

- Goals
- Laplace expansion of \det
 - Existence & uniqueness of \det

Defn For $A \in F^{n \times n}$, $1 \leq i, j \leq n$, set $A^{ij} \in F^{(n-1) \times (n-1)}$ to be the matrix created by deleting the i -th row and j -th column from A .

E.g. If $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}$, then $A^{23} = \begin{pmatrix} 1 & 2 & 4 \\ 9 & 10 & 12 \\ 13 & 14 & 16 \end{pmatrix}$

not 23rd power

Thm (Laplace expansion) Suppose $A \in F^{n \times n}$ and $1 \leq k \leq n$. Then

$$\det A = \sum_{j=1}^n (-1)^{k+j} A_{kj} \det(A^{kj})$$

$\underbrace{\text{with sign } (k_{ij}) \text{ cofactor}}$
 $\underbrace{(k_{ij}) \text{ minor}}$
 expansion of $\det A$ along the k -th row of A

E.g.

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (-1)^{1+1} a \det(d) + (-1)^{1+2} b \det(c)$$

$$= ad - bc \quad \checkmark$$

$$\det \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 0 \\ 5 & 6 & 7 \end{pmatrix} = (-1)^{2+1} \cdot 0 \cdot \det \begin{pmatrix} 2 & 3 \\ 6 & 7 \end{pmatrix} + (-1)^{2+2} \cdot 4 \cdot \det \begin{pmatrix} 1 & 3 \\ 5 & 7 \end{pmatrix}$$

$$+ (-1)^{2+3} \cdot 0 \cdot \det \begin{pmatrix} 1 & 2 \\ 5 & 6 \end{pmatrix}$$

$$= 4(17 - 3 \cdot 5) = -32.$$

Note $(-1)^{k+j}$ makes a "sign checkerboard":

$$\left\{ \begin{array}{ccccccc} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \right\}$$

This can be a helpful mnemonic when expanding along some row.

Cor [Laplace expansion along columns] For $A \in F^{n \times n}$, $1 \leq k \leq n$,

$$\det A = \sum_{i=1}^n (-1)^{i+k} A_{ik} \det(A_{ik})$$

Pf We know $\det A = \det A^T$ and this is Laplace exp'n along the k -th row of A^T .

Question What is the determinant of $\begin{pmatrix} 1 & 3 & 0 \\ 2 & 5 & -6 \\ 4 & 7 & 0 \end{pmatrix}$?

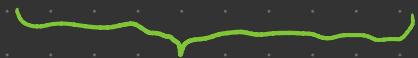
$$-6 \cdot \det \begin{pmatrix} 1 & 3 \\ 4 & 7 \end{pmatrix} = -6 \cdot (-5) = 30$$

Pf sketch for Laplace Expansion Thm By permutation exp'n,

$$\det A = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^n A_{i,\sigma(i)}.$$

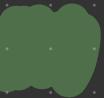
Factor out all the A_{kj} terms:

$$\det A = \sum_{j=1}^n A_{kj} \sum_{\substack{\sigma \in S_n \\ \sigma(k)=j}} \operatorname{sgn}(\sigma) \prod_{\substack{i \in n \\ i \neq k}} A_{i,\sigma(i)}$$



Need to show this is

$$(-1)^{k+j} \det A^{kj}$$

The  part has all the correct A_{ij} terms. Must check
that the sign is off by $(-1)^{k+j}$ (hint: swap rows). \square

Thm [3! det] There is a unique multilinear, alternating,
normalized function $\det: F^{n \times n} \rightarrow F$.

Pf Inspired by Laplace expansion, make the following

recursive definition of a function $d: F^{n \times n} \rightarrow F$:

$$\underline{n=1} \quad d(a) = a$$

$$\underline{n>1} \quad d(A) = \sum_{j=1}^n (-1)^{1+j} A_{1j} d(A^{1j})$$

Check d is multilin, alt, normalized.

Then d is a well-defined determinant function.

Our work with \det & row operations shows that all

determinant functions are determined by a sequence
of row operations to rref:

if E_1, \dots, E_l elementary s.t. $rref(A) = E_1 \cdots E_l A$

then $\det A = \frac{\det(rref(A))}{\det(E_1) \cdots \det(E_l)}$

Thus any two det functions are in fact the same! \square

Time permitting, the general linear group:

let $F^* := F \setminus \{0\}$, $GL_n(F) \subseteq F^{n \times n}$ be the invertible $n \times n$ matrices.

Then $GL_n(F) = \det^{-1} F^* = \{A \in F^{n \times n} \mid \det A \in F^*\}$

$$\begin{array}{ccc} F^{n \times n} & \xrightarrow{\det} & F \\ \text{UI} & & \text{UI} \\ \uparrow & & \uparrow \\ GL_n(F) & \xrightarrow{\det} & F^* \end{array}$$

(group under matrix multiplication)

#3

$$\mathbb{R}^2 \xrightarrow{t} \mathbb{R}^2$$

$$\begin{aligned} e_1 &\longmapsto (1,1) + 3(1,-2) \\ &= (4, -5) \end{aligned}$$

$$\begin{aligned} e_2 &\longmapsto 2(1,1) + 4(1,-2) \\ &= (6, -6) \end{aligned}$$

$$A_{\Sigma}^{\delta}(t) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$\beta = ((1,1), (1,-2))$$

$$\Rightarrow A_{\Sigma}^{\gamma}(t) = \begin{pmatrix} 4 & 6 \\ -5 & -6 \end{pmatrix}$$

$$\delta = ((0,1), (1,1)) \quad \gamma = ((-1,0), (2,1))$$

$$\varepsilon = ((1,2), (1,0)) \quad \zeta = ((1,2), (2,1))$$

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{t} & \mathbb{R}^2 \\ Rep_{\delta} \downarrow & P & Q \downarrow Rep_{\gamma} \\ \mathbb{R}^2 & \xrightarrow{A_{\delta}^{\gamma}(t)} & \mathbb{R}^2 \end{array}$$

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \quad Q = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$Q^{-1} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$\left| \begin{array}{l}
 A_\delta^Y(t) = Q^{-1} A_\varepsilon^\Sigma(t) P \\
 = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 6 \\ -5 & -6 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \\
 = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 6 & 10 \\ -6 & -11 \end{pmatrix} \\
 = \begin{pmatrix} -18 & -32 \\ -6 & -11 \end{pmatrix}
 \end{array} \right| \quad \begin{array}{l}
 t(0,1) = (6, -6) \\
 t(1,1) = t(e_1) + t(e_2) \\
 = (4, -5) + (6, -6) \\
 = (10, -11)
 \end{array}$$

$$\begin{aligned}
 -18 \cdot (-1, 0) - 6(2, 1) &= (6, -6) \quad \checkmark \\
 -32 \cdot (-1, 0) - 11(2, 1) &= (10, -11) \quad \checkmark
 \end{aligned}$$