Lecture 24

Today: • more examples $\vec{x}' = A\vec{x}$
• repeated evals, not enough evecs
• borderline cases: 0 being an eval

Bottom line: $A = 2\times 2$ matrix, λ = rep'd eval

Two possibilities:

(a) nevertheless have 2 lin. ind. evecs w/ eval λ

(b) might only evec w/ eval λ
Then need to find another sol'n.

Prototypical ex

$$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

prototypical ex

$$\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$$

$\underline{\text{Ex}}$: $\vec{x}'(t) = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \vec{x}(t)$

general sol'n:
$\vec{x}(t) = C_1 e^{3t} \begin{pmatrix} 17 \\ 36 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} \pi \\ 5\pi \end{pmatrix}$

evals: $\begin{vmatrix} 3-\lambda & 0 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 \rightsquigarrow \lambda_1 = \lambda_2 = 3$

evecs: $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 3 \begin{pmatrix} a \\ b \end{pmatrix} \Longleftrightarrow \begin{cases} 3a = 3a \\ 3b = 3b \end{cases}$

any $\begin{pmatrix} a \\ b \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is an evec w/ eval 3.

$\underline{\text{Ex}}$: $\vec{x}'(t) = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} \vec{x}(t)$ evals: $\begin{vmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^2$

$\lambda_1 = \lambda_2 = 3$

evecs:

$\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 3 \begin{pmatrix} a \\ b \end{pmatrix} \Longleftrightarrow \begin{cases} 3a + b = 3a \\ 3b = 3b \end{cases} \Longrightarrow b = 0$

$\rightsquigarrow$ so $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an evec w/ eval 3

How to find another soln to system?

$$\vec{x}'(t) = \begin{pmatrix}3 & 1\\ 0 & 3\end{pmatrix}\vec{x}(t) \iff \begin{cases} x_1'(t) = 3x_1(t) + x_2(t) \\ x_2'(t) = 3x_2(t) \end{cases}$$

Firstly, have our soln : $(x_1(t), x_2(t)) = (e^{3t}, 0)$

Now $x_2'(t) = 3x_2(t) \Rightarrow x_2(t) = Ce^{3t}$. Let's take $C = 1$.

Then 1st eqn $\rightsquigarrow$ $x_1'(t) = 3x_1(t) + e^{3t}$. ← use int. factor, or method of undet. coeffs

Soln is $x_1(t) = Ate^{3t}$

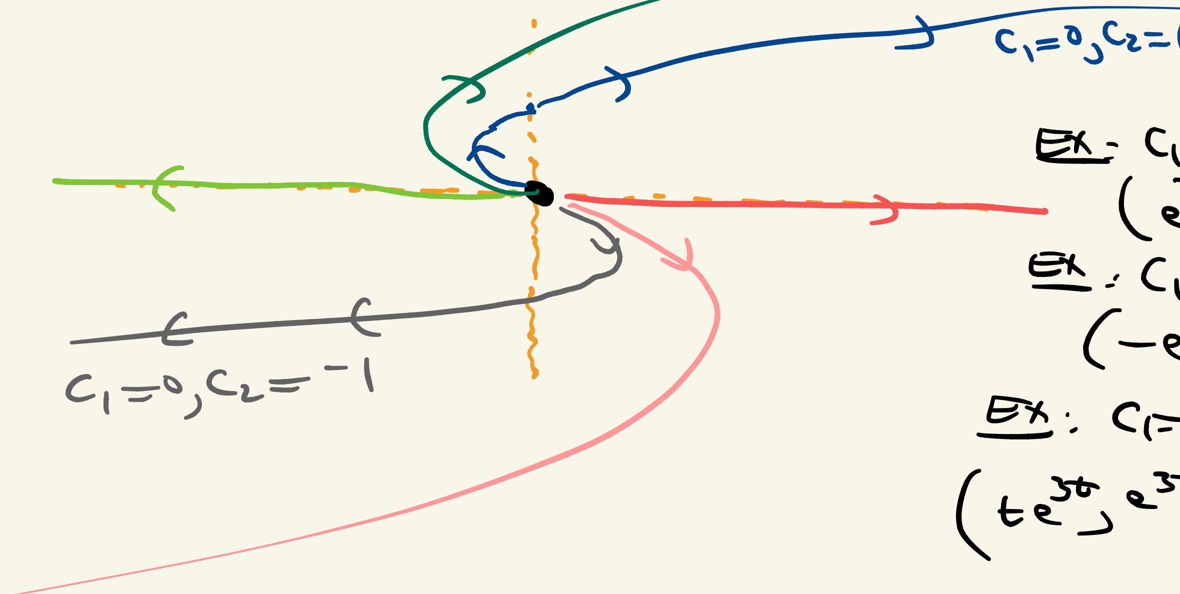
Then $Ae^{3t} + \cancel{At3e^{3t}} = \cancel{3Ate^{3t}} + e^{3t} \Rightarrow A = 1$

So $x_1(t) = te^{3t}$ is a soln.

So $(x_1(t), x_2(t)) = (te^{3t}, e^{3t}) = e^{3t}\left(t\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\right)$

General soln: $C_1 e^{3t}\begin{pmatrix}1\\0\end{pmatrix} + C_2 e^{3t}\left(t\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\right)$.

$= e^{3t}\begin{pmatrix}t\\1\end{pmatrix}$



$$\begin{pmatrix} te^{3t} \\ e^{3t} \end{pmatrix}$$

Ex: $c_1=1,\ c_2=0$

$$(e^{3t}, 0)$$

Ex: $c_1=-1,\ c_2=0$

$$(-e^{3t}, 0)$$

Ex: $c_1=0,\ c_2=1$

$$(te^{3t}, e^{3t}) = e^{3t}\cdot(t,1)$$

Ex: $\vec{x}\,'(t) = \begin{pmatrix} -3 & 5/2 \\ -5/2 & 2 \end{pmatrix}\vec{x}(t)$

evals: $\begin{vmatrix} -3-\lambda & 5/2 \\ -5/2 & 2-\lambda \end{vmatrix}$

$$= (\lambda+3)(\lambda-2) + \frac{25}{4}$$
$$= \lambda^2 + \lambda - 6 + 25/4$$
$$= \lambda^2 + \lambda + 1/4$$
$$= (\lambda + 1/2)^2 \qquad \lambda_1 = \lambda_2 = -1/2$$

evecs:

$$\begin{pmatrix} -3 & 5/2 \\ -5/2 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -1/2\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{cases} -3a + \frac{5b}{2} = -a/2 \\ -5/2\,a + 2b = -b/2 \end{cases} \longrightarrow \begin{cases} 5b/2 = 5a/2 \\ 5b/2 = 5a/2 \end{cases} \Rightarrow b = a$$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Note: rep'd eval, only one evec.

One soln: $e^{-t/2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Second solution: ansatz $\vec{x}(t) = e^{-t/2}\left(t\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix}\right)$

$$\vec{x}'(t) = \left(e^{-t/2} - \frac{1}{2} t e^{-t/2}\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2} e^{-t/2}\begin{pmatrix} c \\ d \end{pmatrix} \qquad = t e^{-t/2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-t/2}\begin{pmatrix} c \\ d \end{pmatrix}$$

Need: $e^{-t/2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2} t e^{-t/2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2} e^{-t/2}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} -3 & 5/2 \\ -5/2 & 2 \end{pmatrix}\begin{pmatrix} t e^{-t/2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ + e^{-t/2}\begin{pmatrix} c \\ d \end{pmatrix} \end{pmatrix}$

Cancel all $e^{-t/2}$ terms:

$$\longrightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{t}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} -3 & 5/2 \\ -5/2 & 2 \end{pmatrix}\left(t\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix}\right)$$

$$\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} c \\ d \end{pmatrix}\right] - \frac{t}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 & 5/2 \\ -5/2 & 2 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} + t\begin{pmatrix} -3 & 5/2 \\ -5/2 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

This is supposed to be for all t, so must have: $(*)$

$$\left[\begin{pmatrix}1\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}c\\d\end{pmatrix}\right] = \begin{pmatrix}-3 & s/2\\ -s/2 & 2\end{pmatrix}\begin{pmatrix}c\\d\end{pmatrix}$$

and $-\frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-3 & s/2\\ -s/2 & 2\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}$ ✓ $(**)$

$(**)$ holds since $\begin{pmatrix}1\\1\end{pmatrix}$ is evec w/ eval $-1/2$

So we just need to choose $\begin{pmatrix}c\\d\end{pmatrix}$ s.t. $(*)$ holds.

$$(*) \iff \begin{pmatrix}1\\1\end{pmatrix} = \left(\begin{pmatrix}-3 & s/2\\ -s/2 & 2\end{pmatrix} - \left(-\frac{1}{2}\right)\mathbb{I}\right)\begin{pmatrix}c\\d\end{pmatrix}$$

$$\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-s/2 & s/2\\ -s/2 & s/2\end{pmatrix}\begin{pmatrix}c\\d\end{pmatrix}$$

$$\iff \begin{cases} -s/2\, c + s/2\, d = 1 \\ -s/2\, c + s/2\, d = 1 \end{cases}$$

$\Longrightarrow \frac{s}{2} d = 1 + s/2\, c$

$\Rightarrow d = c + \frac{2}{s}$

so can take $\begin{pmatrix}c\\ c+2/s\end{pmatrix}$ for any c

So: second soln is: $e^{-t/2}\left(t\begin{pmatrix}1\\1\end{pmatrix}+\begin{pmatrix}c\\c+2/5\end{pmatrix}\right)$

for any choice of c.

$$= e^{-t/2}\left(t\begin{pmatrix}1\\1\end{pmatrix}+\begin{pmatrix}0\\2/5\end{pmatrix}\right)+\underbrace{e^{-t/2}\cdot c\cdot\begin{pmatrix}1\\1\end{pmatrix}}_{\text{this is just our first soln, so might as well take } c=0}$$

So second soln becomes

$$e^{-t/2}\left(t\begin{pmatrix}1\\1\end{pmatrix}+\begin{pmatrix}0\\2/5\end{pmatrix}\right).$$

General soln:

$$\vec{x}(t)=C_1e^{-t/2}\begin{pmatrix}1\\1\end{pmatrix}+C_2e^{-t/2}\left(t\begin{pmatrix}1\\1\end{pmatrix}+\underbrace{\begin{pmatrix}0\\2/5\end{pmatrix}}_{\text{"generalized eigenvector"}}\right)$$

Ex: $\vec{x}\,'(t)=\begin{pmatrix}0&-3\\1&0\end{pmatrix}\vec{x}(t)$

evals: $\begin{vmatrix}-\lambda&-3\\1&-\lambda\end{vmatrix}=\lambda^2+3$

$\lambda_1=i\sqrt{3},\quad \lambda_2=-i\sqrt{3}$

Questions:
- are there evals?
- are any of them cpx?
- are any of evals 0?

<u>evecs</u>:

T_1: $\begin{pmatrix} 0 & -3 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = i\sqrt{3}\begin{pmatrix} a \\ b \end{pmatrix} \rightsquigarrow \begin{cases} -3b = i\sqrt{3}\,a \\ a = i\sqrt{3}\,b \end{cases}$

$\longrightarrow b = \frac{-i}{\sqrt{3}} a$ $\quad \begin{pmatrix} \sqrt{3} \\ -i \end{pmatrix}$ evec w/ eval $i\sqrt{3}$

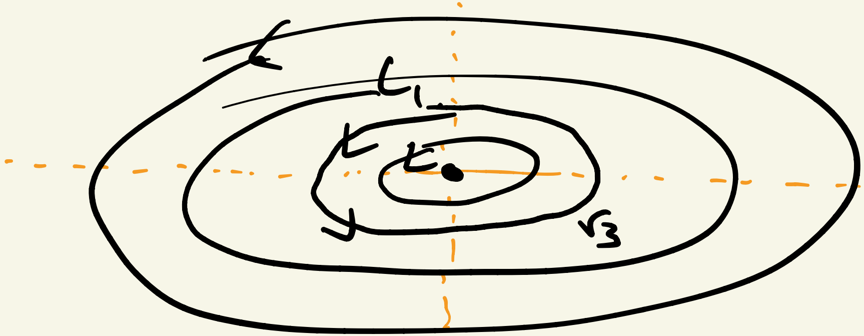
T_2: $\begin{pmatrix} \sqrt{3} \\ i \end{pmatrix}$ evec w/ eval $-i\sqrt{3}$

$$e^{-i\sqrt{3}t}\begin{pmatrix} \sqrt{3} \\ i \end{pmatrix} \longrightarrow \left(\cos(-\sqrt{3}t) + i\sin(-\sqrt{3}t)\right)\begin{pmatrix} \sqrt{3} \\ i \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} \sqrt{3}\cos(-\sqrt{3}t) \\ -\sin(-\sqrt{3}t) \end{pmatrix}}_{\text{real part}} + i \underbrace{\begin{pmatrix} \sqrt{3}\sin(-\sqrt{3}t) \\ -\sin(-\sqrt{3}t) \end{pmatrix}}_{\text{imag part}}$$

So general soln is

$$c_1\begin{pmatrix} \sqrt{3}\cos(\sqrt{3}t) \\ \sin(\sqrt{3}t) \end{pmatrix} + c_2\begin{pmatrix} -\sqrt{3}\sin(-\sqrt{3}t) \\ \sin(\sqrt{3}t \end{pmatrix}$$



Ex: $c_1 = 1$, $c_2 = 0$

$$(x(t), y(t)) = (\sqrt{3}\cos(\sqrt{3}t), \sin(\sqrt{3}t))$$

lies on equation

$$\frac{x^2}{3} + y^2 = 1$$