

## Convergence Tests

*Thm. (Divergence Test).* If  $\lim_{n \rightarrow \infty} a_n \neq 0$ , then  $\sum_{n=1}^{\infty} a_n$  diverges.

*Thm. (p-Series Test).* If  $p > 1$ , the series  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  converges. If  $p \leq 1$ , it diverges.

*Thm. (Integral Test).* Let  $f(n) = a_n$  and suppose  $f$  is positive, decreasing and continuous for  $n > M$ . Then,  $\int_M^{\infty} f(x)dx$  converges if and only if  $\sum_{n=M}^{\infty} a_n$  converges.

*Thm. (Geometric Series Test).* Let  $|r| < 1$ , then

$$\sum_{n=0}^{\infty} cr^n = c + cr + cr^2 + \dots = \frac{c}{1-r}$$

*Thm. (Comparison Test).* Let  $a_n$  and  $b_n$  be sequences with  $0 < a_n < b_n$  for every  $n$ . Then,

1. If  $\sum a_n$  diverges, then  $\sum b_n$  diverges.
2. If  $\sum b_n$  converges, then  $\sum a_n$  converges.

*Thm. (Limit Comparison Test).* Let  $\sum a_n$  and  $\sum b_n$  be series with positive terms. Let  $\lim(\frac{a_n}{b_n}) = L$ .

1. If  $L = 0$  and  $\sum b_n$  converges, then  $\sum a_n$  converges
2. If  $L = \infty$  and  $\sum b_n$  diverges, then  $\sum a_n$  diverges
3. If  $L > 0$  and  $L \in \mathbb{R}$ , then either both series converge or both series diverge

*Thm. (Alternating Series Test).* Let  $\sum a_n$  be a series which can be written in the form  $\sum a_n = \sum (-1)^{n-1} b_n$  for a positive sequence  $b_n$ , then the series  $\sum a_n$  is convergent if

1.  $b_{n+1} \leq b_n$  for all  $n$  ( $b_n$  is decreasing)
2.  $\lim(b_n) = 0$  ( $b_n$  converges to 0)

*Thm. (Ratio Test).* Suppose  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L$  exists. Then,

1. If  $L < 1$ , then  $\sum a_n$  converges absolutely
2. If  $L > 1$ , then  $\sum a_n$  diverges
3. If  $L = 1$ , the test is inconclusive

*Thm. (Root Test).* Suppose  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L$  exists. Then,

1. If  $L < 1$ , then  $\sum a_n$  converges absolutely
2. If  $L > 1$ , then  $\sum a_n$  diverges
3. If  $L = 1$ , the test is inconclusive