CHROMATIC BLUESHIFT OF COMMUTATIVE MU-ALGEBRAS VIA POWER OPERATIONS

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Abstract. We investigate the chromatic blueshift phenomenon in the case of complex-oriented $\mathbb{E}_\infty$-rings to reduce the question to computations with homotopy groups. A recent conjecture of Burklund, Schlank, and Yuan [BSY22, Conjecture 9.9] suggests that taking geometric fixed points of an $\mathbb{E}_\infty$-ring R always lowers chromatic height at all finite abelian p-groups and families of subgroups. Using the power operations of the complex cobordism spectrum MU, we show this is true for the geometric fixed points Φ^{C_p} when R admits an $\mathbb{E}_\infty$ complex orientation and its v_n element in $\pi_*(R)$ has nilpotent degree of 1, after quotienting by the Landweber ideal – under certain regularity assumptions. For higher degrees, we form an algebraic analog of the conjecture that fails, which provides a barrier for the general conjecture to be true.

1. Introduction

We give a partial answer to the following conjecture of Burklund-Schlank-Yuan:

Conjecture 1.0.1 ("Chromatic blueshift conjecture", [BSY22, Conjecture 9.9])**.** *Let R be an $\mathbb{E}_\infty$-ring, equipped with the trivial C_p-action for a prime p. If R has chromatic height $n \geq 0$, then the C_p-Tate cohomology R^{tC_p} has height $n-1$.*

Note that the original conjecture is much broader, and specifies how geometric fixed points along families of subgroups of finite abelian p-groups shift down height, for Borel-complete genuine equivariant $\mathbb{E}_\infty$-rings coming from trivial actions. We study the case of the group C_p with the family consisting only of the trivial subgroup, and R an $\mathbb{E}_\infty$-MU-algebra. For a choice of v_n elements in $\pi_* R$, we establish an algebraic condition on the v_n's which guarantees that blueshift occurs.

We will fix a choice of generators x_i in degree $2i$ such that $\pi_*\mathrm{MU} \simeq \mathbb{Z}[x_1, x_2, \ldots]$. For the rest of the paper, we will fix a prime p and a choice of generators $x_i \in \pi_{2i}\mathrm{MU}$, and we will write v_n for x_{p^n-1}, where $v_0 = p$. For an MU-algebra $f\colon \mathrm{MU} \to R$, we will refer to the images $f_*(v_n)$ as "v_n" too, where it is clear which homotopy groups we work with. The ideal generated by the first n of these is the **Landweber ideal** $\mathcal{I}_n := (p, v_1, \ldots, v_{n-1})$.

Before stating our main result, we will need one more piece of information concerning how chromatic height interacts with v_n elements.

Lemma 1.0.2. *Suppose R is an $\mathbb{E}_\infty$-ring and is complex-oriented (not necessarily an $\mathbb{E}_\infty$-MU-algebra), such that $p, \ldots, v_{n-1}$ form a regular sequence in $\pi_*(R)$. Then, R is of height n if and only if v_{n+1} is nilpotent modulo $\mathcal{I}_{n+1}$.*

With this information in place, our result concerns the case of v_n having a nilpotence degree of exactly 1 modulo $\mathcal{I}_n$.

Theorem A. Let R be an $\mathbb{E}_\infty$-MU-algebra such that $p, v_1, \ldots, v_{n-1}$ is a regular sequence in $\pi_*(R)$. Suppose that $v_{n+1} \in \mathcal{I}_{n+1}$. Then, Tate cohomology shifts chromatic height down by exactly 1. That is, R^{tC_p} is $K(n-1)$-acyclic. ◁

The proof of Theorem A is completely algebraic and involves computations with power operations on MU. Thus, we may consider maps between graded rings $f\colon \mathrm{MU}_* \to R_*$ and make statements about what would happen if R_* is realized as the R-valued coefficient of a point. It is in this sense that we may make speculative statements and disprove they happen on the level of algebra, with no knowledge about whether they get actualized by topology.

Theorem B. Suppose R is an $\mathbb{E}_\infty$-complex-oriented ring, such that:

(1) The homotopy groups of R extend those of MU by adjoining formal halves to all products x_ix_j:

$$\pi_*(R) = \frac{\pi_*(\mathrm{MU})[b_{ij} : 1 \leq i \leq j]}{(x_ix_j - 2b_{ij})}$$

(2) There exists an $\mathbb{E}_\infty$ map $f\colon \mathrm{MU} \to R$ that induces the inclusion map on homotopy groups:

$$f_*\colon \pi_*(\mathrm{MU}) \hookrightarrow \pi_*(R)$$

Then, R is $T(1)$-acyclic, but R^{tC_2} is not $T(0)$-acyclic. ◁

1.1. **Chromatic Blueshift and the Tate Construction.** For a finite group G, recall that Borel G-spectra $\mathsf{Sp}^{\mathrm{B}G}$ are elements of $\mathsf{Fun}(\mathrm{B}G, \mathsf{Sp})$, and that we may regard a non-equivariant spectrum as G-equivariant one with trivial action. There is a natural **norm** map $\mathsf{Nm}\colon (-)_{hG} \to (-)^{hG}$ from the homotopy orbits to the homotopy fixed points of a Borel G-spectrum (which are the homotopy colimit and homotopy limit respectively of the given diagram in Sp indexed by $\mathrm{B}G$). This is analogous to the procedure of summing over orbits to form a fixed point in G-modules. The cofiber of this norm map will be the ***G*-Tate cohomology** of a spectrum, denoted by $(-)^{tG}$:

$$X^{tG} := \mathrm{hocofib}(X_{hG} \xrightarrow{\mathsf{Nm}} X^{hG})$$

Although this is not the most computationally workable definition of the Tate construction, we will not need any computations that are not already known.

The Tate construction on Borel-equivariant spectra with trivial actions has been known to lower chromatic complexity. For example, it:

- reduces v_n-periodicity of complex-oriented spectra:

  Theorem 1.1.1 (Greenlees-Sadofsky [GS96, Corollary 1.7])**.** *For a finite group G, if E is v_n-periodic, in the sense that it is complex-oriented and that v_n is a unit in $\pi_*(\mathbb{S}/(p^{i_0}, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}}) \otimes E)$ for a choice of v_is and suitable indices $i_1, \dots, i_{n-1}$, then E^{tG} is v_{n-1}-periodic.*

- exhibits "Tate vanishing", in that it vanishes $K(n)$-locally or $T(n)$-locally:

  Theorem 1.1.2 ("Tate vanishing" – Greenlees-Sadofsky [GS96] and Hovey-Sadofsky [HS96] / Kuhn [Kuh04])**.** *For a $K(n)$ (resp., $T(n)$)-local spectrum X and G a finite group, X^{tG} is $K(n)$ (resp., $T(n)$)-locally null.*

- lowers Bousfield class:

  Theorem 1.1.3 (Hovey-Sadofsky [HS96, Theorem 1.1])**.** *For a finite spectrum X and finite group G with order a multiple of p, $\langle L_n X^{tG}\rangle = \langle L_{n-1}X\rangle$*

For many more examples, see the introductions of [BR19] or [LLQ19].

A theorem of Hahn [Hah16] allows us to understand chromatic complexity in the case of commutative algebras, which enables us to ask questions about what the Tate construction does in these settings.

Theorem 1.1.4 (Hahn [Hah16, Theorem 1.1])**.** *Let R be an $\mathbb{E}_\infty$-ring that is $K(n)$-acyclic. Then, R is $K(n+1)$-acyclic.*

Then, we can define a notion of chromatic complexity that behaves in the opposite way of type in the sense of Hopkins-Smith [HS98]:

Definition 1.1.5 ([BSY22, Section 9])**.** For R an $\mathbb{E}_\infty$-ring, define its **chromatic height** at a prime p as the largest n such that R is not $T(n)$-acyclic:

$$\mathrm{height}(R) = \max\{n \mid R \otimes T(n) \not\simeq 0\} \in [-1, \infty]$$

◁

Remark 1.1.6. We use the fact that being $T(n)$-acyclic is equivalent to being $K(n)$-acyclic for homotopy ring spectra at $n \geq 1$ ([LMMT24, Lemma 2.3]). We use $T(n)$ as being $T(0)$-acyclic carries p-primary information, and we use the convention that $T(-1) = \mathbb{S}$ to allow heights of -1. Some examples include: $\mathrm{height}(\mathbb{S}) = \mathrm{height}(\mathrm{MU}) = \infty$, $\mathrm{height}(\mathrm{H}\mathbb{Z}) = 0$, $\mathrm{height}(E_n) = n$, and $\mathrm{height}(0) = -1$. ◁

It is easy to see that C_p-Tate cohomology decreases chromatic height of Lubin-Tate theories by exactly 1, such as from the discussion around Section 2 of Ando-Morava-Sadofsky [AMS98]. Combining this with Burklund-Schlank-Yuan's $\mathbb{E}_\infty$-ring maps into Lubin-Tate theories detecting $T(n)$-local information immediately gives the following:

Theorem 1.1.7 (a case of Burklund-Schlank-Yuan [BSY22, Theorem 9.8])**.** *For R an $\mathbb{E}_\infty$-ring of height $n \geq 0$ equipped with the trivial C_p-action, we have that R^{tC_p} has height at least $n-1$.*

This is one "half" of their blueshift conjecture Conjecture 1.0.1 – the canonical map $R \to R^{tC_p}$ forces R^{tC_p} to have height at most height(R), so the conjecture is that Tate cohomology does actually always achieve reduction in chromatic height.

1.2. **Power Operations and Complex Orientations.** As we work with $\mathbb{E}_\infty$-rings, we are able to exploit the structure of being an algebra over the $\mathbb{E}_\infty$-operad. Power operations are able to encode much of this structure algebraically, and we will make use of these to investigate the blueshift conjecture. For a more thorough treatment of power operations, see, for example: Bruner-May-McClure-Steinberger [BMMS86], Nikolaus-Scholze §IV.1 [NS18], or Rezk §1.2 [Rez06].

When we restrict our attention to complex-oriented $\mathbb{E}_\infty$-rings R, we will get a map $\mathrm{MU} \to R$ classifying the complex orientation. If we further require that this map be one of $\mathbb{E}_\infty$-algebras, we will be able to exploit the additional structure imposed by power operations.

First, we review the description of "extended" power operations due to Charles Rezk, written in §IV.1 [NS18]. These extend the classical power operations defined in cohomological degree 0.

Construction 1.2.1. For R an $\mathbb{E}_\infty$-ring and p prime, we construct a natural transformation of functors $\mathsf{Top}^{\mathrm{op}} \to \mathsf{Mon}$:

$$P_p \colon R^*(-) \implies R^*(- \times B\Sigma_p)$$

where Σ_p is the symmetric group on p elements, and the monoid structure is given by multiplication in the cohomology rings.

We represent a cohomology class in $R^n(X)$ by a (homotopy class of a) map $f \in [X, \Omega^\infty \Sigma^n R]$. Taking the p-th fold product gives a Σ_p-equivariant map, with the action permuting the symbols. We postcompose with the canonical map into Σ_p homotopy fixed points:

$$X^{\times p} \xrightarrow{f^{\times p}} (\Omega^\infty \Sigma^n R)^{\times p} \to ((\Omega^\infty \Sigma^n R)^{\times p})^{h\Sigma_p}$$

Further, we may form another composition:

$$X^{\times p} \to ((\Omega^\infty \Sigma^n R)^{\times p})^{h\Sigma_p} \to \Omega^\infty ((\Sigma^n R)^{\otimes p})^{h\Sigma_p}$$

As equivariant spectra, the inner suspension and its Σ_p-action may be described as a smash product with $\mathbb{S}^{n\rho}$, the representation sphere corresponding to the n-fold product of the permutation representation $\rho \colon \Sigma_p \to \mathrm{GL}(\mathbb{R}^p)$:

$$X^{\times p} \to \Omega^\infty (\mathbb{S}^{n\rho} \otimes R^{\otimes p})^{h\Sigma_p}$$

Working carefully, we equivalently have a map:

$$X \times BG \simeq (X^{\times p})_{h\Sigma_p} \to \Omega^\infty (\mathbb{S}^{np} \otimes R) \simeq \Omega^\infty \Sigma^n R$$

This represents a class in $R^{np}(X \times B\Sigma_p)$. ◁

Remark 1.2.2. The power operations as constructed are not ring homomorphisms. While they are multiplicative, they are not additive. ◁

This construction will be too unwieldy for our purposes, due to the difficulties of computing $R^*(X \times B\Sigma_p)$. We will restrict our attention to a choice of cyclic subgroup $C_p \subset \Sigma_p$ where we have more tools at our disposal, at least for complex-oriented rings. As such, we will also just choose X to be a point. Another correction is also in order: by quotienting out the transfer ideal, it is well known that the power operation becomes multiplicative.

Definition 1.2.3. Pick a subgroup $\iota\colon C_p \hookrightarrow \Sigma_p$. We have a cohomological transfer map:

$$\mathrm{Tr}_e^{C_p}\colon R^*(*) \to R^*(BC_p)$$

For the rest of the paper, by a **power operation**, we will mean the map:

$$R^*(*) \to R^{*p}(B\Sigma_p) \xrightarrow{\iota^*} R^{*p}(BC_p) \to R^{*p}(BC_p)/(\mathrm{Tr}_e^{C_p} 1)$$

◁

In the case of complex-oriented spectra, we can say more about the transfer ideal. The complex orientation induces a formal group law F on $R^*[\![x, y]\!]$.

Definition 1.2.4. Under the formal group law F, we denote the n**-series** $[n]_F(\alpha)$ as:

$$[n]_F(\alpha) = \underbrace{\alpha +_F \alpha + \cdots +_F \alpha}_{n \text{ times}}$$

Note that $[n]_F(\alpha)$ is a multiple of α, letting us define a **reduced** n**-series**:

$$\langle n\rangle_F(\alpha) = \frac{[n]_F(\alpha)}{\alpha}$$

We will often drop the subscript F when it is clear, and further may drop the (α). ◁

It is well-known through a Gysin sequence argument that one can calculate the R^*-cohomology of classifying spaces of cyclic groups, in good cases:

Lemma 1.2.5 (see for example [HKR00, Lemma 5.7] and its proof)**.** *For a complex-oriented ring spectrum R such that $[n]$ is not a zero-divisor, we have that:*

$$R^*(BC_n) \simeq R^*[\![\alpha]\!]/[n]$$

with α a generator in degree 2*.*

Remark 1.2.6 ([HKR00, Remark 6.15])**.** With R as in the above theorem, we have that $\mathrm{Tr}_e^{C_p} 1 = \langle n\rangle$. This generates the transfer ideal. ◁

In the same cases, we are able to compute the homotopy ring of R^{tC_p}:

Lemma 1.2.7 (for example, [AMS98, Lemma 2.1])**.** *With R as before, we have that:*

$$\pi_*\big(R^{tC_p}\big) \simeq \alpha^{-1}R^*[\![\alpha]\!]/[p](\alpha)$$

with α a generator in degree -2*.*

In light of these results and noting that $\pi_{-*}R \simeq R^*$, we can force power operations to land in $(R^{tC_p})^*$:

$$R^* \xrightarrow{P_p} R^*[\![\alpha]\!]/[p] \to R^*[\![\alpha]\!]/\langle p\rangle \to \alpha^{-1}R^*[\![\alpha]\!]/[p] \simeq (R^{tC_p})^*$$

By the same duality, we will not notationally distinguish between $v_n \in \pi_*R$ and the topological dual in R^*, or $\alpha \in R_*(BC_p)$ or its dual in $R^*(BC_p)$. Hence, we may view power operations as a map:

$$\pi_*R \xrightarrow{P_p} \pi_*(R^{tC_p})$$

We will exploit this and the naturality of power operations to get information about the Tate construction.

By considering $\mathbb{E}_\infty$ complex orientations $\mathrm{MU} \to R$, we will be able to derive algebraic properties of $\pi_*\big(R^{tC_p}\big)$ that will give us insights into blueshifting.

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2. Algebraic Description of Height

Here, we develop an analog in algebra in which we may convert our results into statements about homotopy groups. We start with a standard result connecting $K(n)$-acyclicity of complex-oriented spectra with their v_i maps:

Proposition 2.0.1 (for example: [DFHH14, Proposition 4.1])**.** *If R is a complex-oriented ring spectrum, we have an equivalence:*

$$L_{K(n)}R \xrightarrow{\sim} \lim_{(i_0,\ldots,i_{n-1})\in\mathbb{N}^n} v_n^{-1}R/(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})$$

We have the following immediate corollary:

Corollary 2.0.2. *If R is a complex-oriented ring spectrum and $(p, \ldots, v_{n-1})$ form a regular sequence in $\pi_*(R)$, then the homotopy ring of the $K(n)$-localization of R is given by $\mathcal{I}_n$-adically completing $v_n^{-1}\pi_*(R)$:*

$$\pi_*(L_{K(n)}R) \simeq \pi_*(v_n^{-1}R)^{\wedge}_{\mathcal{I}_n}$$

Therefore, assuming regularity, we get the two more immediate consequences relating chromatic height to algebra:

Corollary 2.0.3 (Lemma 1.0.2)**.** *Suppose R is an $\mathbb{E}_\infty$-ring and is complex-oriented (not necessarily an $\mathbb{E}_\infty$-MU-algebra), such that $(p, \ldots, v_{n-1})$ form a regular sequence in $\pi_*(R)$. Then, R is of height n if and only if v_n is nilpotent modulo $\mathcal{I}_n$.*

Corollary 2.0.4. *With R as before,* $\mathrm{height}(R) \geq n-1$.

Proof. Assuming regularity, we get that v_{n-1} is not nilpotent modulo $\mathcal{I}_{n-1}$. Applying the previous corollary gives the desired result. $\square$

3. Power Operations

3.1. **Power Operations on** MU**.** In order to do computations with the power operations, we will need to understand how they act on MU. MU_* is rationally generated by the bordism classes of $c_n = [\mathbb{C}P^n] \in \mathrm{MU}_{2n}(*)$. We will need to understand how to relate these back to integral generators, such as the Hazewinkel generators x_i.

Proposition 3.1.1 ([Haz78], Equation 34.4.3)**.** MU_* *is integrally generated by generators x_i in degree $2i$ related to the rational generators c_i via:*

$$\frac{1}{m}v(m)c_{m-1} = x_{m-1} + \sum_{\substack{d|m\\ d\neq 1,k}} \frac{\mu(m,d)v(m)}{(m/d)v(d)} c_{m/d-1}x_{d-1}^{m/d}$$

where

$$v(m) = \begin{cases} q & \text{if } m = q^r, r \geq 1, q \text{ prime} \\ 1 & \text{otherwise} \end{cases}$$

and

$$\mu(m,d) = \prod_{\substack{q|m\\ q \text{ is prime}}} c(q,d)$$

is a product ranging over primes q, with $c(q,d) = 1$ if $v(d) = 1$ or q, and otherwise $c(q,d)$ is some integer such that $c(q,d) \equiv 1 \bmod q$ and $c(q,d) \equiv 0 \bmod v(d)$.

We will use the rational generators c_n to calculate the power operations on the v_n terms, x_{p^n-1}.

Proposition 3.1.2 ([JN10], Proposition 5.21)**.** *Let*

$$\chi = \prod_{i=1}^{p-1}[i](\alpha)$$

and a_i be the coefficients:

$$x\sum_{i\geq 0} a_i x^i = \prod_{i=0}^{p-1}(x +_F [i](\alpha)) \tag{1}$$

Then, the power operations on the generators c_n are calculated as:

$$\chi^{2m}P(c_m) = \chi^{2m+1}\sum_{k=0}^{m} c_{m-k}\operatorname{coeff}\left(\left(\sum_{i\geq 0} a_i z^i\right)^{-(m+1)}, z^k\right)$$

Remark 3.1.3. The result of [JN10] is stated in terms of power operations on BP, however, the formula is still valid for MU by the same arguments. ◁

Remark 3.1.4. Expanding out Equation 1, we see that $a_0 = \chi$. Also, $[i](\alpha) = i\alpha + O(\alpha^2)$, so $\chi = (p-1)!\alpha^{p-1} + O(\alpha^p)$. In the case that $p = 2$, we just have $\chi = \alpha$ exactly. ◁

3.2. **Power Operations and the Landweber Ideal.** The main result of this section is the following.

Theorem 3.2.1. *For R an $\mathbb{E}_\infty$-MU-algebra, if $v_n = 0 \bmod \mathcal{I}_n$ in $\pi_* R$, then for $i \geq 0$, we have that $v_{n+i} = 0 \bmod \mathcal{I}_n$.*

The rest of this subsection is dedicated to proving the results referenced in the above proof.

In general, we will want to calculate the power operations at v_n, which we relate back to the rational generators as follows. First, by Proposition 3.1.1 with $m = k$, we have that

$$c_{k-1} = \binom{k}{v(k)} x_{k-1} + \sum_{\substack{d|k \\ d\neq 1,k}} \binom{d}{v(d)}\mu(k,d)c_{k/d-1}x_{d-1}^{k/d}. \tag{2}$$

Note that $\mu(d,k)k/v(k)$ is always an integer.

We will need to thoroughly study c_{p^n-1}, so we compute it exactly as opposed to approximating:

Lemma 3.2.2. *For $n \geq 1$, we have that:*

$$c_{p^n-1} = \sum_{\substack{i_1,i_2,\ldots,i_k\geq 1 \\ i_1+i_2+\ldots+i_k=n}} p^{n-k}\prod_{j=1}^{k} v_{i_j}^{p^{n-(i_1+i_2+\ldots+i_j)}}$$

Proof. We prove this inductively. At $n = 1$, we simply have, by 2, $c_{p-1} = v_1$. This agrees with the hypothesis, where the only option is $k = 1$ and $i_1 = 1$.

We will proceed by studying $\operatorname{comp}(n)$, the compositions of n - i.e. the ways to form n as a sum of positive integers, as seen in the summation. Note that we can compute compositions of n recursively:

$$\operatorname{comp}(n) = \{(n)\} \cup \bigcup_{m=1}^{n-1}\{\text{adjoin } m \text{ to elements of } \operatorname{comp}(n-m)\}$$

This will form the basis for our proof.

From 2, we have the following recurrence relation:

$$c_{p^n-1} = p^{n-1}v_n + \sum_{\substack{d=p^i \\ 1\leq i<n}} p^{i-1}v_i^{p^{n-i}}c_{p^{n-i}-1}$$

We just need to show that the hypothesis satisfies the same recurrence, so let:

$$\tilde{c}_{p^n-1} = \sum_{(i_1,\ldots,i_k)\in\operatorname{comp}(n)} p^{n-k}\prod_{j=1}^{k} v_{i_j}^{p^{n-(i_1+i_2+\ldots+i_j)}}$$

Then, using the recursion for $\mathrm{comp}(n)$ and adjoining at the start, we have:

$$\begin{aligned}
\tilde{c}_{p^n-1} &= p^{n-1}v_n + \sum_{m=1}^{n-1} \sum_{(i_1,\ldots,i_k)\in\mathrm{comp}(n-m)} p^{n-k-1} v_m^{p^{n-m}} \prod_{j=1}^{k} v_{i_j}^{p^{n-(m+i_1+i_2+\ldots+i_j)}} \\
&= p^{n-1}v_n + \sum_{m=1}^{n-1} p^{m-1} v_m^{p^{n-m}} \sum_{(i_1,\ldots,i_k)\in\mathrm{comp}(n-m)} p^{(n-m)-k} \prod_{j=1}^{k} v_{i_j}^{p^{(n-m)-(i_1+i_2+\ldots+i_j)}} \\
&= p^{n-1}v_n + \sum_{m=1}^{n-1} p^{m-1} v_m^{p^{n-m}} \tilde{c}_{p^{n-m}-1}
\end{aligned}$$

Because c_{p^n-1} and $\tilde{c}_{p^n-1}$ satisfy the same initial condition and recurrence relation, they are the same. □

Corollary 3.2.3. *In particular,*

$$c_{p^n-1} = p^{n-1}v_n \bmod (v_1, \ldots, v_{n-1})$$

Proof. The term picks out the composition (n) of n, and other compositions necessarily have v_{i_j} for $i_j < n$. □

We will not exactly compute arbitrary c_{k-1}, but will need to approximate these terms.

Lemma 3.2.4. *If $p^r \mid k$ for $r \geq 1$, then $c_{k-1} = 0 \bmod (p^r, v_1, \ldots, v_r)$.*

Proof. The claim is true for the base cases $k = p^r$ by Corollary 3.2.3. Then, we will induct over r, and at each step we will further induct over the number of prime factors (counted by multiplicity) of k to prove the layers before r imply the lemma to be true for r.

Assuming we know the result up to $r-1$, we seek to show it is true for r. We induct over the number of prime factors of k, which we call N. The smallest case is $N = r$, which is covered by the above base case. Otherwise, we let $k = Ap^t$ with $t \geq r$ and $\gcd(A, p) = 1$. For the first term, we have three cases to consider:

(1) $\underline{A = 1,\ t = r}$: here, the first term is a multiple of $x_{p^r-1} = v_r$.
(2) $\underline{A = 1,\ t > r}$: here, the first term is a multiple of $k/v(k) = p^{t-1}$ which is a multiple of p^r.
(3) $\underline{A \neq 1}$: $v(k) = 1$, so that the first term is a multiple of k which is a multiple of p^r.

For the index d summand, we have four cases, stated below. Letting $d = Bp^i$ for $1 \leq i \leq t$ with $\gcd(B, p) = 1$ and B dividing into A:

(1) $\underline{d = p^i,\ i \leq r}$: the summand is a multiple of $x_{p^i-1} = v_i$ with $i \leq r$.
(2) $\underline{d = B}$: k/B has fewer than N prime factors and is stil divisible by p^r, so we get that $c_{k/d-1}l = 0 \bmod (p^r, v_1, \ldots, v_r)$.
(3) $\underline{d = Bp^i, \text{ with } i \geq r}$: in this case, $v(d) = 1$ and p^r divides into d.
(4) $\underline{d = Bp^i, \text{ with } 1 \leq i < r}$: $v(d) = 1$ still, so that the term is a multiple of $Bp^i c_{(A/B)p^{t-i}}$. However, $(A/B)p^{t-i}$ is a multiple of p^{r-i}. Because we assume to know the lemma at $r-i$, we deduce that this is a multiple of p^i and some term that is $0 \bmod (p^{r-i}, v_1, \ldots, v_{r-i})$. Therefore, the whole term is $0 \bmod (p^r, v_1, \ldots, v_{r-i})$.

Therefore, we conclude the statement to be true for $N+1$, and so forth. This makes it true for all options at r given lower values, so it is proven for all r, N. Note that when proving $r = 1$, case 4 was never used, so there are no issues with using lower values of r. □

We proceed with a combinatorial lemma we will need for the next result.

Lemma 3.2.5. *Let $b_0, b_1, \ldots, b_N$ be a sequence of N positive integers that add to p^n for some prime p and $n \geq 1$. Define r as the smallest power of p that divides all b_i. Then, we have that p^{n-r} divides the multinomial coefficient:*

$$p^{n-r} \mid \binom{p^n}{b_0\ b_1\ \cdots}$$

Proof. We decompose each b_i as $b_i = q_i p^{r_i}$ with $\gcd(q_i, p) = 1$. Then, we have that:

$$\binom{p^n}{b_0\ b_1\ \ldots} = \prod_{i\geq 0} \binom{p^n - b_0 - \ldots - b_{i-1}}{q_i p^{r_i}}$$
$$= \prod_{i\geq 0} \frac{p^n - b_0 - \ldots - b_{i-1}}{q_i p^{r_i}} \binom{p^n - b_0 - \ldots - b_{i-1} - 1}{b_i - 1}$$

In particular, $p^n - b_0 - \ldots - b_{i-1}$ is a multiple of $p^{\min(r_0, r_1, \ldots, r_{i-1})}$. Let M be the p-adic valuation multinomial coefficient. The upshot is the following:

$$M \geq n + \sum_{i=0}^{N-1} (\min(r_0, \ldots, r_i) - r_i) - r_N$$

However, the value of the multinomial does not depend on the order of the b_is. We are then free to reorder b_i so that r_i are in decreasing order, so that the summation disappears and we are left with:

$$M \geq n - r$$

□

Now, we seek to more concretely understand power operations on these rational generators:

Lemma 3.2.6. *We have that* $\chi^{2(p^n-1)} P(c_{p^n-1}) = p^{n-1} v_n \chi^{p^n-1} \bmod (p^n, v_1, \ldots, v_{n-1}, \langle p\rangle(\alpha))$.

Proof. We make use of Proposition 3.1.2 again. At $m = p^n - 1$, this evaluates to

$$\chi^{2(p^n-1)} P(c_{p^n-1}) = \chi^{2p^n-1} \sum_{k=0}^{p^n-1} c_{p^n-k-1} \operatorname{coeff}((\sum_{i\geq 0} a_i z^i)^{-p^n}, z^k).$$

For $k = 0$, Corollary 3.2.3 tells us there is a term $p^{n-1} v_n$ in the summand, after modding out by $v_1, \ldots, v_{n-1}$. The coefficient attached to it is just $a_0^{-p^n}$. However, by Remark 3.1.4, we have that the entire term becomes $p^{n-1} v_n \chi^{p^n-1}$.

The rest of this proof shows the other terms vanish after modding out the specified ideal. First, write

$$h := \left(\sum_{i\geq 0} a_i z^i\right)^{-1} = \sum_{i\geq 0} w_i z^i$$

for appropriate coefficients w_i. Note that:

$$\operatorname{coeff}(h^{p^n}, z^k) = \sum_{\substack{b_0, b_1, \ldots \geq 0 \\ b_0 + b_1 + \ldots = p^n \\ b_1 + 2b_2 + \ldots = k}} \left(\prod_i w_i^{b_i}\right) \operatorname{coeff}\left(h^{p^n}, z^k \prod_i w_i^{b_i}\right) = \sum_{\substack{b_0, b_1, \ldots \geq 0 \\ b_0 + b_1 + \ldots = p^n \\ b_1 + 2b_2 + \ldots = k}} \left(\prod_i w_i^{b_i}\right) \binom{p^n}{b_0\ b_1\ \ldots}$$

For a choice of $\{b_i\}$, by Lemma 3.2.5 we have that the multinomial term is divisible by p^{n-r} where p^r divides all b_i. Let s be the p-adic valuation of k. Note that $s \geq r$; in particular, we now have that:

$$p^{n-s} \mid \operatorname{coeff}(h^{p^n}, z^k)$$

On the other hand, $p^s \mid p^n - k$, so by Lemma 3.2.4, each term in the original summand satisfies:

$$c_{p^n-k-1} \operatorname{coeff}((\sum_{i\geq 0} a_i z^i)^{-p^n}, z^k) = 0 \bmod (p^n, v_1, \ldots, v_s)$$

with $s < n$. □

With these tools in hand, we can proceed with an approximation of $P(v_n)$:

Lemma 3.2.7.

$$\chi^{2(p^n-1)} P(v_n) = v_n \chi^{p^{n-1}} \bmod (p, v_1, \ldots, v_{n-1}, \langle p\rangle(\alpha))$$

PROOF. We show this inductively on n. $n = 1$ is just the statement of Lemma 3.2.6, through identifying $c_{p-1} = v_1$ by Corollary 3.2.3.

Assuming the lemma for heights i smaller than n, we use the decomposition from Lemma 3.2.2:

$$c_{p^n-1} = p^{n-1}v_n + \sum_{\substack{i_1,i_2,\ldots,i_k\geq 1\\ i_1+i_2+\ldots+i_k=n\\ k\geq 2}} p^{n-k}\prod_{j=1}^{k} v_{i_j}^{p^{n-(i_1+i_2+\ldots+i_k)}}$$

In particular, by applying $\chi^{2(p^n-1)}P(-)$,

$$\chi^{2(p^n-1)}P(c_{p^n-1}) = \chi^{2(p^n-1)}p^{n-1}P(v_n) + \sum_{\substack{i_1,i_2,\ldots,i_k\geq 1\\ i_1+i_2+\ldots+i_k=n\\ k\geq 2}} \chi^{2(k-1)}p^{n-k}\prod_{j=1}^{k}\left(\chi^{\left(p^{i_j}-1\right)}P(v_{i_j})\right)^{p^{n-(i_1+i_2+\ldots+i_k)}}$$

While we only know $\chi^{2\left(p^{i_j-1}\right)}P(v_{i_j})$ up to mod $(p, v_1, \ldots, v_{i_j-1}, \langle p\rangle(\alpha))$, we are taking k products of such terms and multiplying further by p^{n-k}. Therefore, we know the right side up to mod $(p^n, v_1, \ldots, v_{n-1}, \langle p\rangle(\alpha))$, where all summands vanish:

$$\begin{aligned} p^{n-1}v_n\chi^{p^n-1} &= \chi^{2(p^n-1)}P(c_{p^n-1}) \bmod (p^n, v_1, \ldots, v_{n-1}, \langle p\rangle(\alpha)) \\ &= \chi^{2(p^n-1)}p^{n-1}P(v_n) \bmod (p^n, v_1, \ldots, v_{n-1}, \langle p\rangle(\alpha)) \end{aligned}$$

Therefore, we must have that:

$$\chi^{2(p^n-1)}P(v_n) = p^{n-1}v_n\chi^{p^n-1} \bmod (p^n, v_1, \ldots, v_{n-1}, \langle p\rangle(\alpha))$$

□

Lemma 3.2.8. *For $n \geq 1$, $v_n\chi^{p^n-1} - (\chi/\alpha)^{p^n-1}\langle p\rangle(\alpha)$ is divisible by $\alpha^{2(p-1)(p^n-1)}$ after modding out by $(p, v_1, \ldots, v_{n-1}, \langle p\rangle(\alpha))$. Further,*

$$\frac{\chi^{2(p^n-1)}}{\alpha^{2(p-1)(p^n-1)}}P(v_n) = \frac{v_n\chi^{p^n-1} - \dfrac{\chi^{p^n-1}}{\alpha^{p^n-1}}\langle p\rangle(\alpha)}{\alpha^{2(p-1)(p^n-1)}} \bmod \left(p, v_1, \ldots, v_{n-1}, \frac{[p]}{\alpha^{p^n}}\right)$$

PROOF. By Lemma 3.2.7, we have that:

$$\chi^{2(p^n-1)}P(v_n) = v_n\chi^{p^{n-1}} \bmod (p, v_1, \ldots, v_{n-1}, \langle p\rangle(\alpha))$$

χ is divisible by α, and we may freely subtract multiples of the reduced p-series:

$$\chi^{2(p^n-1)}P(v_n) = v_n\chi^{p^n-1} - \frac{\chi^{p^n-1}}{\alpha^{p^n-1}}\langle p\rangle(\alpha) \bmod (p, v_1, \ldots, v_{n-1}, \langle p\rangle(\alpha))$$

However, we can say something stronger: $\langle p\rangle = v_n\alpha^{p^n-1} + O(\alpha^{p^n})$, so we may actually work $\bmod\, [p]/\alpha^{p^n}$. Now, by Remark 3.1.4, the left hand side is divisible by $\alpha^{2(p-1)(p^n-1)}$, so the right must be too:

$$\frac{\chi^{2(p^n-1)}}{\alpha^{2(p-1)(p^n-1)}}P(v_n) = \frac{v_n\chi^{p^n-1} - \dfrac{\chi^{p^n-1}}{\alpha^{p^n-1}}\langle p\rangle(\alpha)}{\alpha^{2(p-1)(p^n-1)}} \bmod \left(p, v_1, \ldots, v_{n-1}, \frac{[p]}{\alpha^{p^n}}\right)$$

□

Finally, we arrive at the following approximation for $P(v_n)$:

Lemma 3.2.9. *We have that:*

$$P(v_n) = -v_{n+1}\alpha^{p-1} + O(\alpha^p) \bmod \left(p, v_1, \ldots, v_n, \frac{[p](\alpha)}{\alpha^{p^n}}\right)$$

PROOF. By Lemma 3.2.8, we see that

$$\frac{\chi^{2(p^n-1)}}{\alpha^{2(p-1)(p^n-1)}}P(v_n) = \frac{v_n\chi^{p^n-1} - \frac{\chi^{p^n-1}}{\alpha^{p^n-1}}\langle p\rangle(\alpha)}{\alpha^{2(p-1)(p^n-1)}} \bmod \left(p, v_1, \dots, v_{n-1}, \frac{[p](\alpha)}{\alpha^{p^n}}\right)$$

The point of the factor in front of $\langle p\rangle(\alpha)$ is that we wish to cancel out v_n. Note that, by the theory of formal group laws,

$$\langle p\rangle(\alpha) = v_n\alpha^{p^n-1} + v_{n+1}\alpha^{p^{n+1}-1} + (O(v_n\alpha^{p^n})) + O(\alpha^{p^{n+1}}) \bmod (p, v_1, \dots, v_{n-1})$$

In particular, after modding out further by v_n, we may simplify as:

$$\begin{aligned}
\frac{\chi^{2(p^n-1)}}{\alpha^{2(p-1)(p^n-1)}}P(v_n) &= \frac{-\chi^{p^n-1}v_{n+1}\alpha^{p^{n+1}-p^n} + \chi^{p^n-1}O(\alpha^{p^{n+1}-p^n+1})}{\alpha^{2(p-1)(p^n-1)}} \\
&= -\left(\frac{(p-1)!\alpha^{p-1} + O(\alpha^p)}{\alpha^{2(p-1)}}\right)^{p^n-1} v_{n+1}\alpha^{p^n(p-1)} + \frac{\left((p-1)!\alpha^{p-1} + O(\alpha^p)\right)^{p^n-1}O(\alpha^{p^n(p-1)+1})}{\alpha^{2(p-1)(p^n-1)}} \\
&= -(p-1)!^{p^n-1}v_{n+1}\alpha^{p-1} + O(\alpha^p) \bmod \left(p, v_1, \dots, v_{n-1}, \frac{[p](\alpha)}{\alpha^{p^n}}\right)
\end{aligned}$$

Therefore, we have that:

$$(p-1)!^{2(p^n-1)}P(v_n) = -(p-1)!^{p^n-1}v_{n+1}\alpha^{p-1} + O(\alpha^p) \bmod \left(p, v_1, \dots, v_n, \frac{[p](\alpha)}{\alpha^{p^n}}\right)$$

Note that $(p-1)! = -1 \bmod p$, and raising it to $1-p^2$ will always be 1 (when $p > 2$, the exponent is even – otherwise, $1 = -1 \bmod 2$). We are left with:

$$P(v_n) = -v_{n+1}\alpha^{p-1} + O(\alpha^p) \bmod \left(p, v_1, \dots, v_n, \frac{[p](\alpha)}{\alpha^{p^n}}\right)$$

$\square$

PROOF OF THEOREM 3.2.1. We induct on i, where we are given the case $i = 0$ by assumption.

Otherwise, suppose the lemma is true up to $i-1$. Then, by Lemma 3.2.9, we have that:

$$P(v_{n+i-1}) = -\alpha^{p-1}v_{n+i} + O(\alpha^p) \bmod \left(p, v_1, \dots, v_{n+i-1}, \frac{[p](\alpha)}{\alpha^{p^{n+i-1}}}\right)$$

and by the inductive assumption, we also have that:

$$P(v_{n+i-1}) = 0 \bmod (p, v_1, \dots, v_{n-1})$$

In particular, we are able to compare these two:

$$P(v_{n+i-1}) = 0 = -\alpha^{p-1}v_{n+i} + O(\alpha^p) \bmod \left(p, v_1, \dots, v_{n+i-1}, \frac{[p](\alpha)}{\alpha^{p^{n+i-1}}}\right)$$

Also by our inductive hypothesis, for $0 < j < i$,

$$v_{n+j} = 0 \bmod (p, v_1, \dots, v_n)$$

so in fact modding out by $v_{n+1}, \dots, v_{n+i-1}$ is redundant after modding out by $p, v_1, \dots, v_n$. Hence in particular

$$0 = -\alpha^{p-1}v_{n+i} + O(\alpha^p) \bmod \left(p, v_1, \dots, v_n, \frac{[p](\alpha)}{\alpha^{p^{n+i-1}}}\right)$$

We have seen that

$$[p](\alpha) = v_{n+i}\alpha^{p^{n+i}} + O(\alpha^{p^{n+i}+1}) \bmod (p, v_1, \dots, v_{n+i-1})$$

Equivalently,

$$\frac{[p](\alpha)}{\alpha^{p^{n+i-1}}} = v_{n+i}\alpha^{p^{n+i}-p^{n+i-1}} + O(\alpha^{p^{n+i}-p^{n+i-1}+1}) \bmod (p, v_1, \dots, v_n)$$

Since we may write

$$\alpha^{p-1}v_{n+i} + O(\alpha^p) = g(\alpha)\frac{[p](\alpha)}{\alpha^{p^{n+i-1}}} \bmod (p, v_1, \dots, v_n)$$

for some power series g in α, and since α is not a zero divisor, comparing coefficients we see that $v_{n+i} = 0 \bmod (p, v_1, \dots, v_n)$, as required. $\square$

Theorem 3.2.10. *If $v_n = 0 \in \pi_* R/\mathcal{I}_n$, then for $i \geq n$,*

$$v_i = 0 \in \pi_*(R^{tC_p})/\mathcal{I}_n$$

Proof. The case of $i = n$ follows from the initial assumption and the canonical ring map $R \to R^{tC_p}$.

Otherwise, we have that $v_n = \sum_{i<n} d_i v_i$. By Lemma 3.2.9, we have:

$$P(v_n) = v_{n+1}(p-1)!^{p^n-1}\alpha^{p-1} + O(\alpha^p) \bmod (p, \dots, v_{n-1}, [p](\alpha)/\alpha^{p^n})$$

We then get:

$$P(v_n) - \sum_{i<n} P(d_i)P(v_i) = 0$$

By comparing coefficients, we get that:

$$v_{n+1}(p-1)!^{p^n-1} = \sum_{i<n} P(d_i)v_{i+1}(p-1)!^{p^n-1}$$

Because we work mod p, we are able to invert $(p-1)! = p-1 \bmod p$. Therefore, we get $v_{n+1} \in (p, \dots, v_n) = (p, \dots, v_{n-1})$.

We inductively apply the same reasoning for v_{n+i} knowing that $v_{n+i-1} \in (p, \dots, v_{n-1})$. $\square$

4. Blueshifting

Using all the structure of the power operations we have developed, we proceed to the proof of the main results. In particular, blueshifting always occurs at nilpotent degree 1 under regularity assumptions, for $\mathbb{E}_\infty$-MU-algebras.

Theorem 4.0.1 (Blueshift at nilpotent degree 1, Theorem A)**.** *Let R be an $\mathbb{E}_\infty$-MU-algebra such that $\mathcal{I}_n \coloneqq (p, v_1, \dots, v_{n-1})$ is a regular sequence in $\pi_*(R)$. Suppose that $v_n \in \mathcal{I}_n$ (a special case of R being $K(n)$-acyclic). Then, R^{tCp} is $K(n-1)$-acyclic.*

Proof. Assume that $v_n = 0 \bmod (p, \dots, v_{n-1})$. Our aim is to show that:

$$[p](\alpha) \text{ is a unit in } v_{n-1}^{-1}\pi_*(R^{tC_p})/(p, \dots, v_{n-2}) = v_{n-1}^{-1}\alpha^{-1}R_*[\![\alpha]\!]/([p](\alpha), p, \dots, v_{n-2})$$

We need to be careful about the order of operations. Namely, the ring $(\mathbb{Z}[1/2])[\![\alpha]\!]$ is distinct from $(\mathbb{Z}[\![\alpha]\!])[1/2]$ – the former has more polynomials than the latter.

As with the $p = 2$ case, we want to decompose this as:

$$[p](\alpha) = v_{n-1}\alpha^{p^{n-1}}(\text{unit}) \in \alpha^{-1}R_*[\![\alpha]\!]/\mathcal{I}_{n-1} \simeq \pi_*(R^{tC_p})$$

The way we will accomplish this is by showing all terms in $[p](\alpha)$ are divisible by v_{n-1}.

If we have any term $c\alpha^n$ in $[p](\alpha) \in R_*[\![\alpha]\!]/(p, \dots, v_{n-2})$, then for some m, $p^m > n$, and so $c = 0 \bmod p, v_1, \dots, v_{m-1}$. So c is some linear combination of $p, v_1, \dots, v_{m-1}$ in $R_*[\![\alpha]\!]$, and hence c is a linear combination of $v_{n-1}, \dots, v_{m-1}$ in $R_*[\![\alpha]\!]/(p, \dots, v_{n-2})$. But in fact, by Theorem 3.2.10, each of $v_n, v_{n+1}, \dots, v_{m-1} = 0 \bmod (p, v_1, \dots, v_{n-1})$. In other words, in $R_*[\![\alpha]\!]/(p, \dots, v_{n-2})$, each of $v_n, v_{n+1}, \dots, v_{m-1} = 0 \bmod v_{n-1}$. So indeed, c is divisible by v_{n-1} in $R_*[\![\alpha]\!]/(p, \dots, v_{n-2})$. We conclude, then, that the p-series is divisible by v_{n-1} and hence $v_{n-1}^{-1}\alpha^{-p^{n-1}}[p](\alpha)$ is a unit in $R_*[\![\alpha]\!]/(p, \dots, v_{n-2})$. It follows that $[p](\alpha) \in v_{n-1}^{-1}\alpha^{-1}R_*[\![\alpha]\!]/(p, v_1, \dots, v_{n-2})$ is a unit, and hence $v_{n-1}^{-1}R^{tC_p}/(v_0, \dots, v_{n-2})$ is the trivial ring. $\square$

In what follows, we will make a construction of a ring R related to $\pi_*\mathrm{MU}$. If R is realized as the homotopy ring of an $\mathbb{E}_\infty$-MU-algebra, it would disprove blueshift at $p = 2$, nilpotent degree 2, and height $n = 1$. If we want to invert the p-series, it is generally necessary to have the ideal $(v_1, v_2, \dots)^N$ vanish modulo p^N. We show that if it is topologically realized, its power operations are uniquely determined by those of MU, and we have that $(v_1, v_2, \dots)^N = 0 \bmod p$, but not mod p^N. We then show that $p^{-1}\alpha^{-1}R_*[\![\alpha]\!]/[p](\alpha)$ is nontrivial, and we conclude that if R can be written as the homotopy groups of an $\mathbb{E}_\infty$ ring spectrum E, then E provides a contradiction to Conjecture 1.0.1.

Construction 4.0.2. We take $R = \mathrm{MU}[b_{ij}]_{i \geq j \geq 1}/(x_i x_j - 2b_{ij})$. In order to define a power operation $\hat{P}$ on R, though, we need to determine $\hat{P}(b_{ij})$. Proceeding in the obvious way, we will define $\hat{P}(b_{ij}) = \frac{1}{2}(f \circ P)(x_i x_j)$. It is not obvious that $(f \circ P)(x_i x_j)$ is divisible by 2, however. But note: if we write

$$P(x_i) = \sum_{k=0}^{\infty} \gamma_k \alpha^k$$
$$P(x_j) = \sum_{k=0}^{\infty} \delta_k \alpha^k$$

we observe that, for grading reasons alone, each γ_k comes from $\mathrm{MU}_{2(k+i)}$, and $\delta_k \in \mathrm{MU}_{2(k+j)}$. Most importantly, this ensures that $\gamma_k, \delta_k \in (x_1, x_2, \dots)$. Certainly, then, we must have

$$\begin{aligned} 2\hat{P}(b_{ij}) &= P(x_i)P(x_j) \\ &= \left(\sum_{k=0}^{\infty} \gamma_k \alpha^k\right)\left(\sum_{k=0}^{\infty} \delta_k \alpha^k\right) \\ &= \sum_{k=0}^{\infty} \alpha^k \sum_{l=0}^{k} \gamma_l \delta_{k-l}. \end{aligned}$$

Each coefficient $\gamma_l \delta_{k-l}$ must, therefore, come from $(x_1, x_2, \dots)^2$. But by construction, this entire ideal is divisible by 2, so in fact it is no problem to define $\hat{P}(b_{ij})$. It suffices, then, to determine commutativity of the diagram

$$\begin{tikzcd} \mathrm{MU}_* \arrow[r, "P"] \arrow[d, "f"'] & \mathrm{MU}_*[\![\alpha]\!]/\langle 2\rangle(\alpha) \arrow[d] \\ R \arrow[r, "\hat{P}"] & R_*[\![\alpha]\!]/\langle 2\rangle(\alpha) \end{tikzcd}$$

which is an immediate consequence of commutativity of the obvious diagram

$$\begin{tikzcd} \mathrm{MU}_* \arrow[r, "P"] \arrow[d, "1"'] & \mathrm{MU}_*[\![\alpha]\!]/\langle 2\rangle(\alpha) \arrow[d] \\ \mathrm{MU}_* \arrow[r] & \mathrm{MU}_*[\![\alpha]\!]/\langle 2\rangle(\alpha) \end{tikzcd}$$

since MU injects into R. $\triangleleft$

Theorem 4.0.3 (Theorem B)**.** *At $p = 2$, R defined in Construction 4.0.2 gives a diagram of the following form:*

$$\begin{tikzcd} \mathrm{MU}_* \arrow[r, "P"] \arrow[d, "f"'] & \mathrm{MU}_*[\![\alpha]\!]/\langle 2\rangle(\alpha) \arrow[d] \\ R \arrow[r, "\hat{P}"] & R_*[\![\alpha]\!]/\langle 2\rangle(\alpha) \end{tikzcd}$$

and has $f(v_1)^2 = 0$ mod 2, but $2^{-1}R^{AtC_p}$ is not the trivial ring. Namely, if R were the homotopy ring of an $\mathbb{E}_\infty$-MU-algebra, then R would be $K(1)$-acyclic but R^{tC_p} would not be $K(0)$-acyclic – it is height 0 still.

Proof. We will show first that the 2-series is not a unit in $2^{-1}\alpha^{-1}R_*[\![\alpha]\!]$. So suppose for the sake of contradiction that some $\alpha^{-n}2^{-m}\sum h_i \alpha^i$ is a multiplicative inverse for $[2](\alpha)$. Because the 2-series has a constant term of 2 and MU_* has no 2-torsion, we can confidently say $[2](\alpha)$ is not a zero divisor. What this means is that $[2](\alpha)$ must have a unique multiplicative inverse. The multiplicative inverse of a power series comes from

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

and so it satisfies:

$$\left(1+\sum_{i\geq 1} g_i\alpha^i\right)^{-1} = 1-\left(\sum_{i\geq 1} g_i\alpha^i\right)+\left(\sum_{i\geq 1} g_i\alpha^i\right)^2-\left(\sum_{i\geq 1} g_i\alpha^i\right)^3+\cdots$$

Then, it must be that the inverse of the 2-series is

$$\alpha^{-n}2^{-m}\sum h_i\alpha^i = \alpha^{-1}2^{-1}\sum_{i\geq 0}(-1)^i\left(\frac{1}{2\alpha}[2](\alpha)-1\right)^i.$$

But then we claim the sum on the right-hand side has arbitrarily high powers of 2 in the denominator. To show this, we look at the sum modulo $(v_2, v_3, \dots)$, the term $\left(\frac{1}{2\alpha}[2](\alpha)-1\right)^i$ becomes

$$\left(\frac{1}{2}(-v_1\alpha+2v_1^2\alpha+\cdots)\right)^i.$$

So the α^j-term of the right-hand side is of the form

$$\left(\frac{v_1^j}{2^j}+\frac{\lambda_1 v_1^j}{2^{j-1}}+\cdots+\frac{\lambda_{j-1}v_1^j}{2}\right)\alpha^j,$$

for some integers λ_i, which we simplify to

$$\left(\frac{1+2\lambda_1+4\lambda_2+\cdots+2^{j-1}\lambda_{j-1}}{2^j}\right)v_1^j\alpha^j.$$

In particular, the numerator is odd and so we cannot cancel any powers of 2 from there. So the only powers of 2 that cancel are from $v_1^2 = 2b_{11}$, which removes at most $\frac{j}{2}$ from the denominator. Thus, there is an inevitable $2^{\frac{j}{2}}$ in the denominator of each h_j, which forces $m \to \infty$. Of course, this is impossible, so such a multiplicative inverse cannot exist. Because of this, it is not the case that $1 \in ([2](\alpha))$, and so 1 and 0 are distinct elements of $2^{-1}\alpha^{-1}R_*[\![\alpha]\!]/[2](\alpha)$. So indeed, $2^{-1}R^{tC_p}$ is not the trivial ring. □

On the level of algebra, all $\mathbb{E}_\infty$ orientations must induce a diagram of the form (Theorem 4.0.3) at $p = 2$. The counterexample produced only has the structure of a pure ring map equipped with a map $\hat{P}\colon R \to R_*[\![\alpha]\!]/\langle 2\rangle(\alpha)$. In order for this to be a true counterexample to Conjecture 1.0.1, we would need R to be the homotopy groups of some ring spectrum E, the map f to descend from an $\mathbb{E}_\infty$ ring spectrum map between MU and E, and $\hat{P}$ to be the power operation on the associated cohomology theory.

Remark 4.0.4. Therefore, while the case of $k = 2$ is still open, there is now a significant barrier to it being true. That is, any proof of the conjecture would have to also show that the map we constructed can never come from an $\mathbb{E}_\infty$ ring map, and that there are additional topological requirements constraining the kind of algebraic maps allowed in these power operation diagrams in a very subtle way. ◁

Remark 4.0.5. We have not checked, but believe similar constructions should yield counterexamples at higher primes and higher values of n. In particular, if an $\mathbb{E}_\infty$-MU-algebra had the homotopy ring:

$$\mathrm{MU}_*[b_i : i \in \mathbb{Z}^n]/(b_i - pv_{i_1}\cdots v_{i_n})$$

then it would be $K(n)$-acyclic at prime p and would have uniquely defined power operations. ◁

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