Lecture 15

Airy's equation : $y'' = ty$

Note: 2nd order, linear, but not constant coefficients.

Ansatz: $y(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + \dots$

(a function admitting a convergent power series is called "analytic")

$$y(t) = \sum_{k=0}^{\infty} a_k t^k$$

$$y'(t) = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + \dots$$

$$= \sum_{k=1}^{\infty} k a_k t^{k-1}$$

$$y''(t) = 2a_2 + 3\cdot 2 a_3 t + 4\cdot 3 a_4 t^2 + 5\cdot 4 a_5 t^3 + \dots$$

∞

$$= \sum_{k=2} k(k-1)t^{k-2}$$

Plug in:

need: $$\sum_{k=2}^{\infty} k(k-1)a_k t^{k-2} - t\sum_{k=0}^{\infty} a_k t^k = 0$$

for all t.

these are both power series, but terms are mismatched.

Goal: write LHS as a single power series $\sum_{k=0} (\quad) t^k$, then set all coeffs to be zero.

Method 1: Keeps things in long form, guess pattern

Method 2: Stick with summand notation

M1: Have $2\cdot 1\cdot a_2 + 3\cdot 2\cdot a_3 t + 4\cdot 3\cdot a_4 t^2 + 5\cdot 4 a_5 t^3 + \ldots$

$$- t(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \ldots) = 0$$

so need

$$(2a_2) + \begin{pmatrix} 3\cdot 2\cdot a_3 \\ -a_0 \end{pmatrix} t + \begin{pmatrix} 4\cdot 3 a_4 \\ -a_1 \end{pmatrix} t^2$$

$$+ (5\cdot 4 a_5 \quad) t^3 +$$

$$+\left(-a_2\right)t+\dots$$

$$+\left((k+2)(k+1)a_{k+2}-a_{k-1}\right)t^k+\dots=0$$

"the general term"

So need:

$$2a_2=0$$
$$3\cdot 2a_3-a_0=0$$
$$4\cdot 3a_4-a_1=0$$
$$5\cdot 4a_5-a_2=0$$
$$\dots$$

System of ∞ly many eqns and ∞ly many unknowns

will solve recursively.

M2: Need

$$\sum_{k=2}^{\infty} k(k-1)a_k t^{k-2} - t\sum_{k=0}^{\infty} a_k t^k = 0$$

$$t\sum_{k=0}^{\infty} a_k t^k = \sum_{k=0}^{\infty} a_k t^{k+1}$$

$$= \sum_{k=1}^{\infty}\left(a_{k-1}\right)t^k$$

$$\sum_{k=2}^{\infty} k(k-1)a_k t^{k-2} = \sum_{k=0}^{\infty}(??)t^k$$

Put $j=k-2$. Then $k=j+2$, so

get $\sum_{j=0}^{\infty}(j+2)(j+1)a_{j+2}t^j = \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}t^k$

when $k=2$, $j=0$

when $k=\infty$, $j=\infty$

Need:

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} t^k - \sum_{k=1}^{\infty} a_{k-1} t^k = 0$$

for all t

Can write as

$$2\cdot 1\cdot a_2 + \sum_{k=1}^{\infty} \left((k+2)(k+1) a_{k+2} - a_{k-1} \right) t^k = 0$$

Set all coeffs to be zero:

$$2a_2 = 0$$

$$(k+2)(k+1) a_{k+2} - a_{k-1} = 0$$

for $k = 1, 2, 3, \dots$

So, need:

$$2a_2 = 0$$

$$3\cdot 2\cdot a_3 - a_0 = 0$$

$$4\cdot 3 a_4 - a_1 = 0$$

$$5\cdot 4 a_5 - a_2 = 0$$

$$\dots$$

Solve recursively:

$$a_2 = 0 \qquad a_5 = \frac{a_2}{5\cdot 4} = 0$$

$$a_3 = \frac{a_0}{3\cdot 2} \qquad a_6 = \frac{a_3}{\dots} = \frac{a_0}{\dots}$$

$$\frac{\quad}{6\cdot 5} = \frac{\quad}{6\cdot 5\cdot 3\cdot 2}$$

$$a_4 = \frac{a_1}{4\cdot 3} \qquad a_7 = \frac{a_4}{7\cdot 6} = \frac{a_1}{7\cdot 6\cdot 4\cdot 3}$$

$\dots$

(1) $a_2 = a_5 = a_8 = a_{11} = \dots = 0$

(2) $a_3 = \dfrac{a_0}{3\cdot 2}, \quad a_6 = \dfrac{a_0}{6\cdot 5\cdot 3\cdot 2},$

$$a_9 = \frac{a_0}{9\cdot 8\cdot 6\cdot 5\cdot 3\cdot 2}, \dots, \quad a_{3k} = \frac{a_0}{2\cdot 3\cdot 5\cdot 6\cdots (3k-1)(3k)}$$

(3) $a_4 = \dfrac{a_1}{4\cdot 3}, \quad a_7 = \dfrac{a_4}{7\cdot 6} = \dfrac{a_1}{7\cdot 6\cdot 4\cdot 3},$

$$a_{10} = \frac{a_{10}}{10\cdot 9} = \frac{a_1}{10\cdot 9\cdot 7\cdot 6\cdot 4\cdot 3}$$

$$a_{3k+1} = \frac{a_1}{3\cdot 4\cdot 6\cdot 7\cdots (3k)(3k+1)}$$

So

$$y(t) = \sum_{k=0}^{\infty} a_k t^k$$

$$= a_0 + \frac{a_0 t^3}{2\cdot 3} + \frac{a_0 t^6}{2\cdot 3\cdot 5\cdot 6} + \frac{a_0 t^9}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9} + \dots$$

$$+ a_1 t + \frac{a_1}{3\cdot 4} t^4 + \frac{a_1}{3\cdot 4\cdot 6\cdot 7} t^7 + \dots$$

$$= a_0 \underbrace{\left(1 + \frac{t^3}{2\cdot 3} + \frac{t^6}{2\cdot 3\cdot 5\cdot 6} + \dots\right)}_{y_1(t)}$$

$$+ a_1 \underbrace{\left(t + \frac{t^4}{3\cdot 4} + \frac{t^7}{3\cdot 4\cdot 6\cdot 7} + \dots \right)}_{y_2(t)}$$

So far: $a_0 y_1(t) + a_1 y_2(t)$ solves Airy's eqn for any choice of a_0, a_1.

Questions:

(1) What are radii of convergence of y_1 and y_2?

(2) Do y_1 and y_2 form a fund set of solutions?

(3) What are these functions?? What do y_1, y_2 look like?

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(1) What is the r.o.c. of y_1?

Recall: for any power series

$$\sum_{k=0}^{\infty} a_k (t - t_c)^k \quad (\text{centered at } t_c)$$

there's a number $R \in [0, \infty]$, the "radius of conv.", such that

- for $|t - t_c| < R$, then the series

converges (absolutely)

- for $|t-t_c|>R$, then the series diverges

Ratio test: If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ exists, $R=\frac{1}{\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|}$

(if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=0$, then $R=\infty$)

Let's apply ratio test to

$$y_1(t)=1+\frac{t^3}{2\cdot 3}+\frac{t^6}{2\cdot 3\cdot 5\cdot 6}+\frac{t^9}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9}+\dots$$

Strictly speaking this is $1+0\cdot t+0\cdot t^2+\frac{t^3}{2\cdot 3}+0t^4+\dots$

Slight problem: what is a_1/a_0? It's 0.

$$\frac{a_3}{a_2}=\frac{(1/6)}{0}$$

In fact, $\frac{a_{n+1}}{a_n}$ oscillates b/w 0 and ∞, so $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ is not defined...

Fix: Put $s=t^3$.

Then $y_1(s)=1+\frac{s}{2\cdot 3}+\frac{s^2}{2\cdot 3\cdot 5\cdot 6}+\frac{s^3}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9}$

$+\dots$

$$= \sum_{k=0}^{\infty} b_n s^n,$$

Then $\frac{b_{k+1}}{b_k} = \frac{2\cdot 3\cdot 5\cdot 6 \dots (3k-1)(3k)}{2\cdot 3\cdot 5\cdot 6\dots(3k-1)(3k)(3k+2)(3k+3)}$

$$= \frac{1}{(3k+2)(3k+3)}$$

So $\lim_{k\to\infty} \left|\frac{b_{k+1}}{b_k}\right| = 0.$

So y_1 converges for $|s| < \infty$,

i.e. $|t^3| < \infty$, i.e. $|t| < \infty$,

i.e. converges for all t.

So r.o.c. of $y_1(t)$ is ∞. ✓

Similarly, can show that r.o.c. of y_2 is ∞.

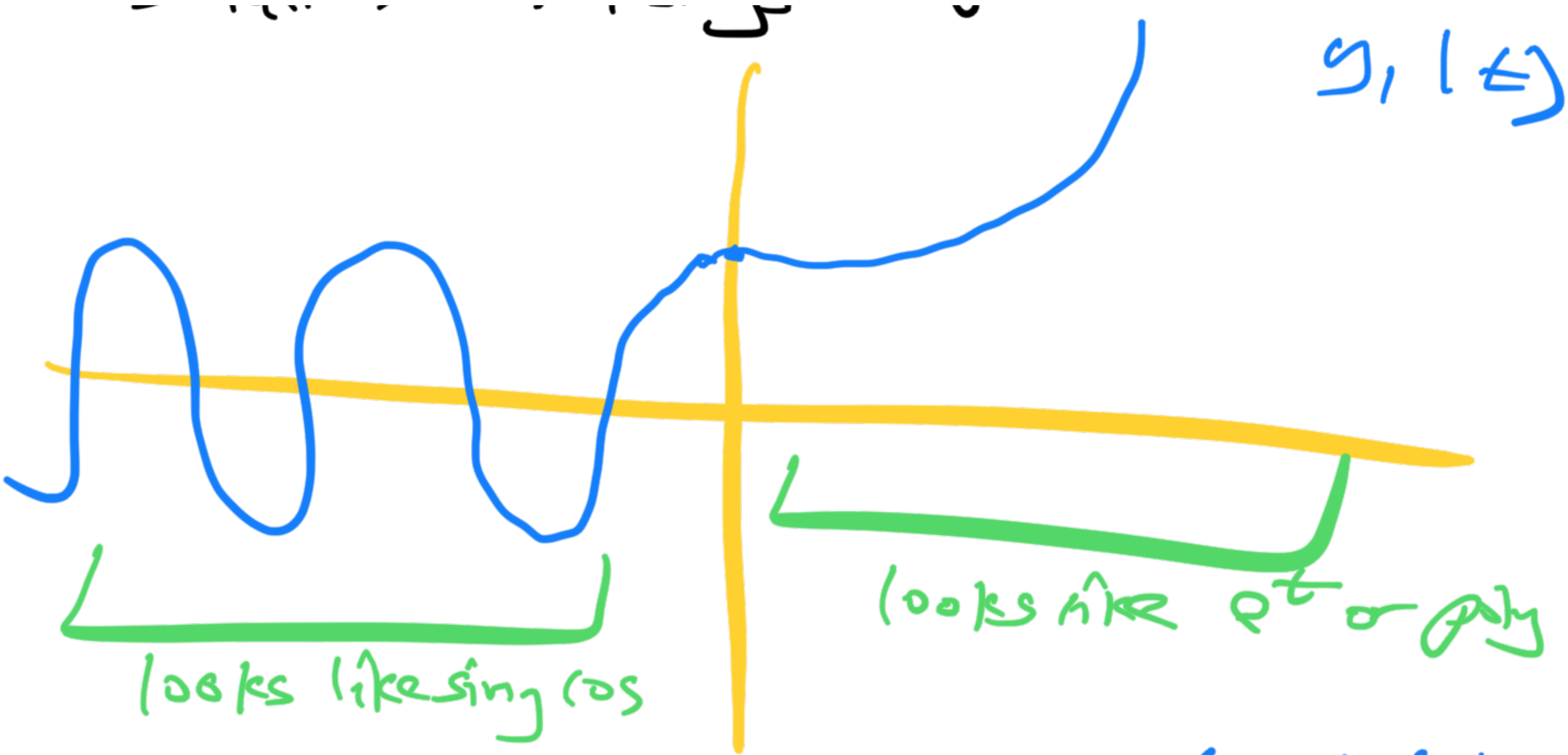
(2) Let's compute $W(y_1, y_2)(0)$.

$y_1(0) = 1$ $\quad$ $y_2(0) = 0$

$y_1'(0) = 0$ $\quad$ $y_2'(0) = 1$

$$W(y_1,y_2)(0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

Conclusion: y_1, y_2 form a fund. set of solutions to Airy's eqn.



"Airy function"

In practice, can approximate y_1, y_2 by looking at first 1000 terms.

$$a_{3k} = \frac{1}{2\cdot 3\cdot 5\cdot 6\cdots(3k-1)(3k)}$$

$$= \frac{4\cdot 7\cdot 10\cdots\cdots}{(3k)!}$$