# Lecture 18

**Recall**: Euler eqn: $t^2 y''(t) + \alpha t y'(t) + \beta y(t) = 0$
for $\alpha, \beta$ constants

**Ex**: $t^2 y''(t) + 3t y'(t) + 4y(t) = 0$
$t < 0$.

**Recall**: last time, ansatz $y(t) = t^r$
but $t^r = e^{r\ln(t)}$ ← only makes sense for $t > 0$.
So $t^r$ ill-defined for $t < 0$.

**Idea**: Put $s = -t$.
ODE becomes:
$$(-s)^2 y''(-s) + 3(-s)y'(-s) + 4y(-s) = 0$$
$s > 0$.

Put $z(t) = y(-t)$.
Then $z'(t) = -y'(-t)$, $z''(t) = y''(-t)$.

So ODE becomes

$$s^2 z''(s) + 3 s z'(s) + 4 z(s) = 0,$$

$$s > 0$$

^just an Euler eqn like in Last lecture!

$$r^2 + 2r + 4 = 0$$

$$\Rightarrow r = \frac{-2 \pm \sqrt{4-16}}{2}$$

$$= \frac{-2 \pm i 2\sqrt{3}}{2} = -1 \pm i\sqrt{3}$$

So, have two solutions

$$z_1(s) = s^{-1} \cos(\sqrt{3} \ln(s))$$

$$z_2(s) = s^{-1} \sin(\sqrt{3} \ln(s)).$$

So can solve as before:

ansatz $z(s) = s^r$ ← well-def since $s > 0$

$\rightsquigarrow$ plug in: $s^r \left( (r)(r-1) + 3r + 4 \right) = 0$

$$\Rightarrow r(r-1) + 3r + 4 = 0.$$

<u>two cpx solutions</u>:

$$\rightsquigarrow s^{-1+i\sqrt{3}}, \quad s^{-1-i\sqrt{3}}.$$

Two real-valued solutions:

$$z_1(s) = s^{-1} \cos(\sqrt{3} \ln(s))$$

$$z_2(s) = s^{-1} \sin(\sqrt{3} \ln(s))$$

<u>recall</u>: $t^{\alpha + i\beta} = t^{\alpha} t^{i\beta}$

$$= t^{\alpha} e^{i\beta \ln(t)}$$

$$= t^{\alpha} \left( \cos(\beta \ln(t)) + i \sin(\beta \ln(t)) \right)$$

$\rightsquigarrow$ $\underbrace{t^{\alpha} \cos(\beta \ln(t))}$ and $\underbrace{t^{\alpha} \sin(\beta \ln(t))}$.

<u>Note</u>: If $z(s) = z_1(s)$, then

$$y(t) = z(-t) = (-t)^{-1} \cos(\sqrt{3} \ln(-t)).$$

Similarly, if $z(s) = z_2(s)$, then

$$y(t) = z(-t) = (-t)^{-1} \sin(\sqrt{3} \ln(-t))$$

Conclude: For $t<0$, have sol'ns $(-t)^{-1}\cos(\sqrt{3}\ln(-t))$ and $(-t)^{-1}\sin(\sqrt{3}\ln(-t))$

(Since $|t|=t$ for $t\geq 0$, $|t|=-t$ for $t<0$)

In fact, shows that:

$$C_1(|t|)^{-1}\cos(\sqrt{3}\ln|t|)+C_2|t|^{-1}\sin(\sqrt{3}\ln|t|)$$

is a valid solution for any $t\neq 0$

this function is symmetric about y axis

Q: How can this be a sol'n to ODE if not differentiable at $t=0$.

Rmk: Since our ODE is linear, $y(t)\equiv 0$ is a solution

If $f(x)=|x|$, $f'(x)=\begin{cases}-1 & \text{for } x<0\\ +1 & \text{for } x>0\end{cases}$

Ex: $(t-\pi)^2y''(t)+5(t-\pi)y'(t)+4y(t)=0$, $t>\pi$

Singular pts? $t=\pi$.

Put $s=t-\pi$. $(t=s+\pi)$

ODE becomes $s^2y''(s+\pi)+5s\,y'(s+\pi)+4y(s+\pi)=0$, $s>0$.

Put $z(s)=y(s+\pi)$.

$z'(s)=y'(s+\pi)$, $z''(s)=y''(s+\pi)$.

$\rightsquigarrow$ get $\underbrace{s^2z''(s)+5sz'(s)+4z(s)=0,\ s>0}_{\text{Euler equation!}}$

Proceed as before: ansatz $z(s) = s^r$

$\rightsquigarrow$ indicial eqn: $r(r-1)+5r+4=0$

$\rightsquigarrow$ $r^2+4r+4=0$

$(r+2)^2=0$

$\rightsquigarrow$ $r_1=r_2=-2$.

So $z(s) = C_1|s|^{-2} + C_2|s|^{-2}\ln|s|$ ← valid soln for all $s\neq 0$

So general soln to orig ODE is:

$$y(t) = z(t-\pi) = C_1|t-\pi|^{-2} + C_2|t-\pi|^{-2}\ln|t-\pi|$$

valid for $t\neq\pi$.

**Note:** In general, if $y_1, y_2$ are solutions to $t^2y''+\alpha t y'+\beta y=0$ for $t>0$, then $y_1(|t|)$ and $y_2(|t|)$ are solns valid for all $t\neq 0$.

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**Recall:** $P(t)y''(t)+Q(t)y'(t)+R(t)y(t)=0$

$\Longleftrightarrow$ $y''(t)+p(t)y'(t)+q(t)y(t)=0$

for $p(t)=Q(t)/P(t)$, $q(t)=R(t)/P(t)$

$t=t_0$ is an ordinary pt if $p(t)$ and $q(t)$ are analytic at $t=t_0$.

Otherwise it's a singular pt.

$t-t_0$

$(t-t_0)^2$

**Def:** A singular pt $t_0$ is regular if $t\,p(t)$ and $t^2 q(t)$ are analytic at $t=t_0$. Otherwise, it's an irregular sing pt.

- ordinary pts ← can already handle, use $\sum a_n(t-t_0)^n$
- singular pts
  - regular
  - irregular ← won't cover

Ex: $t^2y'' + \alpha t y' + \beta y = 0$.

$t=0$ is a regular sing pt.

$$tp(t) = \underbrace{\frac{\alpha t}{t^2}\cdot t}_{\alpha} \qquad t^2q(t) = \underbrace{\frac{\beta}{t^2}\cdot t^2}_{\beta}$$

constants, hence analytic for all $t$

Ex: $t^{2.1}y'' + \alpha t y' + \beta y = 0$

$t=0$ is an irregular pt.

Ex: $t^{100}y'' + 5ty' + 10t^5 y = 0$

$t=0$ is an irregular sing pt.

Write as: $y'' + \frac{5}{t^{99}}y' + \frac{10}{t^{95}}y = 0$

$$tp(t) = \frac{5}{t^{98}} \qquad t^2q(t) = \frac{10}{t^{93}}$$

not analytic at $t=0$.

Ex: $t^2y'' + 5ty' + (\cos(t) + t^7)y = 0$

$t=0$ is a regular singular pt.