CFA Identification & Estimation

Theory Construction and Statistical Modeling

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Outline

South African Personality Inventory Project

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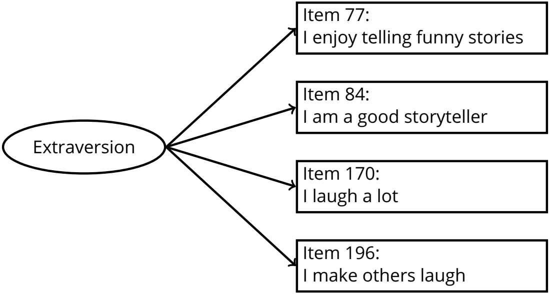
Nel, J. A., Valchev, V. H., Rothmann, S., van de Vijver, F. J. R., Meiring, D., & de Bruin, G. P. (2012). Exploring the personality structure in the 11 languages of South Africa. *Journal of Personality*, *80*, 915–948.

SAPI details

- 1216 participants from 11 official language groups
- From about 50,000 descriptive responses to 262 personality items
- Nine personality clusters:
  - Conscientiousness
  - Emotional Stability
  - Extraversion
  - Facilitating
  - Integrity
  - Intellect
  - Openness
  - Relationship Harmony
  - Soft-Heartedness (Ubuntu)
- Our data: selection of 1000 participants

FLAVORS OF LATENT CONSTRUCT

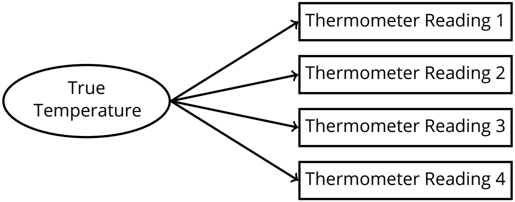
Reflective Constructs



In a reflective measurement model, the items are the dependent variables, and the latent factor is the independent variable.

- The observed items are dependent variables.
- The latent factor is causing the items to take the values we observe.

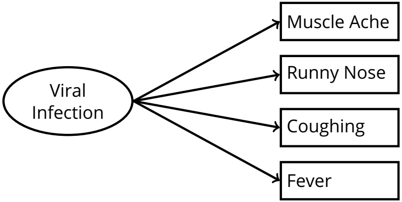
Reflective Constructs



The true temperature is the underlying (unobserved) factor that produces thermometer readings.

- Any given thermometer reading is only an imperfect reflection of the true temperature.
- Multiple readings increase the reliability of our temperature estimate.

Reflective Constructs

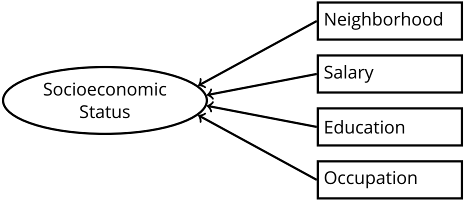


The latent viral infection is the causal factor that gives rise to the observed symptoms of illness.

- Symptoms are the dependent variables.
- Viral infection is the unobserved predictor variable.

Formative Constructs

Flipping the direction of the factor loadings makes a *formative construct*.



SES is an *index* defined as a (weighted) sum of the observed items.

- SES is the (latent) dependent variable, predicted by the items.
- This model is not empirically testable.

Model Estimation

Model-Implied Statistics

Most statistical estimation algorithms operate by minimizing the difference between two key reference points:

1. The *model-implied* statistics/predictions/fitted values
   - The sufficient statistics implied by the structure of your model.
   - Predicted/fitted values produced by your model.
2. The *observed* statistics/values
   - The sufficient statistics calculated from the observed data.
   - The raw outcome values from your dataset.

The predictions/implied statistics produced by a good model must be simpler than the analogous quantities in the observed data.

- A model that exactly replicates the observed data is overfitting.
- The inferences from such models won't generalize to the population.

Model-Implied Statistics

You should already be familiar with this idea from OLS regression.

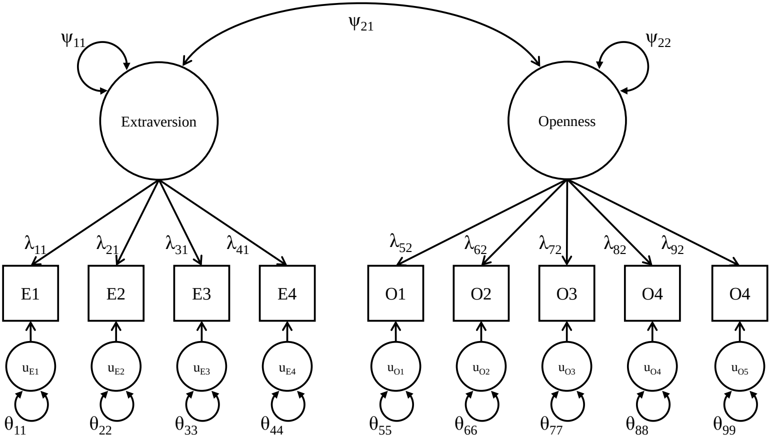
- The fitted values are the model implied statistics.

$$\hat{Y}_n = \hat{\beta}_0 + \sum_{p=1}^{P} \hat{\beta}_p X_{n,p}$$

- The raw outcome variable, Y, contains the observed values.
- Minimize the difference between $\hat{Y}$ and Y to estimate the model.

$$RSS = \sum_{n=1}^{N} \left(Y_n - \hat{Y}_n\right)^2$$

Fully Specified Path Diagram

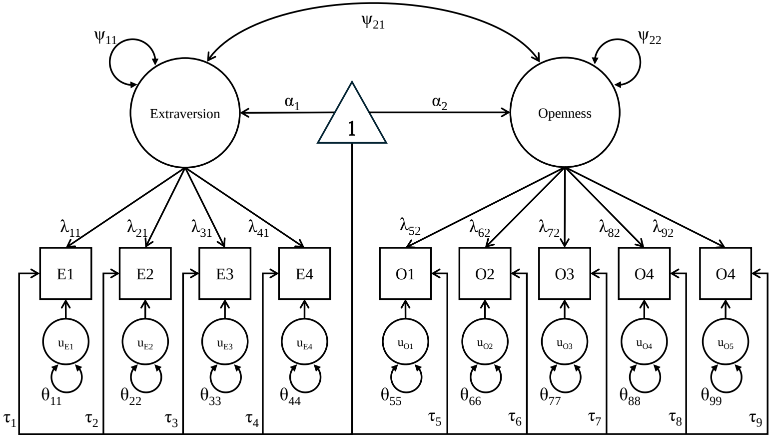


Parameter Matrices

$$\Psi = \begin{bmatrix} \psi_{11} & \\ \psi_{21} & \psi_{22} \end{bmatrix}$$

$$\Theta = \begin{bmatrix} \theta_{11} & & & & & & & & \\ 0 & \theta_{22} & & & & & & & \\ 0 & 0 & \theta_{33} & & & & & & \\ 0 & 0 & 0 & \theta_{44} & & & & & \\ 0 & 0 & 0 & 0 & \theta_{55} & & & & \\ 0 & 0 & 0 & 0 & 0 & \theta_{66} & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \theta_{77} & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \theta_{88} & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \theta_{99} \end{bmatrix} \qquad \Lambda = \begin{bmatrix} \lambda_{11} & 0 \\ \lambda_{21} & 0 \\ \lambda_{31} & 0 \\ \lambda_{41} & 0 \\ 0 & \lambda_{52} \\ 0 & \lambda_{62} \\ 0 & \lambda_{72} \\ 0 & \lambda_{82} \\ 0 & \lambda_{92} \end{bmatrix}$$

Fully Specified Path Diagram



Parameter Matrices

$$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \quad \Psi = \begin{bmatrix} \psi_{11} & \\ \psi_{21} & \psi_{22} \end{bmatrix}$$

$$\tau = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \\ \tau_5 \\ \tau_6 \\ \tau_7 \\ \tau_8 \\ \tau_9 \end{bmatrix} \quad \Theta = \begin{bmatrix} \theta_{11} & & & & & & & & \\ 0 & \theta_{22} & & & & & & & \\ 0 & 0 & \theta_{33} & & & & & & \\ 0 & 0 & 0 & \theta_{44} & & & & & \\ 0 & 0 & 0 & 0 & \theta_{55} & & & & \\ 0 & 0 & 0 & 0 & 0 & \theta_{66} & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \theta_{77} & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \theta_{88} & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \theta_{99} \end{bmatrix} \quad \Lambda = \begin{bmatrix} \lambda_{11} & 0 \\ \lambda_{21} & 0 \\ \lambda_{31} & 0 \\ \lambda_{41} & 0 \\ 0 & \lambda_{52} \\ 0 & \lambda_{62} \\ 0 & \lambda_{72} \\ 0 & \lambda_{82} \\ 0 & \lambda_{92} \end{bmatrix}$$

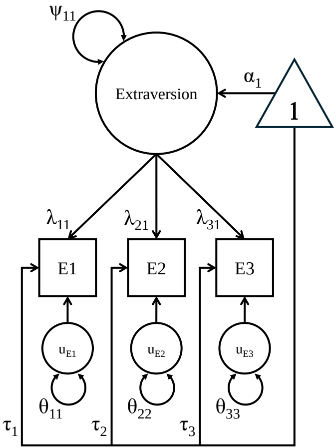
Parameter Matrices

To see what role these parameter matrices play in model estimation, we'll work with a simpler example.

$$\alpha = \begin{bmatrix} \alpha_1 \end{bmatrix} \quad \Psi = \begin{bmatrix} \psi_{11} \end{bmatrix}$$

$$\tau = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} \quad \Lambda = \begin{bmatrix} \lambda_{11} \\ \lambda_{21} \\ \lambda_{31} \end{bmatrix}$$

$$\Theta = \begin{bmatrix} \theta_{11} & & \\ 0 & \theta_{22} & \\ 0 & 0 & \theta_{33} \end{bmatrix}$$



Model-Implied Statistics

The parameter matrices define the model-implied mean vector, $\hat{\mu}$, and covariance matrix, $\hat{\Sigma}$, via the following formulas.

$$\Sigma = \Lambda \Psi \Lambda^T + \Theta$$
$$\mu = \tau + \Lambda \alpha$$

By expanding these formulas, we can see how the model reproduces each mean, variance, and covariance.

$$\hat{\mu} = \begin{bmatrix} \tau_1 + \lambda_{11}\alpha_1 \\ \tau_2 + \lambda_{22}\alpha_1 \\ \tau_3 + \lambda_{33}\alpha_1 \end{bmatrix} \qquad \hat{\Sigma} = \begin{bmatrix} \lambda_{11}\psi_{11}\lambda_{11} + \theta_{11} & & \\ \lambda_{11}\psi_{11}\lambda_{21} & \lambda_{21}\psi_{11}\lambda_{21} + \theta_{22} & \\ \lambda_{11}\psi_{11}\lambda_{31} & \lambda_{21}\psi_{11}\lambda_{31} & \lambda_{31}\psi_{11}\lambda_{31} + \theta_{33} \end{bmatrix}$$

Model-Implied Statistics

The estimating algorithm chooses values for α, τ, Ψ, and Λ that minimize the differences between the model-implied statistics, $\{\hat{\mu}, \hat{\Sigma}\}$, and the sufficient statistics calculated from the observed data, $\{\bar{\mathbf{Y}}, \text{Cov}(\mathbf{Y})\}$.

$$\hat{\mu} = \begin{bmatrix} \tau_1 + \lambda_{11}\alpha_1 \\ \tau_2 + \lambda_{22}\alpha_1 \\ \tau_3 + \lambda_{33}\alpha_1 \end{bmatrix} \qquad \hat{\Sigma} = \begin{bmatrix} \lambda_{11}\psi_{11}\lambda_{11} + \theta_{11} & & \\ \lambda_{11}\psi_{11}\lambda_{21} & \lambda_{21}\psi_{11}\lambda_{21} + \theta_{22} & \\ \lambda_{11}\psi_{11}\lambda_{31} & \lambda_{21}\psi_{11}\lambda_{31} & \lambda_{31}\psi_{11}\lambda_{31} + \theta_{33} \end{bmatrix}$$

$$\bar{\mathbf{Y}} = \begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \bar{y}_3 \end{bmatrix} \qquad \text{Cov}(\mathbf{Y}) = \begin{bmatrix} var(y_1) & & \\ cov(y_2, y_1) & var(y_2) & \\ cov(y_3, y_1) & cov(y_3, y_2) & var(y_3) \end{bmatrix}$$

Optimization Objective

We can formulate the familiar OLS objective as an abstract optimization problem at follows.

$$f(\beta_0, \beta_1, \ldots, \beta_P) = \underset{\beta_0, \beta_1, \ldots, \beta_P}{\arg\min} \left\| Y - \hat{Y} \right\|$$

Applying the same idea to our CFA optimization problem, we get the following formulation.

$$f(\Lambda, \Psi) = \underset{\Lambda, \Psi}{\arg\min} \left\| \text{Cov}(\mathbf{Y}) - \hat{\Sigma} \right\|$$

$$f(\tau, \alpha) = \underset{\tau, \alpha}{\arg\min} \left\| \bar{\mathbf{Y}} - \hat{\mu} \right\|$$

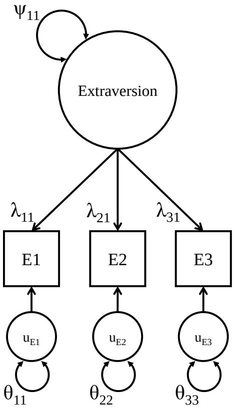
Model Identification

Number of Estimated Parameters

To estimate a CFA model, we must define $\hat{\Sigma}$.

- We must estimate each parameter that defines an element in $\hat{\Sigma}$.
- The constraints on our model determine how many parameter estimates we need.

$$\hat{\Sigma} = \begin{bmatrix} \lambda_{11}\psi_{11}\lambda_{11} + \theta_{11} & & \\ \lambda_{11}\psi_{11}\lambda_{21} & \lambda_{21}\psi_{11}\lambda_{21} + \theta_{22} & \\ \lambda_{11}\psi_{11}\lambda_{31} & \lambda_{21}\psi_{11}\lambda_{31} & \lambda_{31}\psi_{11}\lambda_{31} + \theta_{33} \end{bmatrix}$$

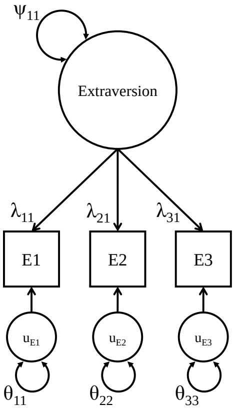


Number of Estimated Parameters

For this example, we need to estimate seven parameters to fully define $\hat{\Sigma}$.

- One latent variance: ψ_{11}
- Three factor loadings: $\{\lambda_{11}, \lambda_{21}, \lambda_{31}\}$
- Three residual variances: $\{\theta_{11}, \theta_{22}, \theta_{33}\}$

$$\hat{\Sigma} = \begin{bmatrix} \lambda_{11}\psi_{11}\lambda_{11} + \theta_{11} & & \\ \lambda_{11}\psi_{11}\lambda_{21} & \lambda_{21}\psi_{11}\lambda_{21} + \theta_{22} & \\ \lambda_{11}\psi_{11}\lambda_{31} & \lambda_{21}\psi_{11}\lambda_{31} & \lambda_{31}\psi_{11}\lambda_{31} + \theta_{33} \end{bmatrix}$$



Available Information

The data only contain a fixed amount of information.

- We can quantify the available information in discrete units.
- Every unique element of $\text{Cov}(\mathbf{Y})$ contributes one unit of information.

$$\text{Cov}(\mathbf{Y}) = \begin{bmatrix} var(y_1) & & \\ cov(y_2, y_1) & var(y_2) & \\ cov(y_3, y_1) & cov(y_3, y_2) & var(y_3) \end{bmatrix}$$

In this example, we have six pieces of available information.

- Three variances: $var(y_1)$, $var(y_2)$, $var(y_3)$
- Three covariances: $cov(y_2, y_1)$, $cov(y_3, y_1)$, $cov(y_3, y_2)$

For a positive-definite $M \times M$ covariance matrix, the number of unique elements will always be $Q = \frac{M(M+1)}{2}$.

Degrees of Freedom

We can only estimate one parameter for each piece of available information, Q.

- We can estimate no more than Q parameters in any one model.

The *degrees of freedom* (df) is the difference between the amount of information available in the data, Q, and the number of parameters estimated in our model, P.

$$df = Q - P$$

If $df < 0$, the model is not estimable.

- A model with $df < 0$ is *not identified*.
- The data do not provide enough information to define a unique solution for all P parameter estimates.

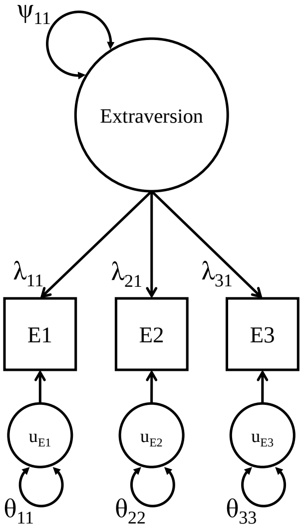
Degrees of Freedom

What are the degrees of freedom for our example?

$$df = Q - P = 6 - 7 = -1$$

This model has negative df.

- We cannot estimate the model in this form.
- We must impose *identifying constraints*.



Identifying Constraints

Consider the following equation:

$$5 = x + y$$

What are the values of x and y?

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What are the values of x and y?

$$y = 5 - x$$

What if we assume that $y = x$?

$$\begin{aligned} 5 &= x + y \\ 0 &= x - y \end{aligned}$$

Identifying Constraints

Consider the following equation:

$$5 = x + y$$

What are the values of x and y?

$$y = 5 - x$$

What if we assume that $y = x$?

$$\begin{aligned} 5 &= x + y \\ 0 &= x - y \end{aligned}$$

Now we have enough information:

$$5 = x + x = 2x \quad \Rightarrow \quad x = y = 2.5$$

Identifying Constraints

We must fix some parameters to identify the model.

- For each construct, we need $df \geq 0$.
- If the construct has three or more indicators:
  - Fix one parameter in the covariance model.
  - Fix one parameter in the mean model.
- If the construct has two indicators:
  - Fix an additional parameter in the covariance model.
- If the construct has only one indicator:
  - Cannot define a latent factor.

Identifying Constraints

These constraints also define the scale of the latent factors.

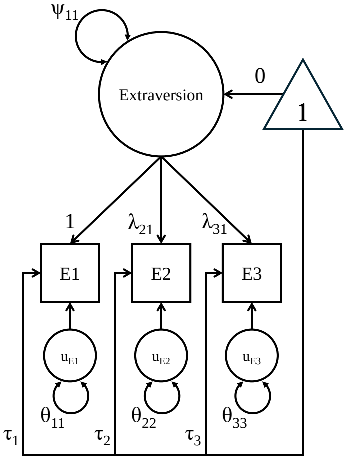
- Latent factors have no direct representation as observed variables in our dataset.
- A latent factor only exists after we've estimated it.
- So latent factors have no inherent scale.
- Identifying constraints are also called *scaling constraints*.

There are two common methods of identifying/scaling CFA models.

1. Marker-variable method
2. Fixed-factor method

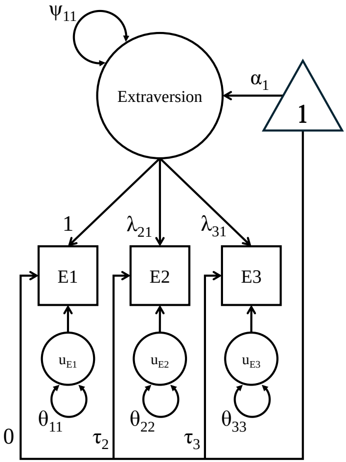
Marker-Variable Method

- Fix one factor loading to 1.
- Fix the latent mean to 0.



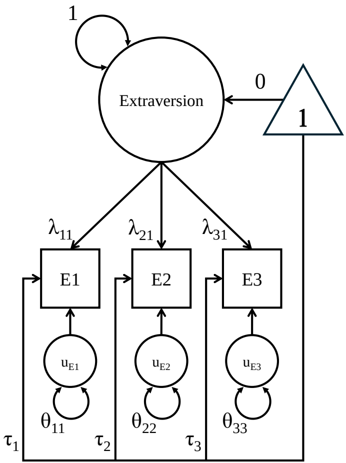
Marker-Variable Method

- Fix one factor loading to 1.
- Estimate the latent mean and fix one item intercept to 0.



Fixed-Factor Method

- Fix the latent variance to 1.
- Fix the latent mean to 0.



Just-Identified: Unconstrained Model

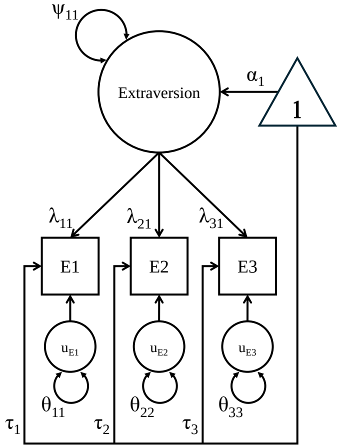
$$\alpha = \begin{bmatrix} \alpha_1 \end{bmatrix} \quad \Psi = \begin{bmatrix} \psi_{11} \end{bmatrix}$$

$$\tau = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} \quad \Lambda = \begin{bmatrix} \lambda_{11} \\ \lambda_{21} \\ \lambda_{31} \end{bmatrix}$$

$$\Theta = \begin{bmatrix} \theta_{11} & & \\ 0 & \theta_{22} & \\ 0 & 0 & \theta_{33} \end{bmatrix}$$

$$Q = \frac{3(3+1)}{2} + 3 = 9$$

$$\begin{aligned} df &= Q - P \\ &= 9 - 11 = -2 \end{aligned}$$



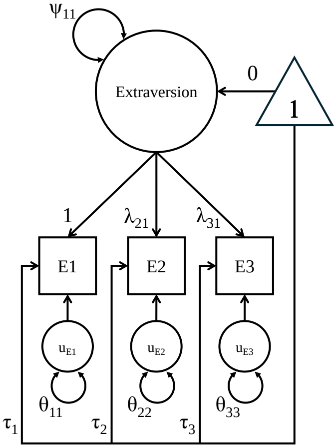
Just-Identified: Marker-Variable

$$\alpha = \begin{bmatrix} 0 \end{bmatrix} \quad \Psi = \begin{bmatrix} \psi_{11} \end{bmatrix}$$

$$\tau = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} \quad \Lambda = \begin{bmatrix} 1 \\ \lambda_{21} \\ \lambda_{31} \end{bmatrix}$$

$$\Theta = \begin{bmatrix} \theta_{11} & & \\ 0 & \theta_{22} & \\ 0 & 0 & \theta_{33} \end{bmatrix}$$

$$\begin{aligned} df &= Q - P \\ &= 9 - 9 = 0 \end{aligned}$$



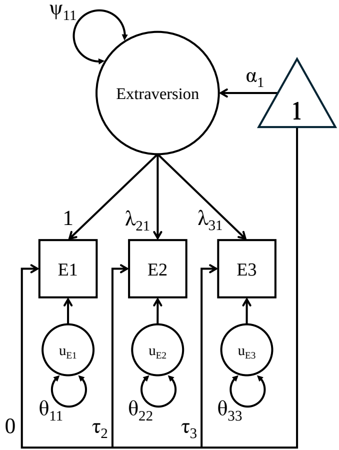
Just-Identified: Marker-Variable

$$\alpha = \begin{bmatrix}\alpha_1\end{bmatrix} \quad \Psi = \begin{bmatrix}\psi_{11}\end{bmatrix}$$

$$\tau = \begin{bmatrix} 0 \\ \tau_2 \\ \tau_3 \end{bmatrix} \quad \Lambda = \begin{bmatrix} 1 \\ \lambda_{21} \\ \lambda_{31} \end{bmatrix}$$

$$\Theta = \begin{bmatrix} \theta_{11} & & \\ 0 & \theta_{22} & \\ 0 & 0 & \theta_{33} \end{bmatrix}$$

$$\begin{aligned} df &= Q - P \\ &= 9 - 9 = 0 \end{aligned}$$



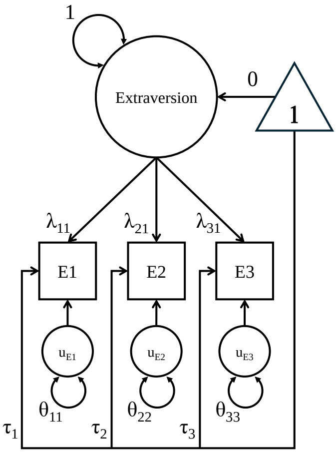
Just-Identified: Fixed-Factor

$$\alpha = \begin{bmatrix} 0 \end{bmatrix} \quad \Psi = \begin{bmatrix} \psi_{11} \end{bmatrix}$$

$$\tau = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} \quad \Lambda = \begin{bmatrix} \lambda_{11} \\ \lambda_{21} \\ \lambda_{31} \end{bmatrix}$$

$$\Theta = \begin{bmatrix} \theta_{11} & & \\ 0 & \theta_{22} & \\ 0 & 0 & \theta_{33} \end{bmatrix}$$

$$\begin{aligned} df &= Q - P \\ &= 9 - 9 = 0 \end{aligned}$$



Under-Identified: Unconstrained Model

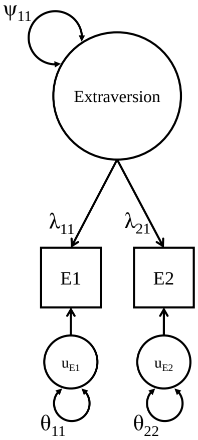
$$\Psi = \begin{bmatrix} \psi_{11} \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} \lambda_{11} \\ \lambda_{21} \end{bmatrix}$$

$$\Theta = \begin{bmatrix} \theta_{11} & \\ 0 & \theta_{22} \end{bmatrix}$$

$$Q = \frac{2(2+1)}{2} = 3$$

$$\begin{aligned} df &= Q - P \\ &= 3 - 5 = -2 \end{aligned}$$



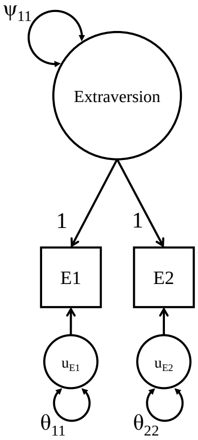
Under-Identified: Marker-Variable

$$\Psi = \begin{bmatrix} \psi_{11} \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Theta = \begin{bmatrix} \theta_{11} & \\ 0 & \theta_{22} \end{bmatrix}$$

$$\begin{aligned} df &= Q - P \\ &= 3 - 3 = 0 \end{aligned}$$



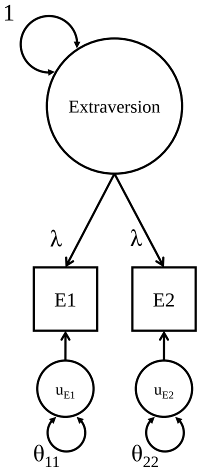
Under-Identified: Fixed-Factor

$$\Psi = \begin{bmatrix} 1 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} \lambda \\ \lambda \end{bmatrix}$$

$$\Theta = \begin{bmatrix} \theta_{11} & \\ 0 & \theta_{22} \end{bmatrix}$$

$$\begin{aligned} df &= Q - P \\ &= 3 - 3 = 0 \end{aligned}$$



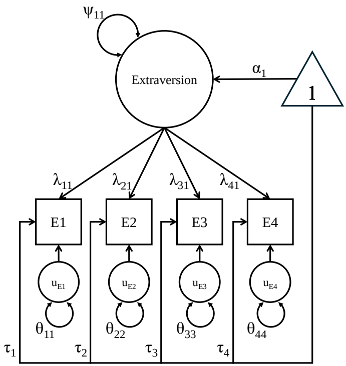
Over-Identified Models

What happens when we have four (or more) indicators?

With 4 indicators, our model contains 14 parameters:

- Four factor loadings
- Four residual variances
- Four item intercepts
- One latent mean
- One latent variance

$$Q = \frac{4(4+1)}{2} + 4 = 14$$
$$df = 14 - 14 = 0$$



Over-Identified Models

With 5 indicators, we have 17 model parameters:

- Five factor loadings
- Five residual variances
- Five item intercepts
- One latent mean
- One latent variance

$$Q = \frac{5(5+1)}{2} + 5 = 20$$
$$df = 20 - 17 = 3$$

With 6 indicators, we have 20 model parameters:

- Six factor loadings
- Six residual variances
- Six item intercepts
- One latent mean
- One latent variance

$$Q = \frac{6(6+1)}{2} + 6 = 27$$
$$df = 27 - 20 = 7$$

Over-Identified Models

With four indicators, we automatically have $df \geq 0$ without imposing any scaling constraints.

- Can we directly estimate the unconstrained model?

```
mod1 <- '
fun =~ Q77 + Q84 + Q170 + Q196
'

fit1 <- lavaan(mod1,
               data           = sapi,
               auto.var       = TRUE,
               meanstructure = TRUE,
               int.ov.free    = TRUE,
               int.lv.free    = TRUE)

Warning:  lavaan->lav_model_vcov():
  Could not compute standard errors!  The information matrix could not be
  inverted.  This may be a symptom that the model is not identified.
```

Hmmm...I guess not.

Over-Identified Models

```
partSummary(fit1, 1:4)

lavaan 0.6-19 ended normally after 13 iterations

  Estimator                                         ML
  Optimization method                           NLMINB
  Number of model parameters                        14

                                                  Used       Total
  Number of observations                           970        1000

Model Test User Model:

  Test statistic                                58.017
  Degrees of freedom                                 0
```

Over-Identified Models

```
partSummary(fit1, 7:8)

Latent Variables:
                   Estimate  Std.Err  z-value  P(>|z|)
  fun =~
    Q77               0.824 NA
    Q84               0.578 NA
    Q170              0.479 NA
    Q196              0.619 NA

Intercepts:
                   Estimate  Std.Err  z-value  P(>|z|)
   .Q77               3.574 NA
   .Q84               3.232 NA
   .Q170              3.955 NA
   .Q196              3.803 NA
    fun               0.000 NA
```

Over-Identified Models

```
partSummary(fit1, 9)

Variances:
                   Estimate  Std.Err  z-value  P(>|z|)
   .Q77               0.491 NA
   .Q84               0.760 NA
   .Q170              0.703 NA
   .Q196              0.370 NA
    fun               1.012 NA
```

Over-Identified Models

In the above example, the data contain enough information to define our model parameters, but that's not enough.

- We still need scaling constraints.
- The estimation algorithm needs an anchor point from which to extrapolate the relative values of the model parameters.
- Without any scaling constraints, an infinite number of parameter matrices will produce the same $\hat{\mu}$ and $\hat{\Sigma}$.
- An infinite number of solutions are equally good.

```
## Fit a model to get some example parameters:
mod1 <- '
fun   =~ Q77 + Q84 + Q170 + Q196
liked =~ Q44 + Q63 + Q76  + Q98
'

fit1 <- cfa(mod1, data = sapi, effect.coding = TRUE)
```

Over-Identified Models

```
## Extract the estimated factor loadings and latent covariance matrix:
(lambda <- inspect(fit1, "est")$lambda)

       fun liked
Q77  1.254 0.000
Q84  0.954 0.000
Q170 0.795 0.000
Q196 0.997 0.000
Q44  0.000 0.764
Q63  0.000 1.155
Q76  0.000 1.133
Q98  0.000 0.949

(psi <- inspect(fit1, "est")$psi)

        fun liked
fun   0.399
liked 0.241 0.311
```

Over-Identified Models

```r
## Extract the estimated residual variances:
(theta <- inspect(fit1, "est")$theta)

       Q77   Q84  Q170  Q196   Q44   Q63   Q76   Q98
Q77  0.548
Q84  0.000 0.727
Q170 0.000 0.000 0.687
Q196 0.000 0.000 0.000 0.364
Q44  0.000 0.000 0.000 0.000 0.662
Q63  0.000 0.000 0.000 0.000 0.000 0.807
Q76  0.000 0.000 0.000 0.000 0.000 0.000 0.966
Q98  0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.469
```

Over-Identified Models

```r
## Manually compute the model-implied covariance matrix:
(sigma  <- lambda %*% psi %*% t(lambda) + theta)

       Q77   Q84  Q170  Q196   Q44   Q63   Q76   Q98
Q77  1.176
Q84  0.477 1.090
Q170 0.398 0.303 0.940
Q196 0.499 0.380 0.316 0.761
Q44  0.231 0.175 0.146 0.183 0.843
Q63  0.349 0.265 0.221 0.277 0.275 1.223
Q76  0.342 0.260 0.217 0.272 0.269 0.407 1.365
Q98  0.287 0.218 0.182 0.228 0.226 0.341 0.335 0.750
```

Over-Identified Models

```
## Randomly sample an arbitrary scaling factor:
(a <- runif(1, 1, 2))

[1] 1.524314

## Rescale all factor loadings by a factor of a:
(lambda2 <- lambda * a)

       fun liked
Q77  1.911 0.000
Q84  1.454 0.000
Q170 1.212 0.000
Q196 1.520 0.000
Q44  0.000 1.164
Q63  0.000 1.760
Q76  0.000 1.727
Q98  0.000 1.447
```

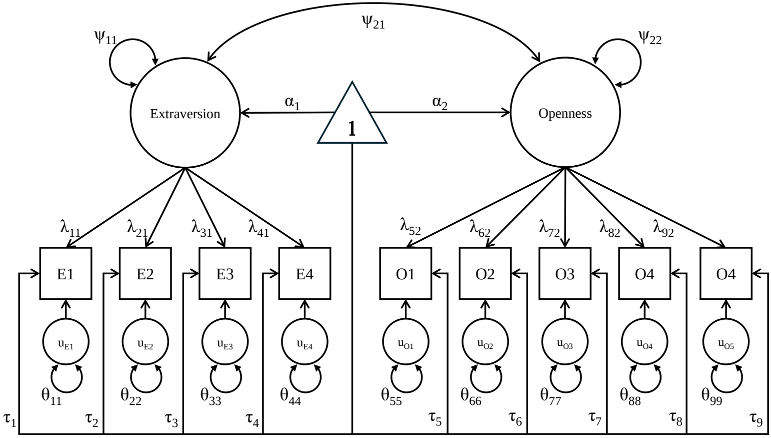
Over-Identified Models

```
## Rescale the latent covariance matrix by a factor of (1 / a^2):
(psi2 <- psi / a^2)

        fun liked
fun   0.172
liked 0.104 0.134

## Compute the model-implied covariance matrix using the rescaled parameters:
(sigma2  <- lambda2 %*% psi2 %*% t(lambda2) + theta)

       Q77   Q84  Q170  Q196   Q44   Q63   Q76   Q98
Q77  1.176
Q84  0.477 1.090
Q170 0.398 0.303 0.940
Q196 0.499 0.380 0.316 0.761
Q44  0.231 0.175 0.146 0.183 0.843
Q63  0.349 0.265 0.221 0.277 0.275 1.223
Q76  0.342 0.260 0.217 0.272 0.269 0.407 1.365
Q98  0.287 0.218 0.182 0.228 0.226 0.341 0.335 0.750
```

Over-Identified Models

```
## Compare the two model-implied covariance matrices:
all.equal(sigma, sigma2)

[1] TRUE

## Repeat the experiment 100 times:
out <- rep(NA, 100)
for(i in 1:100) {
  a <- runif(1, 1, 2)

  lambda2 <- lambda * a
  psi2    <- psi / a^2
  sigma2  <- lambda2 %*% psi2 %*% t(lambda2) + theta

  out[i] <- all.equal(sigma, sigma2)
}

## Always the same result?
all(out)

[1] TRUE
```

Two Construct: Unconstrained Model

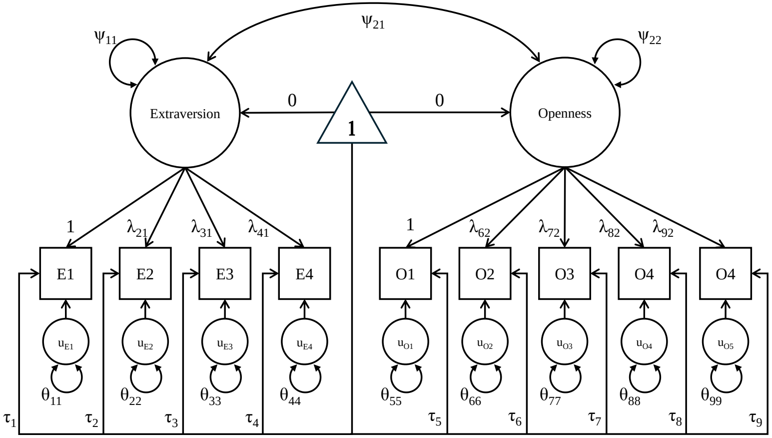


Unconstrained Parameter Matrices

$$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \quad \Psi = \begin{bmatrix} \psi_{11} & \\ \psi_{21} & \psi_{22} \end{bmatrix}$$

$$\tau = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \\ \tau_5 \\ \tau_6 \\ \tau_7 \\ \tau_8 \\ \tau_9 \end{bmatrix} \quad \Theta = \begin{bmatrix} \theta_{11} & & & & & & & & \\ 0 & \theta_{22} & & & & & & & \\ 0 & 0 & \theta_{33} & & & & & & \\ 0 & 0 & 0 & \theta_{44} & & & & & \\ 0 & 0 & 0 & 0 & \theta_{55} & & & & \\ 0 & 0 & 0 & 0 & 0 & \theta_{66} & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \theta_{77} & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \theta_{88} & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \theta_{99} \end{bmatrix} \quad \Lambda = \begin{bmatrix} \lambda_{11} & 0 \\ \lambda_{21} & 0 \\ \lambda_{31} & 0 \\ \lambda_{41} & 0 \\ 0 & \lambda_{52} \\ 0 & \lambda_{62} \\ 0 & \lambda_{72} \\ 0 & \lambda_{82} \\ 0 & \lambda_{92} \end{bmatrix}$$

Two Construct: Marker variable

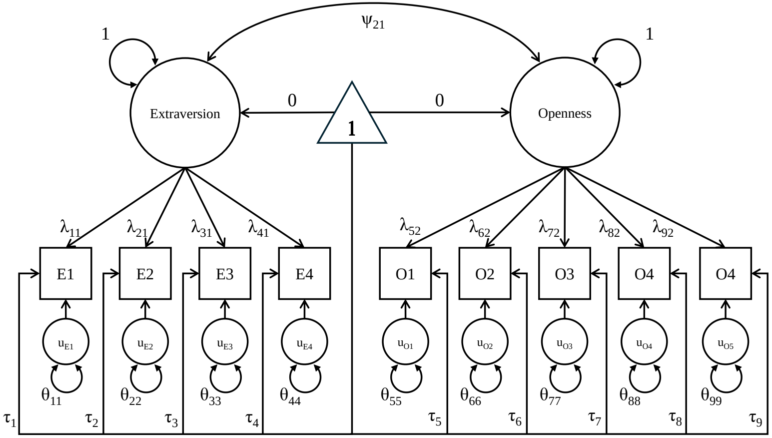


Parameter Matrices: Marker Variable

$$\alpha = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Psi = \begin{bmatrix} \psi_{11} & \\ \psi_{21} & \psi_{22} \end{bmatrix}$$

$$\tau = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \\ \tau_5 \\ \tau_6 \\ \tau_7 \\ \tau_8 \\ \tau_9 \end{bmatrix} \quad \Theta = \begin{bmatrix} \theta_{11} & & & & & & & & \\ 0 & \theta_{22} & & & & & & & \\ 0 & 0 & \theta_{33} & & & & & & \\ 0 & 0 & 0 & \theta_{44} & & & & & \\ 0 & 0 & 0 & 0 & \theta_{55} & & & & \\ 0 & 0 & 0 & 0 & 0 & \theta_{66} & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \theta_{77} & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \theta_{88} & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \theta_{99} \end{bmatrix} \quad \Lambda = \begin{bmatrix} 1 & 0 \\ \lambda_{21} & 0 \\ \lambda_{31} & 0 \\ \lambda_{41} & 0 \\ 0 & 1 \\ 0 & \lambda_{62} \\ 0 & \lambda_{72} \\ 0 & \lambda_{82} \\ 0 & \lambda_{92} \end{bmatrix}$$

Two Construct: Fixed-Factor



Parameter Matrices: Fixed-Factor

$$\alpha = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Psi = \begin{bmatrix} 1 & \\ \psi_{21} & 1 \end{bmatrix}$$

$$\tau = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \\ \tau_5 \\ \tau_6 \\ \tau_7 \\ \tau_8 \\ \tau_9 \end{bmatrix} \quad \Theta = \begin{bmatrix} \theta_{11} & & & & & & & & \\ 0 & \theta_{22} & & & & & & & \\ 0 & 0 & \theta_{33} & & & & & & \\ 0 & 0 & 0 & \theta_{44} & & & & & \\ 0 & 0 & 0 & 0 & \theta_{55} & & & & \\ 0 & 0 & 0 & 0 & 0 & \theta_{66} & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \theta_{77} & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \theta_{88} & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \theta_{99} \end{bmatrix} \quad \Lambda = \begin{bmatrix} \lambda_{11} & 0 \\ \lambda_{21} & 0 \\ \lambda_{31} & 0 \\ \lambda_{41} & 0 \\ 0 & \lambda_{52} \\ 0 & \lambda_{62} \\ 0 & \lambda_{72} \\ 0 & \lambda_{82} \\ 0 & \lambda_{92} \end{bmatrix}$$

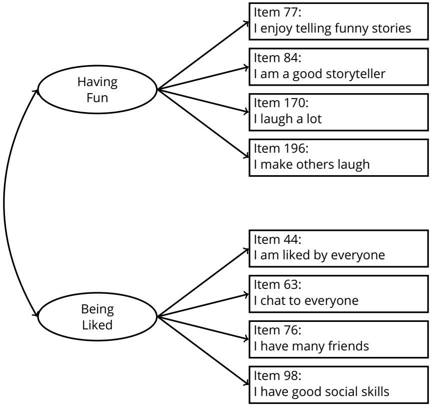
Example

CFA of Extraversion Items

Suppose we hypothesize two distinct dimensions of extraversion underlying 8 of the SAPI items.

1. Having Fun
2. Being Liked by Others

We'll define our measurement model as the two-factor CFA shown to the right.



Example: Marker Variable

Load the SAPI data.

```
dataDir <- "../data/"
sapi <- read.table(paste0(dataDir, "sapi.txt"),
                   header = TRUE,
                   na.strings = "-999")
```

Specify the **lavaan** model syntax for the SAPI extraversion CFA.

```
mod1 <- '
fun   =~ Q77 + Q84 + Q170 + Q196
liked =~ Q44 + Q63 + Q76  + Q98
'
```

Use the `cfa()` function to estimate the model.

```
library(lavaan)
out1 <- cfa(mod1, data = sapi)
```

Example: Marker Variable

```
partSummary(out1, 1:4)

lavaan 0.6-19 ended normally after 30 iterations

  Estimator                                         ML
  Optimization method                           NLMINB
  Number of model parameters                        17

                                                  Used       Total
  Number of observations                           959        1000

Model Test User Model:

  Test statistic                               130.193
  Degrees of freedom                                19
  P-value (Chi-square)                           0.000
```

Example: Marker Variable

```
partSummary(out1, 5:7)

Parameter Estimates:

  Standard errors                             Standard
  Information                                 Expected
  Information saturated (h1) model          Structured

Latent Variables:
                   Estimate  Std.Err  z-value  P(>|z|)
  fun =~
    Q77               1.000
    Q84               0.761    0.051   14.902    0.000
    Q170              0.634    0.047   13.558    0.000
    Q196              0.795    0.046   17.381    0.000
  liked =~
    Q44               1.000
    Q63               1.512    0.147   10.278    0.000
    Q76               1.483    0.149    9.955    0.000
    Q98               1.243    0.119   10.462    0.000
```

Example: Marker Variable

```
partSummary(out1, 8:9)

Covariances:
                   Estimate  Std.Err  z-value  P(>|z|)
  fun ~~
    liked             0.231    0.025    9.234    0.000

Variances:
                   Estimate  Std.Err  z-value  P(>|z|)
   .Q77               0.548    0.038   14.389    0.000
   .Q84               0.727    0.039   18.703    0.000
   .Q170              0.687    0.035   19.572    0.000
   .Q196              0.364    0.025   14.731    0.000
   .Q44               0.662    0.034   19.291    0.000
   .Q63               0.807    0.048   16.943    0.000
   .Q76               0.966    0.054   17.931    0.000
   .Q98               0.469    0.029   16.121    0.000
    fun               0.627    0.056   11.303    0.000
    liked             0.182    0.029    6.290    0.000
```

Example: Marker Variable

```
inspect(out1, "r2")

  Q77   Q84  Q170  Q196   Q44   Q63   Q76   Q98
0.534 0.333 0.268 0.521 0.215 0.340 0.293 0.374

fitMeasures(out1) |> head(22) |> round(3)

             npar              fmin             chisq                df
           17.000             0.068           130.193            19.000
           pvalue    baseline.chisq       baseline.df   baseline.pvalue
            0.000          1574.886            28.000             0.000
              cfi               tli              nnfi               rfi
            0.928             0.894             0.894             0.878
              nfi              pnfi               ifi               rni
            0.917             0.622             0.929             0.928
             logl unrestricted.logl               aic               bic
       -10147.587        -10082.491         20329.175         20411.895
           ntotal              bic2
          959.000         20357.903
```

Example: Marker Variable

```
fitMeasures(out1) |> tail(-22) |> round(3)

               rmsea         rmsea.ci.lower         rmsea.ci.upper
               0.078                  0.066                  0.091
      rmsea.ci.level           rmsea.pvalue         rmsea.close.h0
               0.900                  0.000                  0.050
rmsea.notclose.pvalue    rmsea.notclose.h0                    rmr
               0.421                  0.080                  0.042
          rmr_nomean                   srmr           srmr_bentler
               0.042                  0.043                  0.043
  srmr_bentler_nomean                   crmr            crmr_nomean
               0.043                  0.049                  0.049
          srmr_mplus      srmr_mplus_nomean                  cn_05
               0.043                  0.043                223.037
               cn_01                    gfi                   agfi
             267.582                  0.968                  0.939
                pgfi                    mfi                   ecvi
               0.511                  0.944                  0.171
```

Example: Fixed Factor

We only need to change one option to implement the fixed-factor method.

- The `std.lv = TRUE` option (i.e., standardized latent variables) applies the appropriate constraints.

```
out2 <- cfa(mod1, data = sapi, std.lv = TRUE)
```

Example: Fixed Factor

```
partSummary(out2, 1:4)

lavaan 0.6-19 ended normally after 17 iterations

  Estimator                                         ML
  Optimization method                           NLMINB
  Number of model parameters                        17

                                                  Used       Total
  Number of observations                           959        1000

Model Test User Model:

  Test statistic                               130.193
  Degrees of freedom                                19
  P-value (Chi-square)                           0.000
```

Example: Fixed Factor

```
partSummary(out2, 5:7)

Parameter Estimates:

  Standard errors                             Standard
  Information                                 Expected
  Information saturated (h1) model          Structured

Latent Variables:
                   Estimate  Std.Err  z-value  P(>|z|)
  fun =~
    Q77               0.792    0.035   22.606    0.000
    Q84               0.603    0.035   17.193    0.000
    Q170              0.502    0.033   15.180    0.000
    Q196              0.630    0.028   22.308    0.000
  liked =~
    Q44               0.426    0.034   12.580    0.000
    Q63               0.644    0.040   16.071    0.000
    Q76               0.632    0.043   14.845    0.000
    Q98               0.530    0.031   16.912    0.000
```

Example: Fixed Factor

```
partSummary(out2, 8:9)

Covariances:
                   Estimate  Std.Err  z-value  P(>|z|)
  fun ~~
    liked             0.683    0.033   20.483    0.000

Variances:
                   Estimate  Std.Err  z-value  P(>|z|)
   .Q77               0.548    0.038   14.389    0.000
   .Q84               0.727    0.039   18.703    0.000
   .Q170              0.687    0.035   19.572    0.000
   .Q196              0.364    0.025   14.731    0.000
   .Q44               0.662    0.034   19.291    0.000
   .Q63               0.807    0.048   16.943    0.000
   .Q76               0.966    0.054   17.931    0.000
   .Q98               0.469    0.029   16.121    0.000
    fun               1.000
    liked             1.000
```

Example: Fixed Factor

```
inspect(out2, "r2")

  Q77   Q84  Q170  Q196   Q44   Q63   Q76   Q98
0.534 0.333 0.268 0.521 0.215 0.340 0.293 0.374

fitMeasures(out2) |> head(22) |> round(3)

             npar              fmin             chisq                df
           17.000             0.068           130.193            19.000
           pvalue    baseline.chisq       baseline.df   baseline.pvalue
            0.000          1574.886            28.000             0.000
              cfi               tli              nnfi               rfi
            0.928             0.894             0.894             0.878
              nfi              pnfi               ifi               rni
            0.917             0.622             0.929             0.928
             logl unrestricted.logl               aic               bic
       -10147.587        -10082.491         20329.175         20411.895
           ntotal              bic2
          959.000         20357.903
```

Example: Fixed Factor

```
fitMeasures(out2) |> tail(-22) |> round(3)

                 rmsea          rmsea.ci.lower          rmsea.ci.upper
                 0.078                   0.066                   0.091
       rmsea.ci.level            rmsea.pvalue          rmsea.close.h0
                 0.900                   0.000                   0.050
rmsea.notclose.pvalue      rmsea.notclose.h0                     rmr
                 0.421                   0.080                   0.042
            rmr_nomean                    srmr            srmr_bentler
                 0.042                   0.043                   0.043
   srmr_bentler_nomean                    crmr             crmr_nomean
                 0.043                   0.049                   0.049
            srmr_mplus       srmr_mplus_nomean                   cn_05
                 0.043                   0.043                 223.037
                 cn_01                     gfi                    agfi
               267.582                   0.968                   0.939
                  pgfi                     mfi                    ecvi
                 0.511                   0.944                   0.171
```

Compare Results

```
inspect(out1, "est")$lambda - inspect(out2, "est")$lambda

       fun liked
Q77  0.208 0.000
Q84  0.158 0.000
Q170 0.132 0.000
Q196 0.165 0.000
Q44  0.000 0.574
Q63  0.000 0.868
Q76  0.000 0.851
Q98  0.000 0.713

inspect(out1, "est")$psi - inspect(out2, "est")$psi

         fun  liked
fun   -0.373
liked -0.453 -0.818
```

Compare Results

```
all.equal(fitMeasures(out1), fitMeasures(out2))

[1] TRUE

inspect(out1, "r2") - inspect(out2, "r2")

 Q77  Q84 Q170 Q196  Q44  Q63  Q76  Q98
   0    0    0    0    0    0    0    0

inspect(out1, "est")$theta - inspect(out2, "est")$theta

     Q77 Q84 Q170 Q196 Q44 Q63 Q76 Q98
Q77    0
Q84    0   0
Q170   0   0    0
Q196   0   0    0    0
Q44    0   0    0    0   0
Q63    0   0    0    0   0   0
Q76    0   0    0    0   0   0   0
Q98    0   0    0    0   0   0   0   0
```