THE UNIVERSITY OF RHODE ISLAND

An Investigation of Achromatic and Harmonious Chromatic Numbers of Certain Classes of Graphs

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March 2, 2024

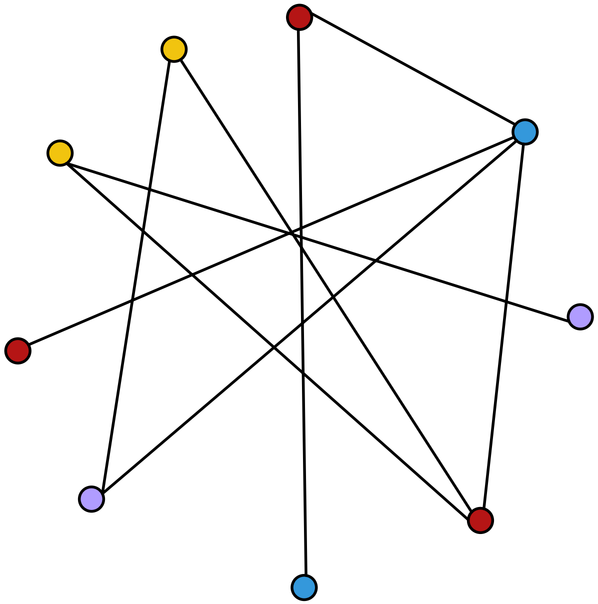
Introduction

Definition 1 (Graph)

A graph, G, is a pair, $G = (V, E)$, where V denotes the set of vertices, E denotes the set of edges, and $E \subseteq \binom{V}{2}$.

Definition 2 (Proper Vertex Coloring)

A proper vertex coloring is a coloring of the vertices of a graph G in which no two adjacent vertices are the same color.

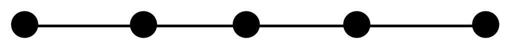


Definition 3 (Path)

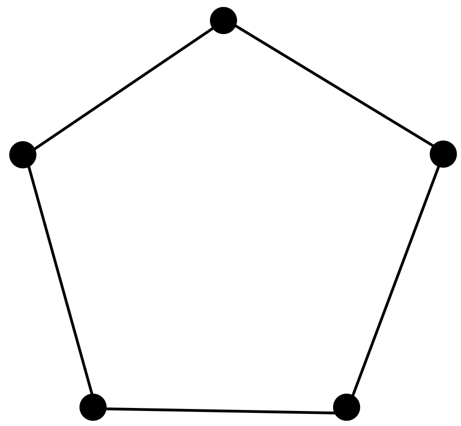
A path, denoted P_n, is a nonempty graph on n vertices, $V = \{x_1, \ldots x_n\}$ with the edges $E = \{x_1x_2, x_2x_3, \ldots x_{n-1}x_n\}$.

Definition 4 (Cycle)

A cycle, denoted C_n, a nonempty graph on n vertices, is $P_n + x_nx_1$.



Path



Cycle

Introduction

We seek to examine the following two types of colorings for cycles and path:

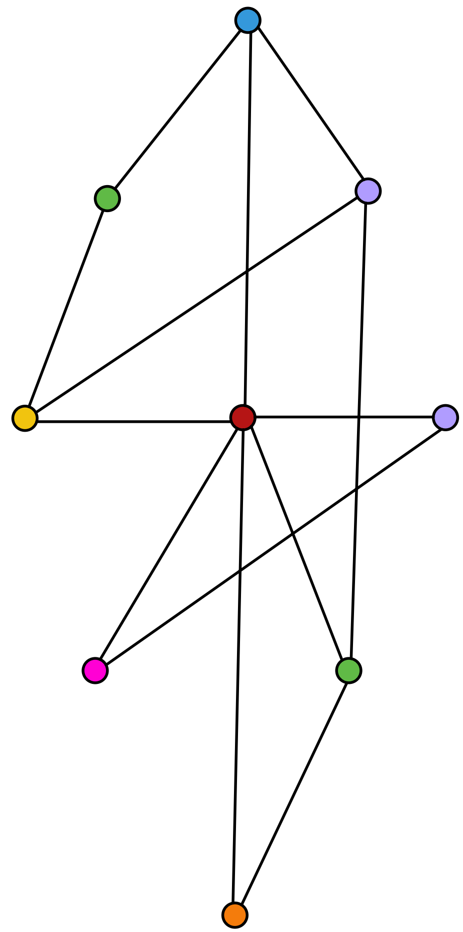
Definition 5 (Harmonious Chromatic Number)

A harmonious coloring is a proper vertex coloring in which each color pair appears on at most one pair of adjacent vertices. The minimum number of colors required for such a coloring is the harmonious chromatic number, $\chi_H(G)$.

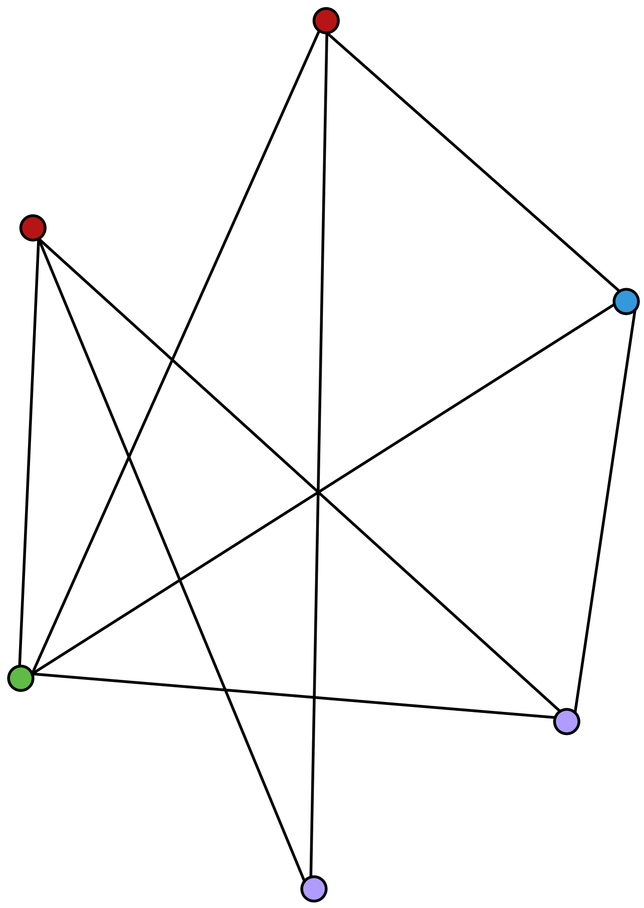
Definition 6 (Achromatic Number)

A complete coloring is a proper vertex coloring in which each color pair appears on at least one pair of adjacent vertices. The maximum number of colors required for such a coloring is the achromatic number, $\psi(G)$.

Example of Achromatic and Complete Colorings



Achromatic Coloring



Complete Coloring

Main Objective

Definition 6 (Harmonious Chromatic Number)

A harmonious coloring is a proper vertex coloring in which each color pair appears on at most one pair of adjacent vertices. The minimum number of colors required for such a coloring is the harmonious chromatic number, $\chi_H(G)$.

Definition 7 (Achromatic Number)

A complete coloring is a proper vertex coloring in which each color pair appears on at least one pair of adjacent vertices. The maximum number of colors required for such a coloring is the achromatic number, $\psi(G)$.

Question: When are $\psi(G) = \chi_H(G)$?

We will answer this for cycles and paths.

Equality for Cycles

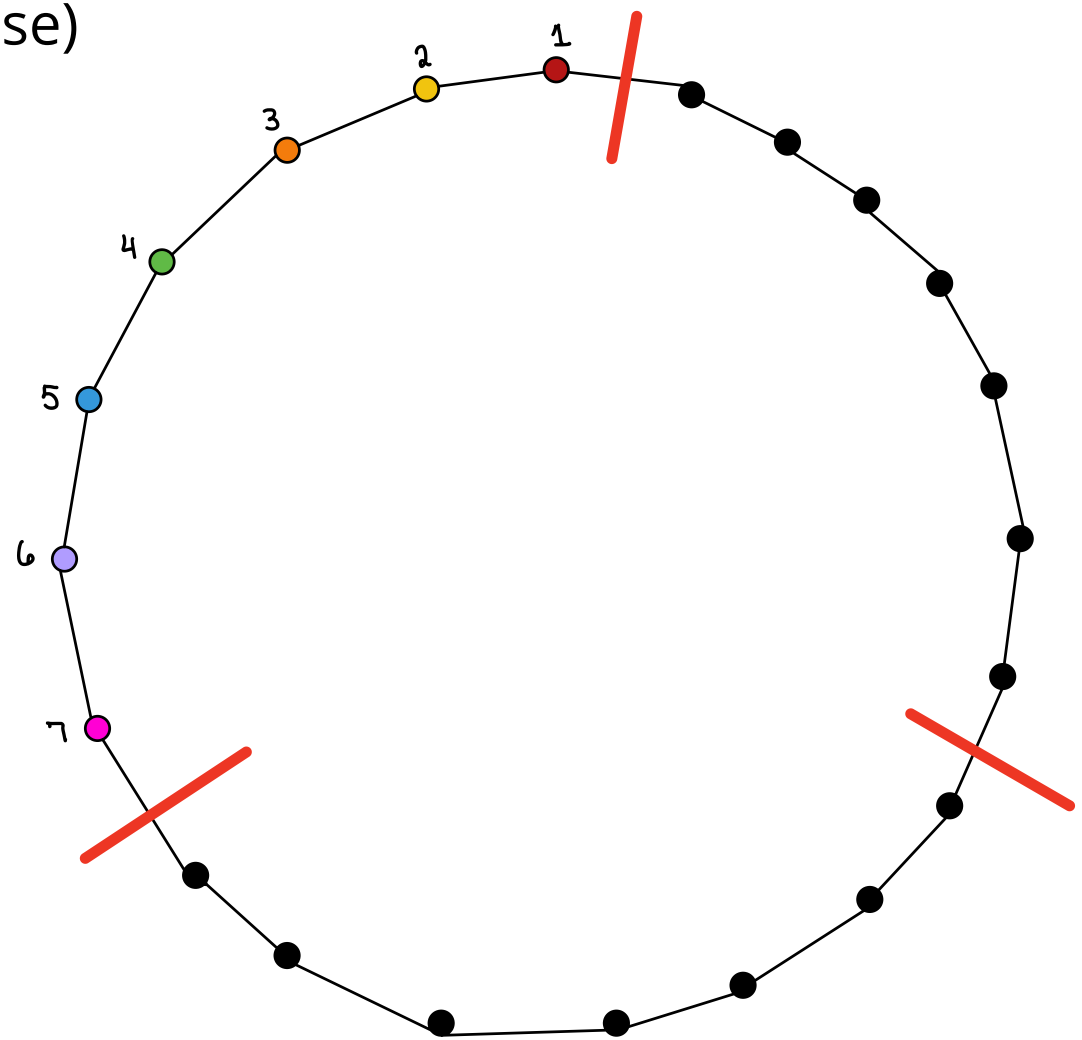
Lemma 7

Let C_n be the cycle graph on n vertices. $\chi_H(C_n) = \psi(C_n) = 2k+1$ whenever $n = k(2k+1)$, $k \in \mathbb{Z}^+$.

To illustrate the algorithm used to prove this, we will consider the case of C_{27} and show $\chi_H(C_{27}) = \psi(C_{27}) = 7$.

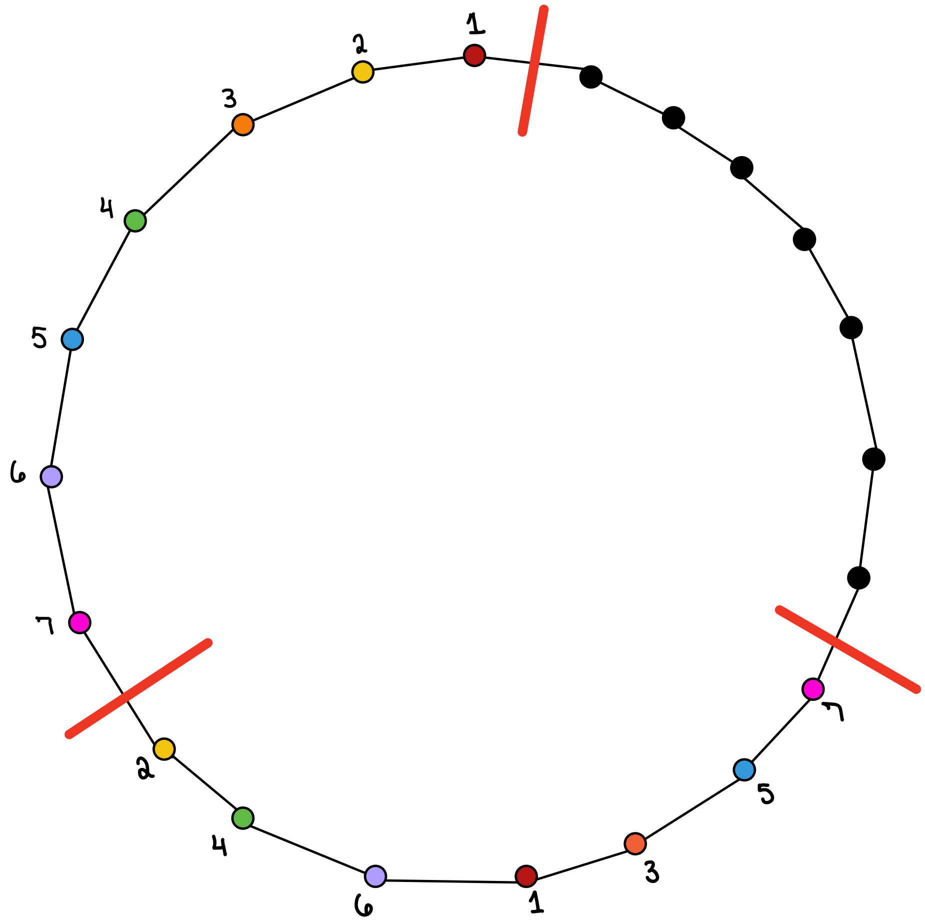
Equality for Cycles

- Partition cycle in k parts ($k = 3$ in this case)
- Color first part using each of the $2k + 1$ colors once in order (7 colors in this case)



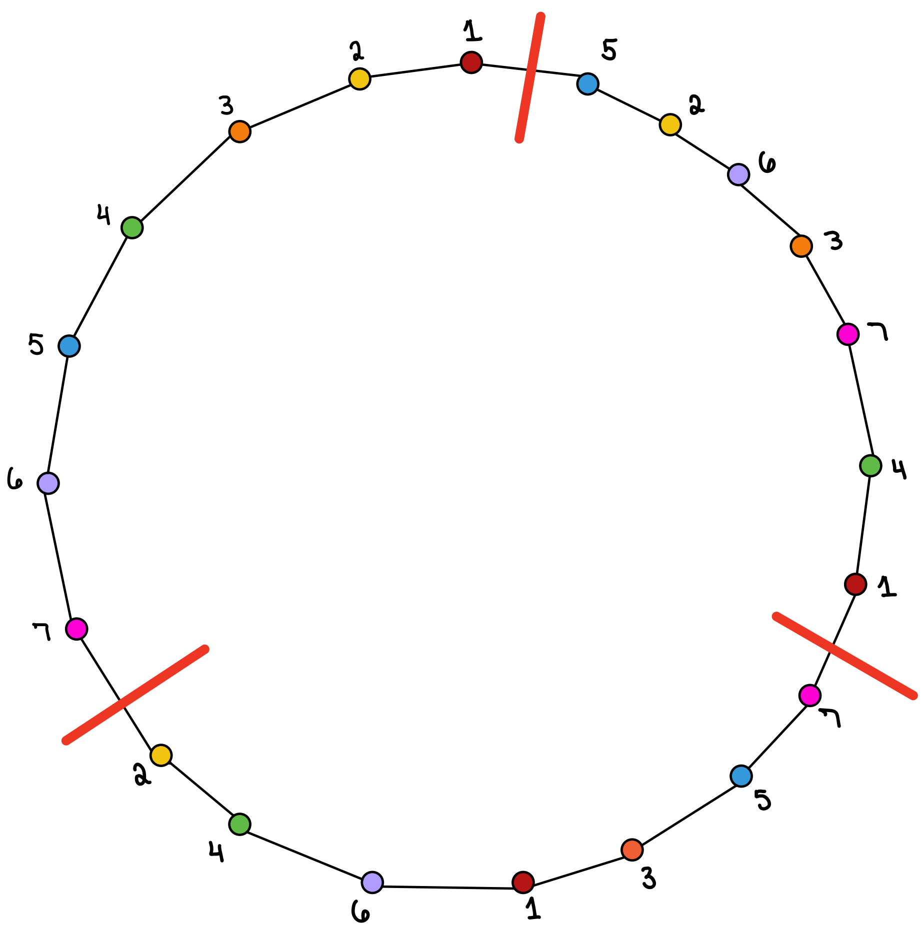
Equality for Cycles

- Color the next part using each of the $2k+1$ colors once starting with color 2 and using every other color
- In general, we repeat this process for every partition (except the last one): start with the color for the numbered partition, say m you're on and the next vertex is colored with $2m$, then $3m$, and so on



Equality for Cycles

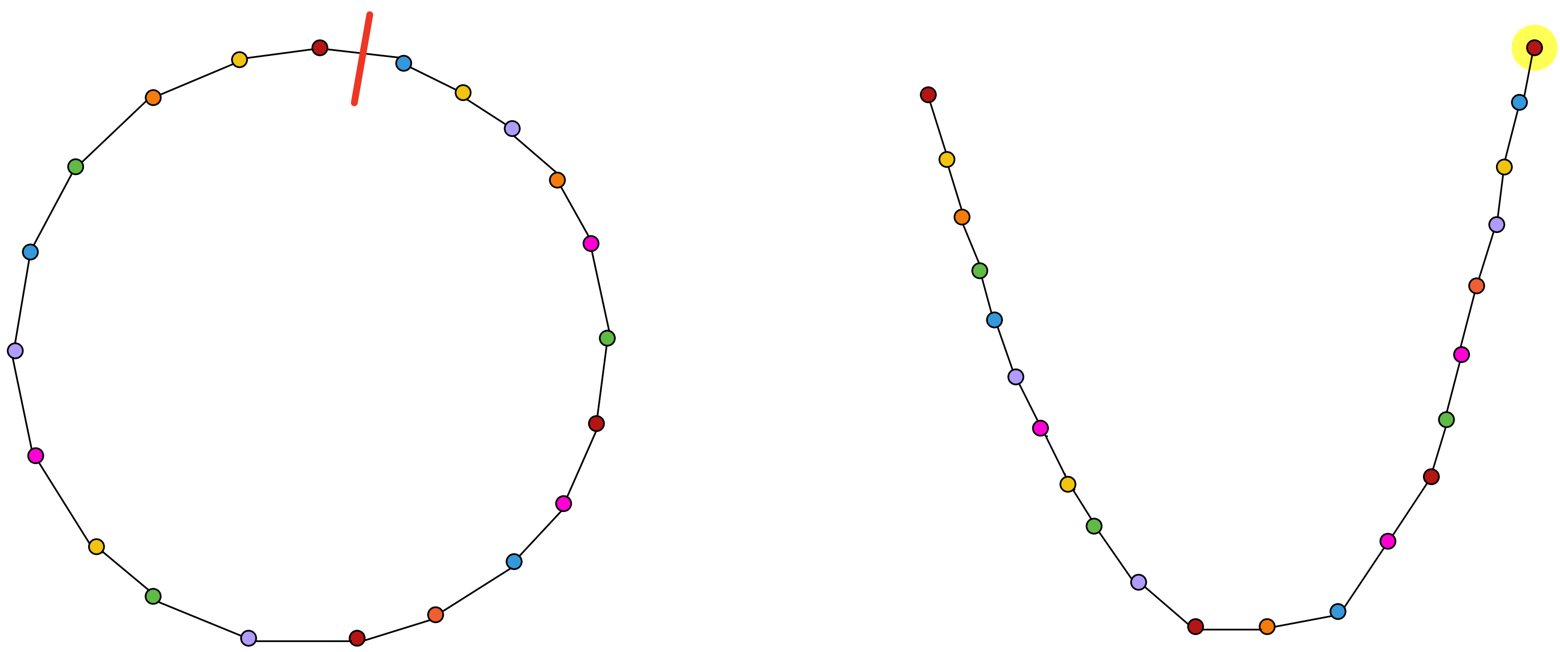
- For the last partition, the colors of the vertices are forced for almost all of the vertices
- In the case, where it is not forced, we choose the lowest color available



Corollary 8

Let P_n be the path graph on n vertices. $\chi_H(P_n) = \psi(P_n) = 2k + 1$ whenever $n = k(2k + 1) + 1$, $k \in \mathbb{Z}^+$.

We will illustrate the proof of this by considering C_{27}.



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