24. X . 2

Goals
- $\ker f = \{0\} \Leftrightarrow f$ injective
- isomorphism type is given by dimension

Throughout, $V, W$ $F$-vs's, $f: V \longrightarrow W$ linear transformation.

**Thm** (ker $f = 0$ detects injectivity)

A linear transformation $f$ is injective iff $\ker f = \underset{\text{now writing } 0 \text{ for triv subspace of } V}{\overset{\{0_V\}}{0}}$.

**Pf** ($\Rightarrow$) Since $f$ is linear, $f(0) = 0$, so $0 \in \ker f$. As $f$ is injective, no other vectors map to $0$ under $f$, so $\ker f = 0$.

($\Leftarrow$) Suppose $v, w \in V$ and $f(v) = f(w)$. Then

$$0 = f(v) - f(w) = f(v-w) \quad \text{[linearity]}$$

so $v-w \in \ker f = 0 \Rightarrow v-w=0 \Rightarrow v=w$. Hence $f$ is inj. $\square$

**Prop** For $S \subseteq V$,

(1) $S$ lin dep $\Rightarrow fS = \{f(s) \mid s \in S\}$ lin dep

(2) $S$ lin ind + $f$ inj $\Rightarrow fS$ lin ind.

Q Why is this a necessary hypothesis?

$f: V \xrightarrow{} W$ destroys lin ind ($V \neq 0$)

$v \longmapsto 0$

$s_1 \longmapsto v$

$s_2, \dots, s_n \longmapsto 2v$

**Pf of Prop** (1) Suppose $\sum \lambda_i s_i = 0$ for $\lambda_i \in F$, $s_i \in S$. Applying $f$,

$f(\sum \lambda_i s_i) = \sum \lambda_i f(s_i) = 0$ so $f$ preserves lin dependencies.

(2) Fix $f(s_1), \dots, f(s_n) \in fS$ and suppose

$$\sum \lambda_i f(s_i) = 0 \implies f\left(\sum \lambda_i s_i\right) = 0 \quad \text{[linearity of } f\text{]}$$

$$\implies \sum \lambda_i s_i \in \ker f$$

Since $f$ is injective, $\ker f = 0 \implies \sum \lambda_i s_i = 0$.

Since $S$ lin ind, all $\lambda_i = 0 \implies fS$ lin ind. $\square$

<u>Recall</u> A lin trans'n $f: V \to W$ is an <u>isomorphism</u> when it

- admits a linear inverse $g: W \to V$

or, equivalently,

- is a bijection.

<u>Note</u> Isomorphism is an equivalence rel'n on any set of vector spaces. [transitive]

$$\begin{tikzcd} V \arrow[r, "f", "\cong"'] & W \arrow[l, bend left] \arrow[r, "g", "\cong"'] & U \arrow[l, bend left] \end{tikzcd} \quad \text{then} \quad \begin{tikzcd} V \arrow[r, "g\circ f"] & U \arrow[l, bend left, "f^{-1}\circ g^{-1} = (g\circ f)^{-1}"] \end{tikzcd}$$

**E.g.** • $F^{2\times 2} \cong F^4$ via $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto (a,b,c,d)$

• Let $F[[x]]$ denote power series with coefficients in $F$.

Then $F[[x]] \cong F^{\mathbb{N}}$ via $\lambda_0 + \lambda_1 x + \lambda_2 x^2 + \cdots \longmapsto (i \mapsto \lambda_i)$.

$\rotatebox{90}{$\subseteq$}$ $\quad$ $\rotatebox{90}{$\subseteq$}$ $\quad$ F-valued sequences ... generating functions!

$F[x] \cong$ $\quad$ sequences that are eventually $0$.

**Prop** A lin trans'n $f : V \to W$ is an isomorphism iff $\underbrace{\ker f = 0}_{\text{inj}}$ and $\underbrace{\operatorname{im} f = W}_{\text{surj}}$. $\square$

$$\boxed{\begin{array}{l}\textbf{Thm} \text{ If } V, W \text{ are finite-dimensional } F\text{-vs's, then} \\[1em] \qquad\qquad V \cong W \iff \dim V = \dim W.\end{array}}$$

$(\Rightarrow)$

**Pf** Since $\cong$ is symmetric, it suffices to prove that

$$\dim V = n \implies V \cong F^n.$$

Choose an ordered basis $\mathcal{B} = (b_1, \dots, b_n)$ of $V$. Then

$$\mathrm{Rep}_{\mathcal{B}} : V \xrightarrow{\cong} F^n, \quad b_i \longmapsto e_i.$$

$(\Leftarrow)$ Isomorphisms $f: V \to W$ preserve bases: $\mathcal{B}$ basis of $V$. Then $f\mathcal{B}$ is lin ind (prop) and if $w \in W$ $\exists!\, v \in V$ s.t. $f(v) = w$ and $f(\sum \lambda_i b_i) = \sum \lambda_i f(b_i)$ so $w \in \operatorname{span} f\mathcal{B}$ $\square$

**Note** Specifying an iso $V \xrightarrow{\cong} F^n$ is equivalent to choosing an ordered basis of $V$ b/c the inverse $g: F^n \longrightarrow V$ gives $g(e_1), \dots, g(e_n)$ as a basis of $V$.

Week 3 #6 Claim $S = \{(a,b), (c,d)\} \subseteq F^2$ lin ind

iff $ad - bc \neq 0$.

★ Given $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, let $\Delta = \Delta(A) = ad - bc$.

$(a,b), (c,d)$ are lin ind iff

$$\operatorname{rref}\left(\begin{array}{cc|c} a & c & 0 \\ b & d & 0 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right)$$

Note (1) elementary row ops take $A := \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ to $\operatorname{rref}(A)$.

(2) observe what each op'n does to $\Delta$:

| | | | | | |
|---|---|---|---|---|---|
| $\underline{A}$ | $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ | $\begin{pmatrix} b & d \\ a & c \end{pmatrix}$ | $\begin{pmatrix} \lambda a & \lambda c \\ b & d \end{pmatrix}$ | $\begin{pmatrix} a & c \\ \lambda b & \lambda d \end{pmatrix}$ | $\begin{pmatrix} a & c \\ b+\lambda a & d+\lambda c \end{pmatrix}$ |
| $\underline{\Delta}$ | $ad-bc$ | $bc-ad$ $= -1\cdot(ad-bc)$ | $\lambda(ad-bc)$ | $\lambda(ad-bc)$ | $ad+\cancel{\lambda ac}$ $-bc-\cancel{\lambda ac}$ $= ad-bc$ |

Thus the effect of a row op on $\Delta$ is to scale by a nonzero $\lambda$

Since rref$(A)$ is given by a sequence of row ops,

$$\Delta(A) = \lambda\cdot\Delta(\text{rref}(A))$$

so $S$ lin ind iff $\mathrm{rref}(A) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\Delta = 1 \implies \Delta(A) = \lambda \cdot 1, \ \lambda \neq 0$$

$$\neq 0$$