Midterm 2
Math 125: Calculus I
University of Southern California Fall 2022
Instructor: Kyler Siegel

Instructions:

- Please write your answers as legibly as possible. You should try to show all of your work within reason. You should include all scratch work with your submission.
- Solve as many of the problems as you can in the allotted time, which is *50 minutes*.
- I recommend first solving the problems you are most comfortable with before moving on to the more challenging ones. Note that the problems are not necessarily ordered by level of difficulty or topic.
- You may *not* use any electronic devices to complete the exam, apart from those used to view and submit the exam. You are *not* allowed to use any textbook, calculator, pre-written notes, the internet, etc, to aid your solutions.
- You may use the restroom if needed but you must leave any electronic devices in the exam room.
- You are expected to follow the honor code. Suspected cases of copying or otherwise cheating will be taken very seriously.
- The exams will be graded on a curve. Therefore the raw score is not important, and you do not necessarily need to solve every problem to achieve a good grade. Just do your best!
- Don't forget to write your name!
- Good luck!!

Question:	1	2	3	4	Total
Points:	15	35	18	28	96
Score:					

1. (15 points) Find the point on the curve $y = \sqrt{x}$ which is closest to the point $(5, 0)$.

Solution: We seek the point $(x, \sqrt{x})$ whose distance to $(5, 0)$ is minimal. This distance is given by

$$\sqrt{(x-5)^2 + (\sqrt{x} - 0)^2} = \sqrt{(x-5)^2 + x}.$$

Recall that it can be more convenient to minimize the squared distance instead of the distance, so that we don't have to deal with a square root. So we seek $x \geq 0$ which minimizes

$$\begin{aligned} f(x) = (x-5)^2 + x &= x^2 - 10x + 25 + x \\ &= x^2 - 9x + 25. \end{aligned}$$

We have $f'(x) = 2x - 9$, so the only critical number is $x = 9/2$. Since x varies over $[0, \infty)$, we also need to check $x = 0$. We have $f(0) = 25$ and $f(9/2) = 81/4 - 81/2 + 25$. Clearly $f(9/2) < f(0)$, so the point closest to the curve is $(9/2, \sqrt{9/2})$.

2. Evaluate the following using any method you wish.

(I) (5 points)

$$\int_0^1 (x^2 + x + 3)dx$$

Solution: This is $(\frac{1}{3}x^3 + \frac{1}{2}x^2 + 3x)|_0^1 = 1/3 + 1/2 + 3 = 3\frac{5}{6}$.

(II) (5 points)

$$\int \sin(3x)dx$$

Solution: This is $-\frac{1}{3}\cos(3x) + C$. This could be done by a u-substitution or simply by inspection.

(III) (5 points)

$$\int x^2 \cos(3x^3 + 5)dx$$

Solution: Make the u-substitution $u = 3x^3 + 5$. Then $du = 9x^2 dx$, so our integral becomes

$$\tfrac{1}{9}\int \cos(u)du = \tfrac{1}{9}\sin(u) + C = \tfrac{1}{9}\sin(3x^3 + 5) + C.$$

(IV) (5 points)

$$\int_1^2 x\sqrt{|x|}dx$$

Solution: While x ranges in $[1, 2]$, we have $|x| = x$. So this is simply

$$\int_1^2 x\sqrt{x}dx = \int_1^2 x^{3/2}dx = \tfrac{2}{5}x^{5/2}|_1^2 = \tfrac{2}{5}(2^{5/2} - 1).$$

(V) (5 points)

$$\int \frac{x^3}{\sqrt{x^2 - 1}}dx$$

Solution: We had a very similar example in a problem set, which we also went over in class. Make the u-substitution $u = x^2 - 1$. Then we have $du = 2xdx$ and $x^2 = u + 1$, so our integral becomes

$$\tfrac{1}{2}\int \tfrac{(u+1)}{\sqrt{u}}du = \tfrac{1}{2}\int (u^{1/2} + u^{-1/2})du = \tfrac{1}{2}(\tfrac{2}{3}u^{3/2} + 2u^{1/2}) + C = \tfrac{1}{3}(x^2 - 1)^{3/2} + \sqrt{x^2 - 1} + C.$$

(VI) (5 points)

$$\frac{d}{dx}\int_x^1 \sin(t^2)dt$$

Solution: This is

$$-\frac{d}{dx}\int_1^x \sin(t^2)dt = -\sin(x^2),$$

using part I of the fundamental theorem of calculus.

(VII) (5 points)

$$\frac{d}{dx}\int_3^{x^3} \sin(t^2)dt$$

Solution: Put $f(x) = \int_3^x \sin(t^2)dt$ and $g(x) = x^3$. Then we wish to compute the derivative of $f(g(x))$, which by the chain rule and part I of the fundamental theorem of calculus is

$$f'(g(x))g'(x) = \sin(g(x)^2)g'(x) = \sin(x^6) \cdot 3x^2.$$

3. (I) (6 points) Calculate the approximate area under the curve $y = x$ between $x = 0$ and $x = 3$ by using a Riemann sum with three rectangles and the midpoint rule.

Solution:
Each of the rectangles will have width 1 since we divide $[0, 3]$ into 3 equal parts. Putting $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3$, the height of the ith rectangle using the midpoint rule is $f(\frac{1}{2}(x_i + x_{i+1}))$ for $i = 0, 1, 2$. Here $f(x) = x$. Then the approximate area is

$$f(1/2) + f(3/2) + f(5/2) = 1/2 + 3/2 + 5/2 = 9/2.$$

(II) (6 points) Now do the same but using the right endpoint rule and using n rectangles.

Solution: Now we put $x_0 = 0$, $x_1 = 3/n$, $x_2 = 2{\cdot}3/n$ etc, up to $x_{n-1} = (n-1){\cdot}3/n$, $x_n = n{\cdot}3/n = 3$.

Using the right endpoint rule, the approximate area is

$$\begin{aligned} 3/n \cdot f(x_1) + 3/n \cdot f(x_2) + \cdots + 3/n \cdot f(x_n) &= 3/n \cdot (3/n + 2 \cdot 3/n + \cdots + n \cdot 3/n) \\ &= (3/n)^2 (1 + 2 + \cdots + n) = \tfrac{9}{n^2} \cdot \tfrac{(n)(n+1)}{2} \\ &= 9 \cdot \frac{n+1}{2n}. \end{aligned}$$

(III) (6 points) Compute the integral $\int_0^3 x dx$ by taking the limit as n goes to ∞ in the previous part. *Note: you will not get credit if you calculate the integral using a different method, although you're welcome to do so to check your answer.*

Solution: The integral is given by

$$\lim_{n \to \infty} 9 \cdot \tfrac{n+1}{2n} = \tfrac{9}{2} \cdot \lim_{n \to \infty} (1 + 1/n) = 9/2.$$

Note that this is the same as our answer to part I. We should expect the answer to part I to be only an approximation to the integral, but in this case it happens to be exact.

4. Consider the function $f(x) = x^3 - 6x^2 + 9x + 1$.

(I) (4 points) Find the critical numbers of $f(x)$.

Solution: Note that $f(x)$ is a polynomial and hence differentiable everywhere, so we don't have to worry about points where the derivative is not defined. We have

$$f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-1)(x-3),$$

so the critical numbers are $x = 1, 3$.

(II) (4 points) On which intervals is f increasing or decreasing?

Solution: We use the increasing / decreasing test. By inspecting $f'(x) = 3(x-1)(x-3)$, we see that $f'(x)$ is positive for $x < 1$ (e.g. try plugging in $x = 0$), negative for $1 < x < 3$, and positive for $x > 3$. So f is

- increasing on $(-\infty, 1)$
- decreasing on $(1, 3)$

- increasing on $(3, \infty)$.

(III) (4 points) On which intervals is f concave up or concave down?

Solution: We use the concavity test. We have

$$f''(x) = 6x - 12 = 6(x - 2),$$

so $f''(x)$ is negative for $x < 2$ and positive for $x > 2$. Therefore f is

- concave down on $(-\infty, 2)$
- concave up on $(2, \infty)$

(IV) (4 points) Find all local maxima and local minima of f.

Solution: Recall that the local maxima and minima must occur at critical numbers of f, so the only possibilities are $x = 1, 3$. To determine what type of extrema these are we can use the first derivative test. Since $f'(x)$ switches from positive to negative at 1, $x = 1$ is a local maximum. Since $f'(x)$ switches from negative to positive at $x = 3$, $x = 3$ is a local minimum.

Alternatively, we could use the second derivative test.

(V) (4 points) Find all inflection points of f.

Solution: Recall that inflection points occur when the function switches from concave up to concave down or vice versa. From our answer to part III, we see that $x = 2$ is the only inflection point.

(VI) (4 points) Give a rough sketch of the graph of f. *Note: your sketch does not need to be perfect but you should try to take into account your answers above.*

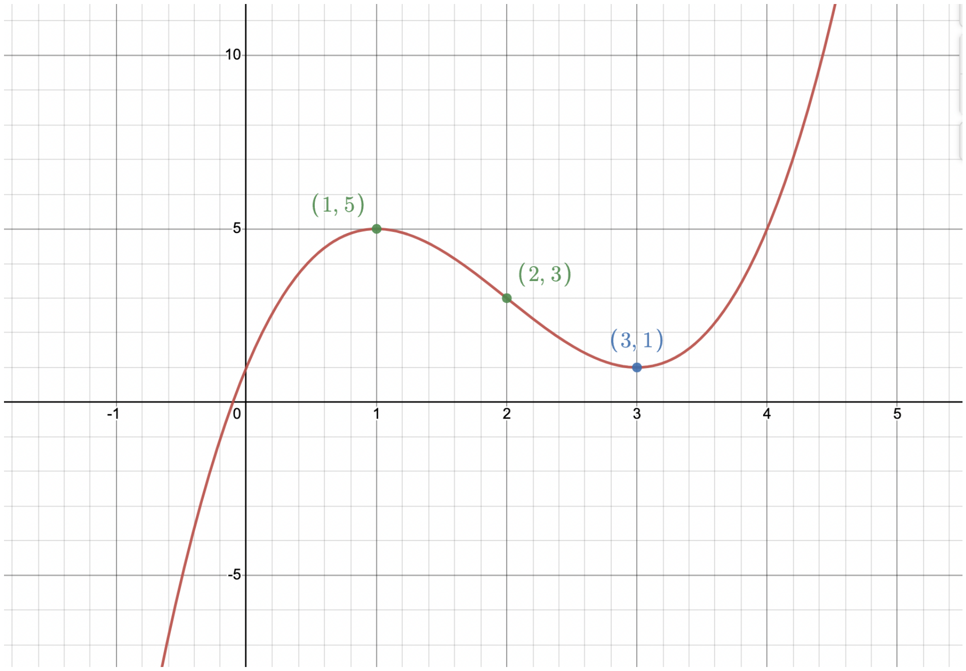

Figure 1: The graph of $f(x)$.

Solution: See Figure 1 for a plot of $f(x)$ generated by Desmos. Note that the critical numbers and the inflection point are labeled. You should satisfy yourself that the intervals of increase / decrease and concave up / down are consistent with our answers above.

(VII) (4 points) What is the maximum value of f when x ranges over $[-1, 0]$?

Solution:
Since $f(x)$ is increasing on $[-1, 0]$, its maximum value on that interval is simply $f(0) = 1$.