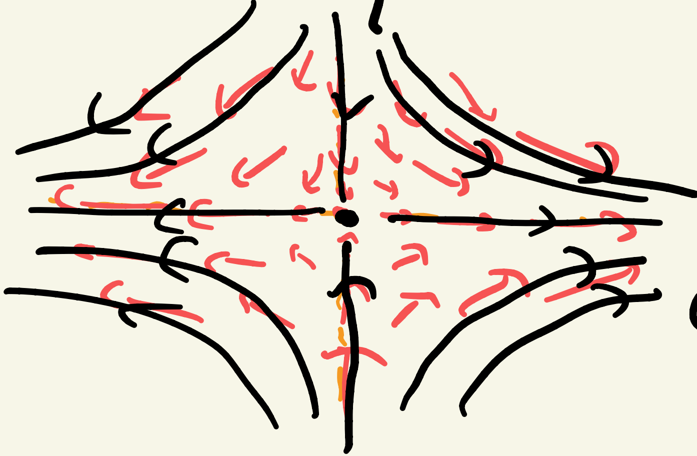
Lecture 23

Ex: $\begin{cases} x_1'(t) = 3x_1(t) \\ x_2'(t) = -4x_2(t) \end{cases} \rightsquigarrow \begin{aligned} x_1(t) &= C_1 e^{3t} \\ x_2(t) &= C_2 e^{-4t} \end{aligned}$



direction field

$\leftarrow$ at each pt (x,y) draw the vector $(3x, -4y)$

Given $x_1(0), x_2(0)$, soln is
$(x_1(t), x_2(t)) = (x_1(0)e^{3t}, x_2(0)e^{-4t})$

"phase portrait": draw a sampling of solution curves

Ex:

$$\begin{cases} x_1'(t) = -x_1/2 + 7x_2/2 \\ x_2'(t) = 7/2\, x_1 - x_2/2 \end{cases} \qquad \vec{x} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

$$\vec{x}'(t) = A\vec{x}(t), \qquad A = \begin{pmatrix} -1/2 & 7/2 \\ 7/2 & -1/2 \end{pmatrix}.$$

Ansatz $\vec{x}(t) = e^{rt}\vec{v}$. $\xrightarrow{\text{need}}$ r eval of A, $\vec{v}$ evec of A.

char poly $|A - \lambda I| = \begin{vmatrix} -1/2-\lambda & 7/2 \\ 7/2 & -1/2-\lambda \end{vmatrix} = (1/2+\lambda)^2 - \frac{49}{4}$

$= \lambda^2 + \lambda + 1/4 - 49/4$

$= \lambda^2 + \lambda - 12$

$= (\lambda + 4)(\lambda - 3)$

evals: $3, -4$

evecs:

$\lambda = 3$: $\begin{pmatrix} -1/2 & 7/2 \\ 7/2 & -1/2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 3\begin{pmatrix} a \\ b \end{pmatrix}$

$\rightsquigarrow -1/2\, a + 7/2\, b = 3a$

$\iff 7/2\, b = 7/2\, a$ evec $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

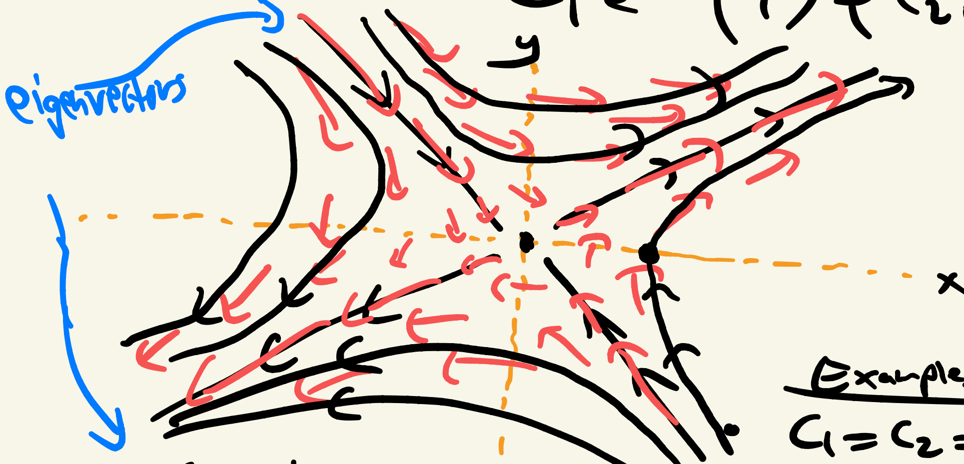
$\iff a = b$

$\lambda = -4$ $\quad \begin{pmatrix} -1/2 & 7/2 \\ 7/2 & -1/2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -4\begin{pmatrix} a \\ b \end{pmatrix} \longrightarrow -\frac{1}{2}a + \frac{7}{2}b = -4a$

$$-a + 7b = -8a$$
$$7b = -7a$$
$$b = -a$$

evec $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

General soln: $\quad C_1 e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-4t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$



direction field: arrow at (x,y) pts in direction $\left(-\frac{1}{2}x + \frac{7}{2}y,\ \frac{7}{2}x - \frac{1}{2}y\right)$

Examples of solutions:

- $C_1 = C_2 = 0$ $\quad (x(t), y(t)) \equiv (0,0)$
- $C_1 = 1, C_2 = 0$ $\quad (x(t), y(t)) = \underbrace{(e^{3t}, e^{3t})}_{\text{lies on line } y=x}$
- $C_1 = 0, C_2 = 1$ $\quad (x(t), y(t)) = (e^{-4t}, -e^{-4t})$

- $C_1 = C_2 = 1$

$(x(t), y(t)) = (e^{3t} + e^{-4t},\ e^{3t} - e^{-4t})$

at $t=0$: $(2, 0)$

This picture is called a "saddle point"

(case one pos, one neg eval)

Ex: $\vec{x}'(t) = \underbrace{\begin{pmatrix} -1/2 & 1 \\ -1 & -1/2 \end{pmatrix}}_{A} \vec{x}(t).$

evals:

$$\begin{vmatrix} -1/2-\lambda & 1 \\ -1 & -1/2-\lambda \end{vmatrix} = (\lambda+1/2)^2 + 1 = \lambda^2 + \lambda + 5/4$$

$$\lambda = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 5/4}}{2}$$

$$= \frac{-1 \pm \sqrt{-4}}{2} = \frac{-1 \pm 2i}{2}$$

$$= -1/2 \pm i$$

evecs:

$\lambda = -1/2 + i$

$$\begin{pmatrix} -1/2 & 1 \\ -1 & -1/2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = (-1/2 + i) \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{cases} -1/2\, a + b = (-1/2 + i) a \\ -a - 1/2\, b = (-1/2 + i) b \end{cases} \implies b = ia$$

can take evec $\begin{pmatrix} 1 \\ i \end{pmatrix}$

$\lambda = -1/2 - i$.

Note: if $A\vec{v} = (\alpha + i\beta)\vec{v}$

then $\overline{A\vec{v}} = \overline{(\alpha+i\beta)\vec{v}}$ ← take cpx conj of everything

$\Leftrightarrow A\overline{\vec{v}} = (\alpha - i\beta)\overline{\vec{v}}$

<u>Upshot</u>: $\begin{pmatrix} 1 \\ -i \end{pmatrix}$ is an evec w/ eval $-1/2 - i$.

So "general solution" $\vec{x}(t) = C_1 e^{(-1/2+i)t}\begin{pmatrix} 1 \\ i \end{pmatrix} + C_2 e^{(-1/2-i)t}\begin{pmatrix} 1 \\ -i \end{pmatrix}$

Not real-valued, but can take real and imaginary parts.

Start w/ $e^{(-1/2+i)t}\begin{pmatrix} 1 \\ i \end{pmatrix} = e^{-t/2} e^{it}\begin{pmatrix} 1 \\ i \end{pmatrix}$

(vector valued function)

$$= e^{-t/2}\left(\cos(t) + i\sin(t)\right)\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$= e^{-t/2}\begin{pmatrix} \cos(t) + i\sin(t) \\ i\cos(t) - \sin(t) \end{pmatrix}$$

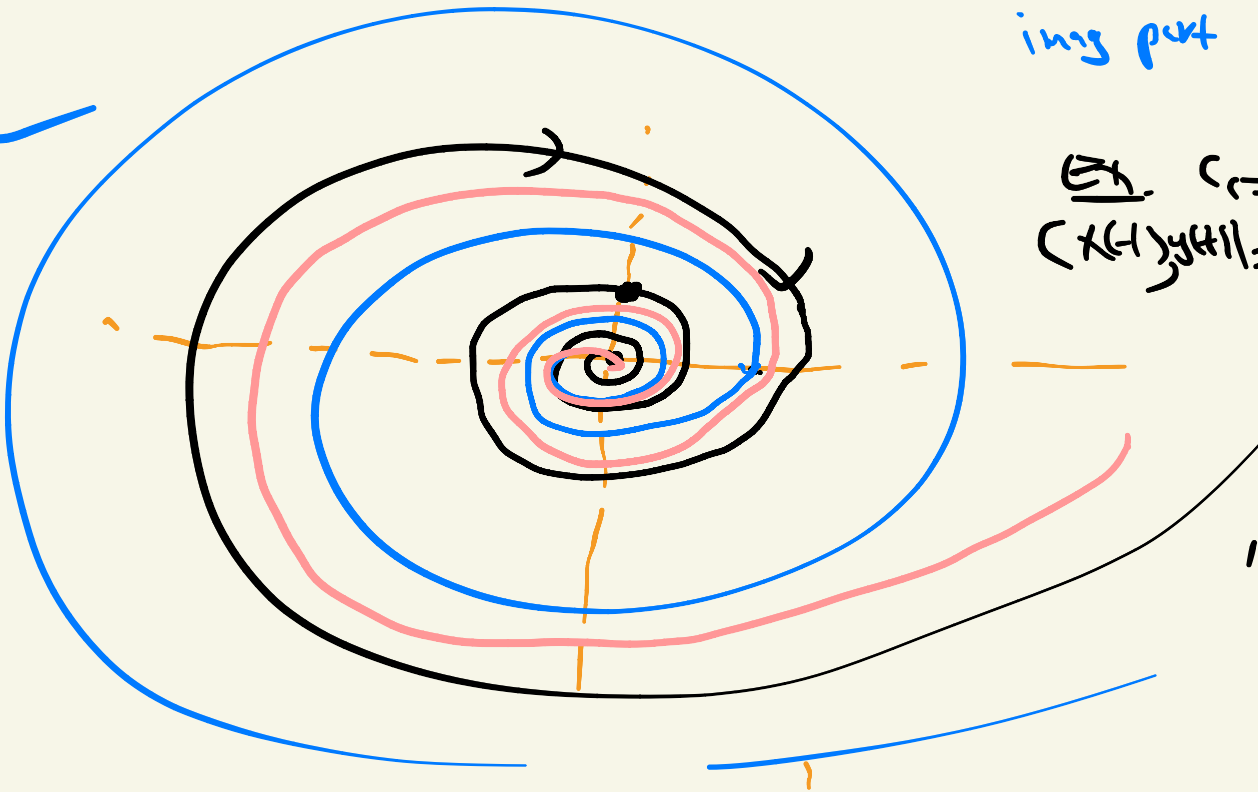
real part: $e^{-t/2}\begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix}$ imag part: $e^{-t/2}\begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}$

Real-valued general solution: $C_1 e^{-t/2}\begin{pmatrix}\cos(t)\\ -\sin(t)\end{pmatrix} + C_2 e^{-t/2}\begin{pmatrix}\sin(t)\\ \cos(t)\end{pmatrix}$

[Rmk: starting $e^{(-1/2-i)t}\begin{pmatrix}1\\ -i\end{pmatrix} \longrightarrow$ real part $e^{-t/2}\begin{pmatrix}\cos(t)\\ -\sin(t)\end{pmatrix}$, imag part $-e^{-t/2}\begin{pmatrix}\sin(t)\\ \cos(t)\end{pmatrix}$]

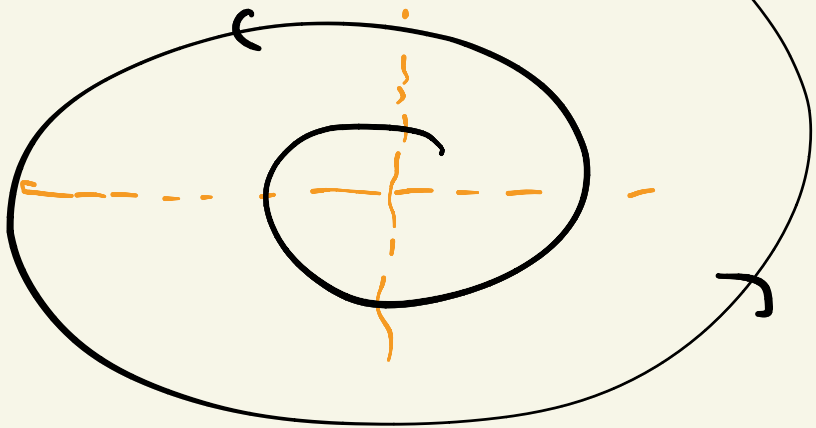
Ex. $C_1 = 0$, $C_2 = 1$

$(x(t), y(t)) = (e^{-t/2}\sin t, e^{-t/2}\cos(t))$



"spiral point"

Q: Is there a system $\vec{x}'(t) = A\vec{x}(t)$ w/ solutions



~~If~~ A has evals $\alpha \pm i\beta$, $\alpha > 0$, get this picture.

Ex: $\vec{x}'(t) = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix} \vec{x}(t)$

evals: $\begin{vmatrix} -3-\lambda & \sqrt{2} \\ \sqrt{2} & -2-\lambda \end{vmatrix} \Rightarrow (\lambda+3)(\lambda+2) - 2$

$= \lambda^2 + 5\lambda + 4$

$= (\lambda+1)(\lambda+4)$

$\lambda = -1, -4$

evecs: $\lambda = -1$

$$\begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -\begin{pmatrix} a \\ b \end{pmatrix} \longrightarrow \begin{aligned} -3a + \sqrt{2}b &= -a \\ -2a &= -\sqrt{2}b \\ b &= \sqrt{2}a \end{aligned}$$

$$\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$$

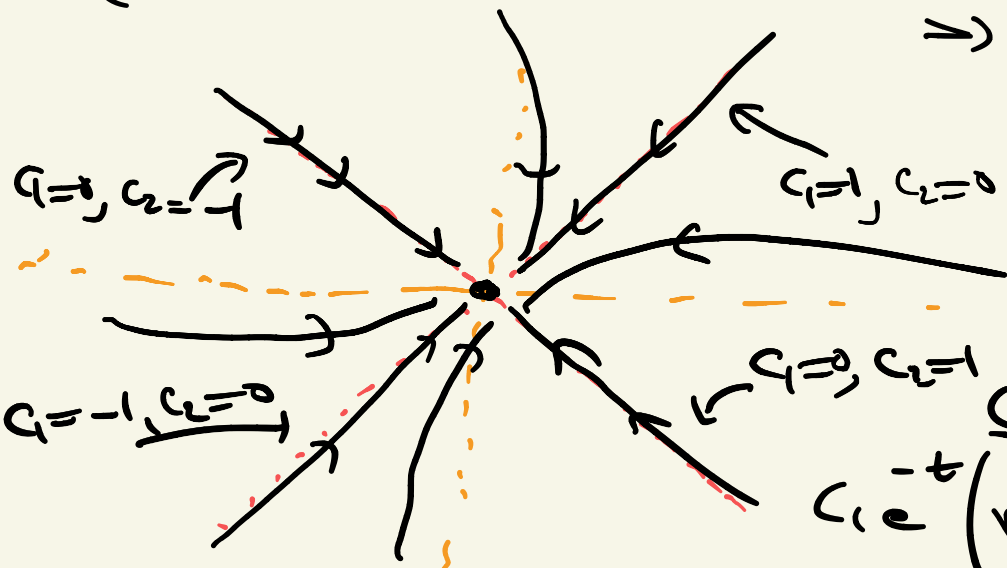
$\lambda = -4$

$$\begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -4\begin{pmatrix} a \\ b \end{pmatrix} \longrightarrow -3a + \sqrt{2}b = -4a$$

$$\Rightarrow a = -\sqrt{2}b$$

$$b = -1/\sqrt{2}\, a$$

$$\cancel{\begin{pmatrix} 1 \\ -1/\sqrt{2} \end{pmatrix}} \quad \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix}$$



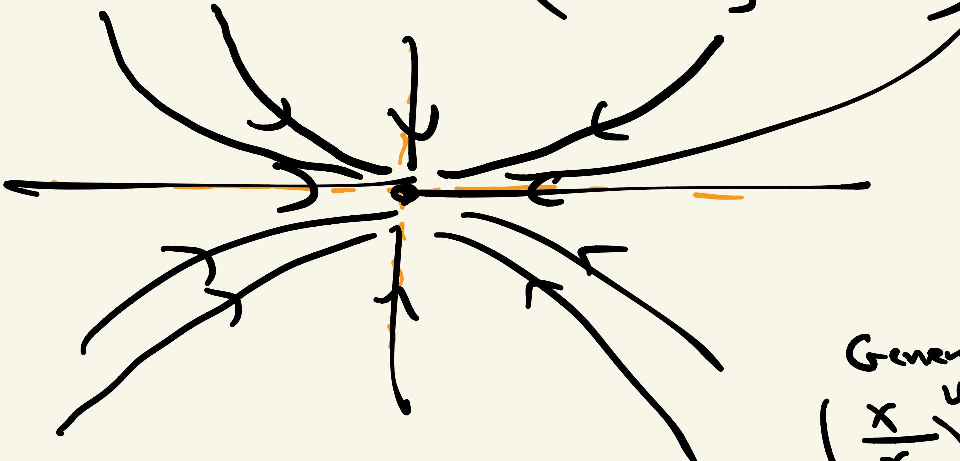
General sol'n:

$$c_1 e^{-t}\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} + c_2 e^{-4t}\begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix}$$

Fact: ~~If~~ A is a symmetric matrix, its evecs are orthogonal.

Analogous system: $\begin{cases} x'(t) = -x(t) \\ y'(t) = -4y(t) \end{cases}$

$$\Longrightarrow (x(t), y(t)) = (x_0 e^{-t}, y_0 e^{-4t})$$



Ex.- $x_0 = y_0 = 1$

$$\underbrace{(x(t), y(t)) = (e^{-t}, e^{-4t})}_{y = x^4}$$

General

$$\left(\frac{x}{x_0}\right)^4 = \frac{y}{y_0} \Rightarrow y = \frac{y_0}{x_0^4} x^4$$

"nodal sink"