Fluid Between Rotating Pipes

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1 Introduction

This document is supposed to accompany the accelfuild.nb Mathematica notebook found on my github.

It stands as a derivation of the problem up until the point where it is solved numerically. I make the assumption of an incompressible fluid.

2 The Problem

Suppose that we have a viscous, incompressible fluid set in between two rotating cylinders of radius R_1 and R_2.

For simplicity, suppose that the pipes are infinite in length and that the fluid starts at rest.

Find the velocity distribution of the system.

3 Walkthrough

Let us start with the equations of motion in cyllindrical coordinates:

$$\frac{\partial v_r}{\partial t} + (\vec{v} \cdot \nabla) v_r - \frac{v_\phi^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu(\nabla^2 v_r - \frac{2}{r^2}\frac{\partial v_\phi}{\partial \phi} - \frac{v_r}{r^2}) \tag{3.1a}$$

$$\frac{\partial v_\phi}{\partial t} + (\vec{v} \cdot \nabla) v_\phi = -\frac{1}{\rho r}\frac{\partial p}{\partial \phi} + \nu(\nabla^2 v_\phi + \frac{2}{r^2}\frac{\partial v_r}{\partial \phi} - \frac{v_\phi}{r^2}) \tag{3.1b}$$

$$\frac{\partial v_z}{\partial t} + (\vec{v} \cdot \nabla) v_z = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu \nabla^2 v_z \tag{3.1c}$$

$$\nabla \cdot \vec{v} = 0 \tag{3.1d}$$

Where we have

$$(\vec{v} \cdot \nabla) f = v_r \frac{\partial f}{\partial r} + \frac{v_\phi}{r}\frac{\partial f}{\partial \phi} + v_z \frac{\partial f}{\partial z} \tag{3.2a}$$

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial f}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2} \tag{3.2b}$$

First, we should consider the fact that the fluid will not move anywhere in the z direction. This is due to the fluid starting at rest.

Then, we should note that the velocity in the ϕ direction, while clearly nonzero, can only depend on r, not on z or ϕ due to the symmetry of the system.

Finally, given that the fluid is incompressible and we are in quite a closed system, the fluid cannot have a velocity in the radial direction (given that the fluid has no velocity in the z direction, this must be true).

So, in summary, we have

$$v_r = 0 \tag{3.3a}$$

$$v_\phi \equiv v = v(r) \tag{3.3b}$$

$$v_z = 0 \tag{3.3c}$$

Then, let's note that by the symmetry of the system, the pressure can only depend on r.

Indeed, it would seem as if the only nonzero velocity of the system is in our ϕ direction and the gradient of the pressure vanishes except in the r direction.

Applying these simplifications, we have

$$\frac{v^2}{r} = \frac{1}{\rho}\frac{\partial p}{\partial r} \tag{3.4a}$$

$$\frac{\partial v}{\partial t} = \nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) - \frac{v}{r^2}\right) \tag{3.4b}$$

Note that the continuity equation is trivially satisfied by our requirements of the velocity (which are in fact only valid for laminar flow).

Now, we know that the fluid velocity at the cylinders of the system will be equal to the velocity of said cylinders. Given that the pipes are rotating, we can say that the velocity of one of the pipes is in the ϕ direction, and in particular, we have

$$v_{pipe} = r_{pipe}\omega_{pipe}(t)\hat{\phi} \tag{3.5}$$

In essence, the velocity of the pipe is equal to the radius from the center of the pipe times the angular velocity of the pipe.

So, by the boundary condition of the pipe, we have

$$v_\phi(R_1, t) = R_1\omega_1(t) \tag{3.6a}$$

$$v_\phi(R_2, t) = R_2\omega_1(t) \tag{3.6b}$$

Where ω_i are the angular velocities of the respective pipes.

And by starting our fluid at rest, we have

$$v_\phi(r, 0) = 0 \tag{3.7}$$

So, in conclusion, we arrive at our final equations of motion and boundary conditions:

$$\frac{1}{\nu}\frac{\partial v}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) - \frac{v}{r^2} \tag{3.8a}$$

$$v(R_1, t) = R_1\omega_1(t) \tag{3.8b}$$

$$v(R_2, t) = R_2\omega_2(t) \tag{3.8c}$$

$$v(r, 0) = 0 \tag{3.8d}$$

We can then determine the pressure (provided an initial condition) from

$$\frac{\partial p}{\partial r} = \rho\frac{v^2}{r} \tag{3.9}$$

Now, this problem is ready to be numerically solved.

It also turns out that it has an analytic solution for ω_i all being constant and we consider steady flow.