Midterm 3 Content

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Bases for Polynomials

Let P_3 be the set of polynomials of degree 3 or less. Explain which of the following are bases for this set.

a) $\{\ (1+x)\ ,\ (x^2-x-1)\ ,\ x^3\ \}$

b) $\{\ 3\ ,\ (5+x^3)\ \}$

c) $\{\ x^2+9x+3\ ,\ 5x\ ,\ 6x^3\ ,\ 12x+3x^2\ ,\ 20\ \}$

d) $\{\ 6x\ ,\ 5x\ ,\ 20x^2\ ,\ 30x^3\ \}$

e) $\{\ 29\ ,\ x^2+1\ ,\ x-4\ ,\ 20x^3\ \}$

Def. (Basis). A subset $\beta \subseteq H$ is a **basis** for H if the following hold:

1. β is linearly independent (has no redundant information)
2. $\text{span}(\beta) = H$ (has sufficient information)

Thm. (Basis Theorem). Suppose V is a p dimensional vector space.

1. A linearly independent set of p vectors is a basis.
2. A spanning set of p vectors is a basis.

Coordinate Systems

Find the coordinates of $\begin{bmatrix} -2 \\ 1 \\ 2 \\ -20 \end{bmatrix}$ in the basis given by the previous question.

Def. (Change of Coordinates). Given a basis $\beta = \{\vec{b}_1, ..., \vec{b}_n\}$, for matrix $P_\beta = [\vec{b}_1|...|\vec{b}_n]$, we can write any $\vec{x} \in \mathbb{R}^n$ as $\vec{x} = P_\beta[\vec{x}]_b$

Eigenvectors and Eigenvalues

Let A be given by the matrix

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix}$$

a) What is the characteristic polynomial of A?

Def. (Characteristic Polynomial). For a matrix A, $\text{char}_A(\lambda) = |A - \lambda I|$

b) What are the eigenvalues and eigenvectors of A?

c) Is this matrix diagonalizable?

Orthogonality

Consider the following statements.

1. Suppose $\vec{u}$, $\vec{v}$, $\vec{w} \in \mathbb{R}^n$ are vectors. If $\vec{u}$ and $\vec{v}$ are orthogonal, then $\vec{u}$ and $\vec{w}$ are orthogonal.
2. For any $\vec{v} \in \mathbb{R}^n$ and any $c \in \mathbb{R}$, we have $||c \cdot \vec{v}|| = c \cdot ||\vec{v}||$
3. Consider $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 4 & 1 & -3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 10 \\ -7 \\ 1 \end{bmatrix}$. Then, $\vec{v}$ is orthogonal to any vector in the column space of A.

Which of the following is true?

a) All of these are false.

b) Statement 1 is true, and 2 and 3 are false.

c) Statement 2 is true, and 1 and 3 are false.

d) Statement 3 is true, and 1 and 2 are false.

e) Statements 1 and 2 are true, and 3 is false.

f) All of these are true.

Gram Schmidt

Let $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 2 & 2 \end{bmatrix}$. Find an orthonormal basis for $\mathrm{Col}(A)$.

Orthogonal Projections

Let W be the subspace of $\mathbb{R}^4$ spanned by the orthogonal set

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0, \end{bmatrix} \quad , \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

a) Find a basis for $W^\perp$, the orthogonal complement of W.

b) Let $\vec{y} = \begin{bmatrix} 3 \\ 0 \\ 5 \\ 2 \end{bmatrix}$. Find $\hat{y} = \text{proj}_W \vec{y}$, the projection of $\vec{y}$ onto W.

c) Find a vector $\vec{z} \in W^\perp$ such that $\vec{y} = \hat{y} + \vec{z}$.

Fundamental Subspaces

Consider the matrix $A = \begin{bmatrix} 2 & 2 & 4 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$

a) What is the RREF of A?

b) What is a basis for the columnspace of A?

c) What is a basis for the nullspace of A?

True/False Questions

(5.1) T/F: A matrix A is invertible if and only if 0 is an eigenvalue of A.

(6.1) T/F: If $||u||^2 + ||v||^2 = ||u + v||^2$, then u and v are orthogonal.

(6.2) T/F: Every linearly independent set is orthogonal.

(6.2) T/F: Every orthogonal set is linearly independent.

(4.4) T/F: A nonzero vector is linearly dependent.

(4.5) T/F: A vector space is infinite dimensional if it is spanned by a set of infinitely many vectors.

Good luck on the test!