Lecture 21

Today: lots of linear algebra

Recall: Ex:

$$\begin{cases} x'(t) = 5x(t) - y(t) \\ y'(t) = -x(t) + 5y(t) \end{cases} \quad (*)$$

Trick: $\tilde{x}(t) = \frac{x+y}{2}$ $\qquad \tilde{y}(t) = \frac{x-y}{2}$

Write $(*)$ in terms of $\tilde{x}, \tilde{y}$:

Note: $\tilde{x} + \tilde{y} = x$
$\tilde{x} - \tilde{y} = y$

$$\begin{cases} \tilde{x}' + \tilde{y}' = 5(\tilde{x} + \tilde{y}) - (\tilde{x} - \tilde{y}) \\ \tilde{x}' - \tilde{y}' = -(\tilde{x} + \tilde{y}) + 5(\tilde{x} - \tilde{y}) \end{cases}$$

$$\rightarrow \begin{cases} \tilde{x}' + \tilde{y}' = 4\tilde{x} + 6\tilde{y} \\ \tilde{x}' - \tilde{y}' = 4\tilde{x} - 6\tilde{y} \end{cases} \rightsquigarrow \begin{cases} 2\tilde{x}' = 8\tilde{x} \\ 2\tilde{y}' = 12\tilde{y} \end{cases}$$

$$\rightsquigarrow \begin{cases} \tilde{x}' = 4\tilde{x} \\ \tilde{y}' = 6\tilde{y} \end{cases}$$

uncoupled system!

equivalent to (*)

Soln: $\tilde{x} = C_1 e^{4t}, \quad \tilde{y} = C_2 e^{6t}$

$$\underbrace{x(t) = C_1 e^{4t} + C_2 e^{6t}, \quad y(t) = C_1 e^{4t} - C_2 e^{6t}}_{\text{general solution}}$$

if we knew $x(0)$ and $y(0)$, could find C_1, C_2

Question:
- Where did the variables $\tilde{x}, \tilde{y}$ come from?
- what's the significance of 4 and 6?

Answer: $\tilde{x}, \tilde{y}$ come from the eigenvectors of

$$A = \begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}$$ (it has eigenvectors $\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}$).

4 and 6 are eigenvalues of A.

$\underbrace{\text{Crash course in linear algebra}}$

- vectors $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \in \mathbb{R}^3$ n-dim'l vectors $\longleftrightarrow$ $n \times 1$ matrix

entries could be real or complex

$\begin{pmatrix} 1+2i \\ 3-2i \end{pmatrix} \in \mathbb{C}^2$

- matrix: $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ (for us, always $n \times n$)

- $$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 1\cdot 5+2\cdot 7 & 1\cdot 6+2\cdot 8 \\ 3\cdot 5+4\cdot 7 & 3\cdot 6+4\cdot 8 \end{pmatrix}$$

  matrix matrix multiplication

- $$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 1\cdot 5+2\cdot 6 \\ 3\cdot 5+4\cdot 6 \end{pmatrix} = \begin{pmatrix} 17 \\ 39 \end{pmatrix}$$

  matrix vector multiplication

- $$\pi\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} \pi & 2\pi \\ 3\pi & 4\pi \end{pmatrix} \qquad \pi\begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 5\pi \\ 6\pi \end{pmatrix}$$

  Scalar · matrix — Scalar · vector

Linear (in) Dependence

A collection of vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k \in \mathbb{C}^n$ is **linearly dependent** if can find scalars $c_1, \dots, c_n \in \mathbb{C}$, not all zero, such that $\quad c_1\vec{v}_1 + \dots + c_k\vec{v}_k = \vec{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$.

means: one of the vectors can be written as a lin. combo of the others

Otherwise, we say they're <u>linearly independent</u>.

<u>Ex</u>: Are $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$ linearly independent?

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = \begin{pmatrix} 8 \\ 10 \\ 12 \end{pmatrix} = 2\cdot\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}.$$

This shows: $\quad \vec{v}_1 - 2\vec{v}_2 + \vec{v}_3 = \vec{0}$

$\Leftrightarrow \quad (1)\vec{v}_1 + (-2)\vec{v}_2 + (1)\vec{v}_3 = \vec{0}$ $\quad$ so linearly dependent.

<u>Fact</u>: ~~If~~ have $\vec{v}_1, \dots, \vec{v}_n \in \mathbb{R}^n$, they're linearly dependent iff $\det(A) = 0$, where $A = \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{pmatrix}$.

Recall: $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0.$

~~Fact~~: Any $n+1$ vectors in $\mathbb{C}^n$ are automatically linearly dependent.

Def: A basis is a collection of n linearly indep vectors in $\mathbb{C}^n$.

Ex: $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$ do not form a basis.

Ex of basis of $\mathbb{C}^3$: $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ "standard basis"

check lin. indep: $c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \vec{0}$

$$\Longleftrightarrow \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longleftrightarrow \begin{aligned} c_1 &= 0 \\ c_2 &= 0 \\ c_3 &= 0. \end{aligned}$$

Another basis of $\mathbb{C}^3$: $\begin{pmatrix}2\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix}$

Check lin indep: $c_1\begin{pmatrix}2\\0\\0\end{pmatrix} + c_2\begin{pmatrix}0\\1\\1\end{pmatrix} + c_3\begin{pmatrix}0\\1\\-1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

$$\iff \begin{pmatrix}2c_1\\c_2+c_3\\c_2-c_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \iff \begin{array}{l}2c_1=0\\c_2+c_3=0\\c_2-c_3=0\end{array}$$

$c_1=0$, $2c_2=0 \Rightarrow c_2=0$
$\Rightarrow c_3=0$

Invertibility: An $n\times n$ matrix A is <u>invertible</u> if can find another $n\times n$ matrix B s.t. $A\cdot B = \mathbb{1}_n$.

$$\mathbb{1}_n = \begin{pmatrix}1 & 0 & \cdots & 0\\ 0 & 1 & \ddots & \vdots\\ \vdots & & \ddots & 0\\ 0 & \cdots & 0 & 1\end{pmatrix}$$

e.g. $\mathbb{1}_3 = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$

Rmk: In general, for $n\times n$ matrices A, B: $A\cdot B \neq B\cdot A$.

But if $A\cdot B = \mathbb{1}$, then $B\cdot A = \mathbb{1}$.

Fact: • A is invertible iff its determinant is nonzero.

• If $A\vec{r}=\vec{w}$, and A invertible, then

$$A^{-1}A\vec{r}=A^{-1}\vec{w} \implies \mathbb{1}\vec{r}=A^{-1}\vec{w}$$
$$\implies \vec{r}=A^{-1}\vec{w}$$

Note:
$$\begin{aligned} ax+by+cz&=c_1\\ dx+ey+fz&=c_2\\ gx+hy+iz&=c_3\end{aligned} \iff \underbrace{\begin{pmatrix} a & b & c\\ d & e & f\\ g & h & i\end{pmatrix}}_{\text{Call this } A}\begin{pmatrix} x\\ y\\ z\end{pmatrix}=\begin{pmatrix} c_1\\ c_2\\ c_3\end{pmatrix}$$

$$\implies \begin{pmatrix} x\\ y\\ z\end{pmatrix}=A^{-1}\begin{pmatrix} c_1\\ c_2\\ c_3\end{pmatrix}$$

How to find inverses?

Ex:
$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}^{-1}=\frac{1}{ad-bc}\begin{pmatrix} d & -b\\ -c & a\end{pmatrix}$$

$1/|A|$

Ex: $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $\quad A^{-1} = \dfrac{1}{-2} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$

$$= \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}}_{A} \cdot \underbrace{\begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}}_{A^{-1}} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbb{1}} \quad \checkmark$$

$$\underbrace{\begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}}_{A^{-1}} \underbrace{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbb{1}}$$

How to find inverse of $n \times n$ matrix?

Form $\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right)$ "augmented matrix"

Identity matrix

Now convert left matrix into $\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ by performing "elementary row operations". The resulting matrix on right is A^{-1}.

"Gaussian elimination"

Elementary row ops:

(1) interchange two rows

(2) multiply a row by a nonzero scalar

(3) add a scalar multiple of one row to another

Ex:

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right) \xrightarrow[r_2 \mapsto r_2 - 3r_1]{} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right)$$

$$\xrightarrow[r_1 \mapsto r_1 + r_2]{} \left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & -2 & -3 & 1 \end{array}\right) \xrightarrow[r_2 \mapsto -\frac{1}{2} r_2]{} \left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right)$$

$$\Longrightarrow A^{-1} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$$

Eigenvectors and eigenvalues

A = $n\times n$ matrix. We say that a scalar λ is an eigenvalue of A with eigenvector $\vec{v}$ if

$$A\vec{v} = \lambda \vec{v}$$

where $\vec{v}$ cannot be the zero vector.

here $\vec{v}$ n-dim'l vector

Note: $A\begin{pmatrix}0\\ \vdots \\ 0\end{pmatrix} = \begin{pmatrix}0\\ \vdots \\ 0\end{pmatrix} = \lambda\begin{pmatrix}0\\ \vdots \\ 0\end{pmatrix}$ so

λ could be any scalar

Rmk: λ can be zero

Ex: $A = \begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}$.

Goal: compute the eigenvectors and eigenvalues of A.

Rmk: Each evec is associated to an eval, but many evecs could have same eval.

Seek $\vec{r} \in \mathbb{C}^2$ and $\lambda \in \mathbb{C}$ s.t.

$A\vec{r} = \lambda\vec{r}$.

write as: $A\vec{r} = \lambda \mathbb{1}\vec{r}$

$$\Longleftrightarrow \underbrace{(A - \lambda\mathbb{1})}_{n \times n \text{ matrix. Is it invertible?}}\vec{r} = \vec{0}$$

If so, have $(A-\lambda\mathbb{1})^{-1}(A-\lambda\mathbb{1})\vec{r} = (A-\lambda\mathbb{1})^{-1}\vec{0}$

$$\leadsto \mathbb{1}\vec{r} = \vec{0} \leadsto \vec{r} = \vec{0}$$

To avoid this, must have $(A-\lambda\mathbb{1})$ is not invertible, i.e. $\det(A-\lambda\mathbb{1}) = 0$.