Math 1ZA3, Calculus 1
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Reference Sheet

Thm. ***(Intermediate Value Theorem).*** Let f be continuous on $[a, b]$ and suppose that $f(a) \neq f(b)$. Then, for any $y_0 \in \big(f(a), f(b)\big)$, there exists some $x_0 \in (a, b)$ such that $f(x_0) = y_0$

Thm. ***(Extreme Value Theorem).*** Let f be continuous on $[a, b]$. Then, there exist c_1 and $c_2 \in [a, b]$ such that $f(c_1)$ and $f(c_2)$ are absolute minima and maxima of f on $[a, b]$. Find these by checking the critical points and endpoints.

Thm. ***(Fermat's Theorem).*** If f has a local extrema at $x = c$, then f has a critical point at c. Recall, a critical point occurs either when $f'(c) = 0$, or if f' does not exist.

Thm. ***(Mean Value Theorem).*** Given an interval $[a, b]$ and a function f that is continuous on $[a, b]$ and differentiable on (a, b), there exists some $c \in [a, b]$ with $f'(c) = \frac{f(b)-f(a)}{b-a}$

Thm. ***(Rolle's Theorem).*** Given an interval $[a, b]$ and a function f that is continuous on $[a, b]$ and differentiable on (a, b), with $f(a) = f(b)$, then there exists some $c \in (a, b)$ with $f'(c) = 0$

Thm. ***(L'Hopital's Rule).*** Consider functions $f(x)$ and $g(x)$ which are differentiable (except possibly at $x = a$). If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or if $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Thm. ***(Squeeze Theorem).*** Suppose that f, g, and h are functions continuous at a, and let $f(x) \leq g(x) \leq h(x)$ near a. It follows that $\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x) \leq \lim_{x\to a} h(x)$. If $\lim_{x\to a} f(x) = \lim_{x\to a} h(x)$, then,

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \lim_{x\to a} h(x)$$

Thm. ***(Gauss's Trick)***. Given an integer n,

$$\sum_{i=1}^{n} i = 1 + 2 + ... + n = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + ... + n^3 = \Big(\frac{n(n+1)}{2}\Big)^2$$

Thm. ***(Fundamental Theorem of Calculus, Part 1)***. Let f be integrable on $[a, b]$ and let F be an antiderivative of f, such that $F'(x) = f(x)$. Then,

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

Thm. ***(Fundamental Theorem of Calculus, Part 2)***. Let f be continuous on $[a, b]$. Then, the area function under f given by $A(x) = \int_a^x f(t)\,\mathrm{d}t$ is differentiable on $[a, b]$. In addition,

$$\frac{d}{dx}A(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x) = A'(x)$$

Thm. ***(Mean Value Theorem, for Integrals)***. Let f be continuous on $[a, b]$. Then, there exists some $c \in (a, b)$ with $f(c) = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$

Thm. ***(Comparison Theorem)***. If f and g are continuous and $f(x) \geq g(x) \geq 0$ for any $x \geq a$ for some $a \in \mathbb{R}$, then:

1. If $\int_a^\infty f(x)\,\mathrm{d}x$ converges, then $\int_a^\infty g(x)\,\mathrm{d}x$ converges.
2. If $\int_a^\infty g(x)\,\mathrm{d}x$ diverges, then $\int_a^\infty f(x)\,\mathrm{d}x$ diverges.

Derivative Rules

Power Rule: If $n \in \mathbb{R}^{\neq 0}$, then $\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}$

Product Rule: If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are functions in x, then $\frac{\mathrm{d}}{\mathrm{d}x}(fg) = f'g + g'f$

Quotient Rule: If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are functions in x, then $\frac{\mathrm{d}}{\mathrm{d}x}(\frac{f}{g}) = \frac{f'g-g'f}{g^2}$

Chain Rule: If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are functions in x, then $\frac{\mathrm{d}}{\mathrm{d}x}f(g) = f'(g)g'$

Definitions

Derivatives:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad \text{and} \qquad f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Riemann Integral:

$$\int_a^b f(x)\,\mathrm{d}x \;=\; \lim_{n \to \infty} \sum_{i=1}^{n} f\left(a + i \cdot \frac{b-a}{n}\right) \cdot \frac{b-a}{n}$$

Useful Formulae

Triangle Inequality: for any $a, b \in \mathbb{R}$, $\quad |a+b| \;\leq\; |a| + |b|$

Magnitude of Inequalities: If $a < b < c$, then $|b| < \max\{|a|, |c|\}$ and $|b| > \min\{|a|, |c|\}$

Magnitude of Integrals: For any f, a, b, $\quad \left| \int_a^b f(x)\,\mathrm{d}x \right| \;\leq\; \int_b^a |f(x)|\,\mathrm{d}x$

Pythagorean Theorem and Circle Relation: $x^2 + y^2 = r^2$

Integrals of Odd and Even Functions: Let $a \in \mathbb{R}$ be arbitrary

1. If f is odd $\big[f(-x) = -f(x)\big]$, then $\int_{-a}^{a} f(x)\,\mathrm{d}x = 0$
2. If f is even $\big[f(-x) = f(x)\big]$, then $\int_{-a}^{a} f(x)\,\mathrm{d}x = 2 \cdot \int_0^a f(x)\,\mathrm{d}x$

Common Limits

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 \qquad \text{and} \qquad \lim_{x\to 0}\frac{1-\cos x}{x} = 0$$

Common Derivatives

$$\frac{d}{dx}\sin(x) = \cos(x)$$
$$\frac{d}{dx}\cos(x) = -\sin(x)$$
$$\frac{d}{dx}\tan(x) = \sec^2(x)$$
$$\frac{d}{dx}\sinh(x) = \cosh(x)$$
$$\frac{d}{dx}\cosh(x) = \sinh(x)$$
$$\frac{d}{dx}e^x = e^x$$
$$\frac{d}{dx}\ln|x| = \frac{1}{x}$$
$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$$
$$\frac{d}{dx}\mathrm{arcsec}(x) = \frac{1}{\sqrt{x^2-1}}$$

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