A TREE FORMULA FOR THE ELLIPSOIDAL SUPERPOTENTIAL OF THE COMPLEX PROJECTIVE PLANE

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ABSTRACT. The ellipsoidal superpotential of the complex projective plane can be interpreted as a count of rigid rational plane curves of a given degree with one prescribed cusp singularity. In this note we present a closed formula for these counts as a sum over trees with certain explicit weights. This is a step towards understanding the combinatorial underpinnings of the ellipsoidal superpotential and its mysterious nonvanishing and nondecreasing properties.

CONTENTS

1. Introduction

Given a closed symplectic manifold M^{2n}, a homology class $A \in H_2(M)$, and a tuple of positive real numbers $\vec{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n_{>0}$, the **ellipsoidal superpotential** $\mathbf{T}^{\vec{a}}_{M,A} \in \mathbb{Q}$ is an enumerative invariant which encodes important information about (a) stabilized symplectic embeddings of ellipsoids into M and (b) singular rational curves in M. In this note we focus on the case of the complex projective plane $M = \mathbb{CP}^2$, and we put $\mathbf{T}^a_d := \mathbf{T}^{(1,a)}_{\mathbb{CP}^2, d[L]}$ for $a \in \mathbb{R}_{>0}$[1] and $d \in \mathbb{Z}_{\geqslant 1}$, where $[L] \in H_2(\mathbb{CP}^2)$ is the line class.

For instance, when $\mathbf{T}^a_d \neq 0$ we get an obstruction to symplectic embeddings of the form $E(\mu, \mu a) \times \mathbb{C}^N \overset{s}{\hookrightarrow} \mathbb{CP}^2 \times \mathbb{C}^N$ for $\mu \in \mathbb{R}_{>0}$ and $N \in \mathbb{Z}_{\geqslant 0}$, and it is expected that together these give a complete set of stable obstructions for symplectic embeddings of ellipsoids into $\mathbb{CP}^2$. Moreover, when $a = p/q$ is a reduced fraction such that $p + q = 3d$, we have $\mathbf{T}^{p/q}_d \neq 0$ if and only if there exists a genus zero degree d singular symplectic curve in $\mathbb{CP}^2$ which has one (p, q) cusp and is otherwise positively immersed (see [MS2, Thm. E]). We call such singular curves **sesquicuspidal**, and their existence in both the algebraic and symplectic categories are subtle problems which are closely linked.

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[1] In the definition it is convenient to assume that a is irrational, and we extend this to rational $a \in \mathbb{R}_{>0}$ by the convention $\mathbf{T}^a_d := \mathbf{T}^{a+\delta}_d$ for $\delta > 0$ sufficiently small.

Following [MS3, §2], to define $\mathbf{T}^{\vec{a}}_{M,A}$, we first consider the compact symplectic manifold with boundary $M_{\vec{a}} := M \setminus \iota(\mathring{E}(\varepsilon\vec{a}))$, where $\iota : E(\varepsilon\vec{a}) \overset{s}{\hookrightarrow} M$ is a symplectic embedding for some small $\varepsilon > 0$. Here $E(\vec{a}) = \{\pi \sum_{i=1}^n \frac{1}{a_i}|z_i|^2 \leqslant 1\} \subset \mathbb{C}^n$ denotes the closed symplectic ellipsoid with area factors $(a_1,\ldots,a_n)$ and $\mathring{E}(\vec{a})$ denotes its interior. Let $\widehat{M}_{\vec{a}} := M_{\vec{a}} \cup (\mathbb{R}_{\leqslant 0} \times \partial M_{\vec{a}})$ denote the symplectic completion of $M_{\vec{a}}$, and let J be admissible almost complex structure on $\widehat{M}_{\vec{a}}$ in the sense of symplectic field theory (SFT). Then $\mathbf{T}^{\vec{a}}_{M,A}$ is the SFT count of index zero finite energy J-holomorphic planes $u : \mathbb{C} \to \widehat{M}_{\vec{a}}$. In general this is a virtual count taking rational values. However, it is shown in [MS2, §2] that in many relevant cases it can be defined using classical pseudoholomorphic curve techniques and takes integer values. For instance, this is the case for $\mathbf{T}^{p/q}_d$ whenever p/q is a reduced fraction satisfying $p + q = 3d$.

As illustrated above, a central question is to understand when $\mathbf{T}^{\vec{a}}_{M,A}$ is nonzero. For example, we have:

Conjecture 1.1. *For any reduced fraction $p/q > 1$ satisfying $p+q = 3d$ and $(p-1)(q-1) \leqslant (d-1)(d-2)$, we have $\mathbf{T}^{p/q}_d \neq 0$.*

This is equivalent to the statement that there exists an index zero (p,q)-sesquicuspidal symplectic curve in $\mathbb{CP}^2$ of genus zero and degree d if and only if it is allowed by the adjunction formula, which states here that the count of singularities excluding the (p,q) cusp is $\frac{1}{2}(d-1)(d-2) - \frac{1}{2}(p-1)(q-1)$. It is known that an affirmative answer would in particular imply optimality of Hind's folding embedding $E(\mu,\mu a) \times \mathbb{C}^N \overset{s}{\hookrightarrow} \mathbb{CP}^2 \times \mathbb{C}^N$ with $\mu = \frac{a+1}{3a}$ for all $N \in \mathbb{Z}_{\geqslant 1}$ and $a > \tau^4$, where $\tau = \frac{1+\sqrt{5}}{2}$ is the golden ratio (see [Hin]). Another closely related question concerns the behavior of $\mathbf{T}^a_d$ as a function of a:

Conjecture 1.2. *The count $\mathbf{T}^a_d \in \mathbb{R}$ is nondecreasing as a function of $a \in \mathbb{R}_{>1}$.*

The recent article [MS3] gives a recursive formula for $\mathbf{T}^{\vec{a}}_{M,A}$, which takes the following form in the case $M = \mathbb{CP}^2$. Firstly, we associate to each $a \in \mathbb{R}_{>0}$ a unit step lattice path $\Gamma^a_0, \Gamma^a_1, \Gamma^a_2, \cdots \in \mathbb{Z}^2_{\geqslant 0}$. Explicitly, for $k \in \mathbb{Z}_{\geqslant 0}$ and a irrational, Γ^a_k is the pair $(i,j) \in \mathbb{Z}^2_{\geqslant 0}$ which minimizes $\max\{i, aj\}$ subject to $i + j = k$. For example, in the case $a = \frac{3}{2} + \delta$ with $\delta > 0$ sufficiently small the first few terms are $(0,0), (1,0), (1,1), (2,1), (3,1), (3,2), (4,2), (4,3)$, and so on (see [MS3, Fig. 1] for an illustration).

Theorem 1.3 ([MS3])**.** *For any $a \in \mathbb{R}_{>0}$ and $d \in \mathbb{Z}_{\geqslant 1}$ we have:*

$$\widetilde{\mathbf{T}}^a_d = \left(\Gamma^a_{3d-1}\right)! \left((d!)^{-3} - \sum_{\substack{k \geqslant 2 \\ d_1,\ldots,d_k \in \mathbb{Z}_{\geqslant 1} \\ d_1+\cdots+d_k = d}} \frac{\widetilde{\mathbf{T}}^a_{d_1} \cdot \cdots \cdot \widetilde{\mathbf{T}}^a_{d_k}}{k! (\sum_{s=1}^k \Gamma^a_{3d_i - 1})!} \right). \tag{1.1}$$

Here we put $\widetilde{\mathbf{T}}^a_d := \mathrm{mult}_a(\Gamma^a_k) \cdot \mathbf{T}^a_d$, where $\mathrm{mult}_a(i,j) = i$ if $i > aj$ and $\mathrm{mult}_a(i,j) = j$ otherwise, and we write (1.1) using $\widetilde{\mathbf{T}}^a_d$ instead of $\mathbf{T}^a_d$ as convenience which yields slightly simpler formulas. We add pairs in $\mathbb{Z}^2_{\geqslant 0}$ componentwise in the usual fashion, and we define the factorial of a pair (i,j) by $(i,j)! := i!j!$. Note that all dependence on a in (1.1) is via the lattice path $\Gamma_0^a, \Gamma_1^a, \Gamma_2^a, \dots \in \mathbb{Z}^2_{\geqslant 0}$. The term $(d!)^{-3}$ arises from the computation of degree zero genus zero stationary gravitational descendant Gromov–Witten invariants of $\mathbb{CP}^2$ (see [MS3, §5.3.2]).

Although Theorem 1.3 and its generalization makes it possible to compute $\mathbf{T}_d^a$ for any fixed $d \in \mathbb{Z}_{\geqslant 1}$ and $a \in \mathbb{R}_{>0}$ given enough computational power, its recursivity somewhat obscures its enumerative essence, and the presence of terms of both positive and negative sign complicates efforts to study e.g. Conjecture 1.1 or Conjecture 1.2. Even more basically, although it is known by geometric arguments [McD, MS1] that we have $\mathbf{T}_d^\infty > 0$ for all $d \in \mathbb{Z}_{\geqslant 1}$[2], this is not a priori clear from Theorem 1.3.[3] Similarly, it follows by automatic transversality and the results in [MS2, §2] that $\mathbf{T}_d^{p/q}$ is a nonnegative integer whenever $p/q > 1$ is a reduced fraction with $p + q = 3d$, but this is not obvious from (1.1) due to the presence of denominators.

The above considerations motivate the search for a positive combinatorial formula for $\mathbf{T}_d^a$, the existence of which could help shed light on the above conjectures. In this note we take a step in this direction by establishing a closed formula for $\mathbf{T}_d^{\vec{a}}$ as a sum over trees with d leaves and certain combinatorially defined weights. Our formula still has some negative terms which we are not able to avoid, but they enter in a fairly transparent way, opening up further avenues for studying $\mathbf{T}_{M,A}^{\vec{a}}$ via combinatorics.

Before stating the formula, we will need some graph theoretic terminology.

Definition 1.4. *For $k \in \mathbb{Z}_{\geqslant 1}$, let $\mathfrak{T}_k^{\mathrm{un}}$ denote the set of (isomorphism classes of) rooted trees with k* unordered *leaves and no bivalent vertices (i.e. no vertices with $|v| = 2$).*

In other words, a tree $T \in \mathfrak{T}_k^{\mathrm{un}}$ has a distinguished root vertex and k leaf vertices, and the remaining vertices are called **internal**. We denote the set of leaf vertices by $V_{\mathrm{leaf}}(T)$ and the set of internal vertices by $V_{\mathrm{in}}(T)$. See Figure 1 for a picture of $\mathfrak{T}_4^{\mathrm{un}}$.

We orient all edges of T towards the root, and we will say that v is "above" w if w lies on the oriented path from v to the root.

Definition 1.5. *For a tree T in $\mathfrak{T}_k^{\mathrm{un}}$, an interval vertex $v \in V_{\mathrm{in}}(T)$ is* **movable** *if there are no internal vertices above it. We denote the set of movable vertices of T by $V_{\mathrm{mov}}(T) \subset V_{\mathrm{in}}(T)$.*

Definition 1.6. *For a vertex v of a tree T in $\mathfrak{T}_k^{\mathrm{un}}$, the* **leaf number** *$\ell(v)$ is the number of leaf vertices lying above v, including possibly v itself (i.e. $\ell(v) = 1$ if v is a leaf vertex).*

We are now ready to state our main result:

[2] Here we put $\mathbf{T}_d^\infty := \mathbf{T}_d^a$ for $a \gg d$. This corresponds to the count from [MS1] of degree d rational plane curves satisfying an order $3d-1$ local tangency constraint, which essentially amounts to fixing the $(3d-2)$-jet at a point.

[3] A fortiori, it follows by the obstruction bundle gluing method of [McD] that $\mathbf{T}_d^\infty$ is nondecreasing as a function of $d \in \mathbb{Z}_{\geqslant 1}$.

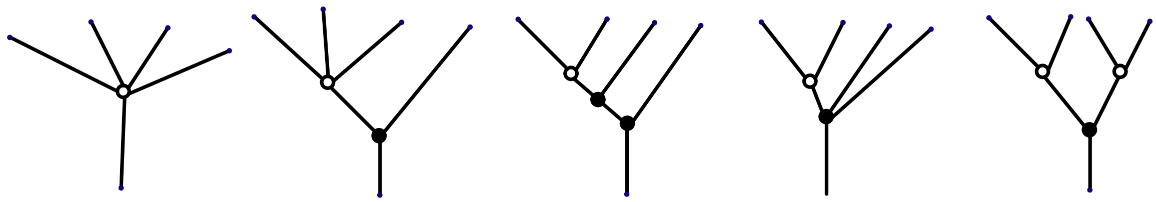

FIGURE 1. The trees comprising $\mathfrak{T}_4^{\mathrm{un}}$. The top four vertices are the leaves and the bottom vertex is the root. The internal vertices are denoted with a large circle, which is open for a movable vertex and solid otherwise. The values of $|\mathrm{Aut}(T)|$ are $24, 6, 2, 4, 8$ respectively.

Theorem A. *For any $d \in \mathbb{Z}_{\geqslant 1}$, we have:*

$$\widetilde{\mathbf{T}}_d^a = (\Gamma_2^a!)^d \sum_{T \in \mathfrak{T}_d^{\mathrm{un}}} \frac{(-1)^{|V_{\mathrm{in}}(T) \setminus V_{\mathrm{mov}}(T)|}}{|\mathrm{Aut}(T)|} \cdot \prod_{v \in V_{\mathrm{in}}(T)} \left(\frac{\Gamma^a_{3\ell(v)-1}!}{\left(\sum\limits_{v' \to v} \Gamma^a_{3\ell(v')-1}\right)!} \right) \cdot \prod_{v \in V_{\mathrm{mov}}(T)} \left((\ell(v)!)^{-2} \frac{(\ell(v)\Gamma_2^a)!}{(\Gamma_2^a!)^{\ell(v)}} - 1 \right). \tag{1.2}$$

Here the sum $\sum\limits_{v' \to v}$ is over all vertices v' which are adjacent to v and lie strictly above it.

Remark 1.7. Note that we have $\Gamma_2^a = 1$ for $1 < a < 2$ and $\Gamma_2^a = 2$ for $a \geqslant 2$. Therefore the last parenthesized term in (1.2) is $\frac{1}{\ell(v)!} - 1$ if $1 < a < 2$ and $2^{-\ell(v)}\binom{2\ell(v)}{\ell(v)} - 1$ if $a \geqslant 2$. In the latter case one can check that the term $2^{-\ell(v)}\binom{2\ell(v)}{\ell(v)} - 1$ is always positive, so a summand in (1.2) is negative if and only if there are an odd number of "unmovable" vertices $v \in V_{\mathrm{in}}(T) \setminus V_{\mathrm{mov}}(T)$. ♢

In the special case $a \gg 1$, we have $\Gamma_k^\infty = (k, 0)$ for all $k \in \mathbb{Z}_{\geqslant 0}$, so we get the following formula for $\widetilde{\mathbf{T}}_d^\infty = (3d-1)\mathbf{T}_d^\infty$:

Corollary B. *For $d \in \mathbb{Z}_{\geqslant 1}$ we have:*

$$\widetilde{\mathbf{T}}_d^\infty = 2^d \sum_{T \in \mathfrak{T}_d^{\mathrm{un}}} \frac{(-1)^{|V_{\mathrm{in}}(T) \setminus V_{\mathrm{mov}}(T)|}}{|\mathrm{Aut}(T)|} \cdot \prod_{v \in V_{\mathrm{in}}(T)} \left(\frac{(3\ell(v)-1)!}{(3\ell(v) - |v| + 1)!} \right) \cdot \prod_{v \in V_{\mathrm{mov}}(T)} \left(2^{-\ell(v)} \binom{2\ell(v)}{\ell(v)} - 1 \right). \tag{1.3}$$

Here $|v|$ denotes the valency (a.k.a. degree) of v. Equivalently, this is the number of incoming edges plus 1, where for $v \in V_{\mathrm{in}}(T)$ the **incoming** (resp. **outgoing**) edges of v are those edges of T which have v as an endpoint and are oriented towards (resp. away from) v.

Example 1.8. In the case $d = 1$, there is just a single tree in $\mathfrak{T}_1^{\mathrm{un}}$, which has no internal vertices. Both products in (1.3) are vacuous, so we get $\widetilde{\mathbf{T}}_1^\infty = 2$ and thus $\mathbf{T}_1^\infty = \frac{1}{2}\widetilde{\mathbf{T}}_1^\infty = 1$. ♢

Example 1.9. In the case $d = 2$, there is also just one tree in $\mathfrak{T}_2^{\mathrm{un}}$, which has one internal vertex, and that vertex is movable with leaf number is 2. Then (1.3) gives

$$\widetilde{\mathbf{T}}_2^{\infty} = 2^2 \cdot \frac{(-1)^0}{2} \cdot \frac{5!}{4!} \cdot \left(2^{-2}\binom{4}{2} - 1\right) = 5,$$

and thus $\mathbf{T}_2^{\infty} = \frac{1}{5}\widetilde{\mathbf{T}}_2^{\infty} = 1$. ◊

Example 1.10. To compute $\widetilde{\mathbf{T}}_3^{\infty}$, put $\mathfrak{T}_3^{\mathrm{un}} = \{T_1, T_2\}$, where T_1 has a single internal vertex of valency 4 and T_2 has two internal vertices each of valency 3. We have $|\mathrm{Aut}(T_1)| = 3!$ and $|\mathrm{Aut}(T_2)| = 2!$. The internal vertex of T_1 is movable and has leaf number 3. The internal vertices of T_2 have leaf numbers $2, 3$, and the first is movable while the second is not. Plugging these into (1.3) gives

$$\begin{aligned}\widetilde{\mathbf{T}}_3^{\infty} &= 2^3\left(\frac{1}{3!}\frac{8!}{6!}\left[2^{-3}\binom{6}{3} - 1\right] - \frac{1}{2!}\frac{5!}{4!}\frac{8!}{7!}\left[2^{-2}\binom{4}{2} - 1\right]\right)\\ &= 2^3(14 - 10) = 32,\end{aligned}$$

and hence $\mathbf{T}_3^{\infty} = \frac{1}{8}\widetilde{\mathbf{T}}_3^{\infty} = 4$. ◊

Remark 1.11. Let $\mathfrak{T}_d^{\mathrm{or}}$ be defined just like $\mathfrak{T}_d^{\mathrm{un}}$, except with *ordered* leaves (see Definition 2.1). Curiously, we have precisely $\mathbf{T}_d^{\infty} = |\mathfrak{T}_d^{\mathrm{or}}|$ for $d = 1, 2, 3, 4$, while experimentally we have $\mathbf{T}_d^{\infty} < |\mathfrak{T}_d^{\mathrm{or}}|$ for $d \geqslant 5$. This perhaps suggests that $\mathbf{T}_d^{\infty}$ counts elements of $\mathfrak{T}_d^{\mathrm{or}}$ with some additional conditions which only become relevant when there are at least 5 leaves. ◊

The main ingredients in the proof of Theorem A are (i) the computation from [MS3] of genus zero punctured stationary descendants in symplectic ellipsoids (see XX below), and (ii) homological perturbation theory for $\mathcal{L}_\infty$ algebras. First, in §2 we recall some relevant formalism for $\mathcal{L}_\infty$ algebras and state an explicit formula for inverting $\mathcal{L}_\infty$ homomorphisms between evenly graded $\mathcal{L}_\infty$ algebras. Then, in §3 we combine this with the computation for punctured stationary descendants in ellipsoids to give a closed formula for $\mathbf{T}_d^a$, which after some manipulations yields (1.2).

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2. Homological perturbation theory

Here we recall XX, and defer to XX for XX. $\mathcal{L}_\infty$ algebras provide an useful framework for organizing the curve counts underlying the ellipsoidal superpotential. An $\mathcal{L}_\infty$ algebra over $\mathbb{Q}$ consists of a $\mathbb{Z}$-graded rational vector space V along multilinear k-to-1 maps ℓ^k for $k \in \mathbb{Z}_{\geqslant 1}$ which are symmetric in a suitable graded sense and satisfy an infinite sequence of quadratic relations. As it happens, in this note we will only need to consider evenly graded $\mathcal{L}_\infty$ algebras over $\mathbb{Q}$, which are nothing but evenly graded rational vector spaces. Indeed, if V is only supported in even degrees then all of the $\mathcal{L}_\infty$ operations $\ell^1, \ell^2, \ell^3, \ldots$ necessarily vanish for degree parity reasons, and moreover the Koszul-type signs all disappear.

Given an evenly graded $\mathcal{L}_\infty$ algebra V, its (reduced) symmetric tensor coalgebra is $\overline{S}V = \oplus_{k=1}^{\infty} \odot^k V$, where $\odot^k V := (\underbrace{V \otimes \cdots \otimes V}_{k})/\Sigma_k$ is the k-fold symmetric tensor power of V. The coproduct on $\Delta_{\overline{S}V} : \overline{S}V \to \overline{S}V \otimes \overline{S}V$ is given by

$$\Delta_{\overline{S}V}(v_1 \odot \cdots \odot v_k) = \sum_{i=1}^{k-1} \sum_{\sigma \in \mathrm{Sh}(i,k-i)} (v_{\sigma(1)} \odot \cdots \odot v_{\sigma(i)}) \otimes (v_{\sigma(i+1)} \odot \cdots \odot v_{\sigma(k)}),$$

where $\mathrm{Sh}(i, k-i)$ is the subset of permutations $\sigma \in \Sigma_k$ satisfying $\sigma(1) < \cdots < \sigma(i)$ and $\sigma(i+1) < \cdots < \sigma(k)$.

Furthermore, given evenly graded $\mathcal{L}_\infty$ algebras V, W over $\mathbb{Q}$, an $\mathcal{L}_\infty$ homomorphism $\Phi : V \to W$ is a sequence of degree zero[4] linear maps $\Phi^k : \odot^k V \to W$ for $k \in \mathbb{Z}_{\geqslant 1}$, i.e. Φ^k is a multilinear map with k symmetric inputs in V and one output in W. The maps $\Phi^1, \Phi^2, \Phi^3, \ldots$ can be uniquely assembled into a degree zero coalgebra map $\widehat{\Phi} : \overline{S}V \to \overline{W}$, i.e. a linear map satisfying

$$(\widehat{\Phi} \otimes \widehat{\Phi}) \circ \Delta_{\overline{S}V} = \Delta_{\overline{S}W} \circ \widehat{\Phi}.$$

Indeed, we have

$$\widehat{\Phi}(v_1, \ldots, v_k) = \sum_{\substack{s \geqslant 1 \\ 1 \leqslant k_1 \leqslant \cdots \leqslant k_s \\ k_1 + \cdots + k_s = k}} \sum_{\sigma \in \overline{\mathrm{Sh}}(k_1, \ldots, k_s)} (\Phi^{k_1} \odot \ldots \odot \Phi^{k_s})(v_{\sigma(1)} \odot \ldots \odot v_{\sigma(k)}).$$

Conversely, given such a coalgebra map $\widehat{\Phi}$ we recover $\Phi^1, \Phi^2, \Phi^3, \ldots$ via the composition

$$\odot^k V \quad \subset \quad \overline{S}V \xrightarrow{\widehat{\Phi}} \overline{S}W \xrightarrow{\pi_k} W.$$

If V_1, V_2, V_3 are evenly graded $\mathcal{L}_\infty$ algebras over $\mathbb{Q}$ and we have $\mathcal{L}_\infty$ homomorphisms $\Phi : V_1 \to V_2$ and $\Psi : V_2 \to V_3$, then the composition $\mathcal{L}_\infty$ homomorphism $\Psi \circ \Phi : V_1 \to V_3$ is characterized by $\widehat{\Psi \circ \Phi} = \widehat{\Psi} \circ \widehat{\Phi}$. In terms of the operations $\Phi^1, \Phi^2, \Phi^3, \ldots$ and $\Psi^1, \Psi^2, \Psi^3, \ldots$ this translates into $(\Psi \circ \Phi)^1(v_1) = (\Psi^1 \circ \Phi^1)(v_1)$, $(\Psi \circ \Phi)^2(v_1, v_2) = \Psi^1(\Phi^2(v_1, v_2)) + \Psi^2(\Phi^1(v_1), \Phi^1(v_2))$, and so on. The identity $\mathcal{L}_\infty$ homomorphism $\mathbb{1} : V \to V$ corresponds to the identity map $\overline{S} \to \overline{S}$, or equivalently we have that $\mathbb{1}^1 : V \to V$ is the identity map and $\mathbb{1}^k \equiv 0$ for $k \in \mathbb{Z}_{\geqslant 2}$.

Trees arise naturally in $\mathcal{L}_\infty$ contexts, and it often more convenient to work with trees with *ordered* leaves.

Definition 2.1. *For $k \in \mathbb{Z}_{\geqslant 1}$, let $\mathfrak{T}_k^{\mathrm{or}}$ denote the set of (isomorphism classes of) rooted trees with k* ordered *leaves and no bivalent vertices.*

Note that the ordering of the leaf vertices of $T \in \mathfrak{T}_k^{\mathrm{or}}$ amounts to a bijection between $V_{\mathrm{leaf}}(T)$ and $\{1, \ldots, k\}$. We will also refer to the edges connected to the leaf vertices as the **leaf edges**, and the edge connected to the root vertex as the **root edge**. The set $\mathfrak{T}_k^{\mathrm{un}}$ is the quotient $\mathfrak{T}_k^{\mathrm{or}}/\Sigma_k$ by the natural symmetric group action on $\mathfrak{T}_k^{\mathrm{or}}$ which reorders the leaves, and the stabilizer of a tree $T \in \mathfrak{T}_k^{\mathrm{or}}$ is the automorphism group $\mathrm{Aut}(T)$ of T.

[4] Note that some references use different grading conventions.

The following proposition is proved using standard techniques from homological perturbation theory. One can formulate a more general version without any grading restrictions, but here we state a simplified version which suffices for our purposes.

Proposition 2.2. *Let V and W be evenly graded $\mathcal{L}_\infty$ algebras over $\mathbb{Q}$, and let $\Phi : V \to W$ be an $\mathcal{L}_\infty$ homomorphism such that the linear map $\Phi^1 : V \to W$ is invertible. Then there exists an $\mathcal{L}_\infty$ homomorphism $\Psi : W \to V$ such that*

$$\Psi \circ \Phi = \mathbb{1}_V \quad \text{and} \quad \Phi \circ \Psi = \mathbb{1}_W.$$

Moreover, Ψ is given explicitly as follows. We first set $\Psi^1 : W \to V$ to be the inverse of $\Phi^1 : V \to W$. For $k \geqslant 2$, $T \in \mathfrak{T}_k^{\mathrm{or}}$, and $w_1, \ldots, w_k \in W$, define $\Psi^T(w_1, \ldots, w_k)$ as follows. Start by labeling the ith leaf edge of T by $\Psi^1(w_i)$ for $i = 1, \ldots, k$. Recursively, for each internal vertex $v \in V_{\mathrm{in}}(T)$, say with j incoming edges, label the outgoing edge by the result after applying $\Psi^1 \circ \Phi^j$ to the corresponding labels of its incoming edges. We define $\Psi^T(w_1, \ldots, w_k)$ to be the resulting label on the root edge, and finally put

$$\Psi^k(w_1, \ldots, w_k) = \sum_{T \in \mathfrak{T}_k^{\mathrm{or}}} (-1)^{|V_{\mathrm{in}}(T)|} \Psi^T(w_1, \ldots, w_k).$$

Example 2.3. We have:

- $\Psi^2(x, y) = -\Psi^1\Phi^2(\Psi^1(x), \Psi^1(y))$
- $\Psi^3(x, y, z) = -\Psi^1\Phi^3(\Psi^1(x), \Psi^1(y), \Psi^1(z)) + \Psi^1\Phi^2(\Psi^1(x), \Psi^1\Phi^2(\Psi^1(y), \Psi^1(z))) + \Psi^1\Phi^2(\Psi^1(y), \Psi^1\Phi^2(\Psi^1(x), \Psi^1(z))) + \Psi^1\Phi^2(\Psi^1(z), \Psi^1\Phi^2(\Psi^1(x), \Psi^1(y)))$.

$\Diamond$

Proof. The relation $\Psi \circ \Phi = \mathbb{1}$ means that for $k \geqslant 2$ we must have $(\Psi \circ \Phi)^k(v_1, \ldots, v_k) = 0$ for any $v_1, \ldots, v_k \in V$, and this amounts to

$$\sum_{\substack{s \geqslant 1 \\ 1 \leqslant k_1 \leqslant \cdots \leqslant k_s \\ k_1 + \cdots + k_s = k}} \sum_{\sigma \in \overline{\mathrm{Sh}}(k_1, \ldots, k_s)} \Psi^s \circ (\Phi^{k_1} \odot \cdots \odot \Phi^{k_s})(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = 0,$$

or equivalently

$$\Psi^k(\Phi^1(v_1), \ldots, \Phi^1(v_k)) = -\sum_{\substack{1 \leqslant s \leqslant k-1 \\ 1 \leqslant k_1 \leqslant \cdots \leqslant k_s \\ k_1 + \cdots + k_s = k}} \sum_{\sigma \in \overline{\mathrm{Sh}}(k_1, \ldots, k_s)} \Psi^s \circ (\Phi^{k_1} \odot \cdots \odot \Phi^{k_s})(v_{\sigma(1)}, \ldots, v_{\sigma(k)}),$$

i.e. for any $w_1, \ldots, w_k \in W$ we must have

$$\Psi^k(w_1, \ldots, w_k) = -\sum_{\substack{1 \leqslant s \leqslant k-1 \\ 1 \leqslant k_1 \leqslant \cdots \leqslant k_s \\ k_1 + \cdots + k_s = k}} \sum_{\sigma \in \overline{\mathrm{Sh}}(k_1, \ldots, k_s)} \Psi^s \circ (\Phi^{k_1} \odot \cdots \odot \Phi^{k_s})(\Psi^1(w_{\sigma(1)}), \ldots, \Psi^1(w_{\sigma(k)})).$$

This is easily seen to agree with the definition of Ψ given in the statement of the proposition, which therefore necessarily satisfies $\Psi \circ \Phi = \mathbb{1}$.

As for the relation $\Phi \circ \Psi = \mathbb{1}$, we need

$$\sum_{\substack{s \geqslant 1 \\ 1 \leqslant k_1 \leqslant \cdots \leqslant k_s \\ k_1 + \cdots + k_s = k}} \sum_{\sigma \in \overline{\mathrm{Sh}}(k_1, \ldots, k_s)} \Phi^s \circ (\Psi^{k_1} \odot \cdots \odot \Psi^{k_s})(w_{\sigma(1)}, \ldots, w_{\sigma(k)}) = 0$$

for any $w_1, \dots, w_k \in W$, or equivalently

$$\Phi^1(\Psi^k(w_1, \dots, w_k)) = - \sum_{\substack{s \geqslant 2 \\ 1 \leqslant k_1 \leqslant \cdots \leqslant k_s \\ k_1 + \cdots + k_s = k}} \sum_{\sigma \in \overline{\mathrm{Sh}}(k_1, \dots, k_s)} \Phi^s \circ (\Psi^{k_1} \odot \cdots \odot \Psi^{k_s})(w_{\sigma(1)}, \dots, w_{\sigma(k)}),$$

i.e.

$$\Psi^k(w_1, \dots, w_k) = - \sum_{\substack{s \geqslant 2 \\ 1 \leqslant k_1 \leqslant \cdots \leqslant k_s \\ k_1 + \cdots + k_s = k}} \sum_{\sigma \in \overline{\mathrm{Sh}}(k_1, \dots, k_s)} \Psi^1 \circ \Phi^s \circ (\Psi^{k_1} \odot \cdots \odot \Psi^{k_s})(w_{\sigma(1)}, \dots, w_{\sigma(k)}),$$

which is equivalent to our definition of Ψ^k. □

3. Tree formula

Before proving Theorem A, we need to recall that computation of punctured stationary descendants in ellipsoids from [MS3]. XX XX

We associate to each $a \in \mathbb{R}_{>0}$ an evenly graded $\mathcal{L}_\infty$ algebra C_a with basis given by formal symbols $\mathfrak{o}_i^a$ with degree $|\mathfrak{o}_i^a| = -2 - 2i$ for $i \in \mathbb{Z}_{\geqslant 1}$. We encode the ellipsoidal superpotential as elements of these $\mathcal{L}_\infty$ algebras by putting $\mathfrak{m}^a_{\mathbb{CP}^2,d} = \widetilde{\mathbf{T}}^a_d \, \mathfrak{o}^a_{3d-1}$ for each $a \in \mathbb{R}_{>0}$ and $d \in \mathbb{Z}_{\geqslant 1}$. For $d \in \mathbb{Z}_{\geqslant 1}$, put

$$\exp_A(\mathfrak{m}^{\vec{a}}_M) := \sum_{\substack{k \geqslant 1 \\ A_1, \dots, A_k \in H_2(M) \\ A_1 + \cdots + A_k = A}} \tfrac{1}{k!} \mathfrak{m}^{\vec{a}}_{M,A_1} \odot \cdots \odot \mathfrak{m}^a_{M,A_k}.$$

This encodes collections of rigid pseudoholomorphic planes in $\widehat{M_{\vec{a}}}$ in homology class A.

We denote the genus zero Gromov–Witten invariant of M in homology class A and carrying a maximal order stationary descendant condition by $N_{M,A}\langle\!\langle \psi^{c_1(A)-2} pt \rangle\!\rangle \in \mathbb{Q}$. In this note we will only need the computation of these invariants in $\mathbb{CP}^2$, namely for all $d \in \mathbb{Z}_{\geqslant 1}$ we have

$$N_{M,A}\langle\!\langle \psi^{3d-2} pt \rangle\!\rangle = (d!)^{-3}.$$

Roughly, $N_{M,A}\langle\!\langle \psi^{c_1(A)-2} pt \rangle\!\rangle$ can be viewed as XX the count of degree d rational pseudoholomorphic curves in M with prescribed $(c_1(A) - 1)$-jet at a point. However, compared with the corresponding local tangency $\langle\!\langle \mathcal{T}^{c_1(A)-2} pt \rangle\!\rangle$, the descendant counts typically receive extra contributions from boundary strata with ghost components.

We introduce one more evenly graded $\mathcal{L}_\infty$ algebra C_o with basis given by formal symbols $\mathfrak{q}_i$ for $i \in \mathbb{Z}_{\geqslant 1}$. Similar to above, put $\mathfrak{m}^o_{M,A} := N_{M,A}\langle\!\langle \psi^{c_1(A)-2} pt \rangle\!\rangle \, \mathfrak{q}_{3d-1}$ and

$$\exp_A(\mathfrak{m}^o_M) := \sum_{\substack{k \geqslant 1 \\ A_1, \dots, A_k \in H_2(M) \\ A_1 + \cdots + A_k = A}} \tfrac{1}{k!} \mathfrak{m}^o_{M,A_1} \odot \cdots \odot \mathfrak{m}^o_{M,A_k}.$$

Theorem 3.1 ([MS3]). *For each $a \in \mathbb{R}_{>0}$ we have*

$$\widehat{\epsilon}_{\vec{a}}(\exp_d(\mathfrak{m}^a_{\mathbb{CP}^2})) = \exp_d(\mathfrak{m}^o_{\mathbb{CP}^2}), \tag{3.1}$$

where $\epsilon_a : C_a \to C_o$ is the $\mathcal{L}_\infty$ homomorphism defined by

$$\epsilon_a^k(\mathfrak{o}_{i_1}^a, \ldots, \mathfrak{o}_{i_k}^a) = \frac{1}{(\Gamma_{i_1}^a + \cdots + \Gamma_{i_k}^a)!} \tag{3.2}$$

for each $k, i_1, \ldots, i_k \in \mathbb{Z}_{\geqslant 1}$.

At this point the statement of Theorem 3.1 is formally self-contained, although we have not explained any geometric interpretation. In essence, (3.1) is the algebraic relation given by neck stretching closed curve stationary descendants in M along the boundary of (a rescaling of) the ellipsoid $E(\vec{a})$. Meanwhile, (3.2) is the computation of punctured curve stationary descendants in the symplectic completion of $E(\vec{a})$. More precisely, for $\vec{a} \in \mathbb{R}_{>0}$ with rationally independent components, we identify $\mathfrak{o}_k^{\vec{a}}$ with the Reeb orbit of kth smallest action (or equivalently Conley–Zehnder index $n - 1 + 2k$) in $\partial E(\vec{a})$. Then $\epsilon_a^k(\mathfrak{o}_{i_1}^{\vec{a}}, \ldots, \mathfrak{o}^a \vec{a}_{i_k})$ encodes the count of rational pseudoholomorphic curves in $\widehat{E}(\vec{a})$ with positive punctures asymptotic to the Reeb orbits $\mathfrak{o}_{i_1}^{\vec{a}}, \ldots, \mathfrak{o}_{i_k}^{\vec{a}}$ and carrying the stationary descendant condition $\lessdot\psi^{c_1(A)-2}pt\gtrdot$.

Given $\vec{a} \in \mathbb{R}_{>0}^n$, let $\eta_{\vec{a}} : C_o \to C_{\vec{a}}$ denote $\mathcal{L}_\infty$ homomorphism inverse to the stationary descendant map $\epsilon_{\vec{a}} : C_{\vec{a}} \to C_o$, whose explicit construction is provided by Proposition 2.2. In particular, $\eta_{\vec{a}}^1$ is the linear inverse of $\epsilon_{\vec{a}}^1$, i.e. for $k \in \mathbb{Z}_{\geqslant 1}$ we have

$$\epsilon_{\vec{a}}^1(\mathfrak{o}_k^{\vec{a}}) = \frac{\mathfrak{q}_k}{(\Gamma_k^{\vec{a}})!} \quad \text{and} \quad \eta_{\vec{a}}^1(\mathfrak{q}_k) = (\Gamma_k^{\vec{a}})!\mathfrak{o}_k^{\vec{a}}.$$

Applying $\pi_1 \circ \widehat{\eta}_{\vec{a}}$ to both sides of (3.1) gives

$$\pi_1(\exp_A(\mathfrak{m}_M^{\vec{a}})) = (\pi_1 \circ \widehat{\eta}_{\vec{a}})(\exp_A(\mathfrak{m}_M^o)),$$

i.e.

$$\begin{aligned}
\mathfrak{m}_{M,A}^{\vec{a}} &= \sum_{\substack{k \geqslant 1 \\ A_1, \ldots, A_k \in H_2(M) \\ A_1 + \cdots + A_k = A}} \tfrac{1}{k!} \eta_{\vec{a}}^k(\mathfrak{m}_{M,A_1}^o, \ldots, \mathfrak{m}_{M,A_k}^o) \\
&= \sum_{\substack{k \geqslant 1 \\ A_1, \ldots, A_k \in H_2(M) \\ A_1 + \cdots + A_k = A}} \tfrac{1}{k!} \sum_{T \in \mathfrak{T}_k^{\mathrm{or}}} (-1)^{|V_{\mathrm{in}}(T)|} \eta_{\vec{a}}^T(\mathfrak{m}_{M,A_1}^o, \ldots, \mathfrak{m}_{M,A_k}^o).
\end{aligned}$$

Using $\mathfrak{m}_{M,A}^{\vec{a}} = \widetilde{\mathbf{T}}_{M,A}^{\vec{a}} \cdot \mathfrak{o}_{c_1(A)-1}^{\vec{a}}$ and $\mathfrak{m}_{M,A}^o = N_{M,A} \lessdot\psi^{c_1(A)-2}pt\gtrdot \cdot \mathfrak{q}_{c_1(A)-1}$, we can rewrite the above as

$$\begin{aligned}
&\widetilde{\mathbf{T}}_{M,A}^{\vec{a}} \cdot \mathfrak{o}_{c_1(A)-1}^{\vec{a}} \\
&= \sum_{\substack{k \geqslant 1 \\ A_1, \ldots, A_k \in H_2(M) \\ A_1 + \cdots + A_k = A}} \tfrac{1}{k!} \left(\prod_{s=1}^k N_{M,A_s} \lessdot\psi^{c_1(A_s)-2}pt\gtrdot \right) \sum_{T \in \mathfrak{T}_k^{\mathrm{or}}} (-1)^{|V_{\mathrm{in}}(T)|} \eta_{\vec{a}}^T(\mathfrak{q}_{c_1(A_1)-1}, \ldots, \mathfrak{q}_{c_1(A_k)-1}).
\end{aligned} \tag{3.3}$$

We now specialize to the case $M = \mathbb{CP}^2$ and $A = d[L]$, so that equation (3.3) becomes

$$\widetilde{\mathbf{T}}^a_d \cdot \mathfrak{o}^a_{3d-1} = \sum_{\substack{k \geqslant 1 \\ d_1,\ldots,d_k \in \mathbb{Z}_{\geqslant 1} \\ d_1+\cdots+d_k = d}} \tfrac{1}{k!(d_1!)^3 \cdots (d_k!)^3} \sum_{T \in \mathfrak{T}^{\mathrm{or}}_k} (-1)^{|V_{\mathrm{in}}(T)|} \eta^T_a(\mathfrak{q}_{3d_1-1}, \ldots, \mathfrak{q}_{3d_k-1}). \qquad (3.4)$$

Our goal is to rewrite the above as a sum over trees with exactly d leaves. We will introduce relevant notation as we need it.

Notation 3.2. *Given $d_1, \ldots, d_k \in \mathbb{Z}_{\geqslant 1}$ with $d_1 + \cdots + d_k = d$, let $\mathcal{P}(d_1, \ldots, d_k)$ denote the set of surjections $h : \{1, \ldots, d\} \twoheadrightarrow \{1, \ldots, k\}$ such that $|h^{-1}(i)| = d_i$ for $i = 1, \ldots, k$.*

Then we can rewrite (3.4) temporarily more redundantly as

$$\widetilde{\mathbf{T}}^a_d \cdot \mathfrak{o}^a_{3d-1} = \sum_{\substack{k \geqslant 1 \\ d_1,\ldots,d_k \in \mathbb{Z}_{\geqslant 1} \\ d_1+\cdots+d_k = d}} \frac{1}{k!(d_1!)^3 \cdots (d_k!)^3} \binom{d}{d_1, \ldots, d_k}^{-1} \sum_{h \in \mathcal{P}(d_1,\ldots,d_k)} \sum_{T \in \mathfrak{T}^{\mathrm{or}}_k} (-1)^{|V_{\mathrm{in}}(T)|} \eta^T_a(\mathfrak{q}_{3d_1-1}, \ldots, \mathfrak{q}_{3d_k-1}).$$

Notation 3.3.

(1) Put

$$\mathfrak{T}^{\mathrm{or}}_k(\{1, \ldots, d\}) := \{(T, h) \mid T \in \mathfrak{T}^{\mathrm{or}}_k,\ h : \{1, \ldots, d\} \twoheadrightarrow \{1, \ldots, k\}\},$$

i.e. $\mathfrak{T}^{\mathrm{or}}_k(\{1, \ldots, d\})$ is the set of trees $T \in \mathfrak{T}^{\mathrm{or}}_k$ equipped with a partition of $\{1, \ldots, d\}$ into k parts labeled by the leaves of T.

(2) Given $(T, h) \in \mathfrak{T}^{\mathrm{or}}_k(\{1, \ldots, d\})$, put

$$C_{(T,h)} := \frac{1}{(d_1!)^2 \cdots (d_k!)^2} \quad \text{and} \quad \epsilon^{(T,h)}_a(\mathfrak{q}_\bullet) := \epsilon^T_a(\mathfrak{q}_{3d_1-1}, \ldots, \mathfrak{q}_{3d_k-1}),$$

where $d_i := |h^{-1}(i)|$ for $i = 1, \ldots, k$.

We then have

$$\widetilde{\mathbf{T}}^a_d \cdot \mathfrak{o}^a_{3d-1} = \sum_{k \geqslant 1} \sum_{(T,h) \in \mathfrak{T}^{\mathrm{or}}_k(\{1,\ldots,d\})} \tfrac{1}{k!d!} \cdot C_{(T,h)} \cdot (-1)^{|V_{\mathrm{in}}(T)|} \eta^{(T,h)}_a(\mathfrak{q}_\bullet).$$

Notation 3.4. *Put*

$$\mathfrak{T}^{\mathrm{un}}_k(\{1, \ldots, d\}) = \{(T, h) \mid T \in \mathfrak{T}^{\mathrm{un}}_k,\ h : \{1, \ldots, d\} \twoheadrightarrow V_{\mathrm{leaf}}(T)\},$$

i.e. an element of $\mathfrak{T}^{\mathrm{un}}_k(\{1, \ldots, d\})$ is a tree $T \in \mathfrak{T}^{\mathrm{un}}_k$ with k unordered leaves equipped with a partition of $\{1, \ldots, d\}$ into k parts labeled by the leaves of T. Put also

$$\mathfrak{T}^{\mathrm{un}}(\{1, \ldots, d\}) := \bigcup_{1 \leqslant k \leqslant d} \mathfrak{T}^{\mathrm{un}}_k(\{1, \ldots, d\}).$$

Note that the symmetric group Σ_k acts freely on $\mathfrak{T}^{\mathrm{or}}_k(\{1, \ldots, d\})$ via $\sigma \cdot (T, h) = (T', \sigma \circ h)$, where T' is given by reordering the leaves of T according to the permutation $\sigma \in \Sigma_k$, and the quotient is identified with $\mathfrak{T}^{\mathrm{un}}_k(\{1, \ldots, d\})$. We then have

$$\widetilde{\mathbf{T}}^a_d \cdot \mathfrak{o}^a_{3d-1} = \sum_{(T,h) \in \mathfrak{T}^{\mathrm{un}}(\{1,\ldots,d\})} \tfrac{1}{d!} \cdot C_{(T,h)} \cdot (-1)^{|V_{\mathrm{in}}(T)|} \eta^{(T,h)}_a(\mathfrak{q}_\bullet). \qquad (3.5)$$

Observe that there is a natural map

$$\zeta : \{(T, \mathcal{S}) \mid T \in \mathfrak{T}^{\mathrm{or}}_d, \mathcal{S} \subset V_{\mathrm{mov}}(T)\} \to \mathfrak{T}^{\mathrm{un}}(\{1, \ldots, d\}),$$

defined as follows. Given $(T, \mathcal{S})$, let T' be the tree obtained by removing all edges and vertices strictly above v, for each $v \in \mathcal{S}$. Note that each $v \in \mathcal{S}$ becomes a leaf in T'. We define a surjection $h : \{1, \ldots, d\} \twoheadrightarrow V_{\mathrm{leaf}}(T')$ in such a way that

- if $v \in V_{\mathrm{leaf}}(T')$ corresponds to a leaf of T, then $h^{-1}(v)$ is the original label of v
- for $v \in \mathcal{S}$, $h^{-1}(v)$ is the set of labels of leaves lying above v in T.

We put $\zeta(T, \mathcal{S}) = (T', h)$.

In fact, ζ is a bijection, with inverse map

$$\zeta^{-1} : \mathfrak{T}^{\mathrm{un}}(\{1, \ldots, d\}) \to \{(T, \mathcal{S}) \mid T \in \mathfrak{T}_d^{\mathrm{or}}, \mathcal{S} \subset V_{\mathrm{mov}}(T)\}$$

described as follows. Given $(T, h) \in \mathfrak{T}^{\mathrm{un}}(\{1, \ldots, d\})$, for each $v \in V_{\mathrm{leaf}}(T)$ such that $|h^{-1}(v)| \geqslant 2$ we add $|h^{-1}(v)|$ new leaf vertices, each joined to v by a new leaf edge. By construction the resulting tree T' comes with a natural bijection $\{1, \ldots, d\} \xrightarrow{\sim} V_{\mathrm{leaf}}(T')$, i.e. $T' \in \mathfrak{T}_d^{\mathrm{or}}$. Also, each $v \in V_{\mathrm{leaf}}(T)$ satisfying $|h^{-1}(v)| \geqslant 2$ naturally corresponds to a movable vertex in T', and we denote the set of these by $\mathcal{S} \subset V_{\mathrm{mov}}(T')$. Then we have $\zeta^{-1}(T, h) = (T', \mathcal{S})$.

Using this bijection, we rewrite (3.5) as:

$$\widetilde{\mathbf{T}}_d^a \cdot \mathfrak{o}_{3d-1}^a = \tfrac{1}{d!} \sum_{T \in \mathfrak{T}_d^{\mathrm{or}}} \sum_{\mathcal{S} \subset V_{\mathrm{mov}}(T)} C_{\zeta(T,\mathcal{S})} \cdot (-1)^{|V_{\mathrm{in}}(\zeta(T,\mathcal{S}))|} \eta_a^{\zeta(T,\mathcal{S})}(\mathfrak{q}_\bullet). \tag{3.6}$$

It will be convenient to extend the notion of leaf number to elements of $\mathfrak{T}^{\mathrm{un}}(\{1, \ldots, d\})$ as follows. For $(T, h) \in \mathfrak{T}^{\mathrm{un}}(\{1, \ldots, d\})$, we define the leaf number $\ell(v)$ to be the cardinality of $\bigcup_w |h^{-1}(w)|$, where the union is over all leaf vertices w lying above v (including possibly v itself).

Proof of Theorem A. According to Proposition 2.2, $\eta_a^T(\mathfrak{q}_{3d_1-1}, \ldots, \mathfrak{q}_{3d_k-1})$ is computed as follows. Recall that $T \in \mathfrak{T}_k^{\mathrm{or}}$ has k ordered leaves. For $i = 1, \ldots, k$, we label the ith leaf edge by $\eta_a^1(\mathfrak{q}_{3d_i-1}) = (\Gamma_{3d_i-1}^a)!\, \mathfrak{o}_{3d_i-1}^a$. We then iteratively label each edge of T, say with source vertex v, by the result of applying $\eta_a^1 \circ \epsilon_a^j$ to the labels of the incoming edges of v (here the valency of v is $j+1$).

Then for $T \in \mathfrak{T}_d^{\mathrm{or}}$ and $\mathcal{S} \subset V_{\mathrm{mov}}(T)$, with $\zeta(T, \mathcal{S}) \in \mathfrak{T}^{\mathrm{un}}(\{1, \ldots, d\})$ as defined above we have:

$$\eta_a^{\zeta(T,\mathcal{S})}(\mathfrak{q}_\bullet) = \prod_{v \in V_{\mathrm{leaf}}(\zeta(T,\mathcal{S}))} \left(\Gamma_{3\ell(v)-1}^a\right)! \cdot \frac{\prod\limits_{v \in V_{\mathrm{in}}(\zeta(T,\mathcal{S}))} \left(\Gamma_{3\ell(v)-1}^a\right)!}{\prod\limits_{v \in V_{\mathrm{in}}(\zeta(T,\mathcal{S}))} \left(\sum\limits_{v' \to v} \Gamma_{3\ell(v')-1}^a\right)!} \cdot \mathfrak{o}_{3d-1}^a. \tag{3.7}$$

Observe that in the case $\mathcal{S} = \varnothing$, $\zeta(T, \varnothing)$ is simply T itself but viewed as an element of $\mathfrak{T}^{\mathrm{un}}(\{1, \ldots, d\})$, and we have

$$\eta_a^{\zeta(T,\varnothing)}(\mathfrak{q}_\bullet) = \eta_a^T(\underbrace{\mathfrak{q}_2, \ldots, \mathfrak{q}_2}_{d}) = \prod_{v \in V_{\mathrm{leaf}}(T)} \left(\Gamma_{3\ell(v)-1}^a\right)! \cdot \frac{\prod\limits_{v \in V_{\mathrm{in}}(T)} \left(\Gamma_{3\ell(v)-1}^a\right)!}{\prod\limits_{v \in V_{\mathrm{in}}(T)} \left(\sum\limits_{v' \to v} \Gamma_{3\ell(v')-1}^a\right)!} \cdot \mathfrak{o}_{3d-1}^a, \tag{3.8}$$

where

$$\Gamma_2^a! = \begin{cases} 1 & \text{if} \quad 1 < a < 2 \\ 2 & \text{if} \quad a \geqslant 2. \end{cases}$$

We have natural a identification $V_{\mathrm{in}}(T) \approx V_{\mathrm{in}}(\zeta(T,\mathcal{S})) \cup \mathcal{S}$, and the leaves of $\zeta(T,\mathcal{S})$ correspond to (a) leaves of T not lying above any $v \in \mathcal{S}$ and (b) elements of $\mathcal{S}$. Then, comparing (3.7) and (3.8), we have

$$\begin{aligned}\eta_a^{\zeta(T,\mathcal{S})}(\mathfrak{q}_\bullet) &= \eta_a^{\zeta(T,\varnothing)}(\mathfrak{q}_\bullet) \cdot \frac{\prod\limits_{v\in\mathcal{S}} \left(\Gamma^a_{3\ell(v)-1}\right)!}{\prod\limits_{v\in\mathcal{S}} (\Gamma_2^a!)^{\ell(v)}} \cdot \frac{\prod\limits_{v\in\mathcal{S}} \left(\sum\limits_{v'\to v} \Gamma^a_{3\ell(v')-1}\right)!}{\prod\limits_{v\in\mathcal{S}} \left(\Gamma^a_{3\ell(v)-1}\right)!} \\ &= \eta_a^{\zeta(T,\varnothing)}(\mathfrak{q}_\bullet) \cdot \prod_{v\in\mathcal{S}} \frac{(\ell(v)\Gamma_2^a)!}{(\Gamma_2^a!)^{\ell(v)}}.\end{aligned}$$

Note that we have $C_{\zeta(T,\mathcal{S})} = \left(\prod\limits_{v\in\mathcal{S}} \ell(v)!\right)^{-2}$, and hence

$$\begin{aligned}\widetilde{\mathbf{T}}_d^a \cdot \mathfrak{o}^a_{3d-1} &= \tfrac{1}{d!} \sum_{T\in\mathfrak{T}_d^{\mathrm{or}}} (-1)^{|V_{\mathrm{in}}(T)|} \sum_{\mathcal{S}\subset V_{\mathrm{mov}}(T)} C_{\zeta(T,\mathcal{S})} \cdot (-1)^{|\mathcal{S}|} \cdot \eta_a^{\zeta(T,\mathcal{S})}(\mathfrak{q}_\bullet) \\ &= \tfrac{1}{d!} \sum_{T\in\mathfrak{T}_d^{\mathrm{or}}} (-1)^{|V_{\mathrm{in}}(T)|} \cdot \eta_a^{\zeta(T,\varnothing)}(\mathfrak{q}_\bullet) \sum_{\mathcal{S}\subset V_{\mathrm{mov}}(T)} \left(\prod_{v\in\mathcal{S}} \ell(v)!\right)^{-2} \cdot (-1)^{|\mathcal{S}|} \cdot \prod_{v\in\mathcal{S}} \frac{(\ell(v)\Gamma_2^a)!}{(\Gamma_2^a!)^{\ell(v)}}.\end{aligned}$$

We also have

$$\eta_a^{\zeta(T,\varnothing)}(\mathfrak{q}_\bullet) = (\Gamma_2^a!)^d \cdot \prod_{v\in V_{\mathrm{in}}(T)} \frac{\left(\Gamma^a_{3\ell(v)-1}\right)!}{\left(\sum\limits_{v'\to v} \Gamma^a_{3\ell(v')-1}\right)!} \cdot \mathfrak{o}^a_{3d-1},$$

and hence

$$\begin{aligned}\widetilde{\mathbf{T}}_d^a &= \tfrac{1}{d!}(\Gamma_2^a!)^d \sum_{T\in\mathfrak{T}_d^{\mathrm{or}}} (-1)^{|V_{\mathrm{in}}(T)|} \cdot \prod_{v\in V_{\mathrm{in}}(T)} \frac{\left(\Gamma^a_{3\ell(v)-1}\right)!}{\left(\sum\limits_{v'\to v} \Gamma^a_{3\ell(v')-1}\right)!} \cdot \sum_{\mathcal{S}\subset V_{\mathrm{mov}}(T)} (-1)^{|\mathcal{S}|} \cdot \prod_{v\in\mathcal{S}} (\ell(v)!)^{-2} \frac{(\ell(v)\Gamma_2^a)!}{(\Gamma_2^a!)^{\ell(v)}} \\ &= \tfrac{1}{d!}(\Gamma_2^a!)^d \sum_{T\in\mathfrak{T}_d^{\mathrm{or}}} (-1)^{|V_{\mathrm{in}}(T)|} \cdot \prod_{v\in V_{\mathrm{in}}(T)} \frac{\left(\Gamma^a_{3\ell(v)-1}\right)!}{\left(\sum\limits_{v'\to v} \Gamma^a_{3\ell(v')-1}\right)!} \cdot \prod_{v\in V_{\mathrm{mov}}(T)} \left(1 - (\ell(v)!)^{-2} \frac{(\ell(v)\Gamma_2^a)!}{(\Gamma_2^a!)^{\ell(v)}}\right).\end{aligned}$$

Finally, we can write this as a sum over $\mathfrak{T}_d^{\mathrm{un}}$ by recalling that there is a natural forgetful map $\mathfrak{T}_d^{\mathrm{or}} \to \mathfrak{T}_d^{\mathrm{un}}$ and the fiber of any $T \in \mathfrak{T}_d^{\mathrm{un}}$ has cardinality $\frac{d!}{|\mathrm{Aut}(T)|}$. ☐

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