Midterm 1
Modern Algebra 1
Columbia University Fall 2019
Instructor: Kyler Siegel

Instructions:

- Please write your answers **in this printed exam**. You may use the back of pages for additional work. You may also use printer paper if you need additional space, but you must hand in all relevant work. Please turn in all scratch work which is relevant to your submitted answers.
- Suspected cases of copying or otherwise cheating will be taken very seriously.
- Solve as many problems of the following problems as you can in the allotted time, which is *one hour and fifteen minutes.* I recommend first solving the problems you are most comfortable with before moving on to the more challenging ones. Note that the problems are not ordered by level of difficulty or topic.
- The exams will be graded on a curve. Therefore the raw score is not important, and you do not necessarily need to solve every problem to achieve a good grade. Just do your best!
- There are twenty true or false questions, four multiple choice questions, four short answer questions, and two short proof questions.
- For true or false questions, you will receive $+2$ points for a correct answer, 0 points for no answer, and -3 points for an incorrect answer. For multiple choice questions, you will receive $+4$ points for a correct answer, 0 points for no answer, and -2 points for an incorrect answer. This means **you should not make random guesses** unless you are reasonably sure that you know the answer.
- You may use any commonly used notation for standard groups, subgroups, and their elements, as long as it is completely unambiguous. If you are using nonstandard notation you must fully explain it for credit.
- Turn off all electronic devices. You may use the restroom if you must, but you may not take any devices with you.
- Good luck!!

Name: ____________________

Uni: ____________________

Question:	1	2	3	4	Total
Points:	30	16	18	20	84
Score:					

Notation reminders:

- $N_G(A) := \{g \in G \ : \ gAg^{-1} = A\}$ denotes the normalizer of a subset $A \subset G$.
- $\ker(\Phi)$ and $\mathrm{im}\,(\Phi)$ denote the kernel and image respectively of a homomorphism Φ
- $D_{2\cdot n}$ denotes the dihedral group corresponding to the symmetries of the regular n-gon.

1. *True or false questions. Circle one. You do not need to provide any justification.*

(I) (2 points) Every group of order 19 is abelian.
A. True B. False

(II) (2 points) The group $\mathbb{Z}/(2\mathbb{Z}) \times \mathbb{Z}/(3\mathbb{Z})$ is isomorphic to $\mathbb{Z}/(6\mathbb{Z})$
A. True B. False

(III) (2 points) The group $\mathbb{Z}/(3\mathbb{Z}) \times \mathbb{Z}/(3\mathbb{Z})$ is isomorphic to $\mathbb{Z}/(9\mathbb{Z})$
A. True B. False

(IV) (2 points) If G is a group and H and K are subgroups, then $HK = \{hk \ : \ h \in H, \ k \in K\}$ is a subgroup of G.
A. True B. False

(V) (2 points) Every permutation $\sigma \in S_{100}$ can be written as a product of 3-cycles.
A. True B. False

(VI) (2 points) $(\mathbb{Z}/11\mathbb{Z})^\times$ is a cyclic group.
A. True B. False

(VII) (2 points) A group all of whose proper subgroups are cyclic is cyclic.
A. True B. False

(VIII) (2 points) If G and H are groups and $\Phi : G \to H$ is a homomorphism, then we have $N_G(\ker(\Phi)) = G$.
A. True B. False

(IX) (2 points) Suppose that H is a subgroup of a group G, and that H is an abelian group. Then H is a normal subgroup of G.
A. True B. False

(X) (2 points) If H is a subgroup of a finite group G with $|G| = 2|H|$, then H is a normal subgroup of G.
A. True B. False

(XI) (2 points) If G is a group of order four, then it is isomorphic to either $(\mathbb{Z}/(5\mathbb{Z}))^\times$ or $(\mathbb{Z}/(8\mathbb{Z}))^\times$.
A. True B. False

(XII) (2 points) If H and K are subgroups of a group G, and we have $H \subset N_G(K)$, then $HK = KH$.
A. True B. False

(XIII) (2 points) There exists a surjective homomorphism from $D_{2\cdot 5}$ to $\mathbb{Z}/(2\mathbb{Z})$. A. True B. False

(XIV) (2 points) The dihedral group $D_{2\cdot 3}$ is isomorphic to the symmetric group S_3.
A. True B. False

(XV) (2 points) If H and K are normal subgroups of G, then $H \cap K$ is also a normal subgroup of G.
A. True B. False

2. *Multiple choice questions. You do not need to provide any justification. In each case,* ***select all that apply****.*

(I) (4 points) Which of the following groups is isomorphic to $(\mathbb{Z}/(10\mathbb{Z}))^{\times}$?
A. $\mathbb{Z}/(10\mathbb{Z})$ B. $\mathbb{Z}/(4\mathbb{Z})$ C. $C_2 \times C_2$ D. $(\mathbb{Z}/(8\mathbb{Z}))^{\times}$

(II) (4 points) What is the remainder when 13^{101} is divided by 17?
A. 1 B. 3 C. 11 D. 13

(III) (4 points) Which of the following groups is isomorphic to a subgroup of $D_{2\cdot 8}$?
A. C_2 B. $C_2 \times C_2$ C. C_8 D. $C_2 \times C_8$

(IV) (4 points) Which of the following groups is isomorphic to the center of $D_{2\cdot 5}$?
A. C_1 B. C_2 C. $C_2 \times C_2$ D. C_4

3. *Short answer questions. You do not need to provide any justification for full credit. However, if you do you might receive some partial credit if your answer is incorrect but well-reasoned.*

(I) (6 points) How homomorphisms are there from C_8 to C_{14}?

(II) (6 points) Let $\sigma \in S_6$ be the permutation given by $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 1 & 6 & 3 & 2 \end{pmatrix}$. Write σ as a product of transpositions, and also determine the sign of σ.

(III) (6 points) Explain at least one way in which the following statement of the Second Isomorphism Theorem is mathematically incorrect:

> Let G be a group, and let A and B be subgroups. Then AB is a subgroup of G, $B \trianglelefteq AB$, $A \cap B \trianglelefteq A$ and $AB/B \cong A/A \cap B$.

4. *Short proofs. Make your arguments as rigorous as possible. You may cite results covered in class provided you are completely clear about what you are citing.*

(I) (10 points) Let G and H be finite groups such that $\gcd(|G|, |H|) = 1$. Let $\phi : G \to H$ be a homomorphism. Prove that ϕ sends every element of G to the identity element of H.

(II) (10 points) Let G be a finite cyclic group of order n, and let k be an integer which is relatively prime to n. Prove that for any element $g \in G$, there exists an element $h \in G$ such that $g = h^k$. *For 3 bonus points: prove that the same holds without assuming that G is cyclic.*