TA
Andrew Sullivan Math 2030

Problem Set 7 Solutions

5.1 1. Determine radius of convergence:
$\sum_{n=0}^{\infty}(x-3)^n$

Start w/ ratio test, comparing n^{th} term to $(n+1)^{th}$ term:
$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{(x-3)^n}\right| = |x-3|\lim_{n\to\infty}\left|\frac{\cancel{(x-3)^n}}{\cancel{(x-3)^n}}\right| = |x-3|$

For series to converge, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$

thus for series to converge: $|x-3|<1$
And the radius of convergence is $\boxed{2<x<4}$

4. $\sum_{n=0}^{\infty} 2^n x^n$ Start w/ ratio test: $\lim_{n\to\infty}\left|\frac{2^{n+1}x^{n+1}}{2^n x^n}\right| =$

$2|x|\lim_{n\to\infty}\left|\frac{\cancel{2^n x^n}}{\cancel{2^n x^n}}\right| = 2|x| < 1$ ← for convergence

so $|x|<\frac{1}{2}$, and the radius of convergence is $\boxed{-\frac{1}{2}<x<\frac{1}{2}}$

7. $\sum_{n=1}^{\infty}\frac{(-1)^n n^2 (x+2)^n}{3^n}$: Ratio test: $\lim_{n\to\infty}\left|\frac{(-1)^{\cancel{n}+1}(n+1)^2(x+2)^{\cancel{n}+1}}{3^{\cancel{n}+1}}\cdot\frac{\cancel{3^n}}{\cancel{(-1)^n} n^2 \cancel{(x+2)^n}}\right|$

$= \left|\frac{(x+2)(-1)^n}{3}\right|\lim_{n\to\infty}\underbrace{\left|\frac{(n+1)^2}{n^2}\right|}_{\text{this lim} \to 1} = \frac{1}{3}|x+2|$

For ratio to converge: $\frac{1}{3}|x+2|<1$, so $|x+2|<3$
The radius of convergence is $\boxed{-5<x<-1}$

13. Determine Taylor series about x_0 and radius of convergence: $\ln x$, $x_0=1$

Definition of Taylor Series: $f(x) = \sum_{n=0}^{\infty}\frac{f^n(x_0)}{n!}(x-x_0)^n$
Expand $\ln x$ by finding first few terms
$f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2}$, $f'''(x) = \frac{2}{x^3}$ ← plug in x_0

thus $\ln(x) \approx 1\cdot(x-1) - \frac{1}{2!}\cdot\frac{1}{1^2}(x-1)^2 + \frac{1}{3!}\cdot\frac{1\cdot 2}{1^3}(x-1)^3 - \frac{1}{4!}\cdot\frac{1\cdot 2\cdot 3}{1^4}(x-1)^4$

As you can see, the series is alternating and there is a factorial pattern that emerges in numerator

so $\boxed{\ln(x) = \sum_{n=1}^{\infty} \frac{(n-1)!}{n!}(x-1)^n}$

Check convergence: $\lim_{n\to\infty} \left| \frac{\cancel{n!}}{(n+1)\cancel{!}} (x-1)^{\cancel{n}+1} \cdot \frac{\cancel{n!}}{\cancel{(n-1)!}\ \cancel{(x-1)^n}} \right|$

$= |x+1| \lim_{n\to\infty} \underbrace{\left| \frac{n}{n+1} \right|}_{\text{goes to } 1} = |x+1| < 1$ for convergences

Thus the radius of convergence is $\boxed{0<x<2}$

16. $\frac{1}{1-x}$ about $x_0 = 2$

Expand function by first few terms w/ derivatives

$f'(x) = \frac{1}{(1-x)^2}$, $f'' = \frac{2}{(1-x)^3}$, $f''' = \frac{3\cdot 2}{(1-x)^4}$, (all by chainrule)

Thus the expansion is as follows

$\frac{1}{1-x} \approx -1 + 1\cdot(x-2) - \frac{2(x-2)^2}{2} + \frac{3\cdot 2(x-2)^3}{3\cdot 2} + \ldots$

Thus pattern is relatively simple given alternating series

$\boxed{\frac{1}{1-x} = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{n!(x-2)^n}{n!}}$ Find radius of convergence w/ ratio test

$\lim_{n\to\infty} \left| \frac{(-1)^{n+2}\cancel{(n+1)!}(x-2)^{\cancel{n}+1}\, n!}{\cancel{(-1)^{n+1}}\ \cancel{n!}\ \cancel{(x-2)^n}\ \cancel{(n+1)!}} \right| = |(x-2)(-1)| \lim_{n\to\infty} |1| = |x-2| < 1$

Thus the radius of convergence is $\boxed{1<x<3}$

17. $y = \sum_{n=0}^{\infty} n x^n$, find y' and y''

Find y' via power rule: $y' = \sum_{n=0}^{\infty} n^2 x^{n-1} = \sum_{n=0}^{\infty} (n+1)^2 x^n$ By shifting indices shifting indices where $n-2 \Rightarrow n$

17 (cont.) y'' via power rule: $y'' = \sum_{n=0}^{\infty} n^2(n-1)x^{n-2} = \sum_{n=0}^{\infty} (n+2)^2(n+1)x^n$

Expand y'' and y' including coefficient of x^n term

$y' \approx 1 + 4x^1 + 9x^2 + 16x^3 + \dots + 1(n+1)^2 x^n + \dots$

$y'' \approx 5 + 18x + 16 \cdot 3x^2 + 100x^3 + \dots + (n+2)^2(n+1)x^n + \dots +$

20. Verify given equation

$$\sum_{k=0}^{\infty} a_{k+1}x^k + \sum_{k=0}^{\infty} a_k x^{k+1} = a_1 + \sum_{k=1}^{\infty} (a_{k+1} + a_{k-1})x^k$$

$a_1 + \sum_{k=1}^{\infty} a_{k+1}x^k + \sum_{k=0}^{\infty} a_k x^{k+1}$ } this can be rewritten by shifting the indicies

$\sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{m=1}^{\infty} a_{m-1}x^m = \sum_{k=1}^{\infty} a_{k-1}x^k$

take $k+1=m$, rewrite

can switch indices

By addition of series

Thus $a_1 + \sum_{k=1}^{\infty} a_{k+1}x^k + \sum_{k=1}^{\infty} a_{k-1}x^k = a_1 + \sum_{k=1}^{\infty} (a_{k+1} + a_{k-1})x^k$ ✓

25. Rewrite the given expression as a sum whose generic term involves x^n:

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + x\sum_{k=1}^{\infty} k a_k x^{k-1}$$

Start by shifting the indices of the first series

set $n = m-2$ and plug in:

thus it converts $\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$

Now move to $x\sum_{k=1}^{\infty} k a_k x^{k-1}$, bring x factor into sum

↳ $\sum_{k=1}^{\infty} k a_k x^k$

↳ The indices of this sum can be rewritten easily to start with 0 as 0^{th} term of series is 0

So $\sum_{k=1}^{\infty} k a_k x^k = \sum_{n=0}^{\infty} n a_n x^n$

25 (cont.) Write whole expression as:

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} n a_n x^n = \sum_{n=0}^{\infty}\left((n+2)(n+1)a_{n+2} + na_n\right)x^n$$

5.2 1. a) Seek power series solution and recurrence relation
b) First four terms of solutions y_1, y_2
c) Evaluate Wronskian
d) Find general solution

$y'' - y = 0, \quad x_0 = 0$

assume $y = \sum_{n=0}^{\infty} a_n x^n$; $y' = \sum_{n=0} a_n n x^{n-1}$; $y'' = \sum_{n=0} (n-1) n a_n x^{n-2}$

y'' can be rewritten as $\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n$

Plugging back in ODE, we arrive at
$\sum_{n=0}^{\infty}\left((n+1)(n+2)a_{n+2} - a_n\right)x^n = 0$ and can find recurrence relation

Thus recurrence relation is $a_{n+2} = \frac{a_n}{(n+1)(n+2)}$

Thus $a_2 = \frac{a_0}{2}$, $a_4 = \frac{a_2}{3\cdot 4} = \frac{a_0}{2\cdot 3\cdot 4}$, $a_6 = \frac{a_4}{5\cdot 6} = \frac{a_0}{2\cdot 3\cdot 4\cdot 5\cdot 6}$

and $a_3 = \frac{a_1}{2\cdot 3}$, $a_5 = \frac{a_3}{4\cdot 5} = \frac{a_1}{2\cdot 3\cdot 4\cdot 5}$, $a_7 = \frac{a_1}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}$

b) $y_1 \approx a_0\left(1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{6!}x^6 + \dots\right)$

$y_2 \approx a_1\left(x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \dots\right)$

using above

c) Find wronskian at $x_0 = 0$ $y_1(0) = a_0$, $y_2(0) = 0$
$y_1'(0) = 0$ and $y_2'(0) = a_1$
Thus $W = a_0 a_1 \neq 0$, so this is a fundamental set of solutions

d) This can be written by a general term
Whereby $a_{2n} = \frac{a_0}{(2n)!}$ for evens and $a_{2n+1} = \frac{a_1}{(2n+1)!}$ for odds

5. $(1-x)y''+y=0,\ x_0=0$

Start w/ $y=\sum_{n=0}^{\infty} a_n x^n$; $y''=\sum_{n=0}^{\infty}(n-1)n a_n x^{n-2}$

Evaluate: $(1-x)\sum_{n=0}^{\infty}(n-1)n a_n x^{n-2} =$

$$=\sum_{n=0}^{\infty}(n-1)n a_n x^{n-2}-\sum_{n=0}^{\infty}(n-1)n a_n x^{n-1} \quad m=n-1$$

This can be rewritten: $\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=0}^{\infty}(n+1)n a_{n+1}x^n$

Thus equation is as follows:

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}-(n+1)n a_{n+1}+a_n\right]x^n=0$$

Thus $a_{n+2}=\frac{n a_{n+1}}{n+2}-\frac{a_n}{(n+2)(n+1)}$ $\qquad a_2=\frac{-a_0}{2}$

$a_3=\frac{a_2}{3}-\frac{a_1}{(n+2)(n+1)}$ $\quad a_3=\frac{-a_0}{2\cdot 3}-\frac{a_1}{(n+2)(n+1)}=\frac{-a_0}{2\cdot 3}-\frac{a_1}{3\cdot 2}$

$a_4=\frac{2a_3}{4}-\frac{a_2}{3\cdot 4}=\frac{-2a_0}{2\cdot 3\cdot 4}-\frac{2a_1}{4\cdot 3\cdot 2}+\frac{a_0}{2\cdot 3\cdot 4}$

$a_5=\frac{3a_4}{5}-\frac{a_3}{5\cdot 4}=\frac{3}{5}\left(\frac{-2a_0}{2\cdot 4\cdot 5}-\frac{2a_1}{4\cdot 3\cdot 2}-\frac{a_0}{2\cdot 3\cdot 4}\right)-\frac{1}{5\cdot 4}\left(\frac{-a_0}{2\cdot 5}-\frac{a_1}{3\cdot 2}\right)$

b) The two solutions arise from the different a_1 and a_0 terms. Thus both solutions will have $n=3,4,5\ldots$ terms

y_1 (a_0 solution): $y_1=a_0\left(1-\frac{1}{2}x^2-\frac{1}{6}x^3+\left(\frac{-2}{4!}+\frac{1}{4!}\right)x^4+\ldots\right)$

y_2 (a_1 solution): $y_2=a_1\left(x-\frac{x^3}{6}-\frac{x^4}{12}+\left(\frac{1}{5!}-\frac{6}{5!}\right)x^5\right)$

arises from a_1 terms in a_5 above

c) $y_1(x=0)=a_0$ $\qquad y_1'(x=0)=0$ $\qquad$ Thus $W=a_0a_1\neq 0$

$y_2(x=0)=0$ $\qquad y_2'(x=0)=a_1$ $\qquad$ so this is a fundamental set of solutions

d) Series is too complex to be written as just a sum

9. $(1+x^2)y'' - 4xy' + 6y = 0$, $x_0 = 0$

Start assuming $y = \sum_{n=0}^{\infty} a_n x^n$

$$y'' = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}, \quad y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

Plug into equation:

$$\underbrace{(1+x^2)\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}}_{} - \underbrace{4x\sum_{n=0}^{\infty} n a_n x^{n-1}}_{\text{multiply } x \text{ into sum}} + 6\sum_{n=0}^{\infty} a_n x^n = 0$$

Multiply $1+x^2$ into sum:

$$\underbrace{\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}}_{} + \sum_{n=0}^{\infty} n(n-1)a_n x^n - 4\sum_{n=0}^{\infty} n a_n x^n + 6\sum_{n=0}^{\infty} a_n x^n = 0$$

Shift indices so that the whole equation is in terms of x^n:

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} + \underbrace{n(n-1)a_n - 4na_n + 6a_n}_{a_n(n^2-5n+6)}\right]x^n = 0$$

Thus obtain recurrence relation:

$$a_{n+2} = \frac{-(n-3)(n-2)a_n}{(n+2)(n+1)} \; : \; a_2 = +3a_0 \; : \; a_4 = 0$$
$$a_3 = -\frac{a_1}{3} \; : \; a_5 = 0$$

This means series terminates

So y_1 follows from even series terms

and y_2 follows from odd series terms

b) $y_1 = a_0(1 - 3x^2)$ $\qquad$ $y_1' = -6a_0 x$

$y_2 = a_1\left(x - \frac{x^3}{3}\right)$ $\qquad$ $y_2' = a_1(1 - x^2)$

c) $y_1(x_0 = 0) = a_0$ $\qquad$ $y_1'(x=0) = 0$

$y_2(x=0) = 0$ $\qquad$ $y_2'(x=0) = a_1$

Thus $W(y_1, y_2) = a_0 a_1 \neq 0$, so this is a fundamental set of solutions

d) Since series terminates, it cannot be written as a simpler sum.