MATH 411: TOPICS IN ADVANCED ANALYSIS
HOMEWORK DUE WEDNESDAY WEEK 8

Problem 1. Complete the inductive proof from class that $\mathrm{eval}\colon A \to \hat{\hat{A}}$ is an isomorphism when A is a finite Abelian group. In particular, you need to prove that if eval is an isomorphism for finite Abelian groups A and B, then it is also an isomorphism for $A \times B$.

Problem 2. Let A be a finite Abelian group and let $V = \mathbb{C}^A$ be the complex vector space of $\mathbb{C}$-valued functions on A. The *convolution* of $f, g \in V$ is defined to be

$$\begin{aligned} f * g\colon A &\longrightarrow \mathbb{C} \\ a &\longmapsto \frac{1}{|A|} \sum_{b \in A} f(b) g(ab^{-1}). \end{aligned}$$

(a) Show that for all $f, g \in V$ and $\chi \in \hat{A}$, one has

$$\widehat{f * g}(\chi) = \hat{f}(\chi)\hat{g}(\chi).$$

(b) Show that if $\chi \in \hat{A}$ and $1 \neq a \in A$, then

$$\sum_{\chi \in \hat{A}} \chi(a) = 0.$$

(You may want to use the fact that $\hat{A}$ is an orthogonal basis for V.)

Problem 3. Define the *Fourier series* of $f \in V$ to be

$$\begin{aligned} Sf\colon A &\longrightarrow \mathbb{C} \\ a &\longmapsto \sum_{\chi \in \hat{A}} \hat{f}(\chi)\chi(a). \end{aligned}$$

(a) Show that

$$Sf = f * D$$

where D is defined by

$$D(a) = \frac{1}{|A|} \sum_{\chi \in \hat{A}} \chi(a) = \begin{cases} 1 & \text{if } a = 1, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Prove that $f * D = f$ and thus $Sf = f$.