POLY-BERNOULLI NUMBERS & MATCHSTICK GAMES ON CYLINDERS

REED COLLEGE COLLABORATIVE MATHEMATICS RESEARCH GROUP 2024

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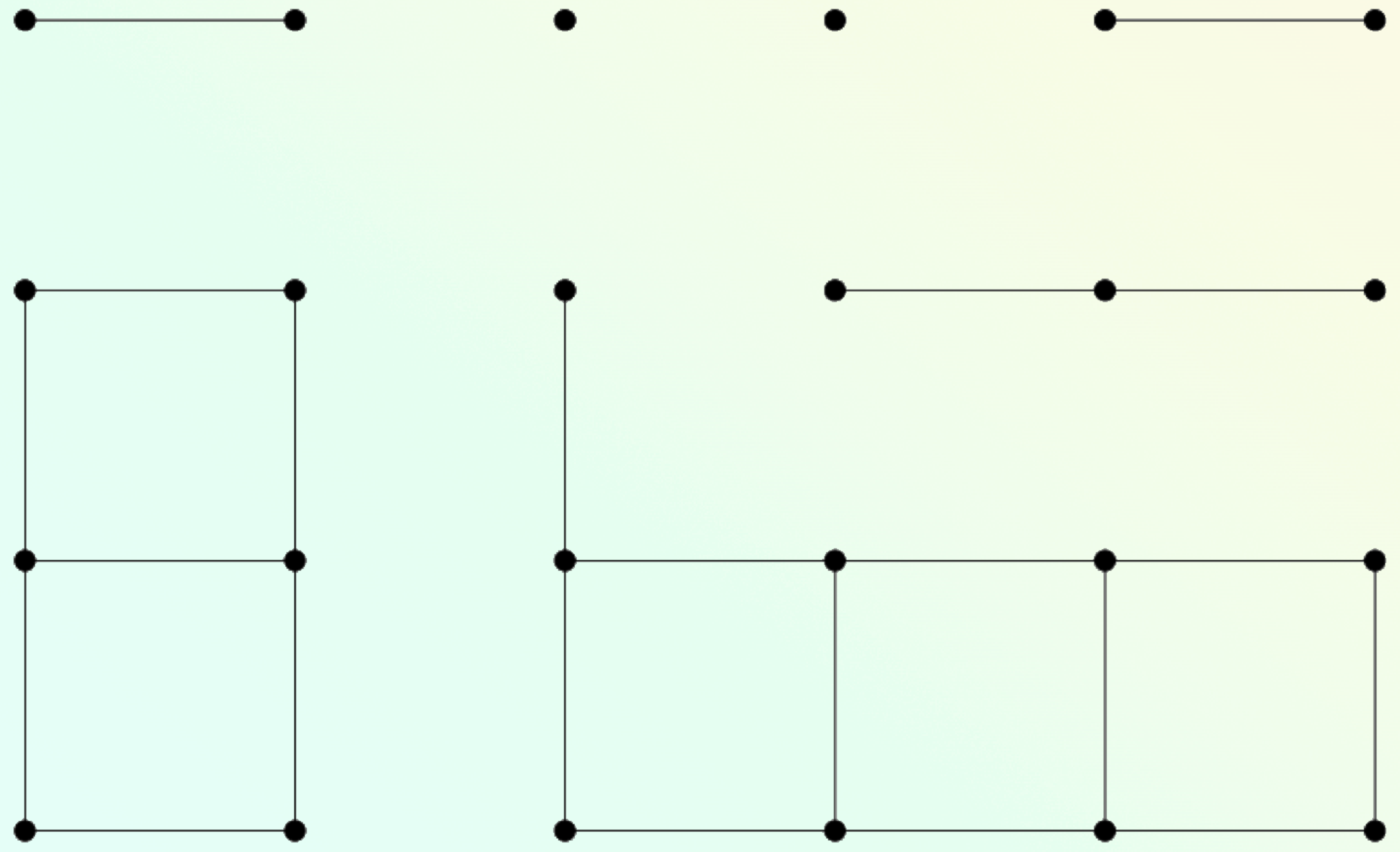
MATCHSTICK GAMES

SATURATED TRANSFER SYSTEMS ON MODULAR LATTICES

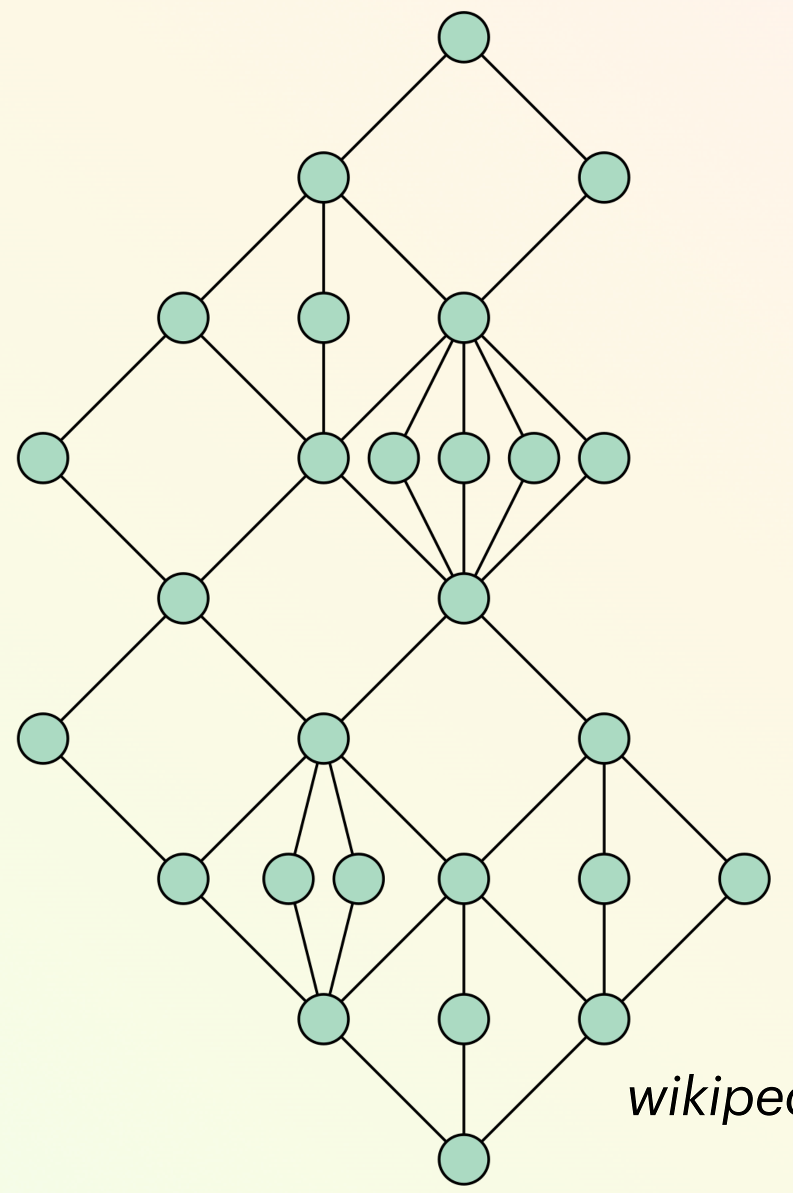
Write $[m] = \{0 < 1 < \cdots < m\}$ so that $[m] \times [n]$ is a **rectangular grid** poset.

MATCHSTICK GAME RULES:

- Vertical stick implies all sticks to its left
- Horizontal stick implies all sticks below it
- $3 \implies 4$ in unit boxes



MODULAR LATTICES



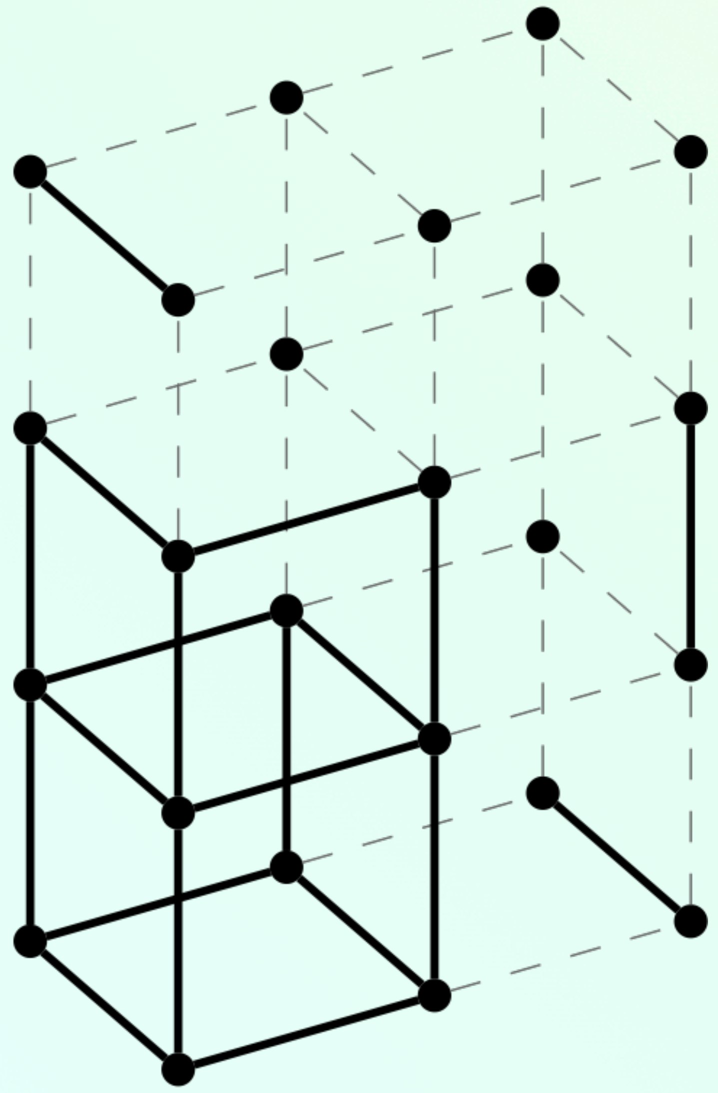
KEY STRUCTURAL PROPERTY OF SUBGROUP LATTICES OF ABELIAN GROUPS

A lattice P is **modular** when $x \leq y$ implies $x \vee (z \wedge y) = (x \vee z) \wedge y$.

Equivalently, $[x \wedge y, y] \cong [x, x \vee y]$ for all x, y (**diamond isomorphism**).

MATCHSTICK GAME RULES:

- Q a subset of covering relations of P
- $x \; Q \; (x \vee y) \implies (x \wedge y) \; Q \; y$
- $3 \Longrightarrow 4$ in covering diamonds



THEOREM. A lattice is modular if and if it contains no pentagonal sublattice.

WHY MATCHSTICK GAMES?

Matchstick games on modular lattice P enumerate

- saturated transfer systems
- submonoids of $(P, \vee)$
- max-closed relations (Knuth)
- interior operators
- coreflective factorization systems
- cofibrant model structures

(See Tien Chih's talk!)

COUNTING GAMES

THEOREM (Hafeez-Marcus-O-Osorno '22). The number of legal matchstick games on $[m] \times [n]$ is

$$\text{games}([m] \times [n]) = \sum_{j=2}^{m+2} (-1)^{m-j} \left\{ {m+1 \atop j-1} \right\} \frac{j!}{2} j^n$$

for $\left\{ {r \atop s} \right\}$ the Stirling number of the second kind counting s-block partitions of an r-element set, and these numbers satisfy the recurrence

$$\text{games}([m] \times [n+1]) = \text{games}([m] \times [n]) + \sum_{j=0}^{m} \binom{m+1}{j} \text{games}([j] \times [n]) \,.$$

POLY-BERNOULLI NUMBERS

BERNOULLI NUMBERS + POLYLOGARITHMS = COMBINATORICS

COROLLARY. Matchstick games on $[m] \times [n]$ are seminumerous with poly-Bernoulli numbers: $2s(m,n) = B_{m+1,n+1}$.

The **poly-Bernoulli numbers** $B_n^{(s)}$ [Kaneko 1997] are defined by

$$\sum_{n \geq 0} B_n^{(s)} \frac{z^n}{n!} = \frac{1}{1-e^{-z}} \sum_{k \geq 1} \frac{(1-e^{-z})^k}{k^s}.$$

Then $B_n^{(1)}$ is the classical Bernoulli numbers, and $B_{m,n} := B_n^{(-m)}$ is a positive integer for $m, n \in \mathbb{N}$.

POLY-BERNOULLI NUMBERS

BERNOULLI NUMBERS + POLYLOGARITHMS = COMBINATORICS

THEOREM [Kaneko; see Knuth 2024]. The (re-indexed) pB numbers satisfy

$$B_{m,n} = \sum_{k \geq 0} (k!)^2 \left\{ {m+1 \atop k+1} \right\} \left\{ {n+1 \atop k+1} \right\}$$

and their exponential generating function is

$$G(x,y) = \sum_{m,n \geq 0} B_{m,n} \frac{x^m y^n}{m!n!} = \frac{e^{x+y}}{e^x + e^y - e^{x+y}}.$$

CAN WE GENERALIZE THESE FORMULÆ SO THEY APPLY TO OTHER MODULAR LATTICES?

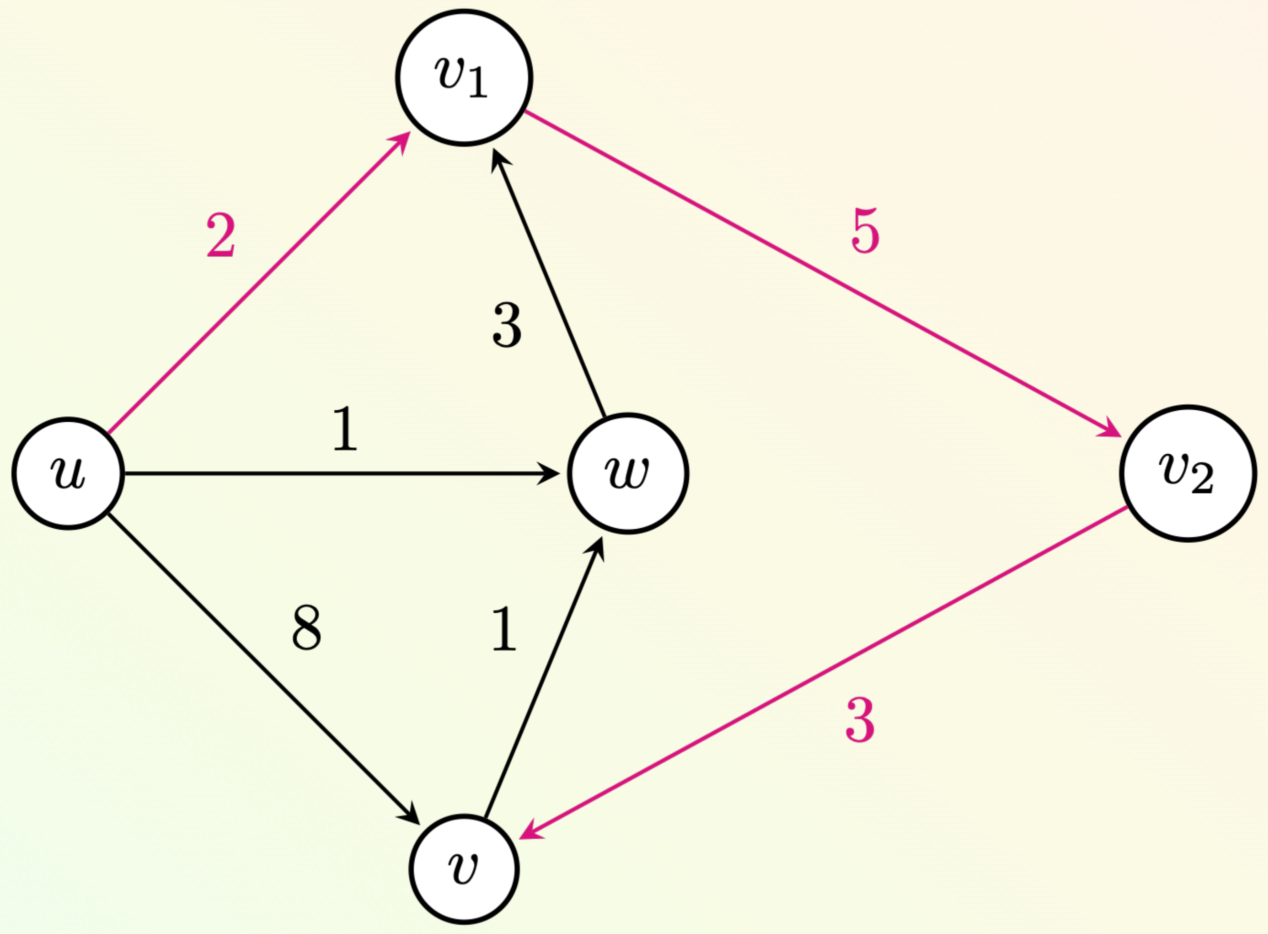
WE WILL FOCUS ON LATTICES OF THE FORM $P \times [n]$ FOR P MODULAR.

TRANSFER MATRIX METHOD

BUILDING $\mathrm{games}(P \times [n])$ **ONE LAYER AT A TIME**

If G is a weighted directed graph with adjacency matrix $A(G)$, then

$$(A(G)^n)_{u,v} = \sum_{\substack{\text{length } n \text{ walks} \\ v_0 v_1 \cdots v_n \\ u = v_0,\ v = v_n}} \prod_{i=0}^{n-1} \mathrm{weight}(v_i v_{i+1}) \, .$$



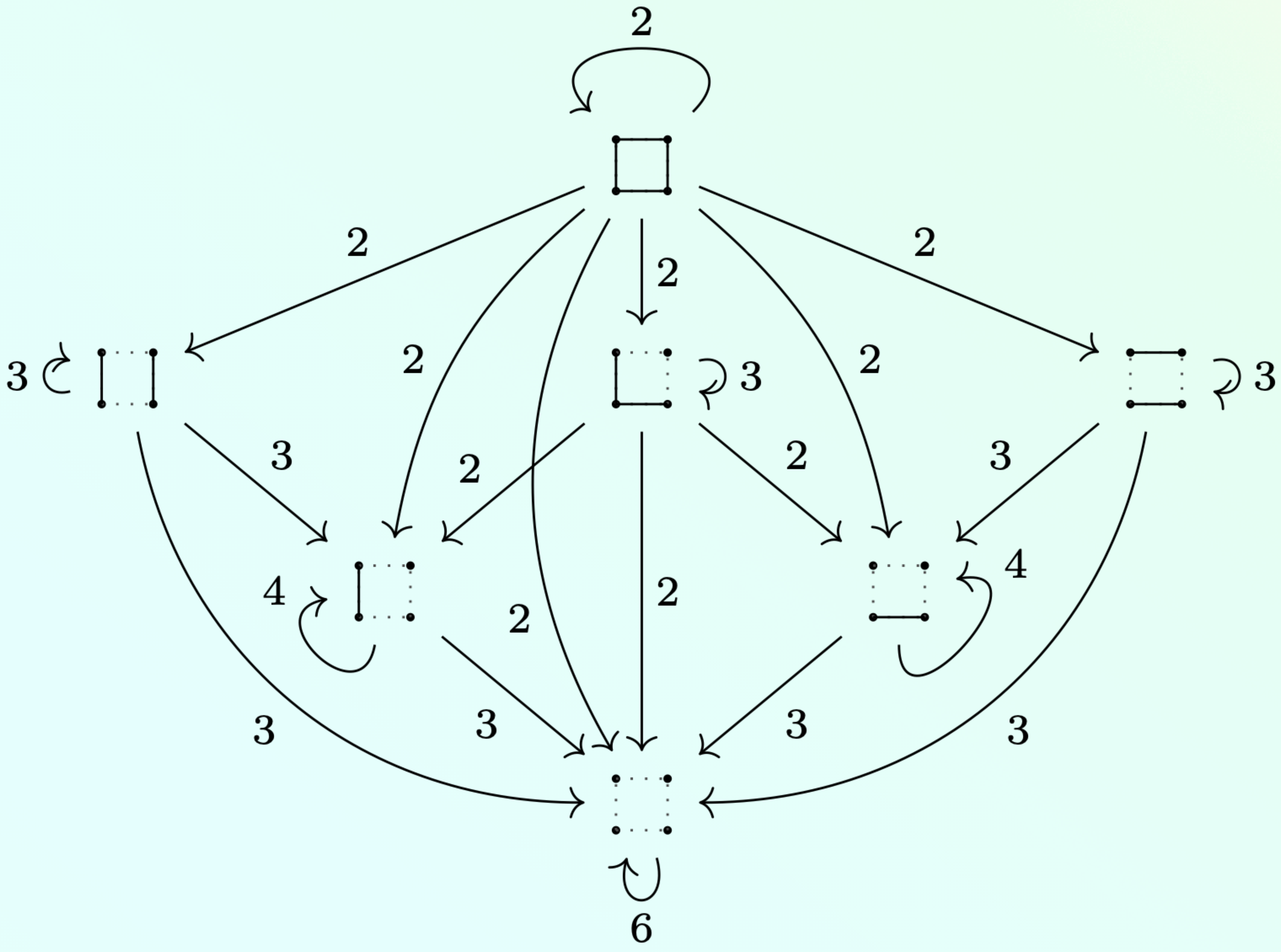
Define $G(P)$ to have vertex set $\mathrm{games}(P)$ and a directed edge QQ' with weight the number of matchstick games on $P \times [1]$ with Q on 'bottom' and Q' on 'top'.

For $A(P) := A(G(P))$, we have

$$\mathrm{games}(P \times [n]) = \sum_{Q,Q' \in \mathrm{games}(P)} (A(P)^n)_{Q,Q'} \, .$$

EXAMPLE

SPECIALIZING TO $P = [1] \times [1]$



$$
\begin{pmatrix}
6 & 3 & 3 & 3 & 2 & 3 & 2 \\
0 & 4 & 0 & 3 & 2 & 0 & 2 \\
0 & 0 & 4 & 0 & 2 & 3 & 2 \\
0 & 0 & 0 & 3 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 3 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 3 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 2
\end{pmatrix}
$$

DIAGONALIZABILITY

LEMMA [CMRG '24]. For any finite modular lattice P there is a linear extension of $\mathrm{games}(P)$ such that $A(P)$ is upper triangular with equal diagonal entries contiguous and each contiguous block diagonal. It follows that $A(P)$ is diagonalizable.

$$\begin{pmatrix} 6 & 3 & 3 & 3 & 2 & 3 & 2 \\ 0 & 4 & 0 & 3 & 2 & 0 & 2 \\ 0 & 0 & 4 & 0 & 2 & 3 & 2 \\ 0 & 0 & 0 & 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

FORMULÆ & ASYMPTOTICS

THEOREM [CMRG '24]. For each finite modular lattice P, there are rational numbers b_i and positive integers λ_i, $1 \leq i \leq m$, such that

$$\#\mathrm{games}(P \times [n]) = \sum_{i=1}^{m} b_i \lambda_i^n \,.$$

COROLLARY [CMRG '24]. The largest eigenvalue of $A(P)$ equals $\mathrm{ac}(P)$, the number of antichains of P, and thus

$$\#\mathrm{games}(P \times [n]) = \Theta(\mathrm{ac}(P)^n)\,.$$

EXAMPLE

SPECIALIZING TO $[1] \times [1] \times [n]$

Set $P = [1] \times [1]$. The number of matchstick games on $P \times [n] = [1] \times [1] \times [n]$ is

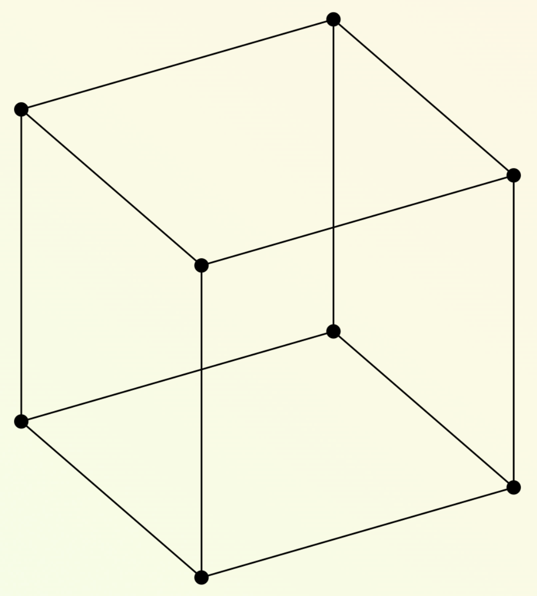
$$\#\mathrm{games}([1] \times [1] \times [n]) = \frac{35}{2} \cdot 6^n - 12 \cdot 4^n + 3^n + \frac{1}{2} \cdot 2^n .$$

(An equivalent formula was discovered independently by Filip Stappers in the context of “max-closed relations”.)

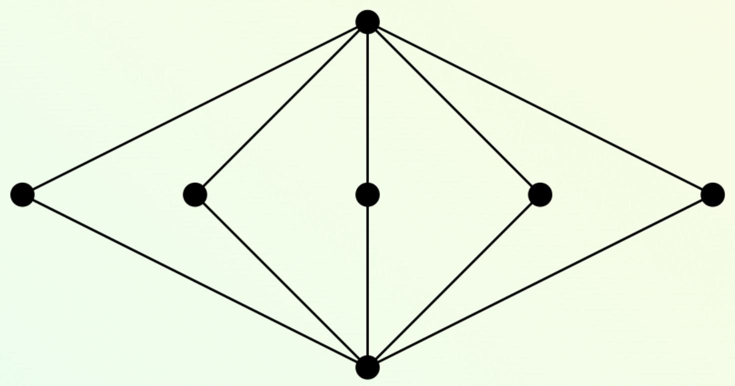
n	0	1	2	3	4	5	6	7
#games	7	61	449	3043	19697	124051	768089	4704523

FORMULÆ

P	games$(P \times [n])$
$[1] \times [2]$	$\frac{15471}{112} \cdot 10^n - \frac{406}{15} \cdot 8^n - 154 \cdot 7^n + \frac{315}{8} \cdot 6^n + 16 \cdot 5^n + 14 \cdot 4^n - \frac{111}{35} \cdot 3^n - \frac{13}{48} \cdot 2^n$
$[1] \times [3]$	$\frac{12800093}{8580} \cdot 15^n - \frac{525746}{2475} \cdot 13^n - \frac{42493}{60} \cdot 12^n - \frac{840037}{336} \cdot 11^n + \frac{735867}{560} \cdot 10^n + \frac{1482}{5} \cdot 9^n + \frac{59599}{75} \cdot 8^n - \frac{45157}{120} \cdot 7^n + \frac{719}{24} \cdot 6^n - \frac{797}{15} \cdot 5^n - \frac{101729}{13860} \cdot 4^n + \frac{28631}{8400} \cdot 3^n + \frac{7213}{34320} \cdot 2^n$
$[1] \times [4]$	$\frac{665038415449}{31039008} \cdot 21^n - \frac{168120903799}{68068000} \cdot 19^n - \frac{147245784937}{21021000} \cdot 18^n - \frac{1273644454}{75075} \cdot 17^n - \frac{26792689523}{540540} \cdot 16^n + \frac{21650713}{572} \cdot 15^n + \frac{15454724597}{1848000} \cdot 14^n + \frac{45969602783}{4158000} \cdot 13^n + \frac{2096204}{75} \cdot 12^n - \frac{796666013}{23520} \cdot 11^n + \frac{22150545}{4928} \cdot 10^n - \frac{17901851}{5250} \cdot 9^n + \frac{250240977}{143000} \cdot 8^n + \frac{799421317}{831600} \cdot 7^n - \frac{423949649}{960960} \cdot 6^n + \frac{9458167}{96096} \cdot 5^n + \frac{32846000389}{3216213000} \cdot 4^n - \frac{5004577}{1501500} \cdot 3^n - \frac{335766173}{1551950400} \cdot 2^n$
$[2] \times [2]$	$\frac{1084132338269}{578918340} \cdot 20^n - \frac{23890508}{51597} \cdot 17^n - \frac{93338401}{32400} \cdot 14^n - \frac{14916213}{53900} \cdot 13^n + \frac{395263}{1575} \cdot 12^n + \frac{1561186}{1701} \cdot 11^n + \frac{3350007}{3920} \cdot 10^n + \frac{43367}{275} \cdot 9^n + \frac{12446}{2025} \cdot 8^n - \frac{367959}{1300} \cdot 7^n - \frac{19267}{784} \cdot 6^n - \frac{320743}{11340} \cdot 5^n + \frac{65901}{9100} \cdot 4^n + \frac{3303091}{749700} \cdot 3^n - \frac{61687}{1069200} \cdot 2^n$
$[1]^3$	$\frac{159923969}{530400} \cdot 20^n - \frac{3336709}{23400} \cdot 15^n - \frac{31273}{2520} \cdot 12^n - \frac{2956707}{11200} \cdot 10^n + \frac{493}{10} \cdot 8^n + \frac{96831}{520} \cdot 7^n - \frac{13385}{288} \cdot 6^n + \frac{108}{175} \cdot 5^n - \frac{351}{32} \cdot 4^n - \frac{8621}{21420} \cdot 3^n + \frac{2453}{12480} \cdot 2^n$
$[2]^{*3}$	$\frac{2745}{112} \cdot 10^n - \frac{105}{8} \cdot 6^n + \frac{3}{7} \cdot 3^n + \frac{3}{16} \cdot 2^n$
$[2]^{*4}$	$\frac{80223}{2240} \cdot 18^n - \frac{2745}{224} \cdot 10^n - \frac{35}{8} \cdot 6^n + \frac{12}{7} \cdot 4^n + \frac{3}{35} \cdot 3^n + \frac{1}{64} \cdot 2^n$
$[2]^{*5}$	$\frac{3559545}{63488} \cdot 34^n - \frac{80223}{7168} \cdot 18^n - \frac{13725}{1792} \cdot 10^n - \frac{25}{32} \cdot 6^n + \frac{12}{7} \cdot 4^n - \frac{19}{217} \cdot 3^n - \frac{125}{2048} \cdot 2^n$



$[1]^3$



$[2]^{*5}$

GENERATING FUNCTIONS

EXPONENTIAL GENERATING FUNCTION FOR HIGHER-DIMENSIONAL GRIDS

Set $G(x_1, \dots, x_k) = \displaystyle\sum_{n_1,\dots,n_k \geq 0} 2\#\mathrm{games}([n_1 - 1] \times \cdots \times [n_k - 1]) \frac{x_1^{n_1} \cdots x_k^{n_k}}{n_1! \cdots n_k!}$.

By [Kaneko ’97], $G(x, y, 0, \dots, 0) = \dfrac{e^{x+y}}{e^x + e^y - e^{x+y}}$.

CONJECTURE [CMRG ’24]. The exponential generating function $G(x_1, \dots, x_k)$ is a rational function in the variables $e^{x_1}, \dots, e^{x_k}$.

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