Lecture 19

Ex: $(t-1)^2 y'' + 5(t-1)y' + 4y = 0$ ← (shifted) Euler equation

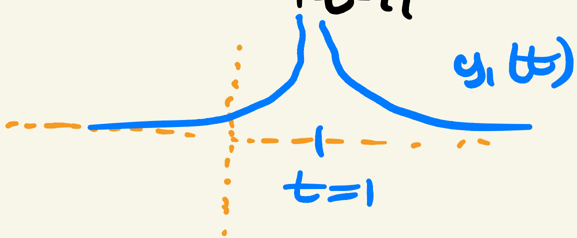
Ansatz: $y(t) = (t-1)^r$ ⟶ $r(r-1) + 5r + 4 = 0$ indicial eqn

$$r^2 + 4r + 4 = 0 \quad \to \quad (r+2)^2 = 0$$

$$r_1 = r_2 = -2$$

→ Soln's: $y_1(t) = |t-1|^{-2}$ $\quad y_2(t) = |t-1|^{-2}\ln|t-1|$

Think: $y_1(t) = \dfrac{1}{|t-1|^2} = \dfrac{1}{(t-1)^2}$ valid for $t \neq 1$.



← looks like $\dfrac{1}{|t-1|}$, except "blows up faster"

Q: Since any $t \neq 1$ is an ordinary pt, why not take ansatz e.g.

$$y(t) = \sum_{n=0}^{\infty} a_n t^n \ ?$$

Indeed, could plug into ODE and solve for $a_0, a_1, a_2, \ldots$ recursively

Or, let's write $\dfrac{1}{(t-1)^2} = \sum_{n=0}^{\infty} a_n t^n$. (Taylor series of $\frac{1}{(t-1)^2}$)

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + t^4 + \ldots$$

take derivative of both sides:

$$\frac{1}{(1-t)^2} = 1 + 2t + 3t^2 + 4t^3 + \ldots$$

So $y_1(t) = 1 + 2t + 3t^2 + 4t^3 + \ldots$ is a soln to our ODE.

ICs: $y_1(0) = 1 \qquad y_1'(0) = 2$ ($y_1(t)$ is the unique soln satisfying these ICs)

Ex: $\underbrace{(t-1)^2 y'' + 5(t-1)y' + (4-t)y' = 0.}$

not an Euler eqn, but $t=1$ is still a regular sing pt.

Recall: $y''+py'+qy=0$, $t=t_0$ is a regular point iff: $(t-t_0)p(t)$, $(t-t_0)^2q(t)$ are analytic at $t=t_0$.

Ex: $$\underbrace{2t^2y''-ty'+(1}_{\text{Euler eqn}}+t)y=0, \quad t>0$$

"perturbation of Euler eqn"

Note: Has a regular singular pt at $t=0$.

Ansatz: $$y(t)=\underbrace{t^r\sum_{n=0}^{\infty}a_nt^n}_{\text{"Frobenius series"}}$$ for some $r\in\mathbb{R}$

$$=a_0t^r+a_1t^{r+1}+a_2t^{r+2}+\dots$$

$$y(t) = \sum_{n=0}^{\infty} a_n t^{n+r} \qquad y'(t) = \sum_{n=0}^{\infty} a_n (n+r) t^{n+r-1}$$

$$y''(t) = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) t^{n+r-2}$$

Plug in: $2t^2 \left(\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) t^{n+r-2} \right) - t \sum_{n=0}^{\infty} a_n (n+r) t^{n+r-1}$

$$+ (1+t) \sum_{n=0}^{\infty} a_n t^{n+r} = 0$$

Should write this as a single Frobenius series.

Goal: Choose $a_0, a_1, a_2, a_3, \dots$ and r s.t. this is true identity

Have

$$\underbrace{\sum_{n=0}^{\infty} 2a_n (n+r)(n+r-1) t^{n+r} - \sum_{n=0}^{\infty} a_n (n+r) t^{n+r} + \sum_{n=0}^{\infty} a_n t^{n+r}}_{\text{comes from Euler eqn part}} + \sum_{n=0}^{\infty} a_n t^{n+r+1} = 0$$

write last term as: Put $j = n+1$. Then, becomes

$$\sum_{j=1}^{\infty} a_{j-1} t^{j+r} = \sum_{n=1}^{\infty} a_{n-1} t^{n+r}.$$

Have:

$$\sum_{n=1}^{\infty}\left(\begin{array}{l}2a_n(n+r)(n+r-1) \\ -a_n(n+r)+a_n \\ \quad +a_{n-1}\end{array}\right)t^{n+r}+\left[\begin{array}{l}2a_0(r)(r-1)-a_0r \\ +a_0\end{array}\right]t^r = 0$$

For this to hold for all t,

(1) t^r term is zero $\rightsquigarrow$ $2a_0 r(r-1)-a_0r+a_0=0$

(2) t^{n+r} term is zero $\rightsquigarrow$ $2a_n(n+r)(n+r-1)-a_n(n+r)+a_n+a_{n-1}=0$ for all $n\ge 1$.

Note: Euler eqn part is $2t^2y''-ty'+y=0$

$$\Leftrightarrow t^2y''-\frac{t}{2}y'+\frac{1}{2}y=0$$

$$\alpha=-1/2 \qquad \beta=1/2$$

$$r(r-1)+\alpha r+\beta=0 \Leftrightarrow r(r-1)-r/2+1/2=0$$

$$\Leftrightarrow 2r(r-1)-r+1=0$$

So eqn (1) is the indicial eqn of Euler part.

$$(2) \Longrightarrow \underbrace{2a_n(n+r)(n+r-1) - a_n(n+r) + a_n = -a_{n-1}}_{\text{determines } a_n \text{ in terms of } a_{n-1}}$$

Let's solve.

$$(1) \iff 2r(r-1) - r + 1 = 0 \iff 2r^2 - 3r + 1 = 0$$

$$r = \frac{3 \pm \sqrt{9-8}}{4}$$

$$= \frac{3 \pm 1}{4}$$

$r_1 = 1, \quad r_2 = 1/2$

from now, assume $r_1 \geq r_2$.

Case (a): $r = 1$.

$$(2) \to 2a_n(n+1)n - a_n(n+1) + a_n = -a_{n-1}$$

$$a_n\left(\underbrace{2n^2+2n-n}-1+1\right) = -a_{n-1}$$

$$2n^2+n = n(2n+1)$$

$$\Longrightarrow a_n = \frac{-a_{n-1}}{n(2n+1)}, \quad \rightsquigarrow \quad a_0$$

$$a_1 = \frac{-a_0}{1\cdot 3}$$

$$a_2 = \frac{-a_1}{2\cdot 5} = \frac{a_0}{1\cdot3\cdot2\cdot5}$$

$$a_3 = \frac{-a_2}{3\cdot 7} = \frac{-a_0}{1\cdot3\cdot2\cdot5\cdot3\cdot7}$$

$$a_4 = \frac{-a_3}{4\cdot 9} = \frac{a_0}{1\cdot3\cdot2\cdot5\cdot3\cdot7\cdot4\cdot9}$$

etc.

So have a solution:

$$y_1(t) = t^1\left(1 - \frac{1}{1\cdot3}t + \frac{1}{1\cdot3\cdot2\cdot5}t^2 - \frac{1}{1\cdot3\cdot2\cdot5\cdot3\cdot7}t^3 + \dots\right)$$

Case (b): $r = 1/2$

$$(2) \longrightarrow 2a_n(n+1/2)(n-1/2) - a_n(n+1/2) + a_n = -a_{n-1}$$

$$\Leftrightarrow 2a_n\left(n^2 - 1/4\right) - n a_n + 1/2\, a_n = -a_{n-1}$$

$$\Leftrightarrow a_n \underbrace{\left(2n^2 \cancel{-1/2} - n \cancel{+1/2}\right)}_{2n^2 - n \,=\, n(2n-1)} = -a_{n-1}$$

$$a_n = \frac{-a_{n-1}}{n(2n-1)}$$

a_0

$$a_1 = \frac{-a_0}{1\cdot 1}$$

$$a_2 = \frac{-a_1}{2\cdot 3} = \frac{a_0}{1\cdot 1\cdot 2\cdot 3}$$

$$a_3 = \frac{-a_2}{2\cdot 5} = \frac{-a_0}{1\cdot 1\cdot 2\cdot 3\cdot 2\cdot 5}$$

etc

So second soln:

$$y_2(t) = t^{1/2}\left(1 - \frac{1}{1\cdot 1}t + \frac{1}{1\cdot 1\cdot 2\cdot 3}t^2 - \frac{1}{1\cdot 1\cdot 2\cdot 3\cdot 2\cdot 5}t^3 + \dots\right)$$

should check that r.o.c. > 0

Can check: y_1, y_2 form fund. set of solns

General soln: $c_1 y_1 + c_2 y_2$

Given: $y'' + p(t)y' + q(t)y = 0$ and suppose $t = t_0$ is a regular singular pt.

Then can seek a soln ~~of form~~

$$y(t) = (t-t_0)^r \sum_{n=0}^{\infty} a_n (t-t_0)^n.$$

Plug in, will end up w/ an indicial eqn

$$(r)(r-1) + \alpha r + \beta = 0$$

Potential issues:

- repeated roots
- roots could be cpx numbers
- $r_1 - r_2$ is an integer