Functions of a Complex Variable

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PREFACE

The object of these Lectures is to give the student who has completed a course in Advanced Calculus, an introduction to the most important methods and results of Modern Analysis. The Cauchy theory, Weierstrass's analytic methods and results, Riemann's geometric and physical approach—this is the material out of which a systematic and harmonious treatment of this great branch of mathematics has grown. It was the Author's privilege to lecture on this subject for many years at Harvard University and, together with his colleagues there, to strive to make the treatment effective for the student of this First Course in Higher Analysis.

I have chosen these words advisedly, for my colleague from after, reading them, says:—"But the theory of functions of real variables is the more elementary subject, and is needed for any adequate appreciation of the theory in the complex domain.". True; a part of that theory is needed, and is provided for in these Lectures. But it must be borne in mind that the student is only just emerging from the Calculus, and it is reasonable to give him first what he can most readily receive. The theory of functions of real variables, if carried beyond its rudiments, soon loses contact, for the beginner, with the broader fields of analysis, geometry, and physics. It is these contacts, this broader knowledge, with which the beginner should become familiar before he specializes too closely in that great field, while for the student of Physics such specialization does not, at least at the present stage, come into consideration. And so I have restricted myself to those concepts and methods of that theory which are actually used in the present subject. The basal definitions and theorems are given in detail in the text; but a small amount of supplementary study in real analysis is suggested by specific references to the Author's *Real Variables*. (*Functions of Real Variables*, The University Press, The National University of Peking, 1936; referred to in the following pages as *Real Variables*.)

Sufficient bibliographical references are included, to provide the student with the requisite historical background, and frequent references to the Author's *Funktionentheorie* enable the reader to pursue a subject further. (*Lehrbuch der Funktionentheorie*, vol. I, 5. ed. 1928, Theubner, Leipzig; referred to in the following pages as *Funktionentheorie* I.)

After the rudiments of the subject have been treated, in Chapters I—VII, there is a wide choice of the closing topic. It might well be an application to the elliptic functions—and indeed this is the choice which the Author made in the *Funktionentheorie*. Or, again, the linear total differential equations of the second order, with special reference to those which occur in Mathematical Physics, would be a highly appropriate subject. A large part of the treatment of differential equations in the *Real Variables*, Chap. XII, can be carried over at once into the complex domain, and this, too, is a useful exercise for the student. Broader than any of these, however, is the Theory of the Potential Function, for in the hands of Riemann it yielded the fundamental theorem in conformal mapping, it opened up a new field in Algebraic Geometry, and it led to the automorphic functions. Furthermore, a study of the rudiments of the Logarithmic Potential forms an excellent introduction to the study of the Newtonian Potential Function.

To my colleague in Mathematics, Professor Kiang Tsai-Han, and to Dean Van Tsee-Chong, for their help in making the publication of these Lectures possible, I wish to express my hearty thanks. The services of my efficient Assistant, Mr. Sun Shu-Peng, in helping to prepare the manuscript for press and to carry about the typographical corrections, have been of the greatest value to me; I fell deep gratitude to him and the genuine interest he has uniformly shown in all these important details. Finally, my warm appreciation of the cooperation of the University Press in all that goes into the making of this book.

The National University of Peking
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TYPESETTER COMMENTS

I purchased this book at a small bookstore in Ottawa while on a short vacation from college. Initially, it was to read for my own knowledge and benefit. After reading the first few sections of the book, I was very engaged with Osgood's explanations and approach to complex variables (something I was rather unfamiliar with at the time). I attempted to locate copies of this book online and came up short. There appears to only be copies in libraries (or bookstores), and even that is limited; certainly there is no online or pdf version.

Therefore, I've decided to type this text into LaTeX to the best of my ability. Being very comfortable with LaTeX already, but no expert in it (or Tikz), I decided that the purpose would be twofold: to learn how to type a book in LaTeX, complete with drawings in Tikz, and to make the textbook more available to second or third year undergraduates. With the paper in my copy becoming brittle and some of the text fading away, I wanted to preserve Osgood's work.

There are a few modifications I made. Almost all of the text is verbatim from the book, with the majority of the changes being in the style of mathematics. For instance, $\lim\limits_{n=\infty}$ became $\lim\limits_{n\to\infty}$ and z in S became $z \in S$ to fit a modern "syntax" of mathematics.

I have decided to make the source code available to the public, which can be found on my GitHub.

CONTENTS

CHAPTER 1

COMPLEX NUMBERS

1.1 Introduction

Mathematicians did not begin by defining numbers: they worked with them. The consciousness of the number system grew, and at the beginning of the seventeenth century—the most important century up to that time in the history of human thought—algebra, as we know it in the requirements for admission to college, was completed. True, it was very far from complete when we look at the definition of number. But the formal processes were recognized, and science could go on. It is hard to imagine what the Greek mind might not have accomplished if Euclid and Archimedes could have passed the entrance examinations for Pei ta in Algebra[1].

Nor did mathematicians worry much over the foundations in algebra. The formal processes were enough for the next step:

$$\begin{gathered} A+B=B+A \\ A+(B+C)=(A+B)+C \\ AB=BA \\ A(BC)=(AB)C \\ A(B+C)=AB+AC \end{gathered}$$

But then came the question of solving equations. The equation

$$ax=b, \quad a \neq 0,$$

could always be solved. The quadratic

$$ax^2+bc+c=0$$

[1]For an interesting account of the beginnings of arithmetic and algebra cf. David Eugene Smith: *T*he Teaching and History of Elementary Mathematics. A systematic development of the number system, so far as real numbers are concerned, is found in Osgood: *F*unctions of Real Variables, Chap. II.

admitted the formal solution

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

But what if $b^2 - 4ac$ is negative? The algebrists of the sixteenth and seventeenth century took the hurdle.

$$\begin{aligned} x^2 + 1 &= 0 \\ x &= \pm\sqrt{-1} \end{aligned}$$

They did not define $\sqrt{-1}$; they treated it like any other letter of algebra, replacing its square, however, by -1:

$$a + b\sqrt{-1} + c + d\sqrt{-1} = (a + c) + (b + d)\sqrt{-1} \tag{1.1}$$

$$\begin{aligned} (a + b\sqrt{-1})(c + d\sqrt{-1}) &= ac + ad\sqrt{-1} + bc\sqrt{-1} + bd\left(\sqrt{-1}\right)^2 \\ &= ac - bd + (ad + bc)\sqrt{-1} \end{aligned} \tag{1.2}$$

The results were of far reaching importance. Algebraic geometry came into being. An ellipse may and may not be cut by a line its plane. It is not an exception for a line to fail to meet the curve. As many lines fail to meet it as meet it, so to speak. But when imaginaries are introduced, and the plane is suitably extended, *every* right line cuts the ellipse, and there are in general two points of intersection.

In algebra, the Fundamental Theorem emerged, namely, the fact that every algebraic equation has a root.

The modern developments of Arithmetic are more concerned with numbers which are not real, than with those which are.

But it was in Analysis that the most spectacular results were achieved. The mathematicians of Euler's time had observed that when $\sqrt{-1}$ is introduced, the trigonometric functions on the one hand, and the logarithm and the exponential on the other, unite to form one family:

$$\sin\varphi = \frac{e^{\varphi i} - e^{-\varphi i}}{2i} \tag{1.3}$$

$$\cos\varphi = \frac{e^{\varphi i} + e^{-\varphi i}}{2} \tag{1.4}$$

where we write with Euler:

$$i = \sqrt{-1}.$$

And again:

$$\tan^{-1} x = \frac{i}{2} \log \frac{i+x}{i-x}.$$

If we set

$$z = x + iy$$

we find:

$$e^x = e^x(\cos y + i \sin y).$$

The student will do well at this point to turn to the Author's Advanced Calculus and study again Chap. XX.

One further illustration of the formal use of imaginaries before leaving this part of the subject. The linear differential equation with constant coefficients:

$$\frac{d^2y}{dx^2} + 2a\frac{dy}{dx} + by = 0 \tag{1.5}$$

can be solved by setting

$$y = e^{mx}$$

and then determining m from the equation:

$$m^2 + 2am + b = 0 \tag{1.6}$$

If m_1 and m_2 are roots of this equation, then

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

is the general solution of 1.5. But what if the roots of 1.6 are imaginary?

Setting

$$c = \sqrt{b - a^2} \tag{1.7}$$

we find two new solutions:

$$y = e^{-ax+cix}, \qquad y_2 = e^{-ax-cix},$$

both of these are imaginary, and so useless for any practical purposes.

Now, the sum of any two solutions of 1.5 is also a solution. If, then, we write:

$$\begin{aligned} y_1 &= e^{-ax}\cos cx + ie^{-ax}\sin cx \\ y_2 &= e^{-ax}\cos cx - ie^{-ax}\sin cx \end{aligned}$$

we find:

$$Y_1 = y_1 + y_2 = 2e^{-ax}\cos cx.$$

And similarly

$$Y_2 = y_1 - y_2 = 2ie^{-ax}\sin cx.$$

Moreover, the product of 1.5 by a constant is also a solution, and so we are led to the two solutions:

$$u_1 = e^{-ax}\cos cx, \qquad u_2 = e^{-ax}\sin cx \tag{1.8}$$

All this is seventeenth and eighteenth century mathematics, and can make no claim to rigor. There is no foundation for it to rest on. It is crass formalism. Nevertheless, this formal work has produced a concrete result. Here are two real functions which have emerged. Now the test of whether a given function is a solution of a differential equation is not the process whereby the function was obtained — we may have found it in the street. The test is: Does it satisfy the differential equation? Let us see.

$$\begin{aligned} u_1 &= e^{-ax}\cos cx \\ \frac{du_1}{dx} &= -ae^{-ax}\cos cx - ce^{-ax}\sin cx \\ \frac{d^2u_1}{dx^2} &= \left(a^2 - c^2\right)e^{-ax}\cos cx + 2ace^{-ax}\sin cx \end{aligned}$$

On multiplying the first of these equations by b, the second by $2a$ and adding, remembering 1.7, the right—hand side reduces identically to 0. Hence u_1 is a solution, in spite of the shady methods by which it was obtained. And similarly for u_2.

This example illustrates one side of the eighteenth century use of imaginaries. A large number of definite integrals, like

$$\int_0^\infty \frac{\sin x}{x}\, dx$$

were evaluated in this manner. In fact, the attempt to put his latter class of formal developments on a firm basis led Cauchy, in one of his earliest papers, in 1814, to lay the foundations of the Theory of Functions.

Two other topics of eighteenth century mathematics should be mentioned before we leave this brief sketch of the main ideas which led up to the modern Theory of Functions of a Complex Variable. First, the problem of *cartography*, or map making. This is the question of transforming one curved surface on another in such a manner that small figures on the one surface will go over in approximately similar figures of the other surface. A necessary condition is, that angles be preserved; i.e, the angles between two intersecting curves on the one surface and the angle between their images on the other surface, must be equal. Conversely, this condition is sufficient. Such a transformation is called *isogonal*, and the map is called *conformal*. There are two principal maps of the earth, which are used in geography, namely, Mercator's Chart and Stenographic Projection; cf. Chaps. II, §10 and III, §6.

In particular, the surfaces may be planes. If a system of Cartesian coordinates (x, y) be chosen in the one plane, and a suitable Cartesian system (u, v) in the other, then the condition for a conformal mapping is that the following differential equations be true:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{1.9}$$

These are known as the *Cauchy—Riemann Differential Equations*. They have come to us as the definition of conformal mapping. They appeared, however, still earlier, in a paper of Clairaut of the year 1743, on the Figure of the Earth, in his study of a two—dimensional flow of an incompressible fluid; cf. Chap II, §4.

Thus even more fundamental than Geometry in the development of Analysis has been the science of Mechanics and Mathematical Physics. Nor was it in the study of hydromechanics alone. The first half of the nineteenth century brought a tremendously stimulating contribution to mathematical physics in the investigation of Fourier on the flow of heat in conducting substances. The flow of electricity in conductors obeys parallel laws, and the two problems are mathematically identical. Thus the work of Fourier helped to pave the way for Maxwell in his study of electricity and magnetism, in the second half of the last century.

When the flow is two—dimensional and is steady, the temperature u satisfies *Laplace's Equation*:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1.10}$$

So important did Maxwell consider this case that he published accurately drawn figures illustrating particular cases of flow, in his great work on Electricity and Magnetism. These and similar examples have proved useful to electrical engineers, and hold a permanent in this applied science.

By an *analytic function of a complex variable*:

$$w = f(z),$$

where

$$z = x + yi, \qquad w = u + vi$$

is meant a function which has a derivative; i.e,

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

shall exist, no matter how Δz approaches 0. A necessary condition for this limit to exist is, that u and v satisfy the Cauchy—Riemann Differential Equations 1.9, the partial derivatives being continuous functions.

This condition is conversely sufficient: If u and v are two real functions which have continuous first partial derivatives in a two—dimensional region S of the (x, y) plane, and if the Cauchy—Riemann Differential Equations are satisfied in S, then the complex function

$$w = u + vi = f(z)$$

will have a derivative at each point of S. Moreover, the functions u and v will then possess derivatives of all orders, and u will satisfy Laplace's Equation 1.10, as is seen at once by eliminating v.

Thus the real part of an analytic function of a complex variable satisfies Laplace's Equation. Conversely, every every solution u of Laplace's Equation leads to a complex conjugate function v and the two together satisfy the Cauchy—Riemann Differential Equations. So the Theory of Functions of a Complex Variables is in principle coextensive with the theory of Laplace's Equation.

It is of historical interest to trace the origins of the Theory of Functions of a Complex Variable to the great branches of Mathematical Physics and of Geometry. It is no less important for a sure sense of scientific values, to recognize how deep these roots go down. Not the theory of unctions alone, but the mathematical physics of the future will take its origin in these same sources. We are studying something universal when we make these theories a part of our mental equipment and our habits of thought.

1.2 The System of Complex Numbers

The system of real numbers having been developed according to the method of Dedekind or Cantor, or in some other way, a new class of objects is defined by the mark (a, b) where a and b are any two real numbers.

Equality: Two of these numbers,

$$A = (a, b), \quad B = (c, d)$$

shall be said to be equal if

$$a = c,\, b = d.$$

We write:

$$A = B.$$

If $A = B$, then $B = A$. Moreover, if $A = B$ and $B = C$, then $A = C$.

The relation of $<$ and $>$ are not defined. But the fact that A and B are not equal may be expressed in the form:

$$A \neq B.$$

By a *combination* of two of these numbers is meant a rule, whereby a third number is determined. These combinations lead eventually to *addition* and *multiplication*; but we will not speak of them as such for the present, since these words have connotations which would confuse the main ideas. We will think of a combination rather as a function of two independent variables:

$$C = f(A, B);$$

a very simple function, which we proceed to define.

Addition: Let

$$A = (a, b), \quad B = (c, d)$$

be any two numbers of the system. By the *first combination* is meant the mark:

$$A \oplus B.$$

To it is attached the value:

$$C = A \oplus B$$

where

$$C = (a + c, b + d).$$

Thus

$$(a, b) \oplus (c, d) = (a + c, b + d). \tag{1.11}$$

The first combination obeys the Commutative Law:

$$A \oplus B = B \oplus A.$$

It also obeys the Associative Law:

$$A \oplus (B \oplus C) = (A \oplus B) \oplus C.$$

Subtraction: The first combination admits an inverse defined as follows. Let $A = (a, b)$ and $B = (c, d)$ be any two numbers of the system, and let it be required to determine a number $X = (x, y)$ such that

$$A \oplus X = B.$$

the solution is given by the equations:

$$a + x = c, \quad b + y = d.$$

Thus one and only one number X exist, satisfying the condition. We write:

$$X = B \ominus A.$$

The number

$$A_0 = (0, 0)$$

has the property that

$$A \oplus A_0 = A_0 \oplus A = A,$$

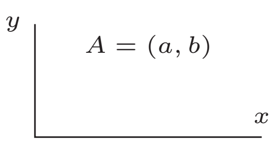
no matter what number A may be. Moreover, it is unique.

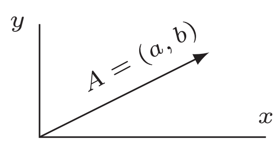
Vectors: The number $A = (a, b)$ admits *two* important geometrical interpretations:

i. It can be represented by the point of the plane of coordinate geometry, whose coordinates are (a, b).

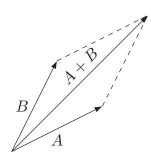
ii. It can be represented by the *vector* whose initial point is the origin of coordinates, and whose terminal point is (a, b): or by any other equal vector.

The representation of complex numbers by the points of a plane, or by vectors in a plane, came into mathematics through the memoire of Argand, Paris, 1806. The method was known to Gauss and is contained implicitly in his doctoral thesis of 1799. The earliest publication is that of Gaspar Wessel, 1797/99. cf. *Funktionentheorie* I, p. 225.





If we use the second interpretation, then the first combination can be interpreted as *vector addition*. The number C is the vector obtained by the parallelogram law; i.e, it is the *vector sum* of A and B.



The term “first combination” has now fulfilled its purpose. Henceforth we shall replace it by addition, meaning thereby precisely what has just been defined as the first combination, and write:

$$C = A + B.$$

Similarly for subtraction. We define

$$B - A \quad \text{as} \quad B \ominus A;$$

i.e, the number which, added to A, gives B:

$$A + (B - A) = B;$$

or

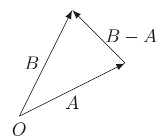
$$X = B - A$$

if

$$A + X = B.$$

It is now easy to interpret the difference $B - A$ geometrically.

Plot the vectors A and B with the same initial point O. Then $B - A$ is the vector whose initial point is the terminal point of A, and whose terminal point is the terminal point of B.



The *negative* of a number $A = (a, b)$ is defined as the number $(-a, -b)$, and is written $-A$:

$$A = (a, b), \quad -A = (-a, -b).$$

We have here the two meanings of the minus sign discussed in the *Real Variables*, p. 42.

Multiplication: The second combination is defined to correspond to the formal law of multiplication of imaginaries 1.2.

Let

$$A = (a, b), \quad B = (c, d)$$

be any two numbers of the system. By the *second combination* is mean the mark:

$$A \otimes B.$$

To it is attached the value

$$C = A \otimes B,$$

where

$$C = (ac - bd, ad + bc).$$

Thus

$$(a, b) \otimes (c, d) = (ac - bd, ad + bc).$$

The second combination obviously obeys the Commutative Law:

$$A \otimes B = B \otimes A.$$

It also obeys the Associative Law:

$$A \otimes (B \otimes C) = (A \otimes B) \otimes C.$$

The Distributive Law combines both the first combination and the second combination:

$$A \otimes (B + C) = (A \otimes B) + (A \otimes C).$$

That it is true, is seen by direct calculation. The details are left to the student.

Division: The second combination admits an inverse defined as follows. Let $A = (a, b)$ and $B = (c, d)$ be any two numbers of the system. It is required to determine a number $X = (x, y)$ such that

$$A \otimes X = B.$$

This is always possible, and the result is unique, when

$$A \neq A_0.$$

For, the equation is equivalent to the pair of equations:

$$\begin{aligned} ax - by &= c \\ bx + ay &= d \end{aligned}$$

The determinant of these equations has the value:

$$a^2 + b^2.$$

Hence the truth of the statement.

When $A = A_0$, there is never a unique solution. If $B \neq A_0$, there is no number, X, which will satisfy the equation. If, on the other hand, $B = A_0$, the equation is satisfied by every number, X.

Idemfactor: The number

$$A_1 = (1, 0)$$

has the property that

$$A_1 \otimes A = A \otimes A_1 = A,$$

no matter what number of the system A may be. It is called the *idemfactor* and is unique. It is the analogue of A_0 in the case of Addition.

If $A \neq A_0$, and if A' is determined by the equation:

$$A \otimes A' = A_1,$$

then A' is called the *reciprocal* of A. It is the analogue of the negative of A in the case of Addition.

The term "second combination" has now fulfilled its purpose. Henceforth we shall replace it by *multiplication*, meaning thereby precisely what has just been defined as the second combination, and write:

$$A \otimes B \quad \text{as} \quad AB.$$

Division corresponds to the equation

$$AX = B, \quad A \neq A_0.$$

We write:

$$X = \frac{B}{A}.$$

Relation of the Present Number System to the System of Real Numbers: Any number (a, b) of the present system can be written in the form:

$$(a, b) = (a, 0) + (0, b).$$

Thus the function of two variables has been broken up into the sum of two functions, each of a single variable.

The sub-class consisting of the numbers $(a, 0)$ constitute essentially a system of real numbers. For if we associate the number $(a, 0)$ with the real number a:

$$(a, 0) \sim a,$$

not only will a one—to—one relation between the members of the above subclass and those of the system of real numbers be set up, but a holohedric isomorphism with reference to the four species will also result. For, from

$$(a,0) \sim a \quad \text{and} \quad (b,0) \sim b$$

follows that

$$(a,0)+(b,0) \sim a+b$$
$$(a,0)-(b,0) \sim a-b$$

and furthermore:

$$(a,0)(b,0) \sim ab$$
$$(b,0)(a,0) \sim \frac{b}{a} \quad a \neq 0$$

It is, therefore, henceforth immaterial whether we write $(a,0)$ or a in any expressions made up of numbers of the system. In particular,

$$(0,b) = (b,0)(0,1)$$

and so we can write:

$$(0,b) = b(0,1).$$

Hence the number (a,b) of the system can be written in the form:

$$(a,b) = a + b(0,1).$$

The Number: $(0,1) = i$. The number $(0,1)$ has the property that

$$(0,1)(0,1) = (-1,0) = -1.$$

Denote this number by i. And now we see that we have a number system in which the algebraic equation:

$$x^2 + 1 = 0$$

has a solution:

$$x = i, \ -i.$$

Thus the number system which we have defined turns out to be identical with the *System of Ordinary Complex Numbers*:

$$a + bi, \quad i = \sqrt{-1}.$$

It remains to mention explicitly a fundamental property of this arithmetic. A product AB vanishes:

$$AB = 0,$$

when and only when one of the factors vanishes:

$$A = 0 \quad \text{or} \quad B = 0.$$

Of course, both may vanish.

We see how easily and naturally this system of numbers was evolved, to meet the formal requirement of the product:

$$(a + b\sqrt{-1})(c + d\sqrt{-1}) = ac - bd + (ad + bc)\sqrt{-1}.$$

It is a system of number—pairs: pairs of real numbers, a and b, just as the fractions were evolved from the natural numbers by forming number—pairs (m, n); and again the negative numbers by forming pairs of negative numbers. There is nothing mystical or imaginary about the process. We see, too, how much simpler is the definition of $\sqrt{-1}$ than was the definition of $\sqrt{2}$. It was the definition of irrationals which was the most remote in all the extensions of the number system.

The definition of ordinary complex numbers as pairs of real numbers is due to Sir William Rowan Hamilton, the inventor of quatrains, and dates from 1837.

1.3 Formulas

A complex number, $z = x+yi$, can be expressed in polar coordinates by the formula:

$$z = r(\cos\varphi + i\sin\varphi).$$

The number r is called the *absolute value* of z and is written as $|z|$:

$$|z| = \sqrt{x^2 + y^2}.$$

The number φ has been called the "amplitude" and the "argument" of z. Neither term is satisfactory, nor has it been generally adopted. The term *angle* would seem to be the simplest and most suggestive name for φ, write:

$$\varphi = \operatorname{arc} z$$

(read: "angle of z"). For a given $z \neq 0$, there are an infinite number of determinations of φ, differing from one another by multiples of 2π. These are comprised, it is true, among the determinations of $\tan^{-1} y/x$, but are not coextensive with them. Thus

$$\operatorname{arc} a = 2n\pi, \quad n = 0, \pm 1, \ldots,$$

where $z = a$ is a positive real number. But

$$\tan^{-1} 0 = n\pi, \quad n = 0, \pm 1, \ldots.$$

The term *pure imaginary* is applied to a number of the form ci, where c is real. When

$$z = x + yi$$

is represented by a point of the plane, the coordinate axes are called the *axis of reals* and the *axis of pure imaginaries*. The circle

$$x^2 + y^2 = 1, \qquad \text{or} \qquad z = \cos\varphi + i\sin\varphi$$

is known as the *unit circle*. It is the locus of points for which $|z| = 1$.

The *conjugate* of $x + yi$ is the number $x - yi$, and is frequently denoted by $\overline{z}$:

$$z = x - yi.$$

The real part of z is denoted by $\Re(z)$:

$$x = \Re(Z), \quad y = \Re\left(\frac{z}{i}\right).$$

Geometrical Interpretation of Product and Quotient: Let

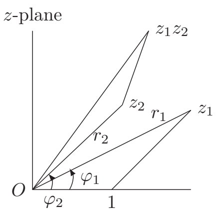
$$\begin{aligned} z_1 &= r_1(\cos\varphi_1 + i\sin\varphi_1) \\ z_2 &= r_2(\cos\varphi_2 + i\sin\varphi_2) \end{aligned}$$

Then the product $z_1 z_2$ is given by the formula:

$$z_1 z_2 = r_1 r_2[\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2)],$$

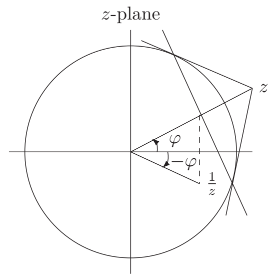
as is shown by trigonometry.

The result admits a simple geometrical interpretation. Construct the triangle whose vertices lie at the points $0, 1, z_1$; and likewise the triangle whose vertices are at $0, z_2, z_1, z_2$. These triangle are similar. For the angles at 0 are equal, and the including sides are proportional.



Conversely, then: Given the numbers z_1 and z_2, construct the first triangle. Then draw a second triangle similar to the first, with vertex at 0 and the side $\overline{0z_2}$ corresponding to the side $\overline{01}$. The third vertex will be at the point z_1z_2.

Thus we have a geometrical construction for a product, just as we had a geometrical construction for a sum. In one respect, however, there is a difference. The geometric sum of two vectors was independent of the unit of length and the choice of axes; the geometric construction for the product of two vectors, however, is impossible until the unit of length and the axes have been determined.



Reciprocals: There is an elegant construction for the reciprocal of a complex number, z. Suppose the point representing z lies outside the unit circle. Draw the tangents, and the chord determined by them. The distance of this chord from O represents the absolute value of the reciprocal of z; and the angle of the reciprocal is the negative of the angle of z. The construction in the other cases can be left to the reader.

Tensors and Rotors: The equation

$$z = r(\cos\varphi + i\sin\varphi) = re^{\varphi i}$$

may be interpreted in an operational sense. To obtain the vector z we may start with the vector $z = 1$, the initial point being the origin, and then stretch it in the ratio of $r : 1$. We thus obtain a vector of length r, lying along the positive axis of reals. Next, rotate this vector through the angle φ. It now comes into coincidence with the vector z we wished generate.

In the above equation, then, the first factor, r, may be thought of as a *stretching factor*, or *tensor*. The second factor, $e^{\varphi i}$, *rotates* the vector to which it is applied, and so is called a *rotor*.

The interpretation can be extended to any product,

$$AB.$$

The effect of multiplying B by A is to *stretch* the vector B in the ratio of $|A| : 1$, and then *rotate* the vector thus obtained through the angle $\varphi = \operatorname{arc} A$:

$$AB = e^{\varphi i}|A|B,$$

the order being:

$$|A|B; \quad \text{then} \quad e^{\varphi i}(|A|B).$$

The order may be reversed:

$$AB = |A|e^{\varphi i}B.$$

Powers and Roots: If

$$z = r(\cos\varphi + i\sin\varphi) = re^{\varphi i},$$

then

$$\begin{aligned} z^2 &= r^2(\cos 2\varphi + i\sin 2\varphi) = r^2e^{2\varphi i} \\ z^3 &= r^3(\cos 3\varphi + i\sin 3\varphi) = r^3e^{3\varphi i} \\ &\vdots \\ z^n &= r^n(\cos n\varphi + i\sin n\varphi) = r^ne^{n\varphi i} \end{aligned}$$

It is now easy to solve the equation:

$$z^n = A.$$

Let

$$A = \mathcal{A}e^{\alpha i},$$

where $\mathcal{A}$ is real and $\mathcal{A} > 0$. Then

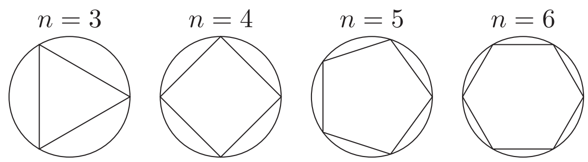
$$z_k = \mathcal{A}^{\frac{1}{n}} e^{\frac{\alpha + 2k\pi}{n} i}, \quad k = 0, 1, \ldots, n-1.$$

These points lie on a circle with radius $\mathcal{A}^{\frac{1}{n}}$, with its centre at the origin, and form the vertices of a regular inscribed polygon of n sides, the angle of one root being α/n.

Roots of Unity: In particular, the roots of unity are given by the equation:

$$x^n = 1.$$

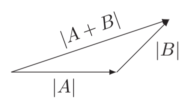
They form the vertices of a regular polygon of n sides inscribed in the unit circle, one vertex being at the point $z = 1$.



Inequalities: The inequality of greatest importance is the following:

$$|A + B| \leqq |A| + |B| \tag{1.12}$$

Geometrically it is once obvious. It says that the length of one side of a triangle is less than the sum of the other two. But if the vectors A and B are collinear and have the same sense, then the lower sign holds. If they have the opposite sense, the upper sign holds.[2]



From 1.12 it follows immediately that

$$|A_1 + \cdots + A_n| \leqq |A_1| + \cdots + |A_n|. \tag{1.13}$$

A further inequality is this:

$$|\{|A| - |B|\}| \leqq |A + B|.$$

It can be deduced from 1.12 by writing

$$|A + C| \leqq |A| + |C|$$

and then setting

$$A + C = -B.$$

[2]For an arithmetic proof cf. *Funktionentherie* I, p. 221.

Thus
$$|B| - |A| \leqq |A + B|.$$
In this last inequality, interchange A and B:
$$|A| - |B| \leqq |A + B|.$$
Hence the theorem.

Exercises

1. Show that the product of any number by its conjugate is equal to the square of its absolute value.

2. Let
$$G(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$$
be a polynomial with real coefficients. Show that the conjugate of $G(z)$ is equal to $G(\overline{z})$.

3. Let
$$R(x) = \frac{f(x)}{\varphi(x)}$$
be a rational function, the polynomials $f(x)$, $\varphi(x)$ having real coefficients. Show that the conjugate of $R(z)$ is $R(\overline{z})$.

4. If the linear function
$$\frac{az + b}{cz + d}, \quad ad - bc \neq 0,$$
takes on real values for three distinct real values of z, show that the coefficients can all be taken as real numbers.

Is it correct to say: "The coefficients are all real"?

5. The function
$$w = \log z$$
is defined by the equation:
$$z = e^w.$$

Show that
$$\log z = \log r + \varphi i,$$
where
$$z = r(\cos\varphi + i\sin\varphi).$$

6. Show that
$$\log(-1) = (2n+1)\pi i, \quad n = 0, \pm 1, \ldots.$$

7. Plot the points
$$\log(-5-12i).$$

8. Show that
$$\tan^{-1} z = \frac{i}{2}\log\frac{i+z}{i-z}.$$

9. Show that
$$\cos^{-1} x = i\log\left(x + \sqrt{x^2-1}\right),$$
and obtain a similar formula for $\sin^{-1} x$.

10. Compute:
$$\sin^{-1} 2; \quad \cos\frac{i}{2}; \quad \tan^{-1} 2i.$$

11. Solve the equation:
$$x^n = -1.$$
Plot the roots for the cases $n = 2, 3, 4, 5, 6$.

12. Solve the equation
$$x^3 = -1$$
by algebraic methods, and show that the results agree with those of Question 11.

13. Solve the equation:

$$x^2 + 2ix - 5 = 0,$$

and plot the roots.

14. The same for

$$x^2 + (4 - 2i)x + (1 - i) = 0.$$

15. Find all the roots of the equation:

$$x^5 = 15,$$

and plot them.

16. Prove the addition theorem:

$$e^{u+v} = e^u e^v,$$

for complex values of the arguments.

Suggestion: write

$$u = u_1 + iu_2, \quad v = v_1 + iv_2.$$

17. Prove the addition theorem:

$$\begin{aligned} \sin(u + v) &= \sin u \cos v + \cos u \sin v \\ \cos(u + v) &= \cos u \cos v - \sin u \sin v \end{aligned}$$

for complex values of the arguments.

18. State accurately under what conditions the equation

$$\log u + \log v = \log uv$$

is true for complex values of the arguments, and prove your statement.

19. If

$$|A| < h \quad \text{and} \quad |B| < h,$$

show that

$$|A+B| < 2h,$$

and, generally,

$$|\pm A \pm B| < 2h,$$

where all four possible combinations of the $\pm$ sign are admitted.

20. Prove that

$$\frac{1}{\sqrt{2}}[|A|+|B|] \leqq |a+bi|,$$

where a and b are real numbers.

21. If

$$|A-B| < \varepsilon \quad \text{and} \quad |B-C| < \varepsilon,$$

show that

$$|A-C| < 2\varepsilon.$$

22. If $|z| \leqq h < 1$, show that

$$|\operatorname{arc}(1+z)| \leqq \sin^{-1} h,$$

where the numerically smallest value of $\operatorname{arc}(1+z)$ is taken.

Suggestion: Draw a circle of radius h about the point 1. The point $1+z$ will lie in this circle.

23. A force can be represented by a vector. If forces $Z_1, \ldots, Z_n$ act at a point and all lie in a plane, show that the resultant force, Z, is represented by the sum:

$$Z = Z_1 + \cdots + Z_n.$$

24. What is the condition that n forces, lying in a plane and acting at a point, will be in equilibrium?

25. The n complex n-th roots of unity can be represented as vectors. Interpret the vectors as n forces acting at a point. Hence show that the sum of the n complex n-th roots of unity is equal to zero.

26. Devise a somewhat similar proof to show that the sum of the k-th powers of the n complex n-th roots of unity is 0.

27. The theorem of partial fractions asserts that any proper fraction can be represented as the sum of the terms of the type:

$$\frac{A_0}{(z-a)^n} + \frac{A_1}{(z-a)^{n-1}} + \cdots + \frac{A_{n-1}}{z-a}, \quad A_0 \neq 0.$$

If the given fraction is the quotient of two polynomials with real coefficients, show that the fraction can be represented as the sum of expressions of the two types:

$$\frac{A_0}{(x-a)^n} + \cdots + \frac{A_{n-1}}{x-a}$$

and

$$\frac{L_0 x + M_0}{(x^2+px+q)^m} + \frac{L_1 x + M_1}{(x^2+px+q)^{m-1}} + \cdots + \frac{L_{m-1} x + M_{m-1}}{x^2+px+q},$$

where all the coefficients are real and

$$p^2 - 4q < 0.$$

Of the three quantities A_0, L_0, M_0, one or two may vanish, but not all three.

CHAPTER 2

ANALYTIC FUNCTIONS & LINEAR TRANSFORMATIONS

2.1 Functions

Let S be a region of the complex z-plane. To each point S shall be assigned a number,

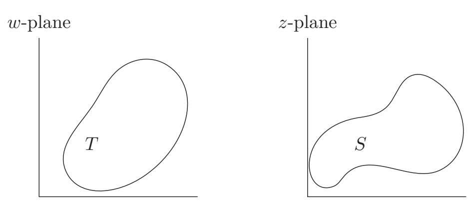
$$w = u + vi.$$

Then w is a *function* of z:

$$w = f(z).$$

Instead of a region,S, we might have a curve,C; or, more generally, any point set $M : \{z\}$ whatever. If to each point z of M a number w is assigned, then w is a function of z.

In the case of a real function of a real variable, $y = f(x)$, the function can be represented by a curve, and a real function of two real variables can be represented by a surface. But a complex function $w = u + vi$ of $z = x + yi$ would require a four-dimensional space—the space of the (u, v, x, y). We can, however, represent the function by means of two planes, plotting z in one plane and w in a second plane.



Thus under suitable conditions a region S of the z-plane will be *mapped* on a region T of the w-plane. Similarly, a curve C of the z-plane will go over into a curve Γ of the w-plane. Again, if s_n denotes the sum of the first n-terms of an infinite series:

$$u_1 + u_2 + \ldots,$$

whose terms are complex numbers, then $s_1, \ldots$ will be represented by isolated points in the complex plane.

A point $z = a$ is said to be an *interior* point of a region S if all points within a certain circle about a:

$$|z - a| < \delta, \tag{2.1}$$

belong to S. By the *neighborhood* of the point a is meant a region having a as an interior point. It may be a region defined by 2.1. Or, if

$$a = \alpha + \beta i, \quad z = x + yi,$$

it may consists of the points z for which

$$|x - \alpha| < \delta, \quad |y - \beta| < \delta. \tag{2.2}$$

In any case, whatever neighborhood be chosen, it is possible to take δ so that the region 2.1 or the region 2.2 lies inside of it.

Let $M : \{z\}$ be any set of points whatever, or a *point set.* By a *cluster point* or *point of condesation*, is meant a point $z = a$ such that there are no points of M distinct from a in every neighborhood of a. Thus no matter how small δ may be chosen, there will be a point z' of M such that

$$0 < |z' - a| < \delta.$$

By a *regular arc* is meant the curve:

$$x = f(t), \quad y = \varphi(t),$$

where $f(t)$, $\varphi(t)$ are continuous, together with their first derivatives, in the closed interval $t_0 \leqq t \leqq t_1$ and $f'(t)$, $\varphi'(t)$ do not vanish simultaneously. Moreover two distinct values of t shall not yield the same point (x, y).

A *regular curve* is composed of a succession of regular arcs joined at their extremities. It may be open or closed.

The regions S here considered shall be bounded by a finite number of simple regular curves.

2.2 Limits

Let M be any point set $\{z\}$ having a as a cluster point, and let $f(z)$ be defined in the points of M. Then $f(z)$ shall be said to *approach the limit* b when z approaches a:

$$\lim_{z \to a} f(z) = b,$$

if, to any arbitrary positive number ε, there corresponds a positive δ such that

$$|b - f(z)| < \varepsilon, \tag{2.3}$$

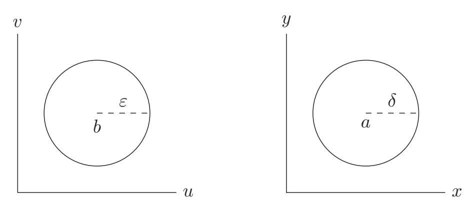
provided that

$$0 < |z - a| < \delta, \quad z \in M.$$

Geometrically the condition has the following meaning. Let a circle of radius ε be drawn about the point $w = b$ in the w-plane, and let a circle of radius δ be drawn about the point $z =$ in the z-plane. Let $z = z' \neq a$ be any point of M lying in the δ-circle, and let

$$w' = f(z').$$

Then w' lies in the ε-circle.



Theorem 2.2.1.
A necessary and sufficient condition that w approach b is that

$$\lim_{(x,y) \to (\alpha,\beta)} u = U, \qquad \lim_{(x,y) \to (\alpha,\beta)} v = V, \tag{2.4}$$

where

$$a = \alpha + \beta i, \quad b = U + Vi.$$

Proof.

i. The condition is necessary. Here, by hypothesis, 2.3 is true. Now,
$$w - b = u - U + i(v - V).$$
By Chapter 1 Exercise 21,
$$\frac{1}{\sqrt{2}}[|u - U| + |v - V|] \leqq |w - b|.$$
Hence
$$|u - U| < \sqrt{2}\varepsilon, \quad |v - V| < \sqrt{2}\varepsilon,$$
and consequently 2.4 holds.

ii. The condition is sufficient. Here, by hypothesis, 2.4 is true; i.e,
$$|u - U| < \varepsilon, \quad |v - V| < \varepsilon,$$
provided
$$|x - a| < \delta, \quad |y - \beta| < \delta.$$
Hence
$$|w - b| < 2\varepsilon,$$
and consequently 2.3 holds.

∎

As a result of 2.3, we observe that if the limit $b \neq 0$, then there exists a certain neighborhood of the point a such that
$$f(z) \neq 0, \quad 0 < |z - a| < h, \quad z \in M.$$
For, choose $\varepsilon < |b|$. Then the point $w = 0$ lies outside the ε-circle, and we need only set h equal to the δ corresponding to this ε. More generally, let C be any positive number less than $|b|$:
$$0 < C < |b|.$$
Then it is possible to determine h so that
$$C < |f(z)|, \quad 0 < |z - a| < h, \quad z \in M.$$

Theorem 2.2.2.
If each of two functions approaches a limit:

$$\lim_{z\to a} f(z) = A, \quad \lim_{z\to a} \varphi(z) = B,$$

their sum approaches a limit, and the limit of the sum is equal to the sum of the limits:

$$\lim_{z\to a}[f(z) + \varphi(z)] = \lim_{z\to a} f(z) + \lim_{z\to a} \varphi(z).$$

Proof.
We wish to show that, to an arbitrary positive ε, corresponding to a positive δ such that

$$|A + B - f(z) - \varphi(z)| < \varepsilon, \tag{2.5}$$

provided

$$0 < |z - a| < \delta, \quad z \in M.$$

By hypothesis,

$$\begin{aligned} |A - f(z)| < \varepsilon', \quad 0 < |z - a| < \delta_1, \quad z \in M \\ |B - \varphi(z)| < \varepsilon', \quad 0 < |z - a| < \delta_2, \quad z \in M \end{aligned}$$

where ε' is an arbitrary positive number. Hence by Chapter 1 Exercise 20,

$$|A + B - f(z) - \varphi(z)| < 2\varepsilon', \tag{2.6}$$

provided

$$0 < |z - a| < \delta, \quad z \in M,$$

where δ is the smaller of the two numbers, δ_1 and δ_2.

Now, ε is the choice of our adversary; but ε' is at our disposal. Let us choose $\varepsilon' = \frac{1}{2}\varepsilon$. Then 2.5 follows from 2.6, and the proof is complete. ∎

Theorem 2.2.3.
If each of two functions approaches a limit:

$$\lim_{z\to a} f(z) = A, \quad \lim_{z\to a} \varphi(z) = B,$$

their product approaches a limit, and the limit of their product is equal to the product of their limits.

Proof.
We wish to show that

$$|AB - f(z)\varphi(z)| < \varepsilon \tag{2.7}$$

provided

$$0 < |z-a| < \delta, \quad z \in M.$$

Let

$$f(z) = A + \zeta_1, \quad \varphi(z) = B + \zeta_2.$$

Then

$$\begin{aligned} |\zeta_1| < \varepsilon', \quad 0 < |z-a| < \delta_1, \quad z \in M \\ |\zeta_2| < \varepsilon', \quad 0 < |z-a| < \delta_2, \quad z \in M \end{aligned}$$

Now,

$$f(z)\varphi(z) - AB = B\zeta_1 + A\zeta_2 + \zeta_1\zeta_2.$$

Hence by Chapter 1 Exercise 20,

$$|AB - f(z)\varphi(z)| \leqq \varepsilon'|A| + \varepsilon'|B| + \varepsilon'^2 \tag{2.8}$$

Choose ε', to begin with, < 1. Then

$$\varepsilon'|A| + \varepsilon'|B| + \varepsilon'^2 < [|A| + |B| + 1]\varepsilon'.$$

We now choose ε' so that

$$[|A| + |B| + 1]\varepsilon' < \varepsilon$$

and take as δ the smaller of the two numbers, δ_1 and δ_2. Hence 2.7 follows from 2.8, and the theorem is proved. ■

A particular case under the theorem is that in which one of the functions is a constant: $\varphi(z) = C$. Thus

$$\lim_{z\to a} Cf(z) = C \lim_{z\to a} f(z),$$

provided $f(z)$ approaches a limit.

Theorem 2.2.4.
If each of the two functions approaches a limit:

$$\lim_{z\to a} f(z) = A, \quad \lim_{z\to a} \varphi(z) = B,$$

their quotient approaches a limit, and the limit of the quotient is equal to the quotient of their limits:

$$\lim_{z\to a} \frac{f(z)}{\varphi(z)} = \frac{\lim\limits_{z\to a} f(z)}{\lim\limits_{z\to a} \varphi(z)},$$

provided

$$\lim_{z\to a} \varphi(z) \neq 0.$$

Proof.
Begin by choosing h so that

$$\frac{1}{2}|B| < |\varphi(z)|, \quad 0 < |z-a| < h, \quad z \in M.$$

Next, observe that

$$\frac{A}{B} - \frac{f(z)}{\varphi(z)} = \frac{A}{B} - \frac{A+\zeta_1}{B+\zeta_2} = \frac{A\zeta_2 - B\zeta_1}{B\varphi(z)}$$

provided

$$0 < |z-a| < h,\, \delta_1,\, \delta_2, \quad z \in M.$$

Hence

$$\left|\frac{A}{B} - \frac{f(z)}{\varphi(z)}\right| < \frac{|A|+|B|}{\frac{1}{2}|B|^2}\varepsilon'. \tag{2.9}$$

and the remainder of the proof presents no difficulty. ■

The Case that z or w Becomes Infinite: If the point set $M : \{z\}$ is not bounded, we say that $f(z)$ *approaches a limit* b when z *becomes infinite*:

$$\lim_{z\to\infty} f(z) = b,$$

if to an arbitrary positive ε there corresponds a positive number G such that

$$|b - f(z)| < \varepsilon$$

provided

$$G < |z|, \quad z \in M.$$

The function $f(z)$ is said to *become infinite* when z approaches a:

$$\lim_{z \to a} f(z) = \infty,$$

if to an arbitrary large positive number G there corresponds a positive δ such that

$$G < |f(z)|,$$

provided

$$0 < |z - a| < \delta, \quad z \in M.$$

The definition:

$$\lim_{z \to \infty} f(z) = \infty$$

is now obvious.

A necessary and sufficient condition that

$$\lim_{z \to a} f(z) = \infty$$

is that

$$\lim_{z \to a} \frac{1}{f(z)} = 0.$$

Finally, the fundamental theorem relating to the existence of a limit; *Real Variables*, Chapter II, §6 and Chapter III, §5; *Funktionentheorie* I, p.30. Stated in the form which corresponds to the positive case it is as follows:

Theorem 2.2.5 (fundamental theorem)**.**
Let $f(z)$ be defined for all points z which lie outside a certain circle. To an arbitrary positive number ε shall correspond a positive number M such that

$$|f(z') - f(z'')| < \varepsilon, \quad M < |z'|, |z''|.$$

Then $f(z)$ approaches a limit, U, as z becomes infinite:

$$\lim_{z \to \infty} f(z) = U.$$

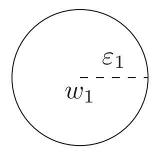
Proof.
Let $\varepsilon_1, \varepsilon_2, \ldots$ be a set of positive numbers such that

$$\varepsilon_1 \geq \varepsilon_2 \geq \ldots; \quad \lim_{n\to\infty} \varepsilon_n = 0.$$

Let z_1 be a point of the region $M_1 < |z|$. Mark the point $w_1 = f(z_1)$, and draw a circle of radius ε_1 about w_1.

$$|w - w_1| < \varepsilon_1 \tag{2.10}$$

Let $M_1, M_2, \ldots$ be the values of M which correspond to them by hypothesis, and let $M_1, \leqq M_2 \leqq \ldots$. Then every point $w = f(z)$, where $M_1 < |z|$, will lie in this circle.



Next, proceed to ε_2. Let z_2 be a point of the region $M_2 < |z|$. Mark the point

$$w_2 = f(z_2).$$

This point lies in the circle 2.10. Draw a circle of radius ε_2 about w_2:

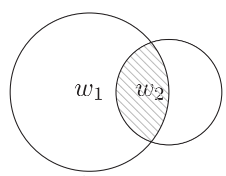
$$|w - w_2| < \varepsilon_2 \tag{2.11}$$

Then every point

$$w = f(z),$$

where $M_2 < |z|$, lies in this circle.

It can happen that the circle 2.11 contains points not included in the circle 2.10. Such points may be suppressed, since no one of them can represent a value of $f(z)$ when $M_2 < |z|$. Thus only the shaded region is retained.



And now repeat the process, again and again. We thus construct a succession of regions—S_1, or the circle 2.10; S_2, the shaded part of the circle 2.11; and so on—where S_n is bounded by a finite number of arcs of circles, and lies in S_{n-1}. Since the maximum diameter of S_n approaches 0 as $n = \infty$, these regions determine a single point, U, which lies within or on the boundary of each, and this is the limit demanded by the theorem. For, if a circle of arbitrarily small radius ε be drawn about U, then all the regions S_n from a definite point on, $m \leqq n$, will lie in the circle, or

$$|f(z) - U| < \varepsilon, \quad M_n < |z|.$$

■

It is obvious that we might have considered a function $f(z)$ defined merely for a set of points extending to infinity, like the integers $z = 1, 2, \ldots$. The theorem and proof still apply, with the one modification that z must each time be a point of the set.

Again, instead of a region, or a set of points, extending to infinity, we might have chosen the neighborhood of a finite point a, this one point excepted; or we could have let z range over the points of a point set which has a as a point of condensation, not belonging to the set.

Exercises

1. Give the arithmetic details of the proof that the S_n determine a single point, U, which lies within or on the boundary of each.

2. Give all the details of the proof of Theorem 2.2.1, introducing each time a suitable ε' and showing what δ is needed and how it is obtained. Illustrate the inequalities geometrically (two sets of figures, each composed of a pair).

2.3 Continuity

Let $f(z)$ be defined in a region S of the z-plane. Then $f(z)$ is said to be *continuous in a point* $z = z_0$ of S if

$$\lim_{z \to z_0} f(z) = f(z_0)$$

or

$$|f(z) - f(z_0)| < \varepsilon,$$

provided

$$|z - z_0| < \delta, \quad z \in S.$$

The function is said to be *continous* in S if it is continuous in each point of S.

If $f(z)$ and $\varphi(z)$ are two functions, each of which is continuous in a point $z = z_0$, then the functions:

$$f(z) + \varphi(z)$$

$$f(z)\varphi(z)$$
$$\frac{f(z)}{\varphi(z)}, \quad \text{provided } \varphi(z_0) \neq 0$$

are continuous in $z = z_0$.

If $f(z)$ is continuous in $z = z_0$ and does not vanish there, then there is a certain neighborhood of z_0:

$$|z - z_0| < h, \quad 0 < h, \tag{2.12}$$

throughout which $f(z)$ is different from 0. Moreover, if

$$0 < C < |f(z_0)|,$$

then h can be so chosen that

$$C < |f(z)|$$

for all points of the region 2.12.

Let

$$w = f(z)$$
$$w = u + vi$$
$$z = x + yi.$$

A necessary and sufficient condition that $f(z)$ be continuous at the point z_0 is, that each of the functions u and v be continuous at the point (x_0, y_0).

All of these theorems follow at once from the theorems of §2.

Theorem 2.3.1.
A continuous function of a continuous function is a continuous function:

$$W = f(w)$$
$$w = \varphi(z)$$
$$W = f(\varphi(z))$$

More precisely, let $f(w)$ be continuous in the neighborhood of a point $w = w_0$ and let $\varphi(z)$ be continuous in the point $z = z_0$.

Let $w_0 = \varphi(z_0)$. Then $W = f(\varphi(z))$ is continuous in $z = z_0$.

Consider what this theorem means geometrically. If z is a point near z_0, then $w = \varphi(z)$ will be a point near w_0. But when w is near w_0, $f(w)$ is near $f(w_0)$.

Let the student put these considerations into ε-form and construct a rigorous proof.

Exercises

1. Show that the function $w = z$ is continuous for all values of z.

2. Show that the function $w = z^n$ is continuous for all values of z, where n is a natural number.

3. Show that the function $w = c$ is continuous for all values of z.

4. Show that a polynomial

$$G(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$$

is continuous for all values of z.

5. Show that a rational function

$$R(z) = \frac{f(z)}{\varphi(z)}$$

where $f(z)$ and $\varphi(z)$ are polynomials relatively prime, is continuous for all values of z for which it is defined.

2.4 Derivatives

Let a function

$$w = f(z)$$

be defined in a region S. Let z_0 be an interior point of the region, and let $z_0 + \Delta z$ be a second point of S. Let

$$w_0 = f(z_0), \quad w_{\Delta} w = f(z_0 + \Delta z).$$

Form the difference quotient

$$\frac{\Delta w}{\Delta z} = \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}.$$

Let Δz approach 0. If the difference quotient approaches a limit, the function is said to have a *derivative* in the point z_0.

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = D_z w = f'(z_0).$$

A necessary condition for a derivative is obtained as follows. Let

$$w = u + vi, \quad z = x + yi.$$

Since $\Delta w / \Delta z$ by hypothesis approaches a limit as Δz approaches 0, the variable will approach the same limit if Δz is restricted to real values, $\Delta z = \Delta x$. Hence

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta x \to 0} \left(\frac{\Delta_x u}{\Delta x} + i \frac{\Delta_x v}{\Delta x} \right) = D_z w.$$

Now,

$$\lim_{\Delta x \to 0} \frac{\Delta_x u}{\Delta x} = \frac{\partial u}{\partial x}, \quad \lim_{\Delta x \to 0} \frac{\Delta_x v}{\Delta x} = \frac{\partial v}{\partial x},$$

and so

$$D_z w = \frac{\partial w}{\partial x} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}. \tag{2.13}$$

Similarly, Δz may be restricted to values that are pure imaginaries. Thus we find

$$\lim_{\Delta y \to 0} \frac{\Delta w}{i \Delta y} = \lim_{\Delta y \to 0} \left(\frac{1}{i} \frac{\Delta_y u}{\Delta y} + \frac{\Delta_y v}{\Delta y} \right) = D_z w,$$

$$D_z w = \frac{1}{i} \frac{\partial w}{\partial y} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} \tag{2.14}$$

Comparing 2.13 and 2.14 we obtain the equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{2.15}$$

These are known as the *Cauchy-Riemann Differential Equations*. We have now obtained them as a necessary condition that the function have a derivative. They can be written in the equivalent form

$$\frac{\partial w}{\partial x} = \frac{1}{i}\frac{\partial w}{\partial y} \tag{2.16}$$

A very simple function of a complex variable may fail to have a derivative. Consider the function

$$w = x - yi.$$

Here,

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial v}{\partial y} = -1,$$

and the Cauchy-Riemann conditions are not satisfied. It is easy to see directly that this function has no derivative. For

$$\frac{\Delta w}{\Delta z} = \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}.$$

If Δz approaches 0 along the axis of reals, the limit of $\Delta w/\Delta z$ has the value 1. But if Δz approaches 0 along the axis of pure imaginaries, the limit is -1.

Conversely, let u and v be two real functions defined in the region S and having first partial derivatives satisfying the Cauchy-Riemann Differential Equations. Let these derivatives, furthermore, be continuous. Then the function

$$w = u + vi = f(z)$$

will have a derivative, $f'(z)$, and the latter will be continuous in S.

Theorem 2.4.1.
Let a function $w = f(z)$ where $w = u + vi$ and $z = x + yi$ be defined in a region S, and let u, v have continuous first derivatives there. Then a necessary and sufficient condition that $f(z)$ have a continuous derivative is, that the Cauchy-Riemann Differential equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

be satisfied.

Proof.
We have the appraisals

$$\Delta u = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \zeta_1 \Delta x + \zeta_2 \Delta y$$
$$\Delta v = \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y + \zeta_3 \Delta x + \zeta_4 \Delta y$$

where $\zeta_1, \ldots, \zeta_4$ are infinitesimals; i.e. Variables which approach 0 when $(\Delta x, \Delta y)$ approaches $(0,0)$. On substituting these values into the equation

$$\frac{\Delta w}{\Delta z} = \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y},$$

and reducing, we find, by first-year algebra

$$\frac{\Delta w}{\Delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + (\zeta_1 + i\zeta_3)\frac{\Delta x}{\Delta z} + (\zeta_2 + i\zeta_4)\frac{\Delta y}{\Delta z}. \tag{2.17}$$

Since

$$\left|\frac{\Delta x}{\Delta z}\right| \leqq 1, \quad \left|\frac{\Delta y}{\Delta z}\right| \leqq 1,$$

and since each parenthesis approaches 0, it is clear that: the right side of 2.17 approaches a limit, when $(\Delta x, \Delta y)$ approaches $(0,0)$, and the value of this limit is given by the first two terms; or

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

∎

Remark. If a function $f(z)$ has a derivative at each point of S, the derivative is necessarily continuous in S (Goursat's theorem). The proof of this theorem belongs to a later stage in the theory. If, on the other hand, the functions u and v possess first partial derivatives which satisfy the Cauchy-Riemann differential equations, it does not follow that the function $w = u + vi$ is analytic. The derivative and in fact the functions u and v themselves, may fail to be continuous. Consequently some further restriction is required. It is enough, as we have seen, to demand the continuity of the derivatives.

Example 2.4.2.
The function e^z has been defined as follows:

$$e^z = e^x(\cos y + i \sin y).$$

It has a derivative; for

$$u = e^x \cos y, \quad v = e^x \sin y,$$

and these functions satisfy the Cauchy-Riemann Differential Equations, since

$$\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -e^x \sin y = -\frac{\partial v}{\partial x}.$$

Exercises

1. Let each of the functions $w_1 = f(z)$, $w_2 = \varphi(z)$ have a derivative at the point $z = z_0$. Show that the functions $f(z) + \varphi(z)$, $f(z)\varphi(z)$, and $f(z)/\varphi(z)$, provided $\varphi(z_0) \neq 0$, have a derivative there, and that

$$\begin{aligned}
&D_z(w_1 + w_2) = D_z w_1 + D_z w_2 \\
&D_z(w_1 w_2) = w_1 D_z w_2 + w_2 D_z w_1 \\
&D_z \frac{w_1}{w_2} = \frac{w_2 D_z w_1 - w_1 D_z w_2}{w_2^2}, \quad \varphi(z_0) \neq 0 \\
&D_z F(w) = D_w F(w) D_z w
\end{aligned}$$

where $w = \varphi(z)$ and $F(w)$ has a derivative in the point $w_0 = \varphi(z_0)$.

2. Show that $D_z c = 0$ and $D_z z = 1$, where c is a constant.

3. Show that $D_z z^n = n z^{n-1}$, where n is a whole number.

4. Prove that a polynomial

$$G(z) = a_0 z^n + \cdots + a_n$$

has a derivative.

5. Show that a rational function has a derivative.

6. Let u, v be continuous, together with their first derivatives, in a region S, and let them satisfy the Cauchy-Riemann Differential Equations. Let $x = r\cos\varphi$ and $y = r\sin\varphi$. Show that

$$\begin{aligned} \frac{\partial u}{\partial r} &= \frac{1}{r}\frac{\partial v}{\partial \varphi} \\ \frac{1}{r}\frac{\partial u}{\partial \varphi} &= -\frac{\partial v}{\partial r} \\ D_z w &= e^{-\varphi i}\frac{\partial w}{\partial r} \end{aligned}$$

7. State and prove the converse of the theorem of the prior exercise.

8. Show that the function $\log z = \log r + \varphi i$, $0 \leqq \varphi < 2\pi$, has a derivative, and that $D_z \log z = \frac{1}{z}$.

9. Show that the function $r^x(\cos\alpha\varphi + i\sin\alpha\varphi)$, $0 \leqq \varphi < 2\pi$, where α is any real constant, has a derivative.

10. Let $w = f(z)$ have a continuous derivative in the region S. Write $w = R(\cos\Theta + i\sin\Theta)$. Show that

$$\frac{\partial R}{\partial x} = R\frac{\partial \Theta}{\partial y}, \quad \frac{\partial R}{\partial y} = -R\frac{\partial \Theta}{\partial x},$$

and prove the converse.

11. If in the prior exercise, $z = r(\cos\theta + i\sin\theta)$, show that

$$\frac{\partial R}{\partial r} = \frac{R}{r}\frac{\partial \Theta}{\partial \theta}, \quad \frac{\partial R}{\partial \theta} = -rR\frac{\partial \Theta}{\partial r},$$

and prove the converse.

2.5 Differentials

Let the function $w = f(z)$ have a derivative in the point $z = z_0$:

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = D_z w.$$

If we set

$$\frac{\Delta w}{\Delta z} = D_z w + \zeta, \tag{2.18}$$

then ζ is an infinitesimal:

$$\lim_{\Delta z \to 0} \zeta = 0.$$

From 2.18 it follows that

$$\Delta w = D_z w \Delta z + \zeta \Delta z.$$

Thus the increment Δw is broken up into the sum of two infinitesimals:

i. $D_z w \Delta z$, a linear function of Δz,

ii. $\zeta \Delta z$, an infinitesimal of higher order.

The first term is the *principal part* of the infinitesimal Δw and is defined as the *differential* of w:

$$dw = D_z w \Delta z. \tag{2.19}$$

The differential of the independent variable z is defined as the increment: $dz = \Delta z$. Hence

$$dw = D_z w dz. \tag{2.20}$$

Equation 2.20 is true, no matter what the independent variable may be. Suppose that z depends on t: $z = \varphi(t)$, where $\varphi(t)$ has a derivative in the point $t = t_0$, and $z_0 = \varphi(t_0)$. Then by §4, Exercise 1 equation 4,

$$D_t w = D_z w D_t z. \tag{2.21}$$

By definition, t now being the independent variable,

$$dw = D_t w \Delta t, \quad dz = D_t z \Delta t.$$

Hence 2.20 follows from 2.21.

Derivative Along a Curve: A function $f(z)$ may be defined merely along a curve

$$x = \varphi(t), \quad y = \psi(t),$$

where $\varphi(t)$ and $\psi(t)$ are continuous, together with their first derivatives, throughout a closed interval $t_0 \leqq t \leqq t_1$, and furthermore

$$0 < \varphi'(t)^2 + \psi'(t)^2.$$

Thus $w = f(z) = u + vi$ becomes a complex function of the real variable t:

$$w = F(t) + i\Phi(t).$$

w is said to have a *derivative* along C if $\Delta w/\Delta t$ approaches a limit:

$$\lim_{\Delta t \to 0} \frac{\Delta w}{\Delta t} = D_t w.$$

A necessary and sufficient condition is, that $F(t)$ and $\Phi(t)$ both have derivatives. Then,

$$D_t w = F'(t) + i\Phi'(t). \tag{2.22}$$

The differential of w is defined by the equation:

$$dw = D_t w dt = F'(t)dt + i\Phi'(t)dt. \tag{2.23}$$

2.6 Analytic Functions

Let $f(z)$ be defined throughout a region S of the complex z-plane, and let $f(z)$ have a derivative, $f'(z)$ at each interior point of S. Let $f'(z)$ be continuous[1] in S. Then $f(z)$ is defined as *analytic* in S.

A function is defined as *analytic in a point* if it is analytic throughout a region which includes the point in its interior.

The term *holomorphic* is also used, meaning the same thing as analytic in a region.

[1]It can be shown that $f'(z)$ is necessarily continuous in S; this is *Goursat's Theorem; Funktionentheorie*, I, p.368. But this is a question which belongs to a later stage of the theory, and so we introduce the hypothesis of continuity at this point.

Exercises

1. Show that the sum, difference, product, and quotient (provided the denominator does not vanish) of two analytic functions is an analytic function.

Take each of these four theorems by itself. State precisely the hypothesis, and the conclusion. And give a *proof*.

2. An analytic function of an analytic function is an analytic function.

State this theorem in detail, and prove it.

2.7 The Inverse Function

Let

$$w = f(z)$$

be analytic in the point $z = z_0$, and let $f'(z_0) \neq 0$. Each point z of the neighborhood of z_0 is carried over into a point w of the neighborhood of w_0. And now I say: *If the neighborhood of z_0 is suitably restricted, the map of this region on a portion of the w-plane will be one-to-one, and consist of a region T of the latter plane.*

The proof of this theorem follows at once from the theorems relating to Implicit Functions and the Inverse of a Transformation, *Real Variables*, Chapter IV, §12. It is here a question of the inverse of the transformation:

$$u = \varphi(x, y), \quad v = \Psi(x, y),$$

where $z = x + yi$ and $w = u + vi$.

The functions $\varphi(x, y)$ and $\Psi(x, y)$ are continuous, together with their first partial derivatives, in the neighborhood of the point (x_0, y_0), and it remains merely to examine the Jacobian:

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}. \tag{2.24}$$

Now, because of the Cauchy-Riemann relations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

we have

$$J = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2.$$

Hence

$$J|_{z=z_0} = |f'(z_0)|^2 \neq 0,$$

and so all the conditions of the theorem are fulfilled. It follows, then, that the transformation defined my $w = f(z)$ is equivalent to a transformation $z = F(w)$, where $F(w)$ is single-valued and continuous throughout a certain region T of the w-plane, including the point $w = w_0$ in its interior.

Let us formulate the result as a theorem, and supplement it by the face that the inverse function is also analytic.

Theorem 2.7.1.
Let $f(z)$ be analytic throughout a circle K:

$$|z - z_0| < h,$$

and let $f'(z_0) \neq 0$. Then it is possible, on suitably restricting h, to find a circle H:

$$|w - w_0| < h', \quad w_0 = f(z_0),$$

such that the equation $w = f(z)$, where w is an arbitrary point of H, admits one and only one root z in K. The function $z = F(w)$ thus defined is analytic in the point $w = w_0$.

To prove that $F(w)$ has a derivative, form the difference quotient

$$\frac{\Delta z}{\Delta w} = \frac{F(w_0 + \Delta w) - F(w_0)}{\Delta w}.$$

Since $\Delta z \neq 0$ when $\Delta w \neq 0$, it follows that

$$\frac{F(w_0 + \Delta w) - F(w_0)}{\Delta w} = \left(\frac{\Delta w}{\Delta z}\right)^{-1}.$$

When Δw approaches 0, Δz also approaches 0, and hence the right hand side approaches a limit, namely $(D_z w)^{-1}$. Moreover, $f'(z)$

vanishes nowhere in K, if h is suitably restricted, and thus the proof applies to every point of H. Hence

$$D_w z = \frac{1}{D_z w}.$$

The right hand side is a continuous function of z, and hence of w. This completes the proof.[2]

Exercises

1. In the theorem cited from the *Real Variables* (2.7.1) the regions about the points (x_0, y_0) and (u_0, v_0) are rectangles. Explain the transition to circles.

2. Prove the existence of a derivative of $F(w)$ by means of the Cauchy-Riemann Differential Equations.

3. If $y = f(x)$ is a real function of the real variable x, where $a < x < b$; and if $f(x)$ has a continuous derivative which does not vanish: $f'(x) \neq 0$, the inverse function is single-valued.

By analogy in the complex case, let $f(z)$ be analytic in a region S — say, inside a circle; and let $f'(z) \neq 0$ there. Should you expect the inverse function to be single-valued? Why?

[2]It might seem as if this theorem should admit a much more elementary proof, especially if we add to our knowledge the fact that the angle between two curves is preserved, §9. The point set K is carried over into a certain region point set $\mathfrak{S}$, and, at least when h is suitably restricted, it is not obvious that $\mathfrak{S}$ must fill out smoothly a certain neighborhood of w_0? There are two questions that have to be met: i. Why should $\mathfrak{S}$ cover every part of the neighborhood of w_0? In other words, may there not be open spaces—lakes, lacunae—near w_0, whose points correspond to no points of K? ii. May not $\mathfrak{S}$ overlap itself? In other words, may not a point of $\mathfrak{S}$ correspond to two or more different points of K?

There seems to be no simpler answers to these questions, than the one given by the Theorem of Implicit Functions—and, indeed, we are fortunate to have so simple an answer as this.

2.8 The Transformation $w = az$

Let the complex z-plane be transformed on the w-plane by means of the transformation

$$w = az, \tag{2.25}$$

where $a \neq 0$ is a constant such that

$$a = \mathcal{A}e^{zi} = \mathcal{A}(\cos\alpha + i\sin\alpha).$$

Write z and w in polar form:

$$z = r(\cos\varphi + i\sin\varphi), \quad w = R(\cos\Psi + i\sin\Psi).$$

Then 2.25 becomes

$$Re^{\Psi i} = \mathcal{A}re^{(\varphi+\alpha)i}. \tag{2.26}$$

Hence

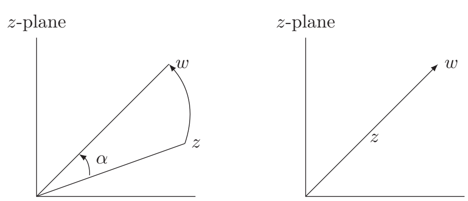
$$R = \mathcal{A}r, \quad \Psi = \varphi + \alpha. \tag{2.27}$$

Consider, first, two particular cases. The general case can then be compounded out of these.

i. $\mathcal{A} = 1$, $\alpha \neq 0$. Here,

$$R = r, \quad \Psi = \varphi + \alpha.$$

Each point z goes over into a point w at the same distance from the origin, but with its angle increased by α. Thus the whole plane is rotated about the origin through an angle α, without any deformation whatever. It is easier to think of the z-plane as *transformed into itself*, rather than on a separate w-plane; cf, the first figure.



ii. $\alpha = 0$. Here,

$$R = \mathcal{A}, \quad \Psi = \varphi.$$

A point $z \neq 0$ is moved along the ray drawn from the origin through the point, and it comes to a point w whose distance from O bears to the distance of z from O the fixed ratio $\mathcal{A}$. If $\mathcal{A} > 1$, w is further from O than z was. If $\mathcal{A} < 1$, w is nearer.

It is easy to visualize the transformation of the plane as a whole. It is as if it were a rubber membrane, which is stretched equally in all directions, if $\mathcal{A} > 1$. If $\mathcal{A} < 1$, the membrane is allowed to contract. In this case, the plane is no longer transformed as a rigid body. It is deformed. Nevertheless, figures go over into *similar* figures—a square goes over into a square; a circle into a circle, etc. But the size is changed, always in the linear ratio of $\mathcal{A} : 1$.

The general case represented by 2.27 can now be compounded out of these two particular cases. Thus we see that the z-plane is rotated through an angle α and stretched in the ratio of $\mathcal{A} : 1$. The order can be reversed — first, the stretching, then the rotation.

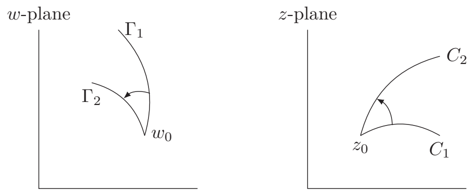
Transformations play a *most important role* in all branches of mathematics, physics, and geometry, and the student can not begin too early to train himself in this domain of thought. He will do well to turn to the general case of projective transformations, even though these will not concern him in this course; cf. Osgood and Graustein, *Plane and Solid Analytic Geometry*, Chap. XV and Chap. XXIII, §10; also *Advanced Calculus*, p. 279, §11.

2.9 Preservation of Angles

Turning now to the general case of an analytic function:

$$w = f(z), \tag{2.28}$$

let $f(z)$ be analytic in the point $z = z_0$, and let $f'(z_0) \neq 0$. We have seen that the neighborhood of the point z_0 is mapped on the neighborhood of the point $w = w_0$ in a one-to-one manner and continuously. We will now show that two curves, C_1 and C_2, intersecting at z_0 under the angle α, go over into curves Γ_1 and Γ_2 intersecting at w_0 under an equal angle. Since the same is true of any other point z' near z_0, the transformation defined by 2.28 is called *isogonal* (*winkeltreu*, having the same angles).



Let $z' = z_0 + \Delta z$ be any second point, and let $w' = w_0 + \Delta w$ be its image. Since

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = D_z w,$$

where

$$D_z w = A = \mathcal{A} e^{\alpha i}, \quad \mathcal{A} \neq 0,$$

it follows that

$$\frac{\Delta w}{\Delta z} = A + \zeta,$$

where z is an infinitesimal such that $\lim\limits_{\Delta z \to 0} \zeta = 0$. Hence

$$\Delta w = \Delta z(A + \zeta). \tag{2.29}$$

This last equation tells the story. It shows that the angle of Δw is nearly equal to the angle of Δz plus the angle of A:

$$\operatorname{arc} \Delta w = \operatorname{arc} \Delta z + \operatorname{arc}(A + \zeta).$$

Let a curve C be drawn from z_0, and let Γ be its image. Let z' be a nearby point of C, and w' its image. Then the angle that the secant z_0, z' makes with the axis of reals, represents $\operatorname{arc} \Delta z$; and similarly in the w-plane. Let z' approach z_0. Then

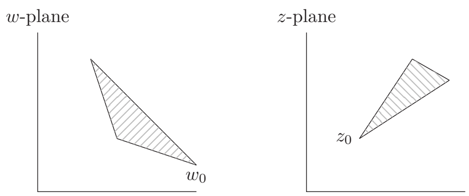
$$\lim \operatorname{arc} \Delta w = \lim \operatorname{arc} \Delta z + \lim \operatorname{arc}(A + \zeta), \tag{2.30}$$

where $\Phi = \varphi + \alpha$.

Equation 2.30 means that, no matter what curve C be drawn from z_0, the angle which its tangent makes with the positive axis of reals, plus the constant angle α, will give the angle which the tangent to its image, Γ, makes with the positive axis of reals in the w-plane. From this result the isogonal property mentioned at the beginning follows immediately.

2.10 Conformal Mapping

The isogonal property thrown light on the map defined by the function $w = f(z)$.



Consider a small scalene triangle drawn in the neighborhood of z_0. It goes over into a curvilinear triangle into the w-plane, which has the same angles. Since the curved sides of the latter triangle look almost like straight lines, the triangle will have the appearance of an ordinary right-line triangle, similar to the first, but magnified in the ratio $\mathcal{A} : 1$ and rotated through the angle α. And what has just been said of small triangles is true of any small figures, for the figure in the z-plane can be covered by a network of small triangles. The smaller the figure, the less relative departure of the image from precise similarity. Because of this property the transformation is called *conformal* (preserving form).

Element of Arc: Let the curve C of §9 be represented in the usual manner by the equations:

$$x = \varphi(t), \quad y = \psi(t),$$

where

$$0 < \varphi'(t)^2 + \psi'(t)^2,$$

and let the arc be denoted by s. The length of the chord is $|\Delta z|$ and hence

$$\lim_{\Delta t \to 0} \frac{|\Delta z|}{\Delta t} = \pm D_t s.$$

Let S denote the arc of Γ. Then

$$\lim_{\Delta t \to 0} \frac{|\Delta w|}{\Delta t} = \pm D_t S.$$

From 2.29 it follows that

$$\lim_{\Delta t \to 0} \frac{|\Delta w|}{\Delta t} = \lim_{\Delta t \to 0} \frac{|\Delta z|}{\Delta t} \lim_{\Delta t \to 0} |A + \zeta|,$$

and hence

$$D_t S = D_t s |D_z w|. \tag{2.31}$$

$|D_z w|$ is a positive real function of the two real variables x, y. Denote it by M. Then 2.31 can be written in the form:

$$dS = M ds.$$

This equation is characteristic of a conformal transformation, and can, indeed, be taken as the definition of such a transformation.

For a detailed discussion cf. *Funktionentheorie* I, Chap. VI, §8, p. 245, and also Chap. II, §7, p.74.

Mercator's map of the world affords a simple and interesting illustration from Cartography; cf. *Advanced Caculus*, p. 169. Stereographic projection will be taken up in the next Chapter, §6, and may well be studied at this point.

Exercises

1. Study the conformal map by means of the two real equations:

$$u = f(x, y), \quad v = \varphi(x, y)$$

where $f(x, y), \varphi(x, y)$ are continuous, together with their first partial derivatives, and

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

some of these derivatives being different from zero.

This is all done in the *Funktionentheorie* I, Chap. II, §7, but the student is warned against making a minute study of that treatment. He may give it a cursory glance when he is sleepy. Then, when he is fully mobilized, he should produce it independently. But it is better to defer this exercise till the next chapter has been completed, and then come back to this study.

CHAPTER 3

CONFORMAL MAPPING

3.1 The Logarithmic Function

Consider the map defined by the function $w = \log z$. Let

$$z = r(\cos\varphi + i\sin\varphi). \tag{3.1}$$

Then

$$\log z = \log r + \varphi i. \tag{3.2}$$

The function is infinitely multiple-valued. Select that branch for which

$$-\pi < \varphi < \pi. \tag{3.3}$$

Thus the z-plane is cut along the negative axis of reals, and we are considering the single-valued function defined in this region, S, by equation 3.2 and the inequalities 3.3. Let

$$w = u + vi.$$

Then

$$u = \log r, \quad v = \varphi. \tag{3.4}$$

The positive axis of reals in the z-plane, $\varphi = 0$, goes over into the entire axis of reals in the w-plane:

$$-\infty < u < \infty, \quad v = 0. \tag{3.5}$$

An arbitrary ray,

$$\varphi = \alpha, \quad -\pi < \alpha < \pi \tag{3.6}$$

goes over into a parallel to the u-axis,

$$v = \alpha. \tag{3.7}$$

On the other hand, the unit circle, $r = 1$, goes over into a segment of the axis of pure imaginaries:

$$u = 0, \quad -\pi < v < \pi. \tag{3.8}$$

And an arbitrary circle,

$$r = \rho, \tag{3.9}$$

goes over into an equal line-segment, displaced horizontally:

$$u = \log \rho, \quad -\pi < v < \pi. \tag{3.10}$$

It appears, then, that the region S of the z-plane is carried over into a region T of the w-plane, consisting of a strip bounded by the parallels,

$$v = \pi, \quad v = -\pi. \tag{3.11}$$

Moreover, the map is *conformal*. We can bring this fact out suggestively by drawing a suitable network of lines in the two regions. Let the strip be divided into a large number of congruent strips, let us say 12. Then draw corresponding line segments equally spaced, beginning with 3.8 and choosing ρ in 3.10 so that the distance between two successive line segments will be the same as the breadth of a strip. Thus for the first line segment to the right,

$$\log \rho_1 = \frac{2\pi}{12} = 0.5236$$

and hence

$$\rho = 1.6881.$$

This means that the circle

$$|z| = 1.6881$$

goes over into the line segment:

$$u = 0.5236, \quad -\pi < v < \pi.$$

And similarly for the other line segments, for which

$$\log \rho_n = 0.5236n, \quad \rho_n = e^{0.5236n}, \quad n = 0, \pm 1, \ldots.$$

Thus while the $u_n's$ form an *arithmetic* series, the ρ_n's form a *geometric* series. The interior of the unit circle $|z| = 1$ corresponds to the part of the strip T to the left of the axis of pure imaginaries, and the images of the little squares in T are the curvlinear quadrilaterals indicated in the figure. The exterior of the unit circle goes over into the part of the strip to the right.

Exercises

1. Into what figure is the quadrant of the unit circle which lies in the first quadrant carried by the above map?

2. What region is the circular ring bounded by the circles $|z| = 1$ and $|z = 2|$ and cut open along the negative axis of reals, carried into?

3. A rectangle in the w-plane is bounded by the lines

$$u = -0.32, \quad u = 1.8, \quad v = -0.25, \quad v = 0.$$

Draw accurately the image in the z-plane.

4. Plot accurately the point of the w-plane into which the point $z = -3 - 2i$ is transformed.

5. Plot accurately the point of the z-plane into which the point $w = -0.5371 - 0.6873i$ goes.

6. A plane area is bounded by two concentric circles of radii 2 in. and 3 in., and by two radii which make an angle of 45° with each other. Show that it can be mapped conformally on a rectangle, and determine the ratio of the sides.

3.2 The Function $w = z^\alpha$

Consider the map defined by the function

$$w = z^\alpha, \tag{3.12}$$

where α is a positive real number. Let

$$z = r(\cos\varphi + i\sin\varphi), \quad w = R(\cos\psi + i\sin\psi).$$

Then

$$R(\cos\psi + i\sin\psi) = r^\alpha(\cos\alpha\psi + i\sin\alpha\psi).$$

Hence

$$R = r^\alpha, \quad \psi = \alpha\psi + 2k\pi. \tag{3.13}$$

Thus a circle about the origin, $z = 0$, goes over into a circle about the origin, $w = 0$.

Consider a sector of a circle:

$$0 \leqq r \leqq r_1, \quad 0 \leqq \varphi \leqq \varphi_1. \tag{3.14}$$

Let $\psi = \alpha\varphi$, and assume that $\alpha\varphi_1 = \pi$, $1 < \alpha$. Let $r_1 = 1$; then $R_1 = 1$. Thus a sector of the unit circle in the z-plane, whose angle is $\varphi_1 = \pi/\alpha$, is opened out like a fan on a semicircle. And yet, not wholly like a fan, for the points of the z-figure are drawn in toward the centre. If, for example, $\alpha = 2$, the points on the circle $r = \frac{1}{2}$ go over into points on the circle $R = \frac{1}{4}$.

Exercises

1. Study in detail the case $\alpha = 3$. Taking 5 cm as the unit, draw accurately the two figures, dividing each of the angles into six equal parts. Recalling the corresponding figure in §1, choose as radii in the one figure:

$$\rho_n = e^{-0.5236n}, \quad n = 0, 1, 2, 3, 4.$$

2. Examine the case $0 < \alpha < 1$. In particular, let $\alpha = \frac{1}{3}$:

$$w = z^{\frac{1}{3}}.$$

How is this map related to the former map? Generalize.

3. Study the case: $1 < \alpha$,

$$\alpha\varphi_1 = \pi, \quad r_1 = \infty.$$

Show that points inside the unit circle $|z| = 1$ are drawn in nearer the origin, but points outside this circle are carried further away. Thus there is a stretching away from the unit circle, along the rays emanating from the origin.

3.3 The Function $w = \sin^{-1} z$