

Lectures on Stabilized Ellipsoid Embeddings

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Abstract

These notes are based on a five-part minicourse on stabilized symplectic embeddings given at a workshop in Les Marécottes, Switzerland during a September 2025 workshop.¹ Our main goal is to explain the recent resolution of the (restricted) stabilized ellipsoid embedding problem by D. McDuff and the author. Along the way we also introduce various other ideas which shed light on the context and hint at possible generalizations. Some of the concepts covered include sesquicuspidal curves, symplectic inflation, multidirectional tangency constraints, well-placed curves, cluster transformations, Looijenga pairs, toric models, scattering diagrams, and the tropical vertex theorem.

Contents

Lecture 1: Overview	2
§1.1 Hamiltonian flows and ellipsoid embeddings	2
§1.2 The symplectic Fibonacci staircase	4
§1.3 Full fillings by long ellipsoids	5
§1.4 Stabilized ellipsoid embeddings	7
§1.5 Unicuspidal and sesquicuspidal curves	8
§1.6 Outline of the remaining lectures	10
Lecture 2: Enumerative geometry and sesquicuspidal curves	10
§2.1 The inner and outer corner curves	10
§2.2 Inflation along sesquicuspidal curves	11
§2.3 Multidirectional tangency constraints	12
Lecture 3: Well-placed curves and their symmetries	12
§3.1 Looijenga pairs	12
§3.2 Toric models	12
§3.3 Elementary transformations	12
§3.4 Well-placed curves	12
§3.5 The shift and reflection symmetries	12
Lecture 4: Scattering diagrams and the tropical vertex	12
§4.1 Scattering diagram basics	12
§4.2 Some toy examples	12
§4.3 The tropical vertex theorem	12
Lecture 5: Putting it all together	12

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§5.1 The change of lattice trick	12
§5.2 Proof of Theorem C	12

Disclaimer. Many of the mathematical statements in these lectures are somewhat informal simplifications; we recommend consulting the original references for more precise formulations whenever possible. Also, we apologize in advance for any missing citations, and we welcome any comments, questions, or corrections.

Lecture 1: Overview

In this lecture, we first briefly introduce Hamiltonian flows, nonsqueezing phenomena, and the ellipsoid embedding problem in §1.1. We then recall the celebrated McDuff–Schlenk Fibonacci staircase in §1.2, and give a brief modern overview of ways to derive it. In §1.3 we recall some connections between four-dimensional ellipsoid embeddings, symplectic ball packings, and symplectic cones, and as an application we establish full symplectic fillings of the four-ball by sufficiently long ellipsoids. In §1.4, we introduce stabilized ellipsoid embeddings and formulate our main result, namely Theorem A. Lastly, in §1.5 we introduce sesquicuspidal curves as a key tool for obstructing stabilized ellipsoid embeddings. We end with a brief sketch in §1.6 of the remaining lectures.

§1.1 Hamiltonian flows and ellipsoid embeddings

Consider a Hamiltonian on \mathbb{R}^{2n} , i.e. a smooth function $H : \mathbb{R}^{2n} \rightarrow \mathbb{R}$. Its *symplectic gradient* (a.k.a. *Hamiltonian vector field*) is defined by $\nabla_\omega H := J\nabla H$, where ∇ is the usual gradient and J is the block matrix $J = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$. Integrating $\nabla_\omega H$ gives rise to the flow $\text{fl}_H^t : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ for $t \in \mathbb{R}$.

There are various interesting questions one can ask about Hamiltonian flows, for instance:

- periodic orbits (existence, lower bounds, density, . . .)
- integrability versus chaos (exact solvability, KAM tori)
- Poincaré recurrence (or “superrecurrence”)

and so on. These lectures are related to the so-called “symplectic nonsqueezing phenomena” and its generalizations. We first recall some basics.

Proposition 1.1 (“Liouville’s theorem”)

The symplectic gradient $\nabla_\omega H$ is divergence-free, i.e. fl_H^t preserves volume.

Notation 1.2. For $\vec{a} = (a_1, \dots, a_n) \in \mathbb{R}_{>0}^n$, put $E(\vec{a}) := \left\{ \pi \sum_{i=1}^n \frac{1}{a_i} (x_i^2 + y_i^2) \leq 1 \right\}$.

Remark 1.3. Note that $E(a, \dots, a)$ is the ball $B^{2n}(a)$ of radius $\sqrt{a/\pi}$. Also, if $\mu : \mathbb{C}^n \rightarrow \mathbb{R}_{\geq 0}^n$ denotes the moment map for the standard \mathbb{T}^n -action, i.e. $\mu(z_1, \dots, z_n) = (\pi|z_1|^2, \dots, \pi|z_n|^2)$, then $\mu(E(a_1, a_2))$ is the right triangle with vertices $(0, 0)$, $(a_1, 0)$, $(0, a_2)$ as in Figure 1.1.

Notation 1.4. Put $E(\vec{a}) \xrightarrow{s} E(\vec{b})$ if $\text{fl}_H^1(E(\vec{a})) \subset E(\vec{b})$ for some (time-dependent) $H : \mathbb{R}^{2n} \times [0, 1] \rightarrow \mathbb{R}$.

Remark 1.5. This turns out to be equivalent to abstract symplectic embedding, i.e. a smooth embedding

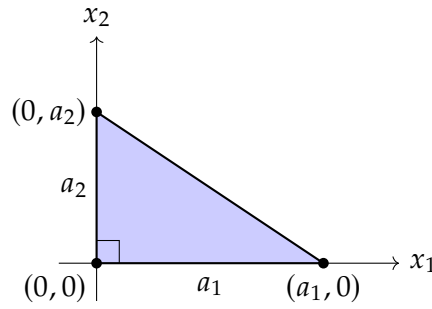


Figure 1: The image of $E(a_1, a_2)$ under the moment map $\mu : \mathbb{C}^2 \rightarrow \mathbb{R}_{\geq 0}^2$.

preserving the standard symplectic form, thanks to the “extension after restriction principle” – see e.g. [Sch18, §4.4]. The discrepancy between time-dependent and time-independent Hamiltonian flows is more subtle – see e.g. [PS16].

Theorem 1.6 (Gromov ‘85 [Gro85])

If $E(\vec{a}) \xrightarrow{s} E(\vec{b})$, then we must have $\min\{a_i\} \leq \min\{b_i\}$.

Definition 1.7. For a symplectic manifold M^{2n} , the **ellipsoid embedding function** is

$$\mathcal{E}_M : \mathbb{R}_{>0}^n \rightarrow \mathbb{R}_{>0}, \quad \mathcal{E}_M(\vec{a}) := \inf\{c \mid E(\frac{1}{c} \cdot \vec{a}) \xrightarrow{s} M\}.$$

Remark 1.8. Note that we have $\mathcal{E}_M(c \cdot \vec{a}) = c\mathcal{E}_M(\vec{a})$ for any scalar $c \in \mathbb{R}_{>0}$, i.e. $\mathcal{E}_M(\vec{a})$ is really a function of just $n - 1$ variables.

Problem 1.9 (“ellipsoid embedding problem” (EEP)). Compute $\mathcal{E}_M(\vec{a})$ for $M = E(\vec{b})$.

Problem 1.10 (“restricted ellipsoid embedding problem” (REEP)). Compute $\mathcal{E}_M(\vec{a})$ for $M = B^{2n}$.

Here are a few initial comments.

- The ellipsoid embedding problem is trivial for $n = 1$.
- The ellipsoid embedding problem is “solved” for $n = 2$ by the work of many people (McDuff, Hutchings, Schlenk, Hofer, Biran, Polterovich, Li, ...). Here “solved” means reduced to a deceptively simple combinatorial criterion.
- The solution to the restricted ellipsoid embedding problem for $n = 2$ was worked out in complete detail by McDuff–Schlenk [MS12].
- To our knowledge there is not even a conjecture for the restricted ellipsoid embedding problem for $n \geq 3$ (i.e. a formula for the function $\mathcal{E}_{B^6}(a_1, a_2, a_3)$).

§1.2 The symplectic Fibonacci staircase

Theorem 1.11 (McDuff–Schlenk [MS12])

The ellipsoid embedding function $\mathcal{E}_{B^4}(1, a)$ of the four-dimensional ball B^4 is as follows:

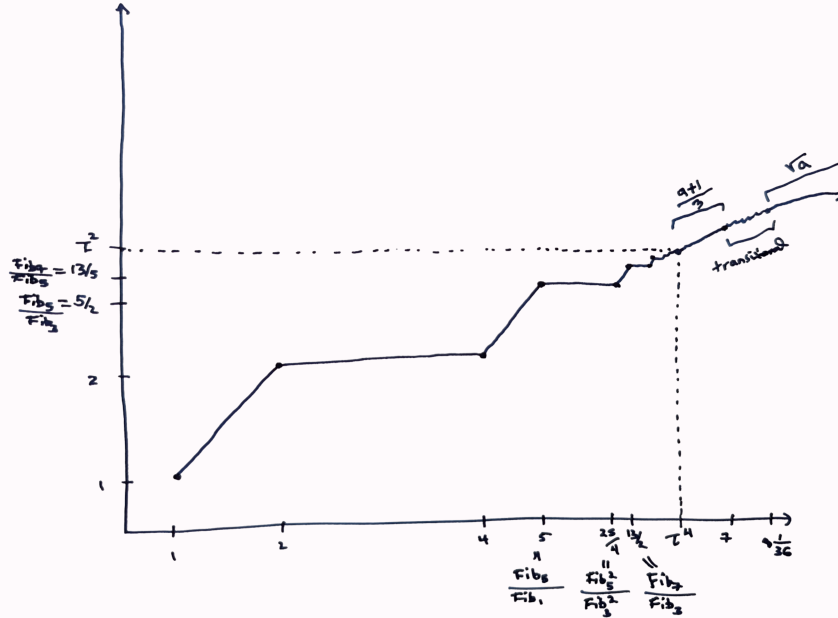


Figure 2: The ellipsoid embedding function $\mathcal{E}_{B^4}(1, a)$.

While Figure 2 is just a cartoon, the function $\mathcal{E}_{B^4}(1, a)$ is explicitly computed in [MS12] and it includes the following noteworthy features:

- it is an “infinite Fibonacci staircase” for $a \in [1, \tau^4]$, with infinitely many steps accumulating at $\tau^4 := \frac{1}{2}(7 + 3\sqrt{5}) \approx 6.85$, where the x and y coordinates of the outer and inner corners are ratios of Fibonacci numbers (or their squares)
- for $a \in [8\frac{1}{36}, \infty)$, we have $\mathcal{E}_{B^4}(1, a) = \sqrt{a}$, i.e. volume is the only embedding obstruction.
- it is the linear function $\frac{1}{3}(a + 1)$ on the segment $a \in [\tau^4, 7]$, and in the range $a \in [7, 8\frac{1}{36}]$ it exhibits transitional behavior.

Let Fib_i denote the i th Fibonacci number:

i	0	1	2	3	4	5	6	7	...
Fib_i	0	1	1	2	3	5	8	13	...

Exercise 1.12. Show that $\mathcal{E}_{B^4}(1, a)$ is nondecreasing and $\frac{1}{a} \cdot \mathcal{E}_{B^4}(1, a)$ is nonincreasing. Conclude that $\mathcal{E}_{B^4}(1, a)$ for $a \in [1, \tau^4]$ is entirely characterized by:

- embedding obstructions:

$$\mathcal{E}_{B^4}\left(1, \frac{\text{Fib}_{2k+5}}{\text{Fib}_{2k+1}}\right) \geq \frac{\text{Fib}_{2k+5}}{\text{Fib}_{2k+3}} \quad (1.1)$$

- embedding constructions:

$$\mathcal{E}_{B^4}\left(1, \frac{\text{Fib}_{2k+3}^2}{\text{Fib}_{2k+1}^2}\right) \leq \frac{\text{Fib}_{2k+5}}{\text{Fib}_{2k+1}} \quad (1.2)$$

There are multiple ways of establishing (1.1) and (1.2). For (1.1), all of the main approaches are based on **pseudoholomorphic curves**.² While the method of extracting symplectic embedding obstructions from pseudoholomorphic curves is by now fairly streamlined, there are many different approaches for *producing* such curves, for instance:

- applying Cremona transformations to basic exceptional curves
- embedded contact homology
- cluster symmetries.

In these lectures we will primarily explain the last bullet above, as it is the most directly connected to our primary goal.

For (1.2), there are several methods for constructing symplectic embeddings, most notably:

- inflation along pseudoholomorphic curves
- almost toric fibrations.

While almost toric fibrations (ATFs) provide a very visually and conceptually appealing approach to the embeddings for (1.2), we will primarily focus on the first bullet above in these lectures, partly because the ATF approach to embeddings is already discussed very thoroughly elsewhere (see e.g. [Sym, Eva23, CGHMP25, CV22]) and partly because it allows us to reduce both (1.1) and (1.2) to the existence of certain singular symplectic curves in the complex projective plane. There are then multiple methods for constructing the curves relevant for inflation approach to (1.2): Seiberg–Witten theory, tropical methods (see e.g. [MS25b, §5.3]), etc.

§1.3 Full fillings by long ellipsoids

We associate to each positive rational number $a \in \mathbb{Q}_{>1}$ its **weight sequence** $\text{wt}(a) = (w_1, \dots, w_k)$, whose definition is more or less explained by the “box diagram” in Figure 3, which divides a rectangle of height 1 and width a into a collection of squares.

Example 1.13. For $a = 17/5$, the corresponding weight sequence is $\text{wt}(17/5) = (1, 1, 1, 2/5, 2/5, 1/5, 1/5)$.

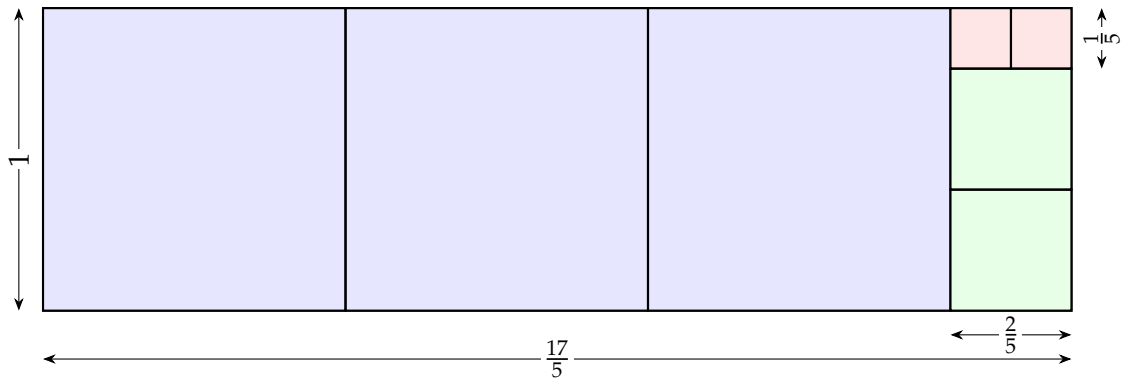
$\underbrace{\hspace{1.5cm}}_3 \quad \underbrace{\hspace{1.5cm}}_2 \quad \underbrace{\hspace{1.5cm}}_2$

Note that:

- $\frac{17}{5} = 3 + \frac{1}{2+\frac{1}{2}}$, i.e. the multiplicities of the entries in the weight sequence give the continued fraction expansion
- $1 \geq w_1 \geq w_2 \geq \dots \geq w_k$
- $w_1^2 + \dots + w_k^2 = a$.

Each of these properties holds for general $a \in \mathbb{Q}_{>1}$.

²It is, however, interesting to ask how much of this story could be recovered using generating functions or microlocal sheaves.

Figure 3: The box diagram giving the weight sequence for $a = 17/5$.**Theorem 1.14 (McDuff [McD09])**

For each $a \in \mathbb{Q}_{>1}$, say with weight sequence $\text{wt}(a) = (w_1, \dots, w_k)$, we have

$$E(1, a) \xrightarrow{s} B^4(c) \iff \bigsqcup_{i=1}^k B^4(w_i) \xrightarrow{s} B^4(c).$$

Notation 1.15. Put:

- $\text{Bl}^k \mathbb{CP}^2 := \mathbb{CP}^2 \# \underbrace{(\overline{\mathbb{CP}^2} \# \dots \# \overline{\mathbb{CP}^2})}_k$, i.e. the k -fold blowup of \mathbb{CP}^2 as a smooth oriented manifold
- $H_2(\text{Bl}^k \mathbb{CP}^2) = \mathbb{Z}\langle \ell, e_1, \dots, e_k \rangle$, with ℓ the line class and e_1, \dots, e_k the obvious exceptional classes
- $C(\text{Bl}^k \mathbb{CP}^2) := \left\{ [\omega] \in H^2(\text{Bl}^k \mathbb{CP}^2) \mid \begin{array}{l} \omega \text{ symplectic form on } \text{Bl}^k \mathbb{CP}^2, \\ c_1(\omega) = \text{PD}(3\ell - e_1 - \dots - e_k) \end{array} \right\}$, the “symplectic cone with standard first Chern class”.

Theorem 1.16 (McDuff–Polterovich [MP94])

We have

$$\bigsqcup_{i=1}^k B^4(w_i) \xrightarrow{s} B^4(c) \iff \text{PD}(c\ell - w_1 e_1 - \dots - w_k e_k) \in C(\text{Bl}^k \mathbb{CP}^2).$$

Notation 1.17. Let $\text{Exc}(\text{Bl}^k \mathbb{CP}^2)$ denote the set of all homology classes $e \in H_2(\text{Bl}^k \mathbb{CP}^2)$ such that e is represented by a symplectically embedded sphere with self-intersection number -1 .

Remark 1.18. Note that *a priori* the definition of $\text{Exc}(\text{Bl}^k \mathbb{CP}^2)$ depends on the symplectic form on $\text{Bl}^k \mathbb{CP}^2$, but in fact one can show that it is independent of the choice of symplectic form in $C(\text{Bl}^k \mathbb{CP}^2)$ (this uses a Gromov compactness argument and the fact that any two $\omega, \omega' \in C(\text{Bl}^k \mathbb{CP}^2)$ are homotopic through symplectic forms).

Remark 1.19. Note that for $e = d\ell - m_1 e_1 - \dots - m_k e_k \in \text{Exc}(\text{Bl}^k \mathbb{CP}^2)$, we have

$$e \cdot e = d^2 - m_1^2 - \dots - m_k^2 = -1 \quad \text{and} \quad c_1(e) = 3d - m_1 - \dots - m_k = 1. \quad (1.3)$$

Theorem 1.20 ([McD98, Bir97, LL01, LL02])

We have

$$C(\mathrm{Bl}^k \mathbb{CP}^2) = \left\{ \alpha \in H^2(\mathrm{Bl}^k \mathbb{CP}^2) \mid \begin{array}{l} \alpha \cdot \alpha > 0, \\ \langle \alpha, e \rangle > 0 \ \forall e \in \mathrm{Exc}(\mathrm{Bl}^k \mathbb{CP}^2) \end{array} \right\}.$$

Corollary 1.21. For $a \in \mathbb{Q}_{>1}$, say with $\mathrm{wt}(a) = (w_1, \dots, w_k)$, we have

$$\mathcal{E}_{B^4}(1, a) = \max \left(\sqrt{a}, \sup \left\{ \frac{m_1 w_1 + \dots + m_k w_k}{d} \mid d\ell - m_1 e_1 - \dots - m_k e_k \in \mathrm{Exc}(\mathrm{Bl}^k \mathbb{CP}^2) \right\} \right)$$

Lemma 1.22. We have $\mathcal{E}_{B^4}(1, a) = \sqrt{a}$ for all $a \geq 9$.

Proof. It suffices to establish $\frac{m_1 w_1 + \dots + m_k w_k}{d} \leq \sqrt{a}$ for all $d\ell - m_1 e_1 - \dots - m_k e_k \in \mathrm{Exc}(\mathrm{Bl}^k \mathbb{CP}^2)$. Observe that we have

$$\frac{m_1 w_1 + \dots + m_k w_k}{d} \leq \frac{m_1 + \dots + m_k}{d} = \frac{3d - 1}{d} < 3 < \sqrt{a}.$$

□

§1.4 Stabilized ellipsoid embeddings

Definition 1.23. For a symplectic manifold M^{2n} and $N \in \mathbb{Z}_{\geq 1}$, the **stabilized ellipsoid embedding function** is

$$\mathcal{E}_M^N : \mathbb{R}_{>0}^n \rightarrow \mathbb{R}_{>0}, \quad \mathcal{E}_M^N(\vec{a}) := \inf \{ c \mid E(\tfrac{1}{c} \cdot \vec{a}) \times \mathbb{R}^{2N} \xhookrightarrow{s} M \times \mathbb{R}^{2N} \}. \quad (1.4)$$

Remark 1.24. Note that by taking the identity map in the extra dimensions, we always have

$$\mathcal{E}_M^N(\vec{a}) \leq \mathcal{E}_M(\vec{a}).$$

Problem 1.25 (“stabilized ellipsoid embedding problem” (SEEP)). Compute $\mathcal{E}_M^N(\vec{a})$ for $M = E(\vec{b})$ and $N \in \mathbb{Z}_{\geq 1}$.

Problem 1.26 (“restricted stabilized ellipsoid embedding problem” (RSEEP)). Compute $\mathcal{E}_M^N(\vec{a})$ for $M = B^{2n}$ and $N \in \mathbb{Z}_{\geq 1}$.

Remark 1.27. Note that we can think of $E(a_1, \dots, a_n) \times \mathbb{R}^{2N}$ as $E(a_1, \dots, a_n, \underbrace{\infty, \dots, \infty}_N)$, and hence

the stabilized ellipsoid embedding problem is essentially a special case of the corresponding unstabilized ellipsoid embedding problem (but now in higher dimensions).

Our primary goal in these lectures is to explain the proof of the following theorem.

Theorem A ([MS24])

For all $N \in \mathbb{Z}_{\geq 1}$, the stabilized ellipsoid embedding of the four-dimensional ball is given by

$$\mathcal{E}_{B^4}^N(1, a) = \begin{cases} \mathcal{E}_{B^4}(1, a) & a \in [1, \tau^4] \\ \frac{3a}{a+1} & a > \tau^4 \end{cases}$$

See Figure 4 for an illustration.

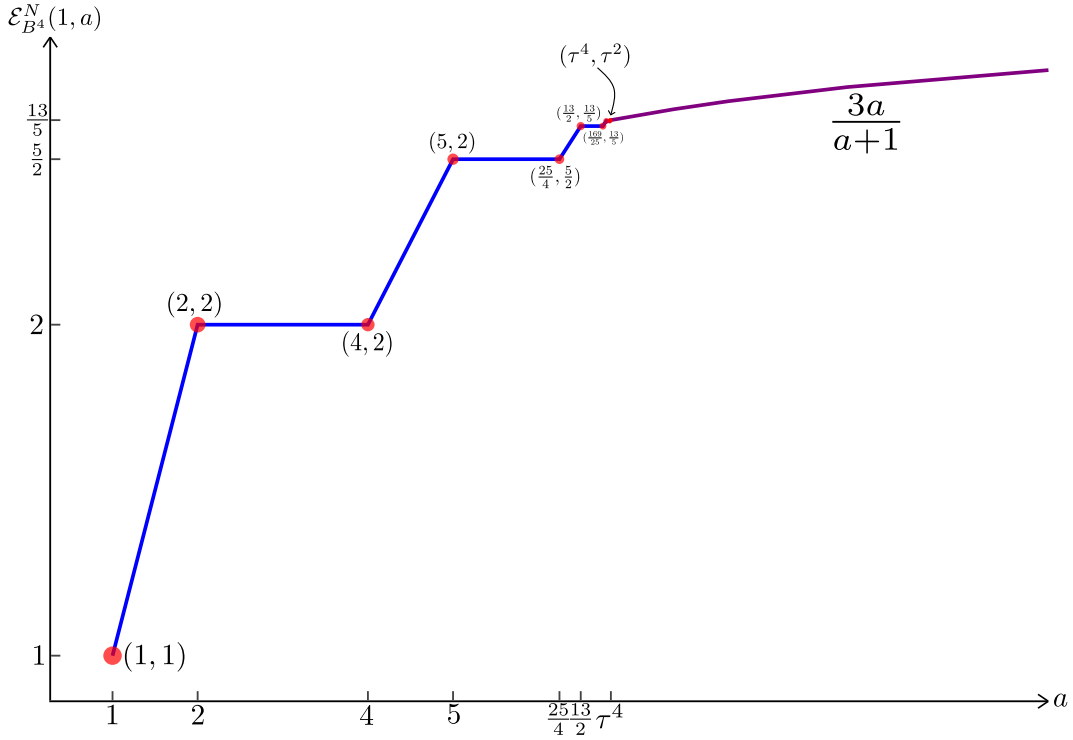


Figure 4: The stabilized ellipsoid embedding function $\mathcal{E}_{B^4}^N(1, a)$.

In brief, the function $\mathcal{E}_{B^4}^N(1, a)$ has a “subcritical regime” which is an infinite staircase and a “supercritical regime” which is a smooth rational function, with the phase transition at $a = \tau^4$. One can check that the rational function $\frac{3a}{a+1}$ meets the infinite Fibonacci staircase precisely at its outer corners and lies otherwise above it for $a \in [1, \tau^4]$, whereas it lies below \sqrt{a} for $a > \tau^4$. Note that the volume bound \sqrt{a} evidently no longer holds for the stabilized problem, i.e. the supercritical behavior is a truly higher dimensional phenomenon.

Remark 1.28. There were important earlier partial results by subsets of Hind, Kerman, Cristofaro-Gardiner, McDuff, and the author (see e.g. [HK14, CGH18, CGHM18, McD18, Sie22]). In particular, the function for $a \in [1, \tau^4]$ was worked out in [CGH18], which by Exercise 1.12 and (1.4) boils down to showing that the inequality (1.1) persists after stabilization.

The upper bounds for Theorem A are given by an explicit “symplectic folding” construction due to Hind, which adapts Guth’s polydisk embedding construction in [Gut08].

Theorem 1.29 (Hind [Hin15])

For each $a \in \mathbb{R}_{\geq 1}$ and $N \in \mathbb{Z}_{\geq 1}$ we have

$$E(1, a) \times \mathbb{R}^{2N} \xrightarrow{s} B^4(c) \times \mathbb{R}^{2N} \quad \text{for} \quad c > \frac{3a}{a+1}. \quad (1.5)$$

Thus, our main goal is to find optimal lower bounds for $\mathcal{E}_{B^4}^N(1, a)$, especially for $a > \tau^4$.

§1.5 Unicuspidal and sesquicuspidal curves

Let p, q be positive integers satisfying $\gcd(p, q) = 1$ and $p > q \geq 2$.

Definition 1.30. A (p, q) -sesquicuspidal symplectic curve in a symplectic four-manifold M^4 is

a subset $C \subset M$ such that

- C has one (p, q) cusp singularity (i.e. modeled locally on $\{x^p + y^q = 0\} \subset \mathbb{C}^2$)
- C has finitely many additional positive ordinary double points (i.e. modeled locally on $\{xy = 0\} \subset \mathbb{C}^2$)
- C is otherwise symplectically embedded.

Remark 1.31. The second condition in Definition 1.30 holds after a generic perturbation, and the number of double points $\underline{\delta}$ away from the distinguished cusp is fixed by adjunction, namely

$$\underline{\delta} = \frac{1}{2}(C \cdot C + \chi(C) - c_1(C) - (p-1)(q-1)). \quad (1.6)$$

In the case $\underline{\delta} = 0$, the curve is called **unicuspidal**.³ Note that we can similarly define (p, q) -sesquicuspidal algebraic curves, which in particular give examples of (p, q) -sesquicuspidal symplectic curves.

Definition 1.32. We will define the (real) **index** of a (p, q) -sesquicuspidal curve $C \subset M$ to be $\text{ind}(C) := 2(c_1(C) - p - q)$.

We will see later that this is the expected (real) dimension of the moduli space of (p, q) -sesquicuspidal curves in a fixed homology class and with fixed “position” of the distinguished cusp.

Proposition B ([MS23, MS25a])

Given a rational index zero (p, q) -sesquicuspidal symplectic curve C in a closed symplectic four-manifold M^4 , we have

$$\mathcal{E}_M^N(p, q) \geq \frac{pq}{\text{area}(C)} \quad (1.7)$$

for all $N \in \mathbb{Z}_{\geq 0}$.

Here $\text{area}(C)$ denotes the symplectic area of C , which depends only on its homology class $[C] \in H_2(M)$. Note that, after rescaling, (1.7) can be written equivalently as $\mathcal{E}_M^N(1, p/q) \geq \frac{p}{\text{area}(C)}$.

The above proposition allows us to reduce Theorem A to the following key construction of singular algebraic curves. As above we assume $\gcd(p, q) = 1$ and $p > q \geq 2$.

Theorem C ([MS24])

There exists a rational index zero (p, q) -sesquicuspidal symplectic curve $C \subset \mathbb{CP}^2$ if and only if

- (a) $p/q = \frac{\text{Fib}_{2k+5}}{\text{Fib}_{2k+1}}$ for $k \in \mathbb{Z}_{\geq 1}$, or
- (b) $p/q > \tau^4$.

Exercise 1.33. Combining Proposition B with Theorem C gives precisely the lower bounds on $\mathcal{E}_{B^4}^N$ needed for Theorem A.

Thus in the remaining lectures we will focus on the proofs of Proposition B and Theorem C, albeit with various detours.

³The term sesquicuspidal comes from Latin sesqui meaning “one and a half”, reflecting the fact that sesquicuspidal curves are only mildly more singular than unicuspidal curves.

§1.6 Outline of the remaining lectures

Here is the plan for the remaining lectures:

- In Lecture 2, we discuss the enumerative geometry and sesquicuspidal curves, and we use this formalism to sketch a proof of Proposition B.
- In Lecture 3, we define well-placed curves and discuss their cluster symmetries, and we give one construction of the “outer corner curves” alluded to in §1.2.
- In Lecture 4, we give a brief crash course in scattering diagrams and formulate the tropical vertex theorem proved in [GPS10].
- Finally, in Lecture 5 we put all of this together to give the proof of Theorem C.

Remark 1.34. While much of the theory developed in this lectures can be extended in various directions, for instance by replacing \mathbb{CP}^2 with other del Pezzo surfaces, for simplicity we will mostly focus our exposition on the case of the complex projective plane.

Lecture 2: Enumerative geometry and sesquicuspidal curves

§2.1 The inner and outer corner curves

Let us begin by recalling the following result as motivation.

Theorem 2.1 ([FdBMLMHN06])

There exists a (p, q) -unicuspidal rational algebraic curve in \mathbb{CP}^2 of degree d with $p > q \geq 2$ and $\gcd(p, q) = 1$ if and only if (d, p, q) is one of the following:

- $(p, q) = (d, d - 1)$ for $d \geq 3$
- $(p, q) = (2d - 1, d/2)$ for $d \geq 4$ even
- $(p, q) = (\text{Fib}_{2k+3}^2, \text{Fib}_{2k+1}^2)$ and $d = \text{Fib}_{2k+3}\text{Fib}_{2k+1}$ for $k \geq 1$
- $(p, q) = (\text{Fib}_{2k+5}, \text{Fib}_{2k+1})$ and $d = \text{Fib}_{2k+3}$ for $k \geq 1$
- $(p, q) = (22, 3)$ and $d = 8$
- $(p, q) = (43, 6)$ and $d = 16$.

Recall that the index of such a curve C is given by

$$\text{ind}(C) = 2(c_1(C) - p - q) = 2(3d - p - q).$$

Exercise 2.2. The indices of the curves in Theorem 2.1 are as follows:

	(a)	(b)	(c)	(d)	(e)	(f)
index	$2d + 2$	$d + 2$	2	0	-2	-2

Observe that, in (c), $\frac{\text{Fib}_{2k+3}^2}{\text{Fib}_{2k+1}^2}$ is precisely the x -value of the k th inner corner point in the Fibonacci staircase from Theorem 1.11, so we will refer to these as **inner corner curves**. Similarly, in (d), $\frac{\text{Fib}_{2k+5}}{\text{Fib}_{2k+1}}$ is precisely the x -value of the k th outer corner point, and we will refer to these as **outer**

corner curves. Note also that the outer corner curves have index zero, so by Proposition B their existence gives

$$\mathcal{E}_{B^4}^N \left(1, \frac{\text{Fib}_{2k+5}}{\text{Fib}_{2k+1}} \right) \geq \frac{\text{Fib}_{2k+5}}{\text{Fib}_{2k+3}},$$

where $\frac{\text{Fib}_{2k+5}}{\text{Fib}_{2k+3}}$ is precisely the y -value of the k th outer corner point.

Remark 2.3. Here are a few more remarks about the curve families (a),(b),(c),(d),(e),(f) in Theorem 2.1:

- family (a) is given by $\{ZY^{d-1} = X^d\}$
- family (b) is given by $\{(ZY - X^2)^{d/2} = XY^{d-1}\}$
- Miyanishi–Sugie [MS81] or Kashiwara [Kas87] construct a pencil with general member (c) and special member (d)
- Orevkov [Ore02] constructs families (d),(e),(f) (we will explain a related construction in §3.5 below)
- families (e),(f) are a bit “surprising” to a symplectic geometer in the sense that they should not persist under a generic perturbation of the almost complex structure.

§2.2 Inflation along sesquicuspidal curves

The following proposition, which is essentially an application of the technique of **symplectic inflation**, constructs four-dimensional symplectic ellipsoid embeddings from singular curves. While it is not directly related to the proof of Theorem C, it does help illustrate the role connection between sesquicuspidal curves and (stabilized) symplectic embeddings.

Proposition D

Let M be a closed symplectic four-manifold and $C \subset M$ a (p, q) -sesquicuspidal symplectic curve such that $[C] = c\text{PD}([\omega_M])$ for some $c \in \mathbb{R}_{>0}$ and $[C] \cdot [C] \geq pq$. Then $\mathcal{E}_M(p, q) \leq c$ (i.e. $\mathcal{E}_M(1, p/q) \leq c/q$).

Remark 2.4. Note that we have

$$\text{vol}(M) = \frac{1}{2} \int \omega_M \wedge \omega_M = \frac{1}{2} \text{PD}([\omega_M]) \cdot \text{PD}([\omega_M]) = \frac{1}{2c^2} [C] \cdot [C].$$

In the case $[C] \cdot [C] = pq$, Proposition D corresponds to an embedding $E\left(\frac{p}{c}, \frac{q}{c}\right) \xrightarrow{s} M$ (up to small error), which is a full filling since $\text{vol}(E\left(\frac{p}{c}, \frac{q}{c}\right)) = \frac{pq}{2c^2} = \text{vol}(M)$.

Exercise 2.5. Show that the inner corner curves (i.e. family (c) from Theorem 2.1) satisfy $[C] \cdot [C] = d^2 = pq$, and these translate into full fillings via Proposition D.

Combining Exercise 1.33 and Exercise 2.5, we see that the entire (stabilized) Fibonacci staircase $\mathcal{E}_{B^4}^N(1, a)$ for $a \in [1, \tau^4]$ and $N \in \mathbb{Z}_{\geq 0}$ follows from the existence of the outer and inner corner unicuspidal curves (i.e. families (c) and (d) in Theorem 2.1).

§2.3 Multidirectional tangency constraints

Lecture 3: Well-placed curves and their symmetries

§3.1 Looijenga pairs

§3.2 Toric models

§3.3 Elementary transformations

§3.4 Well-placed curves

§3.5 The shift and reflection symmetries

Lecture 4: Scattering diagrams and the tropical vertex

§4.1 Scattering diagram basics

§4.2 Some toy examples

§4.3 The tropical vertex theorem

Lecture 5: Putting it all together

§5.1 The change of lattice trick

§5.2 Proof of Theorem C

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