Math 2030
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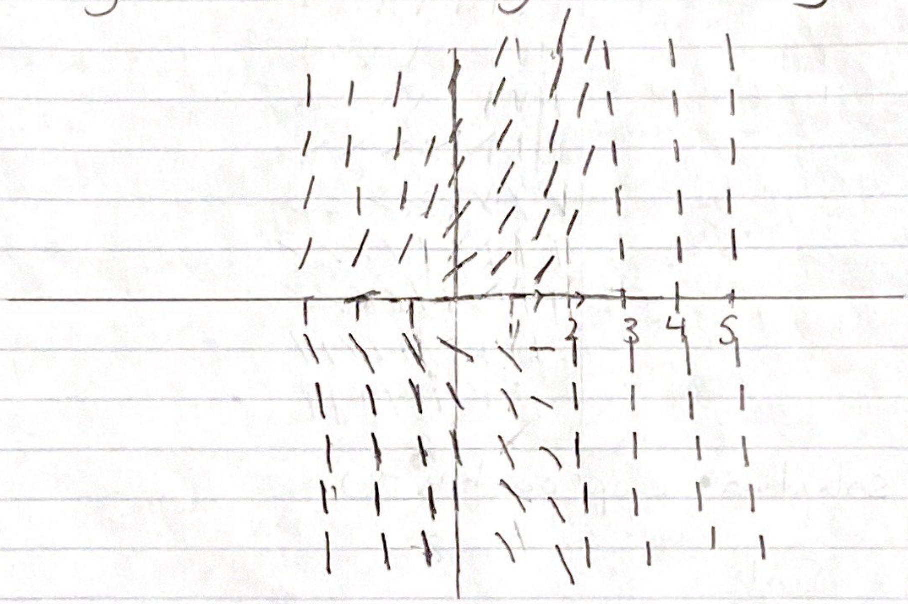
ODE Problem Set 2 Solutions

Section 2.1

2a) $y' - 2y = t^2 e^{2t} \iff y' = t^2 e^{2t} + 2y$



b) The solutions for large values of t approaches infinity.

c) $\frac{dy}{dt} - 2y = t^2 e^{2t}$

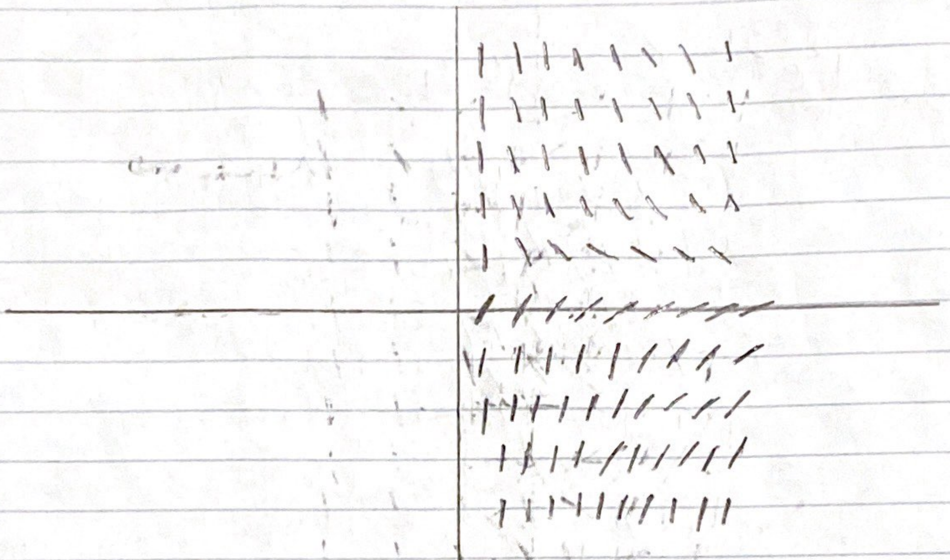
$= e^{-2t}\frac{dy}{dt} - 2ye^{-2t} = t^2$

$= \frac{d}{dt}(e^{-2t}y) = t^2 \implies e^{-2t}y = \frac{1}{3}t^3 + C$

$\implies y = (\frac{1}{3}t^3 + c)e^{2t}$

$\lim_{t \to \infty} (\frac{1}{3}t^3 + c)e^{2t} = \infty$

2.1.6 $ty' + 2y = \sin t \Rightarrow y' = \frac{\sin t - 2y}{t}$ $t \to 0$



b) as $t \to \infty$ solution approaches 0

c) $\frac{dy}{dt} + \frac{2y}{t} = \frac{\sin t}{t}$

$= e^{\int \frac{2}{t}dt} \frac{dy}{dt} + e^{\int \frac{2}{t}dt} \frac{2}{t} y = \frac{\sin t}{t} e^{\int \frac{2}{t}dt}$

$= e^{\ln t^2} \frac{dy}{dt} + e^{\ln t^2}\left(\frac{2}{t}\right) y = \frac{\sin t}{t} e^{\ln(t^2)}$

$= t^2 \frac{dy}{dt} + 2ty = t \sin t$

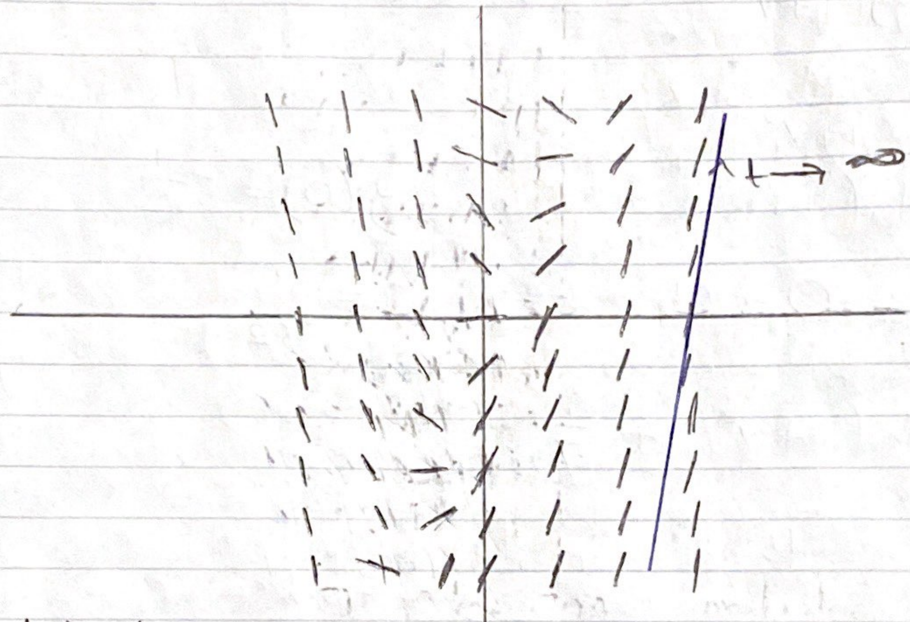
$= \frac{d}{dt}(t^2 y) = t \sin t$

$\Rightarrow t^2 y = -t\cos t + \int \cos t \, dt$

$\Rightarrow t^2 y = -t \cos t + \sin t + C$

$y = -\frac{\cos t}{t} + \frac{\sin t}{t^2} + Ct^{-2}$ as $t \to \infty$ $y \to 0$

2.1.9 $2y' + y = 3t \Leftrightarrow y' = \frac{3t - y}{2}$



b) Solution tangential to $3t-6$

c) $y' + \frac{1}{2}y = \frac{3}{2}t$

$\Rightarrow e^{\frac{1}{2}t} y' + e^{\frac{1}{2}t}\frac{y}{2} = \frac{3}{2} t e^{\frac{1}{2}t}$

$\Rightarrow \frac{d}{dt}\left(e^{\frac{1}{2}t} y\right) = \frac{3}{2} t e^{\frac{1}{2}t}$

$\Rightarrow e^{\frac{1}{2}t} y = 3e^{\frac{1}{2}t} t - 6e^{\frac{1}{2}t} + C$

$\Rightarrow y = 3t - 6 + Ce^{-\frac{1}{2}t}$

by $t \to \infty$ the slope becomes tangential to $3t-6$

2.1.13) $y' - y = 2te^{2t}$ $\quad y(0) = 1$

Integrating factor is e^{-t}.

so $e^{-t}y' - e^{-t}y = 2te^{t}$

$$\frac{d}{dt}\left(e^{-t}y\right) = 2te^{t}$$

Integrating both sides we have:

$$y = 2e^{2t}(t-1) + Ce^{t}$$

Since $y(0) = 1$ we have that $C = 3$.

so $\boxed{y = 2e^{2t}(t-1) + 3e^{t}}$

2.1.30) $y' - y = 1 + \sin t$ $\quad y(0) = y_0$

$e^{-t}y' - e^{-t}y = (1+\sin t)e^{-t}$

$$\frac{d}{dt}\left(e^{-t}y\right) = (1+\sin t)e^{-t}$$

$e^{-t}y = \int (1+\sin t)e^{-t}\,dt$ (Use integration by parts twice)

$$y = Ce^{t} - 1 - \frac{3}{2}(\sin t + \cos t)$$

In order for $\lim_{t\to\infty} y$ to remain bounded we need that $C = 0$

So $y(0) = \boxed{-\frac{5}{2}}$

2.1 35) An example of such an equation would be:

$$y' + y = 2 - t$$

Solving this equation we get:

$$\frac{d}{dt}(e^t y) = e^t - te^t$$

so $y = 3 - t + ce^{-t}$, so as $t \to \infty$

all solutions are asymptomatically $y = 3 - t$

since $ce^{-t} \to 0$ as $t \to \infty$

Section 2.3

8a) We have that $\frac{dS}{dt} = rS + k$

rate of interest.

$$\frac{dS}{dt} - rS = k$$

$$e^{-rt}\frac{dS}{dt} - re^{-rt}S = ke^{-rt}$$

$$\Rightarrow \frac{d}{dt}\left(Se^{-rt}\right) = ke^{-rt}$$

$$\Rightarrow Se^{-rt} = -\frac{k}{r}e^{-rt} + C$$

so $S(t) = Ce^{rt} - \frac{k}{r}$ no initial capital

so $S(0) = 0 \Rightarrow C = \frac{k}{r}$

8b) $S(t) = \frac{k}{r}e^{rt} - \frac{k}{r}$ $r = \frac{7.5}{100}$

$$S(40) = \frac{k\times100}{7.5}e^{\frac{7.5\times40}{100}} - \frac{k\times100}{7.5} = 10^6$$

$$S(40) = 267.81k - 13.\dot{3}k = 10^6 \Rightarrow k \approx 3930$$

8c) $S(t) = \frac{2000}{r}e^{40r} - \frac{2000}{r} = 10^6$

so $\frac{2000}{r}(e^{40r} - 1) = 10^6$

$\Rightarrow e^{40r} - 500r - 1 = 0$ (this can be solved by iteration or by graphical methods)

$r \approx \frac{9.77}{100} = 9.77\%$

Section 2.6

Q1) We have $M(x,y) = 2x+3$ and $N(x,y) = 2y-2$

the equation is exact if $M_y(x,y) = N_x(x,y)$

we have that $M_y(x,y) = N_x(x,y) = 0$

Solving this equation we have:

$$\int 2x+3\,dx = x^2+3x+h(y)$$

$$\int (2y-2)\,dy = y^2-2y+g(x)$$

so let $h(y) = y^2-2y$ and $g(x) = x^2+3x$

So the solution is $x^2+3x+y^2-2y = C$

Q4) $M(x,y) = 2xy^2+2y$ and $N(x,y) = 2x^2y+2x$

$M_y(x,y) = 4xy+2$ $\quad N_x(x,y) = 4xy+2$

so this equation is exact

$$\int 2xy^2+2y\,dx = x^2y^2+2yx+h(y)$$

$$\int 2x^2y+2x\,dy = x^2y^2+2yx+g(x)$$

so $h(y) = g(x) = 0$

So Solution is $x^2y^2+2yx = C$

Q9) $M(x,y) = ye^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x$

$N(x,y) = xe^{xy}\cos 2x - 3$

$M_y(x,y) = \cos 2x\,(e^{xy} + yxe^{xy}) - 2xe^{xy}\sin 2x = N_x(x,y)$

So this equation is exact

We need to compute: $\int ye^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x\,dx$

Note: $\int ye^{xy}\cos 2x\,dx = \dfrac{y^2\cos 2x\,e^{xy} + 2y\sin 2x\,e^{xy}}{y^2+4}$

Similarly $\int -2e^{xy}\sin 2x\,dx = \dfrac{4\cos 2x\,e^{xy} - 2y\sin 2x\,e^{xy}}{y^2+4}$

so $\Psi(x,y) = \cos 2x\,e^{xy} + x^2 + h(y)$

We need to compute: $\int xe^{xy}\cos 2x - 3\,dy$

$= \cos 2x\,e^{xy} - 3y + f(x)$

so $\Psi(x,y) = \cos 2x\,e^{xy} + x^2 - 3y = C$

Q15) The equation is exact if :

$$\frac{\partial}{\partial y}\left(xy^2+bx^2y\right) = \frac{\partial}{\partial x}\left((x+y)x^2\right)$$

$$\frac{\partial}{\partial y}\left(xy^2+bx^2y\right) = 2yx + bx^2$$

$$\frac{\partial}{\partial x}\left((x+y)x^2\right) = 3x^2+2xy$$

and so $b=3$

We need : $\int xy^2+3x^2y \, dx = \frac{y^2x^2}{2} + x^3y + h(y)$

$$\int (x+y)x^2 \, dy = x^2\left(xy+\frac{1}{2}y^2\right) + f(x)$$

So $\Psi(x,y) = \frac{y^2x^2}{2} + x^3y = C$

Q25) There exists an integrating factor that is only a function of x if. $\frac{M_y - N_x}{N}$ is a function of x only

we have that $M(x,y) = 3xy^2 + 2xy + y^3$
$N(x,y) = x^2 + y^2$

So $\frac{M_y - N_x}{N} = \frac{3y^2 + 2x + 3x^2 - 2x}{x^2 + y^2} = \frac{3(x^2+y^2)}{x^2+y^2}$
$= 3$

So indeed there exists an integrating function that is a function of x

Let $u(x)$ be the integrating factor.

We have that $\frac{du}{dx} = \frac{M_y - N_x}{N} u = 3u$

so $\ln|u| = 3x \Rightarrow u = e^{3x}$

Multiplying by e^{3x} we have:

$$(3x^2y + 2xy + y^3)e^{3x} + e^{3x}(x^2+y^2)y' = 0$$

$$\int (x^2+y^2)e^{3x}\, dy = \frac{e^{3x}y^3}{3} + f(x) + x^2ye^{3x}$$

$$\int (3x^2y + 2xy + y^3)e^{3x}\, dx = e^{3x}\left(x^2y + \frac{y^3}{3}\right) + h(y)$$

So $\Psi(x,y) = \frac{e^{3x}y^3}{3} + x^2ye^{3x} = C$