Goals
- Define eigenvectors & eigenvalues of linear trans'ns
- Define the characteristic polynomial
- {eigenvalues} = {roots of characteristic polynomial}

Defn Let $f : V \to V$ be a linear transformation. A nonzero vector $v \in V$ is an eigenvector of f with eigenvalue $\lambda \in F$ when

$$f(v) = \lambda v.$$

If $\alpha = (v_1, \dots, v_n)$ is a basis of eigenvectors of f with eigenvalues $\lambda_1, \dots, \lambda_n$, then

$$A_\alpha^\alpha(f) = \mathrm{diag}(\lambda_1, \dots, \lambda_n) = \begin{pmatrix} \lambda_1 & & \bigcirc \\ & \lambda_2 & & \\ & & \ddots & \\ \bigcirc & & & \lambda_n \end{pmatrix}.$$

Diagonal matrices are much easier to work with, so this is a big win!

Not every lin trans'n has an associated basis of eigenvectors.

<u>Problem</u> Compute the powers of $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$. What about $\operatorname{diag}(\lambda_1, \dots, \lambda_n)^k$?

$$\operatorname{diag}(\lambda_1, \dots, \lambda_n)^k$$

$$\|$$

<u>Answer</u>

$$\operatorname{diag}(\lambda_1^k, \lambda_2^k, \dots, \lambda_n^k)$$

<u>E.g.</u> Let $A = \begin{pmatrix} -1 & 2 \\ -6 & 6 \end{pmatrix}$ with assoc lin trans'n $\mathrm{map}_A : \mathbb{R}^2 \to \mathbb{R}^2$.

Since $\begin{pmatrix} -1 & 2 \\ -6 & 6 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix} = 2 \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

and $\begin{pmatrix} -1 & 2 \\ -6 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} = 3 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

we know $(2,3)$ and $(1,2)$ are eigenvectors of map_A with eigenvalues $2, 3$, resp.

Let $\alpha := ((2,3), (1,2))$. Since $\mathrm{map}_A(2,3) = 2\cdot(2,3)$

$\mathrm{map}_A(1,2) = 3\cdot(1,2)$,

we have $A_\alpha^\alpha(\mathrm{map}_A) = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} =: D$.

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "A"] \arrow[d, "\mathrm{Rep}_\alpha"'] & \mathbb{R}^2 \arrow[d, "\mathrm{Rep}_\alpha"] \\ \mathbb{R}^2 \arrow[r, "D"'] & \mathbb{R}^2 \end{tikzcd}$$

The matrix for Rep_α^{-1} is $P = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$

$\Rightarrow \quad D = P^{-1} A P$.

Summary Eigenvectors $\binom{2}{3}, \binom{1}{2}$ are columns of P. Then

$D = P^{-1}AP$ is diagonal with eigenvalues on the diagonal.

In general, if $f: F^n \longrightarrow F^n$ is a lin trans'n and $\alpha = (v_1, \dots, v_n)$ is an ordered basis of eigenvectors with eigenvalues $\lambda_1, \dots, \lambda_n$,

$$P = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}, \quad \underset{\underset{A^{\mathcal{E}}_{\mathcal{E}}(f)}{\shortparallel}}{A} = \begin{pmatrix} | & & | \\ f(e_1) & \cdots & f(e_n) \\ | & & | \end{pmatrix}, \quad \text{then}$$

$$D = P^{-1}AP = \operatorname{diag}(\lambda_1, \dots, \lambda_n).$$

How to find eigenvectors & eigenvalues:

For $A \in F^{n \times n}$, $v \in F^n$, $\lambda \in F$,

$$Av = \lambda v \iff (A - \lambda I_n)v = 0$$

$$\iff v \in \ker(A - \lambda I_n).$$

Thus λ is an eigenvalue of A iff $\ker(A-\lambda I_n) \neq 0$.

But $\ker(A-\lambda I_n) \neq 0 \iff \operatorname{rank}(A-\lambda I_n) < n$

$\iff \det(A-\lambda I_n) = 0$.

E.g.

$$\det\left(\begin{pmatrix} -1 & 2 \\ -6 & 6 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \det\begin{pmatrix} -1-\lambda & 2 \\ -6 & 6-\lambda \end{pmatrix}$$

$$= (-1-\lambda)(6-\lambda) + 12$$

$$= \lambda^2 - 5\lambda + 6$$

$$= (\lambda-2)(\lambda-3).$$

Thus the eigenvalues of $\begin{pmatrix} -1 & 2 \\ -6 & 6 \end{pmatrix}$ are $\lambda = 2, 3$.

How do we find eigenvectors?

Need to compute $\ker(A - \lambda I_n)$ for eigenvalues λ.

E.g. $$A - 2I_2 = \begin{pmatrix} -1 & 2 \\ -6 & 6 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 2 \\ -6 & 4 \end{pmatrix} \xrightarrow[G\text{-}J]{} \begin{pmatrix} 1 & -2/3 \\ 0 & 0 \end{pmatrix}$$

$$\boxed{\begin{aligned} x \cdot 1 - \tfrac{2}{3} y &= 0 \\ x &= \tfrac{2}{3} y \end{aligned}}$$

So $$\ker(A - 2I_2) = \left\{ \left(\tfrac{2}{3}y, y\right) \mid y \in \mathbb{R} \right\}$$

$$= \operatorname{span}\{(2/3, 1)\} = \operatorname{span}\{(2, 3)\}$$

Similarly,

$$A - 3I_2 = \begin{pmatrix} -1 & 2 \\ -6 & 6 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 2 \\ -6 & 3 \end{pmatrix} \underset{G-J}{\rightsquigarrow} \begin{pmatrix} 1 & -1/2 \\ 0 & 0 \end{pmatrix}$$

so $\ker(A - 3I_2) = \operatorname{span}\{(1/2, 1)\} = \operatorname{span}\{(1,2)\}$

Problem Find a matrix that is not diagonalizable. (Think geometrically.)

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$ rotates by $\pi/2$ ccw

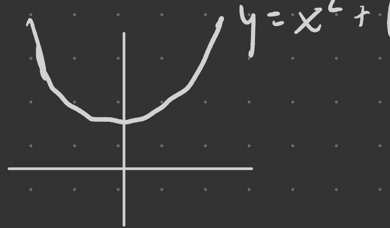
Defn $\chi_A(x) = \det(A - x I_n)$ is the characteristic polynomial of $A \in F^{n \times n}$.

Thm The roots of χ_A are the eigenvalues of A.

$$\chi_{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}(x) = \det\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} -x & -1 \\ 1 & -x \end{pmatrix}$$

$$= x^2 - (-1) = x^2 + 1$$



has no roots over $\mathbb{R}$!

Does have roots over $\mathbb{C}$:

$$\lambda = \pm\sqrt{-1} = \pm i$$

$$\ker\left(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} - \begin{pmatrix}i & 0\\ 0 & i\end{pmatrix}\right) = \ker\begin{pmatrix}-i & -1\\ 1 & -i\end{pmatrix}$$

$$\begin{pmatrix}-i & -1\\ 1 & -i\end{pmatrix} \xrightarrow{r_1 \to i r_1} \begin{pmatrix}1 & -i\\ 1 & -i\end{pmatrix} \xrightarrow{r_2 \to r_2 - r_1} \begin{pmatrix}1 & -i\\ 0 & 0\end{pmatrix}$$

$$\ker\left(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} - i I_2\right) = \operatorname{span}\{(i, 1)\}$$

<u>Check</u> $$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}i\\ 1\end{pmatrix} = \begin{pmatrix}-1\\ i\end{pmatrix} = i\begin{pmatrix}i\\ 1\end{pmatrix}$$

<u>Exc</u> check other.

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

"2×2 Jordan block"

$$\chi_A(x) = \det\begin{pmatrix} 1-x & 1 \\ 0 & 1-x \end{pmatrix}$$

$$= (1-x)^2$$

$\Longrightarrow$ $\lambda = 1$ is the unique eigenvalue.

e_1 is an eigenvector but no others in

$F^2 \setminus \operatorname{span}\{e_1\}$! So not diagonalizable.