Symmetric Monoidal Categories, $\mathscr{P}$-Pseudoalgebras, and Γ-Categories

J. P. May, Jiasen Liu*, Kyle Roke*, Hongyi Zhang*, Keming Zhou*

Abstract

The goal of this paper to obtain an isomorphism of categories that is the heart of a simple multiplicative functor from a category of operadic pseudoalgebras to a category of Segalic pseudoalgebras. This fits into a nonequivariant version of highly structured algebraic K-theory, from symmetric monoidal categories to orthogonal spectra. In particular, it gives a functor from symmetric bimonoidal categories to E_∞ ring spectra.

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1 Introduction

Since as early as 1974, it has been well-understood that certain well-behaved categorical objects can give rise to spectra. This began with Segal [7] and what he called Γ-categories. His category Γ turns out to be exactly the opposite category of the standard skeleton $\mathscr{F}$ of the category of finite sets.

Elaborations on this functor have been completed already in past work by Guillou, May, Merling, and Osorno [2] and Yau [8]. The upshot is twofold. First, this allows us to construct an algebraic K-theory. The starting point of this is the operadic category $\mathscr{P}$**-PsAlg**, where $\mathscr{P}$ is an operad with the property that $\mathscr{P}$-algebras are the same as permutative categories. It turns out also that $\mathscr{P}$-pseudoalgebras are the same as symmetric monoidal categories, as we show in another paper [5]. Our work gives the entry $\mathbb{R}$ in the following composite of functors:

$$\begin{tikzcd}
\mathscr{P}\textbf{-PsAlg} \arrow[rr, dashed, "\mathbb{K}"] \arrow[dd, "\mathbb{R}"'] & & \textbf{Sp} \\
& & \mathscr{F}\text{-spaces} \arrow[u, "\mathbb{S}"'] \\
\mathscr{F}^{sp}\textbf{-PsAlg} \arrow[rr, "\textbf{St}"'] & & \mathscr{F}\textbf{-Alg} \arrow[u, "B"']
\end{tikzcd}$$

Most of this diagram has been thoroughly explored in past work, e.g. in [3], [1], and [2]. The functors that we will largely ignore are $\mathbb{S}$, the Segalic machine; B, the classifying space functor; and **St**, a purely categorical strictification functor from the category of $\mathscr{F}$-pseudoalgebras to the category of $\mathscr{F}$-algebras. The previous work cited above has given a multiplicative analogue of each of these three

* Authors are listed in alphabetical order.

functors.

What remains is the functor $\mathbb{R}\colon \mathscr{P}\text{-}\mathbf{PsAlg} \to \mathscr{F}^{sp}\text{-}\mathbf{PsAlg}$. A functor with the same source and target has been constructed before [2], but it has two major issues. First, this functor loses symmetry. Second, this construction is quite complicated. Given the importance of this algebraic K-theory machine, it is of great interest that we construct a better functor $\mathbb{R}$. This is the work that we undertake in this paper.

We begin by defining the important categories. We review the definitions of operads, and particularly the permutative operad $\mathscr{P}$. We also describe the category $\mathscr{F}$ and some important constructions. This includes the identification of the most important subcategory $\mathscr{F}^{sp}\text{-}\mathbf{PsAlg}^{\mathbb{R}}$. Finally, we proceed with the definition of the $\mathbb{R}$ functor, and in the final section we show that $\mathbb{R}$ is an equivalence of categories by demonstrating an inverse $\mathbb{Q}\colon \mathscr{F}^{sp}\text{-}\mathbf{PsAlg}^{\mathbb{R}} \to \mathscr{P}\text{-}\mathbf{PsAlg}$.

Remark 1.1. The work of this paper began shortly before the start of the UChicago mathematics REU during the summer of 2025. The first author's intuition on the central theorems of this paper became the kernel of this REU research project.

2 Definitions

We begin with a review of the definitions of operads and algebras over them in **Cat**. For more detail, the reader is encouraged to consult [6], where operads were first introduced.

Definition 2.1. An **operad** $\mathscr{O}$ is a sequence $\mathscr{O}(n) \in \mathbf{Cat}$ for $n \in \mathbb{N}$, along with unit maps $\mathrm{id}\colon * \to \mathscr{O}(1)$ and structure maps $\gamma\colon \mathscr{O}(k) \times \mathscr{O}(j_1) \times \cdots \times \mathscr{O}(j_k) \to \mathscr{O}(j_1 + \cdots + j_k)$, satisfying the following identities. Note that we will often write j_+ as shorthand for $j_1 + \cdots + j_k$.

- There is an identification $* \cong \mathscr{O}(0)$.
- The following unit diagrams commute:

$$\begin{tikzcd} \mathscr{O}(k) \times *^k \arrow[r, "\cong"] \arrow[d, "1\times \mathrm{id}^k"'] & \mathscr{O}(k) \\ \mathscr{O}(k) \times \mathscr{O}(1)^k \arrow[ur, "\gamma"'] & \end{tikzcd} \qquad \begin{tikzcd} * \times \mathscr{O}(k) \arrow[r, "\cong"] \arrow[d, "\mathrm{id}\times 1"'] & \mathscr{O}(k) \\ \mathscr{O}(1) \times \mathscr{O}(k) \arrow[ur, "\gamma"'] & \end{tikzcd}$$

- The following associativity diagram commutes:

$$\begin{tikzcd} \mathscr{O}(k) \times \prod_i \mathscr{O}(n_i) \times \prod_{i,j} \mathscr{O}(m_{ij}) \arrow[r, "\gamma\times 1"] \arrow[dd, "\mathrm{id}\times\mathrm{shuffle}"'] & \mathscr{O}(n_+) \times \prod_{i,j} \mathscr{O}(m_{ij}) \arrow[d, "\gamma"] \\ & \mathscr{O}(m_{++}) \\ \mathscr{O}(k) \times \prod_i \Big(\mathscr{O}(n_i) \times \prod_j \mathscr{O}(m_{ij})\Big) \arrow[r, "1\times\gamma"'] & \mathscr{O}(k) \times \prod_i \mathscr{O}(m_{i+}) \arrow[u, "\gamma"'] \end{tikzcd}$$

- Finally, the following equivariance diagrams commute. We take $\sigma \in \Sigma_k$ and $\tau_i \in \Sigma_{n_i}$, and we define $\sigma(n_1, \ldots, n_k)$ to be the permutation in Σ_{n_+} that permutes k blocks of letters by σ. We also define $\tau_1 \oplus \cdots \oplus \tau_k \in \Sigma_{n_+}$ to be the block sum.

$$\begin{tikzcd} \mathscr{O}(k) \times \prod \mathscr{O}(n_{\sigma^{-1}(i)}) \arrow[r, "\sigma\times\sigma^{-1}"] \arrow[d, "\gamma"'] & \mathscr{O}(k) \times \prod \mathscr{O}(n_i) \arrow[d, "\gamma"] \\ \mathscr{O}(n_+) \arrow[r, "\sigma(n_1,\ldots,n_k)"'] & \mathscr{O}(n_+) \end{tikzcd}$$

$$\begin{tikzcd} \mathscr{O}(k) \times \prod \mathscr{O}(n_i) \arrow[r, "1\times\tau_1\times\cdots\times\tau_k"] \arrow[d, "\gamma"'] & \mathscr{O}(k) \times \prod \mathscr{O}(n_i) \arrow[d, "\gamma"] \\ \mathscr{O}(n_+) \arrow[r, "\tau_1\oplus\cdots\oplus\tau_k"'] & \mathscr{O}(n_+) \end{tikzcd}$$

The objects we will be studying are pseudoalgebras over an operad.

Definition 2.2. Let $\mathscr{O}$ be an operad. An *$\mathscr{O}$-pseudoalgebra* is an unbased category X, along with action maps

$$\theta\colon \mathscr{O}(k)\times X^k \to X$$

This defines first a basepoint of X, from the map

$$* \cong \mathscr{O}(0) \to X$$

We also require that these maps be unital, associative, and equivariant, up to a coherent isomorphism where written.

- The unit diagram commutes up to natural isomorphism:

$$\begin{tikzcd} * \times X \arrow[r, "\cong"] \arrow[d, "\mathrm{id}\times 1"'] & X \\ \mathscr{O}(1)\times X \arrow[ur, "\theta"'] & \end{tikzcd}$$

- The following associativity diagram commutes up to a coherent natural isomorphism. For simplicity, we do not reproduce the coherence axioms here, but a good reference is [3, Definition 2.14]. We will make reference to the coherence axioms sparingly over the course of the paper.

$$\begin{tikzcd} \mathscr{O}(k)\times \prod_i \mathscr{O}(n_i)\times X^{n_+} \arrow[rr, "\gamma\times 1"] \arrow[dd, "1\times\mathrm{shuffle}"'] & & \mathscr{O}(n_+)\times X^{n_+} \arrow[d, "\theta"] \\ & \arrow[phantom, "\Downarrow\phi"] & X \\ \mathscr{O}(k)\times \prod_i \left(\mathscr{O}(n_i)\times X^{n_i}\right) \arrow[rr, "1\times\prod\theta"'] & & \mathscr{O}(k)\times X^k \arrow[u, "\theta"'] \end{tikzcd}$$

- Finally, the following equivariance diagram commutes strictly.

$$\begin{tikzcd} \mathscr{O}(k)\times X^k \arrow[rr, "\sigma\times\sigma^{-1}"] \arrow[dr, "\theta"'] & & \mathscr{O}(k)\times X^k \arrow[dl, "\theta"] \\ & X & \end{tikzcd}$$

In order to make this into a category, we need to define pseudomorphisms. We will omit some of the details here.

Definition 2.3. A *pseudomorphism* of $\mathscr{O}$-pseudoalgebras $(\mathcal{A},\theta)$ and $(\mathcal{B},\psi)$ consists of a functor $F\colon \mathcal{A}\to\mathcal{B}$, such that the following diagram commutes up to coherent natural isomorphism. We invite the curious reader to consult [3, Definition 2.23] for the details of the coherence axioms.

$$\begin{tikzcd} \mathscr{O}(n)\times\mathcal{A}^n \arrow[r, "\theta"] \arrow[d, "1\times F^n"'] \arrow[dr, phantom, "\Downarrow"] & \mathcal{A} \arrow[d, "F"] \\ \mathscr{O}(n)\times\mathcal{B}^n \arrow[r, "\psi"'] & \mathcal{B} \end{tikzcd}$$

We now specialize to the specific case we care about, which is the operad $\mathscr{P}$. For brief motivation on why $\mathscr{P}$ is important, it suffices to see that the category $\mathscr{P}$-algebras is equivalent to the category of permutative categories. This is a well-known correspondence, and in fact it extends to $\mathscr{P}$-pseudoalgebras and symmetric monoidal categories, which we show in a different paper [5].

There is a well-understood functor $\mathbf{Ob}\colon \mathbf{Cat}\to\mathbf{Set}$, which sends a small category to its underlying set of objects. This functor has a right adjoint, the indiscrete category functor $\mathcal{E}\colon \mathbf{Set}\to\mathbf{Cat}$, which sends a set S to the category $\mathcal{E}S$ whose objects are S, and such that

$$|\mathcal{E}S(x,y)| = 1$$

for every $x,y\in S$.

Definition 2.4. We reproduce the definition here from [3, Definition 3.2]. The operad $\mathscr{P}$ has

$$\mathscr{P}(n) = \mathcal{E}\Sigma_n$$

The unit map is given by identifying the identity permutation $1 \in \Sigma_n$. Morever, the structure maps are defined by

$$\gamma(\sigma; \tau_1, \ldots, \tau_k) = \sigma(n_1, \ldots, n_k)(\tau_1 \oplus \cdots \oplus \tau_k)$$

It is not a difficult exercise to check that this is a legitimate operad.

Definition 2.5. The category $\mathscr{P}$**-PsAlg** consists of $\mathscr{P}$-pseudoalgebras and psuedomorphisms.

$\mathscr{P}$**-PsAlg** will serve as our first important category in this paper. We next begin the endeavor of exploring the second, $\mathscr{F}$**-PsAlg**.

Definition 2.6. We define the category of based finite sets $\mathscr{F}$ as the category with objects the sets $\mathbf{n} = \{0, \ldots, n\}$ with basepoint 0, and morphisms all based functions $f \colon \mathbf{n} \to \mathbf{m}$ (i.e., those f for which $f(0) = 0$).

Definition 2.7. Π denotes the subcategory of $\mathscr{F}$ with the same objects and those maps $\phi \colon \mathbf{m} \to \mathbf{n}$ such that $\phi^{-1}(j)$ has 0 or 1 element for $1 \leq j \leq n$. Let $\mathscr{E}$ be the subcategory of $\mathscr{F}$ with the same objects and those maps such that $\phi^{-1}(0) = \{0\}$. Let $\Lambda = \Pi \cap \mathscr{E}$.

Definition 2.8. A Π-category is a functor $\Pi \to \mathbf{Cat}_*$. A morphism between Π-categories is a natural transformation of functors, and consequently these form a category Π**-Cat**.

Given any based category $\mathcal{C} \in \mathbf{Cat}_*$, we can naturally define a Π-category structure $\mathbb{R}\mathcal{C}$. Take $\mathbb{R}\mathcal{C}(n) = \mathcal{C}^n \in \mathbf{Cat}_*$, and then for any morphism $\phi \in \Pi(\mathbf{m}, \mathbf{n})$, define $f_i \colon \mathcal{A}^m \to \mathcal{A}$ to be the jth projection $\pi_j \colon \mathcal{A}^m \to \mathcal{A}$ if $\phi^{-1}(i) = \{j\}$, and the composite $\mathcal{A}^m \to * \to \mathcal{A}$ if $\phi^{-1}(i) = \emptyset$. Then define $\mathbb{R}\mathcal{C}(\phi) \colon \mathcal{A}^m \to \mathcal{A}^n$ as the product of the f_i. That $\mathbb{R}$ is a functor $\mathbf{Cat}_* \to \Pi$**-Cat** is a standard calculation.

Now, $\mathbb{R}$ has a left adjoint: namely, the functor $\mathbb{L} \colon \Pi\textbf{-Cat} \to \mathbf{Cat}_*$ sending $\mathcal{D} \colon \Pi \to \mathbf{Cat}_*$ to $\mathcal{D}(1)$. The counit $\epsilon \colon \mathbb{L}\mathbb{R} \implies \mathrm{id}_{\mathbf{Cat}_*}$ is exactly the identity natural isomorphism of $\mathrm{id}_{\mathbf{Cat}_*}$. The unit $\delta \colon \mathrm{id} \implies \mathbb{R}\mathbb{L}$ has components $(\mathcal{D}(n) \to \mathcal{D}(1)^n)_{n \in \mathbb{N}}$, and in general these need not be isomorphisms. In this paper, we will restrict our attention to the case where δ is an isomorphism.

Definition 2.9. The category Π^{sp}**-Alg** of *special* functors consists of those Π-categories for which δ is an isomorphism.

For any $\mathcal{C} \in \mathbf{Cat}_*$, $\mathbb{R}\mathcal{C}$ is naturally a special Π-category, so $\mathbb{R}$ and $\mathbb{L}$ restrict to an adjunction between Π^{sp}**-Alg** and $\mathbf{Cat}_*$. With this restriction, the unit δ becomes an isomorphism. Since ϵ is still the identity, we conclude that these form an adjoint equivalence:

Proposition 2.10. *Π^{sp}**-Alg** and $\mathbf{Cat}_*$ are equivalent categories, with equivalences given by $\mathbb{R}$ and $\mathbb{L}$.*

Proposition 2.11. *The image of $\mathbb{R}$ is exactly the category of Π-algebras for which δ is the identity.*

Proof. For a Π-algebra $\mathcal{C}$, we can see that δ is the product of $\mathcal{C}(\delta_i)$, where $\delta_i \in \Pi(\mathbf{n}, \mathbf{1})$ is the ith projection: it maps i to 1 and everything else to 0. If $\mathcal{C} = \mathbb{R}\mathcal{D}$ for a based category $\mathcal{D}$, then $\mathcal{C}(\delta_i)$ is just the ith projection $\pi_i \colon \mathcal{D}^n \to \mathcal{D}$. Then $\delta = \pi_1 \times \cdots \times \pi_n = \mathrm{id} \colon \mathcal{D}^n \to \mathcal{D}^n$.

Conversely, suppose $\mathcal{C} \in \Pi^{sp}$**-Alg** and that δ is the identity on $\mathcal{C}^n$. We will show that $\mathcal{C} = \mathbb{R}\mathbb{L}(\mathcal{C})$; to this end, it suffices to show that the projections, injections, and permutations of $\mathcal{C}$ are exactly those prescribed by $\mathbb{R}$ on $\mathcal{C}(1)$. Since $\mathrm{id} = \delta = (\mathcal{C}(\delta_1), \ldots, \mathcal{C}(\delta_n))$, we have

$$\begin{aligned}\mathcal{C}(\delta_i) &= \pi_i \circ (\mathcal{C}(\delta_1), \ldots, \mathcal{C}(\delta_n)) \\ &= \pi_i \circ \mathrm{id} \\ &= \pi_i\end{aligned}$$

so that $\mathcal{C}$ sends these projections to their analogous projections on $\mathcal{C}^n$.

Consider next the simple injections $\sigma_i\colon \mathbf{n} \to \mathbf{n{+}1}$ defined by preserving order and missing $i \in \mathbf{n{+}1}$. Then $\mathcal{C}(\sigma_i)$ is a functor $\mathcal{C}(1)^n \to \mathcal{C}(1)^{n+1}$. If we postcompose with one of the projections $\pi_j\colon \mathcal{C}(1)^{n+1} \to \mathcal{C}$, we have $\pi_j \circ \mathcal{C}(\sigma_i) = \mathcal{C}(\delta_j) \circ \mathcal{C}(\sigma_i) = \mathcal{C}(\delta_j \sigma_i)$. Now, there are two cases. If $i \neq j$, then $j \in \operatorname{im} \sigma_i$, so $\delta_j \sigma_i = \delta_{\sigma_i^{-1}(j)}$ (which is either δ_j or δ_{j-1}, depending on whether $j < i$ or $j > i$). Since $\mathcal{C}(\delta_{\sigma_i^{-1}(j)}) = \pi_{\sigma_i^{-1}(j)}$, we see that the jth component of $\mathcal{C}(\sigma_i)$ is the $\sigma_i^{-1}(j)$th coordinate of $\mathcal{C}(1)^n$. The other case is if $i = j$, in which case $\delta_j \sigma_i$ is the zero map $\mathbf{n} \to \mathbf{0} \to \mathbf{1}$. Hence $\mathcal{C}(\delta_j \sigma_i)$ factors in the same way as $0\colon \mathcal{C}(1)^n \to * \to \mathcal{C}(1)$. This confirms that $\mathcal{C}(\sigma_i) = \mathbb{RL}(\mathcal{C})(\sigma_i)$.

Finally, we can check permutations $\sigma\colon \mathbf{n} \to \mathbf{n}$. Postcomposing with projections π_i and following a similar argument as the above, we see that $\pi_i \circ \mathcal{C}(\sigma) = \mathcal{C}(\delta_i \circ \sigma) = \mathcal{C}(\delta_{\sigma^{-1}(i)}) = \pi_{\sigma^{-1}(i)}$. So indeed $\mathcal{C}(\sigma) = \mathbb{RL}(\mathcal{C})(\sigma)$. Since all Π-maps can be built out of these simple projections, injections, and permutations, we can conclude that $\mathcal{C} = \mathbb{RL}(\mathcal{C})$, as required. $\square$

Analogously to our definition of operadic pseudoalgebras after defining operadic algebras, we relax the definition of algebras in the $\mathscr{F}$-context to that of pseudofunctors. For more details, and for a full enumeration of coherence conditions, the curious reader should consult [4], sections 4.1 and 4.2.

Definition 2.12. A *pseudofunctor* $F\colon \mathcal{C} \to \mathcal{D}$, where $\mathcal{C}$ and $\mathcal{D}$ are arbitrary categories, consists of

- a function on objects $F\colon \mathbf{Ob}(\mathcal{C}) \to \mathbf{Ob}(\mathcal{D})$;
- and a function on morphisms $F\colon \mathcal{C}(x,y) \to \mathcal{D}(F(x), F(y))$,

which need not be strictly associative, but for which we have coherent natural isomorphisms

$$\begin{tikzcd}
\mathcal{C}(y,z) \times \mathcal{C}(x,y) \arrow[rr, "\circ"] \arrow[d, "F \times F"'] & \arrow[d, phantom, "\Downarrow"] & \mathcal{C}(x,z) \arrow[d, "F"] \\
\mathcal{D}(F(y),F(z)) \times \mathcal{D}(F(x),F(y)) \arrow[rr, "\circ"'] & & \mathcal{D}(F(x),F(z))
\end{tikzcd}$$

A similar definition can be made for *pseudonatural transformations*: instead of requiring that the usual square of a natural transformation commutes strictly, we only require that it commutes up to coherent natural isomorphism.

Definition 2.13. The category $\mathscr{F}^{sp}$**-PsAlg** consists of pseudofunctors $\mathscr{F} \to \mathbf{Cat}$ whose restriction to Π are special Π-algebras, and pseudonatural transformations whose restriction to Π are strict natural transformations.

Proposition 2.14. *Every $\mathscr{F}$-pseudoalgebra is naturally isomorphic to an $\mathscr{F}$-pseudoalgebra whose restriction to Π is in the image of $\mathbb{R}$.*

Proof. Let $\mathcal{A} \in \mathscr{F}^{sp}$**-PsAlg**, and write $\mathcal{A}_\Pi$ for the restriction of $\mathcal{A}$ to Π. By Proposition 2.10, $\mathcal{A}_\Pi$ is naturally isomorphic to $\mathbb{RL}(\mathcal{A}_\Pi)$, by some natural isomorphism $\xi\colon \mathcal{A}_\Pi \to \mathbb{RL}(\mathcal{A}_\Pi)$. We define a new $\mathscr{F}$-pseudoalgebra $\mathbb{U}\mathcal{A}$ by $\mathbb{U}\mathcal{A}(\mathbf{n}) = \mathbb{RL}(\mathcal{A}_\Pi)(\mathbf{n})$. For $\phi \in \mathscr{F}(\mathbf{m}, \mathbf{n})$, we define $\mathbb{U}\mathcal{A}(\phi)$ to be the composite

$$\mathbb{RL}(\mathcal{A}_\Pi)(\mathbf{m}) \xrightarrow{\xi_m^{-1}} \mathcal{A}(\mathbf{m}) \xrightarrow{\mathcal{A}(\phi)} \mathcal{A}(\mathbf{n}) \xrightarrow{\xi_n} \mathbb{RL}(\mathcal{A}_\Pi)(\mathbf{n})$$

Note that if $\phi \in \Pi(\mathbf{m}, \mathbf{n})$, then $\mathbb{U}\mathcal{A}(\phi) = \mathbb{RL}(\mathcal{A}_\Pi)(\phi)\colon \mathbb{U}\mathcal{A}(\mathbf{m}) \to \mathbb{U}\mathcal{A}(\mathbf{n})$, since ξ is a strict natural isomorphism. Hence the square below commutes strictly:

$$\begin{tikzcd}
\mathcal{A}(\mathbf{m}) \arrow[r, "\mathcal{A}(\phi)"] & \mathcal{A}(\mathbf{n}) \arrow[d, "\xi_m"] \\
\mathbb{RL}(\mathcal{A}_\Pi)(\mathbf{m}) \arrow[u, "\xi_m^{-1}"] \arrow[r, "\mathbb{RL}(\mathcal{A}_\Pi)\phi"'] & \mathbb{RL}(\mathcal{A}_\Pi)(\mathbf{n})
\end{tikzcd}$$

To check that $\mathbb{U}\mathcal{A}$ is a legitimate pseudoalgebra, we need to ensure that $\mathbb{U}\mathcal{A}$ is a pseudofunctor $\mathscr{F} \to \mathbf{Cat}_*$. This follows quickly from pseudofunctoriality of $\mathcal{A}$. If $\phi \in \mathscr{F}(\mathbf{m}, \mathbf{n})$ and $\psi \in \mathscr{F}(\mathbf{n}, \mathbf{k})$, then because $\xi\xi^{-1} = \mathrm{id}$, $\mathbb{U}\mathcal{A}(\psi)\mathbb{U}\mathcal{A}(\phi)$ is exactly

$$\mathbb{RL}(\mathcal{A}_\Pi)(\mathbf{m}) \xrightarrow{\xi_m^{-1}} \mathcal{A}(\mathbf{m}) \xrightarrow{\mathcal{A}(\phi)} \mathcal{A}(\mathbf{n}) \xrightarrow{\mathcal{A}(\psi)} \mathcal{A}(\mathbf{k}) \xrightarrow{\xi_k} \mathbb{RL}(\mathcal{A}_\Pi)(\mathbf{k})$$

Since $\mathcal{A}$ is a pseudoalgebra, we have a natural isomorphism $\phi_{\phi,\psi}$ between $\mathcal{A}(\psi)\mathcal{A}(\phi)$ and $\mathcal{A}(\psi\phi)$, which fits into a diagram

$$\begin{tikzcd}
\mathbb{RL}(\mathcal{A}_\Pi)(\mathbf{m}) \arrow[r, "\xi_m^{-1}"] \arrow[d, equal] & \mathcal{A}(\mathbf{m}) \arrow[r, "\mathcal{A}(\phi)"] \arrow[d, equal] & \mathcal{A}(\mathbf{n}) \arrow[r, "\mathcal{A}(\psi)"] \arrow[d, phantom, "\Swarrow\phi_{\phi,\psi}"] & \mathcal{A}(\mathbf{k}) \arrow[r, "\xi_k"] \arrow[d, equal] & \mathbb{RL}(\mathcal{A}_\Pi)(\mathbf{k}) \arrow[d, equal] \\
\mathbb{RL}(\mathcal{A}_\Pi)(\mathbf{m}) \arrow[r, "\xi_m^{-1}"] & \mathcal{A}(\mathbf{m}) \arrow[rr, "\mathcal{A}(\psi\phi)"] & & \mathcal{A}(\mathbf{k}) \arrow[r, "\xi_k"] & \mathbb{RL}(\mathcal{A}_\Pi)(\mathbf{k})
\end{tikzcd}$$

Hence $\phi'_{\phi,\psi} = \xi_m^{-1} * \phi_{\phi,\psi} * \xi_k$ is a natural isomorphism from $\mathbb{U}\mathcal{A}(\psi)\mathbb{U}\mathcal{A}(\phi)$ to $\mathbb{U}\mathcal{A}(\psi\phi)$. We also need to check the associativity condition for these natural isomorphisms; namely, that for any $f \in \mathscr{F}(\mathbf{m},\mathbf{n})$, $g \in \mathscr{F}(\mathbf{n},\mathbf{k})$, and $h \in \mathscr{F}(\mathbf{k},\mathbf{l})$, that

$$\phi'_{f,hg} \circ (\phi'_{g,h} * 1_{\mathbb{U}\mathcal{A}(f)}) = \phi'_{gf,h} \circ (1_{\mathbb{U}\mathcal{A}(h)} * \phi'_{f,g})$$

But this follows immediately from the analogous condition for $\mathcal{A}$, since whiskering with ξ commutes with everything in sight. Hence $\mathbb{U}\mathcal{A}$ is indeed an $\mathscr{F}$-pseudoalgebra.

Now, all that remains to show is that $\mathbb{U}\mathcal{A}$ is isomorphic to $\mathcal{A}$. On the level of objects, we take the isomorphism $\xi_n\colon \mathcal{A}(n) \to \mathbb{U}\mathcal{A}(n)$. We need to check that this defines a pseudomorphism. For any $f \in \mathscr{F}(\mathbf{m},\mathbf{n})$, we have the square

$$\begin{tikzcd}
\mathcal{A}(\mathbf{m}) \arrow[rrr, "\mathcal{A}(f)"] \arrow[d, "\xi_m"'] & & & \mathcal{A}(\mathbf{n}) \arrow[d, "\xi_n"] \\
\mathbb{RL}(\mathcal{A}_\Pi)(\mathbf{m}) \arrow[r, "\xi_m^{-1}"] & \mathcal{A}(\mathbf{m}) \arrow[r, "\mathcal{A}(f)"] & \mathcal{A}(\mathbf{n}) \arrow[r, "\xi_m"] & \mathbb{RL}(\mathcal{A}_\Pi)(\mathbf{n})
\end{tikzcd}$$

which commutes strictly, and hence ξ is indeed an isomorphism of $\mathcal{A}$ and $\mathbb{U}\mathcal{A}$. $\square$

Definition 2.15. Denote by $\mathscr{F}^{sp}$-$\mathbf{PsAlg}^{\mathbb{R}}$ the subcategory of $\mathscr{F}^{sp}$-$\mathbf{PsAlg}$ given by those $\mathscr{F}$-pseudoalgebras whose restriction to Π is in the image of $\mathbb{R}$.

Proposition 2.16. *$\mathscr{F}^{sp}$-**PsAlg** is equivalent as a 2-category to $\mathscr{F}^{sp}$-**PsAlg**$^{\mathbb{R}}$.*

Proof. To prove this proposition, it suffices to show that the assignment in Proposition 2.14 is suitably functorial. So suppose that $\mathcal{A}$ and $\mathcal{A}'$ are $\mathscr{F}$-pseudoalgebras, and that we have isomorphisms $\xi\colon \mathcal{A}_\Pi \to \mathbb{RL}(\mathcal{A}_\Pi)$ and $\xi'\colon \mathcal{A}'_\Pi \to \mathbb{RL}(\mathcal{A}'_\Pi)$. If we have a morphism $F\colon \mathcal{A} \to \mathcal{A}'$, then we define $\mathbb{U}F\colon \mathbb{U}\mathcal{A} \to \mathbb{U}\mathcal{A}'$ on objects by

$$\begin{tikzcd}
\mathbb{RL}(\mathcal{A}_\Pi)(\mathbf{m}) \arrow[r, "\xi_m^{-1}"] \arrow[rrr, bend right, "(\mathbb{U}\mathscr{F})_m"'] & \mathcal{A}(\mathbf{m}) \arrow[r, "F_m"] & \mathcal{A}'(\mathbf{m}) \arrow[r, "\xi'_m"] & \mathbb{RL}(\mathcal{A}'_\Pi)(\mathbf{m})
\end{tikzcd}$$

We need to check that $\mathbb{U}F$ defines a pseudonatural transformation. If $f \in \mathscr{F}(\mathbf{m},\mathbf{n})$, then from the definition of $\mathbb{U}\mathcal{A}(f)$, the left and right squares of the following must commute:

$$\begin{tikzcd}
\mathbb{RL}(\mathcal{A}_\Pi)(\mathbf{m}) \arrow[r, "\xi_m^{-1}"] \arrow[d, "\mathbb{U}\mathcal{A}(f)"'] & \mathcal{A}(\mathbf{m}) \arrow[r, "F_m"] \arrow[d, "\mathcal{A}(f)"'] \arrow[dr, phantom, "\Swarrow\phi_f"] & \mathcal{A}'(\mathbf{m}) \arrow[r, "\xi'_m"] \arrow[d, "\mathcal{A}'(f)"] & \mathbb{RL}(\mathcal{A}'_\Pi)(\mathbf{m}) \arrow[d, "\mathbb{U}\mathcal{A}'(f)"] \\
\mathbb{RL}(\mathcal{A}_\Pi)(\mathbf{n}) \arrow[r, "\xi_n^{-1}"] & \mathcal{A}(\mathbf{n}) \arrow[r, "F_n"] & \mathcal{A}'(\mathbf{n}) \arrow[r, "\xi'_n"] & \mathbb{RL}(\mathcal{A}'_\Pi)(\mathbf{n})
\end{tikzcd}$$

In particular, we conclude that the natural isomorphism ϕ'_f for $\mathbb{U}F$ is exactly the natural isomorphism $\xi^{-1} * \phi_f * \xi'$. And as before, since whiskering by ξ and ξ' commutes with everything in sight, all of the coherence conditions hold automatically. Since $\mathbb{U}\colon \mathscr{F}^{sp}$-$\mathbf{PsAlg} \to \mathscr{F}^{sp}$-$\mathbf{PsAlg}^{\mathbb{R}}$ restricts to the identity on $\mathscr{F}^{sp}$-$\mathbf{PsAlg}^{\mathbb{R}}$, it is left inverse to the inclusion $\mathscr{F}^{sp}$-$\mathbf{PsAlg}^{\mathbb{R}} \to \mathscr{F}^{sp}$-$\mathbf{PsAlg}$. By Proposition 2.14, we conclude that $\mathbb{U}$ is an equivalence of categories. $\square$

We next demonstrate why we prefer to work in $\mathscr{F}^{sp}$-$\mathbf{PsAlg}^{\mathbb{R}}$, rather than the more general setting of $\mathscr{F}^{sp}$-$\mathbf{PsAlg}$. One way to understand the morphisms of $\mathscr{F}$ are as combinations of projections, Π-maps, and the canonical multiplication maps $\phi_n\colon \mathbf{n} \to \mathbf{1}$, which send everything to 1 (except for 0, of course). We will make this more precise in a moment, but the intuition is that every morphism $\phi \in \mathscr{F}(\mathbf{m},\mathbf{n})$ is the composition of a permutation and the direct sum of n multiplication maps, each sending $\phi^{-1}(j)$ to j for some j.

Definition 2.17. Given two morphisms $f\colon \mathbf{m} \to \mathbf{n}$ and $g\colon \mathbf{m}' \to \mathbf{n}'$ in $\mathscr{F}$, we define their *direct sum* $f \oplus g \in \mathscr{F}(\mathbf{m}{+}\mathbf{m}', \mathbf{n}{+}\mathbf{n}')$ to be the coproduct of the maps

$$\mathbf{m} \xrightarrow{f} \mathbf{n} \xrightarrow{in_n} \mathbf{n}{+}\mathbf{n}'$$

and

$$\mathbf{m}' \xrightarrow{g} \mathbf{n} \xrightarrow{in_{n'}} \mathbf{n}{+}\mathbf{n}'$$

where $in_n\colon \mathbf{n} \to \mathbf{n}{+}\mathbf{n}'$ is the inclusion into the first n coordinates, and $in_{n'}$ is the inclusion into the last n' coordinates.

Definition 2.18. For a based category $\mathcal{A}$, and two functors $f\colon \mathcal{A}^m \to \mathcal{A}^n$ and $g\colon \mathcal{A}^{m'} \to \mathcal{A}^{n'}$, we define their *direct sum* $f \oplus g\colon \mathcal{A}^{m+m'} \to \mathcal{A}^{n+n'}$ as the product of the maps

$$\mathcal{A}^{m+m'} \xrightarrow{proj_m} \mathcal{A}^m \xrightarrow{f} \mathcal{A}^n$$

and

$$\mathcal{A}^{m+m'} \xrightarrow{proj_{m'}} \mathcal{A}^{m'} \xrightarrow{g} \mathcal{A}^{n'}$$

where $proj_m\colon \mathcal{A}^{m+m'} \to \mathcal{A}^m$ is the projection to the first m coordinates, and $proj_{m'}\colon \mathcal{A}^{m+m'} \to \mathcal{A}^{m'}$ is the projection to the last m' coordinates.

It is not hard to check that these definitions are both associative.

Proposition 2.19. *Every morphism in $\mathscr{F}$ is a composite of a projection, a permutation, and a direct sum of canonical multiplication maps.*

Proof. Let $f \in \mathscr{F}(\mathbf{m}, \mathbf{n})$. Consider the set $f^{-1}(0)$, and take the projection $\pi\colon \mathbf{m} \to \mathbf{k}$ sending exactly $f^{-1}(0)$ to 0. It is clear that we have a triangle

$$\begin{tikzcd}
\mathbf{m} \arrow[r, "\pi"] \arrow[dr, "f"'] & \mathbf{k} \arrow[d, "f'"] \\
& \mathbf{n}
\end{tikzcd}$$

for some $f'\colon \mathbf{k} \to \mathbf{n}$ with $f'^{-1}(0) = \{0\}$. Next, we consider the sets $f'^{-1}(j)$ for $j > 0$. There is some permutation $\sigma\colon \mathbf{k} \to \mathbf{k}$ which reorders $\mathbf{k}$ as $f'^{-1}(1) \sqcup f'^{-1}(2) \sqcup \cdots \sqcup f'^{-1}(n)$. Then for $f_j = |f^{-1}(j)| = |f'^{-1}(j)|$ for $j > 0$, we claim that $(\phi_{f_1} \oplus \phi_{f_2} \oplus \cdots \oplus \phi_{f_n}) \circ \sigma = f'$. But this is automatic, since if $i \in f'^{-1}(j)$, then $f_1 + \cdots + f_{j-1} < \sigma(i) \leq f_1 + \cdots + f_{j-1} + f_j$, so the direct sum will map $\sigma(i)$ to j. Diagrammatically:

$$\begin{tikzcd}
\mathbf{m} \arrow[r, "\pi"] \arrow[dr, "f"'] & \mathbf{k} \arrow[r, "\sigma"] \arrow[d, "f'"] & \mathbf{k} \arrow[dl, "\phi_{f_1} \oplus \cdots \oplus \phi_{f_n}"] \\
& \mathbf{n} &
\end{tikzcd}$$

□

There is an analogous statement for morphisms in $\mathscr{E}$: namely, that all morphisms in $\mathscr{E}$ are composites of a permutation and a direct sum of canonical multiplications (in particular, there is no projection).

We now show the following proposition, which suggests why $\mathscr{F}^{sp}$-$\mathbf{PsAlg}^{\mathbb{R}}$ is the category we prefer to work in.

Proposition 2.20. *If $\mathcal{C} \in \mathscr{F}^{sp}$-$\boldsymbol{PsAlg}^{\mathbb{R}}$, then $\mathcal{C}$ commutes with finite direct sums, up to natural isomorphism.*

Proof. Write $\mathcal{C}(1) = \mathcal{D}$. Suppose we have two $\mathscr{F}$-morphisms $f\colon \mathbf{m} \to \mathbf{n}$ and $g\colon \mathbf{m}' \to \mathbf{n}'$. These correspond to functors $\mathcal{C}(f)\colon \mathcal{D}^m \to \mathcal{D}^n$ and $\mathcal{C}(g)\colon \mathcal{D}^{m'} \to \mathcal{D}^{n'}$. We would like to show that the functors $\mathcal{C}(f) \oplus \mathcal{C}(g)\colon \mathcal{D}^{m+m'} \to \mathcal{D}^{n+n'}$ and $\mathcal{C}(f \oplus g)\colon \mathcal{D}^{m+m'} \to \mathcal{D}^{n+n'}$ are naturally isomorphic. Since $\mathcal{C} \in \mathscr{F}^{sp}$-$\mathbf{PsAlg}^{\mathbb{R}}$, $\mathcal{C}$ maps projections in $\mathscr{F}$ to the analogous projections in $\mathbf{Cat}_*$. So for

$proj_n\colon \mathbf{n}{+}\mathbf{n}' \to \mathbf{n}$ the projection to the first n coordinates and $proj_{n'}\colon \mathbf{n}{+}\mathbf{n}' \to \mathbf{n}'$ the projection to the last n' coordinates, we have $\mathcal{C}(proj_n) = proj_n$ and $\mathcal{C}(proj_{n'}) = proj_{n'}$. So we can postcompose with these:

$$\begin{aligned} proj_n \circ (\mathcal{C}(f) \oplus \mathcal{C}(g)) &= \mathcal{C}(f) \circ proj_m \\ &= \mathcal{C}(f) \circ \mathcal{C}(proj_m) \\ &\simeq \mathcal{C}(f \circ proj_m) \end{aligned}$$

$$\begin{aligned} proj_n \circ \mathcal{C}(f \oplus g) &= \mathcal{C}(proj_n) \circ \mathcal{C}(f \oplus g) \\ &\simeq \mathcal{C}(proj_n \circ (f \oplus g)) \\ &= \mathcal{C}(f \circ proj_m) \end{aligned}$$

Similarly:

$$proj_{n'} \circ (\mathcal{C}(f) \oplus \mathcal{C}(g)) \simeq \mathcal{C}(g \circ proj_{m'}) \simeq proj_{n'} \circ \mathcal{C}(f \oplus g)$$

Since we have these natural isomorphisms, we can take their product to get a natural isomorphism

$$\mathcal{C}(f) \oplus \mathcal{C}(g) \simeq \mathcal{C}(f \oplus g)$$

□

3 The Functor $\mathbb{R}$

Definition 3.1. The functor $\mathbb{R}\colon \mathscr{P}\text{-}\mathbf{PsAlg} \to \mathscr{F}\text{-}\mathbf{PsAlg}$ is defined as follows. Let $(\mathcal{A}, \theta)$ be a $\mathscr{P}$-pseudoalgebra. We assign $\mathbb{R}\mathcal{A}(\mathbf{n}) = \mathcal{A}^n$. We can define a Π-algebra structure on this in the standard way, following the functor $\mathbb{R}\colon \mathbf{Cat} \to \Pi\text{-}\mathbf{Cat}$. Then, to the canonical multiplication maps $\phi_n\colon \mathbf{n} \to \mathbf{1}$, we associate the morphism

$$\mathcal{A}^n \xrightarrow{\sim} e_n \times \mathcal{A}^n \hookrightarrow \mathscr{P}(n) \times \mathcal{A}^n \xrightarrow{\theta} \mathcal{A}$$

Remark 3.2. The following must all be checked about $\mathbb{R}$:

- $\mathbb{R}$ extends to morphisms which aren't in Π or multiplication maps.
- $\mathbb{R}\mathcal{A}$ is a pseudofunctor $\mathscr{F} \to \mathbf{Cat}$.
- $\mathbb{R}$ is a functor.

Construction 3.3. We construct the maps for any $\phi \in \mathscr{F}(\mathbf{m}, \mathbf{n})$, by decomposing ϕ in terms of Π-morphisms and the canonical multiplications $\phi_n\colon \mathbf{n} \to \mathbf{1}$. Write $j_r = |\phi^{-1}(r)|$ for $1 \le r \le n$. Let $\sigma \in \Sigma_m$ be the permutation that gives $\mathbf{m}$ the ordering $0 \sqcup \phi^{-1}(1) \sqcup ... \sqcup \phi^{-1}(n)$, where the order of elements in each preimage is induced by the original order. Then we have the following factorization:

$$\begin{tikzcd} \mathbf{m} \arrow[r, "\sigma"] \arrow[dr, "\phi"'] & \mathbf{m} \arrow[d, "0\sqcup\phi^1_{j_1}\sqcup\cdots\sqcup\phi^n_{j_n}"] \\ & \mathbf{n} \end{tikzcd}$$

where ϕ^l_k is the composite

$$\mathbf{k} \xrightarrow{\phi_k} \mathbf{1} \xrightarrow{in_l} \mathbf{n}$$

which sends $\mathbf{k} \setminus \{0\}$ to $l \in \mathbf{n}$. Then we define $\mathbb{R}\mathcal{A}(\phi)$ to be

$$\begin{tikzcd} \mathcal{A}^m \arrow[r, "\sigma"] \arrow[dr, "\mathbb{R}\mathcal{A}(\phi)"'] & \mathcal{A}^m \arrow[d, "0\times\phi_{j_1}\times\cdots\times\phi_{j_n}"] \\ & \mathcal{A}^n \end{tikzcd}$$

We need to check that this definition is independent of choice of permutation σ. We do this in the following proposition.

Proposition 3.4. $\mathbb{R}\mathcal{A}(\phi)$ *does not depend on the choice of permutation, up to natural isomorphism.*

Proof. We need to show that the following diagram commutes, up to natural isomorphism:

$$\begin{tikzcd} \mathcal{A}^m \arrow[r, "\sigma"] \arrow[dr, "\phi_m"'] & \mathcal{A}^m \arrow[d, "\phi_m"] \\ & \mathcal{A} \end{tikzcd}$$

We can obtain such a natural isomorphism by considering the diagram

$$\begin{tikzcd} \mathscr{P}(m) \times \mathcal{A}^m \arrow[r, "\sigma\times 1"] \arrow[dr, "\theta"'] & \mathscr{P}(m) \times \mathcal{A}^m \arrow[d, "\theta"] \\ & \mathcal{A} \end{tikzcd}$$

This does not strictly commute, but the two composites are naturally isomorphic. For observe that $\sigma\colon \mathscr{P}(n) \to \mathscr{P}(n)$ is naturally isomorphic to the identity, by the obvious map $\eta_\tau\colon \tau \to \sigma\tau$, which fits into the following square:

$$\begin{tikzcd} \tau \arrow[r] \arrow[d, "\eta_\tau"'] & \tau' \arrow[d, "\eta_{\tau'}"] \\ \sigma\tau \arrow[r] & \sigma\tau' \end{tikzcd}$$

Then we can take the product of η with the identity on $\mathcal{A}^m$, and whisker with θ, which gives us the natural isomorphism $\theta \iff \theta \circ (\sigma \times 1)$. Finally, we consider the following diagram:

$$\begin{tikzcd} \mathscr{P}(m) \times \mathcal{A}^m \arrow[rr, "\sigma\times 1"] \arrow[ddr, "\theta"'] & & \mathscr{P}(m) \times \mathcal{A}^m \arrow[rr, "\sigma^{-1}\times\sigma"] & & \mathscr{P}(m) \times \mathcal{A}^m \arrow[ddlll, bend left, "\theta"] \\ & \mathcal{A}^m \arrow[ul, hook', "e_m\times 1"'] \arrow[rr, "\sigma"] \arrow[d, "\phi_m"'] \arrow[dr, phantom, "\Swarrow"] & & \mathcal{A}^m \arrow[ur, hook, "e_m\times 1"] \arrow[dll, "\phi_m"] & \\ & \mathcal{A} & & & \end{tikzcd}$$

The fact that the outer diagram commutes up to natural isomorphism is due to the natural isomorphism we just constructed, and the equivariance axiom. Then, commutativity up to natural isomorphism of the inner diagram follows by restriction of the natural isomorphism of the outer diagram, as required. $\square$

Proposition 3.5. $\mathbb{R}\mathcal{A}$ *is a pseudofunctor.*

Proof. We need to check that $\mathbb{R}\mathcal{A}(\psi\phi)$ is naturally isomorphic to $\mathbb{R}\mathcal{A}(\psi)\mathbb{R}\mathcal{A}(\phi)$, where $\phi\colon \mathbf{m} \to \mathbf{n}$ and $\psi\colon \mathbf{n} \to \mathbf{k}$. Decompose ϕ and ψ as the composition of a permutation and a coproduct of multiplication maps as follows:

$$\begin{tikzcd} \mathbf{m} \arrow[r, "\sigma"] \arrow[dr, "\phi"] & \mathbf{m} \arrow[d, "\bigsqcup \phi_{a_i}"] \\ & \mathbf{n} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{n} \arrow[r, "\tau"] \arrow[dr, "\psi"'] & \mathbf{n} \arrow[d, "\bigsqcup \phi_{b_i}"] \\ & \mathbf{k} \end{tikzcd}$$

For simplicity we may assume that the map $\mathbf{n} \to \mathbf{k}$ contains only one multiplication, and that $k = 1$ (for the general case, we simply take a product of the following diagrams). The composition of ϕ and ψ in $\mathscr{F}$ can be expressed similarly, where ρ is a block permutation of blocks of sizes $a_1, \dots, a_n$:

$$\begin{tikzcd} \mathbf{m} \arrow[r, "\sigma"] & \mathbf{m} \arrow[r, "\bigsqcup \phi_{a_i}"] \arrow[d, "\rho"] & \mathbf{n} \arrow[r, "\tau"] & \mathbf{n} \arrow[d, "\phi_n"] \\ & \mathbf{m} \arrow[rr, "\phi_m"'] & & \mathbf{1} \end{tikzcd}$$

On $\mathbb{R}\mathcal{A}$, these translate to

$$\begin{tikzcd}
\mathcal{A}^m \arrow[r, "\sigma"] \arrow[drrr, "\mathbb{R}\mathcal{A}(\psi)\mathbb{R}\mathcal{A}(\phi)"'] & \mathcal{A}^m \arrow[r, "\prod \phi_{a_i}"] & \mathcal{A}^n \arrow[r, "\tau"] & \mathcal{A}^n \arrow[d, "\phi_n"] \\
 & & & \mathcal{A}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{A}^m \arrow[r, "\sigma"] \arrow[drr, "\mathbb{R}\mathcal{A}(\psi\phi)"'] & \mathcal{A}^m \arrow[r, "\rho"] & \mathcal{A}^m \arrow[d, "\phi_m"] \\
 & & \mathcal{A}
\end{tikzcd}$$

Now, observing that we can pick ρ to be exactly the action of τ on the blocks of size $a_1, \dots, a_n$, we can write $\rho = \tau(e_{a_1} \oplus \cdots \oplus e_{a_n})$. Then we see that the above composites being equal up to natural isomorphism is a consequence of the following diagram:

$$\begin{tikzcd}
\mathscr{P}(n) \times \mathscr{P}(a_1) \times \cdots \times \mathscr{P}(a_n) \times \mathcal{A}^m \arrow[r, "\gamma \times 1"] \arrow[dddd, "\text{shuffle}"'] & \mathscr{P}(m) \times \mathcal{A}^m \arrow[rr, "\rho^{-1} \times \rho"] & & \mathscr{P}(m) \times \mathcal{A}^m \arrow[dd, "\theta"] \\
 & \mathcal{A}^m \arrow[ul, hook, "\tau \times e_{a_1} \times \cdots e_{a_n} \times 1"'] \arrow[r, "\rho"] \arrow[dd, "\prod \phi_{a_i}"] & \mathcal{A}^m \arrow[ur, hook, "e_m \times 1"] \arrow[dr, "\phi_m"] \arrow[dd, phantom, "\Downarrow"] & \\
 & & & \mathcal{A} \\
 & \mathcal{A}^n \arrow[r, "\tau"] & \mathcal{A}^n \arrow[ur, "\phi_n"'] \arrow[dr, hook, "e_n \times 1"'] & \\
\mathscr{P}(n) \times \mathscr{P}(a_1) \times \mathcal{A}^{a_1} \times \cdots \times \mathscr{P}(a_n) \times \mathcal{A}^{a_n} \arrow[r, "1 \times \theta^n"'] & \mathscr{P}(n) \times \mathcal{A}^n \arrow[rr, "\tau^{-1} \times \tau"'] & & \mathscr{P}(n) \times \mathcal{A}^n \arrow[uu, "\theta"']
\end{tikzcd}$$

Note that the $\tau^{-1} \times \tau$ maps may be interspersed, with no effect on the mutliplication θ, because of equivariance.

Finally, we check the higher coherence axiom. If we write $\phi_{f,g}$ for the natural isomorphism $\mathbb{R}\mathcal{A}(f)\mathbb{R}\mathcal{A}(g) \implies \mathbb{R}\mathcal{A}(fg)$, we need to show that the following diagram commutes:

$$\begin{tikzcd}
(\mathbb{R}\mathcal{A}h \circ \mathbb{R}\mathcal{A}g) \circ \mathbb{R}\mathcal{A}f \arrow[r, Rightarrow] \arrow[d, Rightarrow, "\phi_{h,g} * 1"'] & \mathbb{R}\mathcal{A}h \circ (\mathbb{R}\mathcal{A}g \circ \mathbb{R}\mathcal{A}f) \arrow[d, Rightarrow, "1 * \phi_{g,f}"] \\
\mathbb{R}\mathcal{A}(hg) \circ \mathbb{R}\mathcal{A}f \arrow[d, Rightarrow, "\phi_{hg,f}"'] & \mathbb{R}\mathcal{A}h \circ \mathbb{R}\mathcal{A}(gf) \arrow[d, Rightarrow, "\phi_{h,gf}"] \\
\mathbb{R}\mathcal{A}((hg)f) \arrow[r, Rightarrow] & \mathbb{R}\mathcal{A}(h(gf))
\end{tikzcd}$$

The left side of this diagram translates to

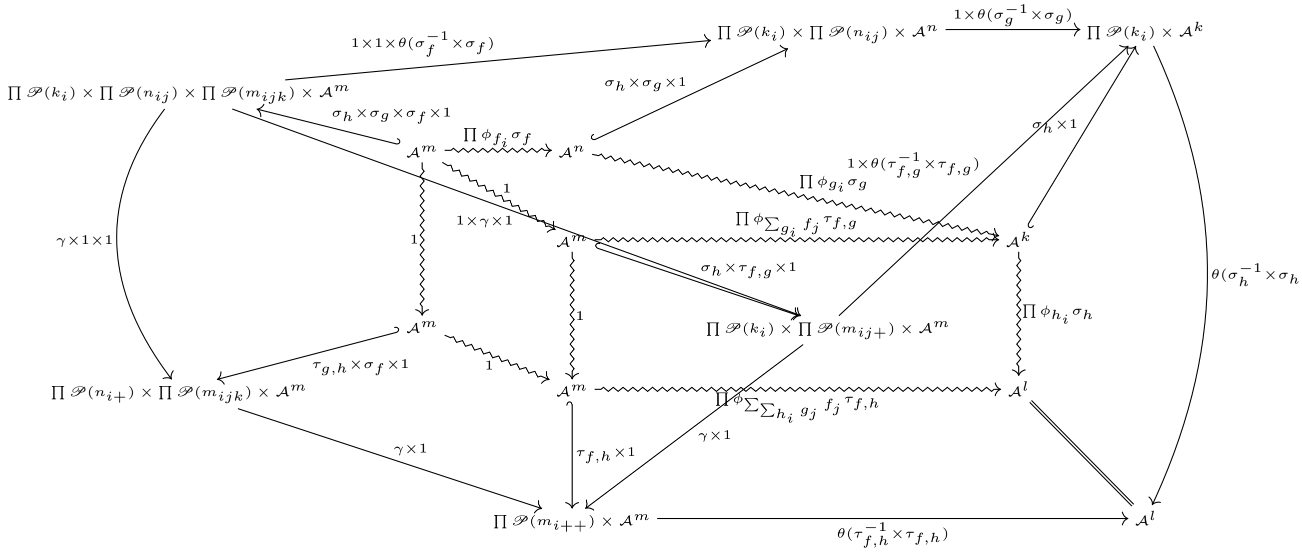
$$\begin{tikzcd}
\mathcal{A}^m \arrow[r, "\sigma_f"] \arrow[ddd, "1"'] & \mathcal{A}^m \arrow[r, "\prod \phi_{f_i}"] & \mathcal{A}^n \arrow[dr, "\sigma_g"] \arrow[ddd, "1"] & & \\
& & & \mathcal{A}^n \arrow[dr, "\prod \phi_{g_i}"] \arrow[dd, phantom, "\Downarrow \phi_{h,g}"] & \\
& & & & \mathcal{A}^k \arrow[d, "\sigma_h"] \\
\mathcal{A}^m \arrow[r, "\sigma_f"] \arrow[ddrr, "1"'] & \mathcal{A}^m \arrow[r, "\prod \phi_{f_i}"] \arrow[d, phantom, "\Leftarrow \phi_{hg,f}"] & \mathcal{A}^n \arrow[dr, "\tau_{g,h}"] & & \mathcal{A}^k \arrow[dd, "\prod \phi_{h_i}"] \\
& {} & & \mathcal{A}^n \arrow[dr, "\prod \phi_{\sum_{h_i} g_j}"] & \\
& & \mathcal{A}^m \arrow[r, "\tau_{f,h}"'] & \mathcal{A}^m \arrow[r, "\prod \phi_{\sum_{\sum_{h_i} g_j} f_j}"'] & \mathcal{A}^l
\end{tikzcd}$$

and the right side to

$$\begin{tikzcd}
\mathcal{A}^m \arrow[r, "\sigma_f"] \arrow[ddd, "1"'] \arrow[ddrr, "1"] & \mathcal{A}^m \arrow[r, "\prod \phi_{f_i}"] \arrow[d, phantom, "\Leftarrow \phi_{g,f}"] & \mathcal{A}^n \arrow[dr, "\sigma_g"] & & \\
& {} & & \mathcal{A}^n \arrow[dr, "\prod \phi_{g_i}"] & \\
& & \mathcal{A}^m \arrow[r, "\tau_{f,g}"] \arrow[dd, "1"] & \mathcal{A}^m \arrow[r, "\prod \phi_{\sum_{g_i} f_j}"] \arrow[d, phantom, "\Leftarrow \phi_{h,gf}"] & \mathcal{A}^k \arrow[d, "\sigma_h"] \\
\mathcal{A}^m \arrow[drr, "1"'] & & & {} & \mathcal{A}^k \arrow[d, "\prod \phi_{h_i}"] \\
& & \mathcal{A}^m \arrow[r, "\tau_{f,h}"'] & \mathcal{A}^m \arrow[r, "\prod \phi_{\sum_{\sum_{h_i} g_j} f_j}"'] & \mathcal{A}^l
\end{tikzcd}$$

Equality of their composition is equivalent to the diagram pasting into the appropriate cube. But these inject exactly into the appropriate pasting diagram of $\mathcal{A}$ as a $\mathscr{P}$-pseudoalgebra, giving us the desired coherence automatically (the original cube is denoted by the squiggle arrows):

$$\begin{tikzcd}[column sep=small]
\prod \mathscr{P}(k_i) \times \prod \mathscr{P}(n_{ij}) \times \prod \mathscr{P}(m_{ijk}) \times \mathcal{A}^m \arrow[rrrrr, "1\times 1\times\theta(\sigma_f^{-1}\times\sigma_f)"] \arrow[dd, "\gamma\times 1\times 1"'] & & & & & \prod \mathscr{P}(k_i) \times \prod \mathscr{P}(n_{ij}) \times \mathcal{A}^n \arrow[dd, "1\times\theta(\sigma_g^{-1}\times\sigma_g)"] \arrow[ddlll, "\gamma\times 1"] \\
& \mathcal{A}^m \arrow[ul, hook, "\sigma_h\times\sigma_g\times\sigma_f\times 1"'] \arrow[rr, squiggly, "\prod \phi_{f_i}\sigma_f"] \arrow[dd, squiggly, "1"] & & \mathcal{A}^n \arrow[urr, hook, "\sigma_h\times\sigma_g\times 1"] \arrow[dd, squiggly, "1"] \arrow[dr, squiggly, "\prod \phi_{g_i}\sigma_g"] & & \\
\prod \mathscr{P}(n_{i+}) \times \prod \mathscr{P}(m_{ijk}) \times \mathcal{A}^m \arrow[rr, "1\times\theta(\sigma_f^{-1}\times\sigma_f)"] \arrow[dddr, "\gamma\times 1"] & & \prod \mathscr{P}(n_{i+}) \times \mathcal{A}^n \arrow[dddrrr, "\theta(\tau_{g,h}^{-1}\times\tau_{g,h})"] & & \mathcal{A}^k \arrow[r, hook, "\sigma_h\times 1"] \arrow[dd, squiggly, "\prod \phi_{h_i}\sigma_h"] & \prod \mathscr{P}(k_i) \times \mathcal{A}^k \arrow[ddd, "\theta(\sigma_h^{-1}\times\sigma_h)"] \\
& \mathcal{A}^m \arrow[ul, hook, "\tau_{g,h}\times\sigma_f\times 1"] \arrow[rr, squiggly, "\prod \phi_{f_i}\sigma_f"] \arrow[dr, squiggly, "1"] & & \mathcal{A}^n \arrow[ul, hook, "\tau_{g,h}\times 1"'] \arrow[dr, squiggly, "\prod \phi_{\sum_{h_i} g_j}\tau_{g,h}"] & & \\
& & \mathcal{A}^m \arrow[dl, hook, "\tau_{f,h}\times 1"] \arrow[rr, squiggly, "\prod \phi_{\sum_{\sum_{h_i} g_j} f_j}\tau_{f,h}"'] & & \mathcal{A}^l \arrow[dr, equal] & \\
& \prod \mathscr{P}(m_{i++}) \times \mathcal{A}^m \arrow[rrrr, "\theta(\tau_{f,h}^{-1}\times\tau_{f,h})"'] & & & & \mathcal{A}^l
\end{tikzcd}$$



Note also that we do not need to check the unity coherence axioms, since $\mathbb{R}\mathcal{A}$ is a strict Π-functor. $\square$

Proposition 3.6. $\mathbb{R}$ *is a functor.*

Proof. We must define the effect of $\mathbb{R}$ on a pseudomorphism of $\mathscr{P}$-pseudoalegebras. Recall that a pseudomorphism $\mathcal{A} \to \mathcal{B}$ consists of a functor $F\colon \mathcal{A} \to \mathcal{B}$, such that the following diagram commutes up to coherent natural isomorphism:

$$\begin{tikzcd}
\mathscr{P}(n) \times \mathcal{A}^n \arrow[r, "\theta"] \arrow[d, "1\times F^n"'] & \mathcal{A} \arrow[d, "F"] \arrow[dl, phantom, "\Downarrow"] \\
\mathscr{P}(n) \times \mathcal{B} \arrow[r, "\psi"] & \mathcal{B}
\end{tikzcd}$$

Using this data, we need to define a pseudonatural transformation $(\zeta_n)_{n\in\mathbb{N}}$ from $\mathbb{R}\mathcal{A}$ to $\mathbb{R}\mathcal{B}$. The answer is (F^n), where $F^n\colon \mathbb{R}\mathcal{A}(n) = \mathcal{A}^n \to \mathcal{B}^n = \mathbb{R}\mathcal{B}(n)$. Then we need to check that the following diagram commutes, up to natural isomorphism, for any $\phi \in \mathscr{F}(\mathbf{m}, \mathbf{n})$:

$$\begin{tikzcd}
\mathcal{A}^m \arrow[r, "\mathbb{R}\mathcal{A}(\phi)"] \arrow[d, "F^m"'] & \mathcal{A}^n \arrow[d, "F^n"] \arrow[dl, phantom, "\Downarrow"] \\
\mathcal{B}^m \arrow[r, "\mathbb{R}\mathcal{B}(\phi)"'] & \mathcal{B}^n
\end{tikzcd}$$

Since ϕ can be broken down into elements of Π and multiplication maps, it suffices to check the above diagram commutes for those maps. If $\phi \in \Pi(\mathbf{m}, \mathbf{n})$, then our desired square commutes strictly (since $\mathbb{R}$ starts as a functor $\mathscr{P}$**-PsAlg** $\to \Pi^{sp}$**-Alg**. And if $\phi = \phi_m\colon \mathbf{m} \to \mathbf{1}$, this follows from

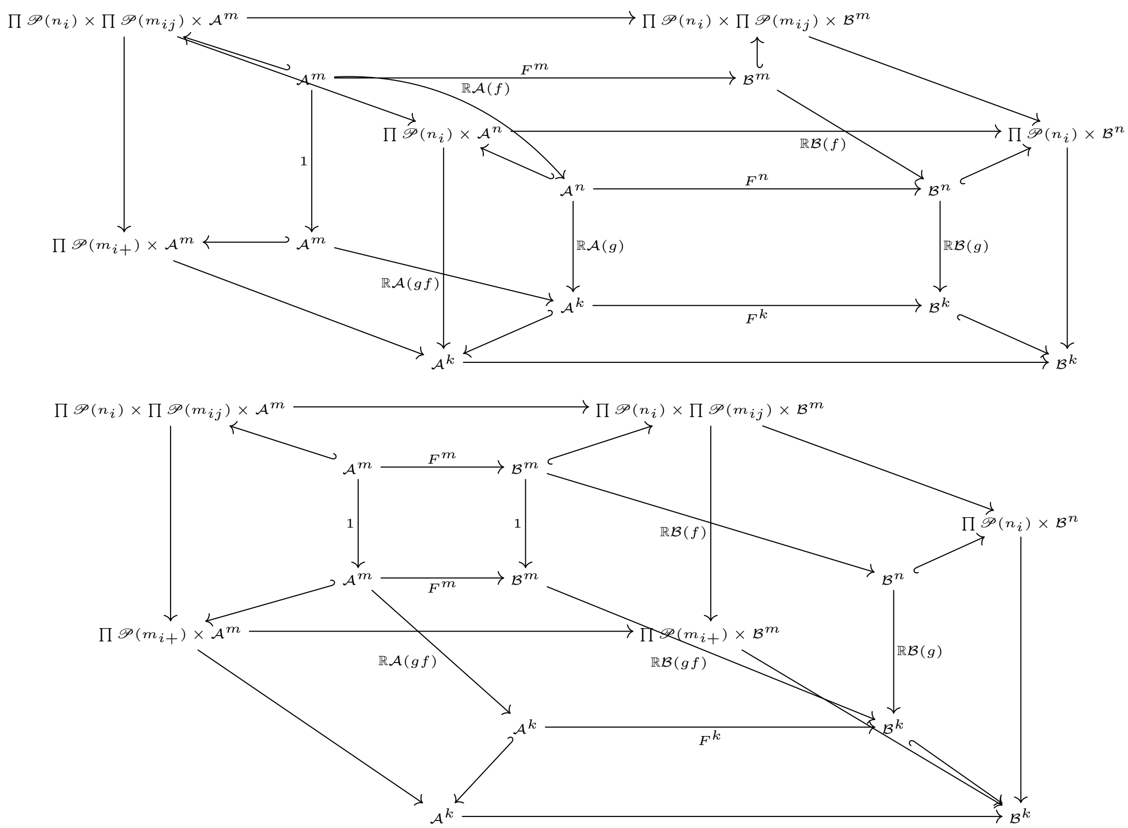
$$\begin{tikzcd}
\mathcal{A}^m \arrow[r, "e_m\times 1"'] \arrow[rr, bend left, "\phi_m"] \arrow[d, "F^m"'] & \mathscr{P}(m) \times \mathcal{A}^m \arrow[r, "\theta"'] \arrow[d, "1\times F^m"'] & \mathcal{A} \arrow[d, "F"] \arrow[dl, phantom, "\Downarrow"] \\
\mathcal{B}^m \arrow[r, "e_m\times 1"] \arrow[rr, bend right, "\phi_m"'] & \mathscr{P}(m) \times \mathcal{B}^m \arrow[r, "\psi"] & \mathcal{B}
\end{tikzcd}$$

where the left square commutes strictly.

We additionally need to check that the appropriate coherence conditions are satisfied for a pseudonatural transformation. If we label the natural isomorphisms for $\mathbb{R}\mathcal{A}$ as α, and those for $\mathbb{R}\mathcal{B}$ as β, and for $f \in \mathscr{F}(\mathbf{m}, \mathbf{n})$, we label our natural isomorphism $F^n\mathbb{R}\mathcal{A}(f) \implies \mathbb{R}\mathcal{B}(f)F^m$ as ζ_f, then we need to show the following equality of pasting diagrams:

$$
\begin{tikzcd}
\mathcal{A}^m \arrow[rr, "\mathbb{R}\mathcal{A}(gf)"] \arrow[d, "F^m"'] \arrow[drr, phantom, "\zeta_{gf}\Downarrow"] & & \mathcal{A}^k \arrow[d, "F^k"] \\
\mathcal{B}^m \arrow[r, "\mathbb{R}\mathcal{B}(f)"'] \arrow[rr, bend left, "\mathbb{R}\mathcal{B}(gf)"] & \mathcal{B}^n \arrow[r, "\mathbb{R}\mathcal{B}(g)"'] & \mathcal{B}^k
\end{tikzcd}
\quad \psi\Uparrow \quad = \quad
\begin{tikzcd}
\mathcal{A}^m \arrow[r, "\mathbb{R}\mathcal{A}(f)"] \arrow[rr, bend left, "\mathbb{R}\mathcal{A}(gf)"] \arrow[d, "F^m"'] \arrow[dr, phantom, "\overset{\Longrightarrow}{\zeta_f}"] & \mathcal{A}^n \arrow[r, "\mathbb{R}\mathcal{A}(g)"] \arrow[d, "F^n"] \arrow[dr, phantom, "\overset{\Longrightarrow}{\zeta_g}"] & \mathcal{A}^k \arrow[d, "F^k"] \\
\mathcal{B}^m \arrow[r, "\mathbb{R}\mathcal{B}(f)"'] & \mathcal{B}^n \arrow[r, "\mathbb{R}\mathcal{B}(g)"'] & \mathcal{B}^k
\end{tikzcd}
\quad (\Uparrow\phi)
$$

But this equality fits neatly inside the equivalent pasting diagram for F as a pseudomorphism of $\mathscr{P}$-pseudoalgebras, as illustrated below:



□

Remark 3.7. In fact, because we defined $\mathbb{R}\mathcal{A}$ as exactly the Π-algebra $\mathbb{R}\mathcal{A}$ (along with extra $\mathscr{F}$-structure), we see that the functor $\mathbb{R}\colon \mathscr{P}\textbf{-PsAlg} \to \mathscr{F}^{sp}\textbf{-PsAlg}$ actually lands in $\mathscr{F}^{sp}\textbf{-PsAlg}^{\mathbb{R}}$. In particular, we will show that this functor is an isomorphism.

4 The Inverse of $\mathbb{R}$

Definition 4.1. The functor $\mathbb{Q}\colon \mathscr{F}^{sp}\textbf{-PsAlg}^{\mathbb{R}} \to \mathscr{P}\textbf{-PsAlg}$ is defined as follows. To a special pseudofunctor $\mathcal{B}\colon \mathscr{F} \to \textbf{Cat}$, we define $\mathbb{Q}\mathcal{B} = \mathcal{B}(\mathbf{1})$, which we abbreviate as $\mathcal{B}$. The structure maps θ are given by the composite

$$
\begin{tikzcd}
\mathscr{P}(n) \times \mathcal{B}^n \arrow[r, "\sim"] \arrow[rrr, bend right=10, "\theta"'] & \mathscr{E}(\mathbf{n},\mathbf{1}) \times \Lambda(\mathbf{n},\mathbf{n}) \times \mathcal{B}^n \arrow[r, "1\times\circ"] & \mathscr{E}(\mathbf{n},\mathbf{1}) \times \mathcal{B}^n \arrow[r, "\circ"] & \mathcal{B}
\end{tikzcd}
$$

The first map here is given by the identification $\mathscr{P}(n) = \mathcal{E}\Sigma_n \simeq * \times \mathcal{E}\Sigma_n = \mathscr{E}(\mathbf{n},\mathbf{1}) \times \Lambda(\mathbf{n},\mathbf{n})$, which is an isomorphism of categories.

In particular, this comes from the following (set-level) identification

$$\Sigma_n \cong \{\phi_n\} \times \Sigma_n = \mathscr{E}(\mathbf{n},\mathbf{1}) \times \Lambda(\mathbf{n},\mathbf{n}) \subset \mathscr{F}(\mathbf{n},\mathbf{1}) \times \mathscr{F}(\mathbf{n},\mathbf{n})$$

and then from applying the indiscrete category functor $\mathcal{E}$. Since $\mathcal{E}$ is a right adjoint, it commutes with products, which gives us an appropriately functorial definition of θ_n.

For simplicity, we will write $\mathscr{E}(n)$ for $\mathscr{E}(\mathbf{n},\mathbf{1})$, and $\Lambda(n)$ for $\Lambda(\mathbf{n},\mathbf{n})$.

Remark 4.2. The following must all be checked about $\mathbb{Q}$:

- $\mathbb{Q}\mathcal{B}$ is a pseudoalgebra.
- $\mathbb{Q}$ is a functor.

Proposition 4.3. *For an $\mathscr{F}$-pseudoalgebra $\mathcal{B}$, θ is naturally isomorphic to the n-fold multiplication map*

$$\mathscr{E}(n) \times \Lambda(n) \times \mathcal{B}^n \xrightarrow{proj} \mathscr{E}(n) \times \mathcal{B}^n \xrightarrow{\circ} \mathcal{B}$$

Proof. Because $\mathcal{B}$ is a pseudofunctor, we have natural transformations

$$\begin{tikzcd}
\mathscr{F}(\mathbf{n},\mathbf{k}) \times \mathscr{F}(\mathbf{m},\mathbf{n}) \times \mathcal{B}^m \arrow[r, "1\times\circ"] \arrow[d, "\circ\times 1"'] \arrow[dr, phantom, "\Downarrow\phi"] & \mathscr{F}(\mathbf{n},\mathbf{k}) \times \mathcal{B}^n \arrow[d, "\circ"] \\
\mathscr{F}(\mathbf{m},\mathbf{k}) \times \mathcal{B}^m \arrow[r, "\circ"'] & \mathcal{B}
\end{tikzcd}$$

Then for $m = n$ and $k = 1$, we can restrict this natural isomorphism to that of functors $\mathscr{E}(\mathbf{n},\mathbf{1}) \times \Lambda(\mathbf{n},\mathbf{n}) \times \mathcal{B}^n \to \mathcal{B}$. Since $\mathscr{E}(n) = *$, composition $\mathscr{E}(n) \times \Lambda(n) \to \mathscr{E}(n)$ is the same as projection to the $\mathscr{E}(n)$-coordinate, as required. □

Proposition 4.4. *$\mathbb{Q}\mathcal{B}$ is a pseudoalgebra.*

Proof. We need to verify the axioms of a pseudoalgebra. First, the unitality diagram:

$$\begin{tikzcd}
* \times \mathcal{B} \arrow[r, "\sim"] \arrow[d, "\mathrm{id}\times 1"'] & \mathcal{B} \\
\mathscr{P}(1) \times \mathcal{B} \arrow[ur, "\theta"'] &
\end{tikzcd}$$

$\mathscr{P}(1) = *$, so id $\times 1$ is exactly the identity. Moreover, θ here is given by

$$\begin{tikzcd}
\mathscr{P}(1) \times \mathcal{B} \arrow[r, "\sim"] \arrow[rrr, bend right, "\theta"'] & \mathscr{E}(1) \times \Lambda(1) \times \mathcal{B} \arrow[r, "1\times\circ"] & \mathscr{E}(1) \times \mathcal{B} \arrow[r, "\circ"] & \mathcal{B}
\end{tikzcd}$$

but since $\mathscr{E}(\mathbf{1},\mathbf{1}) \times \Lambda(\mathbf{1},\mathbf{1})$ is also just a point, this composite is exactly the same as the horizontal map $* \times \mathcal{B} \to \mathcal{B}$.

Next, the equivariance diagram we need is

$$\begin{tikzcd}
\mathscr{P}(n) \times \mathcal{B}^n \arrow[rr, "\sigma\times\sigma^{-1}"] \arrow[dr, "\theta"'] & & \mathscr{P}(n) \times \mathcal{B}^n \arrow[dl, "\theta"] \\
& \mathcal{B} &
\end{tikzcd}$$

Again, this is a direct translation of the following diagram:

$$\begin{tikzcd}
\mathscr{E}(n) \times \Lambda(n) \times \mathcal{B}^n \arrow[rr, "\sigma\times\sigma^{-1}"] \arrow[d, "1\times\circ"'] & & \mathscr{E}(n) \times \Lambda(n) \times \mathcal{B}^n \arrow[d, "1\times\circ"] \\
\mathscr{E}(n) \times \mathcal{B}^n \arrow[dr, "\circ"'] & & \mathscr{E}(n) \times \mathcal{B}^n \arrow[dl, "\circ"] \\
& \mathcal{B} &
\end{tikzcd}$$

This final pentagon commutes because of the triangle

$$
\begin{tikzcd}
\Lambda(n) \times \mathcal{B}^n \arrow[r, "\sigma \times \sigma^{-1}"] \arrow[dr, "\circ"'] & \Lambda(n) \times \mathcal{B}^n \arrow[d, "1 \times \circ"] \\
& \mathcal{B}^n
\end{tikzcd}
$$

Intuitively, the effect of τ on $\mathcal{B}^n$ is identical to the effect of $\tau\sigma$ on $\sigma^{-1}\mathcal{B}$.

Finally, for the associativity diagram, we require the following to commute up to coherent natural isomorphism:

$$
\begin{tikzcd}
\mathscr{P}(k) \times \mathscr{P}(j_1) \times \cdots \times \mathscr{P}(j_k) \times \mathcal{B}^{j_+} \arrow[r, "\gamma \times 1"'] \arrow[dd, "\text{shuffle}"] \arrow[ddr, phantom, "\Downarrow \phi"] & \mathscr{P}(j_+) \times \mathcal{B}^{j_+} \arrow[d, "\gamma"'] \\
& \mathcal{B} \\
\mathscr{P}(k) \times \mathscr{P}(j_1) \times \mathcal{B}^{j_1} \times \cdots \times \mathscr{P}(j_k) \times \mathcal{B}^{j_k} \arrow[r, "1 \times \gamma^k"] & \mathscr{P}(k) \times \mathcal{B}^k \arrow[u, "\gamma"]
\end{tikzcd}
$$

Now, $\gamma\colon \mathscr{P}(k) \times \mathscr{P}(j_1) \times \cdots \times \mathscr{P}(j_k) \to \mathscr{P}(j_+)$ comes from the direct sum of $\mathscr{F}$-morphisms:

$$\mathscr{E}(k) \times \Lambda(k) \times \prod(\mathscr{E}(j_i) \times \Lambda(j_i)) \xrightarrow{shuffle} \mathscr{E}(k) \times \prod \mathscr{E}(j_i) \times \Lambda(k) \times \prod \Lambda(j_i) \xrightarrow{\oplus \times \oplus(\rho)} \mathscr{E}(j_+) \times \Lambda(j_+)$$

where the map $\oplus(\rho)$ is induced by maps

$$\sigma \times \prod \Lambda(j_i) \xrightarrow{\sigma} \prod \Lambda(j_{\sigma(i)}) \xrightarrow{\oplus} \Lambda(j_+)$$

for each $\sigma \in \Lambda(k)$. Looking just at $\mathscr{E}$-morphisms, we have a square

$$
\begin{tikzcd}
\mathscr{E}(k) \times \prod \mathscr{E}(j_i) \times \mathcal{B}^{j_+} \arrow[r, "1 \times \circ"] \arrow[d, "\oplus \times 1"'] \arrow[dr, phantom, "\Downarrow \phi_\oplus"] & \mathscr{E}(k) \times \mathcal{B}^k \arrow[d, "\circ"] \\
\mathscr{E}(j_+) \times \mathcal{B}^{j_+} \arrow[r, "\circ"'] & \mathcal{B}
\end{tikzcd}
$$

which commutes up to natural isomorphism in light of Proposition 2.20 and the fact that the map labeled $\circ\colon \prod \mathscr{E}(j_i) \times \mathcal{B}^{j_+} \to \mathcal{B}^k$ is really the adjoint of the following composition:

$$\prod \mathscr{E}(j_i) \longrightarrow \prod \mathrm{Hom}(\mathcal{B}^{j_i}, \mathcal{B}) \xrightarrow{\oplus} \mathrm{Hom}(\mathcal{B}^{j_+}, \mathcal{B}^k)$$

We make use of the natural isomorphism in Proposition 4.3, which we label $\phi_n\colon \theta_n \implies \circ(proj)$, where $\theta_n\colon \mathscr{E}(n) \times \Lambda(n) \times \mathcal{B}^n \to \mathcal{B}$. Then we have the following extensive diagram:

$$
\begin{tikzcd}
\mathscr{E}(k) \times \Lambda(k) \times \prod(\mathscr{E}(j_i) \times \Lambda(j_i)) \times \mathcal{B}^{j_+} \arrow[r, "1 \times 1 \times \prod(1 \times \circ)"] \arrow[dr, "1 \times 1 \times \prod proj"'] \arrow[dddd, "\oplus \times \oplus(\rho) \times 1"'] & \mathscr{E}(k) \times \Lambda(k) \times \prod(\mathscr{E}(j_i) \times \mathcal{B}^{j_i}) \arrow[rr, "1 \times 1 \times \prod \circ"] \arrow[d, Rightarrow, "1 * 1 * \prod \phi_{j_i}"] & & \mathscr{E}(k) \times \Lambda(k) \times \mathcal{B}^k \arrow[ddl, "proj"] \arrow[dd, "1 \times \circ"] \\
& \mathscr{E}(k) \times \Lambda(k) \times \prod(\mathscr{E}(j_i) \times \mathcal{B}^{j_i}) \arrow[urr, "1 \times 1 \times \prod \circ"'] \arrow[d, "proj \times \prod 1"'] & & \\
& \mathscr{E}(k) \times \prod(\mathscr{E}(j_i) \times \mathcal{B}^{j_i}) \arrow[r, "1 \times \prod \circ"'] \arrow[d, "\oplus \times 1"'] & \mathscr{E}(k) \times \mathcal{B}^k \arrow[dl, Rightarrow, "\phi_\oplus"] \arrow[ddr, "\circ"] & \mathscr{E}(k) \times \mathcal{B}^k \arrow[l, Rightarrow, "\phi_k"] \arrow[dd, "\circ"] \\
& \mathscr{E}(j_+) \times \mathcal{B}^{j_+} \arrow[drr, "\circ"] & & \\
\mathscr{E}(j_+) \times \Lambda(j_+) \times \mathcal{B}^{j_+} \arrow[ur, "proj"] \arrow[r, "1 \times \circ"'] & \mathscr{E}(j_+) \times \mathcal{B}^{j_+} \arrow[u, Rightarrow, "\phi_{j_+}"'] \arrow[rr, "\circ"'] & & \mathcal{B}
\end{tikzcd}
$$

This composite provides us with the natural isomorphism we need in the associativity diagram: $\phi_{j_+}^{-1} \circ \phi_{\oplus} \circ (\phi_k * \prod \phi_{j_i})$. We need to check the operadic associativity coherence diagram, which is equality of the two pasting diagrams that follow (in which ϕ denotes the associativity natural isomorphism):

$$\begin{tikzcd}
\mathscr{P}(n) \times \prod \mathscr{P}(m_i) \times \prod \mathscr{P}(k_{ij}) \times \mathcal{B}^k \arrow[r, "1\times1\times\theta"] \arrow[dd, "\gamma\times1\times1"'] & \mathscr{P}(n) \times \prod \mathscr{P}(m_i) \times \mathcal{B}^m \arrow[dr, "1\times\theta"] \arrow[dd, "\gamma\times1"'] \arrow[ddr, phantom, "\Downarrow\phi"] & \\
 & & \mathscr{P}(n) \times \mathcal{B}^n \arrow[dd, "\theta"] \\
\mathscr{P}(m_+) \times \prod \mathscr{P}(k_{ij}) \times \mathcal{B}^k \arrow[r, "1\times\theta"'] \arrow[dr, "\gamma\times1"'] & \mathscr{P}(m_+) \times \mathcal{B}^m \arrow[d, phantom, "\Downarrow\phi"] \arrow[dr, "\theta"] & \\
 & \mathscr{P}(k_{++}) \times \mathcal{B}^k \arrow[r, "\theta"'] & \mathcal{B}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathscr{P}(n) \times \prod \mathscr{P}(m_i) \times \prod \mathscr{P}(k_{ij}) \times \mathcal{B}^k \arrow[r, "1\times1\times\theta"] \arrow[dr, "1\times\gamma\times1"] \arrow[dd, "\gamma\times1\times1"'] & \mathscr{P}(n) \times \prod \mathscr{P}(m_i) \times \mathcal{B}^m \arrow[d, phantom, "\Downarrow\mathrm{id}*\phi"] \arrow[dr, "1\times\theta"] & \\
 & \mathscr{P}(n) \times \prod \mathscr{P}(k_{i+}) \times \mathcal{B}^k \arrow[r, "1\times\theta"] \arrow[dd, "\gamma\times1"'] \arrow[ddr, phantom, "\Downarrow\phi"] & \mathscr{P}(n) \times \mathcal{B}^n \arrow[dd, "\theta"] \\
\mathscr{P}(m_+) \times \prod \mathscr{P}(k_{ij}) \times \mathcal{B}^k \arrow[dr, "\gamma\times1"'] & & \\
 & \mathscr{P}(k_{++}) \times \mathcal{B}^k \arrow[r, "\theta"'] & \mathcal{B}
\end{tikzcd}$$

Observe that each of the ϕ's in the above pasting diagrams is a natural isomorphism of the following form, or at least is a product of such natural isomorphisms.

$$\begin{tikzcd}
\mathscr{P}(n) \times \prod \mathscr{P}(m_i) \times \mathcal{B}^m \arrow[rrr] \arrow[dr, "proj"'] \arrow[ddd] & & & \mathscr{P}(n) \times \mathcal{B}^n \arrow[dl, "proj"] \arrow[ddd] \\
 & \mathscr{E}(n) \times \prod \mathscr{E}(m_i) \times \mathcal{B}^m \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow\phi_\oplus"] & \mathscr{E}(n) \times \mathcal{B}^n \arrow[u, phantom, "\Downarrow"] \arrow[d] & \\
 & \mathscr{E}(m) \times \mathcal{B}^m \arrow[r] & \mathcal{B} \arrow[ur, phantom, "\Leftarrow"] \arrow[d, phantom, "\Uparrow"] \arrow[dr, equal] & \\
\mathscr{P}(m_+) \times \mathcal{B}^m \arrow[ur, "proj"'] \arrow[rrr] & & & \mathcal{B}
\end{tikzcd}$$

In particular, adjacent faces begin with the same natural isomorphism. E.g., along the arrow $\theta\colon \mathscr{P}(m_+)\times \mathcal{A}^m \to \mathcal{A}$, both adjacent faces begin with the natural isomorphism $\phi_{m_+}\colon \theta \implies \circ(proj)$, and along the edge $\gamma \times 1\colon \mathscr{P}(n) \times \prod \mathscr{P}(m_i) \times \mathcal{A}^m$, both adjacent faces begin with the identity natural isomorphism. What this lets us do is paste these inset squares together, to form a smaller central cube in the pasting diagram. Since both cubes finish at $\mathcal{B}$, each edge composite in the pasting diagram factors through our smaller cube. So it will suffice to show equality of the direct sum pasting diagrams:

$$\begin{tikzcd}
\mathscr{E}(n)\times\prod\mathscr{E}(m_i)\times\prod\mathscr{E}(k_{ij})\times\mathcal{B}^k \arrow[rr, "1\times1\times\circ"] \arrow[dd, "\oplus\times1\times1"'] & & \mathscr{E}(n)\times\prod\mathscr{E}(m_i)\times\mathcal{B}^m \arrow[dr, "1\times\circ"] \arrow[dd, "\oplus\times1"'] & \\
 & & & \mathscr{E}(n)\times\mathcal{B}^n \arrow[dd, "\circ"] \\
\mathscr{E}(m_+)\times\prod\mathscr{E}(k_{ij})\times\mathcal{B}^k \arrow[rr, "1\times\circ"'] \arrow[drr, "\oplus\times1"'] & & \mathscr{E}(m_+)\times\mathcal{B}^m \arrow[d, phantom, "\Downarrow\phi_{\oplus}"] \arrow[dr, "\circ"] \arrow[ur, phantom, "\Downarrow\phi_{\oplus}"] & \\
 & & \mathscr{E}(k_{++})\times\mathcal{B}^k \arrow[r, "\circ"'] & \mathcal{B}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathscr{E}(n)\times\prod\mathscr{E}(m_i)\times\prod\mathscr{E}(k_{ij})\times\mathcal{B}^k \arrow[rr, "1\times1\times\circ"] \arrow[dd, "\oplus\times1\times1"'] \arrow[drr, "1\times\oplus\times1"'] & & \mathscr{E}(n)\times\prod\mathscr{E}(m_i)\times\mathcal{B}^m \arrow[dr, "1\times\circ"] \arrow[d, phantom, "1*\phi_{\oplus}\Downarrow"] & \\
 & & \mathscr{E}(n)\times\prod\mathscr{E}(k_{i+})\times\mathcal{B}^k \arrow[r, "1\times\circ"'] \arrow[dd, "\oplus\times1"'] \arrow[ddr, phantom, "\phi_{\oplus}\Downarrow"] & \mathscr{E}(n)\times\mathcal{B}^n \arrow[dd, "\circ"] \\
\mathscr{E}(m_+)\times\prod\mathscr{E}(k_{ij})\times\mathcal{B}^k \arrow[drr, "\oplus\times1"'] & & & \\
 & & \mathscr{E}(k_{++})\times\mathcal{B}^k \arrow[r, "\circ"'] & \mathcal{B}
\end{tikzcd}$$

The routine verification of this fact is left for the reader. □

Proposition 4.5. $\mathbb{Q}$ *is a functor.*

Proof. A pseudomorphism of $\mathscr{F}$-pseudoalgebras $\mathcal{A},\mathcal{B}$ is given by a pseudonatural transformation: in other words, for each n, a morphism $\eta_n\colon \mathcal{A}^n\cong\mathcal{A}(\mathbf{n})\to\mathcal{B}(\mathbf{n})\cong\mathcal{B}^n$ so that the following commutes, up to natural isomorphism, for any $\phi\in\mathscr{F}(\mathbf{n},\mathbf{m})$:

$$\begin{tikzcd}
\mathcal{A}^n \arrow[r, "\mathcal{A}(\phi)"] \arrow[d, "\eta_n"'] \arrow[dr, phantom, "\Downarrow"] & \mathcal{A}^m \arrow[d, "\eta_m"] \\
\mathcal{B}^n \arrow[r, "\mathcal{B}(\phi)"] & \mathcal{B}^m
\end{tikzcd}$$

Observe, in particular, that η_n has to be identical to η_1^n, so we will write the two interchangeably. This follows from considering the maps $in_i\colon \mathbf{1}\to\mathbf{n}$, given by sending 1 to i, and the diagram that η must respect:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "in_i"] \arrow[d, "\eta_1"'] & \mathcal{A}^n \arrow[d, "\eta_n"] \\
\mathcal{B} \arrow[r, "in_i"'] & \mathcal{B}^n
\end{tikzcd}$$

Since $in_i\in\Pi(\mathbf{1},\mathbf{n})$, this diagram commutes strictly. Then, because $\prod_i in_i=\mathrm{id}\colon\mathcal{A}^n\to\mathcal{A}^n$, we have

$$\begin{tikzcd}
\mathcal{A}^n \arrow[r, "\prod_i in_i", "\sim"'] \arrow[d, "\eta_1^n"'] & \mathcal{A}^n \arrow[d, "\eta_n"'] \\
\mathcal{B}^n \arrow[r, "\prod_i in_i", "\sim"'] & \mathcal{B}^n
\end{tikzcd}$$

Given such an η, we define $\mathbb{Q}\eta\colon\mathbb{Q}\mathcal{A}\to\mathbb{Q}\mathcal{B}$ by $\eta_1\colon\mathcal{A}\to\mathcal{B}$. We need to check that this induces a map of pseudoalgebras. In particular, we need the square

$$\begin{tikzcd}
\mathscr{P}(n)\times\mathcal{A}^n \arrow[r, "\theta"] \arrow[d, "1\times\eta_1^n"] \arrow[dr, phantom, "\Downarrow"] & \mathcal{A} \arrow[d, "\eta_1"] \\
\mathscr{P}(n)\times\mathcal{B}^n \arrow[r, "\theta"] & \mathcal{B}
\end{tikzcd}$$

This is a direct consequence of the following diagram:

$$\begin{tikzcd}
\mathscr{E}(\mathbf{n},\mathbf{1}) \times \mathscr{E}(\mathbf{n},\mathbf{n}) \times \mathcal{A}^n \arrow[r, "1\times\circ"] \arrow[d, "1\times 1\times\eta_n"] & \mathscr{E}(\mathbf{n},\mathbf{1}) \times \mathcal{A}^n \arrow[r, "\circ"] \arrow[d, "1\times\eta_n"] \arrow[dr, phantom, "\Downarrow"] & \mathcal{A} \arrow[d, "\eta_1"] \\
\mathscr{E}(\mathbf{n},\mathbf{1}) \times \mathscr{E}(\mathbf{n},\mathbf{n}) \times \mathcal{B}^n \arrow[r, "1\times\circ"] & \mathscr{E}(\mathbf{n},\mathbf{1}) \times \mathcal{B}^n \arrow[r, "\circ"] & \mathcal{B}
\end{tikzcd}$$

Now that we have a consistent definition of $\mathbb{Q}\eta$ for pseudonatural transformations η, we can quickly check functoriality of $\mathbb{Q}$. If we have two pseudomorphisms $\eta\colon \mathcal{A} \to \mathcal{B}$ and $\zeta\colon \mathcal{B} \to \mathcal{C}$, then the composite $(\mathbb{Q}\zeta)(\mathbb{Q}\eta)\colon \mathcal{A} \to \mathcal{B}$ is exactly $\zeta_1\eta_1 = (\zeta\eta)_1$. Moreover, the following diagram commutes, so we have a legitimate pseudomorphism of $\mathscr{P}$-pseudoalgebras:

$$\begin{tikzcd}
\mathscr{E}(\mathbf{n},\mathbf{1}) \times \mathscr{E}(\mathbf{n},\mathbf{n}) \times \mathcal{A}^n \arrow[rr, "1\times\circ"] \arrow[dr, "1\times 1\times\eta_n"] \arrow[dd, "1\times 1\times(\zeta\eta)_n"] & & \mathscr{E}(\mathbf{n},\mathbf{1}) \times \mathcal{A}^n \arrow[r, "\circ"] \arrow[d, "1\times\eta_n"] \arrow[dr, phantom, "\Downarrow"] & \mathcal{A} \arrow[d, "\eta_1"] \arrow[dd, bend left, "(\zeta\eta)_1"] \\
 & \mathscr{E}(\mathbf{n},\mathbf{1}) \times \mathscr{E}(\mathbf{n},\mathbf{n}) \times \mathcal{B}^n \arrow[r, "1\times\circ"] \arrow[dl, "1\times 1\times\zeta_n"] & \mathscr{E}(\mathbf{n},\mathbf{1}) \times \mathcal{B}^n \arrow[r, "\circ"] \arrow[d, "1\times\zeta_n"] \arrow[dr, phantom, "\Downarrow"] & \mathcal{B} \arrow[d, "\zeta_1"] \\
\mathscr{E}(\mathbf{n},\mathbf{1}) \times \mathscr{E}(\mathbf{n},\mathbf{n}) \times \mathcal{C}^n \arrow[rr, "1\times\circ"] & & \mathscr{E}(\mathbf{n},\mathbf{1}) \times \mathcal{C}^n \arrow[r, "\circ"] & \mathcal{C}
\end{tikzcd}$$

The proof that the coherence axioms hold is omitted here. $\square$

Proposition 4.6. $\mathbb{R}$ *and* $\mathbb{Q}$ *are inverses.*

Proof. Suppose $\mathcal{A} \in \mathscr{P}$-**PsAlg**. Then we see

$$\begin{aligned} \mathbb{Q}\mathbb{R}\mathcal{A} &= \mathbb{Q}(\mathcal{A}^*) \\ &= \mathcal{A} \end{aligned}$$

and the maps $\mathscr{P}(n) \times \mathcal{A}^n \to \mathcal{A}$ are

$$\begin{tikzcd}
\mathscr{P}(n) \times \mathcal{A}^n \arrow[r, "\sim"] \arrow[rrr, bend right, "\theta"'] & \mathscr{E}(\mathbf{n},\mathbf{1}) \times \mathscr{E}(\mathbf{n},\mathbf{n}) \times \mathcal{A}^n \arrow[r, "1\times\circ"] & \mathscr{E}(\mathbf{n},\mathbf{1}) \times \mathcal{A}^n \arrow[r, "\circ"] & \mathcal{A}
\end{tikzcd}$$

where the map $\mathscr{E}(\mathbf{n},\mathbf{n}) \times \mathcal{A}^n \to \mathcal{A}^n$ is defined by

$$\begin{tikzcd}
\sigma \times \mathcal{A}^n \arrow[r, "\sigma"] \arrow[dr, "\sigma^{-1}\times\sigma"'] & \mathcal{A}^n \\
 & e_n \times \mathcal{A}^n \arrow[u, "\pi_2"']
\end{tikzcd}$$

and $\mathscr{E}(\mathbf{n},\mathbf{1}) \times \mathcal{A}^n \to \mathcal{A}$ is defined by

$$\begin{tikzcd}
\mathcal{A}^n \arrow[r, "e_n\times 1"] & \mathscr{P}(n) \times \mathcal{A}^n \arrow[r, "\theta"] & \mathcal{A}
\end{tikzcd}$$

From the equivariance axiom of $\mathcal{A}$, we see that the composite is exactly $\mathscr{P}(n) \times \mathcal{A}^n \to \mathcal{A}$. Now, suppose $F\colon \mathcal{A} \to \mathcal{B}$ is a pseudomorphism of $\mathscr{P}$-pseudoalgebras. Then $\mathbb{Q}\mathbb{R}F$ is defined to be exactly $(\mathbb{R}F)_1$, which we defined to be exactly F. Hence, $\mathbb{Q}\mathbb{R} = \mathrm{id}$.

For the other direction, suppose that $\mathcal{A} \in \mathscr{F}^{sp}$-**PsAlg**. As an object,

$$\begin{aligned} \mathbb{R}\mathbb{Q}\mathcal{A} &= \mathbb{R}(\mathcal{A}(1)) \\ &= \mathcal{A}(1)^* \end{aligned}$$

and by the condition of specialness, $\mathcal{A}(1)^* \simeq \mathcal{A}$ naturally. Now, we need to check $\mathbb{R}\mathbb{Q}\mathcal{A}(\phi) = \mathcal{A}(\phi)$ for any $\phi \in \mathscr{F}(\mathbf{m},\mathbf{n})$. There are two cases we need to consider. First, if $\phi \in \Pi(\mathbf{m},\mathbf{n})$, then $\mathbb{R}\mathbb{Q}\mathcal{A}(\phi)$

is defined automatically by the functor $\mathbf{Cat} \to \Pi^{sp}\textbf{-Alg}$, and we see immediately that it must agree with $\mathcal{A}(\phi)$. On the other hand, if $\phi = \phi_n \colon \mathbf{n} \to \mathbf{1}$, then $\mathbb{R}\mathbb{Q}\mathcal{A}(\phi)$ is defined as

$$\mathcal{A}^n \xrightarrow{e_n \times 1} \mathscr{P}(n) \times \mathcal{A}^n \xrightarrow{\theta} \mathcal{A}$$

where θ is taken to be the action map of $\mathbb{Q}\mathcal{A}$. Now, this action map is exactly

$$\mathscr{P}(n) \times \mathcal{A}^n \xrightarrow{\sim} \mathscr{E}(\mathbf{n},\mathbf{1}) \times \mathscr{E}(\mathbf{n},\mathbf{n}) \times \mathcal{A}^n \xrightarrow{1\times\circ} \mathscr{E}(\mathbf{n},\mathbf{1}) \times \mathcal{A}^n \xrightarrow{\circ} \mathcal{A}$$

The map $e_n \times \mathcal{A}^n \to \mathcal{A}^n$ is just the identity, and $\mathscr{E}(\mathbf{n},\mathbf{1}) \times \mathcal{A}^n \to \mathcal{A}$ is exactly ϕ_n, so the composite just gives us back ϕ_n:

$$\begin{tikzcd}
\mathcal{A}^n \arrow[r, "e_n\times 1"] \arrow[drrr, "\sim", "\mathrm{id}"'] & \mathscr{P}(n)\times\mathcal{A}^n \arrow[r, "\sim"] & \mathscr{E}(\mathbf{n},\mathbf{1})\times\mathscr{E}(\mathbf{n},\mathbf{n})\times\mathcal{A}^n \arrow[r, "1\times\circ"] & \mathscr{E}(\mathbf{n},\mathbf{1})\times\mathcal{A}^n \arrow[r, "\circ"] \arrow[d, "\sim"] & \mathcal{A} \\
& & & \mathcal{A}^n \arrow[ur, "\phi_n"'] &
\end{tikzcd}$$

Finally, if $\eta \colon \mathcal{A} \to \mathcal{B}$ is a pseudonatural transformation of pseudofunctors $\mathscr{F} \to \mathbf{Cat}$, then $(\mathbb{R}\mathbb{Q}\eta)_n = (\mathbb{Q}\eta)^n = \eta_1^n \simeq \eta_n$, so in fact $\mathbb{R}\mathbb{Q}$ is naturally isomorphic to the identity, as required.

There are additional coherence conditions to check, but the details are left to the reader. $\square$

This gives us the following total result:

Theorem 4.7. *$\mathscr{P}$-**PsAlg** and $\mathscr{F}^{sp}$-**PsAlg** are equivalent categories, and $\mathscr{P}$-**PsAlg** and $\mathscr{F}^{sp}$-**PsAlg**$^{\mathbb{R}}$ are isomorphic categories.*

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