Lecture 21

Today:
- motivational example
- lots of linear algebra

Recall: **Ex:**

$$\begin{cases} x'(t) = 5x(t) - y(t) \\ y'(t) = -x(t) + 5y(t) \end{cases} \quad (*)$$

Idea: define new variables. $\tilde{x}(t) = \frac{x+y}{2}$

$\tilde{y}(t) = \frac{x-y}{2}$

In terms of new vars, $(*)$ becomes:

$\tilde{x} + \tilde{y} = x$

$\tilde{x} - \tilde{y} = y$

$$\begin{cases} \tilde{x}' + \tilde{y}' = 5(\tilde{x} + \tilde{y}) - (\tilde{x} - \tilde{y}) \\ \tilde{x}' - \tilde{y}' = -(\tilde{x} + \tilde{y}) + 5(\tilde{x} - \tilde{y}) \end{cases}$$

$$\rightsquigarrow \begin{cases} \tilde{x}' + \tilde{y}' = 4\tilde{x} + 6\tilde{y} \\ \tilde{x}' - \tilde{y}' = 4\tilde{x} - 6\tilde{y} \end{cases}$$

$$\rightsquigarrow \begin{cases} 2\tilde{x}' = 8\tilde{x} \\ 2\tilde{y}' = 12\tilde{y} \end{cases} \rightsquigarrow \begin{cases} \tilde{x}' = 4\tilde{x} \\ \tilde{y}' = 6\tilde{y} \end{cases}$$

uncoupled system!

$$\longrightarrow \tilde{x}(t) = C_1 e^{4t} \quad \tilde{y}(t) = C_2 e^{6t}$$

$$\underbrace{x(t) = C_1 e^{4t} + C_2 e^{6t}, \quad y(t) = C_1 e^{4t} - C_2 e^{6t}}_{\text{general solution}}$$

Questions: • what did the new variables $\tilde{x}$ and $\tilde{y}$ come from?

• What's the significance of 4 and 6?

A: $\tilde{x}, \tilde{y}$ come from the eigenvectors of $\begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}$, and $4, 6$ are its eigenvalues.

Crash course in linear algebra:

- vector $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ $\leftarrow$ think of as $n \times 1$ matrix

entries can be real #s or cpx #s

e.g. $\begin{pmatrix} 1 \\ 1+i \\ \sqrt{2}+2i \end{pmatrix}$

- square matrix: $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

- $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 1\cdot5+2\cdot7 & 1\cdot6+2\cdot8 \\ 3\cdot5+4\cdot7 & 3\cdot6+4\cdot8 \end{pmatrix}$

matrix-matrix mult

- $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 1\cdot5+2\cdot6 \\ 3\cdot5+4\cdot6 \end{pmatrix} = \begin{pmatrix} 17 \\ 39 \end{pmatrix}$

matrix - vector mult

- $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \quad \vec{v}_3 = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$

  3 vectors in $\mathbb{R}^3$

- $\vec{v} = \begin{pmatrix} 1+i \\ 2+5i \end{pmatrix} \in \mathbb{C}^2$

- <u>linear (in)dependence</u>: A collection of vectors $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{C}^n$ are <u>linearly independent</u> if whenever $c_1 \vec{v}_1 + \dots + c_k \vec{v}_k = \vec{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ for $c_1, \dots, c_k \in \mathbb{C}$, then $c_1 = \dots = c_k = 0$.

  Says none of the $\vec{v}_i$ can be written as linear combo of the others

Otherwise, we say they're <u>linearly dependent</u>.

Ex: Are $\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\9\end{pmatrix}$ linearly independent?

$$2\begin{pmatrix}4\\5\\6\end{pmatrix} - \begin{pmatrix}7\\8\\9\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix}, \text{ so } \begin{pmatrix}1\\2\\3\end{pmatrix} - 2\begin{pmatrix}4\\5\\6\end{pmatrix} + \begin{pmatrix}7\\8\\9\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

So they're <u>Dependent</u>.

<u>Fact</u>: Given n vectors $\vec{v}_1, \dots, \vec{v}_n \in \mathbb{R}^n$, they're linearly dependent if and only if $A = \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{pmatrix}$ has determinant 0.

<u>Def</u>: A <u>basis</u> of $\mathbb{R}^n$ is a set of n linearly independent vectors.

<u>Fact</u>: Any $n+1$ vectors in $\mathbb{C}^n$ (or $\mathbb{R}^n$) are linearly Dependent.

<u>Ex</u>: Basis for $\mathbb{R}^3$: $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$ "standard basis"

<u>linearly indep</u>: $c_1\begin{pmatrix}1\\0\\0\end{pmatrix} + c_2\begin{pmatrix}0\\1\\0\end{pmatrix} + c_3\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

$\Longrightarrow \begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Longrightarrow c_1 = c_2 = c_3 = 0.$

Another basis for $\mathbb{R}^3$: $\begin{pmatrix}2\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix}$

<u>linearly indep</u>: $c_1\begin{pmatrix}2\\0\\0\end{pmatrix} + c_2\begin{pmatrix}0\\1\\1\end{pmatrix} + c_3\begin{pmatrix}0\\1\\-1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

$\Longleftrightarrow \begin{pmatrix}2c_1\\c_2+c_3\\c_2-c_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Longleftrightarrow \begin{aligned} 2c_1 &= 0 \\ c_2+c_3 &= 0 \\ c_2-c_3 &= 0 \end{aligned}$

$c_1 = 0 \qquad 2c_2 = 0 \qquad c_2 = 0$

$c_3 = 0.$

Not a basis for $\mathbb{R}^3$: $\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\9\end{pmatrix}$

$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$

independent?

$$c_1\begin{pmatrix}1\\0\\0\end{pmatrix} + c_2\begin{pmatrix}2\\0\\0\end{pmatrix} + c_3\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \iff \begin{pmatrix}c_1+2c_2\\0\\c_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$c_3 = 0 \qquad c_1 = -2c_2$$

So can take $c_1 = -2$, $c_2 = 1$, $c_3 = 0$ gives a nontrivial lin. comb. which equals $\begin{pmatrix}0\\0\\0\end{pmatrix}$.

Invertibility: $n\times n$ matrix A is **invertible** if there exists another $n\times n$ matrix B s.t. $AB = \mathbb{1}_n$.

(Here $\mathbb{1}_n = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&&1\ddots&0\\0&&&0\\0&0&0\;0&1\end{pmatrix}$

ex $\mathbb{1}_3 = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$).

Fact: For any matrix A, $A \cdot \mathbb{1}_n = A$

$\mathbb{1}_n \cdot A = A$.

Warning: in general, for $n \times n$ matrices A, B

$A \cdot B \neq B \cdot A$.

Ex: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

(inverse of A) $\quad = \frac{1}{|A|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Check: $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $\quad A^{-1} = \frac{1}{-2} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$

$= \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$

$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓

More generally, to find inverse of $n \times n$ matrix:

$$\left(\underbrace{\begin{matrix}a&b&c\\d&e&f\\g&h&i\end{matrix}}_{\text{matrix whose inverse want}}\;\middle|\;\underbrace{\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix}}_{\text{identity matrix}}\right)$$

"augmented matrix"

Apply "elementary row operations" to reduce left matrix to the identity. Then the resulting right matrix is A^{-1}.

<u>Elementary row opns</u>:

(1) interchange two rows

(2) multiply a row by a nonzero scalar

(3) add a multiple of one row to another

Ex: $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_2 \mapsto R_2 - 3R_1]{} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right)$$

"Gaussian elimination"

$$\xrightarrow[R_1 \mapsto R_1 + R_2]{} \left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & -2 & -3 & 1 \end{array}\right)$$

$$\xrightarrow[R_2 \mapsto -\frac{1}{2}R_2]{} \left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right)$$

$$\Longrightarrow A^{-1} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}, \text{ as before.}$$

Consider linear system:
$$\begin{aligned} ax+by+cz &= c_1 \\ dx+ey+fz &= c_2 \\ gx+hy+iz &= c_3 \end{aligned}$$

$$\iff \underbrace{\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

Let's assume A is invertible.

<u>Note</u>: $A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \iff A^{-1}A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$

$$\iff \mathbb{1}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$\iff \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

<u>Recall</u>: by Cramer's rule,
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \cdots \qquad x = \frac{\begin{vmatrix} c_1 & b & c \\ c_2 & e & f \\ c_3 & h & i \end{vmatrix}}{|A|}, \quad y = \cdots \quad z = \cdots.$$

Ex: A matrix which is not invertible: $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

$\begin{pmatrix} | & | \\ | & | \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Eigenvectors / eigenvalues:

$\boxed{A\vec{v} = \text{const}\,\vec{v}}$

Given A $n \times n$ matrix, λ is an eigenvalue of A with eigenvector $\vec{v}$ if $A\vec{v} = \lambda \vec{v}$.

Slight issue: $A\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$,

So then λ could be anything ...

A = $n \times n$ matrix
$\vec{v}$ = n dim'l vector
λ = scalar

Caveat:

$\vec{v}$ cannot be the zero vector.

λ can be 0 (scalar)

Ex: 17 is a scalar, (17) is a 1-dim'l vector.

$\underbrace{\hspace{3em}}_{\text{pretty useless}}$

Ex: $A = \begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}$

Goal: find eigenvalues and eigenvectors.

Seek $\vec{r}$, λ such that
($\vec{r}$: vec, λ: scalar)

$$A\vec{r} = \lambda\vec{r}.$$

Note: each eigenvec has an associated eval.

Warning: could be several evecs w/ same eval.

$$\iff A\vec{r} = \lambda \mathbb{I}\vec{r}$$

(recall $\mathbb{I}\vec{r} = \vec{r}$)

$$\iff (A - \lambda\mathbb{I})\vec{r} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Note: If $A - \lambda\mathbb{I}$ is invertible then

$$\vec{r} = (A - \lambda\mathbb{I})^{-1}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

so $\vec{r}$ would be the zero vector.

To avoid this, must assume $A - \lambda \mathbb{I}$ not invertible.

Hence, we can impose $|A - \lambda \mathbb{I}| = 0$.