Lecture 18

Recall: Euler eqn:

$$t^2 y''(t) + \alpha t y'(t) + \beta y(t) = 0.$$

Ex: $t^2 y''(t) + 3t y'(t) + 4y(t) = 0$

$t < 0.$

Recall: Ansatz: $y(t) = t^r$ for some $r \in \mathbb{R}$.

$t^r = e^{r\ln(t)}$ only defined for $t > 0$.

Q: How to handle $t < 0$, since then t^r not defined...

Idea: let $s = -t$.

Then (*) becomes

$$(-s)^2 y''(-s) + 3(-s)y'(-s) + 4y(-s) = 0,$$
$$s > 0.$$

Have $s^2 y''(-s) - 3sy'(-s) + 4y(-s) = 0,$
$s > 0.$

Now, proceed as before:

Ansatz: $z(s) = s^r$. ← now valid since $s > 0$.

$$\rightsquigarrow \left(r(r-1) + 3r + 4\right)s^r = 0, \; s > 0.$$

So indicial eqn: $r^2 + 2r + 4 = 0$

$\Longrightarrow$ two cpx solutions

$$z(s) = s^{r_1}, \; z(s) = s^{r_2}.$$

Put $z(t) = y(-t)$.

Then $z'(t) = -y'(-t)$, $z''(t) = y''(-t)$.

So (*) becomes

$$\underbrace{s^2 z''(s) + 3s z'(s) + 4z(s) = 0, \quad s > 0.}$$

precisely an Euler eqn as in last lecture

(reflects the fact that our ODE has a symmetry)

$$r = \frac{-2 \pm \sqrt{4 - 16}}{2}$$
$$= \frac{-2 \pm i2\sqrt{3}}{2}$$

$\rightarrow r_1 = -1 + i\sqrt{3}$, $r_2 = -1 - i\sqrt{3}$.

Two real solutions:

$$z_1(s) = s^{-1}\sin(\sqrt{3}\ln(s))$$

$$z_2(s) = s^{-1}\cos(\sqrt{3}\ln(s))$$

<u>Recall</u>: $s=-t$, $z(t)=y(-t)$.

If $z(s)=z_1(s)$, then

$$y(t)=z(-t)=(-t)^{-1}\sin(\sqrt{3}\ln(-t))$$

Similarly, if $z(s)=z_2(s)$, then

$$y(t)=z(-t)=(-t)^{-1}\cos(\sqrt{3}\ln(-t))$$

<u>Observe</u>: for $t\neq 0$, have soln

$$y(t)=C_1|t|^{-1}\sin(\sqrt{3}\ln(|t|)) + C_2|t|^{-1}\cos(\sqrt{3}\ln(|t|))$$

(if $t>0$, $|t|=t$,
if $t<0$, $|t|=-t$)

<u>Recall</u>: $t^{\alpha+i\beta}=t^{\alpha}e^{i\beta\ln(t)}$
$= t^{\alpha}(\cos(\beta\ln(t)) + i\sin(\beta\ln(t))$

<u>Conclusion</u>: Have solutions to $(*)$:

$$y_1(t)=(-t)^{-1}\sin(\sqrt{3}\ln(-t))$$

$$y_2(t)=(-t)^{-1}\cos(\sqrt{3}\ln(-t)),$$

for $t<0$.

exactly what we'd have if we started with $t>0$, except t replaced with $-t$

[<u>Note</u>: If $z(s)=s^{-1}\cos(\sqrt{3}\ln(s))$,
then $z(t)=t^{-1}\cos(\sqrt{3}\ln(t))$
$z(☺)=☺^{-1}\cos(\sqrt{3}\ln(☺))$]

Claim: If $t^2 y''(t) + \alpha t y'(t) + \beta y(t) = 0$, $t>0$ has f.s.o.s. $y_1(t)$, $y_2(t)$ then $y_1(|t|)$ and $y_2(|t|)$ is a f.s.o.s. for any $t \neq 0$.

Rmk: This shows that Euler eqns have a special symmetry: all solns are symmetric about y-axis.

Idea: $t^2 y''(t) + \alpha t y'(t) + \beta y(t) = 0$, $t>0$.

Put $s = -t$

$\rightsquigarrow$ $s^2 y''(-s) + \alpha(-s) y'(-s) + \beta y(-s) = 0$, $s>0$

Put $z(s) = y(-s)$.

$\rightsquigarrow$ $s^2 z''(s) + \alpha s z'(s) + \beta z(s) = 0$, $s>0$.

EX: $(t-\pi)^2 y''(t) + 5(t-\pi) y'(t) + 4y(t) = 0$, $t>\pi$.

Note: singular pts: $t=\pi$

Ansatz: $y(t) = (t-\pi)^r$.

$\left((t-\pi)^r = e^{r\ln(t-\pi)}, \text{ so only defined for } t>\pi \right)$

$y'(t) = r(t-\pi)^{r-1}$, $y''(t) = r(r-1)(t-\pi)^{r-2}$

$\rightsquigarrow$ $(t-\pi)^2 r(r-1)(t-\pi)^{r-2} + 5r(t-\pi)(t-\pi)^{r-1} + 4(t-\pi)^r = 0$

$\rightsquigarrow$ $\left((r)(r-1) + 5r + 4\right)(t-\pi)^r = 0$

$\Rightarrow$ $r^2 + 4r + 4 = 0$. $(r+2)^2$

So $r_1 = r_2 = -2$.

$\rightsquigarrow$ $y_1(t) = (t-\pi)^{-2}$

$y_2(t) = (t-\pi)^{-2}\ln(t-\pi)$

~~or $\ln(t)$??~~

Claim: can convert this example to a std Euler eqn by change of vars.

Put $s = t - \pi$

Have $s^2 y''(s+\pi) + 5 s y'(s+\pi) + 4 y(s+\pi) = 0.$

Consider $P(t) y''(t) + Q(t) y'(t) + R(t) y(t) = 0$

$\Longleftrightarrow y''(t) + p(t) y'(t) + q(t) y(t) = 0$

$p(t) = Q(t)/P(t) \quad q(t) = R(t)/P(t).$

Recall: $t = t_0$ is an ordinary pt if $p(t)$ and $q(t)$ are analytic at t_0.

Ex: $t^2 y'' + \alpha t y' + \beta y = 0$

$\Longleftrightarrow y'' + \dfrac{\alpha y'}{t} + \dfrac{\beta y}{t^2}$

$p(t) = \alpha/t \; , \; q(t) = \beta/t^2$

Put $z(s) = y(s+\pi)$.

$z'(s) = y'(s+\pi), \; z''(s) = y''(s+\pi)$

Have $s^2 z''(s) + 5 s z'(s) + 4 z(s) = 0, \; s > 0.$

Back at std Euler eqn!

General soln: $z(s) = C_1 |s|^{-2} + C_2 |s|^{-2} \ln(|s|)$ valid for $s \neq 0.$

So general soln for $y(t) = z(t-\pi)$ is $C_1 |t-\pi|^{-2} + C_2 |t-\pi| \ln(|t-\pi|).$

Def. A singular t_0 is regular if $t p(t)$ and $t^2 q(t)$ are analytic at t_0. Otherwise, t_0 is irregular.

$(t-t_0)p(t)$

$(t-t_0)^2 q(t)$

$t=0$ is a singular pt. (for an Euler eqn).

$t p(t) = \alpha \; , \; t^2 q(t) = \beta$ are constants, so analytic at $t=0$.

$\underline{\text{Ex}}$: $t^{100}y'' + 5ty' + 10y = 0.$

$\Leftrightarrow y'' + \frac{5}{t^{99}}y' + \frac{10}{t^{100}}y = 0.$

So $t=0$ is an sing pt.

$p(t) = \frac{5}{t^{99}}, \quad q(t) = \frac{10}{t^{100}}.$

Consider $t\,p(t) = \frac{5}{t^{98}}$ ← not analytic at $t=0$

$t^2 q(t) = \frac{10}{t^{98}}$

Conclusion: $t=0$ is an irregular singular pt.

$\underline{\text{Ex}}$: $t^2y'' + ty' + (7\cos(t) + 17t^3)y = 0$

has a reg. sing pt at $t=0$.

$p(t) = \frac{t}{t^2} \cdot t = 1$

$q(t) = \frac{7\cos(t) + 17t^3}{t^2} \cdot t^2 = 7\cos(t) + 17t^3$

analytic function at $t=0$