475 SPRING 2025 PROBLEM SET #5

Problem 1. Compute the integral $\int_{-\infty}^{\infty} e^{-x^2} dx$. *Hint: relate this to the double integral* $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)} dxdy$, *and evaluate this double integral by switching to polar coordinates* r, θ.

Problem 2. Use Cauchy's theorem[1] to show that $\int_C f(z)dz = 0$ when C is the unit circle in the complex plane (with either orientation) and f is given by

(a) $f(x) = \frac{z^2}{z-3}$
(b) $f(z) = ze^{-z}$
(c) $f(z) = \frac{1}{z^2+2z+2}$
(d) $f(z) = \operatorname{sech}(z)$
(e) $f(z) = \tan(z)$
(f) $f(z) = \operatorname{Log}(z+2)$.

Problem 3. Let C_1 be the square $\{x = \pm 1, y = \pm 1\}$ with side length 2 centered at the origin, and let C_2 be the circle of radius 4 centered at the origin, both with counterclockwise orientation. Explain why we have $\inf_{C_1} f(z)dz = \int_{C_2} f(z)dz$, where

(a) $f(z) = \frac{1}{3z^2+1}$
(b) $f(z) = \frac{z+2}{\sin(z/2)}$
(c) $f(z) = \frac{z}{1-e^z}$.

Problem 4. Show that we have

$$\int_0^{\infty} e^{-x^2} \cos(2bx)dx = \frac{\sqrt{\pi}}{2} e^{-b^2}.$$

for $b > 0$. More precisely, compute this integral by applying Cauchy's theorem to the function $f(z) = e^{-z^2}$ over the rectangle with vertices $-a, a, a + bi, -a + bi$ and take the limit $a \to \infty$. *Hint: if you get stuck follow the outline in Brown and Churchill (9th edition) section 53 problem 4.*

Problem 5. Show that if C is a positively oriented simple closed contour, then the area of the region encloded by C can be written as

$$\frac{1}{2i}\int_C \overline{z} dz.$$

Hint: try applying Green's theorem as we did in the proof of Cauchy's theorem in the case that $f'(z)$ is continuous.

[1]Note that this is also sometimes called the Cauchy–Gorsat theorem, e.g. in Brown and Churchill.