Midterm 2

Modern Algebra 1

Columbia University Fall 2019

Instructor: Kyler Siegel

Instructions:

- Please write your answers **in this printed exam**. You may use the back of pages for additional work. You may also use printer paper if you need additional space, but you must hand in all relevant work. Please turn in all scratch work which is relevant to your submitted answers.
- Suspected cases of copying or otherwise cheating will be taken very seriously.
- Solve as many problems of the following problems as you can in the allotted time, which is *one hour and fifteen minutes.* I recommend first solving the problems you are most comfortable with before moving on to the more challenging ones. Note that the problems are not ordered by level of difficulty or topic.
- The exams will be graded on a curve. Therefore the raw score is not important, and you do not necessarily need to solve every problem to achieve a good grade. Just do your best!
- For true or false questions, you will receive $+2$ points for a correct answer, 0 points for no answer, and -3 points for an incorrect answer. This means **you should not make random guesses** unless you are reasonably sure that you know the answer. For the short answer questions, there is no penalty for wrong answers, and you do not need to justify your answers for full credit. For the short proof questions, you should be as precise and rigorous and possible.
- You may use any commonly used notation for standard groups, subgroups, and their elements, as long as it is completely unambiguous. If you are using nonstandard notation you must fully explain it for credit.
- Turn off all electronic devices. You may use the restroom if you must, but you may not take any devices with you.
- Good luck!!

Name: ______________________________

Uni: ______________________________

Question:	1	2	3	Total
Points:	22	36	20	78
Score:				

Notation reminders:

- $D_{2\cdot n}$ denotes the dihedral group corresponding to the symmetries of the regular n-gon.
- For $k \in \mathbb{Z}_{\geq 1}$, S_k denotes the symmetric group on k letters, and A_k denotes the alternating group on k letters.
- For G a group and p a prime subgroup, a p-subgroup of G is a subgroup whose order is a prime power of p.

1. **True or false questions. Circle one. You do not need to provide any justification. There is a guessing penalty.**

(I) (2 points) Every group of order 100 is isomorphic to a subgroup of S_{101}. A. True B. False

(II) (2 points) Let G be a group of order 55. Then G is abelian. A. True B. False

(III) (2 points) Let G be a group of order 22. Then G is abelian. A. True B. False

(IV) (2 points) Any two subgroups of S_6 of order 4 are isomorphic. A. True B. False

(V) (2 points) Any two subgroups of S_6 of order 9 are isomorphic. A. True B. False

(VI) (2 points) There is no simple group of order 360. A. True B. False

(VII) (2 points) Let G be a group of order 175. Every subgroup of G of order 35 is normal. A. True B. False

(VIII) (2 points) Every group of order 121 is abelian. A. True B. False

(IX) (2 points) Let G be a group of order 20. Then there exists a transitive action of G on a set of 4 elements. A. True B. False

(X) (2 points) Let G be a group of order 20. Then there exists a surjective homomorphism from G to a group of order 4. A. True B. False

(XI) (2 points) The symmetric group S_5 has a subgroup which is isomorphic to Q_8. A. True B. False

2. **Short answer questions. You do not need to provide any justification. There is no penalty for wrong answers but you must box your final answer if it is not clear.**

(I) (4 points) What is the order of a Sylow 5-subgroup of S_{17}?

(II) (4 points) Let $G_0 = \{e\} \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \ldots \trianglelefteq G_r = S_4$ be a composition series of S_4. What is r? Describe the composition factors $G_1/G_0, G_2/G_1, \ldots, G_r/G_{r-1}$ up to isomorphism.

(III) (4 points) Let G be a noncyclic group of order 93. How many elements of order 3 does G have?

(IV) (4 points) Consider the partition of the quaternion group Q_8 into conjugacy classes. How many conjugacy classes are there? Describe the conjugacy classes.

(V) (4 points) Let S_5 act on itself by left multiplication. What is the size of the orbit of the transposition $(1\ 2) \in S_5$?

(VI) (4 points) Let S_5 act on itself by conjugation. What is the order of the stabilizer of the transposition $(1\ 2) \in S_5$?

(VII) (4 points) How many conjugacy classes of A_5 are there?

(VIII) (4 points) How many subgroups of S_4 are isomorphic to D_8?

(IX) (4 points) Let H denote the subgroup of S_7 generated by the 7-cycle $(1\ 3\ 2\ 4\ 6\ 5\ 7)$. What is the order of the normalizer $N_{S_7}(H)$ of H in S_7?

3. **Short proofs. Make your arguments as precise and rigorous as possible.**

(I) (10 points) Prove that there is no simple group of order 108. *You may invoke general results such as Lagrange's theorem, the orbit-stabilizer theorem, the class equation, and Sylow's theorems, but any specific classification results for groups of given orders should be proved.*

(II) (10 points) Prove that every group of order 1024 has a nontrivial center. *You may invoke general results such as Lagrange's theorem, the orbit-stabilizer theorem, the class equation, and Sylow's theorems, but any specific classification results for groups of given orders should be proved.*