Midterm 1

Modern Algebra 1

Columbia University Fall 2019

Instructor: Kyler Siegel

Instructions:

- Please write your answers **in this printed exam**. You may use the back of pages for additional work. You may use also use printer paper if you need additional space, but you must hand in all relevant work. Please turn in all scratch work which is relevant to your submitted answers.
- Suspected cases of copying or otherwise cheating will be taken very seriously.
- Solve as many problems of the following problems as you can in the allotted time, which is *one hour and fiften minutes*. I recommend first solving the problems are you most comfortable with before moving on to the more challenging ones. Note that the problems are not ordered by level of difficulty or topic.
- The exams will be graded on a curve. Therefore the raw score is not important, and you do not necessarily need to solve every problem to achieve a good grade. Just do your best!
- For true / false questions, you will receive $+2$ points for a correct answer, 0 points for an incorrect answer, and -3 points for an incorrect answer. For multiple choice questions, you will receive $+4$ points for a correct answer, 0 points for an incorrect answer, and -2 points for an incorrect answer. This means **you should not make random guesses** unless you are reasonably sure that you know the answer.
- You may use any commonly used notation for standard groups, subgroups, and their elements, as long as it is completely unambiguous. If you are using nonstandard notation you must fully explain it for credit.
- Turn off all electronic devices. You may use the restroom if you must, but you may not take any devices with you.
- Good luck!!

Name: ____________________

Uni: ____________________

Question:	1	2	3	4	Total
Points:	40	16	24	20	100
Bonus Points:	0	0	0	0	0
Score:					

1. *True or false questions. Circle one. You do not need to provide any justification.*

(I) (2 points) Every group of order 19 is abelian.
A. True B. False

(II) (2 points) The group $\mathbb{Z}/(2\mathbb{Z}) \times \mathbb{Z}/(3\mathbb{Z})$ is isomorphic to $\mathbb{Z}/(6\mathbb{Z})$
A. True B. False

(III) (2 points) The group $\mathbb{Z}/(3\mathbb{Z}) \times \mathbb{Z}/(3\mathbb{Z})$ is isomorphic to $\mathbb{Z}/(9\mathbb{Z})$
A. True B. False

(IV) (2 points) If G is a group and H and K are subgroups, then $HK = \{hk \,:\, h \in H,\ k \in K\}$ is a subgroup of G.
A. True B. False

(V) (2 points) The symmetric group S_4 admits an injective homomorphism to S_5
A. True B. False

(VI) (2 points) The symmetric group S_4 admits a surjective homomorphism to $\mathbb{Z}/(5\mathbb{Z})$
A. True B. False

(VII) (2 points) Every permutation $\sigma \in S_{100}$ can be written as a product of 3-cycles.
A. True B. False

(VIII) (2 points) $(\mathbb{Z}/13\mathbb{Z})^\times$ is a cyclic group.

A. True B. False

(IX) (2 points) A group all of whose proper subgroups is cyclic is cyclic.
A. True B. False

(X) (2 points) If G and H are groups and $\Phi : G \to H$ is a homomorphism, then we have $N_G(\ker(\Phi)) = G$.[1]
A. True B. False

(XI) (2 points) If G and H are groups and $\Phi : G \to H$ is a homomorphism, then we have $N_G(\operatorname{im}\Phi)) = G$.
A. True B. False

(XII) (2 points) If H is a subgroup of a finite group G with $|G| = 2|H|$, then H is a normal subgroup.
A. True B. False

(XIII) (2 points) There exists a homomorphism Φ from $D_{2\cdot 5}$[2] to $\mathbb{Z}/(2\mathbb{Z})$ such that $D_{2\cdot 5}/\ker(\Phi)$ is isomorphic to $\mathbb{Z}/(2\mathbb{Z})$.
A. True B. False

(XIV) (2 points) There exists a homomorphism Φ from $D_{2\cdot 5}$ to $\mathbb{Z}/(3\mathbb{Z})$ such that $D_{2\cdot 5}/\ker(\Phi)$ is isomorphic to $\mathbb{Z}/(3\mathbb{Z})$.
A. True B. False

[1] Recall that $N_G(A) := \{g \in G \ : \ gAg^{-1} = A\}$ denotes the normalizer of a subset $A \subset G$. Also, recall that $\ker(\Phi)$ and $\operatorname{im}(\Phi)$ denote the kernel and image respectively of the homomorphism Φ.

[2] Recall that $D_{2\cdot n}$ denotes the dihedral group corresponding to symmetries of the regular n-gon.

(XV) (2 points) If G is a group of order four, then it is isomorphic to either $(\mathbb{Z}/(5\mathbb{Z}))^{\times}$ or $(\mathbb{Z}/(8\mathbb{Z}))^{\times}$.
A. True B. False

(XVI) (2 points) If G is a group of order eight, then it is isomorphic to C_8,[3] $C_2 \times C_4$, $C_2 \times C_2 \times C_2$, or $D_{2\cdot 4}$.
A. True B. False

(XVII) (2 points) If H and K are subgroups of a group G, and we have $H \subset N_G(K)$, then $HK = KH$.
A. True B. False

(XVIII) (2 points) If H and K are normal subgroups of a group G such that $H \subset K$, then H is a normal subgroup of K.
A. True B. False

(XIX) (2 points) The subgroup of S_5 generated by $(1\ 3\ 2)(4)(5)$ is normal (note: here we are using cycle decomposition notation).
A. True B. False

(XX) (2 points) The dihedral group $D_{2\cdot 3}$ is isomorphic to the symmetric group S_3.
A. True B. False

2. *Multiple choice questions. You do not need to provide any justification. In each case,* ***select all that apply****.*

(I) (4 points) Which of the following groups is isomorphic to $(\mathbb{Z}/(10\mathbb{Z}))^{\times}$?
A. $\mathbb{Z}/(10\mathbb{Z})$ B. $\mathbb{Z}/(4\mathbb{Z})$ C. $C_2 \times C_2$ D. $(\mathbb{Z}/(8\mathbb{Z}))^{\times}$

(II) (4 points) What is the remainder when 13^{101} is divided by 17?

[3] Recall that C_n denotes the multiplicative group of nth roots of unity.

A. 1   B. 3   C. 11   D. 13

(III) (4 points) Which of the following groups is isomorphic to a subgroup of $D_{2\cdot 8}$?
A. C_2 B. $C_2 \times C_2$ C. C_8 D. $C_2 \times C_8$

(IV) (4 points) Which of the following groups is isomorphic to the center of $D_{2\cdot 5}$?
A. C_1 B. C_2 C. $C_2 \times C_2$ D. C_4

3. *Short answer questions. You do not need to provide any justification. Some partial credit may be given if your answer is nearly correct.*

(I) (6 points) How homomorphisms are there from C_8 to C_{14}?

(II) (6 points) Let $\sigma \in S_6$ be the permutation given by $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 1 & 6 & 3 & 2 \end{pmatrix}$. Write σ as a product of transpositions, and also determine the sign of σ.

(III) (6 points) Explain at least one way in which the following statement of the Second Isomorphism Theorem is mathematically incorrect:

> Let G be a group, and let A and B be subgroups. Then AB is a subgroup of G, $B \trianglelefteq AB$, $A \cap B \trianglelefteq A$ and $AB/B \cong A/A \cap B$.

(IV) (6 points) How many cyclic subgroups does $C_2 \times S_3$ have? How many of them are normal?

4. *Short proofs. Make your arguments as rigorous as possible. You may cite results covered in class provided you are completely clear about what you are citing.*

(I) (10 points) Let G and H be finite groups such that $\gcd(|G|, |H|) = 1$. Let $\phi : G \to H$ be a homomorphism. Prove that ϕ sends every element of G to the identity element of H.

(II) (10 points) Let G be a finite group of order n, and let k be an integer which is relatively prime to n. Prove that for any element $g \in G$, there exists another element $h \in G$ such that $g = h^k$.