

# CPSC 340 Assignment 0

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1. Assume  $\|\cdot\|_{\frac{1}{2}}$  is a norm. Then it must be the case that  $\|u+v\|_{\frac{1}{2}} \leq \|u\|_{\frac{1}{2}} + \|v\|_{\frac{1}{2}}$ . Let  $u = [1, 0]^T, v = [0, 1]^T$ .

$$\begin{aligned}\|u+v\|_{\frac{1}{2}} &= (2 \cdot \sqrt{1})^2 = 4 \\ \|u\|_{\frac{1}{2}} &= \|v\|_{\frac{1}{2}} = 1 \\ \|u+v\|_{\frac{1}{2}} &= 4 > \|u\|_{\frac{1}{2}} + \|v\|_{\frac{1}{2}} = 2\end{aligned}$$

So our assumption was incorrect and  $\|\cdot\|_{\frac{1}{2}}$  cannot be a norm

2. (a)  $\|A\| = \sqrt{\lambda_{max}(A^T A)} = \sqrt{\lambda_{max}(A A^T)} = \|A^T\|$   
 (b)  $\|A\|_F = \sum_{i=1}^n \sum_{j=1}^n (a_{i,j})^2$ . This is the sum of squares of every value of the matrix A which corresponds  $Tr(A^T A) = \sum_{i=1}^n \lambda_i(A^T A)$
3. Let  $g(v) = \|x\|^2 - 1$ . So we wish to maximize  $f(x) = x^T Ax$  subject to  $g(v) = 0$

$$\nabla f = \lambda \nabla g$$

$$\nabla g = 2x$$

$$\begin{aligned}\nabla f = \nabla x^T Ax &= \frac{\partial(x^T Ax)}{\partial x} + \frac{\partial(Ax)^T}{\partial x} \cdot \frac{\partial(x^T Ax)}{\partial y} = Ax + \frac{\partial(x^T A^T)}{\partial x} \cdot \frac{\partial(y^T x)}{\partial y} = Ax + A^T x = 2Ax \\ &\Rightarrow 2Ax = \lambda 2x \Rightarrow Ax = \lambda x\end{aligned}$$

This implies that x must be an eigenvector of A.

$$f(x) = x^T Ax = x^T \lambda x = \lambda x^T x = \lambda \|x\|^2 = \lambda$$

Therefore, the max of  $f(x)$  subject to  $\|x\|^2 = 1$  is equivalent to  $\lambda_{max}$

4. (a) Let  $\lambda, x$  be anyt eigenvector, eigenvalue pair of A, then:

$$\begin{aligned}Ax &= \lambda x \\ \Rightarrow x^T Ax &= \lambda x^T x \\ \Rightarrow \lambda &= \frac{x^T Ax}{\|x\|^2} = \frac{x^T BB^T x}{\|x\|^2} = \frac{(B^T x)^T B^T x}{\|x\|^2} = \frac{\|Bx\|^2}{\|x\|^2} \geq 0\end{aligned}$$

- (b) First we will prove if A is positive definite then B will have full row rank. Since A is positive definite all eigen values are positive. This implies that the matrix A will have full row/column rank (since it is symmetric). Since  $A = BB^T$  and  $B \in \mathbb{R}^{n \times k}$  B must have full row-rank.

Now proof if B has full row rank, A will be positive definite. We know  $\text{rank}(B) = \text{rank}(B^T)$ . Furthermore, we know  $\text{rank}(A) = \text{rank}(BB^T) = \min\{\text{rank}(B), \text{rank}(B^T)\} = \text{rank}(B)$ . Since  $B \in \mathbb{R}^{n \times k}$  and is full row rank,  $\text{rank}(B) = n$  which implies  $\text{rank}(A) = n$ . Since  $A \in \mathbb{R}^{n \times n}$  A must be full rank and have a nullity of 0. This implies all eigenvalues are greater than 0 so A is positive definite

5. We wish to show  $D \succeq 0 \Leftrightarrow c - b^T A^{-1}b \geq 0$

If  $D \succeq 0 \rightarrow \forall z \in \mathbb{R}^{n+1}, z^T D z \geq 0$

Let  $x \in \mathbb{R}^n, y \in \mathbb{R}, z = (x, y)^T$

This implies that  $z^T D z = f(x, y) = x^T Ax + 2yb^t x + cy^2$

Consider two possibilities

$$(a) y = 0 \rightarrow f(x, 0) = x^T Ax > 0 \text{ since } A \succ 0 \rightarrow D \succ 0$$

(b)  $y \neq 0$  then after computing the gradient and Hessian we obtain:

$$\nabla f(x, y) = 2Ax + 2yb \text{ and } \nabla^2 f(x, y) = 2A \succ 0 \text{ since } A \succ 0$$

This implies that the stationary point must be a global minimum and the minimizer will be:

$$\nabla f(x, y) = 2Ax + 2yb = 0 \rightarrow Ax = -yb \rightarrow x^* = -yA^{-1}b$$

Evaluating at the minimum gives:

$$\begin{aligned} f(x^*, y) &= (-yA^{-1}b)^T A(-yA^{-1}b) + 2yb^T(-yA^{-1}b) + cy^2 \\ &= y^2(b^T A^{-1} A A^{-1} b) - 2y^2(b^T A^{-1} b) + y^2 c \\ &= y^2(c - b^T A^{-1} b) \end{aligned}$$

So if  $f(x, y) \geq 0 \rightarrow c - b^T A^{-1} b \geq 0$

Assume  $(c - b^T A^{-1} b) \geq 0$ . From the previous proof, we know  $y^2(c - b^T A^{-1} b)$  is the global minimum  $x^*$  for  $f(x, y) = x^T Ax + 2yb^t x + cy^2$ . This implies that  $\forall_{x,y}, f(x, y) \geq 0$ . Recall that  $f(x, y) = (x, y)D(x, y)^T = z^T D z$ . Since we know  $z^T D z \geq 0$  it follows that  $D \succeq 0$ .

6. (a)  $\frac{\partial f(x_1, x_2)}{\partial x_1} = 16x_1(4x_1^2 - x_2)$   
 $\frac{\partial f(x_1, x_2)}{\partial x_2} = -2(4x_1^2 - x_2)$

Roots :  $(x_1, 4x_1)$

Since  $f(x_1, x_2)$  is squared we know these roots will correspond to the non-strict global minimum which in this case is 0

(b)  $\frac{\partial f(x_1, x_2, x_3)}{\partial x_1} = 4x_1^3 - 4x_1$   
 $\frac{\partial f(x_1, x_2, x_3)}{\partial x_2} = 2x_2 + 2x_3$   
 $\frac{\partial f(x_1, x_2, x_3)}{\partial x_3} = 2x_2 + 4x_3$

Roots:  $(0, 0, 0), (1, 0, 0), (-1, 0, 0)$

$$\frac{\partial^2 f(x_1, x_2, x_3)}{\partial x_1^2} = 12x_1^2 - 4$$

$$\frac{\partial^2 f(x_1, x_2, x_3)}{\partial x_2^2} = 2$$

$$\frac{\partial^2 f(x_1, x_2, x_3)}{\partial x_3^2} = 2$$

$$\frac{\partial^2 f(x_1, x_2, x_3)}{\partial x_1 x_2} = 0$$

$$\frac{\partial^2 f(x_1, x_2, x_3)}{\partial x_1 x_3} = 0$$

$$\frac{\partial^2 f(x_1, x_2, x_3)}{\partial x_2 x_3} = 2$$

Evaluating the Hessian at the roots gives the following:

Saddle point:  $(0, 0, 0)$  Non-strict global minimizers:  $(1, 0, 0), (-1, 0, 0)$

(c)  $\frac{\partial f(x_1, x_2)}{\partial x_1} = 6x_1 x_2$

$$\frac{\partial f(x_1, x_2)}{\partial x_2} = 6x_2^2 - 12x_2 + 3x_1^2$$

Roots:  $(0, 0), (0, 2)$

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_1^2} = 6x_2$$

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_2^2} = 12x_2 - 12$$

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_1 \partial x_2} = 6x_1$$

Evaluating the Hessian at the roots gives the following:

Strict local minimizer:  $(0, 2)$  Saddle point:  $(0, 0)$

$$(d) \quad \frac{\partial f(x_1, x_2)}{\partial x_1} = 4x_1^3 + 4x_1x_2 - 8x_1 - 8$$

$$\frac{\partial f(x_1, x_2)}{\partial x_2} = 2x_1^2 + 2x_2 - 8$$

Roots:  $(1, 3)$

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_1^2} = 12x_1^2 + 4x_2 - 8$$

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_2^2} = 2$$

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_1 \partial x_2} = 4x_1$$

Evaluating the Hessian at the roots gives the following:

Strict global minimizer:  $(1, 3)$

$$(e) \quad \frac{\partial f(x_1, x_2)}{\partial x_1} = 4(x_1 - 2x_2)^3 + 64x_2$$

$$\frac{\partial f(x_1, x_2)}{\partial x_2} = -8(x_1 - 2x_2)^3 + 64x_1$$

Roots:  $(0, 0), (1, -\frac{1}{2}), (-1, \frac{1}{2})$

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_1^2} = 12(x_1 - 2x_2)^2$$

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_2^2} = 48(x_1 - 2x_2)$$

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_1 \partial x_2} = 64 - 24(x_1 - 2x_2)^2$$

Evaluating the Hessian at the roots gives the following:

Saddle point:  $(0, 0)$  Non-strict global minimizer:  $(1, -\frac{1}{2}), (-1, \frac{1}{2})$

$$(f) \quad \frac{\partial f(x_1, x_2)}{\partial x_1} = 4x_1 - 2x_2 + 2$$

$$\frac{\partial f(x_1, x_2)}{\partial x_2} = 6x_2 - 2x_1 - 3$$

Roots:  $(-\frac{3}{10}, \frac{2}{5})$

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_1^2} = 4$$

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_2^2} = 6$$

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_1 \partial x_2} = -2$$

Evaluating the Hessian at the roots gives the following:

Strict Global minimizer:  $(-\frac{3}{10}, \frac{2}{5})$

$$(g) \quad \frac{\partial f(x_1, x_2)}{\partial x_1} = 2x_1 + 4x_2 + 1$$

$$\frac{\partial f(x_1, x_2)}{\partial x_2} = 4x_1 + 2x_2 - 1$$

Roots:  $(\frac{1}{2}, -\frac{1}{2})$

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_1^2} = 2$$

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_2^2} = 2$$

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_1 \partial x_2} = 4$$

Evaluating the Hessian at the roots gives the following:

Saddle point at:  $(\frac{1}{2}, -\frac{1}{2})$

7. Assume  $b \in Range(A)$ , we wish to prove that  $f(x)$  is bounded below

$f(x) = x^T Ax + 2b^T x + c \rightarrow \nabla f(x) = 2Ax + 2bx$ . Assume  $b \in Range(A) \rightarrow \exists_y, s.t., Ay = b$ . Let  $x = -y \rightarrow \nabla f(-y) = -2Ay + 2b = -2b + 2b = 0$ . This implies that at  $x = -y$  we have a stationary point. Now examining the Hessian we find  $\nabla^2 f(x) = 2A \succeq 0$  since  $A \succeq 0$ . This implies that our stationary point is a global minimum and  $f(x)$  is bounded below.

Assume  $f(x)$  is bounded below, we need to show  $b \in Range(A)$ .

We know from the previous proof that since  $A \succeq 0$ ,  $\nabla^2 f(x) = 2A \succeq 0$ . This means there exists some global minimizer  $x^*$  where  $\nabla f(x^*) = 2Ax^* + 2b \rightarrow Ax^* = -b \rightarrow b \in Range(A)$