Midterm 2

Modern Algebra 1

Columbia University Fall 2019

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Instructions:

- Please write your answers **in this printed exam**. You may use the back of pages for additional work. You may also use printer paper if you need additional space, but you must hand in all relevant work. Please turn in all scratch work which is relevant to your submitted answers.
- Suspected cases of copying or otherwise cheating will be taken very seriously.
- Solve as many problems of the following problems as you can in the allotted time, which is *one hour and fifteen minutes.* I recommend first solving the problems you are most comfortable with before moving on to the more challenging ones. Note that the problems are not ordered by level of difficulty or topic.
- The exams will be graded on a curve. Therefore the raw score is not important, and you do not necessarily need to solve every problem to achieve a good grade. Just do your best!
- For true or false questions, you will receive $+2$ points for a correct answer, 0 points for no answer, and -3 points for an incorrect answer. This means **you should not make random guesses** unless you are reasonably sure that you know the answer. For the short answer questions, there is no penalty for wrong answers, and you do not need to justify your answers for full credit. For the short proof questions, you should be as precise and rigorous and possible.
- You may use any commonly used notation for standard groups, subgroups, and their elements, as long as it is completely unambiguous. If you are using nonstandard notation you must fully explain it for credit.
- Turn off all electronic devices. You may use the restroom if you must, but you may not take any devices with you.
- Good luck!!

Name: ______________________________

Uni: ______________________________

Question:	1	2	3	Total
Points:	22	10	3	35
Score:				

Notation reminders:

- $D_{2\cdot n}$ denotes the dihedral group corresponding to the symmetries of the regular n-gon.
- For $k \in \mathbb{Z}_{\geq 1}$, S_k denotes the symmetric group on k letters, and A_k denotes the alternating group on k letters.
- For G a group and p a prime subgroup, a p-subgroup of G is a subgroup whose order is a prime power of p.

1. **True or false questions. Circle one. You do not need to provide any justification. There is a guessing penalty.**

(I) (2 points) Every group of order 100 is isomorphic to a subgroup of S_{101}. A. True B. False

Solution: True. By Cayley's theorem, every group of order 100 is isomorphic to a subgroup of S_{100}, which in turn is isomorphic to a subgroup of S_{101}.

(II) (2 points) Let G be a group of order 55. Then G is abelian. A. True B. False

Solution: True. Since $55 = 5 * 11$, we must have $n_5 = n_{11} = 1$. This means that G is isomorphic to $\mathbb{Z}/(5\mathbb{Z}) \times \mathbb{Z}/(11\mathbb{Z})$, which is abelian (and in fact cyclic).

(III) (2 points) Let G be a group of order 22. Then G is abelian. A. True B. False

Solution: False. Since $22 = 2 * 11$ and 2 divides $11 - 1$, there is a nonabelian group of order 22 (see problem #2 from problem set #7).

(IV) (2 points) Any two subgroups of S_6 of order 4 are isomorphic. A. True B. False

Solution: False. Any 4-cycle generates a cyclic subgroup of order four. On the other hand, we can find a subgroup of S_6 isomorphic to S_4, which in turn contains a subgroup isomorphic to $\mathbb{Z}/(2\mathbb{Z}) \times \mathbb{Z}/(2\mathbb{Z})$.

(V) (2 points) Any two subgroups of S_6 of order 9 are isomorphic. A. True B. False

Solution: True. Since $|S_6| = 6 * 5 * 4 * 3 * 2$, any subgroup of order 9 is a Sylow 3-subgroup. By Sylow's second theorem, any two Sylow 3-subgroups are conjugate, and hence isomorphic.

(VI) (2 points) There is no simple group of order 360. A. True B. False

Solution: False. A_6 has order 360 and we proved in class that it is simple.

(VII) (2 points) Let G be a group of order 175. Every subgroup of G of order 35 is normal. A. True B. False

Solution: True. We have $175 = (5^2)(7)$, so a subgroup of order 35 has index 5. Since 5 is the smallest prime dividing 175, this subgroup is automatically normal.

(VIII) (2 points) Every group of order 121 is abelian. A. True B. False

Solution: True. See problem #2 of problem set #6.

(IX) (2 points) Let G be a group of order 20. Then there exists a transitive action of G on a set of 4 elements. A. True B. False

Solution: True. Let H be a Sylow 5-subgroup of G, and let X denote the set of left cosets of H in G. Then G acts on X by left multiplication, and it is easy to check that this action is transitive.

(X) (2 points) Let G be a group of order 20. Then there exists a surjective homomorphism from G to a group of order 4. A. True B. False

Solution: True. Let H be a Sylow 5-subgroup of G. By Sylow's third theorem we must have $n_5 = 1$, hence H is a normal subgroup. Then G surjects onto G/H, which has order 4.

(XI) (2 points) The symmetric group S_5 has a subgroup which is isomorphic to Q_8. A. True B. False

Solution: False. Since D_8 admits a natural injective homomorphism into S_4 and hence also S_5, it follows that every Sylow 2-subgroup of S_5 is isomorphic to D_8.

2. **Short answer questions. You do not need to provide any justification. There is no penalty for wrong answers but you must box your final answer if it is not clear.**

(I) (2 points) What is the order of a Sylow 5-subgroup of S_{17}?

Solution: Since $|S_{17}| = 17!$ is divisible by 5^3 but not 5^4, Sylow 5-subgroups have order 125.

(II) (1 point) Let $G_0 = \{e\} \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \ldots \trianglelefteq G_r = S_4$ be a composition series of S_4. What is r? Describe the composition factors $G_1/G_0, G_2/G_1, \ldots, G_r/G_{r-1}$ up to isomorphism.

Solution:

S_4 has the index two normal subgroup A_4, which has the index three normal subgroup $\langle (1\ 2)(3\ 4), (1\ 3)(2\ 4)\rangle$, which has the index two normal subgroup $\langle (1\ 2)(3\ 4)\rangle$. So a composition series is

$$\{e\} \trianglelefteq \langle (1\ 2)(3\ 4)\rangle \trianglelefteq \langle (1\ 2)(3\ 4), (1\ 3)(2\ 4)\rangle \trianglelefteq A_4 \trianglelefteq S_4.$$

Then $r = 4$, and the composition factors are isomorphic to $\mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/3, \mathbb{Z}/2$.

(III) (1 point) Let G be a noncyclic group of order 93. How many elements of order 3 does G have?

Solution: We have $|G| = 93 = 3 * 31$. By Sylow's third theorem, we must have $n_{31} = 1$. Since there are no elements of order 93, we must have $n_3 = 31$, which gives $31 * 2 = 62$ elements of order 3. Note that we also have 30 elements of order 31 and 1 element of order 1, so these indeed add up to 93.

(IV) (1 point) Consider the partition of the quaternion group Q_8 into conjugacy classes. How many conjugacy classes are there? Describe the conjugacy classes.

Solution: The center $Z(Q_8)$ is $\{1, -1\}$. Recall that the centralizer of i is $\langle i \rangle$, so by the orbit stabilizer theorem the orbit has size 2. Since $jij^{-1} = ji(-j) = (-k)(-j) = -jk = -i$, the conjugacy class of i is $\{i, -i\}$. Similarly, the conjugacy class of j is $\{j, -j\}$, and that of k is $\{k, -k\}$. So there are 5 conjugacy classes, which are $\{1\}, \{-1\}, \{i, -i\}, \{j, -j\}, \{k, -k\}$.

(V) (1 point) Let S_5 act on itself by left multiplication. What is the size of the orbit of the transposition $(1\ 2) \in S_5$?

Solution: This action is easily seen to be transitive, so the size of the orbit is $|S_5| = 5! = 120$.

(VI) (1 point) Let S_5 act on itself by conjugation. What is the order of the stabilizer of the transposition $(1\ 2) \in S_5$?

Solution: The orbit consists of all transpositions, of which are there $\binom{5}{2} = 10$. By the orbit-stabilizer theorem, the order of the stabilizer is $5!/10 = 12$.

(VII) (1 point) How many conjugacy classes of A_5 are there?

Solution: There are 7 possible cycle types in S_5: $5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1$, and $1+1+1+1+1$. The ones giving even permutations are $5, 3+1+1, 2+2+1$, and $1+1+1+1+1$.

As we saw in class, the 5-cycles actually form two conjugacy classes, so this makes 5.

(VIII) (1 point) How many subgroups of S_4 are isomorphic to D_8?

Solution: Three. We get a natural subgroup of S_4 isomorphic to D_8 from the definition of the dihedral group, and it is easy to check that it is not normal. By Sylow's third theorem, $n_2 = 3$.

(IX) (1 point) Let H denote the subgroup of S_7 generated by the 7-cycle $(1\,3\,2\,4\,6\,5\,7)$. What is the order of the normalizer $N_{S_7}(H)$ of H in S_7?

Solution: The number of 7-cycles in S_7 is 6!, since there are 7! ways of ordering the numbers $1, \ldots, 7$, and there is a 7-fold redundancy when we write the cycle decomposition. This means that are 5! subgroups of order 7, since each subgroup contains precisely 6 elements of order 6. Since any two 7-cycles are conjugate, there shows that the orbit of H under the conjugation action of S_7 on its subsets has size 5!. By the orbit-stabilizer theorem, the normalizer $N_{S_7}(H)$ must have order $7!/5! = 42$.

3. **Short proofs. Make your arguments as precise and rigorous as possible.**

(I) (1 point) State and prove Cayley's theorem for finite groups.

Solution: Cayley's theorem: Let G be a finite group. G is isomorphic to a subgroup of $S_{|G|}$.
Proof: Consider the action of G on itself by left multiplication. The kernel of this action is the set of all $g \in G$ such that $gg' = g'$ for all $g' \in G$, but since $gg' = g'$ implies $g = e$ the kernel consists only of the identity element. Then the action corresponds to an injective homomorphism $\Phi : G \to S_{|G|}$. By the first isomorphism theorem, G is isomorphic to $\mathrm{im}\,(\Phi)$, which is a subgroup of $S_{|G|}$.

(II) (1 point) Prove that there is no simple group of order 108. *You may invoke general results such as Lagrange's theorem, the orbit-stabilizer theorem, the class equation, and Sylow's theorems, but any specific classification results for groups of given orders should be proved.*

Solution: Let G be a group of order 108. We have $108 = (2^2)(3^3)$. Let H be a Sylow 3-subgroup of G, which has order index 4 and exists by Sylow's first theorem. Let X denote the set of left cosets of H in G, and let G act on X by left multiplication. We consider the action of G on X by left multiplication. Let $\Phi : G \to S_X$ denote the homomorphism corresponding to this action. Since for any $g \in G \setminus H$ we have $g \cdot H \neq H$, it follows that this action is not trivial, i.e. $\ker(\Phi) \neq G$. We also cannot have $|\ker(\Phi)| = 1$, since then it would follow from the first isomorphism theorem that G is isomorphic to $\mathrm{im}\,(\Phi)$, and hence by Lagrange's theorem $|G|$ divides $|S_X| = 4!$, which is impossible. Since kernels of homomorphisms are always normal subgroups and $\ker(\Phi) \neq G$ and $\ker(\Phi) \neq \{e\}$, it follows that G is not a simple group.

(III) (1 point) Prove that every group of order 1024 has a nontrivial center. *You may invoke general results such as Lagrange's theorem, the orbit-stabilizer theorem, the class equation, and Sylow's theorems, but any specific classification results for groups of given orders should be proved.*

Solution: Get G be a group of order 1024. Let $r \in \mathbb{Z}_{\geq 1}$ denote the number of non-singleton conjugacy classes of G, and let $g_1, \ldots, g_r 1G$ be elements representing each of the non-singleton conjugacy classes. By the class equation, we have

$$|G| = |Z(G)| + \sum_{i=1}^{r} |G|/|C_G(g_i)|,$$

where $C_G(g_i)$ denotes the centralizer of g_i in G. Since $|G| = 2^{10}$, we have that $|G|/|C_G(g_i)|$ is a power of two for $i = 1, \ldots, r$. Moreover, $|G|/|C_G(g_i)|$ cannot be 1, since then we would have $C_G(g_i) = G$, but $\{g_i\}$ is not a singleton conjugacy class. Therefore 2 divides $|G|/|C_G(g_i)|$ for $i = 1, \ldots, r$. Since 2 also divides $|G|$, we find that 2 divides $|Z(G)|$, so $|Z(G)| > 1$.