PLAN Math 1XX3 Tutorial Notes

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Challenge Problems!

Q: Without using L'Hopital's Rule, calculate $\lim_{x\to 0} \frac{\sin x}{x}$

Q: Without using L'Hopital's Rule, calculate $\lim_{x\to 0} \frac{\cos x - 1 + \frac{x^2}{2}}{x^4}$

Q: Suppose f is a function with $f''(x) \geq 0$ for every $x \in \mathbb{R}$. Prove that the graph of $f(c)$ is never below the tangent line at $x = c$.

Q: Suppose f and g are continuous functions and their first n derivatives exist at a point c. Suppose that each of their first $n-1$ derivatives are 0, i.e. $f'(c) = f''(c) = ... = f^{(n-1)}(c) = 0$ and $g'(c) = g''(c) = ... = g^{(n-1)}(c) = 0$, but $g^{(n)} \neq 0$. Show that $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{f^{(n)}(c)}{g^{(n)}(c)}$

Q: Let a_n be a positive sequence and suppose $\sum a_n$ is convergent. Let $b_n = \frac{1}{n}(a_1 + ... + a_n)$. Show that $\sum b_n$ diverges.

Q: Evaluate $\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)}$

Q: Can you think of a convergent series $\sum a_n$ with $\sum a_n^2$ divergent?

A: Choose $a_n = \frac{(-1)^n}{\sqrt{n}}$

Q: Show that $\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + ...$ converges to 0. How many terms are needed such that the error is less than 0.01?

NEW!

Q: Consider the differential equation $y' = y^2$. This is a nonlinear differential equation. Can you find a substitution $v = g(y)$ that will transform this into a linear differential equation?

Solution: If we let $v = y^{-1}$, then $y = v^{-1}$ and $y' = -v^{-2}v'$. Substituting these into the original differential equation gives us $-v^{-2}v' = v^{-2}$, which simplifies to $v' = -1$. This is a linear differential equation.

Q: Prove that if $y_1(t)$ is a solution to the first order linear differential equation $y'+p(t)y = g(t)$, then the function $u(t) = y_1(t) \int \frac{g(t)}{y_1(t)} e^{\int p(t)dt} dt$ is also a solution.

Solution: This can be shown by direct substitution into the differential equation. The key is to recognize that $u'(t) = y_1'(t) \int \frac{g(t)}{y_1(t)} e^{\int p(t)dt} dt + y_1(t) \frac{g(t)}{y_1(t)} e^{\int p(t)dt}$, and that $y_1'(t)+p(t)y_1(t) = 0$ by the assumption that $y_1(t)$ is a solution to the homogeneous equation.

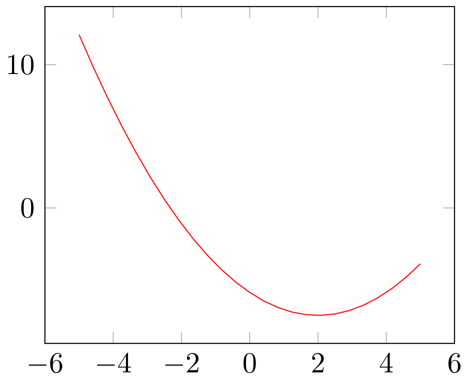
Q: Show that if $y_1(t)$ and $y_2(t)$ are solutions to the homogeneous first order linear differential equation $y' + p(t)y = 0$, then any linear combination $c_1y_1(t) + c_2y_2(t)$ is also a solution.

Solution: This can be shown by direct substitution into the differential equation. The key is to recognize that the derivative of a linear combination of functions is the linear combination of the derivatives, and that $y_1'(t) + p(t)y_1(t) = 0$ and $y_2'(t) + p(t)y_2(t) = 0$ by the assumption that $y_1(t))and(y_2(t)$ are solutions to the homogeneous equation.

Q: Consider the initial value problem $y' = y^2, y(0) = 1$. Show that the solution $y(t)$ is unbounded as (t) approaches a finite value.

Solution: This is a separable differential equation, and its solution can be found by separating variables and integrating: $\int dy/y^2 = \int dt$, which gives $-1/y = t + C$. Using the initial condition $y(0) = 1$, we find that $C = -1$. Therefore, $y(t) = -1/(t - 1)$, and $y(t)$ approaches $-\infty$ as t approaches 1.

Consider the graph of the function f shown below



What are the signs of the first three terms in its Taylor Series centered at $a = 0$?

☐ The 0^{th} degree term is positive

☐ The 0^{th} degree term is negative

☐ The 1^{st} degree term is positive

☐ The 1^{st} degree term is negative

☐ The 2^{nd} degree term is positive

☐ The 2^{nd} degree term is negative

Week 1

Ex. Evaluate the integral

$$\int \frac{3x^2}{\sqrt[4]{16-15x^3}}dx$$

Ex. Evaluate the integral

$$\int x^2 \ln(x)dx$$

Ex. Evaluate the integral

$$\int \ln(x)dx$$

Week 2

Def. *(Separable).* A differential equation is separable if it can be written in the form $\frac{dy}{dx} = f(x)g(y)$ for some functions f and g.

True or False. The differential equation $\frac{dy}{dx} = x^3 - yx$ is separable.

True or False. The differential equation $\frac{dy}{dx} = x^5 - x\sin(y)$ is separable.

True or False. Suppose $\frac{dy}{dx} = -1 - x^2 - y^4 - e^x$. All solutions to this differential equation are decreasing.

Ex. Sketch the slope field for $\frac{dy}{dx} = \frac{-x}{y}$.

Ex. Find a general solution to the equation $\frac{dy}{dx} = y^2 - 4y$.

Ex. Is the differential equation $e^x y' - xy = x$ separable? Linear?

Ex. Show that $\frac{d^2y}{dt^2} + \frac{dy}{dt} = 6y$ has solutions $y = 2e^{2t}$ and $y = -3e^{-3t}$.

Ex. Determine the convergence of

$$\int_0^\infty \frac{\sin(x)}{\sqrt{x^4+1}}dx$$

Ex. Marina loves the piano. Their favourite song has the number of keys played per second given by the integrand below. If they play the song forever, how many key presses are in the song?

$$\int_3^\infty \frac{1}{x - e^{-x}}dx$$

Ex. Mary loves sea animals! They go to an aquarium, and at each second x, they see a number of new sea animals given by the integrand below. If they stay in the aquarium forever, how many varieties of sea animals are there?

$$\int_3^\infty \frac{1}{x - e^{-x}}dx$$

Ex. Evaluate the improper integral

$$\int_{-\infty}^\infty xe^{-x^2}dx$$

Ex. Solve the initial value problem

$$\frac{dy}{dx} = \frac{xy^3}{\sqrt{1+x^2}} \qquad y(0) = -1$$

Week 3

Sequences

Def. *(Limit).* A sequence a_n converges to a limit L if, for every $\varepsilon > 0$, there exists some number N, such that if $n > N$, then $|a_n - L| < \varepsilon$

Ex. Using the definition of a limit, prove that

$$\lim \left(\frac{n}{n^2 + 1} \right) = 0$$

Ex. Using the definition of a limit, prove that

$$\lim \left(\frac{2n}{n + 1} \right) = 2$$

Claim. $\lim (\qquad\qquad) =$

Proof. Let $\varepsilon > 0$ be arbitrary.

Choose $N =$

Then, for any $n > N$, we have

$$|a_n - L| = \qquad\qquad\qquad\qquad = \varepsilon$$

□

Ex. Prove that the sequence $a_n = \frac{2n^2+3}{n^2+1}$ converges.

Ex. Prove that the sequence $a_n = \frac{\sin n}{n}$ converges.

Series

Ex. Calculate the value of $\sum_{n=2}^{\infty} (\frac{2}{7})^n$

Ex. Calculate the value of $\sum_{n=1}^{\infty} (\frac{1}{3})^{2n}$

Ex. Suppose that (a_n) is a decreasing sequence and $\sum a_n$ is a convergent series. Which of the following statements is true about the sequence (a_n)?

- □ (a_n) is bounded below by zero
- □ (a_n) is bounded above by the first term (a_1)
- □ (a_n) converges to zero

Week 4

Convergence Tests

Thm. (Divergence Test). If $\lim\limits_{n\to\infty} a_n \neq 0$, then $\sum\limits_{n=1}^{\infty} a_n$ diverges.

Thm. (p-Series Test). If $p > 1$, the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ converges. If $p \leq 1$, it diverges.

Thm. (Integral Test). Let $f(n) = a_n$ and suppose f is positive, decreasing and continuous for $n > M$. Then, $\int_M^{\infty} f(x)dx$ converges if and only if $\sum_{n=M}^{\infty} a_n$ converges.

Thm. (Geometric Series Test). Let $|r| < 1$, then

$$\sum_{n=0}^{\infty} cr^n = c + cr + cr^2 + ... = \frac{c}{1-r}$$

Thm. (Comparison Test). Let a_n and b_n be sequences with $0 < a_n < b_n$ for every n. Then,

1. If $\sum a_n$ diverges, then $\sum b_n$ diverges.
2. If $\sum b_n$ converges, then $\sum a_n$ converges.

Thm. (Limit Comparison Test). Let $\sum a_n$ and $\sum b_n$ be series with positive terms. Let $\lim(\frac{a_n}{b_n}) = L$.

1. If $L = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges
2. If $L = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges
3. If $L > 0$ and $L \in \mathbb{R}$, then either both series converge or both series diverge

Thm. (Alternating Series Test). Let $\sum a_n$ be a series which can be written in the form $\sum a_n = \sum (-1)^{n-1} b_n$ for a positive sequence b_n, then the series $\sum a_n$ is convergent if

1. $b_{n+1} \leq b_n$ for all n (b_n is decreasing)
2. $\lim(b_n) = 0$ (b_n converges to 0)

Thm. (Ratio Test). Suppose $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$ exists. Then,

1. If $L < 1$, then $\sum a_n$ converges absolutely
2. If $L > 1$, then $\sum a_n$ diverges
3. If $L = 1$, the test is inconclusive

Thm. (Root Test). Suppose $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L$ exists. Then,

1. If $L < 1$, then $\sum a_n$ converges absolutely
2. If $L > 1$, then $\sum a_n$ diverges
3. If $L = 1$, the test is inconclusive

Ex. Determine the convergence of $\sum_{n=0}^{\infty}\frac{3(-1)^n}{(n+1)^3}$. Find the smallest possible m such that the alternating series error $|s - s_m| < 0.03$

Ex. Determine the convergence of $\frac{0}{1} - \frac{1}{2} + \frac{2}{3} - \frac{3}{4} + ...$

Ex. Determine the convergence of $\sum\limits_{n=-2}^{\infty}\left(-\frac{4}{9}\right)^n$

Ex. Determine the convergence of $\sum\limits_{n=2}^{\infty}\frac{1}{n\ln n}$

Ex. Determine the convergence of $\sum\limits_{n=1}^{\infty}\frac{1}{n^n}$

Ex. Determine the convergence of $\sum_{n=1}^{\infty}\frac{2^{n^2}}{n!}$

Ex. Determine the convergence of $\sum\limits_{n=1}^{\infty}\frac{n^3}{3^{1+2n}}$

Week 5

Ex. For which values of x does $\sum_{n=0}^{\infty} (\frac{x-2}{4})^n$ converge?

Ex. Find the radius and interval of convergence for $\sum_{n=1}^{\infty} \frac{x^n}{2n-1}$.

Ex. Find the radius and interval of convergence for $\sum_{n=1}^{\infty} \frac{(5x-4)^n}{n^3}$.

Ex. Consider the function $f(x) = \frac{4}{3+2x^2}$.

Find a power series representation of f centered at $x = 0$ and find the interval of convergence.

Ex. Given that the power series representation of a function f is $\sum_{n=0}^{\infty} (-1)^n \cdot 8^{n+1} \cdot x^n$, find its closed form.

Ex. Find a power series representation for $\arctan x$.

Week 6

Thm. *(Taylor Series).* Let f be a function, then its Taylor Series is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Thm. *(Maclaurin Series).* Let f be a function, then its Maclaurin Series is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

Ex. Find the fourth degree Maclaurin polynomial for $f(x) = x \sin x$

Thm. *(Known Taylor Series).* The following functions have known

Taylor Series and intervals of convergence.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \qquad \mathbb{R}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n+1} \qquad \mathbb{R}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{(n)!} \qquad \mathbb{R}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad (-1,1)$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n \qquad (-1,1)$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-x)^n}{n} \qquad (-1,1]$$

Ex. Find $T_2(x)$ centered at $c = 4$ for the function $f(x) = \sqrt{x}$. Use it to estimate $\sqrt{4.1}$. What is the maximum error?

Ex. Find a quadratic approximation for e^x at a point that can be used to estimate e, $e^{0.1}$ and $e^{0.01}$. Check your answer with a calculator.

Ex. Find a third degree approximation for the function $\tan x$ at a point that can be used to estimate $\tan 1$, $\tan 0.5$ and $\tan 0.25$. Check your answer with a calculator.

Week 7

Ex. Sketch the curve given by the parametric equations $x(t) = \sqrt{t}$ and $y(t) = 2 - t$.

Ex. Find $\frac{dy}{dx}$ at $t = 0$ given by the parametric equations $x(t) = e^t \sin t$ and $y(t) = t \cos t$, where $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Ex. Find the arc length of the curve given by the parametric equations $x(t) = 1 + 3t^2$ and $y(t) = 4 + 2t^3$ on the interval $t \in [0, 1]$.

Ex. Consider the curve given by $x = 2\cos(t)$ and $y = 5\sin(2t)$. What is the value of $\frac{dy}{dx}$ at $t = \frac{\pi}{6}$?

Ex. Consider the curve given by $x = 3e^{2t}$ and $y = e^{3t} - 1$. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Ex. Graph the polar curve $r = 4\cos\theta$

Ex. Graph the polar curve $r = 1 - \sin\theta$

Ex. Graph the polar curve $r = \cos(3\theta)$

Q1. Can we reproduce familiar graphs with polar curves?

$$x = a \quad \longrightarrow \quad \theta = a \tag{1}$$
$$y = c \quad \longrightarrow \quad r = c \tag{2}$$
$$y = mx + b \quad \longrightarrow \quad r = a + b\theta \tag{3}$$

Week 8

Ex. [Review Power Series]. Find a power series representation of the function $f(x) = \ln(5 - x)$.

Thm. [Taylor Error Bound]. The worst-case scenario for the difference between the estimated value of the function as provided by the Taylor polynomial and the actual value of the function is the maximum value of the $(n+1)$th term of the Taylor expansion, where M is an upper bound of the $(n+1)$th derivative.

$$|f(x) - T_n(x)| = M\frac{|x-a|^{n+1}}{(n+1)!}$$

Ex. (Review Taylor Error Bound).

Ex. [Parameteric Curves]. Find a parametric curve $c = (x, y)$ for the triangle given by the points (0,0), (13,0), (6,5).

1 Week 8 OLD

Ex. Calculate the linear combination $(3i + j) - 6j + 2(j - 4i)$.

Ex. Describe the shape bounded by the curves $x^2 + y^2 = 7$ for $|z| \leq 7$.

Ex. Find a parameterization for a line passing through the points $P = (1, 1, 1)$ and $Q = (3, -5, 2)$.

Ex. Show that the lines r_1 and r_2 do not intersect.
$r_1(t) = \langle -1, 0, 2\rangle + t\langle 4, -2, -1\rangle$
$r_2(t) = \langle 0, 1, 1\rangle + t\langle 2, 0, 1\rangle$

Ex. Find values of b which make the vectors $\langle b, 3, 2\rangle$ and $\langle 1, b, 1\rangle$ orthogonal.

Ex. Find the decomposition $a = a||b + a \perp b$ with respect to b for the vectors $a = \langle 4, -1, 0\rangle$ and $b = \langle 0, 1, 1\rangle$

Draw the region given by

Week 9

Ex. True or False

(a) If $\vec{u}$ and $\vec{v}$ are vectors with $\vec{u} \cdot \vec{v} = 0$, then $\vec{u} = 0$ or $\vec{v} = 0$.

(b) If $\vec{u}$ and $\vec{v}$ are vectors and $k \in \mathbb{R}$, then $k(\vec{u} \cdot \vec{v}) = (k\vec{u}) \cdot \vec{v}$

(c) Two planes in $\mathbb{R}^3$ may intersect at a single point.

(d) Three points in $\mathbb{R}^3$ uniquely define a plane.

Thm. (Planes in $\mathbb{R}^3$). A plane in $\mathbb{R}^3$ may be uniquely defined by

1. Three Points
2. A normal vector and a point in the plane
3. Two distinct parallel lines

Ex. Find a parametric equation of the line passing through the point $(0, -1, 2)$ perpendicular to the plane passing through the points $(3, 0, -2)$, $(1, 1, 2)$, and $(0, 2, -2)$.

Ex. For vectors $\vec{u}, \vec{v}, \vec{w}, \vec{x} \in \mathbb{R}^3$, which of the following make sense? What type of variable are they?

1. $\vec{u} \cdot (\vec{v} \times \vec{w})$
2. $\vec{u} \cdot (\vec{v} \cdot \vec{w})$
3. $(\vec{u} \cdot \vec{v}) \cdot (\vec{w} \cdot \vec{x})$
4. $(\vec{u} \times \vec{v}) \cdot (\vec{w} \times \vec{x})$

Ex. Compute the angle between the planes $2x + 3y - 6z = 5$ and $4x - 4y - 7z = -2$.

Ex. Find the area of the triangle with vertices $(2, 0)$, $(3, 4)$, $(-1, 2)$.

Week 10

True or False. If the limit of a function f exists at a point a, then the function is continuous at a.

Ex. Find and sketch the domain of $f(x, y) = \ln(x + y + 1)$.

Ex. Find and sketch the domain of $f(x, y) = \sqrt{4 - x^2 - y^2} + \sqrt{1 + x^2}$.

Ex. Evaluate the limit or show the limit does not exist.

$$\lim_{(x,y)\to(2,1)} \frac{x^2 - 4xy + 4y^2}{x^2 - 4y^2}$$

Ex. Evaluate the limit or show the limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x - y}{6x + 7y}$$

Ex. Evaluate the limit or show the limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^6 + y^2}$$

Ex. Evaluate the limit or show the limit does not exist.

$$\lim_{(x,y)\to(0,0)} xy \arctan\left(\frac{1}{x^2 + y^2}\right)$$

Ex. Find an equation of the tangent plane at the point $(2, 1)$ for the function $f(x, y) = x^2 y + xy^3$.

Ex. Find points on the graph of $z = 3x^2 - 4y^2$ at which the vector $\vec{n} = \langle 3, 2, 2 \rangle$ is normal to the tangent plane.'

Week 11

True or False: A function $f(x, y)$ is differentiable at a point (a, b) if and only if all of its partial derivatives exist at (a, b).

True or False: If the partial derivatives of a function $f(x, y)$ are continuous at a point (a, b), then $f(x, y)$ is differentiable at (a, b).

True or False: If a function $f(x, y)$ is differentiable at a point (a, b), then its tangent plane at (a, b) is the best approximation to $f(x, y)$ near (a, b).

True or False: A function $f(x, y)$ can have continuous partial derivatives at a point (a, b) but still fail to be differentiable at (a, b).

True or False: The directional derivative of a function $f(x, y)$ in the direction of the gradient vector gives the rate of change of f in the direction of steepest ascent.

Ex. Find points on the graph of $z = 3x^2 - 4y^2$ at which the vector $v = \langle 3, 2, 2\rangle$ is normal to the tangent plane.

Def. (Linearization). Let f be differentiable at (a, b). Then, for (x, y) close to (a, b), then

$$f(x, y) \approx f(a, b) + f_x(a, b)\big(x - a\big) + f_y(a, b)\big(y - b\big)$$

Ex. Approximate $\sqrt{\frac{4.1}{9.2}}$ by linearizing $f(x, y) = \sqrt{\frac{x}{y}}$ at $(4, 9)$.

Ex. Linearize $f(x, y) = \operatorname{sech}(x-y) = \frac{2}{e^x+e^{-x}}$ near the point $(\ln 4, \ln 2)$.

Ex. Let $f(x, y) = x^3y^{-4}$. Estimate the change $f(2.03, 0.9) - f(2, 1)$.

Hint. Use the equation $\Delta f = f_x(a, b)\Delta x + f_y(a, b)\Delta y$.

Ex. Approximate $\sqrt{3.01^2 + 3.99^2}$.

Ex. Find the directional derivative of $f(x, y) = \ln(x^2 + y^2)$ in the direction of $v = (3, -2)$ at the point $P = (1, 10)$.

Week 12

Theorem. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. Then, the gradient ∇f is orthogonal to the level sets $f(x) = k \in \mathbb{R}$.

Proof.

Theorem. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. Then, the gradient ∇f points in the direction of greatest rate of change.

Proof.

Ex. You are standing at the point $(0, 3, 9)$ on a mountain whose elevation is given by the function h where $h(x, y) = \frac{y^2}{x^2+1}$. In which direction should you walk to (initially) maintain your current elevation?

Def. *(Gradient).* The gradient ∇ of a function f, written as ∇f, is given by the vector $\nabla f = \langle f_x, f_y \rangle$, and gives the direction of steepest ascent.

Def. *(Directional Derivative).* Suppose f is differentiable at a point P and $\vec{u}$ is a unit vector. Then, the directional derivative of f in the direction of $\vec{u}$ is given by

$$D_{\vec{u}} f(P) = \nabla f(P) \cdot \vec{u}$$

Ex. Find directions in which the directional derivatives of f defined by $f(x, y) = x^2 + xy^3$ at the point $(2, 1)$ is equal to 2.

Chain Rule

Theorem. Suppose

Ex. Let $w = xy\sin(z^2)$. Suppose $x = s - t,\ y = s^2,\ z = t^2$. Find $\frac{\partial w}{\partial s}$.

Week 13

Ex. Compute the double integral

$$\int_0^1 \int_x^1 \sin(y^2)\,\mathrm{d}y\,\mathrm{d}x$$

Ex. Compute the double integral

$$\int_0^1 \int_y^1 \frac{1}{1+x^4}\,\mathrm{d}x\,\mathrm{d}y$$

Ex. Find the sum $\frac{2}{3} - \frac{2}{9} + \frac{2}{27} - \ldots$

Ex. If the partial sums S_n are increasing, which of the following must be true?

1. (a_n) is increasing.
2. (a_n) is positive.
3. $\sum(a_n)$ diverges.

Ex. Suppose $R = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 3 \text{ and } 0 \leq y \leq 8\}$
Evaluate

$$\iint_R xy \cdot e^{\frac{xy^2}{64}}\,\mathrm{d}A$$

Ex. Let $f(x,y) = x^2 + y^2$. In which direction is f increasing the most at $(1,2)$? At $(2,3)$?