

Math 635

USC Spring 2026

*Cluster Varieties:
Algebra, Topology, Geometry, Duality*

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Handwritten Lecture Notes

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Cluster varieties: algebra, topology, geometry, duality

Lecture 1

1/12/26

Roughly speaking:

- a cluster variety is a complex algebraic variety obtained by gluing together many copies of $(\mathbb{C}^*)^n$ where the gluing maps take a very particular form
- a cluster algebra is the algebra of regular functions $f: V \rightarrow \mathbb{C}$ on a cluster variety

Fomin-Zelevinsky, early '00s: introduced cluster algebras
Arise in many parts of math and physics as kind of "universal model" for mutation/wall-crossing phenomena:

- quiver representation theory
- ~~Dehn~~ Teichmüller theory
- Poisson geometry
- Grassmannians
- total positivity
- QFT scattering amplitudes (amplitude amplituhedron)
- integrable systems
- string theory (BPS states), etc

Gross-Hacking-Kontsevich 1/9:

- constructed canonical bases for cluster algebras
- established ~~positivity of the Laurent phenomenon~~ positive Laurent phenomenon
- proof uses mirror symmetry for log Calabi-Yau varieties

many strong applications
in representation theory, e.g.
canonical bases for
finite-dimensional irreducible
representations of $SL_n(\mathbb{C})$

can think of as generalization
of toric varieties

(related to almost toric
fibrations in symplectic geometry)

originally found independently
by Lusztig and
Kashiwara in early 90s
using quantum groups

amazingly, the construction
of GTKK uses only
general geometry - no
rep. theory!

Total positivity

Def : $A \in \text{Mat}_{n \times n}(\mathbb{R})$ is totally positive (TP) if all of its minors are positive.

Gantmacher-Krein '30's : $A \text{ TP} \Rightarrow$ eigenvalues are real, positive, and distinct

Binet-Cauchy theorem : The TP matrices in $G = \text{SL}_n(\mathbb{C}) \times \text{GL}_n(\mathbb{C})$ are closed under multiplication, and hence form a multiplicative semigroup $G_{>0}$.

Lusztig : Extended definition of $G_{>0}$ for other semisimple Lie groups G .

More generally : If a given complex algebraic variety Z ~~comes~~ has a distinguished family Δ of regular functions $Z \rightarrow \mathbb{C}$, we define the TP variety by

$$Z_{>0} := \{ z \in Z \mid \begin{matrix} \text{for } z \in Z \\ f(z) > 0 \end{matrix} \forall f \in \Delta \}$$

Ex : For $Z = \text{Mat}_{n \times n}(\mathbb{C})$, $\text{GL}_n(\mathbb{C})$, $\text{SL}_n(\mathbb{C})$, we recover above notion of TP, $\Delta = \text{minors}$,

Ex : Grassmannian $\text{Gr}_{k \times m}(\mathbb{C}) = \{ k\text{-dim linear subspaces of } \mathbb{C}^m \}$
 $\Delta = \text{Plücker coordinates}$

Ex : partial flag manifolds, homogeneous spaces for semisimple complex Lie groups, etc. Slight scaling ambiguity

Lemma : $A \in \text{Mat}_{n \times n}$ has $\binom{2n}{n} - 1$ minors

$$\text{pf} : \# = \sum_{k=1}^n \binom{n}{k} \binom{n}{k}$$

Vandermonde's identity : $\binom{m+w}{r} = \sum_{k=0}^n \binom{m}{k} \binom{w}{r-k}$

$$\text{Setting } m=w=r=n \Rightarrow \binom{2n}{n} = \sum_{k=0}^n \binom{n}{k} \binom{n}{k}$$

(both sides count:
 Given committee with n men
 \sim women,
 how many subcommittees with r members?)

Q : Can we check that $A \in \text{Mat}_{n \times n}$ is TP testing a subset of the $\binom{2n}{n} - 1$ minors? How many tests are needed?

by only

i.e. want
 "efficient
 TP
 testing"

$$\text{Ex} : A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Mat}_{2 \times 2}$$

$$\delta := ad - bc \Rightarrow d = \frac{a + bc}{a}.$$

So if $a, b, c, \delta > 0$, so is δ .

Reduce $\binom{4}{2} - 1 = 5$ checks to 4 checks.

Plücker coordinates on Grassmannians:

Given $A \in \text{Mat}_{k \times m}$ $\rightarrow \text{row span } [A] \in \text{Gr}_{k, m}$
 If rank k

For $J \subset \{1, \dots, m\}$ \rightarrow Plücker coordinates
 $|J|=k$ $P_J(A) := k \times k$ minor of A corresponding
 to J

Note: For $A, B \in \text{Mat}_{k \times m}$ with $[A] = [B]$ (i.e. same row spans)
 $(P_J(A))_{|J|=k}$ and $(P_J(B))_{|J|=k}$ agree up to common rescaling, i.e. get
 $\text{Gr}_{k, m} \rightarrow \mathbb{CP}^N$ for $N = \binom{m}{k} - 1$.

In fact this is an embedding, the Plücker embedding.

Let $\mathbb{C}[\text{Mat}_{k \times m}] = \text{word. ring of } \text{Mat}_{k \times m}$, i.e. the polynomial algebra in variables ~~x_{ij}~~ x_{ij} for $1 \leq i \leq k$
 $1 \leq j \leq m$

Def: The Plücker ring $R_{k, m}$ is the subring of $\mathbb{C}[\text{Mat}_{k \times m}]$ generated by P_J over $J \in \{1, \dots, m\}, |J|=k$.

Claim: the ideal of relations in $R_{k, m}$ is generated by certain quadratic relations called the Grassmann-Plücker relations.

Def: The totally positive Grassmannian $\text{Gr}_{k, m}^+$ is the subset of $\text{Gr}_{k, m}$ of those pts whose Plücker coords are all positive (up to common scaling).

Note: For $A \in \text{Mat}_{k \times m}(\mathbb{R})$, $[A] \in \text{Gr}_{k, m}^+$ iff all $k \times k$ minors of A have the same sign.

Q: For $A \in \text{Mat}_{k \times m}(\mathbb{R})$, can we verify that all $k \times k$ minors are positive by only checking a subset of the $\binom{m}{k}$ minors?
 How many tests are needed? positive wlog

Positivity testing for $\text{Gr}_{2,m}$

Claim: Given $A \in \text{Mat}_{2 \times m}$, put $P_{ij} := P_{\{i,j\}}$ for $1 \leq i, j \leq m$.
 To check that all 2×2 minors $P_{ij}(A) \geq 0$, suffices to check only $2m-3$ special ones.

Note: $2m-3 = \dim \text{Gr}_{2,m} + 1$

Lemma: For $1 \leq i_1 < i_2 < i_3 \leq m$, have three-term Grassmann-Plücker relations:

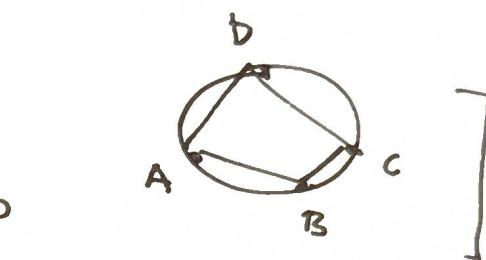
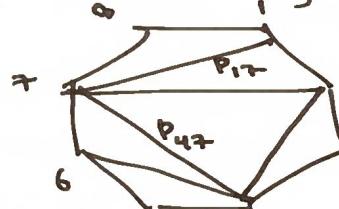
$$P_{i_1 k} P_{i_2 l} = P_{i_1 i_2} P_{k l} + P_{i_1 l} P_{i_2 k}$$

Rmk: For inscribed quadrilateral Ptolemy's thm (2nd century) gives

$$AC \cdot BD = AB \cdot CD + BC \cdot AD$$

Ex: $A = \begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix}$ v/s $P_{13} P_{24} = P_{12} P_{34} + P_{14} P_{23}$, i.e.
 $(ag-ce)(bh-df) = (af-be)(ch-dg) + (ah-de)(bg-cf)$ ✓

Put $P_m = \text{regular } m\text{-gon}$, $T = \text{triangulation}$.



To each side or diagonal associate P_{ij} , where i, j are the end pts

Cluster variables: P_{ij} ranging over diagonals
frozen variables: P_{ij} ranging over sides
extended cluster: $\{ \text{cluster vars} \} \cup \{ \text{frozen vars} \} =: \tilde{x}(T)$

Note: extended cluster has $2m-3$ vars, and we claim that these are algebraically independent.

Ex: In above picture, have cluster variables $P_{13}, P_{23}, P_{15}, P_{45}$
 frozen variables $P_{12}, P_{34}, \dots, P_{28}, P_{16}$

Thm: Each P_{ij} for $1 \leq i < j \leq n$ subtraction-free rational expression can be written as a rational expression in the elements of a given extended cluster $\tilde{x}(T)$.

Cor: For $A \in \text{Mat}_{2 \times m}$, positively evaluating it each $P_{ij} \in \tilde{x}(T)$ on gives $A \in \text{Mat}_{2 \times m}$, then all of the 2×2 minors of A are positive.

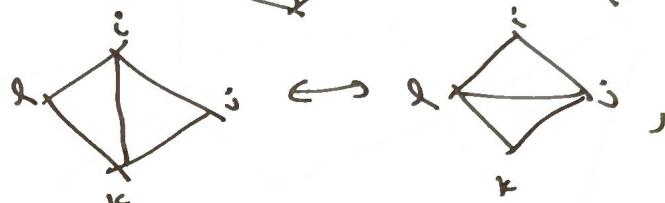
(2) of these $2m-3$ of these

Pf of thm: Follows by combining

- (1) each P_{ij} appears as an elt of an extended cluster $\tilde{x}(T)$ for some triangulation T of \mathbb{P}_m
- (2) any two triangulations of \mathbb{P}_m are related by a sequence of flips



(3) For a flip



replace P_{ik} with P_{li} .

Using three-term GP relation, have $P_{ik} = \frac{P_{ij}P_{lk} + P_{il}P_{jk}}{P_{il}}$

Rank: In fact, each Plücker coordinate P_{ij} can be written as a Laurent polynomial with positive coefficients in the Plücker coordinates from $\tilde{x}(T)$.

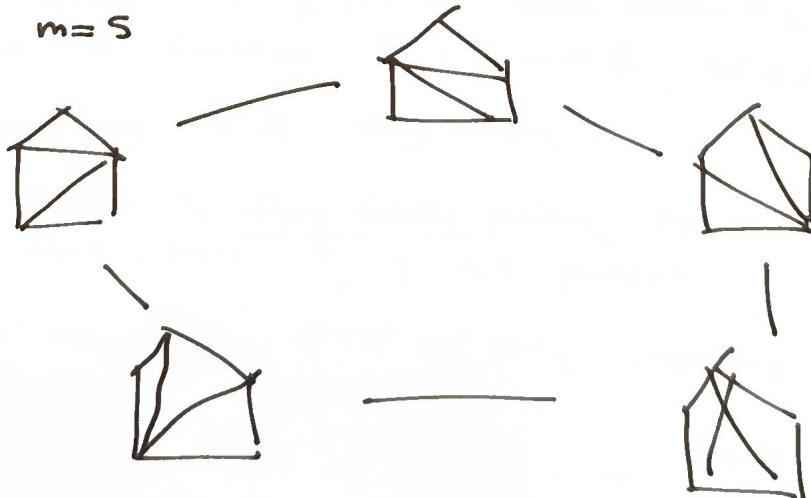
Example of ~~possibly~~ positive Laurent phenomenon.

Combinatorics of flips encoded by graph:

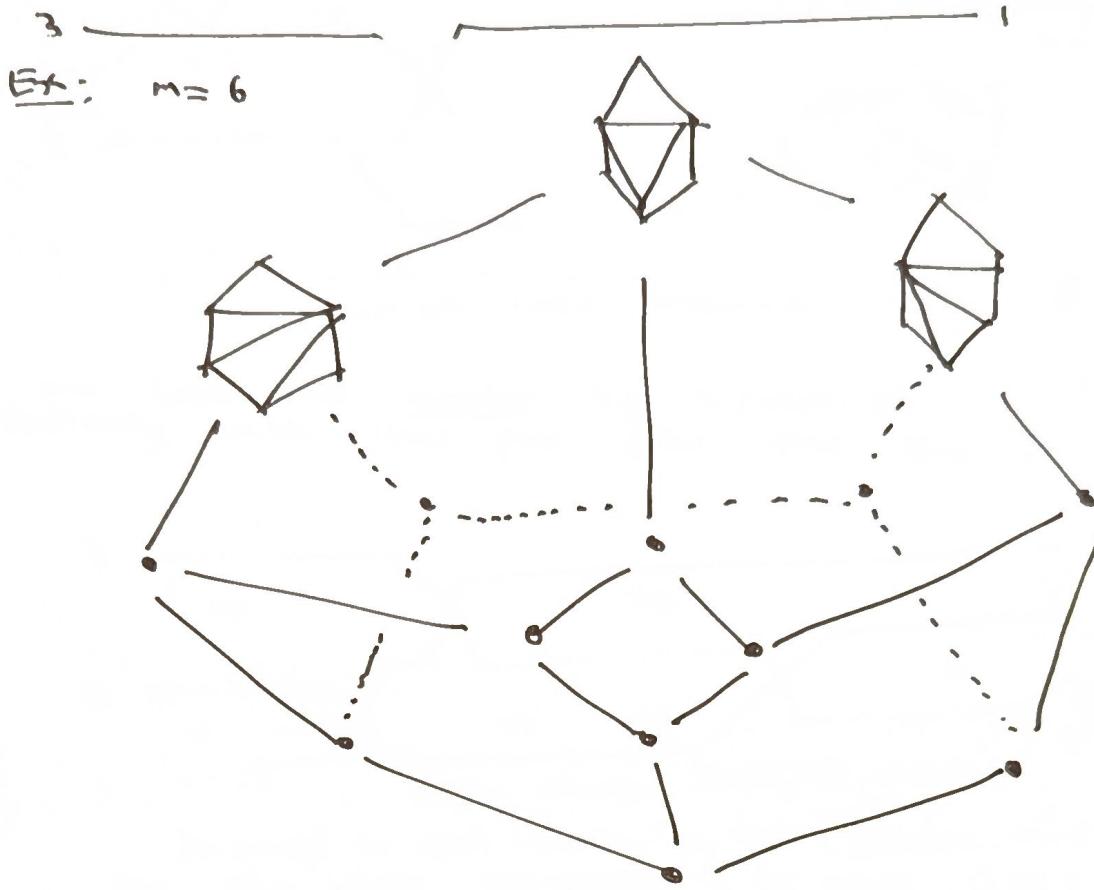
- vertices are ~~triangulations~~ triangulations
- edges are flips

Each vertex has degree $m-3$. In fact, this is the 1-skeleton of an $(m-3)$ -diml convex polytope called the associahedron (discovered by Stasheff).

Ex: $m=5$



Wiring diagrams:



Def: A cluster monomial is a monomial in the variables of a given extended cluster $\tilde{x}(\tau)$.

Thm (19th century invariant theory): The set of all cluster monomials give a linear basis for the Plücker ring $R_{2,n}$.

Lecture 2

11/11/25

Before moving to TP for non matrices, we discuss an intermediate notion called "flag positivity". Put $G = SL_n$.

Def. Given $J \subsetneq \{1, \dots, n\}$ non empty, the flag minor P_J is the function $P_J: G \rightarrow \mathbb{Q}$, $z = (z_{ij}) \mapsto \det(z_{ij}) \mid i \in |J|, j \in J$

Note: there are $2^n - 2$ flag minors.

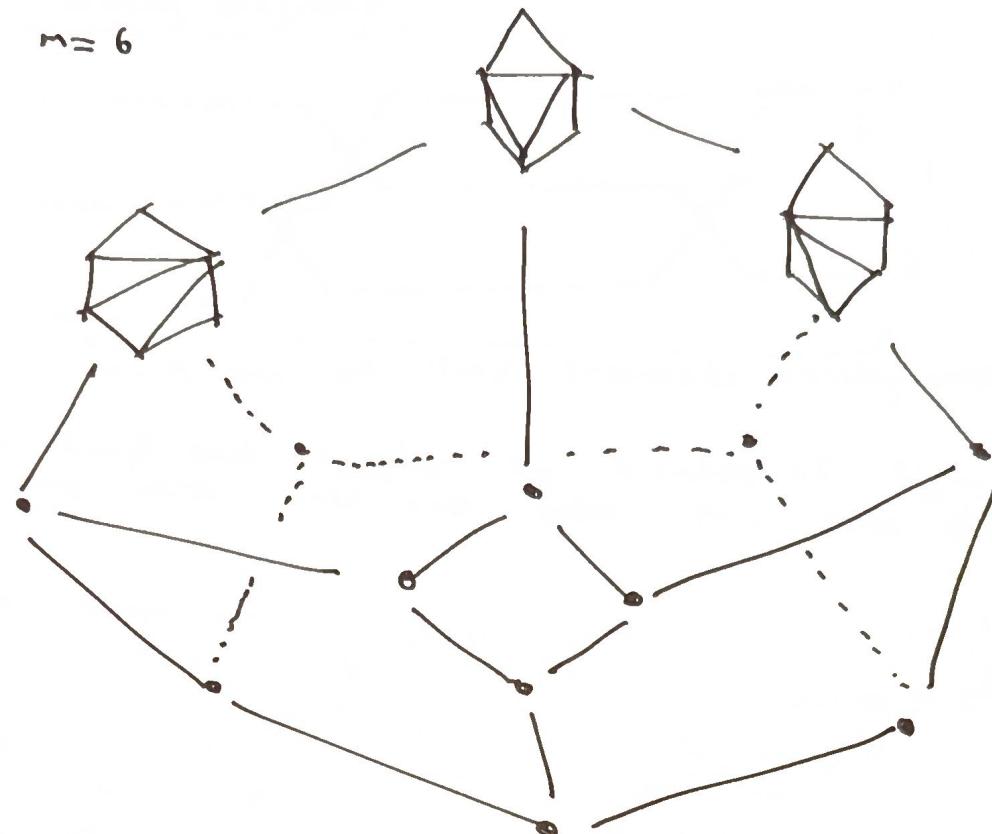
Def: $z \in G$ is flag totally positive (FTP) if all flag minors $P_J(z)$ are positive.

$|J| \times |J|$ minor which is "top-justified"

Q: Can we check FTP by only checking a subset of the $2^n - 2$ flag minors.

Claim: It suffices to check only $\frac{(n-1)(n+2)}{2}$ special flag minors.

Ex: $n=6$



Def: A cluster monomial is a monomial in the variables of a given extended cluster $\tilde{x}(T)$.

Thm (19th century invariant theory): The set of all cluster monomials give a linear basis for the Plücker ring $R_{3, n}$.

Lecture 2

11/11/26

Before moving to TP for $n \times n$ matrices, we discuss an intermediate notion called "flag positivity". Put $G = SL_n$.

Def: Given $J \subseteq \{1, \dots, n\}$ nonempty, the flag minor P_J is the function $P_J: G \rightarrow \mathbb{Q}$, $z = (z_{ij}) \mapsto \det(z_{ij})$ for $i \in |J|$, $j \in J$.
Note: there are $2^n - 2$ flag minors.

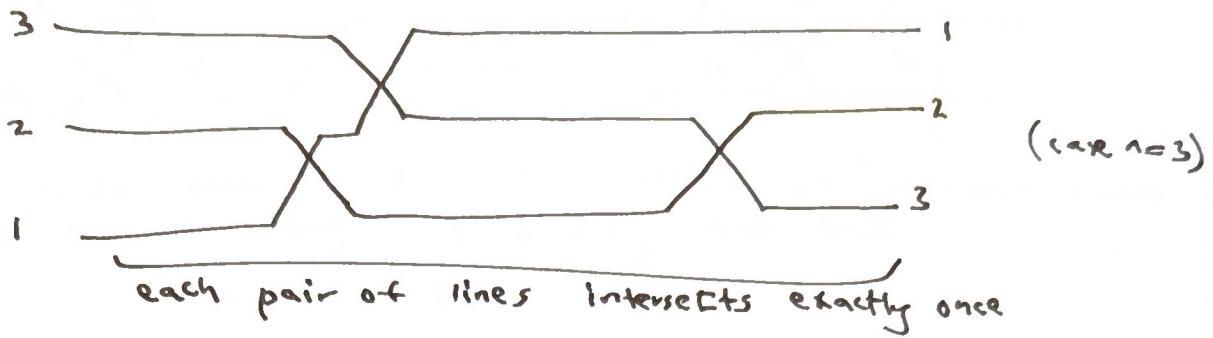
Def: $z \in G$ is flag totally positive (FTP) if all flag minors $P_J(z)$ are positive.

$|J| \times |J|$ minor
which is
"top-justified"

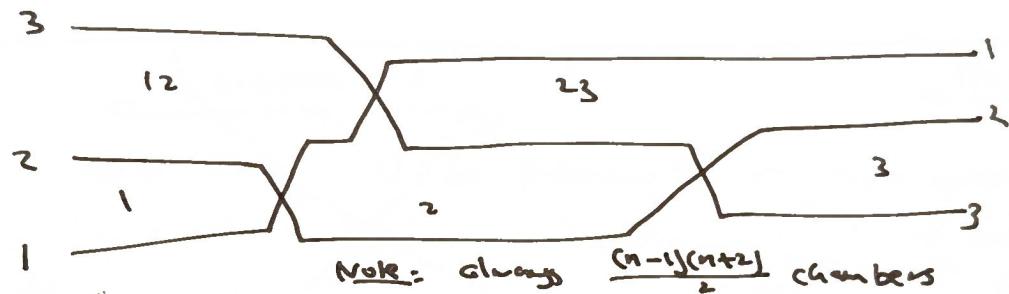
Q: Can we check FTP by only checking a subset of the $2^n - 2$ flag minors.

Claim: It suffices to check only $\frac{(n-1)(n+2)}{2}$ special

Wiring diagrams:



We label each chamber indicating which lines pass below that chamber



Associated to each chamber is its chamber minor P_J the flag minor corresponding to its subset $J \subseteq \{1, \dots, n\}$.

extended cluster: all chamber minors of a wiring diagram
cluster variables: the chamber minors for bounded chambers
frozen variables: the chamber minors for unbounded chambers

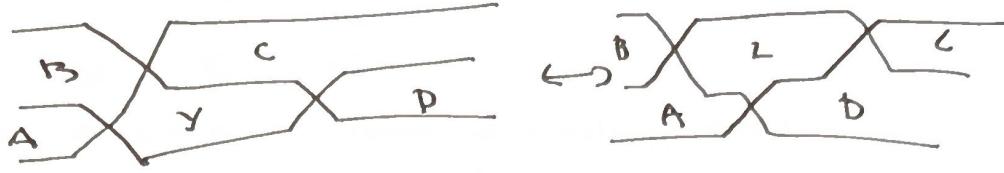
$\frac{n-1}{2}$ of these

$\binom{n-1}{2}$

Thm Every flag minor can be written as a subtraction-free rat'l expr in the chamber minors of a given wiring diag.
Car: If there $\frac{(n-1)(n+2)}{2}$ evaluate positively at a matrix $z \in \text{SL}_n$, then z is FTFP.

Prf: Follows by

- (1) each flag minor appears as a chamber minor in some wiring diagram
- (2) any two wiring diagrams can be transformed into each other by a sequence of local braid moves



(3) Under each braid move, collection of chamber minors changes by exchanging $Y \leftrightarrow Z$, and have
 $YZ = AC + BD$

Point: In fact, each flag minor can be written as a Laurent poly with pos. coeffs in the chamber minors of a given ~~wire~~ wiring diagram.

Lecture 3

17/2/26

Put $G = SL_n$, $U \times G$ subgroup of unipotent lower-triangulars
i.e. lower triangular matrices with 1s on diagonal

$U \times G$ left multiplication action

$\rightarrow U \times G[G] =$ ring of polynomials in the matrix entries of $A \in G$

$\mathbb{C}[G]^U =$ ring of U -invariant polynomials

Note: $\begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a + \beta c & \alpha b + \beta d \\ \gamma a & \gamma b \end{pmatrix}$

... $\begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} -v_1 \\ -v_2 \end{pmatrix} = \begin{pmatrix} -\alpha v_1 + \beta v_2 \\ \gamma v_1 \end{pmatrix}$

Similarly,

$$\begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \delta & \varepsilon \\ 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} -v_1 \\ -v_2 \\ -v_3 \end{pmatrix} = \begin{pmatrix} -\alpha v_1 + \beta v_2 + \gamma v_3 \\ -\delta v_1 + \varepsilon v_2 \\ \gamma v_1 \end{pmatrix}$$

i.e. $P \in \mathbb{C}[G]$
s.t. $P(yz) = P(z)$
 $y \in U, z \in G$

Def: The full flag variety in \mathbb{C}^n is

$\{ \sum c_i v_i \in V_1, c_i v_i \in V_{i+1} \subset \mathbb{C}^n \text{ for } i=1, \dots, n-1 \}$

This can be identified with the homogeneous space

G/B , where

$B \subset G$ is the subgroup

Lecture 3

1/23/26

Put $G = \mathrm{SL}_n(\mathbb{C})$

$B \subset G$ subgroup of lower triangular matrices

$V \subset G$ subgroup of unipotent lower triangular matrices

i.e. 1's on
diagonal

Borel
subgroup

Note:

$$\begin{pmatrix} \alpha & 0 \\ \beta & \gamma \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha b \\ \beta a + \gamma c & \beta b + \gamma d \end{pmatrix}, \quad \text{i.e.}$$

$$\begin{pmatrix} \alpha & 0 \\ \beta & \gamma \end{pmatrix} \begin{pmatrix} -r_1 \\ -r_2 \end{pmatrix} = \begin{pmatrix} -\alpha r_1 \\ -\beta r_1 + \gamma r_2 \end{pmatrix}$$

Similarly,

$$\begin{pmatrix} \alpha & 0 & 0 \\ \beta & \gamma & 0 \\ 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} -r_1 \\ -r_2 \\ -r_3 \end{pmatrix} = \begin{pmatrix} -\alpha r_1 \\ -\beta r_1 + \gamma r_2 \\ -\gamma r_1 + \gamma r_2 + \gamma r_3 \end{pmatrix} \quad \text{etc}$$

Def: The (full) flag variety

$\{ \{v_i\} \subset V, v_i \subset \dots \subset V_{n-1} \subset \mathbb{C}^n \mid v_i \text{ is an } i\text{-dimensional subspace for } i=1, \dots, n-1 \}$

Exercise: This is identified with the homogeneous space

Def: The basic affine space is ~~\mathbb{C}^n~~ \mathbb{C}/G

Note that we have

the basic affine space $\mathbb{C}^n \hookrightarrow \mathbb{C}/G \rightarrow B/G$, i.e.

Here $V \times G$ action by left multiplication

$\rightsquigarrow V \times G[G] = \text{ring of polynomials in the entries of } A \in \mathrm{SL}_n$

$\mathbb{C}[G]^V = \text{ring of } V\text{-invariant polynomials}$

Claim:

(1) the flag minors generate $\mathbb{C}[G]^V$ by First and Second Fundamental Theorems of invariant theory

(2) the ideal of quadratic relations among flag minors is generated by quadratic relations called "generalized Plücker relations"

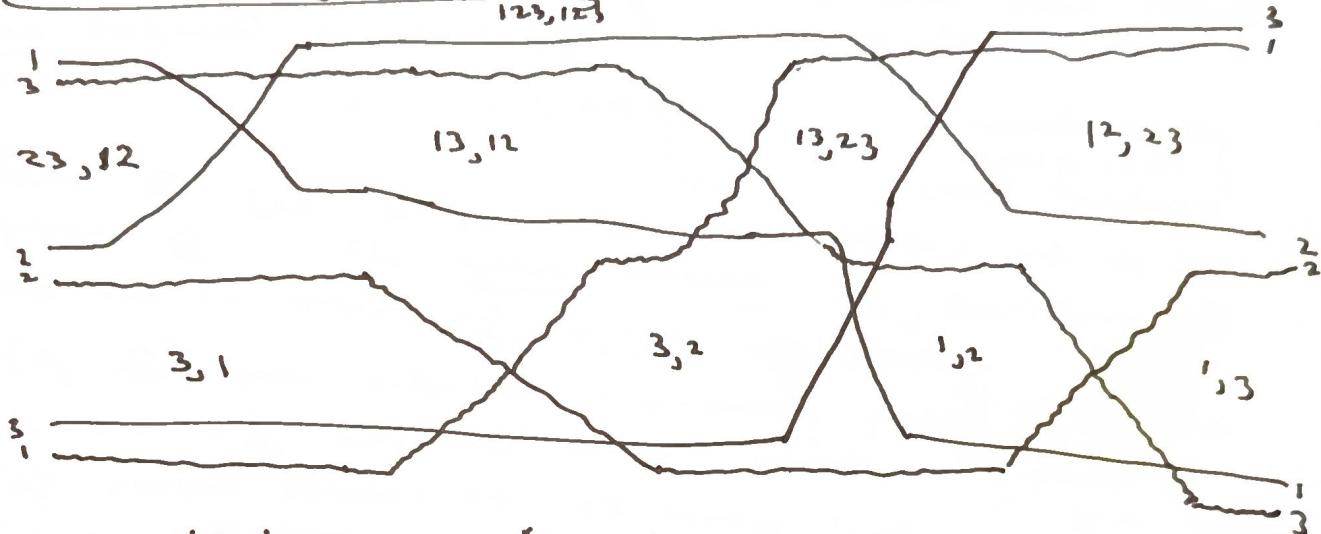
i.e. $P \in \mathbb{C}[G]$
s.t. $P(yz) = P(xy) + P(yz)$
 $y \in V, z \in G$

Checking TP for general non matrices

Given $I, J \subset \{1, \dots, n\}$ of same cardinality, put
 $\Delta_{I,J} :=$ minor determined by rows in I and columns in J

Thus $\mathbf{z} \in \text{Mat}_{nn}$ is TP $\iff \Delta_{I,J} (\pm) > 0$ for all
 $I, J \subset \{1, \dots, n\}$ with $|I| = |J|$

Double wiring diagrams:



\implies chamber minors $\Delta_{3,1}, \Delta_{3,2}, \Delta_{1,2}, \Delta_{1,3}, \Delta_{23,12}, \Delta_{13,23}, \Delta_{13,12}, \Delta_{12,23}, \Delta_{12,13}, \Delta_{123,123}$

Claim: number of chamber minors for a double wiring diagram is always n^2 minors

Then: Every minor of an nxn matrix can be written as a subtraction-free rational expression in the chamber minors of a given double wiring diagram.

Cor: Only need n^2 tests for positivity.

pt idea:

- (1) every minor is a chamber minor for some double wiring diagram
- (2) any two double wiring diagrams are related by sequence of local moves of three different kinds
- (3) each local move results in an exchange of minors $\gamma \leftrightarrow \gamma'$, where $\gamma' = \gamma - A\gamma + B\gamma$.

Rank: In fact in this we really have Laurent polynomials with positive coefficients.

Rank: The graph with vertices double wiring diagrams and edges local moves is not regular, but this will be rectified by the theory of cluster algebras.

Quivers and their mutations

Def: A quiver is a finite oriented graph with no loops or oriented 2-cycles.

Ex:



Def: An ice quiver is a quiver in which some vertices are designated as "frozen", and no arrows between two frozen vertices.



Def:

Let \mathbb{K} be a quiver. The quiver mutation into new ice quiver $\mathbb{Q}' = \mu_k(\mathbb{Q})$ or

- (1) for each oriented two-arrow path $i \rightarrow k \rightarrow j$, add new arrow $i \rightarrow j$ (unless i, j both frozen)
- (2) reverse direction of all arrows incident to k
- (3) repeatedly reverse any oriented 2-cycles until none left

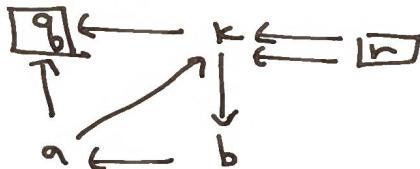
non-frozen vertices will be called "mutable"

vertex of an ice quiver transforms \mathbb{Q} follows:

~~if $i \rightarrow k \rightarrow j$~~

~~unless i, j both frozen~~

Ex:



μ_k



Exercise:

- (1) mutation is an involution i.e. $\mu_k(\mu_k(\mathbb{Q})) = \mathbb{Q}$
- (2) mutation commutes with reversing orientation of all arrows

(3) if k, l then

mutable vertices with no arrows between $\mu_l(\mu_k(\mathbb{Q})) = \mu_k(\mu_l(\mathbb{Q}))$

Rank: If \mathbb{Q} has k arrows incident

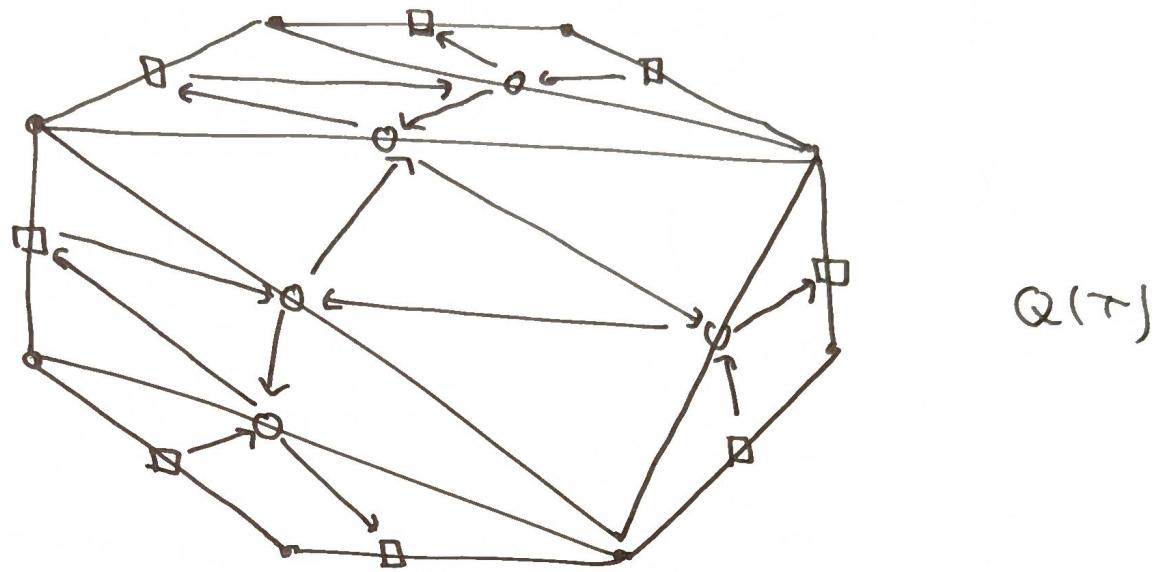
to k , μ_k simply reverses all

Exercise: If \mathbb{Q} is a tree with no frozen, can get from any orientation to any other by a sequence of mutations.

at sinks and sources.

Triangulation and quiver

Can define a quiver from a ~~triangulation~~ triangulation T of P_m .



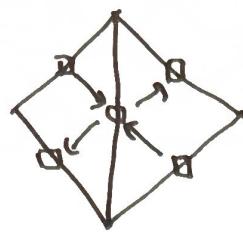
Exercise: If T is a triangulation of P_m and T' obtained by flip along diagonal γ , then

$$Q(T') = \mu_\gamma(Q(T))$$

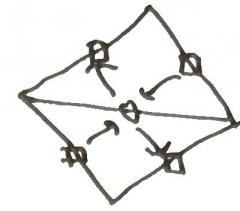
Lecture 34

1/25/26

Ex:

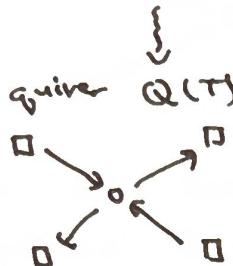


flip

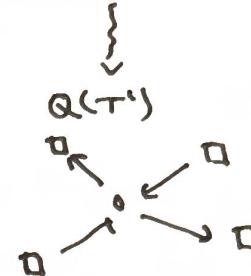


T'

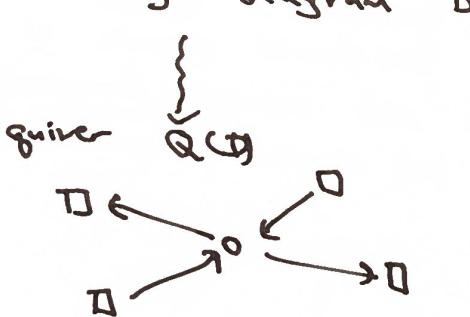
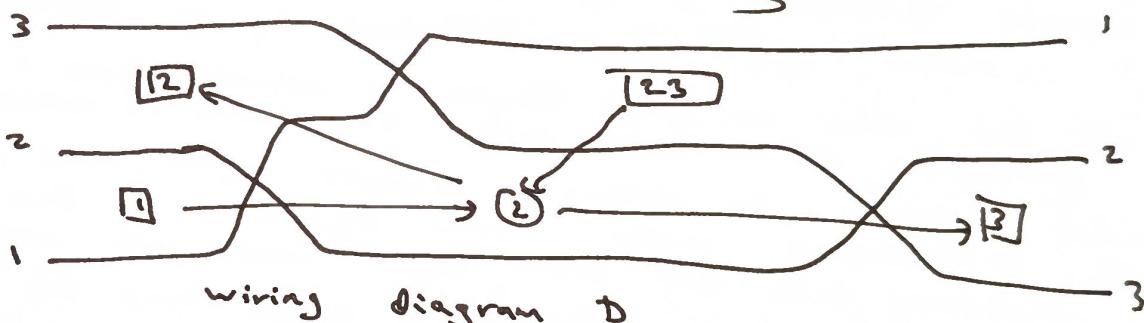
$T = \text{triangulation of } RP_4$



mutation



wiring diagram \rightsquigarrow quiver



vertices: chambers of \mathfrak{p}
(mutable if bounded,
else frozen)

arrows: for chambers c, c'
here $c \rightarrow c'$ in $Q(D)$ if
one of following holds.

- (i) right end of c = left end of c'
- (ii) left end of c is directly above c' ,
right end of c is directly below c'

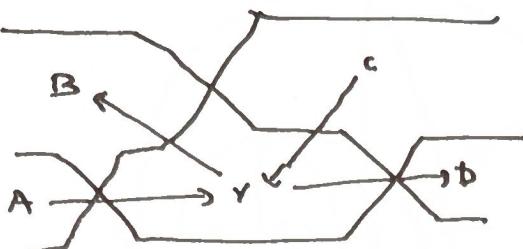


- (iii) left end of right end of c' is directly below c'
directly above c

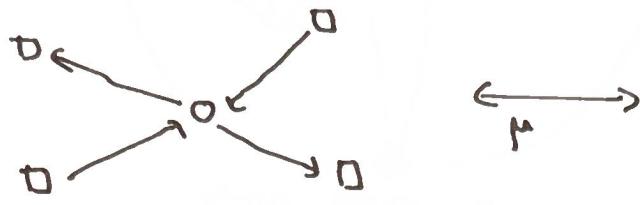
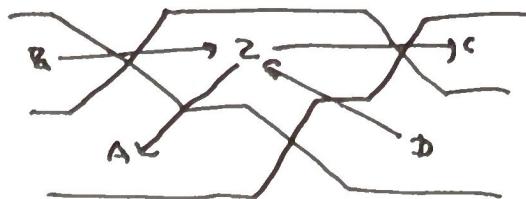


Exercise: If D, D' wiring diagrams related by a braid move at chamber Y , then $(Q(D))' = \mu_Y(Q(D))$.

Ex:



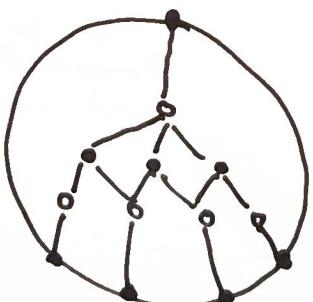
brail
move



Rmk: Also have double wiring diagram \longleftrightarrow quiver $Q(G)$
 Description is more complicated, but quiver associated to a planar bipartite graph is a special case of

Def: A plabic graph G is a connected planar bipartite graph embedded in a disk, where:

- each vertex is colored black or white and lies either in interior of disk or on its boundary
- each edge connects vertices of different colors and is a simple curve whose interior is disjoint from the other edges and the disk boundary
- for each face closure is simply connected
- each internal vertex is simply connected
- each boundary vertex has degree ≥ 2
- each boundary vertex has degree 1



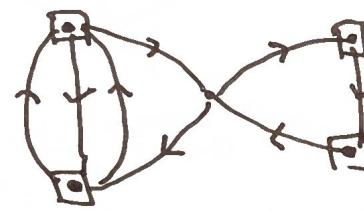
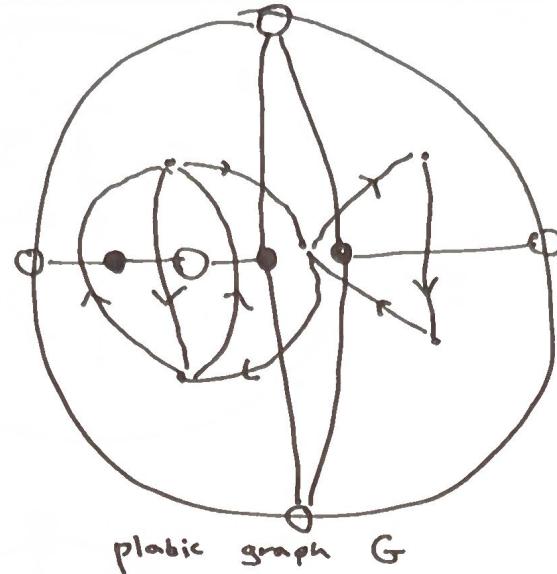
Note: we consider plabic graphs up to isotopy.

plabic graph G \longleftrightarrow quiver $Q(G)$

- vertices are faces of G (frozen if incident to disk boundary, else mutable)
- for each edge of G , have arrow joining the two faces it separates using rule
- remove oriented 2-cycles



Ex:



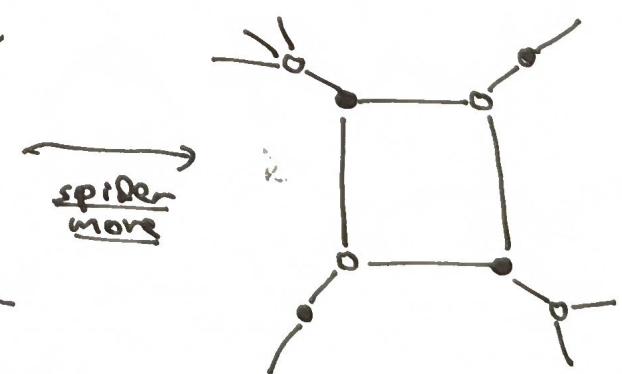
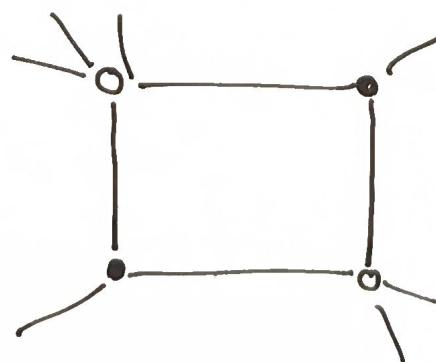
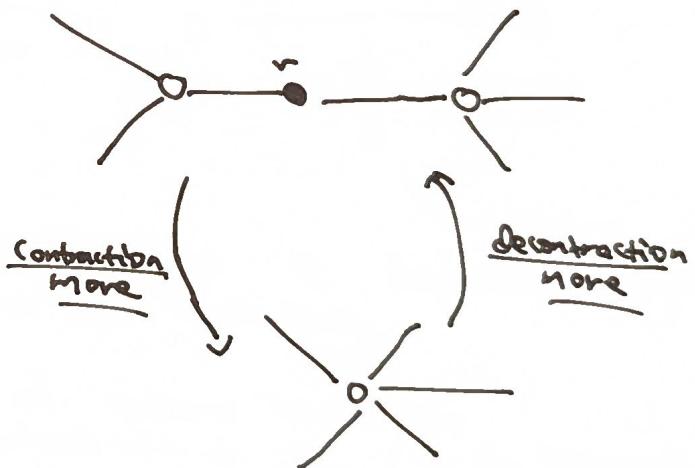
remove
oriented 2-cycles
(and arrows
between frozen)



Def: Say v bivalent vertex
adjacent to two interior
vertices

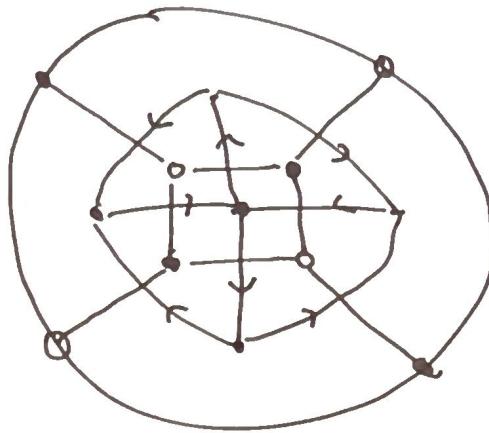
Rule: does not change
associated quiver

Def: Say
~~quadrilater~~
face whose
degree ≥ 3 .
G has a
quadrilateral
vertices have

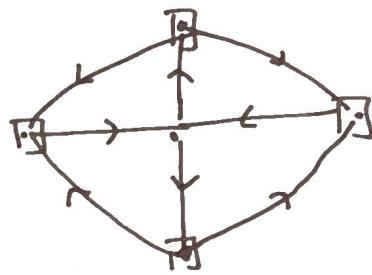
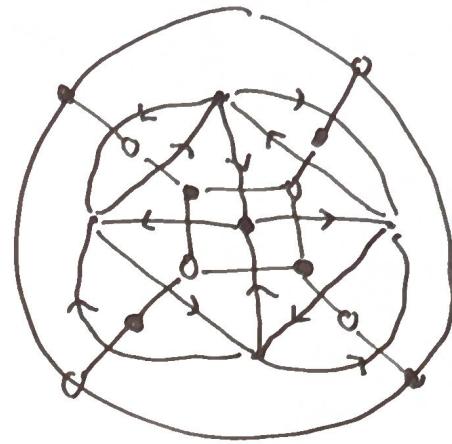


Exercise: If G, G' related by
~~Q(G), Q(G')~~ related by spider move, then
mutation

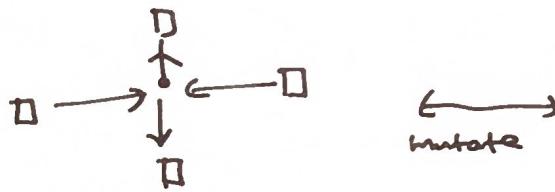
Ex



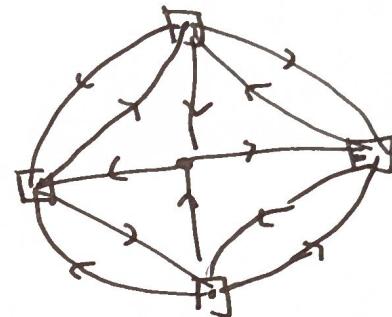
spider move



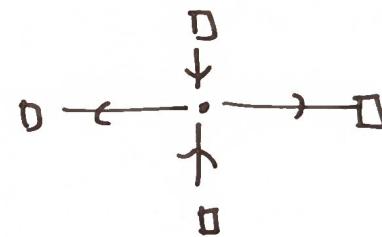
remove $D \rightarrow D$



mutate



remove $D \rightarrow D$ (and
cancel 2-cycles)



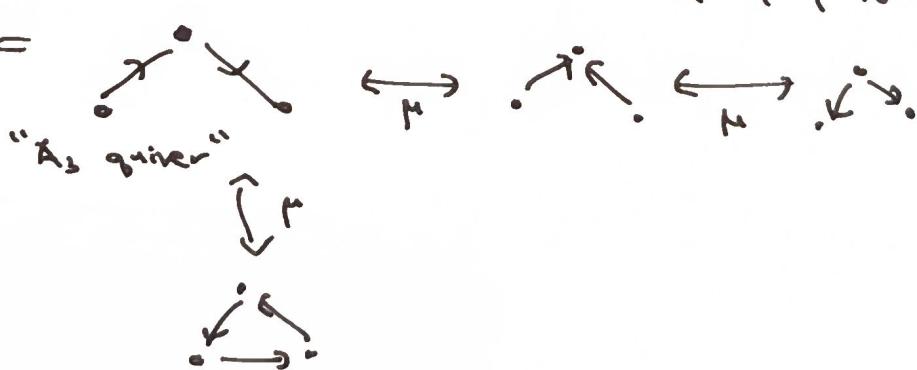
mutation equivalence

Def: Q, Q', Q'' are mutation equivalent if Q becomes isomorphic after a sequence of mutations.

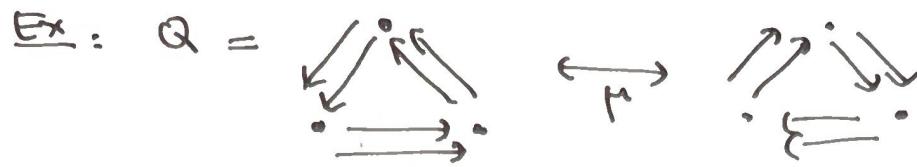
Put $[Q] :=$ set of all quivers which are mutation equivalent to Q (up to isomorphism)

Ex:

$Q =$



Exercise: $[Q]$ has 4 elements



"Markov quiver"

In fact, $[Q]$ is just a single element.

Def. Q has finite mutation type if $[Q]$ is finite.

Rmk: there is a classification theorem for quivers with no frozen vertices and finite mutation type.

Def: Q acyclic if no oriented cycles.

Thm (Caldero-Keller '06): If Q, Q' acyclic and mutation ~~equivalent~~ equivalent, then we can transform Q into Q' by a sequence of mutations at sources and sinks. In particular, Q, Q' have the same underlying undirected graphs.

Lecture 5

11/28/26

Def : Q quiver with vertices labeled by $1, \dots, m$, such that $1, \dots, n$ are the mutable ones ($n \leq m$).

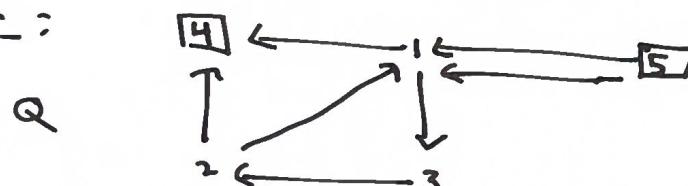
The extended exchange matrix is

$\tilde{B}(Q) = (b_{ij})_{\substack{1 \leq i, m \\ 1 \leq j \leq n}}$, where $b_{ij} = \begin{cases} 1 & \text{if } \text{arrows } i \rightarrow j \\ -1 & \text{if } \text{arrows } j \rightarrow i \\ 0 & \text{else} \end{cases}$

$m \times n$ matrix

The exchange matrix is the submatrix $B(Q) := (b_{ij})_{1 \leq i, j \leq n}$

Ex :



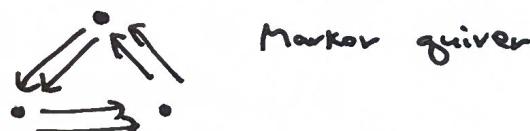
$n \times n$ skew-symmetric matrix

$\tilde{B}(Q) =$

$$\begin{pmatrix} 0 & 2 & 3 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \\ -1 & -1 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

$$B(Q) = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

Ex : $Q =$



Marker quiver



$$\tilde{B}(Q) = \begin{pmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{pmatrix}$$



$$\tilde{B}(Q) = \begin{pmatrix} 0 & -2 & 2 \\ 2 & 0 & -2 \\ -2 & 2 & 0 \end{pmatrix}$$

Rank : Rearranging the vertices of Q results in simultaneously rearranging the rows and columns $1, \dots, n$ and reordering the rows $1, \dots, m$.

Lemma : For quiver Q with $\tilde{B}(Q) = (b_{ij})$ and $Q' = \mu_k(Q)$ for a mutable vertex k of Q , have $\tilde{B}(Q') = (b'_{ij})$, with $b'_{ij} = \begin{cases} -b_{ij} & \text{if } i=k \text{ or } j=k \\ b_{ij} + b_{ik}b_{kj} & \text{if } b_{ik}b_{kj} > 0 \\ b_{ij} - b_{ik}b_{kj} & \text{if } b_{ik}b_{kj} < 0 \\ b_{ij} & \text{else} \end{cases}$

Note : Can replace middle two cases with $b'_{ij} = b_{ij} + |b_{ik}|b_{kj}$ if $b_{ik}b_{kj} > 0$

Ex : 

$$\begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 3 \\ 0 & -3 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -2 & 6 \\ 2 & 0 & -3 \\ -6 & 3 & 0 \end{pmatrix}$$

Def : An $n \times n$ matrix $B = (b_{ij}) \in \text{Mat}_{n \times n}(\mathbb{Z})$ is skew-symmetrizable if for some $\alpha_1, \dots, \alpha_n \in \mathbb{Z}_{>0}$ we have $\alpha_i b_{ij} = -\alpha_j b_{ji}$

Def : An $n \times n$ matrix is extended skew-symmetrizable if the top $n-1$ submatrix is skew-symmetrizable.

i.e. becomes skew-symmetric after rescaling the rows by positive integers

Def : For $\tilde{B} = (b_{ij})$ extended skew-sym. $n \times n$ matrix, $k \in \{1, \dots, n\}$, we define $\mu_k(\tilde{B}) = (b'_{ij})$ using same formula $(*)$.

Exercise :

- $\mu_k(\tilde{B})$ is again extended skew-sym., using same $\alpha_1, \dots, \alpha_n$
- $\mu_k(\mu_k(\tilde{B})) = \tilde{B}$
- $\mu_k(-\tilde{B}) = -\mu_k(\tilde{B})$
- ~~$\mu_k(\tilde{B}) = \tilde{B}$~~
- if $b_{ij} = b_{ji} = 0$, then $\mu_i \mu_j \tilde{B} = \mu_j \mu_i \tilde{B}$

Def: For a skew-symmetrizable $n \times n$ matrix $B = (b_{ij})$, its Diagram is the weighted directed graph $\Gamma(B)$ with vertices $1, \dots, n$ and $i \rightarrow j$ iff $b_{ij} > 0$, with weight $|b_{ij}|b_{ji}|$.

Lemma: If the diagram $\Gamma(B)$ of an $n \times n$ skew-symmetrizable matrix B is connected then the skew-symmetrizing vector $(\theta_1, \dots, \theta_n)$ is unique up to rescaling.

pf: By connectedness, there is an ~~strict~~ ordering i_1, \dots, i_n of $\{1, \dots, n\}$ s.t. ~~not~~ for each $j \geq 2$ we have $b_{i_1, i_j} \neq 0$ for some $i < j$.

If $(\theta_1, \dots, \theta_n)$ and $(\theta'_1, \dots, \theta'_n)$ skew-symmetrizing vectors, have $\theta_i \theta_j = -\theta_j \theta_i$ and $\theta'_i \theta_j = -\theta'_j \theta_i \neq \theta_i \theta'_j$.

If $b_{i_1, i_2} \neq 0$, have $\frac{\theta_{i_1}}{\theta_{i_2}} = -\frac{\theta'_1}{\theta'_2} = -\frac{\theta'_j}{\theta'_i}$

$$\Rightarrow \frac{\theta_1}{\theta_2} = \frac{\theta'_1}{\theta'_2}.$$

Def: Two extended

are mutation equivalent if can get from B to B' by a sequence of mutations, followed by a reordering of the rows and columns in the sense from before.

Put $[B] := \text{mutation equivalence class of } B$.

Prop: For an $n \times n$ skew-symmetrizable matrix, its rank and determinant are preserved by mutations.

pf: Can write $b_{ij} = \begin{cases} -b_{ji} & \text{if } k \in \{i, j\} \\ b_{ij} + \max(0, -b_{ik})b_{ki} + b_{ik} \max(0, b_{kj}) & \text{otherwise} \end{cases}$

$$\begin{aligned} \text{Have } f_K(B) &= J_{m, k} \tilde{B} J_{n, k} + J_{m, k} \tilde{B} F_k + E_k \tilde{B} J_{n, k} \\ &= (J_{m, k} + E_k) \tilde{B} (J_{n, k} + F_k) \end{aligned}$$

where $J_{m, k}$ (resp. $J_{n, k}$) is diagonal $m \times m$ (resp. $n \times n$) and has 1s on diagonal except for -1 in (k, k) entry. $E_k = (e_{ij})$ is $m \times n$ matrix with $e_{ik} = \max(0, -b_{ik})$ and all other entries 0.

$F_{1, \kappa} = (f_{ij})$ is the $n \times n$ matrix with $f_{kj} = \max(0, b_{kj})$ and all other entries 0.

Note: $E_{1, \kappa} \tilde{B} F_{\kappa}$ since $b_{ii} = 0$

Have $\det(I_{n, \kappa} + E_{1, \kappa}) = \det(I_{n, \kappa} + F_{\kappa}) = -1$.

Def: A labeled seed of geometric type in $\mathcal{G} = \mathbb{C}(x_1, \dots, x_n)$ is a pair (\tilde{x}, \tilde{B}) where

- $\tilde{x} = (x_1, \dots, x_n)$ is an adapted n -tuple of elts of \mathcal{G} which form a free generating seed
 - $\tilde{B} = (b_{ij})$ is an $n \times n$ extended matrix
- ie $\mathcal{G} = \mathbb{C}(x_1, \dots, x_n)$ and x_1, \dots, x_n alg. indep. skew-symmetrizable integer

We say:

- \tilde{x} is the labeled extended cluster
- $x = (x_1, \dots, x_n)$ is the (labeled) cluster
- x_1, \dots, x_n are the cluster variables
- x_{n+1}, \dots, x_m are the frozen variables
- \tilde{B} is the extended exchange matrix
- its top $n \times n$ submatrix B is the exchange matrix

	Σ	Σ'
extended cluster	$\tilde{x} = (x_1, x_2, x_3)$	$\tilde{x}' = (x_1, \frac{x_1+x_2}{x_2}, x_3)$
cluster vars	x_1, x_2	$x_1, \frac{x_1+x_3}{x_2}$
frozen vars	x_3	x_3
extended exchange matrix	$\tilde{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}$	$\tilde{B}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$
exchange matrix	$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$B' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Here $n=3$, $\kappa=2$.



Lecture 6

Recall: $\mathcal{L} = \mathbb{Q}(q_1, \dots, q_m)$ field of rational functions, $m \geq n$. Say $x_1, \dots, x_m \in \mathcal{L}$ a free generating set if algebraically independent and $\mathcal{L} = \mathbb{Q}(x_1, \dots, x_m)$.

Def: A labeled seed of geometric type in \mathcal{L} is (\tilde{x}, \tilde{B}) , where:

- $\tilde{x} = (x_1, \dots, x_m)$ free generating set of \mathcal{L}
- $\tilde{B} = (b_{ij})$ $m \times n$ extended skew-symmetrizable integer matrix

Terminology:

- \tilde{x} extended cluster
- $x = (x_1, \dots, x_n)$ cluster, x_1, \dots, x_n cluster variables
- x_{n+1}, \dots, x_m frozen variables
- $\tilde{B} \leftrightarrow$ ~~exchange matrix~~ extended exchange matrix
- top $n \times n$ submatrix B is the exchange matrix

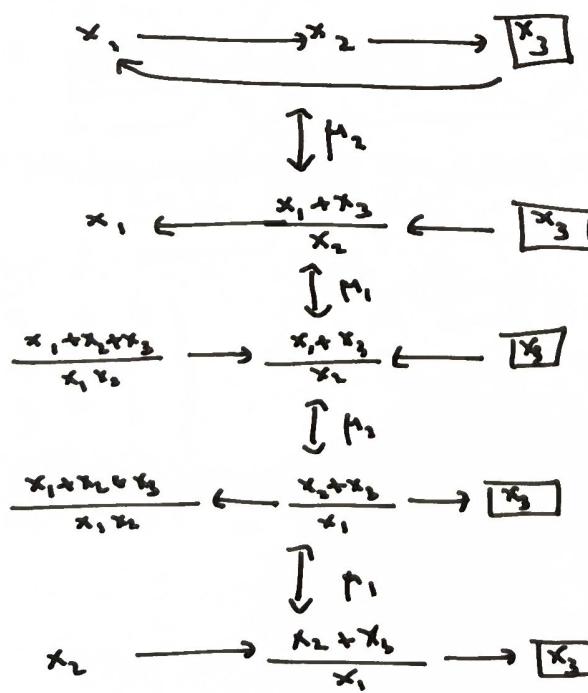
Def: Given (\tilde{x}, \tilde{B}) labeled seed, $k \in \{1, \dots, n\}$, define a new labeled seed $\mu_k(\tilde{x}, \tilde{B}) = (\tilde{x}', \tilde{B}')$, where

- $\tilde{B}' = \mu_k(\tilde{B})$
- $\tilde{x}' = (x'_1, \dots, x'_m)$, where $x'_j = x_j$ for $j \neq k$ and

$$x_k x'_k = \prod_{b_{ik} > 0} x_i^{b_{ik}} + \prod_{b_{ik} < 0} x_i^{-b_{ik}} \quad \text{exchange relation}$$

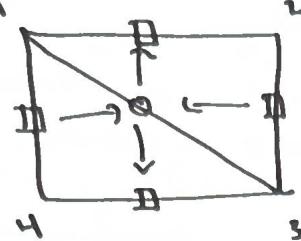
Rmk: When \tilde{B} comes from a quiver, the first product is over arrows ending at k and the second product is over arrows starting at k .

Ex:

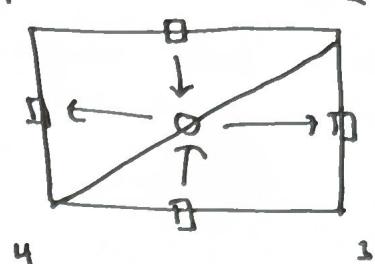


Note: the last seed agrees with the first one up to relabelling.

Ex:



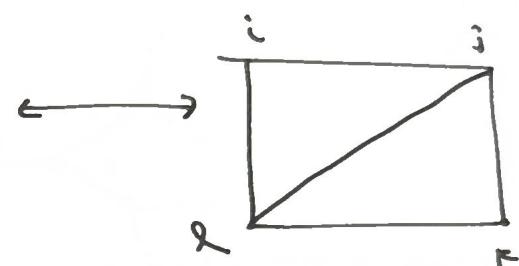
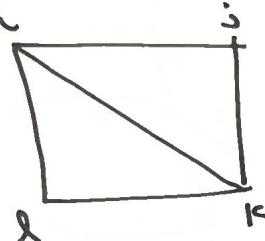
↔ flip



$$\begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix} \quad P_{13} = ag - ce \quad P_{24} = bh - df$$

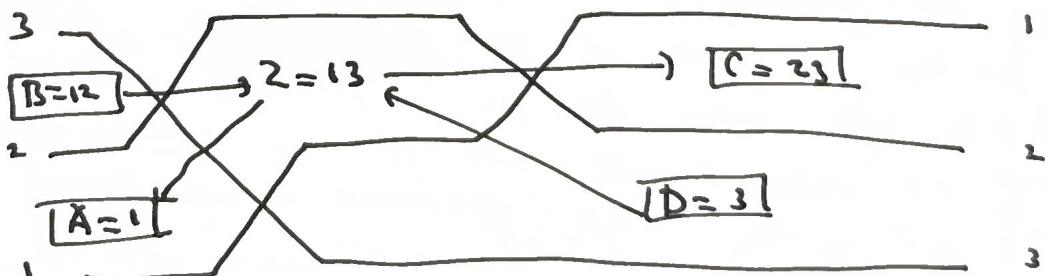
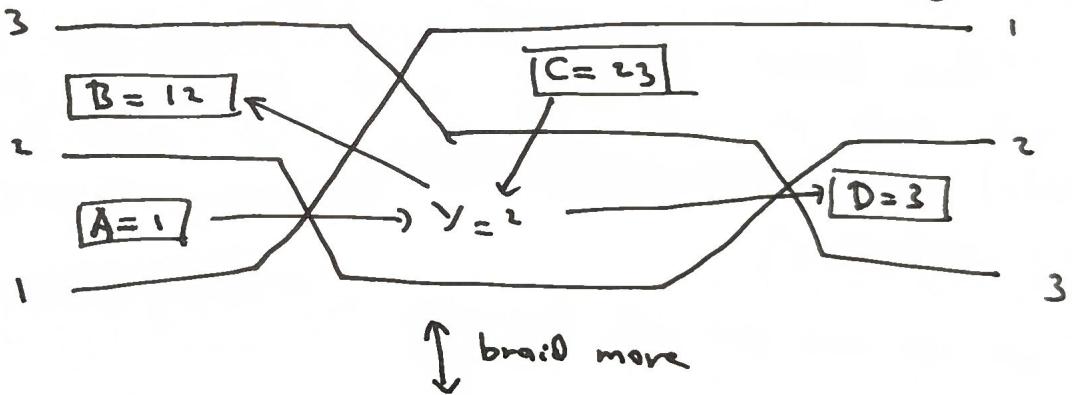
$$\text{Recall: } P_{13} P_{24} = P_{12} P_{34} + P_{14} P_{23}$$

More generally,



$$\rightsquigarrow P_{ik} P_{jl} = P_{ij} P_{lk} + P_{il} P_{jk} \quad \text{special case of the exchange relation}$$

Ex:



$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$\begin{aligned} A &\leftrightarrow a \\ B &\leftrightarrow ae - bf \\ C &\leftrightarrow bf - ce \end{aligned} \quad)$$

etc

$$\text{Have } Y_2 = AC + BD$$

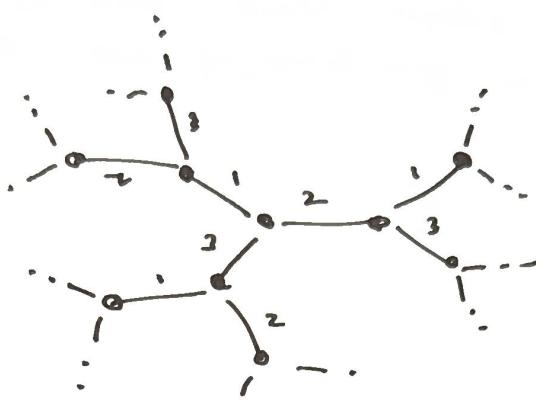
special case of the exchange relation

Notation: Let Π_n denote the n -regular tree with edges labeled by l_1, \dots, l_n such that the edges incident to each vertex carry distinct labels.

$$\Pi_1 \quad \bullet - 1 -$$

$$\Pi_2 \quad \dots - 2 - \bullet - 1 - \bullet - 2 - \bullet - 1 - \bullet - 2 - \bullet - \dots$$

$$\Pi_3$$



Def.: A seed pattern is a choice of labeled seeds $(\tilde{x}(t), \tilde{B}(t))$ for each vertex $t \in \Pi_n$, so that for each labeled edge $t \xrightarrow{\kappa} t'$ the corresponding labeled seeds $(\tilde{x}(t), \tilde{B}(t)), (\tilde{x}(t'), \tilde{B}(t'))$ differ by μ_κ .

Note: a seed pattern is determined by any one of its seeds.

Def: Let $(\tilde{x}(t), \tilde{B}(t))_{t \in \Pi_n}$ be a seed pattern, and put $R := \mathbb{Q}[x_{n_1, \dots, n_m}]$. Let \mathcal{X} be the set of all cluster variables appearing in the seeds $x(t)$ for $t \in \Pi_n$. The cluster algebra A is the R -subalgebra of \mathcal{L} generated by all cluster variables i.e. $A = R[\mathcal{X}]$.

Terminology: The rank n of a cluster algebra is the cardinality of any cluster.

Rank: Note that there is an isomorphism of any free generating set to any other. In particular, up to isomorphism A depends only on \tilde{B}_0 for any initial seed $(\tilde{x}_0, \tilde{B}_0)$, and in fact only on the mutation equivalence class of \tilde{B} . In particular, each (i.e) give Q defines an extended exchange matrix \tilde{B} and hence a cluster algebra.

Ex: For T a triangulation of the regular n -gon P_m , the associated cluster algebra is the Plücker ring $P_{2,m}$.

Ex: For a wiring diagram on k strands, the associated cluster algebra is the algebra generated by flag minors of a $k \times k$ matrix, i.e. the ring of invariants $\mathbb{C}[\mathrm{SL}_k]^U$ (here $U = \text{group of lower-triangular matrices with } 1s \text{ on the diagonal}$)

Ex: For a double wiring diagram on k strands, the associated cluster algebra is $\mathbb{C}[\mathrm{GL}_n]$, i.e. the polynomial ring in k^2 variables. i.e. functions on the basic affine space