Subspaces

> **Goals** • Subspaces
> • Linear combinations • Span

24. IX. 13

$$\mathbb{R}^{\mathbb{R}}$$
$$\cup\!\!\mid$$
$$C(\mathbb{R}) = \{f \mid f:\mathbb{R}\to\mathbb{R} \text{ is continuous}\}$$
$$\cup\!\!\mid$$
$$C'(\mathbb{R}) = \{f \mid f:\mathbb{R}\to\mathbb{R} \text{ with } f' \text{ continuous}\}$$
$$\cup\!\!\mid$$
$$C^{\infty}(\mathbb{R}) = \{f \mid f:\mathbb{R}\to\mathbb{R} \text{ has continuous derivatives of all orders}\}$$
$$\cup\!\!\mid$$
$$\mathbb{R}[x] = \{f \mid f:\mathbb{R}\to\mathbb{R} \text{ is a polynomial function}\}$$

Each of these satisfies $f,g\in W \Rightarrow f+g\in W,\ \lambda f\in W,\ 0\in W$

In fact, each is a vector space in its own right.

Defn A subset $W \subseteq V$ of an F-vs is a **subspace** of V when W is an F-vs with addition and scalar multiplication inherited from V; write $W \leq V$.

$$\begin{array}{ccc} V \times V & \xrightarrow{+} & V \\ \cup\!\mathsf{I} & & \cup\!\mathsf{I} \\ W \times W & \xrightarrow[+]{} & W \end{array} \qquad \begin{array}{ccc} F \times V & \xrightarrow{\cdot} & V \\ \cup\!\mathsf{I} & & \cup\!\mathsf{I} \\ F \times W & \xrightarrow[\cdot]{} & W \end{array}$$

Prop A subset $W \subseteq V$ is a subspace iff

(1) $0 \in W$, (2) $u, v \in W \Rightarrow u + v \in W$, & (3) $u \in W \Rightarrow \lambda u \in W$

iff

(1) $0 \in W$ & (2') $u, v \in W \Rightarrow u + \lambda v \in W$.

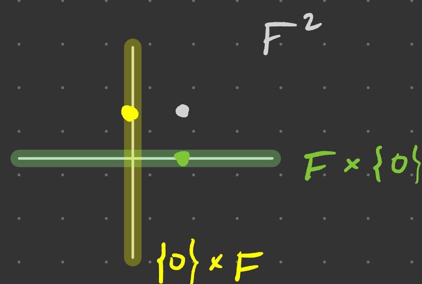
Pf Two. I Lemma 2.9 $\square$

E.g. $\{(x,0) \mid x\in F\} \le F^2 \ge \{(0,y) \mid y\in F\}$

E.g. $\{0\}, V \le V$

Q Is $F\times\{0\} \cup \{0\}\times F$ a subspace of F^2?

$F=\mathbb{R}$



NO!

Q If $U, W \le V$, is $U\cap W \le V$? YES!

(1) $0\in U$ and $0\in W \implies 0\in U\cap W$

(2') Suppose $u, v\in U\cap W$, $\lambda\in F$. Then $u+\lambda v\in U$ and $u+\lambda v\in W$ (b/c U, W subspaces) so $u+\lambda v\in U\cap W$. $\square$

Fix V an F-vs.

Defn For $S \subseteq V$, a linear combination of vectors in S is a vector of the form $\sum_{i=1}^{n} \lambda_i u_i = \lambda_1 u_1 + \dots + \lambda_n u_n$ for some $\lambda_i \in F$, $u_i \in S$, $n \in \mathbb{N}$.

Always finitely many terms!

Note $n = 0$ get empty sum $= 0$

E.g. Is $(-1,4)$ a linear combination of vectors in $\{(3,2), (2,-1)\} \subseteq \mathbb{R}^2$?

Looking for $a, b \in \mathbb{R}$ s.t. $a(3,2) + b(2,-1) = (-1,4)$, i.e.

$$\begin{aligned} 3a + 2b &= -1 \\ 2a - b &= 4 \end{aligned} \rightsquigarrow \left(\begin{array}{cc|c} 3 & 2 & -1 \\ 2 & -1 & 4 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -2 \end{array}\right)$$

so $a = 1$, $b = -2$ works. Indeed, $1 \cdot (3,2) - 2 \cdot (2,-1) = (3-4, 2+2) = (-1,4)$.

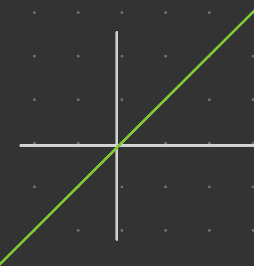
Defn For $S \subseteq V$, the span of S is

$$\text{span}(S) := \{\text{linear combos of vectors in } S\}.$$

Note $\text{span}(\varnothing) = \{0\}$.

E.g.

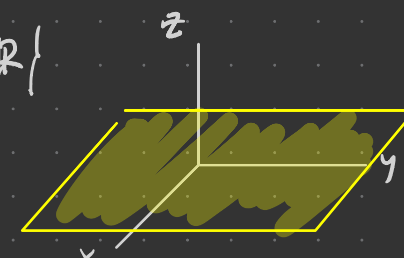
- $\text{span}\{v\} = \{\lambda v \mid \lambda \in F\}$



- $\text{span}_{\mathbb{R}}\{(1,1)\} = \{(a,a) \mid a \in \mathbb{R}\}$

- $\text{span}_{\mathbb{R}}\{(1,0,0),(0,1,0)\} = \{a(1,0,0) + b(0,1,0) \mid a,b \in \mathbb{R}\}$

$$= \{(a,b,0) \mid a,b \in \mathbb{R}\}$$



Spanning sets are not unique!

$$\operatorname{span}\{(1,0,0),\ (0,1,0)\} = \operatorname{span}\{(1,0,0),\ (0,2,0)\}$$

$$= \operatorname{span}\{(1,0,0),(0,1,0),(2,3,0)\}.$$

$$= \operatorname{span} \mathbb{R}^2 \times \{0\}$$

Prop Let $S \subseteq V$. Then

(1) $\operatorname{span} S \leq V$

(2) If $S \subseteq W \leq V$, then $\operatorname{span} S \leq W$

(3) Every subspace of V is the span of some subset of V.

Pf (1) If $S = \emptyset$, then $\operatorname{span} S = \{0\} \leq V$.

If $S \neq \emptyset$, then $\exists u \in S$ so $0 \cdot u = 0 \in \operatorname{span} S$. Given $\sum \lambda_i u_i$, $\sum \mu_j v_j \in \operatorname{span} S$ with $\lambda_i, \mu_j \in F$, $u_i, v_j \in S$, get

$$\sum \lambda_i u_i + \lambda \sum_j \mu_j v_j = \sum \lambda_i u_i + \sum (\lambda \mu_j) v_j \in \operatorname{span} S$$

for all $\lambda \in F$, so $\operatorname{span} S \leq V$.

(2) If $S \subseteq W \leq V$, then $\operatorname{span} S \leq W$

(2) Justify this statement:

$$S \underset{①}{\subseteq} \operatorname{span} S \underset{③}{\subseteq} \operatorname{span} W \underset{②}{=} W.$$

① If $u \in S$, then $u = 1 \cdot u$ is a lin combo of elts of S so $u \in \operatorname{span} S$.

② Have $W \subseteq \operatorname{span} W$ by ①. Suppose $\sum \lambda_i u_i \in \operatorname{span} W$ with $\lambda_i \in F, u_i \in W$. Then this is in W b/c W is closed under scalar mult'n and add'n. Thus $\operatorname{span} W \subseteq W$ as well, thus $\operatorname{span} W = W$.

③ By (1), $\operatorname{span} S$ is a vector space. Apply span to $S \subseteq W$ to get containment.

(3) $W = \operatorname{span} W$ for $W \leq V$. $\square$

Say $S \subseteq V$ generates $W \leq V$ when $\operatorname{span} S = W$.

E.g. (1) $\{1, x, x^2, \ldots\}$ generates $F[x]$

(2) $\{(1,0), (0,1)\}$ generates F^2

(3) Set $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in F^n$. Then $\{e_1, \ldots, e_n\}$ generates F^n.

i-th pos'n

$(a_1, \ldots, a_n) = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n$

(4) For set S and $s \in S$, define

$$\begin{aligned} \chi_s : S &\longrightarrow F \\ t &\longmapsto \begin{cases} 1 & \text{if } t = s \\ 0 & \text{if } t \neq s \end{cases} \end{aligned}$$

Then for S finite, $\{\chi_s \mid s \in S\}$ generates F^S.

Q What if S is infinite?