

# Automorphisms of a covering

$q: E \rightarrow X$  covering

$$\text{Aut}_q(E) := \left\{ \begin{array}{c} E \xrightarrow{\cong} E \\ \downarrow q \quad \downarrow q \end{array} \right\}$$

aka deck transformations  
or covering transformations

Prop  $q: \tilde{E} \rightarrow X$  covering,  $\varphi, \psi \in \text{Aut}_q(\tilde{E})$

(a)  $\varphi = \psi \Leftrightarrow \exists e \in \tilde{E} \text{ s.t. } \varphi(e) = \psi(e)$

(b)  $\varphi|_{q^{-1}\{x\}}$  is a  $\pi_1(X, x)$ -equivariant automorphism of  $q^{-1}\{x\}$

(c)  $U \subseteq X$  evenly covered open  $\Rightarrow \varphi$  permutes components of  $q^{-1}U$

(d)  $\text{Aut}_q(\tilde{E}) \subset E$  freely □

E.g.  $\text{Aut}_{\mathbb{Z}_n}(\mathbb{R}^n) \cong \mathbb{Z}^n$

$$\begin{array}{ccc} x & \xrightarrow{\quad} & (k_1, \dots, k_n) = k \\ \downarrow & & \\ x+k & & \end{array}$$

E.g.  $q: S^n \rightarrow \mathbb{RP}^n$  has  $\text{Aut}_q(S^n) = \{\text{id}, \text{antip}\}$ .

Thm (orbit criterion for covering automorphisms)  $q: E \rightarrow X$  covering

If  $e_1, e_2 \in q^{-1}\{x\}$ , then  $\exists \varphi \in \text{Aut}_q(E)$  s.t.  $\varphi(e_1) = e_2$  iff

$$q_* \pi_1(E, e_1) = I_* \pi_1(E, e_2). \quad \square$$

Cor  $\text{Aut}_q(E)$  acts transitively on each fiber iff  $q$  is normal  
 (i.e. some/all  $q_* \pi_1(E, e) \cong \pi_1(X, x)\}$ ).  $\square$

Thm  $q: E \rightarrow X$  covering,  $x \in X$ . Then

$$\text{Aut}_q(E) \xrightarrow{\cong} \text{Aut}_{\pi_1(X, x)}(q^{-1}\{x\}) = \{ \eta : q^{-1}\{x\} \rightarrow q^{-1}\{x\} \mid \begin{array}{l} \eta \circ \pi_1^{-1} \\ \text{is } f\text{-iso} \\ \text{and } \pi_1(X, x)\text{-equivariant} \\ \text{automorphisms of fiber} \end{array} \}$$

$\varphi \longmapsto \varphi|_{q^{-1}\{x\}}$

Pf Prop (b)  $\Rightarrow$  homomorphism.

Prop (a)  $\Rightarrow$  injective.

$$\left\{ \begin{array}{l} f: A \rightarrow B \quad G\text{-iso} \\ a \mapsto b \\ G_b = G_a \quad g \cdot a = a \end{array} \right.$$

For surjective, suppose  $\eta : q^{-1}\{x\} \xrightarrow{\cong} q^{-1}\{x\}$ . For  $e_i \in q^{-1}\{x\}$ ,  $f(e_i) = b$   
 $e_2 = \eta(e_1)$ , the isotropy of  $e_1, e_2$  are the same.

thus  $q_* \pi_1(E, e_1) = q_* \pi_1(E, e_2) \Rightarrow \exists \psi \in \text{Aut}_q(E)$  with  $\psi|_b$

$\varphi(e_1) = e_2$ . Then  $\eta, \varphi|_{q^{-1}\{x\}}$  are  $\pi_1(X, x)$ -equivariant for  
of  $q^{-1}\{x\}$  agreeing at  $e_1$ , so they are equal.  $\square$

Thm (Covering Aut Group Structure)  $\tilde{\iota}: \tilde{E} \rightarrow X$  covering.

$G = \pi_1(X, x)$ ,  $H = q_* \pi_1(\tilde{E}, e) \leq G$ . For each  $\gamma \in N_G(H)$

$\exists! \varphi_\gamma \in \text{Aut}_{\tilde{q}}(\tilde{E})$  s.t.  $\varphi_\gamma(e) = e \cdot \gamma$ . The map

$N_G(H) \rightarrow \text{Aut}_{\tilde{q}}(\tilde{E})$  is a surjective group hom with  
 $\gamma \mapsto \varphi_\gamma$

kernel  $H \Rightarrow N_G(H) \cong \text{Aut}_{\tilde{q}}(\tilde{E})$

$\uparrow$  i.e.  $N_{\pi_1(X, x)}(q_* \pi_1(\tilde{E}, e)) / q_* \pi_1(\tilde{E}, e)$

Pf We have  $\omega_G(H) \xrightarrow{\cong} \text{Aut}_G(q^{-1}\{x\}) \xleftarrow{\cong} \text{Aut}_q(\bar{E})$ .

$$\begin{array}{ccc}
 & & \\
 H\gamma & \longmapsto & \left( q^{-1}\{x\} \xrightarrow{\cong} \bar{\gamma}^{-1}\{x\} \right) \\
 & \text{s.t. } e \mapsto e\gamma & \\
 & & \\
 & \varphi|_{q^{-1}\{x\}} & \longleftarrow \varphi \\
 & & \\
 H\gamma & \longmapsto & \varphi_\gamma
 \end{array}$$

□

Cor (Normal case) If  $q: E \rightarrow X$  is a normal covering,

then  $\forall x \in X, e \in q^{-1}\{x\}$ ,  $\pi_1(X, x)/\pi_1(E, e) \xrightarrow{\cong} \text{Aut}_q(\bar{E})$

$$(q_*\pi_1(E, e))\gamma \longmapsto \varphi_\gamma$$

□

Cor (Universal cover case) If  $q: \bar{E} \rightarrow X$  covering w/  $\bar{E}$  simply

connected, then  $\pi_1(X, x) \xrightarrow{\cong} \text{Aut}_q(E)$

$$g \mapsto \varphi_g$$

i.e.  $\text{Aut}_q(\tilde{X}) \cong \pi_1(X, x)$

□

### Quotients by group actions

- $\text{Aut}_q(E) \curvearrowright E$  freely by homeomorphisms
- $\text{Aut}_q(E) \curvearrowright q^{-1}\{x\}$  transitively for  $q$  normal

$\Rightarrow X \cong E/\text{Aut}_q(E)$  with  $e_1 \sim e_2$  iff  $e_2 = \varphi(e_1)$  for some  $\varphi \in \text{Aut}_q(E)$   
for  $q$  normal

Now suppose  $\Gamma$  a group  $\curvearrowright E$

When is  $q: E \rightarrow E/\Gamma$  a covering map?

Note  $E \rightarrow E/\Gamma$  necessarily normal since  $\Gamma$  acts transitively on fibers.

aka properly discontinuous

Defn  $\Gamma \times E$  is a covering space action if it acts by homeomorphisms (i.e. is cts) and every  $e \in E$  has a nbhd  $U$  s.t.

$$\forall g \in \Gamma, \quad U \cap (g \cdot U) = \emptyset \text{ unless } g = 1. \quad (U \cap (g \cdot U) \neq \emptyset \text{ iff } g = 1)$$

Call the action affectionate when  $g \cdot e = e \forall e \in E \Rightarrow g = 1$  (so only the unit of  $\Gamma$  acts trivially).

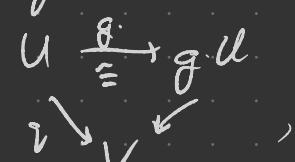
Thm (Covering space quotient theorem)  $E$  conn'd, locally path conn'd,  $\Gamma \times E$  effectively by homeos. Then the quotient  $q: E \rightarrow E/\Gamma$  is a covering map iff the action is a covering space action.

In this case,  $q$  is normal and  $\text{Aut}_q(E) = \Gamma$ .

Pf First assume  $q$  is covering. Each  $g \in \Gamma$  induces a covering automorphism, so  $\Gamma \leq \text{Aut}_q(E)$ . Check that  $\text{Aut}_{q\Gamma}(E)$  is covering, and the restriction of a covering action to a subgroup is covering.

For the converse, suppose  $\Gamma \trianglelefteq E$  is covering. Have  $q: E \rightarrow E/\Gamma$ cts, surjective, open. Take  $x \in E/\Gamma$  and choose  $e \in q^{-1}\{x\}$ . Take  $U$  a nbhd of  $e$  s.t.  $\forall g \in \Gamma$ ,  $U \cap (g \cdot U) = \emptyset$  unless  $g = 1$ .

WLOG,  $U$  path conn'd. Take  $V = q(U)$ , necessarily a path conn'd nbhd of  $x$ . Then  $q^{-1}V = \bigsqcup_{g \in \Gamma} g \cdot U$ . Since  $U \xrightarrow{\cong} g \cdot U$

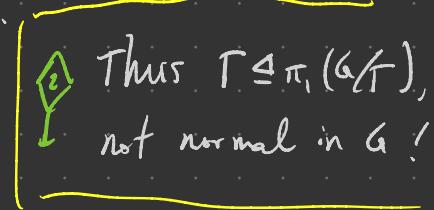


suffices to show  $g|_U: U \xrightarrow{\cong} V$ .  $g|_U$  is surj,cts,open, and it is injective b/c  $g(e) = g(e')$  for  $e, e' \in U \Rightarrow e = e'$  for some  $g \in \Gamma$ .

Since  $\Gamma \backslash E$  is covering, get  $e = e'$ . Thus  $g$  is a covering map.  $\square$

Prop  $\Gamma \leq G \Rightarrow G \curvearrowright \Gamma$  is a covering space action  $\Rightarrow g: G \rightarrow G/\Gamma$

$\begin{array}{c} / \\ \text{discrete} \\ \text{sub gp} \end{array}$   $\begin{array}{c} \backslash \\ \text{conn'd} \\ \text{top'l} \\ \text{group} \end{array}$  is a normal covering map.

 Thus  $\Gamma \trianglelefteq_{\pi_1}(G/\Gamma)$ ,  
not normal in  $G$ !

Pf  $\Gamma$  discrete  $\Rightarrow \exists$  nbhd  $V$  of 1 in  $G$  s.t.

$V \cap \Gamma = \{1\}$ . Define  $F: G \times G \rightarrow G$  Then  $F^{-1}V$  is  $\subset$  nbhd of  $(g, h) \mapsto g^{-1}h$ .

$(1, 1) \Rightarrow \exists$  product nbhd  $U_1 \times U_2$  s.t.  $(1, 1) \in U_1 \times U_2 \subseteq F^{-1}V$

Set  $U = U_1 \cap U_2$ . Then  $g, h \in U \Rightarrow g^{-1}h \in V$ .

WTS  $U$  satisfies covering action condition:  $U \cap (Ug) \neq \emptyset \Rightarrow g = 1$ .

Suppose  $g \in \Gamma$  s.t.  $U \cap (Ug) \neq \emptyset$ . Then  $\exists h \in U$  s.t.  $hg \in U$ .

By construction,  $g = h^{-1}(hg) \in V \cap \Gamma \Rightarrow g = 1$  as desired.  $\square$

Cor Suppose  $G, H$  conn'd, loc path conn'd top'l gps,  $\varphi: G \rightarrow H$  surj cts homomorphism with discrete kernel. If  $\varphi$  is an open or closed map, then it is a normal covering.  $\square$

↳ p.314

E.g.  $SU(2) \rightarrow SO(3)$

$$\mathbb{H}^2$$

$$S^3$$

$$\mathbb{H}^2$$

$$\mathbb{RP}^3$$