Convergence Tests

Thm. (Divergence Test). If $\lim\limits_{n\to\infty} a_n \neq 0$, then $\sum\limits_{n=1}^{\infty} a_n$ diverges.

Thm. (p-Series Test). If $p > 1$, the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ converges. If $p \leq 1$, it diverges.

Thm. (Integral Test). Let $f(n) = a_n$ and suppose f is positive, decreasing and continuous for $n > M$. Then, $\int_M^\infty f(x)dx$ converges if and only if $\sum_{n=M}^\infty a_n$ converges.

Thm. (Geometric Series Test). Let $|r| < 1$, then

$$\sum_{n=0}^{\infty} cr^n = c + cr + cr^2 + ... = \frac{c}{1-r}$$

Thm. (Comparison Test). Let a_n and b_n be sequences with $0 < a_n < b_n$ for every n. Then,

1. If $\sum a_n$ diverges, then $\sum b_n$ diverges.
2. If $\sum b_n$ converges, then $\sum a_n$ converges.

Thm. (Limit Comparison Test). Let $\sum a_n$ and $\sum b_n$ be series with positive terms. Let $\lim(\frac{a_n}{b_n}) = L$.

1. If $L = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges
2. If $L = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges
3. If $L > 0$ and $L \in \mathbb{R}$, then either both series converge or both series diverge

Thm. (Alternating Series Test). Let $\sum a_n$ be a series which can be written in the form $\sum a_n = \sum(-1)^{n-1}b_n$ for a positive sequence b_n, then the series $\sum a_n$ is convergent if

1. $b_{n+1} \leq b_n$ for all n (b_n is decreasing)
2. $\lim(b_n) = 0$ (b_n converges to 0)

Thm. (Ratio Test). Suppose $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$ exists. Then,

1. If $L < 1$, then $\sum a_n$ converges absolutely
2. If $L > 1$, then $\sum a_n$ diverges
3. If $p = 1$, the test is inconclusive

Thm. (Root Test). Suppose $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$ exists. Then,

1. If $L < 1$, then $\sum a_n$ converges absolutely
2. If $L > 1$, then $\sum a_n$ diverges
3. If $p = 1$, the test is inconclusive