Reference Sheet (Main)
Last Updated: April 21, 2024

Def. [*Separable*]. A differential equation is separable if it can be written in the form $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y)$ for some functions f and g.

Def. [*Order*]. The order of a differential equation is the highest order derivative that appears in the equation

Def. [*General Solution*]. A general solution is an expression for the family of solutions to a differential equation

Def. [*Particular Solution*]. A particular solution is a solution to a differential equation passing through a given point

Def. [*Initial Value Problem*]. An IVP is an ODE $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$ with initial condition $y(x_0) = y_0$.

Def. [*1st Order Linear ODE*]. A first order ODE is linear if in the form $a_1(x)y' + a_0(x)y = g(x)$.

Def. [*1st Order Linear ODE Standard Form*]. A first order linear ODE is in standard form if in the form $y' + P(x)y = Q(x)$.

Def. [*Integrating Factor*]. In a first order linear ODE in standard form, the integrating factor is a function $\alpha(x) = e^{\int P(x)dx}$ which can be used to solve the ODE.

Def. [*Pure-Time ODEs*]. An n-th order ODE of the form $y^{(n)} = f(x)$ is called a pure-time ODE.

Thm. [*Autonomous ODE*]. A first order autonomous ODE is of the form $y' = g(y)$.

Thm. [*Newton's Law of Cooling*]. The rate at which the temperature of a body changes, $\frac{\mathrm{d}T}{\mathrm{d}t}$, is proportional to the difference between the temperature of the body, T, and the temperature of the ambient temperature, A. We have

$$\frac{\mathrm{d}T}{\mathrm{d}t} = k(T - A) \qquad T(t) = A + (T(0) - A)e^{kt}$$

Def. [*Limit*]. A sequence a_n converges to a limit L if, for every $\varepsilon > 0$, there exists some number N, such that if $n > N$, then $|a_n - L| < \varepsilon$

Thm. [*Divergence Test*]. If $\lim\limits_{n\to\infty} a_n \neq 0$, then $\sum\limits_{n=1}^{\infty} a_n$ diverges.

Thm. [*p-Series Test*]. If $p > 1$, the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ converges. If $p \leq 1$, it diverges.

Thm. [*Integral Test*]. Let $f(n) = a_n$ and suppose f is positive, decreasing and continuous for $n > M$. Then, $\int_M^\infty f(x)dx$ converges if and only if $\sum_{n=M}^\infty a_n$ converges.

Thm. [*Geometric Series Test*]. Let $|r| < 1$, then

$$\sum_{n=0}^{\infty} cr^n = c + cr + cr^2 + ... = \frac{c}{1-r}$$

Thm. [*Comparison Test*]. Let a_n and b_n be sequences with $0 < a_n < b_n$ for every n. Then,

1. If $\sum a_n$ diverges, then $\sum b_n$ diverges.
2. If $\sum b_n$ converges, then $\sum a_n$ converges.

Thm. [*Limit Comparison Test*]. Let $\sum a_n$ and $\sum b_n$ be series with positive terms. Let $\lim(\frac{a_n}{b_n}) = L$.

1. If $L = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges
2. If $L = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges
3. If $L > 0$ and $L \in \mathbb{R}$, then either both series converge or both series diverge

Thm. [*Alternating Series Test*]. Let $\sum a_n$ be a series which can be written in the form $\sum a_n = \sum(-1)^{n-1}b_n$ for a positive sequence b_n, then the series $\sum a_n$ is convergent if

1. $b_{n+1} \leq b_n$ for all n (b_n is decreasing)
2. $\lim(b_n) = 0$ (b_n converges to 0)

Thm. [*Ratio Test*]. Suppose $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$ exists. Then,

1. If $L < 1$, then $\sum a_n$ converges absolutely
2. If $L > 1$, then $\sum a_n$ diverges
3. If $L = 1$, the test is inconclusive

Thm. [*Root Test*]. Suppose $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$ exists. Then,

1. If $L < 1$, then $\sum a_n$ converges absolutely
2. If $L > 1$, then $\sum a_n$ diverges
3. If $L = 1$, the test is inconclusive

Thm. [*Taylor Series*]. Let f be a function, then its Taylor Series at $c \in \mathbb{R}$ is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Thm. [*Maclaurin Series*]. Let f be a function, then its Maclaurin Series is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

Thm. [*Known Taylor Series*]. The following functions have known Taylor Series and intervals of convergence.

$$\begin{aligned}
\sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1} && \mathbb{R} \\
\cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n+1} && \mathbb{R} \\
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{(n)!} && \mathbb{R} \\
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n && (-1,1) \\
\frac{1}{1+x} &= \sum_{n=0}^{\infty} (-x)^n && (-1,1) \\
\ln(1+x) &= \sum_{n=1}^{\infty} \frac{(-x)^n}{n} && (-1,1]
\end{aligned}$$

Thm. [*Taylor Error Bound*]. The worst-case scenario for the difference between the estimated value of the function as provided by the Taylor polynomial and the actual value of the function is the maximum value of the $(n+1)$th term of the Taylor expansion, where M is an upper bound of the $(n+1)$th derivative.

$$|f(x) - T_n(x)| = M\frac{|x-a|^{n+1}}{(n+1)!}$$

Def. [*Parametric Curve*]. A parametric curve is a vector-valued function $c : \mathbb{R} \to \mathbb{R}^2$ given by $c(t) = \big(x(t), y(t)\big)$

Def. [*Arc Length*]. Let $c : \mathbb{R} \to \mathbb{R}^n$ be a parametric curve. The arc length (or distance travelled along the curve traced by c), for t between a and b, is $\int_a^b \|c'(t)\| dt$

Thm. [*Planes in* $\mathbb{R}^3$]. A plane in $\mathbb{R}^3$ may be uniquely defined by

1. Three Points
2. A normal vector and a point in the plane
3. Two distinct parallel lines

Def. [*Continuity*]. Let $F : \mathbb{R}^2 \to \mathbb{R}$. Say F is continuous at $\vec{a} \in \mathbb{R}^2$ if $\lim_{\vec{x} \to \vec{a}} F(\vec{x}) = F(\vec{a})$.

Thm. [*Squeeze Theorem*]. Let $f : \mathbb{R}^2 \to \mathbb{R}$, $g : \mathbb{R}^2 \to \mathbb{R}$, and $h : \mathbb{R}^2 \to \mathbb{R}$ be three functions such that for every $\vec{x}$ nearby of $\vec{a} \in \mathbb{R}^2$ (except possibly at $\vec{a}$), we have

$$f(\vec{x}) \leq g(\vec{x}) \leq h(\vec{x})$$

If $\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = \lim_{\vec{x} \to \vec{a}} h(\vec{x}) = L$ for $L \in \mathbb{R}$, then

$$\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = L$$

Cor. If $\lim_{\vec{x} \to \vec{a}} F(\vec{x}) = \vec{b}$, then the limit of F along every path approaching $\vec{a}$ exists and is equal to $\vec{b}$.

Def. [*Partial Derivative*]. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function. The partial derivative of f with respect to x_i at a point $\vec{a} \in A$ is defined as

$$\frac{\partial f}{\partial x}(\vec{a}) = \lim_{h \to 0} \frac{f(\vec{a} + h\vec{e}_i) - f(\vec{a})}{h}$$

Def. [*Linearization*]. Let f be differentiable at (a, b). Then, for (x, y) close to (a, b), then

$$f(x, y) \approx f(a, b) + f_x(a, b)\big(x - a\big) + f_y(a, b)\big(y - b\big)$$

Def. [*Level Set*]. The level set of a function $g : \mathbb{R}^n \to \mathbb{R}$ at level R is the set

$$S_R = \{\vec{x} \in \mathbb{R}^n : g(\vec{x}) = R\}$$

Rmk. Level sets are subsets of the domain (not the graphs) of a function.

Def. [*Gradient*]. The gradient ∇ of a function f, written as ∇f, is given by the vector $\nabla f = \langle f_x, f_y \rangle$, and gives the direction of steepest ascent.

Def. [*Directional Derivative*]. Suppose f is differentiable at a point P and $\vec{u}$ is a unit vector. Then, the directional derivative of f in the direction of $\vec{u}$ is given by

$$D_{\vec{u}}f(P) = \nabla f(P) \cdot \vec{u}$$

Thm. Existence of $D_{\vec{v}}f(\vec{p})$ does not guarantee f is differentiable at $\vec{p}$.

Thm. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. Then, the gradient ∇f is orthogonal to the level sets $f(x) = k \in \mathbb{R}$.

Thm. Let $g : \mathbb{R}^n \to \mathbb{R}$ be differentiable. Let $\vec{a} \in \mathbb{R}^n$ with $k = g(\vec{a})$. Let $S_k = \{\vec{x} \in \mathbb{R}^n : g(\vec{x}) = \vec{k}\}$. Then, $\nabla g(\vec{a})$ is orthogonal to the tangent plane to S_k at $\vec{a}$.

Thm. [*Fubini's Theorem*]. Suppose $[a, b] \times [c, d] \subset \mathbb{R}^2$ is a rectangle and $f : \mathbb{R}^2 \to \mathbb{R}$ is integrable. Then,

$$\iint_R f(x, y)\mathrm{d}A = \int_a^b \int_c^d f(x, y)\mathrm{d}y\mathrm{d}x = \int_c^d \int_a^b f(x, y)\mathrm{d}x\mathrm{d}y$$