Lecture 17

Recall: $t^2y'' - 6y = 0$.

Ansatz: $y(t) = \sum_{n=0}^{\infty} a_n t^n$.

$y' = \sum_{n=0}^{\infty} a_n n t^{n-1}$, $y'' = \sum_{n=0}^{\infty} a_n n(n-1) t^{n-2}$

$\rightsquigarrow \sum_{n=0}^{\infty} a_n n(n-1) t^n - 6\sum_{n=0}^{\infty} a_n t^n = 0$

$\rightsquigarrow \sum_{n=0}^{\infty} (a_n n^2 - n a_n - 6a_n) t^n = 0$

$\Rightarrow a_n(n^2 - n - 6) = 0$ for all $n \geq 0$

$n=0$: $a_0 (-6) = 0 \Rightarrow a_0 = 0$

$n=1$: $a_1 (1-1-6) = 0 \Rightarrow a_1 = 0$

$n=2$: $a_2 (2^2-2-6) = 0 \Rightarrow a_2 = 0$

etc.

$\rightsquigarrow y(t) \equiv 0$.

Note: $t = 0$ is a singular point of this ODE.

Write as $t^2y'' - 3\cdot 2 \cdot y = 0$

Try $y(t) = t^3$.

$y' = 3t^2$, $y'' = 6t$

$\to t^2(6t) - 6t^3 = 0$ ✓

Try $y(t) = 1/t^2$

$y' = -2t^{-3}$, $y'' = 6t^{-4}$

$t^2(6t^{-4}) - 6t^{-2} = 0$ ✓

So $y_1(t) = t^{-2}$ is a solution!

$y_2(t) = t^3$ is another solution.

Note: both of these sol'ns are of form t^r for some r.

General soln: $C_1 t^{-2} + C_2 t^3$.

EX: $t^2y'' + 6ty' + 4y = 0$

Ansatz: $y(t) = t^r$.

$y' = rt^{r-1}$, $y'' = r(r-1)t^{r-2}$

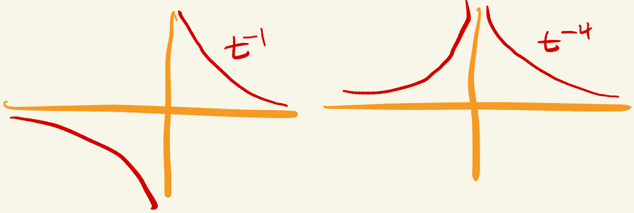
$\rightsquigarrow r(r-1)t^r + 6rt^r + 4t^r = 0$.

$(r^2 - r + 6r + 4)(t^r) = 0$

need: $r^2 + 5r + 4 = 0$

$(r+1)(r+4)=0 \rightsquigarrow r_1=-1, r_2=-4$

$\Rightarrow r=-1,-4$ Get two solns: $y_1(t)=t^{-1}$, $y_2(t)=t^{-4}$



Ex: $t^2y''+7ty'+4y=0$

Ansatz: $y(t)=t^r$

$\rightsquigarrow (r(r-1)+7r+4)t^r=0.$

$\Rightarrow r^2+6r+4=0.$

$r=\frac{-6\pm\sqrt{36-16}}{2}=-3\pm\frac{\sqrt{20}}{2}$

$=-3\pm\sqrt{5}$

$r_1=-3-\sqrt{5} \quad r_2=-3+\sqrt{5}$

$\to y_1(t)=t^{-3-\sqrt{5}}, \quad y_2(t)=t^{-3+\sqrt{5}}$

Question: What does $17^{-3-\sqrt{5}}$ mean??

Before: $e^{1.23}$??

Said $e^t=\underbrace{1+t+t^2/2!+t^3/3!+\dots}_{\text{makes sense for any value of } t}$

$t^r=(e^{\ln(t)})^r$

$=e^{r\ln(t)}$ problem

$\ln(\text{neg values})=$ not defined

Therefore, t^r only is defined for $t>0$.

Definition: $t^2y''+\alpha ty'+\beta y=0$ for $\alpha,\beta\in\mathbb{R}$ constants is called an **Euler equation**.

Note: $t=0$ is a singular point.

Ansatz: $y(t)=t^r$

plug in $\rightsquigarrow (r(r-1)+r\alpha+\beta)t^r=0$

$\Rightarrow \underbrace{r(r-1)+r\alpha+\beta=0}_{\text{"indicial equation"}}$

Naively, two roots r_1, r_2,

general solution: $c_1t^{r_1}+c_2t^{r_2}$.

Issues:

- $r_1=r_2$ then only get one solution.
- if r_1 or r_2 is complex valued, then t^{r_1} or t^{r_2} would not be a real-valued function.
- if say $r<0$, t^r blows up at $t=0$
- t^r is not defined for $t<0$

Ex: $t^2y''+5ty'+4y=0.$ Assume $t>0$

indicial eqn: $r(r-1)+5r+4=0$

$\rightsquigarrow r^2+4r+4=0$

$(r+2)^2=0$

$\rightsquigarrow r_1=r_2=-2.$

So $y_1(t) = t^{-2}$ is one solution. **Reduction of order:**

Try: $y(t) = t \cdot t^{-2} = t^{-1}$.

$y' = -t^{-2}$, $y'' = 2t^{-3}$

$$\underbrace{t^2(2t^{-3}) + 5t(-t^{-2}) + 4(t^{-1})}_{(1)t^{-1} \neq 0} \quad \times$$

Ansatz: $y(t) = u(t)\, y_1(t)$

$y' = u'y_1 + uy_1'$, $y'' = u''y_1 + 2u'y_1' + uy_1''$

$$\rightsquigarrow t^2(u''y_1 + 2u'y_1' + \cancel{uy_1''}) + 5t(u'y_1 + \cancel{uy_1'}) + 4\cancel{uy_1} = 0$$

(cancel since y_1 is a solution)

$$\rightsquigarrow t^2u''y_1 + t^2 2u'y_1' + 5tu'y_1 = 0$$

$y_1(t) = t^{-2}$, $y_1'(t) = -2t^{-3}$

Need: $t^2u''t^{-2} + 2t^2u'(-2t^{-3}) + 5tu't^{-2} = 0$

$$\rightsquigarrow u'' + (u')t^{-1} = 0$$

Put $r(t) = u'(t)$.

Have $r' + t^{-1}r = 0$

Separable: $\frac{r'}{r} = -t^{-1}$

$\ln(r) = -\ln(t)$

$\Rightarrow r = t^{-1}$

$$\rightsquigarrow u(t) = \ln(t)$$

$\Rightarrow y_2(t) := y_1(t)\ln(t) = t^{-2}\ln(t)$ is also a solution!

Fact: If $r_1 = r_2$ is a repeated root of the indicial eqn of an Euler eqn, then the general soln for $t > 0$ is $C_1 t^{r} + C_2 t^{r}\ln(t)$

Proof: Use reduction of order.

Alternatively: put $L[y] = t^2y'' + \alpha t y' + \beta y$. If r_1 is a repeated root, then $L[t^r] = t^r(r-r_1)^2$

Consider $\frac{d}{dr}L[t^r] = rt^{r-1}(r-r_1)^2 + t^r 2(r-r_1)$

$$\Rightarrow \frac{d}{dr}\Big|_{r=r_1} L[t^r] = 0.$$

But, says $0 = L\left[\frac{d}{dr}\Big|_{r=r_1} t^r\right] = 0$.

Question: What $\frac{d}{dr} t^r$?

$$\frac{d}{dr}\left(e^{r\ln(t)}\right) = \ln(t)\, e^{r\ln(t)} = \ln(t) \cdot t^r$$

Upshot: $L[\ln(t)t^{r_1}] = 0$, i.e. $\ln(t)\, t^{r_1}$ is also a soln to ODE.

Ex: $t^2y'' + 2ty' + y = 0$.

$\rightsquigarrow r(r-1) + 2r + 1 = 0$

$\to r^2 + r + 1 = 0$.

$$r = \frac{-1 \pm \sqrt{1^2 - 4}}{2} = -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}$$

$r_1 = -1/2 + i\sqrt{3}/2$, $r_2 = -1/2 - i\sqrt{3}/2$.

So $t^{-1/2 + i\sqrt{3}/2}$ and $t^{-1/2 - i\sqrt{3}/2}$ are solns

Put $\alpha = -1/2$, $\beta = \sqrt{3}/2$.

How to extract two real solutions from $t^{\alpha + i\beta}$ and $t^{\alpha - i\beta}$?

$$t^{\alpha + i\beta} = t^\alpha t^{i\beta} = t^\alpha e^{i\ln(t)\beta}$$

$$= t^\alpha \left[\cos(\ln(t)\beta) + i\sin(\ln(t)\beta)\right]$$

So real and imag parts must individually solve the same ODE.

$\rightsquigarrow y_1(t) = t^\alpha \cos(\ln(t)\beta)$

$y_2(t) = t^\alpha \sin(\ln(t)\beta)$.

Next time:

- what about $t < 0$?
- what do these solns look like?
- what about e.g. $(t-2)^2 y'' + (t-3)y' + 7y = 0$