

Confirmatory Factor Analysis

Theory Construction and Statistical Modeling



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Outline

SAPI

EFA and CFA

Confirmatory or Exploratory?

CFA in R

Scaling

Model Estimation

- Maximum Likelihood

Model Fit

- Degrees of Freedom

- Fit Indices

Model Evaluation



South African Personality Inventory Project



Carin Hill
Leon Jackson
Deon Meiring
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Michael Temane
Velichko H. Valchev
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Nel, J. A., Valchev, V. H., Rothmann, S., van de Vijver, F. J. R., Meiring, D., & de Bruin, G. P. (2012). Exploring the personality structure in the 11 languages of South Africa. *Journal of Personality*, 80, 915–948.

SAPI details

- 1216 participants from 11 official language groups
- From about 50,000 descriptive responses to 262 personality items
- Nine personality clusters:
 - Conscientiousness
 - Emotional Stability
 - Extraversion
 - Facilitating
 - Integrity
 - Intellect
 - Openness
 - Relationship Harmony
 - Soft-Heartedness (Ubuntu)
- Our data: selection of 1000 participants

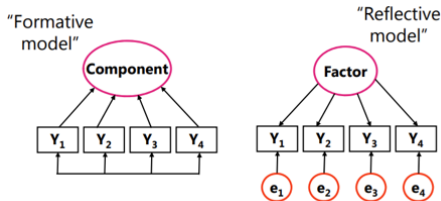


Factor Analysis

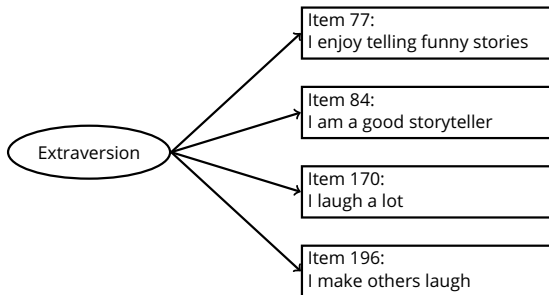
Factor Analysis: Modeling measurement of a latent variable

- EFA: Exploratory Factor Analysis.
- CFA: Confirmatory Factor Analysis.

Both EFA and CFA use a “reflective” measurement model, not a “formative” model.

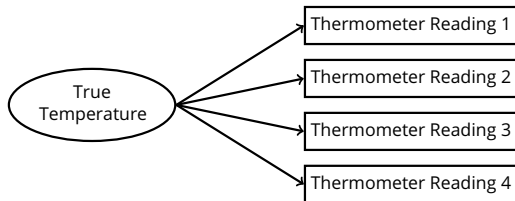


Reflective Constructs

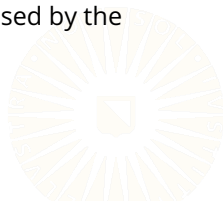


- Items are dependent variables, caused by the factor!
- Latent variable 'extraversion' explains item correlations:
The factor is the reason for the covariances/correlations.

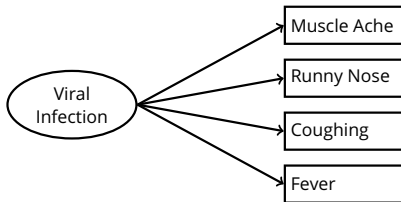
Reflective Constructs



Thermometer readings are the dependent variables, caused by the temperature!



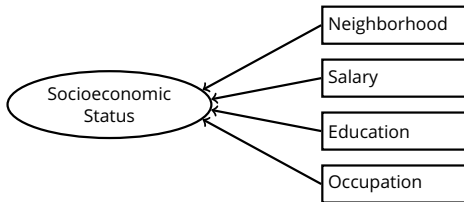
Reflective Constructs



Symptoms are the dependent variables, caused by the viral infection!

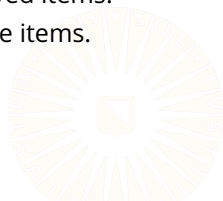


Formative Constructs



SES is an *index* defined as a (weighted) sum of the observed items.

- SES is the (latent) dependent variable, predicted by the items.
- This model is not empirically testable.



Interesting read

Interesting read on theory & latent variables:

Borsboom, D., Mellenbergh, G.J., & Van Heerden, J. (2003). The theoretical status of latent variables. *Psychological review*, 110(2), 203.



CONFIRMATORY OR EXPLORATORY?



Two Subscales of Extraversion

HAVING FUN

- Item 77: I enjoy telling funny stories
- Item 84: I am a good storyteller
- Item 170: I laugh a lot
- Item 196: I make others laugh

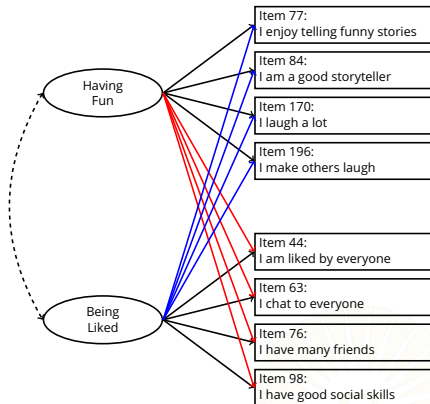
BEING LIKED

- Item 44: I am liked by everyone
- Item 63: I chat to everyone
- Item 76: I have many friends
- Item 98: I have good social skills



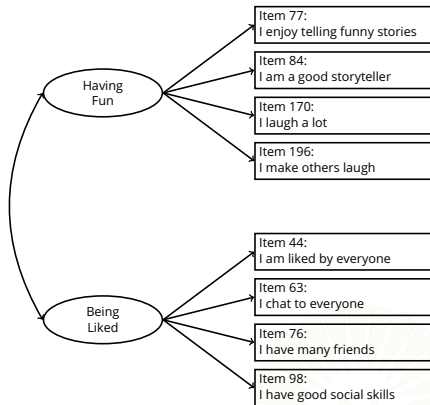
EFA

- All items load onto all factors
- No hypothesized measurement model
- Estimating latent covariances is optional
 - Oblique factors → Estimated
 - Orthogonal factors → Fixed
- Solution is not unique
- Use rotation to improve interpretability



CFA

- The statistical model represents the hypothesized measurement model
- No cross-loadings unless they're predicted by theory
- Almost always estimate the latent covariances
- A unique solution exists



CFA IN R



Example: Estimate a CFA Model

Load the SAPI data.

```
dataDir <- "../data/"
sapi <- read.table(paste0(dataDir, "sapi.txt"),
                  header = TRUE,
                  na.strings = "-999")
```

Specify the **lavaan** model syntax for the SAPI extraversion CFA.

```
mod1 <- '
fun    =~ Q77 + Q84 + Q170 + Q196
liked =~ Q44 + Q63 + Q76 + Q98
'
```

Use the `cfa()` function to estimate the model.

```
library(lavaan)
out1 <- cfa(mod1, data = sapi)
```


Example: Summarize the Fitted CFA

```
partSummary(out1, 1:4)
```

lavaan 0.6-18 ended normally after 30 iterations

Estimator	ML	
Optimization method	NLMINB	
Number of model parameters	17	
	Used	Total
Number of observations	959	1000

Model Test User Model:

Test statistic	130.193
Degrees of freedom	19
P-value (Chi-square)	0.000



Example: Summarize the Fitted CFA

```
partSummary(out1, 5:7)
```

Parameter Estimates:

Standard errors	Standard
Information	Expected
Information saturated (h1) model	Structured

Latent Variables:

	Estimate	Std.Err	z-value	P(> z)
fun =~				
Q77	1.000			
Q84	0.761	0.051	14.902	0.000
Q170	0.634	0.047	13.558	0.000
Q196	0.795	0.046	17.381	0.000
liked =~				
Q44	1.000			
Q63	1.512	0.147	10.278	0.000
Q76	1.483	0.149	9.955	0.000
Q98	1.243	0.119	10.462	0.000

Example: Summarize the Fitted CFA

```
partSummary(out1, 8:9)
```

Covariances:

	Estimate	Std.Err	z-value	P(> z)
fun ~~				
liked	0.231	0.025	9.234	0.000

Variances:

	Estimate	Std.Err	z-value	P(> z)
.Q77	0.548	0.038	14.389	0.000
.Q84	0.727	0.039	18.703	0.000
.Q170	0.687	0.035	19.572	0.000
.Q196	0.364	0.025	14.731	0.000
.Q44	0.662	0.034	19.291	0.000
.Q63	0.807	0.048	16.943	0.000
.Q76	0.966	0.054	17.931	0.000
.Q98	0.469	0.029	16.121	0.000
fun	0.627	0.056	11.303	0.000
liked	0.182	0.029	6.290	0.000

Example: Model Fit Statistics



Example: Model Fit Statistics

```
fitMeasures(out1)
```

npars	fmin	chisq
17.000	0.068	130.193
df	pvalue	baseline.chisq
19.000	0.000	1574.886
baseline.df	baseline.pvalue	cfi
28.000	0.000	0.928
tli	nnfi	rfi
0.894	0.894	0.878
nfi	pnfi	ifi
0.917	0.622	0.929
rni	logl	unrestricted.logl
0.928	-10147.587	-10082.491
aic	bic	ntotal
20329.175	20411.895	959.000
bic2	rmsea	rmsea.ci.lower
20357.903	0.078	0.066
rmsea.ci.upper	rmsea.ci.level	rmsea.pvalue
0.091	0.900	0.000
rmsea.close.h0	rmsea.notclose.pvalue	rmsea.notclose.h0
0.050	0.421	0.080

Example: Visualize the Fitted CFA

```
library(lavaanPlot)
lavaanPlot(model = out1,
  node_options = list(shape = "box",
                       fontname = "Helvetica"),
  edge_options = list(color = "grey"),
  coefs = TRUE,
  stand = TRUE,
  covs = TRUE)
```



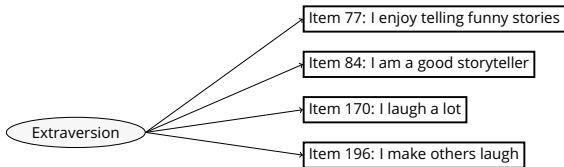
Example: Visualize the Fitted CFA

```
Error in path.expand(path):  invalid 'path' argument
```

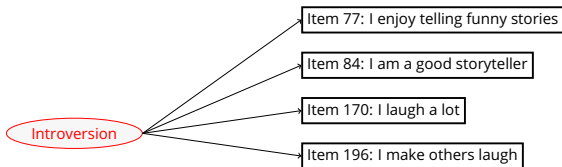


Latent variable scaling

Latent variables are not observed, thus no inherent scale.



Latent variable scaling Ctd.



Therefore, set up model such that scale of latent variable is clear.



Two common ways

1. Marker-variable method

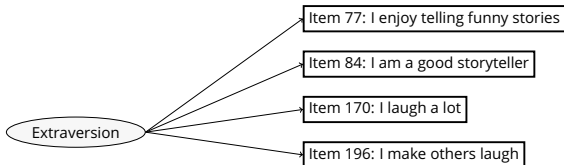
Constrain one of the factor loadings (default).

2. Reference group method:

Constrain the factor variance.

3. Effect coding:

Constrain the average of the loadings.



1. Marker-variable method (default)

Default parameterization:

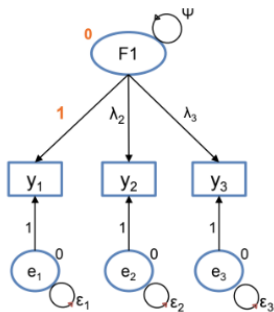
- First factor loading constrained at 1.
- Factor mean constrained at 0.

Other defaults:

- Mean of residuals is by definition 0.
- Residuals have a loading of 1.

Estimated:

- factor variance (Ψ),
- 'other' factor loadings (λ_2, λ_3),
- all item intercepts (ν_1, ν_2, ν_3),
- all residual variances ($\epsilon_1, \epsilon_2, \epsilon_3$).



1. Default marker-variable method – lavaan

```
# Model
model.1CFA <- '
  Extraversion =~ Q77 + Q84 + Q170 + Q196
'

# Fit model
fit_1CFA <- cfa(model.1CFA, data=sapi,
  missing='fiml', fixed.x=F) # use FIML
```

- First factor loading constrained at 1:

```
Extraversion =~
  Q77              1.000
```

- Factor mean constrained at 0:

```
Extraversion      0.000
```



1. Default marker-variable method – lavaan Ctd

```
parameterEstimates(fit_1CFA)[1:4,-c(5,6,7)]
```

	lhs	op	rhs	est	ci.lower	ci.upper
1	Extraversion	=~	Q77	1.000	1.000	1.000
2	Extraversion	=~	Q84	0.708	0.616	0.799
3	Extraversion	=~	Q170	0.567	0.466	0.668
4	Extraversion	=~	Q196	0.742	0.640	0.845

Factor loading of first indicator fixed to 1.
all other loadings are relative to that.

If reference category changed, other loadings also change.



2. Reference-group method

Parameterization:

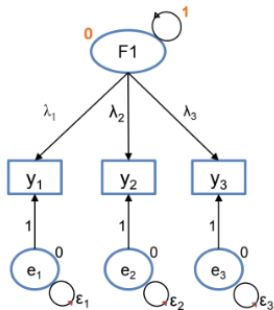
- Factor variance constrained at 1.
- Factor mean constrained at 0.

Defaults:

- Mean of residuals is by definition 0.
- Residuals have a loading of 1.

Estimated:

- all factor loadings ($\lambda_1, \lambda_2, \lambda_3$),
- all item intercepts (ν_1, ν_2, ν_3),
- all residual variances ($\epsilon_1, \epsilon_2, \epsilon_3$).



2. Reference-group method – lavaan

```
# Model
model.1CFA_RefGr <- '
  # Free first factor loading, using: NA*
  Extraversion =~ NA*Q77 + Q84 + Q170 + Q196

  # Set factor variance to 1, using: 1*
  Extraversion ~~ 1*Extraversion
'

# Fit model
fit_1CFA_RefGr <- cfa(model.1CFA_RefGr, data=sapi,
  missing='fiml', fixed.x=F) # use FIML
```

- Factor variance constrained at 1:

Extraversion	1.000
--------------	-------

- Factor mean constrained at 0:

Extraversion	0.000
--------------	-------



2. Reference-group method – lavaan Ctd

```
parameterEstimates(fit_1CFA_RefGr)[1:4,-c(5,6,7)]
```

	lhs	op	rhs	est	ci.lower	ci.upper
1	Extraversion	=~	Q77	0.835	0.759	0.910
2	Extraversion	=~	Q84	0.591	0.520	0.662
3	Extraversion	=~	Q170	0.473	0.404	0.543
4	Extraversion	=~	Q196	0.619	0.559	0.680

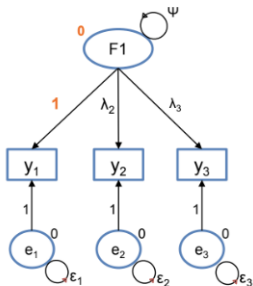
Advantage:

All factor loadings and scores on standardized metric.

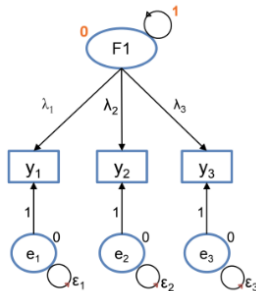


Which method to choose?

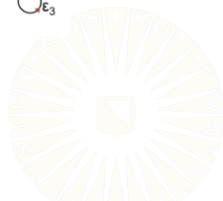
1. Marker-variable method



2. Reference-group method



Does not matter for substantive conclusions.
Sometimes, pragmatic reasons.



MODEL ESTIMATION



Maximum Likelihood Estimation

ML estimation simply finds the parameter values that are “most likely” to have given rise to the observed data.

- The *likelihood* function is just a probability density (or mass) function with the data treated as fixed and the parameters treated as random variables.
- Having such a framework allows us to ask: “Given that I’ve observed these data values, what parameter values most probably describe these data?”



Maximum Likelihood Estimation

ML estimation is usually employed when there is no closed form solution for the parameters we seek.

- This is why you don't usually see ML used to fit general linear models.

After choosing a likelihood function, we iteratively optimize the function to produce the ML estimated parameters.

- In practice, we nearly always work with the natural logarithm of the likelihood function (i.e., the *loglikelihood*).



ML Intuition

Let's say we have the following $N = 10$ observations.

- We assume these data come from a normal distribution with a known variance of $\sigma^2 = 1$.
- We want to estimate the mean of this distribution, μ .

```
(y <- rnorm(n = 10, mean = 5, sd = 1))
```

```
[1] 5.060983 3.364836 4.968344 6.696222 3.610013  
[6] 6.627266 4.165329 4.615346 4.537332 6.024850
```

ML Intuition

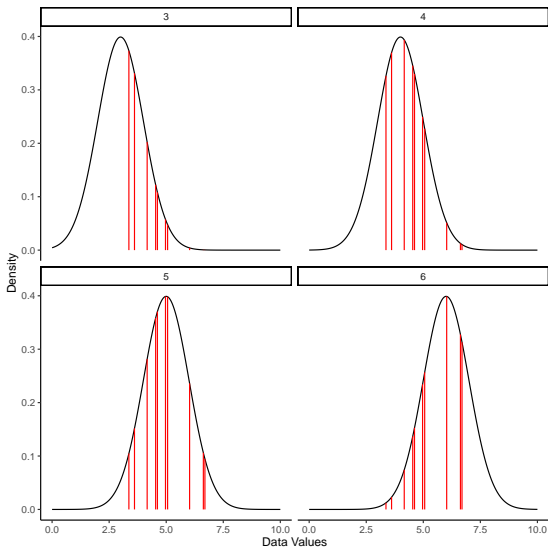
In ML estimation, we would define different normal distributions.

- Every distribution would have $\sigma^2 = 1$.
- Each distribution would have a different value of μ .

We then compare the observed data to those distributions and see which distribution best fits the data.



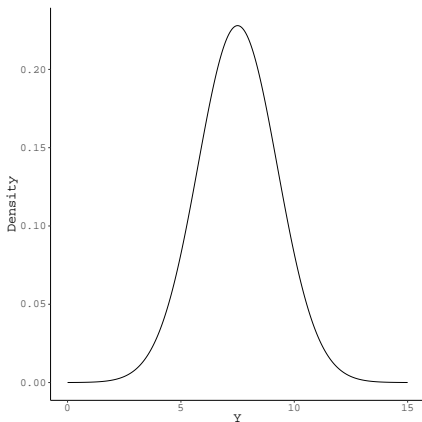
ML Intuition



Likelihoods

Suppose we have the following model:

$$Y \sim N(\mu, \sigma^2).$$



Likelihoods

For a given Y_n , we have:

$$P(Y_n | \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(Y_n - \mu)^2}{2\sigma^2}}. \quad (1)$$

If we plug estimated parameters into Equation 1, we get the probability of observing Y_n given $\hat{\mu}$ and $\hat{\sigma}^2$:

$$P(Y_n | \hat{\mu}, \hat{\sigma}^2) = \frac{1}{\sqrt{2\pi\hat{\sigma}^2}} e^{-\frac{(Y_n - \hat{\mu})^2}{2\hat{\sigma}^2}}. \quad (2)$$

Applying Equation 2 to all N observations and multiplying the results produces a *likelihood*:

$$\hat{L}(\hat{\mu}, \hat{\sigma}^2) = \prod_{n=1}^N P(Y_n | \hat{\mu}, \hat{\sigma}^2).$$



Likelihoods

We generally want to work with the natural logarithm of Equation 2. Doing so gives the *loglikelihood*:

$$\begin{aligned}\hat{\mathcal{L}}(\hat{\mu}, \hat{\sigma}^2) &= \ln \prod_{n=1}^N P(Y_n | \hat{\mu}, \hat{\sigma}^2) \\ &= -\frac{N}{2} \ln 2\pi - N \ln \hat{\sigma} - \frac{1}{2\hat{\sigma}^2} \sum_{n=1}^N (Y_n - \hat{\mu})^2\end{aligned}$$

ML tries to find the values of $\hat{\mu}$ and $\hat{\sigma}^2$ that maximize $\hat{\mathcal{L}}(\hat{\mu}, \hat{\sigma}^2)$.

- Find the values of $\hat{\mu}$ and $\hat{\sigma}^2$ that are *most likely*, given the observed values of Y .

Likelihoods

Suppose we have a linear regression model:

$$Y = \beta_0 + \beta_1 X + \varepsilon,$$

$$\varepsilon \sim N(0, \sigma^2).$$

This model can be equivalently written as:

$$Y \sim N(\beta_0 + \beta_1 X, \sigma^2)$$

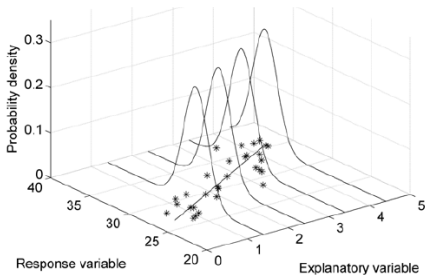


Image retrieved from:

<http://www.seaturtle.org/mtn/archives/mtn122/mtn122p1.shtml>

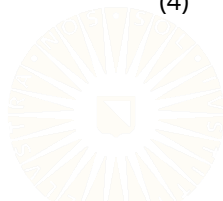
Likelihoods

For a given $\{Y_n, X_n\}$, we have:

$$P(Y_n|X_n, \beta_0, \beta_1, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(Y_n - \beta_0 - \beta_1 X_n)^2}{2\sigma^2}}. \quad (3)$$

If we plug our estimated parameters into Equation 3, we get the probability of observing Y_n given $\hat{Y}_n = \hat{\beta}_0 + \hat{\beta}_1 X_n$ and $\hat{\sigma}^2$.

$$P(Y_n|X_n, \hat{\beta}_0, \hat{\beta}_1, \hat{\sigma}^2) = \frac{1}{\sqrt{2\pi\hat{\sigma}^2}} e^{-\frac{(Y_n - \hat{\beta}_0 - \hat{\beta}_1 X_n)^2}{2\hat{\sigma}^2}} \quad (4)$$



Likelihoods

So, our final loglikelihood function would be the following:

$$\begin{aligned}\hat{\mathcal{L}}(\hat{\beta}_0, \hat{\beta}_1, \hat{\sigma}^2) &= \ln \prod_{n=1}^N P(Y_n | X_n, \hat{\beta}_0, \hat{\beta}_1, \hat{\sigma}^2) \\ &= -\frac{N}{2} \ln 2\pi - N \ln \hat{\sigma} - \frac{1}{2\hat{\sigma}^2} \sum_{n=1}^N (Y_n - \hat{\beta}_0 - \hat{\beta}_1 X_n)^2.\end{aligned}$$



Example

```
## Fit a model:
out1 <- lm(ldl ~ bp + glu + bmi, data = diabetes)

## Extract the predicted values and estimated residual standard error:
yHat <- predict(out1)
s     <- summary(out1)$sigma

## Compute the row-wise probabilities:
pY <- dnorm(diabetes$ldl, mean = yHat, sd = s)

## Compute the loglikelihood, and compare to R's version:
sum(log(pY)); logLik(out1)[1]

[1] -2109.939
[1] -2109.93
```

Multivariate Normal Distribution

The PDF for the multivariate normal distribution is:

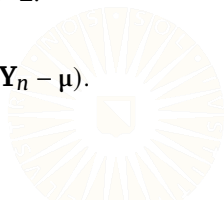
$$P(\mathbf{Y}|\boldsymbol{\mu}, \Sigma) = \frac{1}{\sqrt{(2\pi)^P |\Sigma|}} e^{-\frac{1}{2}(\mathbf{Y}-\boldsymbol{\mu})^T \Sigma^{-1}(\mathbf{Y}-\boldsymbol{\mu})}.$$

So, the multivariate normal loglikelihood is:

$$\mathcal{L}(\boldsymbol{\mu}, \Sigma) = - \left[\frac{P}{2} \ln(2\pi) + \frac{1}{2} \ln |\Sigma| + \frac{1}{2} \right] \sum_{n=1}^N (\mathbf{Y}_n - \boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{Y}_n - \boldsymbol{\mu}).$$

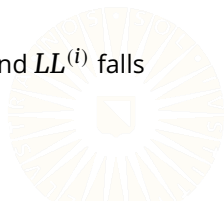
Which can be further simplified if we multiply through by -2:

$$-2\mathcal{L}(\boldsymbol{\mu}, \Sigma) = [P \ln(2\pi) + \ln |\Sigma|] \sum_{n=1}^N (\mathbf{Y}_n - \boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{Y}_n - \boldsymbol{\mu}).$$



Steps of ML

1. Choose a probability distribution, $f(Y|\theta)$, to describe the distribution of the data, Y , given the parameters, θ .
2. Choose some estimate of θ , $\hat{\theta}^{(i)}$.
3. Compute each row's contribution to the loglikelihood function by evaluating: $\ln \left[f \left(Y_n | \hat{\theta}^{(i)} \right) \right]$.
4. Sum the individual loglikelihood contributions from Step 3 to find the loglikelihood value, $\hat{\mathcal{L}}$.
5. Choose a "better" estimate of the parameters, $\hat{\theta}^{(i+1)}$, and repeat Steps 3 and 4.
6. Repeat Steps 3 – 5 until the change between $LL^{(i-1)}$ and $LL^{(i)}$ falls below some trivially small threshold.
7. Take $\hat{\theta}^{(i)}$ as your estimated parameters.



MODEL FIT



MODEL EVALUATION

