A TREE FORMULA FOR THE ELLIPSOIDAL SUPERPOTENTIAL OF COMPLEX PROJECTIVE SPACE

KYLER SIEGEL

Contents

Let M^{2n} be a closed symplectic manifold and $A \in H_2(M)$ a homology class. Given a tuple $\vec{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n_{>0}$, let

$$E(\vec{a}) = \{\pi \sum_{i=1}^{n} |z_i|^2/a_i \leqslant\} \subset \mathbb{C}^n$$

denote the symplectic ellipsoid with area factors $a_1, \ldots, a_n$, and let $\mathring{E}(\vec{a})$ denote its interior. We consider the compact symplectic cap $M_{\vec{a}} := M \setminus \iota(\mathring{E}(\varepsilon\vec{a}))$, where ι is a symplectic embedding $E(\varepsilon\vec{a}) \overset{s}{\hookrightarrow} M$ for some $\varepsilon > 0$ sufficiently small, with symplectic completion $\widehat{M}_{\vec{a}} = M_{\vec{a}} \cup (\mathbb{R}_{\leqslant 0} \times \partial M_{\vec{a}})$. The **ellipsoidal superpotential** $\mathbf{T}^{\vec{a}}_{M,A} \in \mathbb{Q}$ is the count of index zero finite energy pseudoholomorphic planes in $\widehat{M}_{\vec{a}}$. These counts play an important role in obstructing stabilized symplectic embeddings of ellipsoids into M, and in the case $\dim M = 4$ they also encode information about singular symplectic curves in M. A central question is to understand when $\mathbf{T}^{\vec{a}}_{M,A}$ is nonzero.

In the article [MS] a recursive formula is given for $\mathbf{T}^{\vec{a}}_{M,A}$ in terms stationary descendant Gromov–Witten invariants of M, which are readily computable e.g. if M is a smooth toric Fano variety. In this note, a companion to [MS], we give an explicit closed formula for $\mathbf{T}^{\vec{a}}_{M,A}$ in the case $M = \mathbb{CP}^2$ as a sum over trees. This is a step towards the goal of finding a positive formula of $\mathbf{T}^{\vec{a}}_{M,A}$.

Fix M^{2n} be a closed symplectic manifold, $A \in H_2(M)$ a homology class, and $a > 1$ an irrational number. We put $M_a := M \setminus \iota(\mathring{E}(\varepsilon, a\varepsilon))$

A broad goal for this paper is to develop purely combinatorial methods for approaching the problems discussed in §**??** and §**??**. Although Theorem **??** and its generalization make it possible to compute any given $\mathbf{T}^{\vec{a}}_{M,A}$ given enough computational power, its recursivity somewhat obscures its enumerative nature, and the

presence of terms of both positive and negative sign complicates efforts to prove general nonvanishing results. On the other hand, geometric considerations such as automatic transversality imply that $\mathbf{T}_{M,A}^{\vec{a}}$ is frequently nonnegative at least in dimension four, and this raises the possibility of a positive bijective formula underlying Theorem **??**. As a step in this direction, in §**??** we prove a closed formula for $\mathbf{T}_{M,A}^{\vec{a}}$ as a sum over trees, with "relatively few" negative terms.

Before stating the formula, we need some graph theoretic terminology (see also §1 and §2). For $d \in \mathbb{Z}_{\geqslant 1}$, let $\mathfrak{T}_d^{\mathrm{un}}$ denote the set of rooted trees with d (unordered) leaves and no bivalent vertices (i.e. no vertices with $|v| = 2$). For $T \in \mathfrak{T}_d^{\mathrm{un}}$, let $V_{\mathrm{in}}(T)$ denote its set of internal vertices, and let $V_{\mathrm{mov}}(T)$ denote its set of *movable vertices*, i.e. those $v \in V_{\mathrm{in}}(T)$ such that there are no internal vertices lying above v. Let $\ell(v)$ denote the *leaf number* of v, i.e. the number of leaf vertices lying above v. Finally, let $|\mathrm{Aut}(T)|$ denote the number of symmetries of T. See Figure 1 for a picture of $\mathfrak{T}_4^{\mathrm{un}}$.

For simplicity in this introduction we formulate just the special case $\mathbf{T}_d := \mathbf{T}_d^{\infty}$.

Theorem A (= Corollary 2.0.6)**.** *For $d \in \mathbb{Z}_{\geqslant 1}$ we have:*

$$\widetilde{\mathbf{T}}_d = 2^d \sum_{T \in \mathfrak{T}_d^{\mathrm{un}}} \frac{(-1)^{|V_{\mathrm{in}}(T) \setminus V_{\mathrm{mov}}(T)|}}{|\mathrm{Aut}(T)|} \cdot \prod_{v \in V_{\mathrm{in}}(T)} \left(\frac{(3\ell(v) - 1)!}{(3\ell(v) - |v| + 1)!} \right) \cdot \prod_{v \in V_{\mathrm{mov}}(T)} \left(2^{-\ell(v)} \binom{2\ell(v)}{\ell(v)} - 1 \right).$$

$$(0.0.1)$$

The proof incorporates techniques homological perturbation theory for $\mathcal{L}_\infty$ algebras, and the formula for general $a \in \mathbb{R}_{>0}$ involves the lattice path Γ^a. Note that the term $2^{-\ell(v)} \binom{2\ell(v)}{\ell(v)} - 1$ is always positive, so all negativity comes from the nonmovable vertices.

Example 0.0.1. In the case $d = 1$, there is just a single tree in $\mathfrak{T}_1^{\mathrm{un}}$, which has no internal vertices. Both products in (0.0.1) are vacuous, so we get $\widetilde{\mathbf{T}}_1 = 1$.

In the case $d = 2$, there is also just one tree in $\mathfrak{T}_2^{\mathrm{un}}$, which has one internal vertex, which is movable and has leaf number is 2. Then (0.0.1) gives

$$\widetilde{\mathbf{T}}_2 = 2^2 \cdot \frac{(-1)^0}{2} \cdot \frac{5!}{4!} \cdot \left(2^{-2} \binom{4}{2} - 1 \right) = 5.$$

$\Diamond$

1. Homological perturbation theory

In this subsection we formulate our main algebraic tool for inverting $\mathcal{L}_\infty$ homomorphisms. We include a simple proof for completeness, although the technique is well-known to experts.

Definition 1.0.1.

(1) *Let $\mathfrak{T}_k^{\mathrm{or}}$ denote the set of (isomorphism classes of) trees with $k + 1$ ordered univalent vertices, such that all other vertices have valency at least three.*

(2) *Similarly, let $\mathfrak{T}_k^{\mathrm{un}}$ denote the set of (isomorphism classes of) rooted trees with $k + 1$ unordered univalent vertices, such that all other vertices have valency at least three.*

For $T \in \mathfrak{T}_k^{\mathrm{or}}$, the first univalent vertex is called the **root vertex** and the remaining univalent vertices are called the **leaf vertices**. We denote the set of leaf vertices by $V_{\mathrm{leaf}}(T)$. Note that the ordering of the leaf vertices amounts to a bijection between $V_{\mathrm{leaf}}(T)$ and $\{1, \ldots, k\}$. We will also refer to the edges connected to the leaf vertices as the **leaf edges**, and the edge connected to the root vertex as the **root edge**. The set $\mathfrak{T}_k^{\mathrm{un}}$ is the quotient $\mathfrak{T}_k^{\mathrm{or}}/\Sigma_k$ by the natural symmetric group action on $\mathfrak{T}_k^{\mathrm{or}}$ which reorders the leaves. The stabilizer of a tree $T \in \mathfrak{T}_k^{\mathrm{or}}$ is the automorphism group $\mathrm{Aut}(T)$ of T.

We orient all edges of T towards the root, and we will say that v is "above" w if w lies on the oriented path from v to the root. We denote by $V_{\mathrm{in}}(T)$ the set of internal vertices of T, i.e. those having valency at least three. For $v \in V_{\mathrm{in}}(T)$, the **incoming** (resp. **outgoing**) edges of v are those edges of T which have v as an endpoint and are oriented towards (resp. away from) v.

An $\mathcal{L}_\infty$ algebra is called **abelian** if all of the $\mathcal{L}_\infty$ operations $\ell^1, \ell^2, \ell^3, \ldots$ vanish identically. Note that, with our degree conventions, any $\mathcal{L}_\infty$ algebra supported entirely in even degrees is automatically abelian. The following proposition is proved using standard techniques from homological perturbation theory. One can formulate a more general version without any grading restrictions, but here we state a simplified version which suffices for our purposes. Note that for any evenly graded $\mathcal{L}_\infty$ algebra V all of the signs $\Diamond(V, \sigma; v_1, \ldots, v_k)$ are automatically positive, and hence can be ignored.

Proposition 1.0.2. *Let V and W be evenly graded $\mathcal{L}_\infty$ algebras over $\mathbb{Q}$, and let $\Phi : V \to W$ be an $\mathcal{L}_\infty$ homomorphism such that the linear map $\Phi^1 : V \to W$ is invertible. Then there exists an $\mathcal{L}_\infty$ homomorphism $\Psi : W \to V$ such that*

$$\Psi \circ \Phi = \mathbb{1}_V \quad and \quad \Phi \circ \Psi = \mathbb{1}_W.$$

Moreover, Ψ is given explicitly as follows. We first set $\Psi^1 : W \to V$ to be the inverse of $\Phi^1 : V \to W$. For $k \geqslant 2$, $T \in \mathfrak{T}_k^{\mathrm{or}}$, and $w_1, \ldots, w_k \in W$, define $\Psi^T(w_1, \ldots, w_k)$ as follows. Start by labeling the ith leaf edge of T by $\Psi^1(w_i)$ for $i = 1, \ldots, k$. Recursively, for each internal vertex $v \in V_{\mathrm{in}}(T)$, say with j incoming edges, label the outgoing edge by the result after applying $\Psi^1 \circ \Phi^j$ to the corresponding labels of its incoming edges. We define $\Psi^T(w_1, \ldots, w_k)$ to be the resulting label on the root edge, and finally put

$$\Psi^k(w_1, \ldots, w_k) = \sum_{T \in \mathfrak{T}_k^{\mathrm{or}}} (-1)^{|V_{\mathrm{in}}(T)|} \Psi^T(w_1, \ldots, w_k).$$

Example 1.0.3. We have:

- $\Psi^2(x, y) = -\Psi^1 \Phi^2(\Psi^1(x), \Psi^1(y))$
- $\Psi^3(x, y, z) = -\Psi^1 \Phi^3(\Psi^1(x), \Psi^1(y), \Psi^1(z)) + \Psi^1 \Phi^2(\Psi^1(x), \Psi^1 \Phi^2(\Psi^1(y), \Psi^1(z))) + \Psi^1 \Phi^2(\Psi^1(y), \Psi^1 \Phi^2(\Psi^1(x), \Psi^1(z))) + \Psi^1 \Phi^2(\Psi^1(z), \Psi^1 \Phi^2(\Psi^1(x), \Psi^1(y)))$.

$\Diamond$

Proof. The relation $\Psi \circ \Phi = \mathbb{1}$ means that for $k \geqslant 2$ we must have $(\Psi \circ \Phi)^k(v_1, \ldots, v_k) = 0$ for any $v_1, \ldots, v_k \in V$, and this amounts to

$$\sum_{\substack{s \geqslant 1 \\ 1 \leqslant k_1 \leqslant \cdots \leqslant k_s \\ k_1 + \cdots + k_s = k}} \sum_{\sigma \in \overline{\mathrm{Sh}}(k_1, \ldots, k_s)} \Psi^s \circ (\Phi^{k_1} \odot \cdots \odot \Phi^{k_s})(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = 0,$$

or equivalently

$$\Psi^k(\Phi^1(v_1), \ldots, \Phi^1(v_k)) = -\sum_{\substack{1 \leqslant s \leqslant k-1 \\ 1 \leqslant k_1 \leqslant \cdots \leqslant k_s \\ k_1 + \cdots + k_s = k}} \sum_{\sigma \in \overline{\mathrm{Sh}}(k_1, \ldots, k_s)} \Psi^s \circ (\Phi^{k_1} \odot \cdots \odot \Phi^{k_s})(v_{\sigma(1)}, \ldots, v_{\sigma(k)}),$$

i.e. for any $w_1, \ldots, w_k \in W$ we must have

$$\Psi^k(w_1, \ldots, w_k) = -\sum_{\substack{1 \leqslant s \leqslant k-1 \\ 1 \leqslant k_1 \leqslant \cdots \leqslant k_s \\ k_1 + \cdots + k_s = k}} \sum_{\sigma \in \overline{\mathrm{Sh}}(k_1, \ldots, k_s)} \Psi^s \circ (\Phi^{k_1} \odot \cdots \odot \Phi^{k_s})(\Psi^1(w_{\sigma(1)}), \ldots, \Psi^1(w_{\sigma(k)})).$$

This is easily seen to agree with the definition of Ψ given in the statement of the proposition, which therefore necessarily satisfies $\Psi \circ \Phi = \mathbb{1}$.

As for the relation $\Phi \circ \Psi = \mathbb{1}$, we need

$$\sum_{\substack{s \geqslant 1 \\ 1 \leqslant k_1 \leqslant \cdots \leqslant k_s \\ k_1 + \cdots + k_s = k}} \sum_{\sigma \in \overline{\mathrm{Sh}}(k_1, \ldots, k_s)} \Phi^s \circ (\Psi^{k_1} \odot \cdots \odot \Psi^{k_s})(w_{\sigma(1)}, \ldots, w_{\sigma(k)}) = 0$$

for any $w_1, \ldots, w_k \in W$, or equivalently

$$\Phi^1(\Psi^k(w_1, \ldots, w_k)) = -\sum_{\substack{s \geqslant 2 \\ 1 \leqslant k_1 \leqslant \cdots \leqslant k_s \\ k_1 + \cdots + k_s = k}} \sum_{\sigma \in \overline{\mathrm{Sh}}(k_1, \ldots, k_s)} \Phi^s \circ (\Psi^{k_1} \odot \cdots \odot \Psi^{k_s})(w_{\sigma(1)}, \ldots, w_{\sigma(k)}),$$

i.e.

$$\Psi^k(w_1, \ldots, w_k) = -\sum_{\substack{s \geqslant 2 \\ 1 \leqslant k_1 \leqslant \cdots \leqslant k_s \\ k_1 + \cdots + k_s = k}} \sum_{\sigma \in \overline{\mathrm{Sh}}(k_1, \ldots, k_s)} \Psi^1 \circ \Phi^s \circ (\Psi^{k_1} \odot \cdots \odot \Psi^{k_s})(w_{\sigma(1)}, \ldots, w_{\sigma(k)}),$$

which is equivalent to our definition of Ψ^k. $\qquad\square$

2. Tree formula

We now prove an explicit formula for $\widetilde{\mathbf{T}}_d^a$ as a sum over trees. It is likely that similar techniques could be used to study curves in more general closed symplectic manifolds M, but for concreteness we restrict the discussion to $\mathbb{CP}^2$.

We first need some definitions. Recall that the sets of trees $\mathfrak{T}_k^{\mathrm{or}}$ and $\mathfrak{T}_k^{\mathrm{un}}$ with k ordered and unordered leaves respectively were defined in Definition 1.0.1.

Definition 2.0.1. *For a tree T in $\mathfrak{T}_k^{\mathrm{or}}$ or $\mathfrak{T}_k^{\mathrm{un}}$, an interval vertex $v \in V_{\mathrm{in}}(T)$ is* **movable** *if there are no internal vertices above it. We denote the set of movable vertices of T by $V_{\mathrm{mov}}(T) \subset V_{\mathrm{in}}(T)$.*
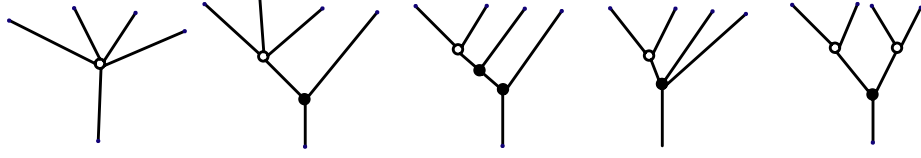
4

FIGURE 1. The trees comprising $\mathfrak{T}_4^{\mathrm{un}}$. The top vertices are the leaves and the bottom vertex is the root. The internal vertices are denoted with a large circle, which is open for a movable vertex and solid otherwise.

Definition 2.0.2. *For a vertex v of a tree T in $\mathfrak{T}_d^{\mathrm{or}}$ or $\mathfrak{T}_d^{\mathrm{un}}$, the **leaf number** $\ell(v)$ is the number of leaf vertices lying above v (including possibly v itself). In particular, $\ell(v) = 1$ if v is a leaf vertex.*

Example 2.0.3. Figure 1 shows the elements of $\mathfrak{T}_4^{\mathrm{un}}$ with the movable vertices labeled by open circles. $\diamond$

Theorem 2.0.4. *For any $d \in \mathbb{Z}_{\geqslant 1}$, we have:*

$$\widetilde{\mathbf{T}}_d^a = (\Gamma_2^a!)^d \sum_{T \in \mathfrak{T}_d^{\mathrm{un}}} \frac{(-1)^{|V_{\mathrm{in}}(T)|}}{|\mathrm{Aut}(T)|} \cdot \prod_{v \in V_{\mathrm{in}}(T)} \left(\frac{\Gamma_{3\ell(v)-1}^a!}{\left(\sum_{v' \to v} \Gamma_{3\ell(v')-1}^a\right)!} \right) \cdot \prod_{v \in V_{\mathrm{mov}}(T)} \left(1 - (\ell(v)!)^{-2} \frac{(\ell(v)\Gamma_2^a)!}{(\Gamma_2^a!)^{\ell(v)}} \right).$$

Here the sum $\sum\limits_{v' \to v}$ is over all vertices v' which are adjacent to v and lie strictly above it.

Remark 2.0.5. More explicitly, for $1 < a < 2$ we have

$$\widetilde{\mathbf{T}}_d^a = \sum_{T \in \mathfrak{T}_d^{\mathrm{un}}} \frac{(-1)^{|V_{\mathrm{in}}(T)|}}{|\mathrm{Aut}(T)|} \cdot \prod_{v \in V_{\mathrm{in}}(T)} \left(\frac{\Gamma_{3\ell(v)-1}^a!}{\left(\sum_{v' \to v} \Gamma_{3\ell(v')-1}^a\right)!} \right) \cdot \prod_{v \in V_{\mathrm{mov}}(T)} \left(1 - \frac{1}{\ell(v)!} \right),$$

while for $a \geqslant 2$ we have

$$\widetilde{\mathbf{T}}_d^a = 2^d \sum_{T \in \mathfrak{T}_d^{\mathrm{un}}} \frac{(-1)^{|V_{\mathrm{in}}(T)|}}{|\mathrm{Aut}(T)|} \cdot \prod_{v \in V_{\mathrm{in}}(T)} \left(\frac{\Gamma_{3\ell(v)-1}^a!}{\left(\sum_{v' \to v} \Gamma_{3\ell(v')-1}^a\right)!} \right) \cdot \prod_{v \in V_{\mathrm{mov}}(T)} \left(1 - 2^{-\ell(v)} \binom{2\ell(v)}{\ell(v)} \right)$$

$$= 2^d \sum_{T \in \mathfrak{T}_d^{\mathrm{un}}} \frac{(-1)^{|V_{\mathrm{in}}(T) \setminus V_{\mathrm{mov}}(T)|}}{|\mathrm{Aut}(T)|} \cdot \prod_{v \in V_{\mathrm{in}}(T)} \left(\frac{\Gamma_{3\ell(v)-1}^a!}{\left(\sum_{v' \to v} \Gamma_{3\ell(v')-1}^a\right)!} \right) \cdot \prod_{v \in V_{\mathrm{mov}}(T)} \left(2^{-\ell(v)} \binom{2\ell(v)}{\ell(v)} - 1 \right).$$

$\diamond$

Note that the term $2^{-\ell(v)}\binom{2\ell(v)}{\ell(v)} - 1$ is always positive, so all negativity in the above expression comes from the "unmovable" vertices $v \in V_{\mathrm{in}}(T) \setminus V_{\mathrm{mov}}(T)$. In the special case $a \gg 1$, we have $\Gamma_k^\infty = (k, 0)$, so we get the following formula for $\widetilde{\mathbf{T}}_d = (3d - 1)\mathbf{T}_d$:

Corollary 2.0.6. *For $d \in \mathbb{Z}_{\geqslant 1}$ we have:*

$$\widetilde{\mathbf{T}}_d = 2^d \sum_{T \in \mathfrak{T}_d^{\mathrm{un}}} \frac{(-1)^{|V_{\mathrm{in}}(T) \setminus V_{\mathrm{mov}}(T)|}}{|\mathrm{Aut}(T)|} \cdot \prod_{v \in V_{\mathrm{in}}(T)} \left(\frac{(3\ell(v) - 1)!}{(3\ell(v) - |v| + 1)!} \right) \cdot \prod_{v \in V_{\mathrm{mov}}(T)} \left(2^{-\ell(v)} \binom{2\ell(v)}{\ell(v)} - 1 \right).$$

$$(2.0.1)$$

Here $|v|$ denotes the valency of v, i.e. the number of incoming edges plus 1.

Example 2.0.7. To compute $\widetilde{\mathbf{T}}_3$ using Corollary 2.0.6, we put $\mathfrak{T}_3^{\mathrm{un}} = \{T_1, T_2\}$, where T_1 has a single internal vertex of valency 4 and T_2 has two internal vertices each of valency 3. We have $|\mathrm{Aut}(T_1)| = 3!$ and $|\mathrm{Aut}(T_2)| = 2!$. The internal vertex of T_1 is movable and has leaf number 3. The internal vertices of T_2 have leaf numbers $2, 3$, and the first is movable while the second is not. Plugging these into (2.0.1) gives $\widetilde{\mathbf{T}}_3 = 32$. $\diamond$

Remark 2.0.8. Curiously, we have precisely $\mathbf{T}_d = |\mathfrak{T}_d^{\mathrm{or}}|$ for $d = 1, 2, 3, 4$, while experimentally we have $\mathbf{T}_d < |\mathfrak{T}_d^{\mathrm{or}}|$ for $d \geqslant 5$. This perhaps suggests that there could be a positive formula for $\mathbf{T}_d$ which counts elements of $\mathfrak{T}_d$ with some additional conditions which only become relevant starting in degree 5. $\diamond$

Given $\vec{a} \in \mathbb{R}_{>0}^n$, recall that $\eta_{\vec{a}} : C_o \to C_{\vec{a}}$ is the $\mathcal{L}_\infty$ homomorphism inverse to the stationary descendant map $\epsilon_{\vec{a}} : C_{\vec{a}} \to C_o$, and the explicit construction of $\eta_{\vec{a}}$ is provided by Proposition 1.0.2. In particular, $\eta_{\vec{a}}^1$ is the linear inverse of $\epsilon_{\vec{a}}^1$, i.e. for $k \in \mathbb{Z}_{\geqslant 1}$ we have

$$\epsilon_{\vec{a}}^1(\mathfrak{o}_k^{\vec{a}}) = \frac{\mathfrak{q}_k}{(\Gamma_k^{\vec{a}})!} \quad \text{and} \quad \eta_{\vec{a}}^1(\mathfrak{q}_k) = (\Gamma_k^{\vec{a}})! \mathfrak{o}_k^{\vec{a}}.$$

Applying $\pi_1 \circ \widehat{\eta}_{\vec{a}}$ to both sides of (**??**) gives

$$\pi_1(\exp_A(\mathfrak{m}_M^{\vec{a}})) = (\pi_1 \circ \widehat{\eta}_{\vec{a}})(\exp_A(\mathfrak{m}_M^o)),$$

i.e.

$$\mathfrak{m}_{M,A}^{\vec{a}} = \sum_{\substack{k \geqslant 1 \\ A_1, \ldots, A_k \in H_2(M) \\ A_1 + \cdots + A_k = A}} \tfrac{1}{k!} \eta_{\vec{a}}^k(\mathfrak{m}_{M,A_1}^o, \ldots, \mathfrak{m}_{M,A_k}^o)$$

$$= \sum_{\substack{k \geqslant 1 \\ A_1, \ldots, A_k \in H_2(M) \\ A_1 + \cdots + A_k = A}} \tfrac{1}{k!} \sum_{T \in \mathfrak{T}_k^{\mathrm{or}}} (-1)^{|V_{\mathrm{in}}(T)|} \eta_{\vec{a}}^T(\mathfrak{m}_{M,A_1}^o, \ldots, \mathfrak{m}_{M,A_k}^o).$$

Using $\mathfrak{m}_{M,A}^{\vec{a}} = \widetilde{\mathbf{T}}_{M,A}^{\vec{a}} \cdot \mathfrak{o}_{c_1(A)-1}^{\vec{a}}$ and $\mathfrak{m}_{M,A}^o = N_{M,A}\langle\!\langle \psi^{c_1(A)-2} pt \rangle\!\rangle \cdot \mathfrak{q}_{c_1(A)-1}$, we can rewrite the above as

$$\widetilde{\mathbf{T}}^{\vec{a}}_{M,A} \cdot \mathfrak{o}^{\vec{a}}_{c_1(A)-1}$$

$$= \sum_{\substack{k \geqslant 1 \\ A_1,\dots,A_k \in H_2(M) \\ A_1+\cdots+A_k=A}} \frac{1}{k!} \left(\prod_{s=1}^{k} N_{M,A_s} \ll \psi^{c_1(A_s)-2} pt \gg \right) \sum_{T \in \mathfrak{T}^{\mathrm{or}}_k} (-1)^{|V_{\mathrm{in}}(T)|} \eta^T_{\vec{a}} \left(\mathfrak{q}_{c_1(A_1)-1}, \dots, \mathfrak{q}_{c_1(A_k)-1} \right).$$

$$(2.0.2)$$

We now specialize to the case $M = \mathbb{CP}^2$ and $A = d[L]$, so that equation (2.0.2) becomes

$$\widetilde{\mathbf{T}}^a_d \cdot \mathfrak{o}^a_{3d-1} = \sum_{\substack{k \geqslant 1 \\ d_1,\dots,d_k \in \mathbb{Z}_{\geqslant 1} \\ d_1+\cdots+d_k=d}} \frac{1}{k!(d_1!)^3\cdots(d_k!)^3} \sum_{T \in \mathfrak{T}^{\mathrm{or}}_k} (-1)^{|V_{\mathrm{in}}(T)|} \eta^T_a \left(\mathfrak{q}_{3d_1-1}, \dots, \mathfrak{q}_{3d_k-1} \right). \quad (2.0.3)$$

Our goal is to rewrite the above as a sum over trees with exactly d leaves. We will introduce relevant notation as we need it.

Notation 2.0.9. *Given $d_1, \dots, d_k \in \mathbb{Z}_{\geqslant 1}$ with $d_1 + \cdots + d_k = d$, let $\mathcal{P}(d_1, \dots, d_k)$ denote the set of surjections $h : \{1, \dots, d\} \twoheadrightarrow \{1, \dots, k\}$ such that $|h^{-1}(i)| = d_i$ for $i = 1, \dots, k$.*

Then we can rewrite (2.0.3) temporarily more redundantly as

$$\widetilde{\mathbf{T}}^a_d \cdot \mathfrak{o}^a_{3d-1} = \sum_{\substack{k \geqslant 1 \\ d_1,\dots,d_k \in \mathbb{Z}_{\geqslant 1} \\ d_1+\cdots+d_k=d}} \frac{1}{k!(d_1!)^3\cdots(d_k!)^3} \binom{d}{d_1,\dots,d_k}^{-1} \sum_{h \in \mathcal{P}(d_1,\dots,d_k)} \sum_{T \in \mathfrak{T}^{\mathrm{or}}_k} (-1)^{|V_{\mathrm{in}}(T)|} \eta^T_a \left(\mathfrak{q}_{3d_1-1}, \dots, \mathfrak{q}_{3d_k-1} \right).$$

Notation 2.0.10.

(1) Put

$$\mathfrak{T}^{\mathrm{or}}_k(\{1,\dots,d\}) := \{ (T,h) \mid T \in \mathfrak{T}^{\mathrm{or}}_k, \ h : \{1,\dots,d\} \twoheadrightarrow \{1,\dots,k\} \},$$

i.e. $\mathfrak{T}^{\mathrm{or}}_k(\{1,\dots,d\})$ is the set of trees $T \in \mathfrak{T}^{\mathrm{or}}_k$ equipped with a partition of $\{1,\dots,d\}$ into k parts labeled by the leaves of T.

(2) Given $(T,h) \in \mathfrak{T}^{\mathrm{or}}_k(\{1,\dots,d\})$, put

$$C_{(T,h)} := \frac{1}{(d_1!)^2 \cdots (d_k!)^2} \quad \text{and} \quad \epsilon^{(T,h)}_a(\mathfrak{q}_\bullet) := \epsilon^T_a \left(\mathfrak{q}_{3d_1-1}, \dots, \mathfrak{q}_{3d_k-1} \right),$$

where $d_i := |h^{-1}(i)|$ for $i = 1, \dots, k$.

We then have

$$\widetilde{\mathbf{T}}^a_d \cdot \mathfrak{o}^a_{3d-1} = \sum_{k \geqslant 1} \sum_{(T,h) \in \mathfrak{T}^{\mathrm{or}}_k(\{1,\dots,d\})} \frac{1}{k!d!} \cdot C_{(T,h)} \cdot (-1)^{|V_{\mathrm{in}}(T)|} \eta^{(T,h)}_a(\mathfrak{q}_\bullet).$$

Notation 2.0.11. *Put*

$$\mathfrak{T}^{\mathrm{un}}_k(\{1,\dots,d\}) = \{ (T,h) \mid T \in \mathfrak{T}^{\mathrm{un}}_k, \ h : \{1,\dots,d\} \twoheadrightarrow V_{\mathrm{leaf}}(T) \},$$

i.e. an element of $\mathfrak{T}_k^{\mathrm{un}}(\{1, \ldots, d\})$ *is a tree* $T \in \mathfrak{T}_k^{\mathrm{un}}$ *with* k *unordered leaves equipped with a partition of* $\{1, \ldots, d\}$ *into* k *parts labeled by the leaves of* T. *Put also*

$$\mathfrak{T}^{\mathrm{un}}(\{1, \ldots, d\}) := \bigcup_{1 \leqslant k \leqslant d} \mathfrak{T}_k^{\mathrm{un}}(\{1, \ldots, d\}).$$

Note that the symmetric group Σ_k acts freely on $\mathfrak{T}_k^{\mathrm{or}}(\{1, \ldots, d\})$ via $\sigma \cdot (T, h) = (T', \sigma \circ h)$, where T' is given by reordering the leaves of T according to the permutation $\sigma \in \Sigma_k$, and the quotient is identified with $\mathfrak{T}_k^{\mathrm{un}}(\{1, \ldots, d\})$. We then have

$$\widetilde{\mathbf{T}}_d^a \cdot \mathfrak{o}_{3d-1}^a = \sum_{(T,h) \in \mathfrak{T}^{\mathrm{un}}(\{1,\ldots,d\})} \frac{1}{d!} \cdot C_{(T,h)} \cdot (-1)^{|V_{\mathrm{in}}(T)|} \eta_a^{(T,h)}(\mathfrak{q}_\bullet). \tag{2.0.4}$$

Observe that there is a natural map

$$\zeta : \{(T, \mathcal{S}) \mid T \in \mathfrak{T}_d^{\mathrm{or}}, \mathcal{S} \subset V_{\mathrm{mov}}(T)\} \to \mathfrak{T}^{\mathrm{un}}(\{1, \ldots, d\}),$$

defined as follows. Given $(T, \mathcal{S})$, let T' be the tree obtained by removing all edges and vertices strictly above v, for each $v \in \mathcal{S}$. Note that each $v \in \mathcal{S}$ becomes a leaf in T'. We define a surjection $h : \{1, \ldots, d\} \twoheadrightarrow V_{\mathrm{leaf}}(T')$ in such a way that

- if $v \in V_{\mathrm{leaf}}(T')$ corresponds to a leaf of T, then $h^{-1}(v)$ is the original label of v
- for $v \in \mathcal{S}$, $h^{-1}(v)$ is the set of labels of leaves lying above v in T.

We put $\zeta(T, \mathcal{S}) = (T', h)$.

In fact, ζ is a bijection, with inverse map

$$\zeta^{-1} : \mathfrak{T}^{\mathrm{un}}(\{1, \ldots, d\}) \to \{(T, \mathcal{S}) \mid T \in \mathfrak{T}_d^{\mathrm{or}}, \mathcal{S} \subset V_{\mathrm{mov}}(T)\}$$

described as follows. Given $(T, h) \in \mathfrak{T}^{\mathrm{un}}(\{1, \ldots, d\})$, for each $v \in V_{\mathrm{leaf}}(T)$ such that $|h^{-1}(v)| \geqslant 2$ we add $|h^{-1}(v)|$ new leaf vertices, each joined to v by a new leaf edge. By construction the resulting tree T' comes with a natural bijection $\{1, \ldots, d\} \xrightarrow{\sim} V_{\mathrm{leaf}}(T')$, i.e. $T' \in \mathfrak{T}_d^{\mathrm{or}}$. Also, each $v \in V_{\mathrm{leaf}}(T)$ satisfying $|h^{-1}(v)| \geqslant 2$ naturally corresponds to a movable vertex in T', and we denote the set of these by $\mathcal{S} \subset V_{\mathrm{mov}}(T')$. Then we have $\zeta^{-1}(T, h) = (T', \mathcal{S})$.

Using this bijection, we rewrite (2.0.4) as:

$$\widetilde{\mathbf{T}}_d^a \cdot \mathfrak{o}_{3d-1}^a = \frac{1}{d!} \sum_{T \in \mathfrak{T}_d^{\mathrm{or}}} \sum_{\mathcal{S} \subset V_{\mathrm{mov}}(T)} C_{\zeta(T,\mathcal{S})} \cdot (-1)^{|V_{\mathrm{in}}(\zeta(T,\mathcal{S}))|} \eta_a^{\zeta(T,\mathcal{S})}(\mathfrak{q}_\bullet). \tag{2.0.5}$$

It will be convenient to extend the notion of leaf number to elements of $\mathfrak{T}^{\mathrm{un}}(\{1, \ldots, d\})$ as follows. For $(T, h) \in \mathfrak{T}^{\mathrm{un}}(\{1, \ldots, d\})$, we define the leaf number $\ell(v)$ to be the cardinality of $\bigcup_w |h^{-1}(w)|$, where the union is over all leaf vertices w lying above v (including possibly v itself).

Proof of Theorem 2.0.4. According to Proposition 1.0.2, $\eta_a^T(\mathfrak{q}_{3d_1-1}, \ldots, \mathfrak{q}_{3d_k-1})$ is computed as follows. Recall that $T \in \mathfrak{T}_k^{\mathrm{or}}$ has k ordered leaves. For $i = 1, \ldots, k$, we label the ith leaf edge by $\eta_a^1(\mathfrak{q}_{3d_i-1}) = (\Gamma_{3d_i-1}^a)! \, \mathfrak{o}_{3d_i-1}^a$. We then iteratively label each edge of T, say with source vertex v, by the result of applying $\eta_a^1 \circ \epsilon_a^j$ to the labels of the incoming edges of v (here the valency of v is $j + 1$).

Then for $T \in \mathfrak{T}_d^{\mathrm{or}}$ and $\mathcal{S} \subset V_{\mathrm{mov}}(T)$, with $\zeta(T,\mathcal{S}) \in \mathfrak{T}^{\mathrm{un}}(\{1,\ldots,d\})$ as defined above we have:

$$\eta_a^{\zeta(T,\mathcal{S})}(\mathfrak{q}_\bullet) = \prod_{v \in V_{\mathrm{leaf}}(\zeta(T,\mathcal{S}))} \left(\Gamma_{3\ell(v)-1}^a\right)! \cdot \frac{\prod\limits_{v \in V_{\mathrm{in}}(\zeta(T,\mathcal{S}))} \left(\Gamma_{3\ell(v)-1}^a\right)!}{\prod\limits_{v \in V_{\mathrm{in}}(\zeta(T,\mathcal{S}))} \left(\sum\limits_{v' \to v} \Gamma_{3\ell(v')-1}^a\right)!} \cdot \mathfrak{o}_{3d-1}^a. \quad (2.0.6)$$

Observe that in the case $\mathcal{S} = \varnothing$, $\zeta(T,\varnothing)$ is simply T itself but viewed as an element of $\mathfrak{T}^{\mathrm{un}}(\{1,\ldots,d\})$, and we have

$$\eta_a^{\zeta(T,\varnothing)}(\mathfrak{q}_\bullet) = \eta_a^T(\underbrace{\mathfrak{q}_2,\ldots,\mathfrak{q}_2}_{d}) = \prod_{v \in V_{\mathrm{leaf}}(T)} \left(\Gamma_{3\ell(v)-1}^a\right)! \cdot \frac{\prod\limits_{v \in V_{\mathrm{in}}(T)} \left(\Gamma_{3\ell(v)-1}^a\right)!}{\prod\limits_{v \in V_{\mathrm{in}}(T)} \left(\sum\limits_{v' \to v} \Gamma_{3\ell(v')-1}^a\right)!} \cdot \mathfrak{o}_{3d-1}^a,$$
$$(2.0.7)$$

where

$$\Gamma_2^a! = \begin{cases} 1 & \text{if} \quad 1 < a < 2 \\ 2 & \text{if} \quad a \geqslant 2. \end{cases}$$

We have natural a identification $V_{\mathrm{in}}(T) \approx V_{\mathrm{in}}(\zeta(T,\mathcal{S})) \cup \mathcal{S}$, and the leaves of $\zeta(T,\mathcal{S})$ correspond to (a) leaves of T not lying above any $v \in \mathcal{S}$ and (b) elements of $\mathcal{S}$. Then, comparing (2.0.6) and (2.0.7), we have

$$\eta_a^{\zeta(T,\mathcal{S})}(\mathfrak{q}_\bullet) = \eta_a^{\zeta(T,\varnothing)}(\mathfrak{q}_\bullet) \cdot \frac{\prod\limits_{v \in \mathcal{S}} \left(\Gamma_{3\ell(v)-1}^a\right)!}{\prod\limits_{v \in \mathcal{S}} (\Gamma_2^a!)^{\ell(v)}} \cdot \frac{\prod\limits_{v \in \mathcal{S}} \left(\sum\limits_{v' \to v} \Gamma_{3\ell(v')-1}^a\right)!}{\prod\limits_{v \in \mathcal{S}} \left(\Gamma_{3\ell(v)-1}^a\right)!}$$

$$= \eta_a^{\zeta(T,\varnothing)}(\mathfrak{q}_\bullet) \cdot \prod_{v \in \mathcal{S}} \frac{(\ell(v)\Gamma_2^a)!}{(\Gamma_2^a!)^{\ell(v)}}.$$

Note that we have $C_{\zeta(T,\mathcal{S})} = \left(\prod\limits_{v \in \mathcal{S}} \ell(v)!\right)^{-2}$, and hence

$$\widetilde{\mathbf{T}}_d^a \cdot \mathfrak{o}_{3d-1}^a = \frac{1}{d!} \sum_{T \in \mathfrak{T}_d^{\mathrm{or}}} (-1)^{|V_{\mathrm{in}}(T)|} \sum_{\mathcal{S} \subset V_{\mathrm{mov}}(T)} C_{\zeta(T,\mathcal{S})} \cdot (-1)^{|\mathcal{S}|} \cdot \eta_a^{\zeta(T,\mathcal{S})}(\mathfrak{q}_\bullet)$$

$$= \frac{1}{d!} \sum_{T \in \mathfrak{T}_d^{\mathrm{or}}} (-1)^{|V_{\mathrm{in}}(T)|} \cdot \eta_a^{\zeta(T,\varnothing)}(\mathfrak{q}_\bullet) \sum_{\mathcal{S} \subset V_{\mathrm{mov}}(T)} \left(\prod_{v \in \mathcal{S}} \ell(v)!\right)^{-2} \cdot (-1)^{|\mathcal{S}|} \cdot \prod_{v \in \mathcal{S}} \frac{(\ell(v)\Gamma_2^a)!}{(\Gamma_2^a!)^{\ell(v)}}.$$

We also have

$$\eta_a^{\zeta(T,\varnothing)}(\mathfrak{q}_\bullet) = (\Gamma_2^a!)^d \cdot \prod_{v \in V_{\mathrm{in}}(T)} \frac{\left(\Gamma_{3\ell(v)-1}^a\right)!}{\left(\sum\limits_{v' \to v} \Gamma_{3\ell(v')-1}^a\right)!} \cdot \mathfrak{o}_{3d-1}^a,$$

9

and hence

$$\widetilde{\mathbf{T}}_d^a = \frac{1}{d!}(\Gamma_2^a!)^d \sum_{T \in \mathfrak{T}_d^{\mathrm{or}}} (-1)^{|V_{\mathrm{in}}(T)|} \cdot \prod_{v \in V_{\mathrm{in}}(T)} \frac{\left(\Gamma_{3\ell(v)-1}^a\right)!}{\left(\sum\limits_{v' \to v} \Gamma_{3\ell(v')-1}^a\right)!} \cdot \sum_{\mathcal{S} \subset V_{\mathrm{mov}}(T)} (-1)^{|\mathcal{S}|} \cdot \prod_{v \in \mathcal{S}} (\ell(v)!)^{-2} \frac{(\ell(v)\Gamma_2^a)!}{(\Gamma_2^a!)^{\ell(v)}}$$

$$= \frac{1}{d!}(\Gamma_2^a!)^d \sum_{T \in \mathfrak{T}_d^{\mathrm{or}}} (-1)^{|V_{\mathrm{in}}(T)|} \cdot \prod_{v \in V_{\mathrm{in}}(T)} \frac{\left(\Gamma_{3\ell(v)-1}^a\right)!}{\left(\sum\limits_{v' \to v} \Gamma_{3\ell(v')-1}^a\right)!} \cdot \prod_{v \in V_{\mathrm{mov}}(T)} \left(1 - (\ell(v)!)^{-2} \frac{(\ell(v)\Gamma_2^a)!}{(\Gamma_2^a!)^{\ell(v)}}\right).$$

Finally, we can write this as a sum over $\mathfrak{T}_d^{\mathrm{un}}$ by recalling that there is a natural forgetful map $\mathfrak{T}_d^{\mathrm{or}} \to \mathfrak{T}_d^{\mathrm{un}}$ and the fiber of any $T \in \mathfrak{T}_d^{\mathrm{un}}$ has cardinality $\frac{d!}{|\mathrm{Aut}(T)|}$. $\qquad\square$

References

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