Midterm exam
Math 535a: Differential Geometry
University of Southern California Spring 2021
Instructor: Kyler Siegel

Instructions:

- You are allowed to use our main textbook *Introduction to Smooth Manifolds* by John Lee as much as you wish, but you must not consult any other textbook and you must not consult the internet or communicate with other people about any material related to this exam.
- You are welcome to type up with solutions in LaTeX, or write them by hand. Either way you should strive to make your answers are clear, comprehensive, and legible as possible.
- If you have any pressing questions about the wording of a problem, you may email Kyler. He obviously won't be able to help with the actual content of any problem.
- At the top of your exam, please write your name, student id, and the following sentence: "I have adhered to all of the above rules.", followed by your signature.
- Good luck!!

Question:	1	2	3	4	5	Total
Points:	20	20	15	20	25	100
Score:						

1. (20 points) Let M be a smooth manifold which is compact. Prove that there is no smooth submersion from M to $\mathbb{R}^n$ for any $n \geq 1$. *Hint: show that such a submersion would necessarily be an open map, and recall that $\mathbb{R}^n$ is connected.*

Solution: Suppose by contradiction that $F : M \to \mathbb{R}^n$ is a smooth submersion, and put $m = \dim(M)$. Note that $\operatorname{im}(F) \subset \mathbb{R}^n$ is compact, since M is compact and F is continuous, and continuous maps send compact subsets to compact subsets. In particular, $\operatorname{im}(F) \subset \mathbb{R}^n$ is closed, since $\mathbb{R}^n$ is Hausdorff and compact subsets of Hausdorff spaces are closed.

We claim that $\operatorname{im}(F) \subset \mathbb{R}^n$ is also open, meaning that $\operatorname{im}(F) \subset \mathbb{R}^n$ is both closed and open. Since $\mathbb{R}^n$ is connected, the only nonempty closed and open subset is $\mathbb{R}^n$ itself, but F cannot be surjective since $\operatorname{im}(F)$ is compact (whereas $\mathbb{R}^n$ for $n \geq 1$ is not compact), so this gives the desired contradiction.

To justify the above claim, we can invoke the Rank Theorem. Given $q \in \operatorname{im}(F)$, we must show that there exists an open neighborhood of q in $\mathbb{R}^n$ which is contained in $\operatorname{im}(F)$. Given any fixed $p \in F^{-1}(\{q\})$, we can find a smooth chart (U, ϕ) for M centered at p and a smooth chart (V, ψ) for $\mathbb{R}^n$ centered at q such that $F(U) \subset V$ and we have

$$\psi \circ F \circ \phi^{-1}|_{\phi(U)}(x^1, \ldots, x^m) = (x^1, \ldots, x^n)$$

(note that $m \geq n$ since F is a submersion). Since $\phi(U) \subset \mathbb{R}^m$ is an open neighborhood of $\phi(p) = 0$, we can find open neighborhoods A, B of 0 in $\mathbb{R}^n, \mathbb{R}^{m-n}$ respectively such that the Cartesian product $A \times B \subset \mathbb{R}^n \times \mathbb{R}^{m-n}$ is an open neighborhood of 0 which is contained in $\phi(U)$. Then $\operatorname{im}(\psi \circ F \circ \phi^{-1}|_{A\times B}) = A$, and therefore we have $\psi^{-1}(A) \subset \operatorname{im}(F)$. Since $\psi^{-1}(A) \subset \mathbb{R}^n$ is an open neighborhood of q contained in $\operatorname{im}(F)$, this justifies the claim.

2. (20 points) Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth map for some $n \geq 1$. Let

$$\mathrm{Gr}(F) := \{(a, b) \in \mathbb{R}^n \times \mathbb{R}^n \mid b = F(a)\} \subset \mathbb{R}^{2n}$$

denote its graph, and let

$$\Delta := \{(a, b) \in \mathbb{R}^n \times \mathbb{R}^n \mid a = b\} \subset \mathbb{R}^{2n}$$

denote the diagonal. Under what conditions on F do $\mathrm{Gr}(F)$ and Δ intersect transversely as submanifolds of $\mathbb{R}^{2n}$?

Solution: We will give the answer for any smooth map $F : M \to M$ where M is a smooth manifold. Firstly, a point $(a, b) \in M \times M$ lies in $\mathrm{Gr}(F) \cap \Delta$ if and only if $b = F(a) = a$. In other words, the intersection points are in bijective correspondence with the fixed points of F.

The tangent space $T_{(a,b)}(M \times M)$ is naturally isomorphic to $T_aM \times T_bM$, and the tangent spaces $T_{(a,b)}\mathrm{Gr}(F)$ and $T_{(a,b)}\Delta$ are naturally viewed as subspaces of $T_{(a,b)}(M \times M)$. To identify these, note that Δ is the image of the smooth embedding $G : M \to M \times M$ defined by $G(p) = (p, p)$, hence $dG_p(v) = (v, v)$, so we have

$$T_{(a,a)}\Delta = \{(v, w) \in T_aM \times T_aM \mid v = w\}.$$

Similarly, $\mathrm{Gr}(F)$ is the image of the smooth embedding $H : M \to M \times M$ defined by $H(p) = (p, F(p))$, hence $dH_p(v) = (v, dF_p(v))$, so we have

$$T_{(a,F(a))}\mathrm{Gr}(F) = \{(v, w) \in T_aM \times T_{F(a)}M \mid w = dF_a(v)\}.$$

An intersection point $(a, b) \in \mathrm{Gr}(F) \cap \Delta$ is transverse if and only if we have

$$T_{(a,b)}\Delta \cap T_{(a,b)}\mathrm{Gr}(F) = \{0\} \subset T_aM \times T_bM.$$

This fails if and only if there is some $v \in T_aM$ such that $v = dF_a v$, i.e. dF_a has 1 as an eigenvalue.

In summary, $\mathrm{Gr}(F)$ and Δ intersect transversely if and only if for each fixed point p of F, $dF_p : T_pM \to T_pM$ does not have 1 as an eigenvalue.

Aside: a fixed point whose differential does not have 1 as an eigenvalue is called nondegenerate*, and these play a fundamental role in fixed point theory and dynamics. A typical goal is to give an a priori lower bound on the number of fixed points of a map $F : M \to M$. The case of all fixed points being nondegenerate is usually the best case scenario for obtaining such a lower bound, whereas degenerate fixed points sometimes ought to count as multiple fixed points, in the same way that the root of $f(x) = x^3$ should really count as three roots.*

3. (15 points) Prove that $\{x^3 - y^3 + xyz - xy = 1\}$ is a smooth submanifold of $\mathbb{R}^3$. Describe the tangent space at the point $(1, 0, 2)$.

Solution: We appeal to the Regular Level Set Theorem. Put $F : \mathbb{R}^3 \to \mathbb{R}$ defined by $F(x, y, z) = x^3 - y^3 + xyz - xy$. Then we have

$$DF_{(x,y,z)} = \begin{pmatrix} 3x^2 + yz - y & -3y^2 + xz - x & xy \end{pmatrix}.$$

The point (x, y, z) is a critical point if and only if this matrix vanishes, which corresponds to the system

$$\begin{cases} 3x^2 + yz - y & = 0 \\ -3y^2 + xz - x & = 0 \\ xy & = 0. \end{cases}$$

For any solution it is easy to see that we must have $x = y = 0$, and then z is unconstrained. We have $F(0,0,z) = 0$, so 0 is a critical value and all other values are regular values. In particular, 1 is a regular value, so $S := F^{-1}(1)$ is a smoothly embedded submanifold of $\mathbb{R}^3$.

Put $p := (1,0,2)$. We describe the tangent space T_pS as a subspace of $T_p\mathbb{R}^3 = \mathbb{R}\langle \frac{\partial}{\partial x}\big|_p, \frac{\partial}{\partial y}\Big|_p, \frac{\partial}{\partial z}\big|_p\rangle$. Recall that we have $T_pS = \ker(dF_p)$. Specializing the above computation to p, we have

$$DF_p = \begin{pmatrix} 3 & 1 & 0 \end{pmatrix},$$

and hence

$$T_pS = \{a \frac{\partial}{\partial x}\Big|_p + b\frac{\partial}{\partial y}\Big|_p + c\frac{\partial}{\partial z}\Big|_p \mid a,b,c \in \mathbb{R},\ 3a + b = 0\}.$$

4. (20 points) Prove that the map $F : \mathbb{RP}^2 \to \mathbb{RP}^5$ given in projective coordinates by

$$F([x : y : z]) = [x^2 : y^2 : z^2 : yz : xz : xy]$$

is a smooth embedding.

Solution: Since $\mathbb{RP}^2$ is compact, it suffices to show that F is a smooth injective immersion, since then it is automatically a smooth embedding. Let us first check that F is an injective map. Suppose that we have

$$[x_1^2 : y_1^2 : z_1^2 : y_1z_1 : x_1z_1 : x_1y_1] = [x_2^2 : y_2^2 : z_2^2 : y_2z_2 : x_2z_2 : x_2y_2],$$

i.e.

$$(x_1^2, y_1^2, z_1^2 : y_1z_1, x_1z_1, x_1y_1) = \lambda \cdot (x_2^2, y_2^2, z_2^2 : y_2z_2, x_2z_2, x_2y_2)$$

for some $\lambda \in \mathbb{R} \setminus \{0\}$. Evidently we must have $\lambda > 0$, and hence we have $x_1 = ax_2\sqrt{\lambda}$, $y_1 = by_2\sqrt{\lambda}$, and $z_1 = cz_2\sqrt{\lambda}$ for some $a, b, c \in \{1, -1\}$.

Suppose first that $x_1 = y_1 = 0$. Then we have $x_2 = y_2 = 0$ and hence $[x_1 : y_1 : z_1] = [0 : 0 : 1] = [x_2 : y_2 : z_2]$. Similarly, if $x_1 = z_1 = 0$ or $y_1 = z_1 = 0$, then we must have $[x_1 : y_1 : z_1] = [x_2 : y_2 : z_2]$.

Now suppose that $x_1 = 0$ but y_1 and z_1 are nonzero. Then y_2 and z_2 are also nonzero and from $y_1/y_2 = \lambda z_2/z_1$ we see that $b = c$, and hence

$$[x_1 : y_1 : z_1] = [0 : y_1 : z_1] = [0 : by_2\sqrt{\lambda} : bz_2\sqrt{\lambda}] = [0 : y_2 : z_2].$$

Similarly, the same is true if we assume y_1 is zero but x_1, z_2 are nonzero, or if z_1 is zero but x_1, y_1 are nonzero.

Finally, suppose that x_1, y_1, z_1 are nonzero. Then the same is true for x_2, y_2, z_2, and we have that x_1/x_2, y_1/y_2, and z_1/z_2 all have the same sign, i.e. $a = b = c$, and hence we have

$$[x_1 : y_1 : z_1] = [ax_2\sqrt{\lambda} : ay_2\sqrt{\lambda} : az_2\sqrt{\lambda}] = [x_2 : y_2 : z_2].$$

To see that F is a smooth immersion, let (U_i, ϕ_i), $i = 1, 2, 3$ denote the standard coordinate charts on $\mathbb{RP}^2$, where U_i consists of all points $[x^1 : x^2 : x^3]$ such that $x^i \neq 0$ and $\phi_i : U_i \to \mathbb{R}^n$ sends $[x^1 : x^2 : x^3]$ to the result after omitting the *ith* coordinate and dividing the remaining coordinates by x^i. Similarly, let (V_i, ψ_i), $i = 1, \ldots, 6$ denote the analogous standard coordinate charts on $\mathbb{RP}^5$.

Observe that $F(U_1) \subset V_1$, and we have $\psi_1 \circ F \circ \phi_1^{-1}(a,b) = \psi_1([1 : a^2 : b^2 : ab : b : a]) = (a^2, b^2, ab, b, a)$. This is a polynomial function, hence smooth, and its Jacobian is the transpose of

$$\begin{pmatrix} 2a & 0 & b & 0 & 1 \\ 0 & 2b & a & 1 & 0 \end{pmatrix},$$

which has rank two for any $a, b \in \mathbb{R}$, hence $F|_{U_1}$ is an immersion. Note that if we permute the components of $[x : y : z]$, then $F([x : y : z])$ changes only by permuting its components, i.e. by postcomposing by a diffeomorphism of $\mathbb{RP}^5$. Since the latter does not affect whether a map is an immersion, this shows that $F|_{U_2}$ and $F|_{U_3}$ are also immersions, and hence F itself is an immersion.

Note: this map F is called the Veronese embedding*. It can be generalized to an embedding of any $\mathbb{RP}^m$ into $\mathbb{RP}^n$ for some n, where the components are all possible monomials of a given degree d.*

5. (25 points) Let M be a smoothly embedded m-dimensional submanifold of $\mathbb{R}^n$ for some $1 \leq m < n$. For each $d < n - m$, prove that there exists a d-dimensional affine subspace of $\mathbb{R}^n$ which is disjoint from M. Is the same true if $d = n - m$?

Solution: We consider affine subspaces S_c of $\mathbb{R}^n$ of the form $\mathbb{R}^d \times \{c\}$ for $c \in \mathbb{R}^{n-d}$. Then S_c intersects M if and only if c lies in the image of the projection map $\pi : M \to \mathbb{R}^{n-d}$ onto the last $n-d$ components. By assumption $n - d > m$, and hence every point in $\mathbb{R}^{n-d}$ is a critical value of π, and hence by Sard's theorem the image of π has measure zero. In particular, π is not surjective, so we can indeed find $c \in \mathbb{R}^{n-d} \setminus \operatorname{im}(\pi)$.

In the case $d = n-m$, this argument does not go through. For example in the case $n = 2$ and $d = m = 1$, consider the image M of the smooth embedding $\mathbb{R} \to \mathbb{R}^2$, $t \mapsto (e^t \cos(t), e^t \sin(t))$. M is a spiral, and it intersects every affine line in $\mathbb{R}^2$. Indeed, let $L \subset \mathbb{R}^2$ be some affine line, and let $v \in \mathbb{R}^2 \setminus \{0\}$ be a vector pointing in a direction orthogonal to L. We can $t_1 \in \mathbb{R}$ such that $(e^{t_1} \cos(t_1), e^{t_1} \sin(t_1))$ is an arbitrarily large positive multiple of v, and we can also find t_2 such that $(e^{t_2} \cos(t_2), e^{t_2} \sin(t_2))$ is an arbitrarily large negative multiple of v. Using the Intermediate Value Theorem applied to the projection to along L, it follows that M must intersect L.