

CPSC 406 Assignment 1

Kyle Leeners, 30591119, w0u0b

January 18, 2018

1. Assume $\|\cdot\|_{\frac{1}{2}}$ is a norm. Then it must be the case that $\|u+v\|_{\frac{1}{2}} \leq \|u\|_{\frac{1}{2}} + \|v\|_{\frac{1}{2}}$. Let $u = [1, 0]^T$, $v = [0, 1]^T$.

$$\|u+v\|_{\frac{1}{2}} = (2 \cdot \sqrt{1})^2 = 4$$

$$\|u\|_{\frac{1}{2}} = \|v\|_{\frac{1}{2}} = 1$$

$$\|u+v\|_{\frac{1}{2}} = 4 > \|u\|_{\frac{1}{2}} + \|v\|_{\frac{1}{2}} = 2$$

So our assumption was incorrect and $\|\cdot\|_{\frac{1}{2}}$ cannot be a norm

2. (a) $\|A\| = \sqrt{\lambda_{\max}(A^T A)} = \sqrt{\lambda_{\max}(A A^T)} = \|A^T\|$
(b) $\|A\|_F = \sum_{i=1}^n \sum_{j=1}^n (a_{i,j})^2$. This is the sum of squares of every value of the matrix A which corresponds $Tr(A^T A) = \sum_{i=1}^n \lambda_i(A^T A)$
3. Let $g(v) = \|x\|^2 - 1$. So we wish to maximize $f(x) = x^T A x$ subject to $g(v) = 0$

$$\nabla f = \lambda \nabla g$$

$$\nabla g = 2x$$

$$\begin{aligned} \nabla f &= \nabla x^T A x = \frac{\partial(x^T A x)}{\partial x} + \frac{\partial(A x)^T}{\partial x} \cdot \frac{\partial(x^T A x)}{\partial y} = A x + \frac{\partial(x^T A^T)}{\partial x} \cdot \frac{\partial(y^T x)}{\partial y} = A x + A^T x = 2A x \\ &\Rightarrow 2A x = \lambda 2x \Rightarrow A x = \lambda x \end{aligned}$$

This implies that x must be an eigenvector of A.

$$f(x) = x^T A x = x^T \lambda x = \lambda x^T x = \lambda \|x\|^2 = \lambda$$

Therefore, the max of $f(x)$ subject to $\|x\|^2 = 1$ is equivalent to λ_{\max}

4. (a) Let λ, x be any eigenvector, eigenvalue pair of A, then:

$$A x = \lambda x$$

$$\Rightarrow x^T A x = \lambda x^T x$$

$$\Rightarrow \lambda = \frac{x^T A x}{\|x\|^2} = \frac{x^T B B^T x}{\|x\|^2} = \frac{(B^T x)^T B^T x}{\|x\|^2} = \frac{\|B x\|^2}{\|x\|^2} \geq 0$$

- (b) First we will prove if A is positive definite then B will have full row rank. Since A is positive definite all eigen values are positive. This implies that the matrix A will have full row/column rank (since it is symmetric). Since $A = B B^T$ and $B \in \mathbb{R}^{n \times k}$ B must have full row-rank.

Now proof if B has full row rank, A will be positive definite. We know $\text{rank}(B) = \text{rank}(B^T)$. Furthermore, we know $\text{rank}(A) = \text{rank}(BB^T) = \min\{\text{rank}(B), \text{rank}(B^T)\} = \text{rank}(B)$. Since $B \in \mathbb{R}^{n \times k}$ and is full row rank, $\text{rank}(B) = n$ which implies $\text{rank}(A) = n$. Since $A \in \mathbb{R}^{n \times n}$ A must be full rank and have a nullity of 0. This implies all eigenvalues are greater than 0 so A is positive definite

5. We wish to show $D \succeq 0 \leftrightarrow c - b^T A^{-1} b \geq 0$

If $D \succeq 0 \rightarrow \forall_{z \in \mathbb{R}^{n+1}}, z^T D z \geq 0$

Let $x \in \mathbb{R}^n, y \in \mathbb{R}, z = (x, y)^T$

This implies that $z^T D z = f(x, y) = x^T A x + 2y b^T x + c y^2$

Consider two possibilities

(a) $y = 0 \rightarrow f(x, 0) = x^T A x > 0$ since $A \succ 0 \rightarrow D \succ 0$

(b) $y \neq 0$ then after computing the gradient and Hessian we obtain:

$\nabla f(x, y) = 2Ax + 2yb$ and $\nabla^2 f(x, y) = 2A \succ 0$ since $A \succ 0$

This implies that the stationary point must be a global minimum and the minimizer will be:

$\nabla f(x, y) = 2Ax + 2yb = 0 \rightarrow Ax = -yb \rightarrow x^* = -yA^{-1}b$

Evaluating at the minimum gives:

$$\begin{aligned} f(x^*, y) &= (-yA^{-1}b)^T A (-yA^{-1}b) + 2yb^T (-yA^{-1}b) + cy^2 \\ &= y^2(b^T A^{-1} A A^{-1} b) - 2y^2(b^T A^{-1} b) + y^2 c \\ &= y^2(c - b^T A^{-1} b) \end{aligned}$$

So if $f(x, y) \geq 0 \rightarrow c - b^T A^{-1} b \geq 0$

Assume $(c - b^T A^{-1} b) \geq 0$. From the previous proof, we know $y^2(c - b^T A^{-1} b)$ is the global minimum x^* for $f(x, y) = x^T A x + 2yb^T x + cy^2$. This implies that $\forall_{x, y}, f(x, y) \geq 0$. Recall that $f(x, y) = (x, y) D (x, y)^T = z^T D z$. Since we know $z^T D z \geq 0$ it follows that $D \succeq 0$.

6. (a) $\frac{\partial f(x_1, x_2)}{\partial x_1} = 16x_1(4x_1^2 - x_2)$

$$\frac{\partial f(x_1, x_2)}{\partial x_2} = -2(4x_1^2 - x_2)$$

Roots : $(x_1, 4x_1)$

Since $f(x_1, x_2)$ is squared we know these roots will correspond to the non-strict global minimum which in this case is 0

(b) $\frac{\partial f(x_1, x_2, x_3)}{\partial x_1} = 4x_1^3 - 4x_1$

$$\frac{\partial f(x_1, x_2, x_3)}{\partial x_2} = 2x_2 + 2x_3$$

$$\frac{\partial f(x_1, x_2, x_3)}{\partial x_3} = 2x_2 + 4x_3$$

Roots: $(0, 0, 0), (1, 0, 0), (-1, 0, 0)$

$$\frac{\partial^2 f(x_1, x_2, x_3)}{\partial x_1^2} = 12x_1^2 - 4$$

$$\frac{\partial^2 f(x_1, x_2, x_3)}{\partial x_2^2} = 2$$

$$\frac{\partial^2 f(x_1, x_2, x_3)}{\partial x_3^2} = 2$$

$$\frac{\partial^2 f(x_1, x_2, x_3)}{\partial x_1 x_2} = 0$$

$$\frac{\partial^2 f(x_1, x_2, x_3)}{\partial x_1 x_3} = 0$$

$$\frac{\partial^2 f(x_1, x_2, x_3)}{\partial x_2 x_3} = 2$$

Evaluating the Hessian at the roots gives the following:

Saddle point: $(0, 0, 0)$ Non-strict global minimizers: $(1, 0, 0), (-1, 0, 0)$

(c) $\frac{\partial f(x_1, x_2)}{\partial x_1} = 6x_1 x_2$

$$\frac{\partial f(x_1, x_2)}{\partial x_2} = 6x_2^2 - 12x_2 + 3x_1^2$$

Roots: $(0, 0), (0, 2)$

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_1^2} = 6x_2$$

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_2^2} = 12x_2 - 12$$

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_1 x_2} = 6x_1$$

Evaluating the Hessian at the roots gives the following:

Strict local minimizer: $(0, 2)$ Saddle point: $(0, 0)$

(d) $\frac{\partial f(x_1, x_2)}{\partial x_1} = 4x_1^3 + 4x_1x_2 - 8x_1 - 8$

$$\frac{\partial f(x_1, x_2)}{\partial x_2} = 2x_1^2 + 2x_2 - 8$$

Roots: $(1, 3)$

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_1^2} = 12x_1^2 + 4x_2 - 8$$

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_2^2} = 2$$

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_1 x_2} = 4x_1$$

Evaluating the Hessian at the roots gives the following:

Strict local minimizer: $(1, 3)$

(e) $\frac{\partial f(x_1, x_2)}{\partial x_1} = 4(x_1 - 2x_2)^3 + 64x_2$

$$\frac{\partial f(x_1, x_2)}{\partial x_2} = -8(x_1 - 2x_2)^3 + 64x_1$$

Roots: $(0, 0), (1, -\frac{1}{2}), (-1, \frac{1}{2})$

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_1^2} = 12(x_1 - 2x_2)^2$$

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_2^2} = 48(x_1 - 2x_2)$$

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_1 x_2} = 64 - 24(x_1 - 2x_2)^2$$

Evaluating the Hessian at the roots gives the following:

Saddle point: $(0, 0)$ Non-strict local minimizer: $(1, -\frac{1}{2}), (-1, \frac{1}{2})$

(f) $\frac{\partial f(x_1, x_2)}{\partial x_1} = 4x_1 - 2x_2 + 2$

$$\frac{\partial f(x_1, x_2)}{\partial x_2} = 6x_2 - 2x_1 - 3$$

Roots: $(-\frac{3}{10}, \frac{2}{5})$

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_1^2} = 4$$

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_2^2} = 6$$

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_1 x_2} = -2$$

Evaluating the Hessian at the roots gives the following:

Strict Global minimizer: $(-\frac{3}{10}, \frac{2}{5})$

(g) $\frac{\partial f(x_1, x_2)}{\partial x_1} = 2x_1 + 4x_2 + 1$

$$\frac{\partial f(x_1, x_2)}{\partial x_2} = 4x_1 + 2x_2 - 1$$

Roots: $(\frac{1}{2}, -\frac{1}{2})$

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_1^2} = 2$$

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_2^2} = 2$$

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_1 x_2} = 4$$

Evaluating the Hessian at the roots gives the following:

Saddle point at: $(\frac{1}{2}, -\frac{1}{2})$

7. Assume $b \in \text{Range}(A)$, we wish to prove that $f(x)$ is bounded below

$f(x) = x^T Ax + 2b^T x + c \rightarrow \nabla f(x) = 2Ax + 2bx$. Assume $b \in \text{Range}(A) \rightarrow \exists y, s.t., Ay = b$.
Let $x = -y \rightarrow \nabla f(-y) = -2Ay + 2b = -2b + 2b = 0$. This implies that at $x = -y$ we have a stationary point. Now examining the Hessian we find $\nabla^2 f(x) = 2A \succeq 0$ since $A \succeq 0$. This implies that our stationary point is a global minimum and $f(x)$ is bounded below.

Assume $f(x)$ is bounded below, we need to show $b \in \text{Range}(A)$.

We know from the previous proof that since $A \succeq 0$, $\nabla^2 f(x) = 2A \succeq 0$. This means there exists some global minimizer x^* where $\nabla f(x^*) = 2Ax^* + 2b \rightarrow Ax^* = -b \rightarrow b \in \text{Range}(A)$