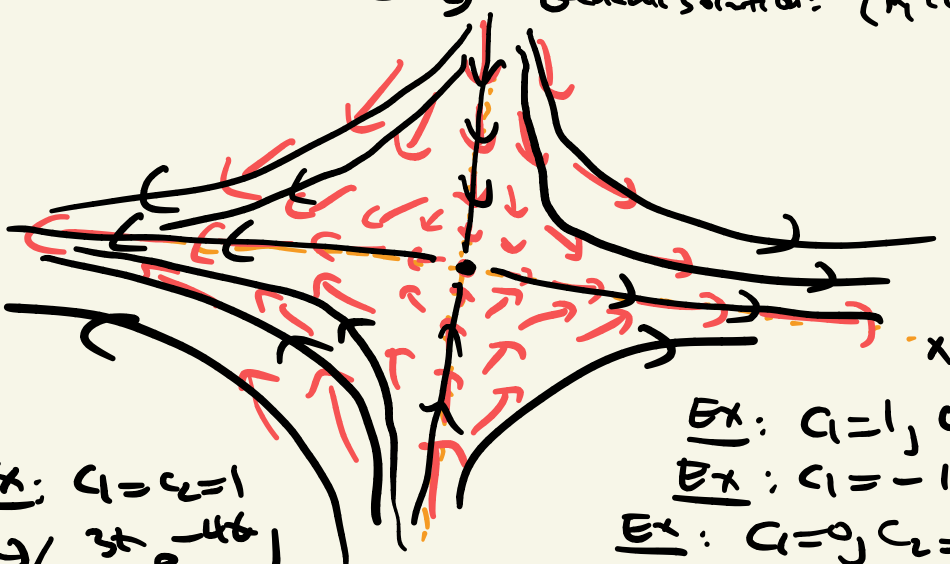
Lecture 23

Ex:
$$\begin{cases} x_1'(t) = 3x_1(t) \\ x_2'(t) = -4x_2(t) \end{cases} \rightsquigarrow \begin{matrix} x_1(t) = c_1 e^{3t} \\ x_2(t) = c_2 e^{-4t} \end{matrix}$$

General solution: $(x_1(t), x_2(t)) = (c_1 e^{3t}, c_2 e^{-4t})$

direction field: at (x,y) draw arrow in direction $(3x, -4y)$



phase portrait: sample of solutions

Ex: $c_1 = 1, c_2 = 0 \to (e^{3t}, 0)$

Ex: $c_1 = -1, c_2 = 0 \to (-e^{3t}, 0)$

Ex: $c_1 = 0, c_2 = 1 \to (0, e^{-4t})$

Ex: $c_1 = 0, c_2 = -1 \to (0, -e^{-4t})$

Ex: $c_1 = c_2 = 1 \to (e^{3t}, e^{-4t})$

This picture is called a "saddle point"!

Ex:
$$\begin{cases} x_1'(t) = -x_1/2 + 7x_2/2 \\ x_2'(t) = 7/2\, x_1 - x_2/2 \end{cases}$$

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

$$\vec{x}'(t) = A\vec{x}(t)$$

$$A = \begin{pmatrix} -1/2 & 7/2 \\ 7/2 & -1/2 \end{pmatrix}$$

Ansatz: $\vec{x}(t) = e^{rt}\vec{v}$ $\rightsquigarrow$ r has to be an eval, w/ evec $\vec{v}$.

Find evals and evecs:

$$\left| \begin{pmatrix} -1/2 & 7/2 \\ 7/2 & -1/2 \end{pmatrix} - \lambda \mathbb{1}_2 \right| = \begin{vmatrix} -1/2-\lambda & 7/2 \\ 7/2 & -1/2-\lambda \end{vmatrix}$$

$$= (-1/2-\lambda)^2 - 49/4$$

$$= (\lambda+1/2)^2 - 49/4$$

$$= \lambda^2 + \lambda + 1/4 - 49/4$$

$$= \lambda^2 + \lambda - 12$$

$$= (\lambda - 3)(\lambda + 4)$$

evals: $3, -4$

Find evecs:

$\lambda = 3$

$$\begin{pmatrix} -1/2 & 7/2 \\ 7/2 & -1/2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 3\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\to \begin{cases} -\tfrac{1}{2}a + \tfrac{7}{2}b = 3a \\ \tfrac{7}{2}a - \tfrac{b}{2} = 3b \end{cases} \rightsquigarrow \tfrac{7}{2}b = \tfrac{7}{2}a \Rightarrow b = a \qquad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

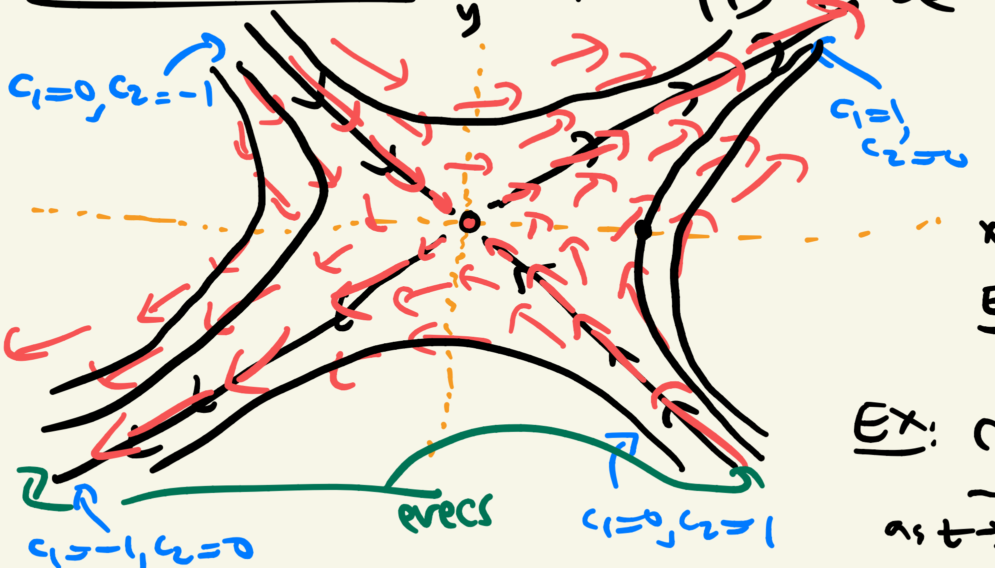
$\lambda = -4$

$$\begin{pmatrix} -\tfrac{1}{2} & \tfrac{7}{2} \\ \tfrac{7}{2} & -\tfrac{1}{2} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -4\begin{pmatrix} a \\ b \end{pmatrix} \rightsquigarrow -\tfrac{1}{2}a + \tfrac{7}{2}b = -4a$$

$$\tfrac{7}{2}a = -\tfrac{7}{2}b$$

$$b = -a \qquad \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

General soln: $c_1 e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-4t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$



Ex: $c_1 = c_2 = 0 \rightsquigarrow \vec{x}(t) = (0, 0)$

Ex: $c_1 = 1, c_2 = 0 \rightsquigarrow \vec{x}(t) = \underbrace{(e^{3t}, e^{3t})}_{\text{lies on line } y = x}$

Ex: $c_1 = 0, c_2 = 1$

$\vec{x}(t) = (e^{-4t}, -e^{-4t})$

Ex: $c_1 = c_2 = 1$

$\to \vec{x}(t) = (e^{3t} + e^{-4t}, e^{3t} - e^{-4t})$

as $t \to \infty$, asymptote to (e^{3t}, e^{3t})

"saddle point"

$$A = \begin{pmatrix} -1/2 & 1 \\ -1 & -1/2 \end{pmatrix}$$

Ex: $\vec{x}'(t) = \begin{pmatrix} -1/2 & 1 \\ -1 & -1/2 \end{pmatrix} \vec{x}(t)$

evals: $\left| \begin{pmatrix} -1/2 & 1 \\ -1 & -1/2 \end{pmatrix} - \lambda \mathbb{1}_2 \right| = \begin{vmatrix} -1/2-\lambda & 1 \\ -1 & -1/2-\lambda \end{vmatrix}$

$$= (\lambda + 1/2)^2 + 1 = \lambda^2 + \lambda + 5/4$$

$$\lambda = \frac{-1 \pm \sqrt{1 - 4 \cdot 5/4}}{2} = \frac{-1 \pm \sqrt{-4}}{2} = -1/2 \pm i$$

Recall: if A is real matrix, then complex evals appear in c.c. conj. pairs.

Suppose $A\vec{v} = (\alpha + i\beta)\vec{v}$

take conj. of both sides

$$\overline{A\vec{v}} = \overline{(\alpha + i\beta)\vec{v}}$$

$$A\overline{\vec{v}} = (\alpha - i\beta)\overline{\vec{v}}$$

so $\overline{\vec{v}}$ is an evec w/ eval $\alpha - i\beta$

<u>evecs</u>:

$\lambda = -1/2 + i \longrightarrow \begin{pmatrix} -1/2 & 1 \\ -1 & -1/2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = (-1/2 + i)\begin{pmatrix} a \\ b \end{pmatrix}$

$\longrightarrow \cancel{-1/2\, a} + b = (\cancel{-1/2} + i)a$

$b = ia$

can take $\begin{pmatrix} 1 \\ i \end{pmatrix}$

$\lambda = -1/2 - i \longrightarrow$ can take $\begin{pmatrix} 1 \\ -i \end{pmatrix}$

"General solution" $\vec{x}(t) = C_1 e^{(-1/2+i)t}\begin{pmatrix} 1 \\ i \end{pmatrix} + C_2 e^{(-1/2-i)t}\begin{pmatrix} 1 \\ -i \end{pmatrix}$

To get a real-valued general sol'n, must take real and imag parts.

Start w/ $e^{(-1/2+i)t}\begin{pmatrix} 1 \\ i \end{pmatrix} = e^{-t/2} e^{it}\begin{pmatrix} 1 \\ i \end{pmatrix}$

$= e^{-t/2}\left(\cos(t) + i\sin(t)\right)\begin{pmatrix} 1 \\ i \end{pmatrix}$

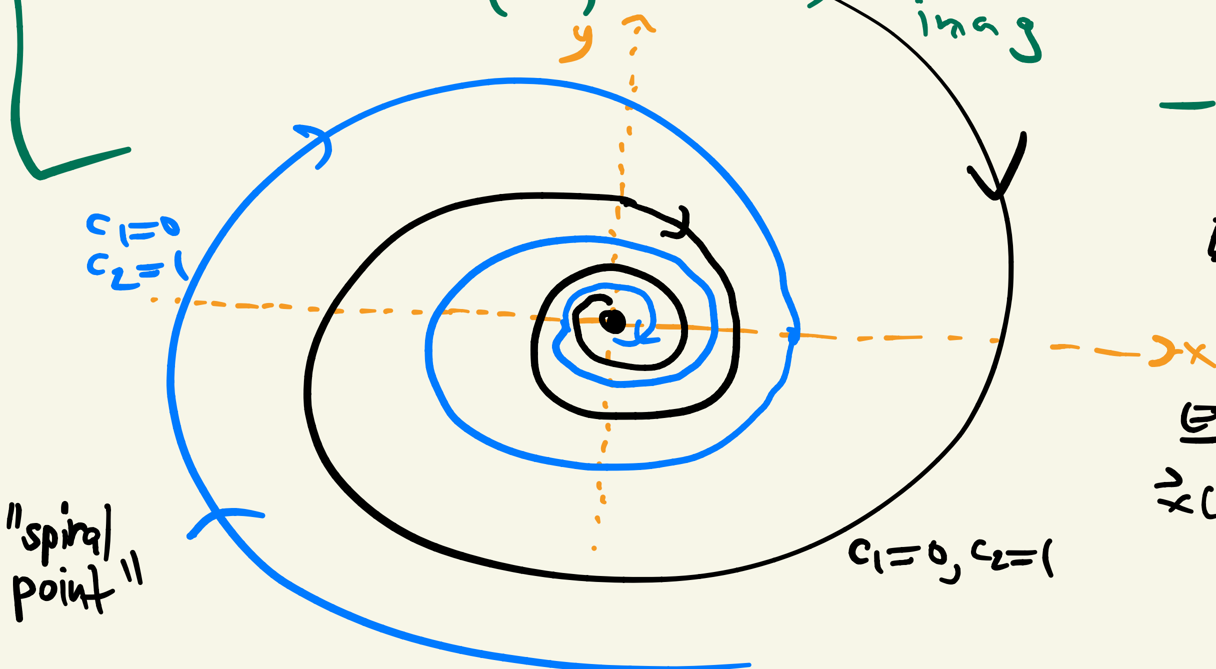
$= e^{-t/2}\begin{pmatrix} \cos(t) + i\sin(t) \\ i\cos(t) - \sin(t) \end{pmatrix}$

real part: $e^{-t/2}\begin{pmatrix}\cos(t)\\-\sin(t)\end{pmatrix}$ imag part: $e^{-t/2}\begin{pmatrix}\sin(t)\\\cos(t)\end{pmatrix}$

General soln: $\underbrace{C_1 e^{-t/2}\begin{pmatrix}\cos(t)\\-\sin(t)\end{pmatrix} + C_2 e^{-t/2}\begin{pmatrix}\sin(t)\\\cos(t)\end{pmatrix}}$

Note: $e^{(-1/2-i)t}\begin{pmatrix}1\\-i\end{pmatrix} \longrightarrow$ real part $e^{-t/2}\begin{pmatrix}\cos(t)\\-\sin(t)\end{pmatrix}$

imag $-e^{-t/2}\begin{pmatrix}\sin(t)\\\cos(t)\end{pmatrix}$



Ex: $c_1=1,\ c_2=0$

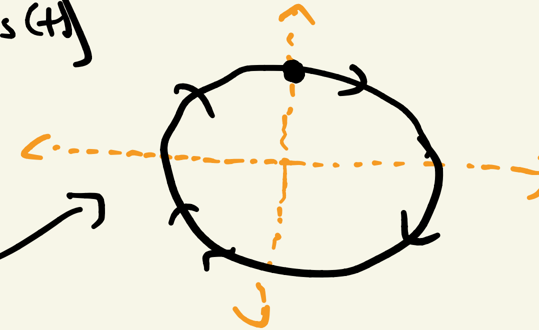
$\vec{x}(t) = e^{-t/2}\begin{pmatrix}\cos(t)\\-\sin(t)\end{pmatrix}$

Ex. $c_1=0,\ c_2=1$

$\vec{x}(t) = e^{-t/2}\begin{pmatrix}\sin(t)\\\cos(t)\end{pmatrix}$

Ex: "Warm up" $\vec{x}(t) = (C\sin(t), C\cos(t))$

circle of radius C



Q: Is there a system $\vec{x}' = A\vec{x}$ which spirals outwards?

If A has evals $\alpha \pm i\beta$ with $\alpha > 0$, then solutions spiral outwards.

Ex: $\vec{x}'(t) = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix} \vec{x}(t)$

evals:

$$\begin{vmatrix} -3-\lambda & \sqrt{2} \\ \sqrt{2} & -2-\lambda \end{vmatrix} = (\lambda+2)(\lambda+3) - 2$$

$$= \lambda^2 + 5\lambda + 4$$

$$= (\lambda+1)(\lambda+4)$$

evals: $-1, -4$

evecs:

$\lambda=-1$: $\begin{pmatrix}-3 & \sqrt{2}\\ \sqrt{2} & -2\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = -\begin{pmatrix}a\\ b\end{pmatrix} \to -3a+\sqrt{2}b=-a$

$\sqrt{2}b=2a$

$b=\sqrt{2}a$ $\quad \begin{pmatrix}1\\ \sqrt{2}\end{pmatrix}$

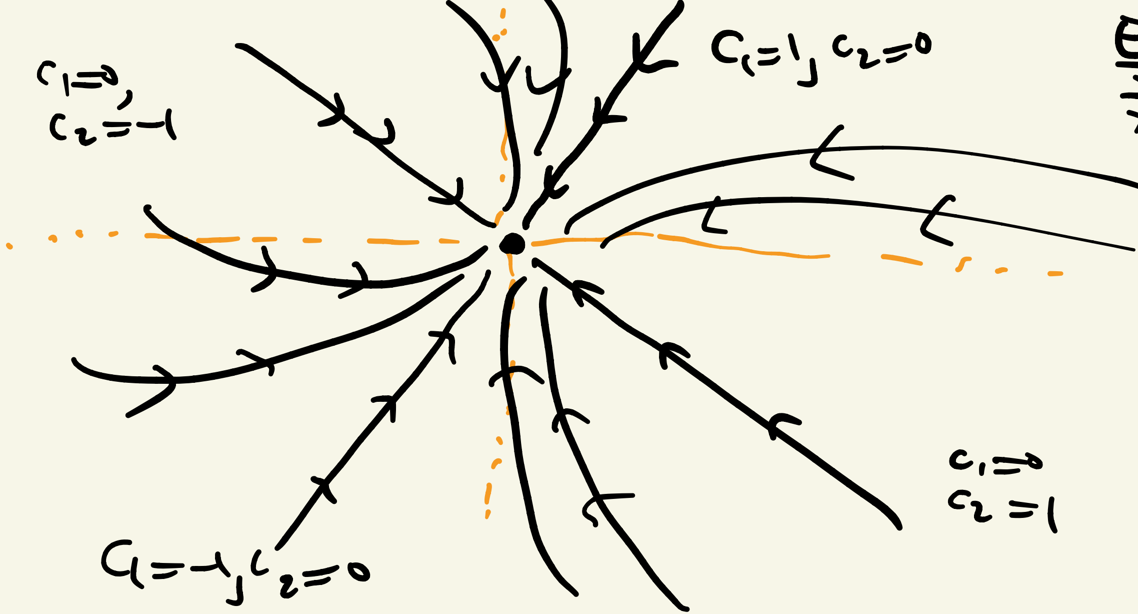
$\lambda=-4$: $\begin{pmatrix}-3 & \sqrt{2}\\ \sqrt{2} & -2\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = -4\begin{pmatrix}a\\ b\end{pmatrix}$

$-3a+\sqrt{2}b=-4a \to a=-\sqrt{2}b \Rightarrow b=-a/\sqrt{2}$ $\quad \cancel{\begin{pmatrix}1\\ -1/\sqrt{2}\end{pmatrix}} \quad \begin{pmatrix}\sqrt{2}\\ -1\end{pmatrix}$

Note: $\begin{pmatrix}1\\ \sqrt{2}\end{pmatrix}$ is orthogonal to $\begin{pmatrix}\sqrt{2}\\ -1\end{pmatrix}$.

In fact, if A is a symmetric matrix, then the evecs are automatically orthogonal

General solution: $\vec{x}(t) = c_1 e^{-t}\begin{pmatrix}1\\ \sqrt{2}\end{pmatrix} + c_2 e^{-4t}\begin{pmatrix}\sqrt{2}\\ -1\end{pmatrix}$



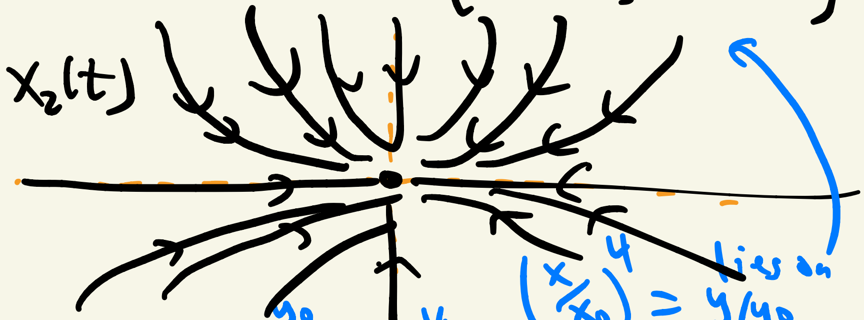
Ex: $c_1=1, \ c_2=0$

$\vec{x}(t)= e^{-t}\begin{pmatrix}1\\ \sqrt{2}\end{pmatrix}$

"nodal sink"

Analogy: $\begin{cases} x_1'(t) = -x_1(t) \\ x_2'(t) = -4x_2(t) \end{cases} \longrightarrow \vec{x}(t) = \left(x_0 e^{-t}, y_0 e^{-4t}\right)$

Ex: $\vec{x}(t) = \underbrace{\left(e^{-t}, e^{-4t}\right)}_{\text{lies on } y=x^4}$



"nodal sink": 2 distinct negative evals

2 distinct positive evals

"nodal source"

Q: What's the picture for two equal negative evals? (and two distinct evecs)

Ex: $\begin{cases} x_1'(t) = -3x_1(t) \\ x_2'(t) = -3x_2(t) \end{cases}$

evals $\lambda_1 = \lambda_2 = -3$.

Any nonzero vec is an evec.