Evolution of Cooperation (Section 3)

14 February 2017

Repeated interactions

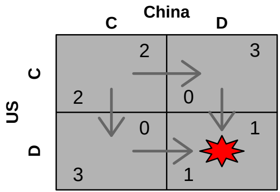
- "Folk Theorem": any level of cooperation can be achieved (as NE) in infinitely repeated game provided players are "sufficiently patient".
- Applies to *any* normal form game, not just the Prisoner's Dilemma.
- Lots of equilibria possible even if single-period game only has one!
- **Problem:** if repeated interactions are *finite*, then people may not cooperate in any period.

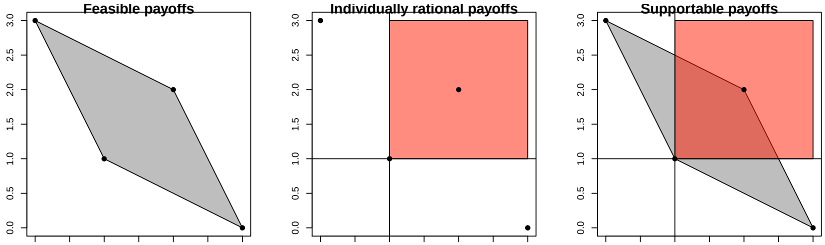
Repeated interactions

Example: Greenhouse gas emissions

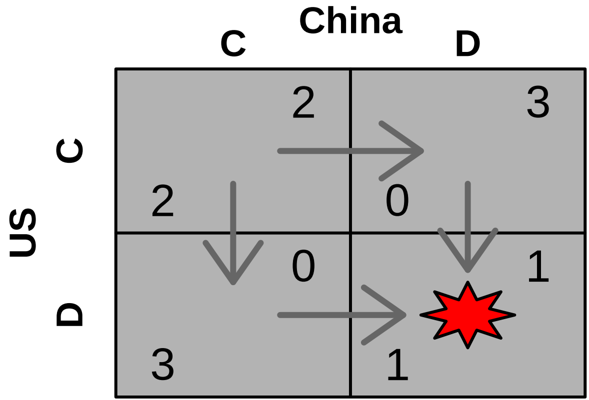
- US and China account for 1/2 of world's emissions.
- Clean air is a public good, pollution is bad for everyone.
- Why can't we reduce emissions?
- "America is not a planet. And we are not even the largest carbon producer anymore, China is [...] I am not in favor of any policies that make America a harder place for people to live." - Marco Rubio
- But if US/China are playing a repeated game, maybe there is hope!

Repeated interactions





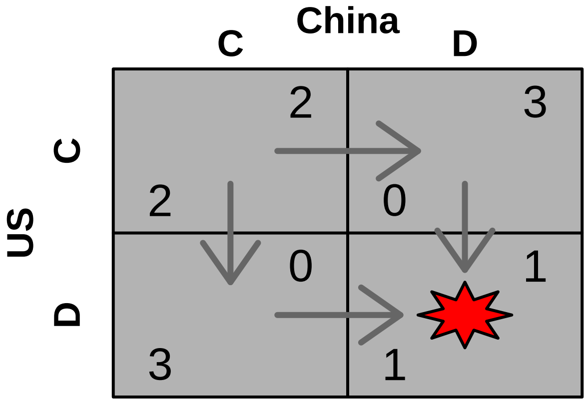
Repeated interactions



Always cooperate v. Always defect:

- ALLC: C C C C C C = 0
- ALLD: D D D D D D = 18

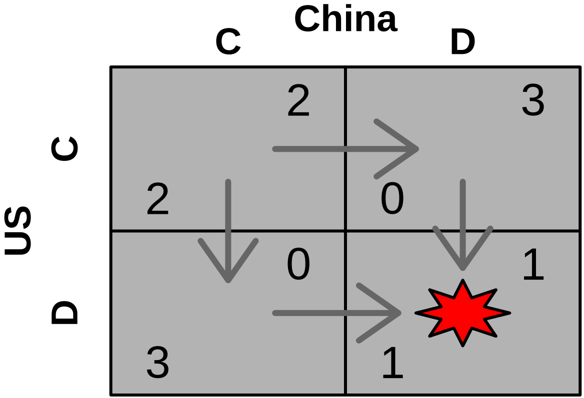
Repeated interactions



Tit for Tat v. Always defect:

- TFT: C D D D D D = 5
- ALLD: D D D D D D = 8

Repeated interactions



Tit for Tat v. Tit for Tat:

- TFT: C C C C C C = 12
- TFT: C C C C C C = 12

Repeated interactions

- Payoff matrix for **repeated game** $(t = 6)$:

		China		
		TFT	ALLD	ALLC
US	TFT	12 12	8 5	12 12
	ALLD	5 8	6 6	0 18
	ALLC	12 12	18 0	12 12

Repeated interactions

- US best responses (vertical arrows)

		China		
		TFT	ALLD	ALLC
US	TFT	12 12	8 5	12 12
	ALLD	5 8	6 6	0 18
	ALLC	12 12	18 0	12 12

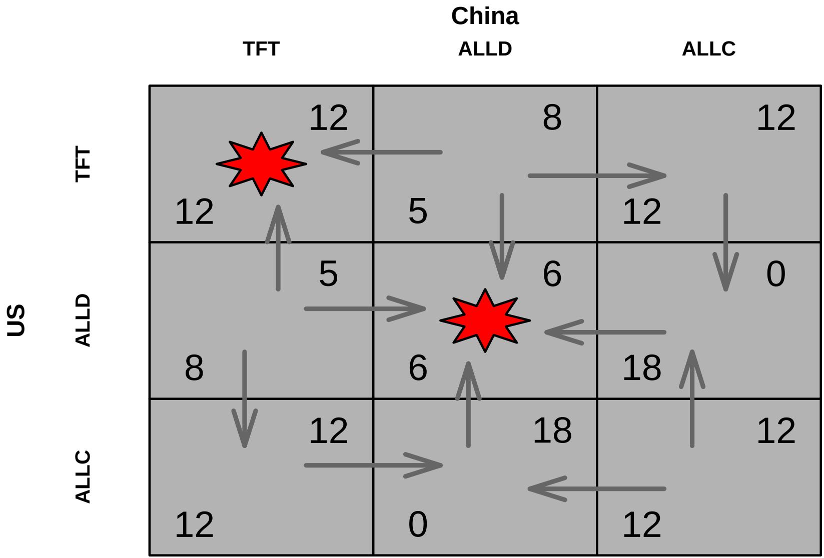
Repeated interactions

- China best responses (horizontal arrows)

		China		
		TFT	ALLD	ALLC
US	TFT	12, 12	5, 8	12, 12
	ALLD	8, 5	6, 6	18, 0
	ALLC	12, 12	0, 18	12, 12

Repeated interactions

- TFT not *strict* Nash equilibria.



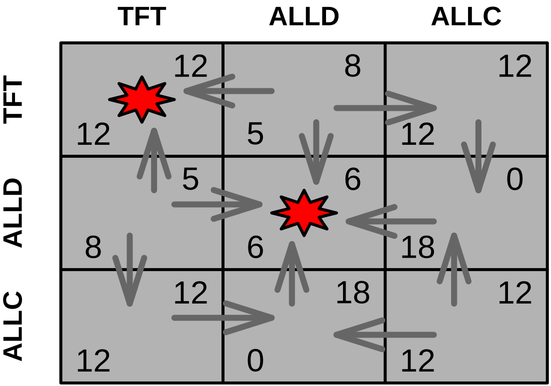
Repeated interactions (evolution)

- Finite example from before ($t = 6$), now w/ generic evolutionary dynamics

	TFT	ALLD	ALLC
TFT	12, 12	5, 8	12, 12
ALLD	8, 5	6, 6	18, 0
ALLC	12, 12	0, 18	12, 12

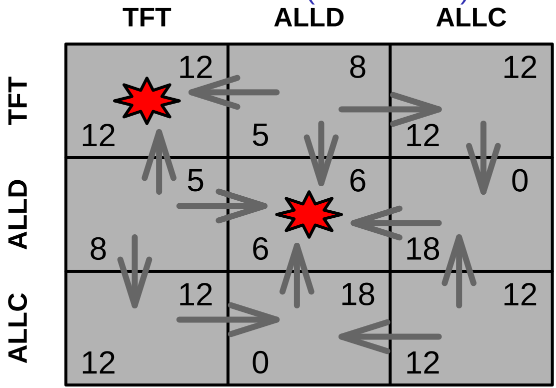
- Suppose p play TFT, q play ALLD, $1 - q - p$ play ALLC

Repeated interactions (evolution)



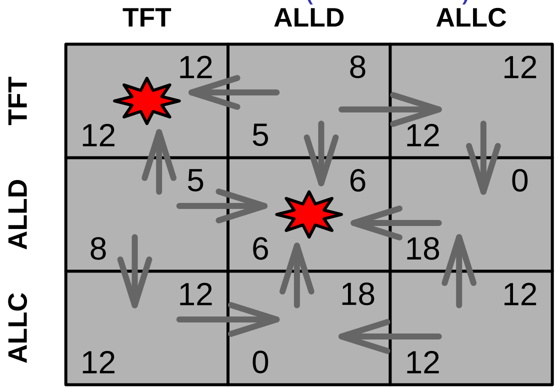
- $\pi_{TFT} = 12p + 5q + 12(1-q-p) = 12 - 7q$
- $\pi_{ALLD} = 8p + 6q + 18(1-q-p) = 18 - 12q - 10p$
- $\pi_{ALLC} = 12p + 12(1-q-p) = 12 - 12q$

Repeated interactions (evolution)



- Let $p = q = 1/3$
- $\pi_{TFT} = 12 - 7/3 \approx 9.67$
- $\pi_{ALLD} = 18 - 12/3 - 10/3 \approx 10.67$
- $\pi_{ALLC} = 12 - 12/3 \approx 8$
- ALLD is risk dominant, largest basin of attraction. Also *strict* NE $\Rightarrow$ ESS.

Repeated interactions (evolution)



- **“Neutral drift”** - ALLC and TFT have same payoff in 2x2 game.
- In ALLC v. TFT population, an ALLC mutation cannot be eliminated.
- In mixed population of ALLC and TFT, TFT is not an ESS.
- In 3x3 game, given enough ALLC for ALLD to prey on, TFT dies out!

Repeated interactions (backward induction)

- *Continuation probability*: probability of next round occuring $w \in [0, 1]$.
- Assume w is fixed and independent across rounds.
- Then probability of still playing after t rounds is w^t.
- *Expected* or "mean"number of rounds is:

$$
\begin{aligned}
1 + w + w^2 + w^3 + \dots &= 1 + w(1 + w + w^2 + \dots) \\
\text{Math trick: let } x &= 1 + w + w^2 + w^3 + \dots \\
\Rightarrow x &= 1 + wx \\
x - wx &= 1 \\
x(1 - w) &= 1 \\
x &= \frac{1}{1 - w} \\
1 + w + w^2 + w^3 + \dots &= \frac{1}{1 - w}
\end{aligned}
$$

Repeated interactions (backward induction)

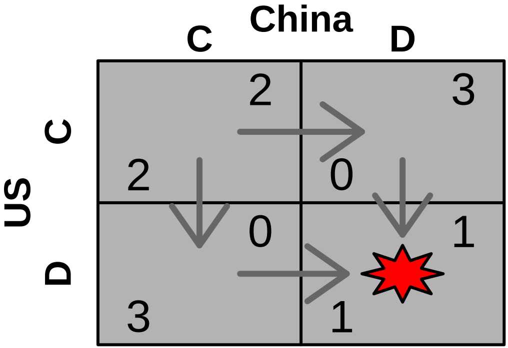
- Payoff of π in every **future** round of infinite game is:

$$\begin{aligned}
\pi(w + w^2 + w^3 + \dots) &= \pi w(1 + w^2 + w^3 + \dots) \\
&= \pi w \left(\frac{1}{1-w}\right) \\
&= \pi \left(\frac{w}{1-w}\right)
\end{aligned}$$

- Depending on what strategies are at play, we might get π_0 in first period and $\pi\left(\frac{w}{1-w}\right)$ in all other periods

Repeated interactions (backward induction)

Example: both play Grim Trigger



- *Expected payoffs to cooperation*: $\frac{2}{1-w}$
- *Expected payoffs for defection*: $3 + 1 \times \left(\frac{w}{1-w}\right)$
- NE if $\frac{2}{1-w} > 3 + \left(\frac{w}{1-w}\right)$ (if $w > 1/2$).
- More generally, we need $w > \frac{T-R}{T-P}$.

Repeated interactions (backward induction)

Example: both play Tit-for-Tat (TFT)

US \ China	C	D
C	R, R	S, T
D	T, S	P, P

- *Expected payoffs to cooperation*: $\frac{R}{1-w}$
- *Expected payoffs for defection in all periods*: $T + w\frac{P}{1-w}$
- *Expected payoffs for 1 period of defection (oscillation)*:

$$T + wS + w^2T + w^3S + \ldots = T + wS + w^2(T + wS) + \ldots$$
$$= (T + wS)/(1 - w^2)$$

Repeated interactions (backward induction)

- TFT is *strict* NE if:

$$\frac{R}{1-w} > \max\left\{\frac{(T+wS)}{(1-w^2)}, T + w\frac{P}{1-w}\right\}$$

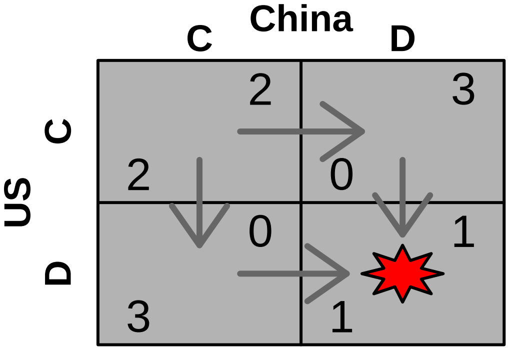
- Some algebra gives:

$$\begin{aligned}
\frac{R}{1-w} &> \frac{(T+wS)}{(1-w^2)} \\
w &> \frac{T-R}{R-S} \\
\frac{R}{1-w} &> T + w\frac{P}{1-w} \\
w &> \frac{T-R}{T-P}
\end{aligned}$$

- Therefore: $w > \max\left\{\frac{T-R}{R-S}, \frac{T-R}{T-P}\right\}$

Repeated interactions (backward induction)

- Back to running example of stage game.



- Here, $R = 2$, $T = 3$, $S = 0$, $P = 1$. Need:
  $w > \max\left\{\frac{T-R}{R-S}, \frac{T-R}{T-P}\right\}$
- To guarantee TFT as strict NE here, $w > \max\{\frac{1}{2}, \frac{1}{2}\}$

Repeated interactions (backward induction)

- Suppose $w = 3/4$ and both play TFT.
- Payoff to [TFT,TFT] is $\frac{2}{1-0.75} = 8$
- Payoff from *one mistake*: $(3 + 0.75 \times 0)/(1 - 0.75^2) \approx 6.89$
- Payoff from switch to ALLD after first period: $3 + \frac{0.75}{(1-0.75)} = 6$
- **Mistakes are costly!** Solution?

Repeated interactions (backward induction)

- Suppose $w = 3/4$ and both play TFT.
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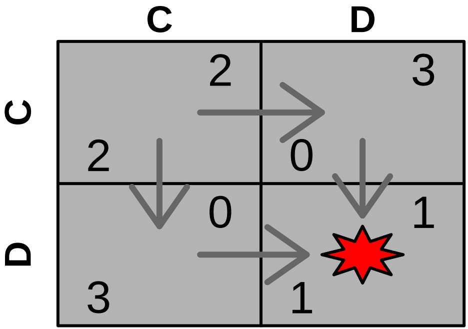
- Payoff from switch to ALLD after first period: $3 + \frac{0.75}{(1-0.75)} = 6$
- **Mistakes are costly!** Solution?
- Correct for mistakes:

Reactive strategies

- Let q denote the probability of cooperating in current period if your opponent cooperated in the previous one.
- Let p denote the probability of cooperating in current period if your opponent defected in previous one.
- There are **infinitely many** types of reactive strategies.
- Try making up your own!
- Example: TFT has $q = 1$ and $p = 0$.
- Example: A "Generous TFT" (GTFT) that has $q = 1$ and $p = 1/3$.

Reactive strategies

Illustration:



- GTFT: $C\ C\ C\ \overset{*}{D}\ C\ D\ C\ C\ C\ C\ldots$
- GTFT: $C\ C\ C\ C\ D\ C$ **C** $C\ C\ C\ldots$
- TFT is better than GTFT against ALLD, but if a cooperative population is reached, GTFT will eventually outcompete TFT.
- If p is too high, however, then GTFT cannot resist invasion from ALLD.