mathbf{u} &= 0, \end{aligned} \tag{5.2}$$

which describes the process of phase separation of incompressible binary fluid flows in porous media. A direct scaling argument shows that the global well-posedness of large-data classical solutions to (5.2) in $\mathbb{R}^n$ is energy sub-critical when $n \leq 3$, critical when $n = 4$, and super-critical when $n \geq 5$. In a recent joint work [9], we solved the energy critical problem in $\mathbb{R}^4$, and hence unified the previous results concerning the global well-posedness of the model in lower dimensional spaces. In particular, the global well-posedness of the Cahn-Hilliard

equation in $\mathbb{R}^4$ follows as a special case of our result. This is the first time that such a result is established for the Cahn-Hilliard equation since its initiation in the 1950s. The energy method devised in our study can be adapted to a family of PDE models describing phase separation of incompressible binary fluid flows when convection is insignificant.

6 Biological Fluid Flow

Apart from the fundamental research disciplines, I have also actively engaged in the qualitative analysis of continuum PDE models arising in cross-disciplinary areas. One such area that I have been delving into is biological fluid flows. Recent developments in science and engineering stimulated the rapid growth of cross-disciplinary research and created tremendous opportunities for conversations among seemingly unrelated fields. One such example is the mingling of mathematical biology and fluid dynamics. Numerous continuum PDE models have been invented to understand the complex structures of biological fluid flows, and rigorous mathematical analysis of the modeling equations has been actively carried out in the past two decades, with the frequency of publications continuing to increase. Among the large family of bio-fluid models, I have been focusing on two specific ones, namely, the Navier-Stokes-Keller-Segel system and a mechanochemical model of vasculogenesis.

6.1 Interplay Between Chemotaxis and Hydrodynamics

The coupled Navier-Stokes-Keller-Segel system:

$$\begin{aligned} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla\pi &= \eta\Delta\mathbf{u} - n\nabla\phi, \\ \nabla\cdot\mathbf{u} &= 0, \\ \partial_t n + \mathbf{u}\cdot\nabla n - D_1\Delta n &= -\nabla\cdot(n\, r(p)\nabla p), \\ \partial_t p + \mathbf{u}\cdot\nabla p - D_2\Delta p &= -n\, f(p), \end{aligned} \tag{6.1}$$

was designed in the mid 2000s to describe the dynamics of swimming bacteria, *Bacillus subtilis*, as part of the effort to understand the dynamical interactions of oxygen diffusion/consumption, chemotaxis and viscous incompressible fluid flows. One of the special features of (6.1) is that both the Patlak-Keller-Segel model, (2.1), and the Navier-Stokes-Boussinesq equations, (3.1), are embedded in such a large system. Mathematically, the coupling and strong nonlinearities in (6.1) bring significant challenges to the qualitative analysis of the model. In a joint work [3], we rigorously justified that the bacteria diffusion coefficient, D_1, plays a prominent role as the governing mechanism of the large-time dynamics of the process, which is consistent with prior numerical simulations of the model

on bounded domains. This is the first rigorous mathematical result concerning the large-time behavior of the model on bounded physical domains.

Another mathematical feature of (6.1) consists of the non-uniform analytic nonlinearities in the equations, which leaves the model outside of the general framework of PDE models with uniform analytic nonlinearities. In a joint work [18], by renovating the existing conventional approaches, we established the first rigorous mathematical result concerning the spatial analyticity of solutions to the model in any space dimension, which corroborated prior numerical studies.

6.2 Mechanochemical Model of Vasculogenesis

Vasculogenesis is the physiological process through which blood vessels form in the embryo. It is regarded as the initial stage of the formation of vascular networks, closely followed by angiogenesis - the leading mechanism of tumor growth. Mathematical modeling of vasculogenesis using continuum mechanochemical PDE models appeared in various contexts. The model that I am getting into is a non-locally coupled system, consisting of the damped compressible Euler equations and a reaction-diffusion equation:

$$\begin{aligned} \partial_t \rho + \nabla \cdot (\rho \mathbf{v}) &= 0, \\ \partial_t (\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla P(\rho) &= -\alpha \rho \mathbf{v} + \beta \rho \nabla \Phi, \\ \tau \partial_t \Phi &= d \Delta \Phi - a \Phi + b \rho, \end{aligned} \tag{6.2}$$

which has rich mathematical structures and encompasses some well-known models in distinct areas. Indeed, when $\tau = a = 0$, one obtains the famous Euler-Poisson equations with damping. On the other hand, if the inertia is neglected, one recovers the Keller-Segel type model of chemotaxis with linear sensitivity and potentially nonlinear diffusion:

$$\begin{aligned} \partial_t \rho &= \frac{1}{\alpha} \Delta P(\rho) - \frac{\beta}{\alpha} \nabla \cdot (\rho \nabla \Phi), \\ \tau \partial_t \Phi &= d \Delta \Phi - a \Phi + b \rho. \end{aligned}$$

Collectively, the biological/physical background and mathematical features of (6.2) present an intriguing problem for qualitative analysis. In a recent joint work [22], we identified a set of parameter values and boundary conditions, under which the model generates non-constant steady states, which can be calculated explicitly in any space dimension. Moreover, we established the exponential stability of the steady states within the regime of small-amplitude classical solutions. Our analytical results showed consistency with experimental and numerical observations, which may help explain the underlying mechanisms of the formation of vascular networks.

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