Math 635: Cluster Varieties

Algebra, Topology, Geometry, Duality

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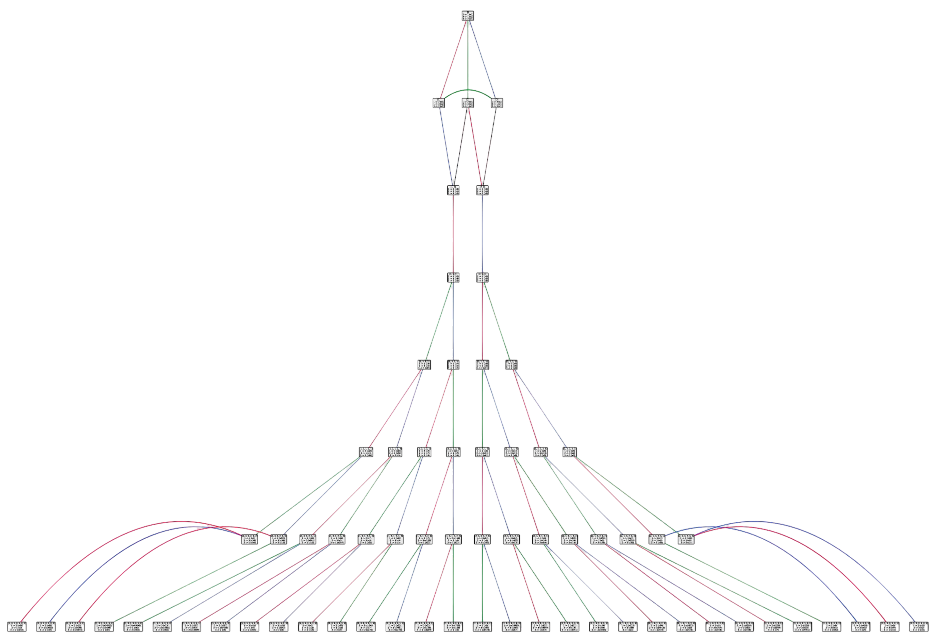

Disclaimer: These notes are based on handwritten lecture notes which were typeset and lightly edited with AI assistance. This typesetting process is not perfect and could have introduced some errors.

Contents

1 Lecture 1

Date: January 12, 2026
Main reference: [FWZ21], §1–2.

1.1 Introduction

Roughly speaking:

- A **cluster variety** is a complex algebraic variety obtained by gluing together many copies of $(\mathbb{C}^*)^n$, where the gluing maps take a very particular form.
- A **cluster algebra** is the algebra of regular functions $f\colon V \to \mathbb{C}$ on a cluster variety.

Fomin–Zelevinsky, early 2000s: Introduced cluster algebras. They arise in many parts of mathematics and physics as a kind of "universal model" for mutation/wall-crossing phenomena:

- Quiver representation theory
- Teichmüller theory
- Poisson geometry
- Grassmannians
- Total positivity
- QFT scattering amplitudes (amplituhedron)
- Integrable systems
- String theory (BPS states)
- etc.

Gross–Hacking–Keel–Kontsevich (GHKK) [Gro+18]:

- Constructed canonical bases for cluster algebras.
- Established positivity of the Laurent phenomenon.
- Proof uses mirror symmetry for log Calabi–Yau varieties (which can be thought of as a generalization of toric varieties, related to almost toric fibrations in symplectic geometry).
- Many strong applications in representation theory, e.g., canonical bases for finite-dimensional irreducible representations of $\mathrm{SL}_n(\mathbb{C})$.

Remark 1.1. The canonical bases were originally found independently by Lusztig and Kashiwara in the early 1990s using quantum groups. Amazingly, the construction of GHKK uses only general geometry—no representation theory!

1.2 Total Positivity

Definition 1.2. A matrix $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ is **totally positive** (TP) if all of its minors are positive.

Gantmacher–Krein (1930s): If A is TP, then the eigenvalues of A are real, positive, and distinct.

Binet–Cauchy theorem: The TP matrices are closed under multiplication, and hence form a multiplicative semigroup $G_{>0}$.
Lusztig: Extended the definition of $G_{>0}$ to other semisimple Lie groups G.

More generally: If a given complex algebraic variety Z has a distinguished family Δ of regular functions $Z \to \mathbb{C}$, we define the **TP variety** by

$$Z_{>0} := \{z \in Z \mid f(z) > 0 \text{ for all } f \in \Delta\}.$$

Example 1.3. For $Z = \mathrm{Mat}_{n\times n}(\mathbb{C})$, $\mathrm{GL}_n(\mathbb{C})$, or $\mathrm{SL}_n(\mathbb{C})$, we recover the above notion of TP, where $\Delta = \{\text{minors}\}$.

Example 1.4. The **Grassmannian** $\mathrm{Gr}_{k,m}(\mathbb{C}) = \{k\text{-dimensional linear subspaces of } \mathbb{C}^m\}$, with $\Delta = \{\text{Plücker coordinates}\}$.

Example 1.5. Partial flag manifolds, homogeneous spaces for semisimple complex Lie groups, etc. (slight scaling ambiguity).

Lemma 1.6. *A matrix $A \in \mathrm{Mat}_{n\times n}$ has $\binom{2n}{n} - 1$ minors.*

Proof. The number of minors is

$$\# = \sum_{k=1}^{n} \binom{n}{k}^2.$$

By Vandermonde's identity:

$$\binom{m+w}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{w}{r-k}.$$

Setting $m = w = r = n$ gives

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2,$$

from which the result follows. $\square$

Remark 1.7. To verify Vandermonde's identity, note that both sides count the number of subcommittees with r members, given a committee with m men and w women.

Question 1.8. Can we check that $A \in \mathrm{Mat}_{n\times n}$ is TP by only testing a subset of the $\binom{2n}{n} - 1$ minors? How many tests are needed?

Example 1.9. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_{2\times 2}$. Define $\delta := ad - bc$, so $d = \frac{\delta + bc}{a}$. Thus, if $a, b, c, \delta > 0$, then d is automatically positive. This reduces $\binom{4}{2} - 1 = 5$ checks to 4 checks.

The goal is "efficient TP testing."

1.3 Plücker Coordinates on Grassmannians

Given $A \in \mathrm{Mat}_{k\times m}$ of rank k, we have $\mathrm{rowspan}(A) =: [A] \in \mathrm{Gr}_{k,m}$.

For $J \subseteq \{1, \dots, m\}$ with $|J| = k$, the **Plücker coordinate** is

$$P_J(A) := k \times k \text{ minor of } A \text{ corresponding to columns } J.$$

Note 1.10. For $A, B \in \mathrm{Mat}_{k\times m}$ with $[A] = [B]$ (i.e., same row spans), the tuples $(P_J(A))_{|J|=k}$ and $(P_J(B))_{|J|=k}$ agree up to common rescaling. We thus get a map

$$\mathrm{Gr}_{k,m} \longrightarrow \mathbb{CP}^{N-1}, \quad N = \binom{m}{k}.$$

In fact, this is an embedding, called the **Plücker embedding**.

Let $\mathbb{C}[\mathrm{Mat}_{k\times m}]$ denote the coordinate ring of $\mathrm{Mat}_{k\times m}$, i.e., the polynomial algebra in variables x_{ij} for $1 \le i \le k$, $1 \le j \le m$.

Definition 1.11. The **Plücker ring** $R_{k,m}$ is the subring of $\mathbb{C}[\mathrm{Mat}_{k\times m}]$ generated by P_J over all $J \in \{1, \dots, m\}$ with $|J| = k$.

Claim 1.12. *The ideal of relations in $R_{k,m}$ is generated by certain quadratic relations called the* ***Grassmann–Plücker relations****.*

Definition 1.13. The **totally positive Grassmannian** $\mathrm{Gr}^+_{k,m}$ is the subset of $\mathrm{Gr}_{k,m}$ consisting of those points whose Plücker coordinates are all positive (up to common scaling).

Note 1.14. For $A \in \mathrm{Mat}_{k\times m}(\mathbb{R})$, we have $[A] \in \mathrm{Gr}^+_{k,m}$ if and only if all $k \times k$ minors of A have the same sign.

Question 1.15. For $A \in \mathrm{Mat}_{k\times m}(\mathbb{R})$, can we verify that all $k \times k$ minors are positive by only checking a subset of the $\binom{m}{k}$ minors? How many tests are needed?

(We may assume positive WLOG by rescaling.)

1.4 Positivity Testing for $\mathrm{Gr}_{2,m}$

Claim 1.16. *Given $A \in \mathrm{Mat}_{2\times m}$, put $P_{ij} := P_{\{i,j\}}$ for $1 \le i < j \le m$. To check that all 2×2 minors $P_{ij}(A) > 0$, it suffices to check only the $2m - 3$ special ones.*

Note 1.17. $2m - 3 = \dim \mathrm{Gr}_{2,m} + 1$.

Lemma 1.18. *For $1 \le i < j < k < \ell \le m$, we have the three-term Grassmann–Plücker relation:*

$$P_{ik}P_{j\ell} = P_{ij}P_{k\ell} + P_{i\ell}P_{jk}.$$

Remark 1.19. For an inscribed quadrilateral (Figure 1), Ptolemy's theorem (2nd century) gives

$$AC \cdot BD = AB \cdot CD + BC \cdot AD.$$

Example 1.20. Let $A = \begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix}$. We verify $P_{13}P_{24} = P_{12}P_{34} + P_{14}P_{23}$, i.e.,

$$(ag - ce)(bh - df) = (af - be)(ch - dg) + (ah - de)(bg - cf). \quad \checkmark$$

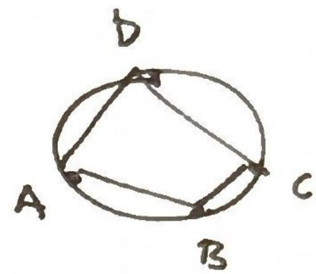

Figure 1: Inscribed quadrilateral for Ptolemy's theorem.

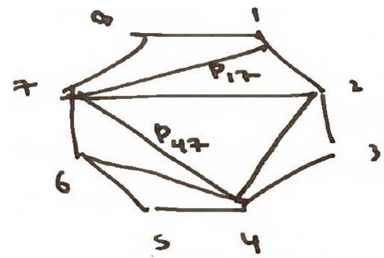

Figure 2: A triangulated polygon $\mathbb{P}_m$ with vertices labeled $1, \ldots, m$.

Put $\mathbb{P}_m$ = regular m-gon, and let T be a triangulation.
To each side or diagonal, associate P_{ij}, where i, j are the endpoints.

- **Cluster variables**: P_{ij} ranging over diagonals.
- **Frozen variables**: P_{ij} ranging over sides.
- **Extended cluster**: $\{\text{cluster vars}\} \cup \{\text{frozen vars}\} =: \widetilde{x}(T)$.

Note 1.21. The extended cluster has $2m - 3$ variables, and we claim that these are algebraically independent.

Example 1.22. In Figure 2, we have cluster variables $P_{17}, P_{27}, P_{47}, P_{24}$ and frozen variables $P_{12}, P_{23}, \ldots, P_{78}, P_{18}$.

Theorem 1.23. *Each P_{ij} for $1 \leq i < j \leq n$ can be written as a subtraction-free rational expression in the elements of a given extended cluster $\widetilde{x}(T)$.*

Corollary 1.24. *If each $P_{ij} \in \widetilde{x}(T)$ evaluates positively on a given $A \in \mathrm{Mat}_{2\times m}$, then all of the $2m - 3$ of the $\binom{m}{2}$ minors of A are positive.*

Proof of Theorem. Follows by combining:

(1) Each P_{ij} appears as an element of an extended cluster $\widetilde{x}(T)$ for some triangulation T of $\mathbb{P}_m$.

(2) Any two triangulations of $\mathbb{P}_m$ are related by a sequence of **flips** (see Figure 3).

(3) For a flip, replace P_{ik} with $P_{j\ell}$. Using the three-term GP relation, we have

$$P_{ik} = \frac{P_{ij}P_{k\ell} + P_{i\ell}P_{jk}}{P_{j\ell}}.$$

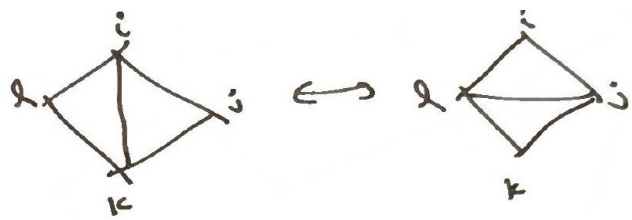

Figure 3: A flip replaces one diagonal with another in a quadrilateral.

Remark 1.25. In fact, each Plücker coordinate P_{ij} can be written as a Laurent polynomial with positive coefficients in the Plücker coordinates from $\widetilde{x}(T)$. This is an example of the **positive Laurent phenomenon**.

The combinatorics of flips is encoded by a graph:

- Vertices are triangulations.
- Edges are flips.

Each vertex has degree $m-3$. In fact, this is the 1-skeleton of an $(m-3)$-dimensional convex polytope called the **associahedron** (discovered by Stasheff); see Figures 4 and 5.

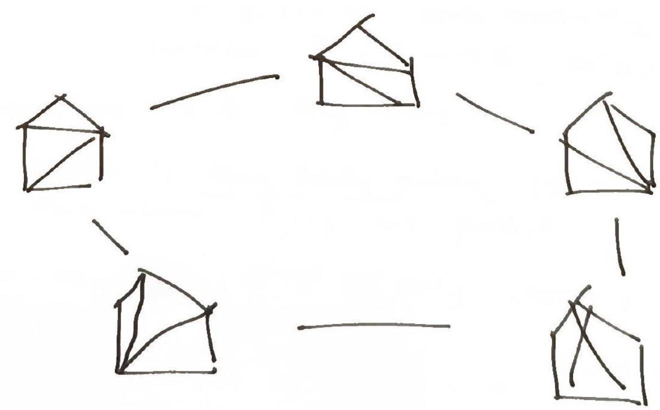

Figure 4: The associahedron for $m = 5$ (a pentagon).

Definition 1.26. A **cluster monomial** is a monomial in the variables of a given extended cluster $\widetilde{x}(T)$.

Theorem 1.27 (19th century invariant theory)**.** *The set of all cluster monomials gives a linear basis for the Plücker ring $R_{2,m}$.*

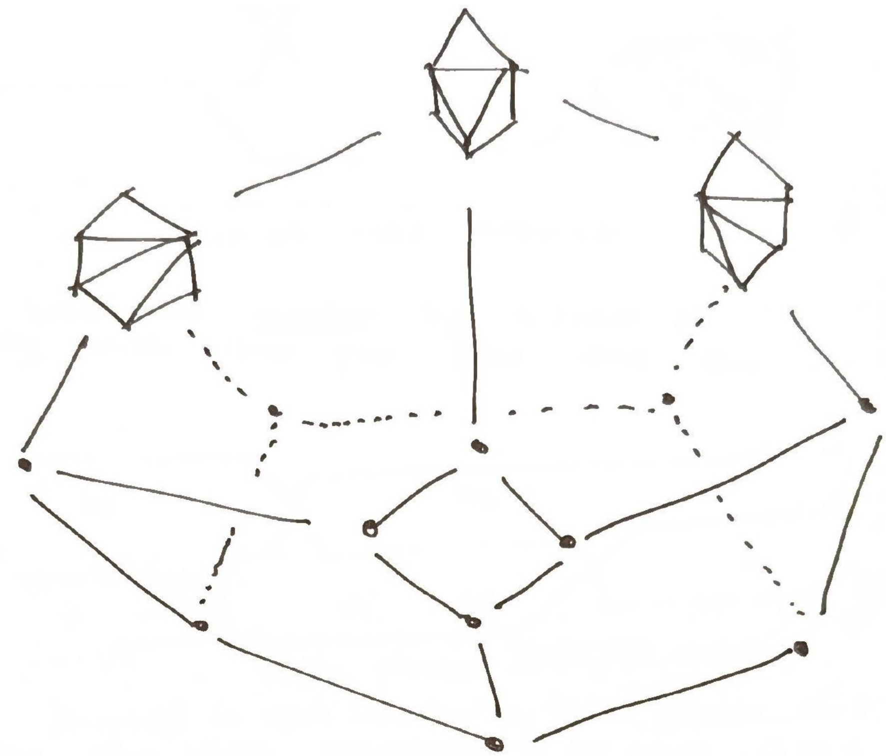

Figure 5: The associahedron for $m = 6$ (a 3-dimensional polytope).

2 Lecture 2

Date: January 14, 2026
Main reference: [FWZ21], §2–3.

2.1 Flag Positivity

Before moving to TP for $n \times n$ matrices, we discuss an intermediate notion called "flag positivity." Put $G = \mathrm{SL}_n$.

Definition 2.1. Given $J \subsetneq \{1, \ldots, n\}$ nonempty, the **flag minor** P_J is the function $P_J \colon G \to \mathbb{C}$ defined by

$$P_J(z) := z(\vec{e}_J) \mapsto \det\bigl(z_{\alpha\beta} \mid \alpha \le |J|,\, \beta \in J\bigr),$$

i.e., the $|J| \times |J|$ minor which is "top-justified."

Note 2.2. There are $2^n - 2$ flag minors.

Definition 2.3. An element $z \in G$ is **flag totally positive** (FTP) if all flag minors $P_J(z)$ are positive.

Question 2.4. Can we check FTP by only checking a subset of the $2^n - 2$ flag minors?

Claim 2.5. *It suffices to check only $\frac{(n-1)(n+2)}{2}$ special flag minors.*

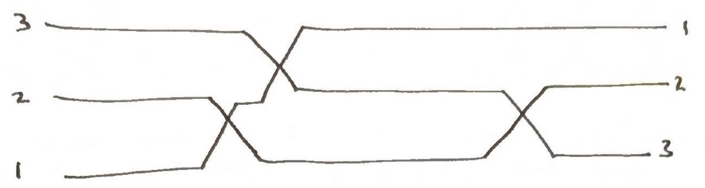

Figure 6: A wiring diagram for $n = 3$: each pair of lines intersects exactly once.

2.2 Wiring Diagrams

Each pair of lines intersects exactly once (Figure 6).

We label each **chamber** by a subset of $\{1, \ldots, n\}$ indicating which lines pass below that chamber (see Figure 7).

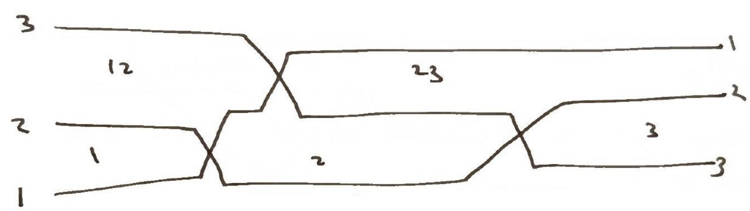

Figure 7: A wiring diagram with chamber labels.

Note 2.6. There are always $\frac{(n-1)(n+2)}{2}$ chambers.

Associated to each chamber is its **chamber minor** P_J, the flag minor corresponding to its subset $J \subsetneq \{1, \ldots, n\}$.

Extended cluster: All chamber minors of a wiring diagram.

- **Cluster variables**: the chamber minors for bounded chambers. There are $\frac{(n-1)n}{2}$ of these.
- **Frozen variables**: the chamber minors for unbounded chambers. There are $2n - 2$ of these.

Theorem 2.7. *Every flag minor can be written as a subtraction-free rational expression in the chamber minors of a given wiring diagram.*

Corollary 2.8. *If the* $\frac{(n-1)(n+2)}{2}$ *chamber minors evaluate positively at a matrix* $z \in \mathrm{SL}_n$*, then* z *is FTP.*

Proof outline. Follows by:

(1) Each flag minor appears as a chamber minor in some wiring diagram.

(2) Any two wiring diagrams can be transformed into each other by a sequence of local **braid moves** (see Figure 8).

(3) Under each braid move, the collection of chamber minors changes by exchanging $Y \leftrightarrow Z$, and we have

$$YZ = AC + BD.$$

□

Remark 2.9. In fact, each flag minor can be written as a Laurent polynomial with positive coefficients in the chamber minors of a given wiring diagram.

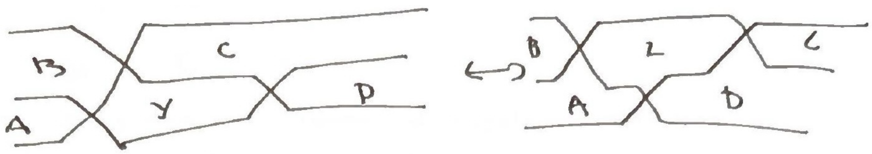

Figure 8: A braid move exchanges two adjacent crossings.

3 Lecture 3

Date: January 23, 2026
Main reference: [FWZ21], §1.3, §1.4, §2.1.

3.1 The Flag Variety and Basic Affine Space

Put $G = \mathrm{SL}_n(\mathbb{C})$. Let $B \subset G$ denote the subgroup of lower triangular matrices (the Borel subgroup), and let $U \subset G$ denote the subgroup of unipotent lower triangular matrices, i.e., lower triangular matrices with 1's on the diagonal:

$$U = \left\{ \begin{pmatrix} 1 & 0 & \cdots & 0 \\ * & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & 1 \end{pmatrix} \right\}.$$

Note 3.1. As a variety, $U \cong \mathbb{C}^{n(n-1)/2}$.

Similarly, let U^+ denote the subgroup of unipotent upper triangular matrices.

Definition 3.2. The (complete) **flag variety** is

$$\mathcal{F}\ell = B\backslash G = \{V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n \mid \dim V_i = i\}.$$

This is identified with the homogeneous space $B\backslash G$, where B acts on G by left multiplication.

Definition 3.3. The **basic affine space** is $U\backslash G$, where U acts on G by left multiplication.

Note 3.4. There is a natural projection $U\backslash G \to B\backslash G$, which is a $(\mathbb{C}^*)^{n-1}$-bundle (a torus bundle) over the flag variety.

Let $\mathbb{C}[G]$ denote the coordinate ring of $G = \mathrm{SL}_n(\mathbb{C})$, and let $\mathbb{C}[G]^U$ denote the ring of U-invariant polynomials, where U acts by left multiplication on matrix entries.

Claim 3.5 (First and Second Fundamental Theorems of Invariant Theory)**.**

(1) $\mathbb{C}[G]^U$ is generated by flag minors.

*(2) The ideal of relations among flag minors in $\mathbb{C}[G]^U$ is generated by the **generalized Plücker relations**.*

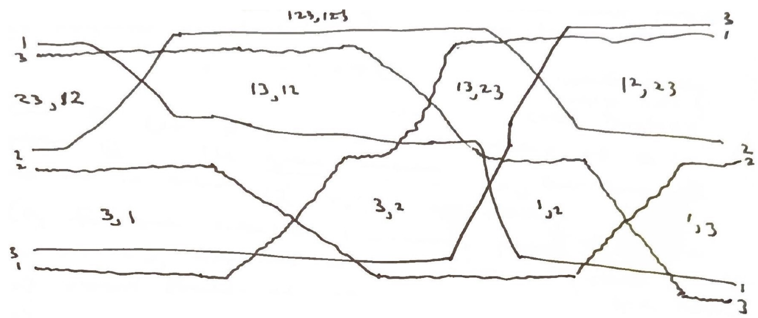

Figure 9: A double wiring diagram for $n = 3$.

3.2 Checking Total Positivity for $n \times n$ Matrices

Given $I, J \subseteq \{1, \ldots, n\}$ of some cardinality, let Δ^I_J denote the minor of an $n \times n$ matrix determined by rows in I and columns in J. This extends to flag minors when $|I| = |J|$.

Double wiring diagrams: These are a generalization of the wiring diagrams from Lecture 2, used to study total positivity for $n \times n$ matrices (see Figure 9).

Claim 3.6. *Every minor Δ^I_J of a chamber can be written as a subtraction-free rational expression in the chamber minors of a given double wiring diagram.*

Claim 3.7. *Every minor is a chamber minor for some double wiring diagram.*

The proof follows from:

(1) Any two double wiring diagrams can be linked by local moves.

(2) Each local move relates chamber minors of different diagrams.

(3) Each local double move satisfies a relation of the form $YZ = AC + BD$.

Remark 3.8. The graph with vertices given by double wiring diagrams and edges given by local moves is related to the theory of cluster algebras.

Remark 3.9. In fact, each minor can be written as a Laurent polynomial with positive coefficients in the chamber minors.

3.3 Quivers and Their Mutation

Definition 3.10. A **quiver** Q is a finite directed graph (see Figure 10) with:

- No loops (no arrows $i \to i$).
- No 2-cycles (no pairs of arrows $i \rightrightarrows j$ going both directions).

Figure 10: Examples of quivers (valid examples marked ✓, invalid example marked ×).

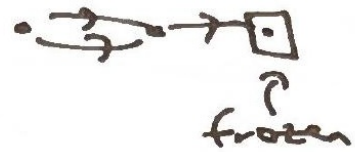

Figure 11: An ice quiver with frozen vertices indicated by boxes.

Definition 3.11. An **ice quiver** is a quiver in which some vertices are designated as "frozen" (see Figure 11), and there are no arrows between two frozen vertices. The non-frozen vertices are called **mutable**.

Definition 3.12. Let Q be an ice quiver and let k be a mutable vertex. The **mutation** $\mu_k(Q) = Q'$ at vertex k is defined as follows (see Figure 12):

(1) For each path $i \to k \to j$, add an arrow $i \to j$ (unless i, j are both frozen).

(2) Reverse the direction of all arrows incident to k.

(3) Remove any 2-cycles that were created.

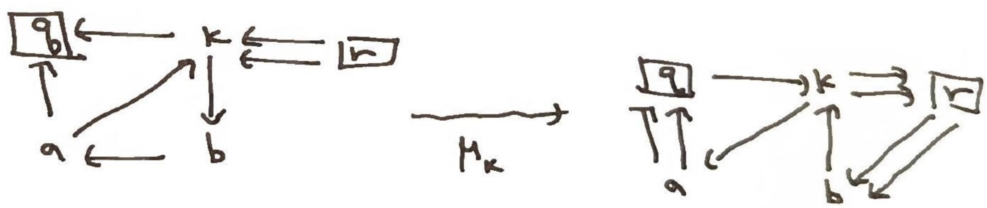

Figure 12: Illustration of quiver mutation at a vertex.

Exercise 3.13.

(1) Mutation is an involution, i.e., $\mu_k(\mu_k(Q)) = Q$.

(2) Mutation commutes with reversing the orientations of all arrows.

(3) If k, ℓ are mutable vertices with no arrows between them, then mutations commute:

$$\mu_k(\mu_\ell(Q)) = \mu_\ell(\mu_k(Q)).$$

Remark 3.14. If k is a sink or source, then μ_k simply reverses all arrows incident to k.

Exercise 3.15. For any quiver Q that is a tree with no frozen vertices, show that one can get from any orientation to any other orientation by a sequence of mutations at sources and sinks.

3.4 Triangulations and Quivers

We can assign to each triangulation T of the polygon $\mathbb{P}_m$ a quiver $Q(T)$ (see Figure 13).

Exercise 3.16. If T' is obtained from T by a flip along diagonal γ, then

$$Q(T') = \mu_\gamma(Q(T)).$$

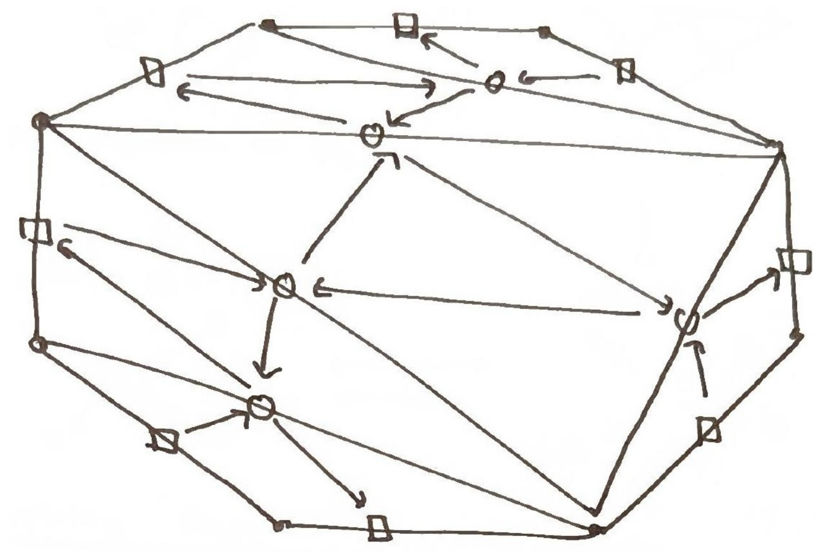

Figure 13: A triangulation T of $\mathbb{P}_m$ and its associated quiver $Q(T)$.

4 Lecture 4

Date: January 26, 2026
Main reference: [FWZ21], §2.2, §2.3, §2.4, §2.5, §2.6.

4.1 Review: Triangulations and Quivers

Example 4.1. Let T be a triangulation of $\mathbb{P}_4$. Then a flip along a diagonal gives a new triangulation T' (see Figure 14):

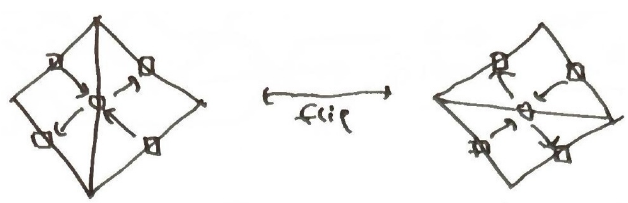

Figure 14: A flip between triangulations T and T' of $\mathbb{P}_4$, and the corresponding quivers $Q(T)$ and $Q(T')$ related by mutation.

4.2 Wiring Diagrams and Quivers

Given a wiring diagram D, we can associate a quiver $Q(D)$ (see Figure 15).
Vertices: The vertices of $Q(D)$ are the chambers of D. A vertex is mutable if the corresponding chamber is bounded, and frozen otherwise.

Arrows: For chambers c, c', we have an arrow $c \to c'$ in $Q(D)$ if one of the following holds (see Figure 16):

(i) The right end of c equals the left end of c'.

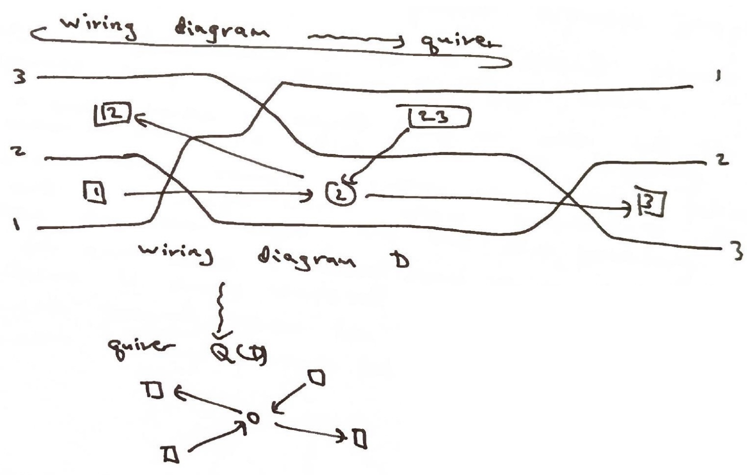

Figure 15: A wiring diagram D and its associated quiver $Q(D)$.

(ii) The left end of c is directly above c', and the right end of c' is directly below c.

(iii) The left end of c is directly below c', and the right end of c' is directly above c.

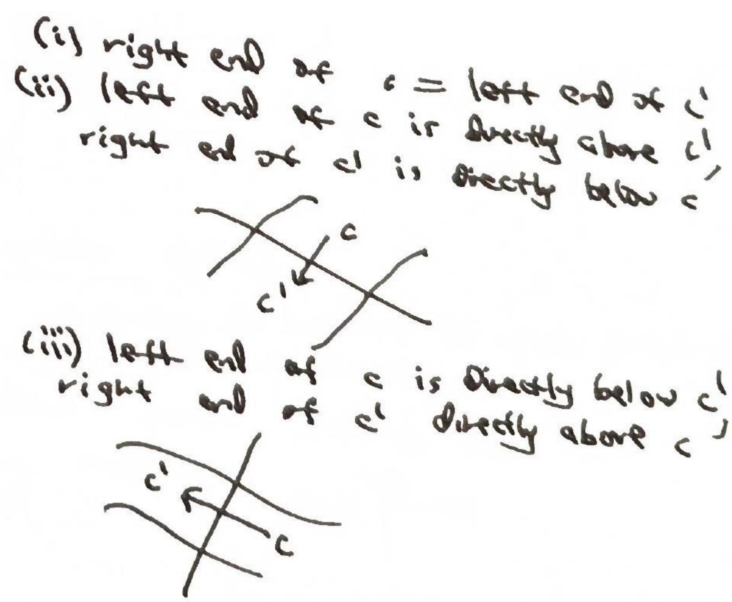

Figure 16: The arrow rules for chambers in a wiring diagram.

Exercise 4.2. If D, D' are wiring diagrams related by a braid move at chamber Y, then

$$Q(D') = \mu_Y(Q(D)).$$

Example 4.3. Figure 17 shows two wiring diagrams related by a braid move, and the corresponding quivers related by mutation at the central chamber.

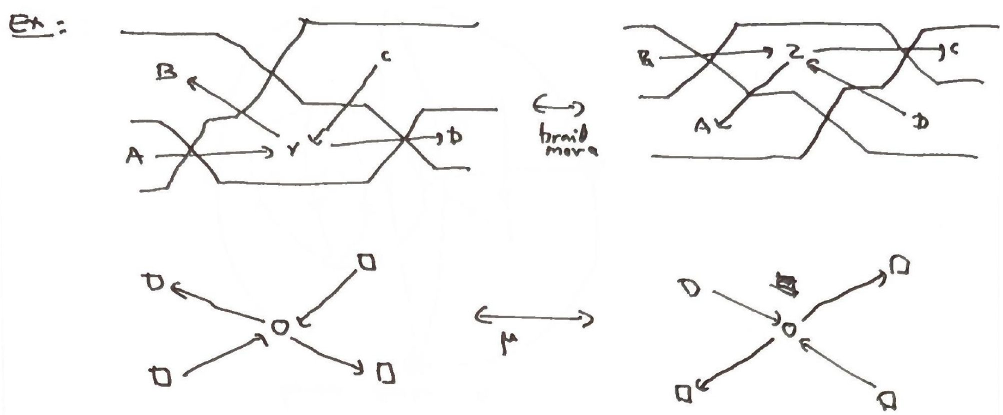

Figure 17: A braid move on wiring diagrams and the corresponding quiver mutation.

4.3 Plabic Graphs

Remark 4.4. We also have an assignment

$$\text{double wiring diagram } D \quad \rightsquigarrow \quad \text{quiver } Q(D).$$

The description is more complicated, but it is a special case of the quiver associated to a planar bipartite graph.

Definition 4.5. A **plabic graph** G is a connected planar bipartite graph embedded in a disk, where:

- Each vertex is colored black or white and lies either in the interior of the disk or on its boundary.
- Each edge connects vertices of different colors and is a simple curve whose interior is disjoint from the other edges and the disk boundary.
- For each face (connected component of complement), the closure is simply connected.
- Each interior vertex has degree ≥ 2.
- Each boundary vertex has degree 1.

Note 4.6. We consider plabic graphs up to isotopy; see Figure 18 for an example.

4.4 Quivers from Plabic Graphs

Given a plabic graph G, we can associate a quiver $Q(G)$:

Vertices: The vertices of $Q(G)$ are the faces of G. A vertex is frozen if the corresponding face is incident to the disk boundary, and mutable otherwise.

Arrows: For each edge of G, we have an arrow joining the two faces it separates, using the orientation rule shown in Figure 19:
Finally, remove oriented 2-cycles.

Example 4.7. Figure 20 shows a plabic graph G and the construction of its quiver $Q(G)$.

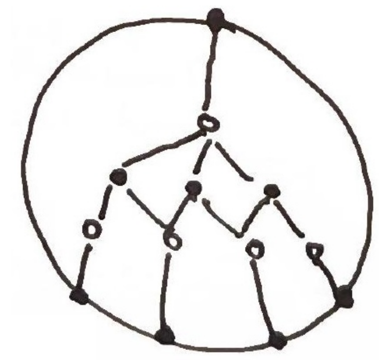

Figure 18: An example of a plabic graph.

Figure 19: The orientation rule for arrows: the arrow points so that the white vertex is on the left.

4.5 Moves on Plabic Graphs

Definition 4.8. Say a vertex v is **bivalent** if it is adjacent to two interior vertices.

Remark 4.9. Contracting or decontracting a bivalent vertex (Figure 21) does not change the associated quiver.

Definition 4.10. Say G has a **quadrilateral** if it has a face whose vertices have degree ≥ 3.

Exercise 4.11. If G, G' are related by a spider move (Figure 22), then $Q(G), Q(G')$ are related by mutation.

Example 4.12. Figure 23 shows two plabic graphs related by a spider move, and the corresponding quivers.

4.6 Mutation Equivalence

Definition 4.13. Two quivers Q, Q' are **mutation equivalent** if Q becomes isomorphic to Q' after a sequence of mutations.

Definition 4.14. Put

$$[Q] := \{\text{all quivers which are mutation equivalent to } Q\}/\text{isomorphism}.$$

Example 4.15. Let Q be the A_3 quiver (three vertices in a line):

$$\bullet \to \bullet \to \bullet$$

Then $[Q]$ has 4 elements (Figure 24):

Exercise 4.16. Show that $[Q]$ has exactly 4 elements for Q the A_3 quiver.

Example 4.17. Let Q be the "Markov quiver" (Figure 25):
In fact, $[Q]$ is just a single element (the Markov quiver is mutation equivalent only to itself).

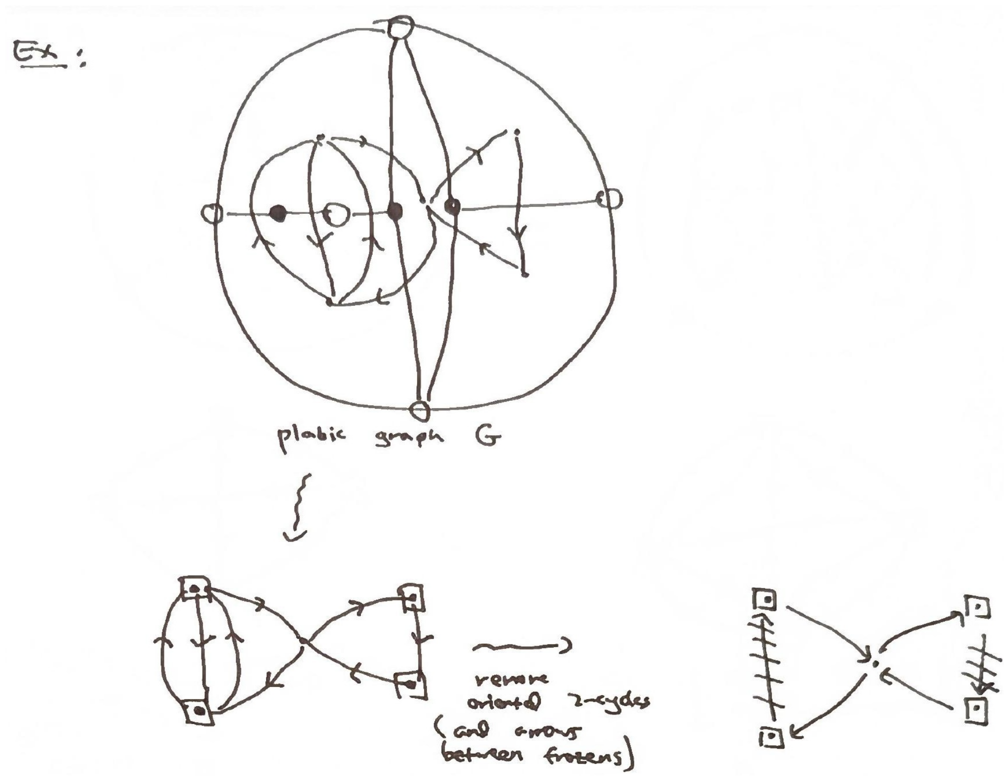

Figure 20: A plabic graph G and its associated quiver $Q(G)$, after removing oriented 2-cycles and arrows between frozen vertices.

4.7 Finite Mutation Type

Definition 4.18. A quiver Q has **finite mutation type** if $[Q]$ is finite.

Remark 4.19. There is a classification theorem for quivers with no frozen vertices and finite mutation type.

Definition 4.20. A quiver Q is **acyclic** if it has no oriented cycles.

Theorem 4.21 (Caldero–Keller '06)**.** *If Q, Q' are acyclic and mutation equivalent, then we can transform Q into Q' by a sequence of mutations at sources and sinks. In particular, Q and Q' have the same underlying undirected graphs.*

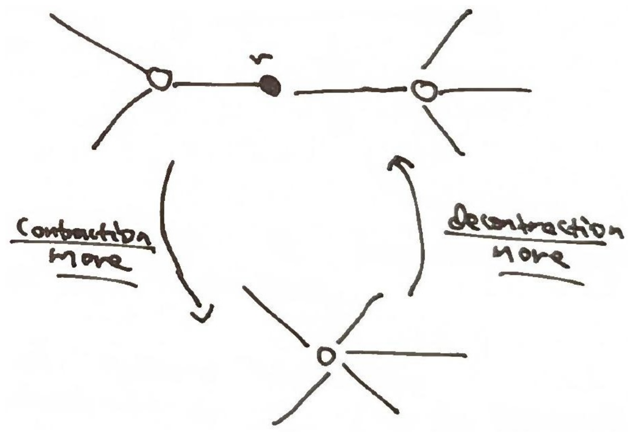

Figure 21: Contraction and decontraction moves on a bivalent vertex.

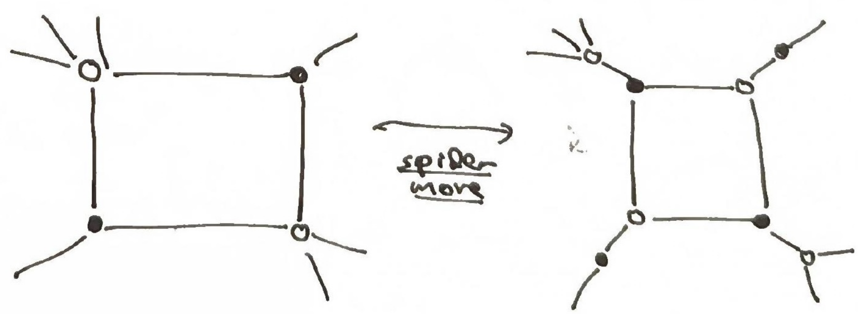

Figure 22: The spider move on a quadrilateral face.

5 Lecture 5

Date: January 28, 2026
Main reference: [FWZ21], §2.7, §2.8.

5.1 Extended Exchange Matrices

Definition 5.1. Let Q be a quiver with vertices labeled by $1, \ldots, m$, such that $1, \ldots, n$ are the **mutable** vertices (with $n \leq m$). The **extended exchange matrix** is

$$\widetilde{B}(Q) = (b_{ij})_{\substack{1 \leq i \leq m, \\ 1 \leq j \leq n}} \quad \text{where} \quad b_{ij} = \begin{cases} \ell & \text{if } \ell \text{ arrows } i \to j \\ -\ell & \text{if } \ell \text{ arrows } j \to i \\ 0 & \text{else} \end{cases}$$

This is an $m \times n$ matrix. The **exchange matrix** is the submatrix

$$B(Q) := (b_{ij})_{1 \leq i,j \leq n},$$

which is an $n \times n$ skew-symmetric matrix.

Example 5.2. Consider the quiver Q with mutable vertices $1, 2, 3$ and frozen vertices $4, 5$ (Figure 26):

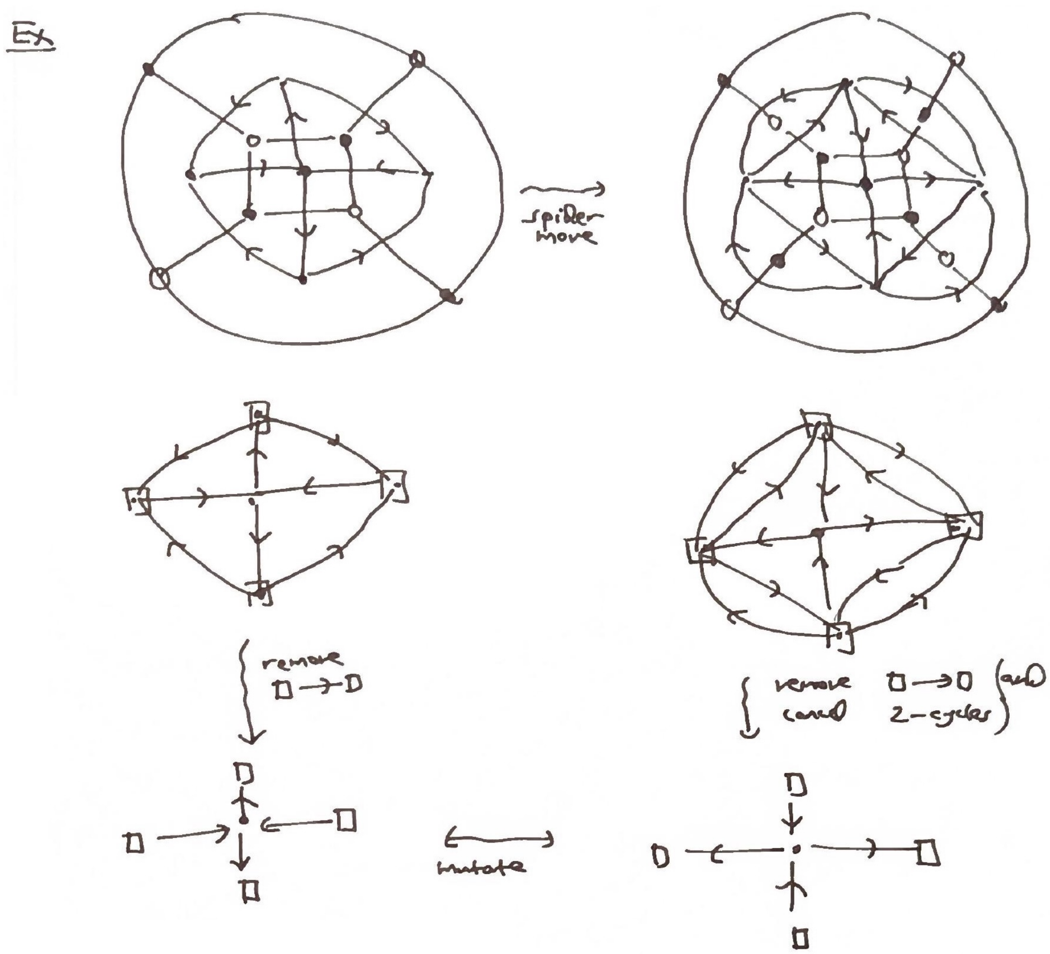

Figure 23: Two plabic graphs related by a spider move, and their quivers related by mutation.

The extended exchange matrix is

$$\widetilde{B}(Q) = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \\ -1 & -1 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \quad B(Q) = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}.$$

Example 5.3. Let Q be the Markov quiver. Figure 27 shows the extended exchange matrices for Q and two of its mutations.

Remark 5.4. Reordering the vertices of Q results in simultaneously reordering the rows $1, \dots, n$ and reordering the columns $1, \dots, m$.

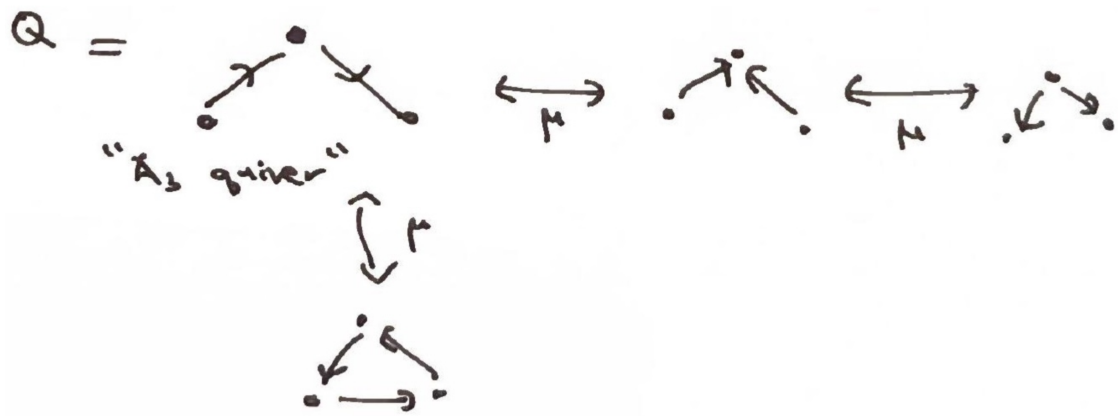

Figure 24: The mutation equivalence class of the A_3 quiver.

Figure 25: The Markov quiver.

5.2 Matrix Mutation

Lemma 5.5. *For a quiver Q with $\widetilde{B}(Q) = (b_{ij})$ and $Q' = \mu_k(Q)$ for a mutable vertex k of Q, we have $\widetilde{B}(Q') = (b'_{ij})$ with*

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + b_{ik}b_{kj} & \text{if } b_{ik}b_{kj} > 0 \\ b_{ij} - b_{ik}b_{kj} & \text{if } b_{ik}b_{kj} < 0 \\ b_{ij} & \text{else} \end{cases} \tag{$*$}$$

Note 5.6. One can replace the middle two cases with

$$b'_{ij} = b_{ij} + |b_{ik}|b_{kj} \quad \text{if } b_{ik}b_{kj} > 0.$$

Example 5.7. Figure 28 shows an example of matrix mutation.

5.3 Skew-Symmetrizable Matrices

Definition 5.8. An $n \times n$ matrix $B = (b_{ij}) \in \mathrm{Mat}_{n\times n}(\mathbb{Z})$ is **skew-symmetrizable** if for some $d_1, \ldots, d_n \in \mathbb{Z}_{>0}$ we have

$$d_i b_{ij} = -d_j b_{ji}.$$

(I.e., B becomes skew-symmetric after rescaling the rows by positive integers.)

Definition 5.9. An $m \times n$ matrix is **extended skew-symmetrizable** if the top $n \times n$ submatrix is skew-symmetrizable.

Definition 5.10. For $\widetilde{B} = (b_{ij})$ an extended skew-symmetrizable $m \times n$ matrix and $k \in \{1, \ldots, n\}$, we define $\mu_k(\widetilde{B}) = (b'_{ij})$ using the same formula $(*)$.

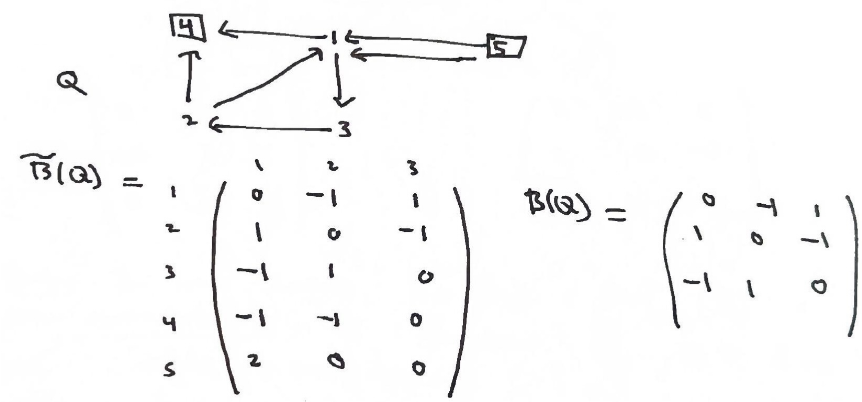

Figure 26: A quiver with frozen vertices 4 and 5 (boxed), and its extended and exchange matrices.

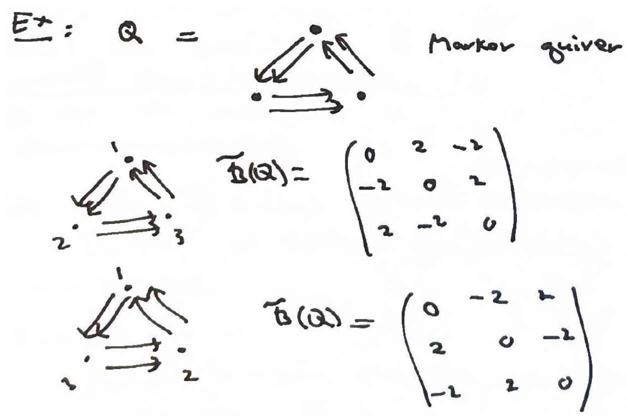

Figure 27: The Markov quiver and extended exchange matrices for mutations.

Exercise 5.11. (1) $\mu_k(\widetilde{B})$ is again extended skew-symmetrizable, using the same $d_1, \ldots, d_n$.

(2) $\mu_k(\mu_k(\widetilde{B})) = \widetilde{B}$.

(3) $\mu_k(-\widetilde{B}) = -\mu_k(\widetilde{B})$.

(4) If $b_{ij} = b_{ji} = 0$, then $\mu_i \mu_j \widetilde{B} = \mu_j \mu_i \widetilde{B}$.

5.4 Diagrams and Uniqueness

Definition 5.12. For a skew-symmetrizable $n \times n$ matrix $B = (b_{ij})$, its **diagram** is the weighted directed graph $\Gamma(B)$ with vertices $1, \ldots, n$ and $i \to j$ if and only if $b_{ij} > 0$, with weight $|b_{ij} b_{ji}|$.

Lemma 5.13. *If the diagram $\Gamma(B)$ of an $n \times n$ skew-symmetrizable matrix B is connected, then the skew-symmetrizing vector $(d_1, \ldots, d_n)$ is unique up to rescaling.*

Proof. By connectedness, there is an ordering $l_1, \ldots, l_n$ of $\{1, \ldots, n\}$ such that for each $j \geq 2$ we have $b_{l_i l_j} \neq 0$ for some $i < j$.

If $(d_1, \ldots, d_n)$ and $(d'_1, \ldots, d'_n)$ are skew-symmetrizing vectors, we have $d_i b_{ij} = -d_j b_{ji}$ and $d'_i b_{ij} = -d'_j b_{ji}$ for all i, j.

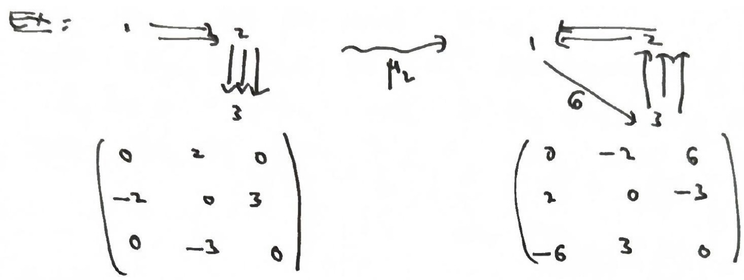

Figure 28: An example of quiver mutation μ_2 and the corresponding matrix mutation.

If $b_{ij} \neq 0$, we have

$$\frac{b_{ij}}{b_{ji}} = \frac{-d_j}{d_i} = \frac{-d'_j}{d'_i}.$$

Thus $\frac{d_j}{d'_j} = \frac{d_i}{d'_i}$. $\square$

5.5 Mutation Equivalence for Matrices

Definition 5.14. Two extended skew-symmetrizable matrices $\widetilde{B}, \widetilde{B}'$ are **mutation equivalent** if one can get from $\widetilde{B}$ to $\widetilde{B}'$ by a sequence of mutations followed by a reordering of the rows and columns in the sense from before. Put

$$[B] := \text{mutation equivalence class of } B.$$

Proposition 5.15. *For an $n \times n$ skew-symmetrizable matrix, its rank and determinant are preserved by mutations.*

Proof. One can write

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } k \in \{i, j\} \\ b_{ij} + \max(0, -b_{ik})b_{kj} + b_{ik}\max(0, b_{kj}) & \text{otherwise} \end{cases}$$

We have

$$\begin{aligned} \mu_k(B) &= J_{m,k}\widetilde{B}J_{n,k} + J_{m,k}\widetilde{B}F_k + E_k\widetilde{B}J_{n,k} \\ &= (J_{m,k} + E_k)\widetilde{B}(J_{n,k} + F_k) \end{aligned}$$

where:

- $J_{m,k}$ (resp. $J_{n,k}$) is a diagonal $m \times m$ (resp. $n \times n$) matrix with 1s on the diagonal except for -1 in the (k, k) entry.
- $E_k = (e_{ij})$ is an $m \times m$ matrix with $e_{ik} = \max(0, -b_{ik})$ and all other entries 0.
- $F_k = (f_{ij})$ is an $n \times n$ matrix with $f_{kj} = \max(0, b_{kj})$ and all other entries 0.

Note: $E_k\widetilde{B}F_k = 0$ since $b_{kk} = 0$.
We have $\det(J_{m,k} + E_k) = \det(J_{n,k} + F_k) = -1$. $\square$

5.6 Labeled Seeds

Definition 5.16. A **labeled seed of geometric type** in $\mathcal{F} = \mathbb{C}(x_1, \ldots, x_m)$ (the field of rational functions) is a pair $(\mathbf{x}, \widetilde{B})$ where:

- $\mathbf{x} = (x_1, \ldots, x_m)$ is an m-tuple of elements of $\mathcal{F}$ which form a free generating set (i.e., $\mathcal{F} = \mathbb{C}(x_1, \ldots, x_m)$ and $x_1, \ldots, x_m$ are algebraically independent).
- $\widetilde{B} = (b_{ij})$ is an $m \times n$ extended skew-symmetrizable matrix.

We say:

- $\mathbf{x}$ is the (labeled) **extended cluster** of $(\mathbf{x}, \widetilde{B})$.
- $(x_1, \ldots, x_n)$ is the (labeled) **cluster**.
- $x_1, \ldots, x_n$ are the **cluster variables**.
- $x_{n+1}, \ldots, x_m$ are the **frozen variables**.
- $\widetilde{B}$ is the **extended exchange matrix**.
- Its top $n \times n$ submatrix B is the **exchange matrix**.

Example 5.17. Figure 29 shows two labeled seeds Σ and Σ' related by mutation, with $m = 3$ and $n = 2$.

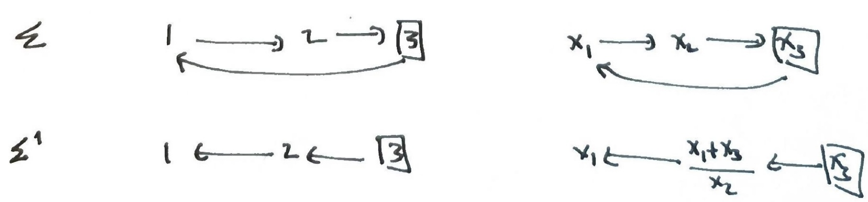

Figure 29: Two labeled seeds Σ and Σ' related by mutation at vertex 1.

For Σ: the extended cluster is $\mathbf{x} = (x_1, x_2, x_3)$, the cluster is (x_1, x_2), the cluster variables are x_1, x_2, the frozen variable is x_3, and

$$\widetilde{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For Σ': the extended cluster is $\mathbf{x}' = \left(x_1', \frac{x_1+x_3}{x_2}, x_3\right)$, the cluster variables are $x_1', \frac{x_1+x_3}{x_2}$, and

$$\widetilde{B}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

6 Lecture 6

Date: February 4, 2026
Main reference: [FWZ21], §3.1.

6.1 Labeled Seeds and Seed Mutation

Recall: $\mathcal{F} = \mathbb{C}(y_1, \ldots, y_m)$ is a field of rational functions, $m \geq n$. Say $x_1, \ldots, x_m \in \mathcal{F}$ is a **free generating set** if it is algebraically independent and $\mathcal{F} = \mathbb{C}(x_1, \ldots, x_m)$.

Definition 6.1. A **labeled seed** of geometric type in $\mathcal{F}$ is $(\widetilde{x}, \widetilde{B})$ where:

- $\widetilde{x} = (x_1, \ldots, x_m)$ is a free generating set of $\mathcal{F}$.
- $\widetilde{B} = (b_{ij})$ is an $m \times n$ extended skew-symmetrizable integer matrix.

Terminology:

- $\widetilde{x}$ is the **extended cluster**.
- $x = (x_1, \ldots, x_n)$ is the **cluster**; $x_1, \ldots, x_n$ are the **cluster variables**.
- $x_{n+1}, \ldots, x_m$ are the **frozen variables**.
- $\widetilde{B}$ is the **extended exchange matrix**; the top $n \times n$ submatrix B is the **exchange matrix**.

Definition 6.2. Given $(\widetilde{x}, \widetilde{B})$ a labeled seed, $k \in \{1, \ldots, n\}$, define a new labeled seed $\mu_k(\widetilde{x}, \widetilde{B}) = (\widetilde{x}', \widetilde{B}')$, where:

- $\widetilde{B}' = \mu_k(\widetilde{B})$
- $\widetilde{x}' = (x'_1, \ldots, x'_m)$, where $x'_j = x_j$ for $j \neq k$ and

$$x_k x'_k = \prod_{b_{ik}>0} x_i^{b_{ik}} + \prod_{b_{ik}<0} x_i^{-b_{ik}} \qquad \text{(exchange relation)}$$

Remark 6.3. When $\widetilde{B}$ comes from a quiver, the first product is over arrows ending at k and the second product is over arrows starting at k. See Figure 30 for an example.

6.2 Examples

Recall the Plücker relation (Figure 31):

$$P_{13}P_{24} = P_{12}P_{34} + P_{14}P_{23}.$$

More generally, a flip gives

$$P_{ik}P_{j\ell} = P_{ij}P_{\ell k} + P_{i\ell}P_{jk},$$

which is a special case of the exchange relation; see also Figure 32 for the wiring diagram case.

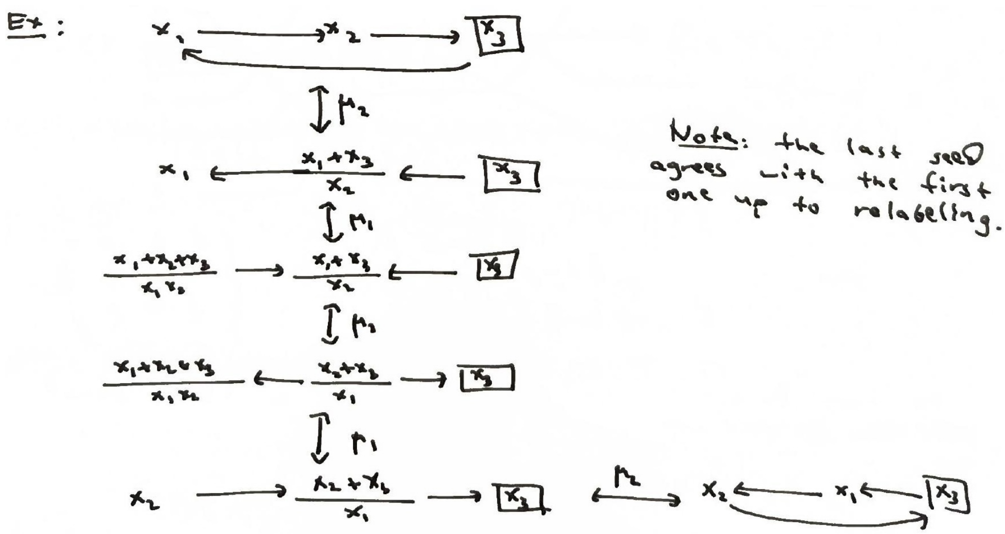

Figure 30: Example of a sequence of seed mutations. Note: the last seed agrees with the first one up to relabeling.

6.3 Seed Patterns and Cluster Algebras

Notation: Let $\mathbb{T}_n$ denote the n-regular tree (Figure 33) with edges labeled by $1, \ldots, n$, such that the edges incident to each vertex carry distinct labels.

Definition 6.4. A **seed pattern** is a choice of labeled seed $(\widetilde{x}(t), \widetilde{B}(t))$ for each vertex $t \in \mathbb{T}_n$, so that for each labeled edge $t \overset{k}{—} t'$, the corresponding labeled seeds $(\widetilde{x}(t), \widetilde{B}(t))$ and $(\widetilde{x}(t'), \widetilde{B}(t'))$ differ by μ_k.

Note 6.5. A seed pattern is determined by any one of its seeds.

Definition 6.6. Let $(\widetilde{x}(t), \widetilde{B}(t))_{t\in\mathbb{T}_n}$ be a seed pattern, and put $R := \mathbb{C}[x_{n+1}, \ldots, x_m]$. Let $\mathcal{X}$ be the set of all cluster variables appearing in the seeds $x(t)$ for $t \in \mathbb{T}_n$. The **cluster algebra** $\mathcal{A}$ is the R-subalgebra of $\mathcal{F}$ generated by all cluster variables, i.e., $\mathcal{A} = R[\mathcal{X}]$.

Terminology: The **rank** n of any cluster is the rank of a cluster algebra.

Remark 6.7. Note that there is an isomorphism of $\mathcal{F}$ mapping any free generating set to any other. In particular, up to isomorphism $\mathcal{A}$ depends only on $\widetilde{B}_0$ for any initial seed $(\widetilde{x}_0, \widetilde{B}_0)$, and in fact only on the mutation equivalence class of $\widetilde{B}$. In particular, each (ice) quiver Q determines an extended exchange matrix $\widetilde{B}$ and hence a cluster algebra.

6.4 Examples of Cluster Algebras

(1) **Triangulations:** The associated cluster algebra is the Plücker ring.

(2) **Wiring diagrams:** For a wiring diagram, the associated cluster algebra is the algebra of regular functions on $\mathrm{Flag}(\mathrm{SL}_n)$ (i.e., on the Borel), generated by flag minors with the Plücker relations.

(3) **Double wiring diagrams:** For a double wiring diagram, the associated cluster algebra is $\mathbb{C}[G]^U$ for $G = \mathrm{SL}_n$, i.e., the ring of regular functions on the basic affine space.

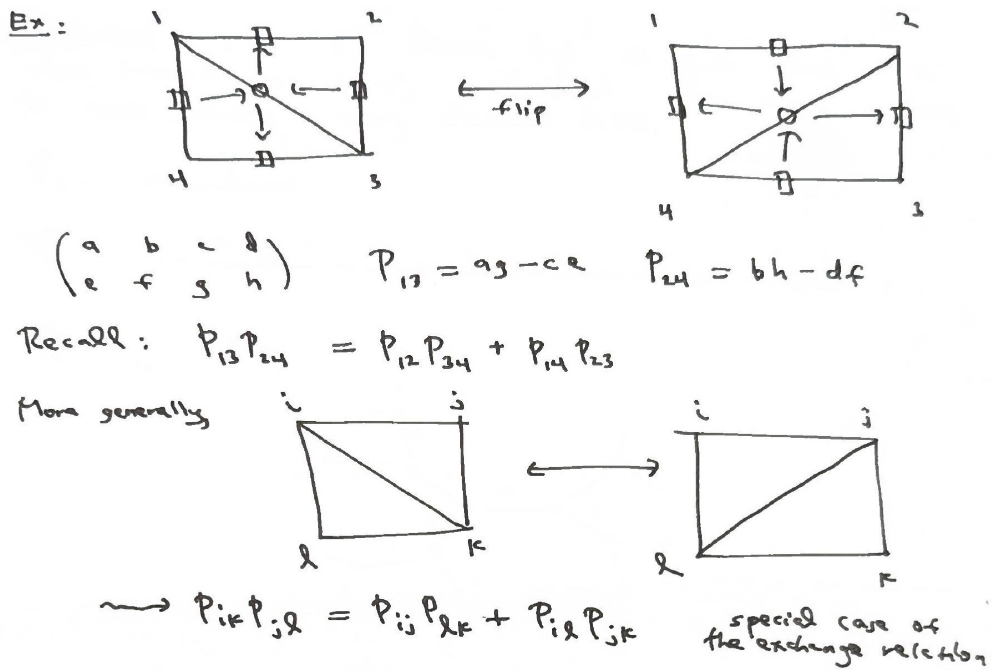

Figure 31: Triangulation flip and Plücker coordinates.

7 Lecture 7

Date: February 6, 2026
Main reference: [FWZ21], §3.2.

Recall: a labeled seed $(\widetilde{x}_0, \widetilde{B}_0)$ determines a seed pattern $(\widetilde{x}(t), \widetilde{B}(t))_{t\in\mathbb{T}_n}$, and hence a cluster algebra $\mathcal{A} \subset \mathcal{F}$ generated by all cluster variables and the frozen variables. Here $\widetilde{x}_0 = (x_1, \dots, x_m)$ is a free generating set of $\mathcal{F} = \mathbb{C}(y_1, \dots, y_m)$, $x_1, \dots, x_n$ are the cluster variables, $x_{n+1}, \dots, x_m$ are the frozen variables, and the rank of $\mathcal{A}$ is n.

7.1 Rank 1 Cluster Algebras

Example 7.1. (Rank $n = 1$.) The 1-regular tree is $\mathbb{T}_1 = \bullet\!\!-\!\!\bullet$. The extended exchange matrix is

$$\widetilde{B}_0 = \begin{pmatrix} b_{21} \\ \vdots \\ b_{m1} \end{pmatrix}.$$

The exchange relation is

$$x_1 x_1' = \prod_{b_{i1}>0} x_i^{b_{i1}} + \prod_{b_{i1}<0} x_i^{-b_{i1}} = M_1 + M_2,$$

where M_1, M_2 are monomials in the frozen variables $x_2, \dots, x_m$. The cluster algebra is

$$\mathcal{A} = \mathbb{C}[x_1, x_1', x_2, \dots, x_m] \subset \mathcal{F} = \mathbb{C}(x_1, x_2, \dots, x_m),$$

which has the presentation

$$\mathcal{A} \cong \mathbb{C}[z_1, z_1', z_2, \dots, z_m]/(z_1 z_1' = M_1 + M_2),$$

where M_1, M_2 are the corresponding monomials in $z_2, \dots, z_m$.

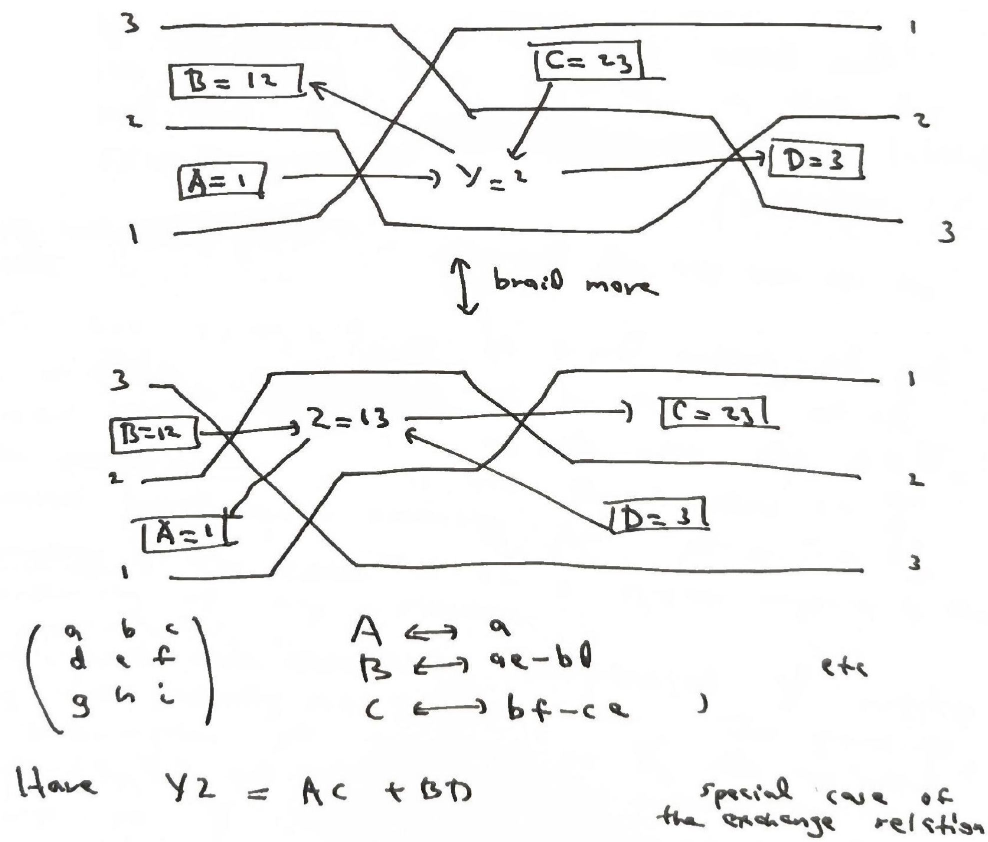

Figure 32: Wiring diagram braid move example. The relation $YZ = AC + BD$ is a special case of the exchange relation.

Example 7.2. Let $G = \mathrm{SL}_3(\mathbb{C})$ and let U be the subgroup of unipotent lower triangular 3×3 matrices. Then $\mathbb{C}[G]^U$ is a cluster algebra of rank 1.

Recall: $\mathbb{C}[G]^U$ is generated by flag minors P_J, $J \subsetneq \{1, 2, 3\}$. Here:

- $\mathcal{F} = \mathbb{C}(P_1, P_2, P_3, P_{12}, P_{23})$
- Frozen variables: P_1, P_3, P_{12}, P_{23}
- Cluster variables: P_2, P_{13}
- Single exchange relation: $P_2 P_{13} = P_1 P_{23} + P_3 P_{12}$

See Figure 34 for the corresponding wiring diagrams, where a braid move exchanges the cluster variables P_2 and P_{13}.

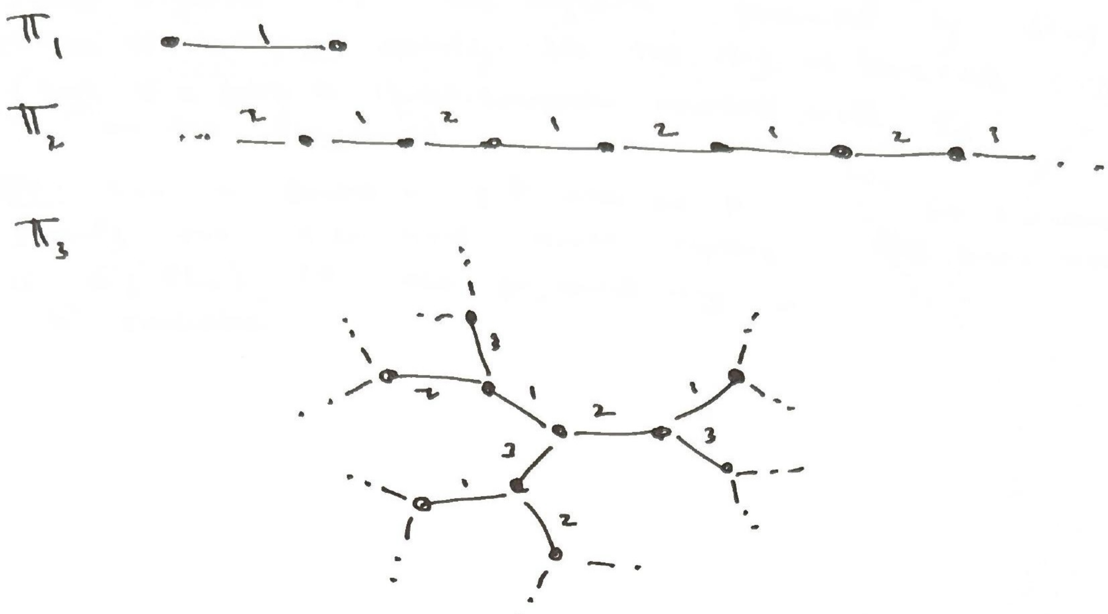

Figure 33: The n-regular trees $\mathbb{T}_1$, $\mathbb{T}_2$, and $\mathbb{T}_3$.

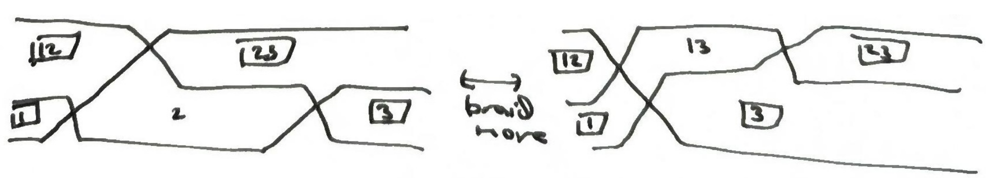

Figure 34: Wiring diagrams for SL_3: the braid move exchanges the cluster variables P_2 and P_{13}, corresponding to the exchange relation $P_2P_{13} = P_1P_{23} + P_3P_{12}$.

7.2 Rank 2 Cluster Algebras

Example 7.3. (Rank $n = 2$.) The extended exchange matrix has the form

$$\widetilde{B}_0 = \begin{pmatrix} 0 & \pm b \\ \mp c & 0 \\ b_{31} & b_{32} \\ \vdots & \vdots \\ b_{m1} & b_{m2} \end{pmatrix},$$

where either $b, c > 0$ or $b = c = 0$.

Suppose there are no frozens, i.e., $n = m$, so $\widetilde{B}_0 = \begin{pmatrix} 0 & \pm b \\ \mp c & 0 \end{pmatrix}$. Then $\mu_1(\widetilde{B}_0) = \mu_2(\widetilde{B}_0) = -\widetilde{B}_0$.

The exchange pattern along $\mathbb{T}_2$ has seeds

$$\cdots \overset{2}{\text{---}} (z_1, z_0) \overset{1}{\text{---}} (z_1, z_2) \overset{2}{\text{---}} (z_3, z_2) \overset{1}{\text{---}} (z_3, z_4) \overset{2}{\text{---}} \cdots$$

with exchange matrices alternating between $\begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}$, and the exchange relation gives

$$z_{k-1}z_{k+1} = \begin{cases} z_k^c + 1 & \text{if } k \text{ is even,} \\ z_k^b + 1 & \text{if } k \text{ is odd.} \end{cases}$$

Example 7.4. When $b = c = 0$, the extended exchange matrix is

$$\widetilde{B} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ b_{31} & b_{32} \\ \vdots & \vdots \\ b_{m1} & b_{m2} \end{pmatrix}.$$

Note that μ_k flips the sign of the kth column for $k = 1, 2$. The exchange relations are

$$x_1 x_1' = M_1 + M_2, \qquad x_2 x_2' = M_3 + M_4,$$

where M_1, M_2, M_3, M_4 are monomials in the frozen variables. The cluster variables are x_1, x_1', x_2, x_2', and this reduces to two rank 1 exchange patterns.

Notation: Let $\mathcal{A}(b, c)$ denote the cluster algebra of rank 2 with exchange matrices $\begin{pmatrix} 0 & \pm b \\ \mp c & 0 \end{pmatrix}$ and no frozen variables.

Example 7.5. $\mathcal{A}(1,1)$: The exchange relation becomes $z_{k-1}z_{k+1} = z_k + 1$. We compute:

$$\begin{aligned} z_3 &= \frac{z_2 + 1}{z_1}, \\ z_4 &= \frac{z_3 + 1}{z_2} = \frac{z_1 + z_2 + 1}{z_1 z_2}, \\ z_5 &= \frac{z_1 + 1}{z_2}, \\ z_6 &= z_1, \quad z_7 = z_2, \quad \text{etc.} \end{aligned}$$

So the sequence of cluster variables is **5-periodic**.

Example 7.6. Consider $\widetilde{B}_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ p & q \end{pmatrix}$ with rank 2 and 1 frozen variable y, where $p, q \geq 0$ are integers. The seed pattern is:

$$\begin{array}{ccccccccc} (z_1, z_2) & \overset{1}{\text{---}} & (z_3, z_2) & \overset{2}{\text{---}} & (z_3, z_4) & \overset{1}{\text{---}} & (z_5, z_4) & \overset{2}{\text{---}} & (z_5, z_6) \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ p & q \end{pmatrix} & & \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ -p & p+q \end{pmatrix} & & \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ q & -(p+q) \end{pmatrix} & & \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ -q & -p \end{pmatrix} & & \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ -q & p \end{pmatrix} \end{array}$$

We have:

$$\begin{aligned} z_3 &= \frac{z_2 + y^p}{z_1}, \\ z_4 &= \frac{y^{p+q} z_1 + z_2 + y^p}{z_1 z_2}, \\ z_5 &= \frac{y^q z_1 + 1}{z_2}, \\ z_6 &= z_1, \quad z_7 = z_2, \quad \text{etc.} \end{aligned}$$

So the cluster variables are still **5-periodic**.

Remark 7.7. Although we assumed $p, q \geq 0$ above, up to mutating and swapping columns, every $(b, c) \in \mathbb{Z}^2$ can be written in one of the forms

$$(p, q), \quad (p+q, -p), \quad (q, -p-q), \quad (-p, -q), \quad (-q, p).$$

See Figure 35. Later we will view this as a simple example of a **scattering diagram**.

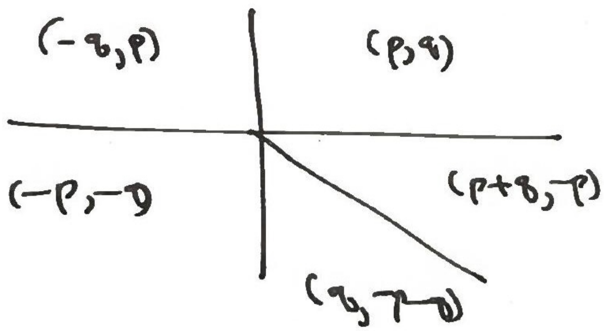

Figure 35: The five mutation forms of the frozen row (b, c) for a rank 2 cluster algebra with one frozen variable, viewed as a scattering diagram.

8 Lecture 8

Date: February 9, 2026
Main reference: [FWZ21], §3.3, §3.4.

8.1 Rank 2 examples (continued)

Example 8.1. $\mathcal{A}(1,2)$: The exchange relation is

$$z_{k-1}z_{k+1} = \begin{cases} z_k^2+1 & \text{if } k \text{ is even,} \\ z_k+1 & \text{if } k \text{ is odd.} \end{cases}$$

We compute:

$$\begin{aligned}
z_3 &= \frac{z_2^2+1}{z_1}, \\
z_4 &= \frac{z_3+1}{z_2} = \frac{z_1+z_2^2+1}{z_1z_2}, \\
z_5 &= \frac{z_1^2+2z_1+1+z_2^2}{z_1z_2^2}, \\
z_6 &= \frac{z_1+1}{z_2}, \\
z_7 &= z_1, \quad z_8 = z_2, \quad \text{etc.}
\end{aligned}$$

So the sequence of cluster variables is **6-periodic**.

Example 8.2. $\mathcal{A}(1,3)$: The exchange relation is

$$z_{k-1}z_{k+1} = \begin{cases} z_k^3+1 & \text{if } k \text{ is even,} \\ z_k+1 & \text{if } k \text{ is odd.} \end{cases}$$

Set $z_1 = z_2 = 1$. Then:

$$\begin{aligned}
z_3 &= \frac{z_2^3+1}{z_1} = 2, \\
z_4 &= \frac{z_3+1}{z_2} = \frac{2+1}{1} = 3, \\
z_5 &= \frac{z_4^3+1}{z_3} = \frac{28}{2} = 14, \\
z_6 &= \frac{z_5+1}{z_4} = \frac{15}{3} = 5, \\
z_7 &= \frac{z_6^3+1}{z_5} = \frac{126}{14} = 9, \\
z_8 &= \frac{z_7+1}{z_6} = \frac{10}{5} = 2, \\
z_9 &= \frac{z_8^3+1}{z_7} = \frac{9}{9} = 1, \\
z_{10} &= \frac{z_9+1}{z_8} = \frac{2}{2} = 1.
\end{aligned}$$

So it is **8-periodic** at least after specializing $z_1 = z_2 = 1$, and we claim that it is 8-periodic even without this specialization.

Example 8.3. $\mathcal{A}(1,4)$: Setting $z_1 = z_2 = 1$ gives the sequence

$$1, 1, 2, 3, 41, 14, 937, 67, 21506, 321, \ldots$$

This is *not* periodic. However, all terms are integers, and in fact each z_k is a **Laurent polynomial** in z_1, z_2.

8.2 The Laurent phenomenon

Theorem 8.4 (Laurent phenomenon)**.** *Let $(\widetilde{x}_0, \widetilde{B}_0)$ be a labeled seed, with $\widetilde{x}_0 = (x_1, \ldots, x_m)$, and associated cluster algebra $\mathcal{A}$. Every cluster variable of $\mathcal{A}$ is a Laurent polynomial with integer coefficients in the variables $x_1, \ldots, x_m$. Moreover, $x_{n+1}, \ldots, x_m$ do not appear in the denominators.*

Remark 8.5. Note that we can replace $\widetilde{x}_0$ equally with any other extended cluster of $\mathcal{A}$.

Proof idea. Say $t_0 \in \mathbb{T}_n$ is the initial vertex with $(\widetilde{x}_0, \widetilde{B}_0)$ the initial (labeled) seed. Let $x = x(t)$ be a cluster variable in the seed at some vertex $t \in \mathbb{T}_n$, where $\widetilde{x}_0 = (x_1, \ldots, x_m)$. We want to show that x is a Laurent polynomial in $x_1, \ldots, x_m$. We use induction on $d = \operatorname{dist}(t, t_0)$.

Base cases:

- If $d = 1$, then $x(t_1) = (x_1, \ldots, x_j', \ldots, x_m)$ where

  $$x_j' = \frac{\displaystyle\prod_{b_{ij}>0} x_i^{b_{ij}} + \prod_{b_{ij}<0} x_i^{-b_{ij}}}{x_j},$$

  which is already a Laurent polynomial.

- If $d = 2$, then $x(t_2) = (x_1, \ldots, x_j', \ldots, x_k', \ldots, x_m)$ where

  $$x_k' = \frac{\text{poly in } x_1, \ldots, x_j', \ldots, x_m}{x_k} = \frac{\text{Laurent poly in } x_1, \ldots, x_m}{x_k}$$

  (or swap j and k).

Inductive step: Now assume $d \geq 3$, and assume for simplicity that $b^0_{jk} = b^0_{kj} = 0$ where $\widetilde{B}_0 = (b^0_{ij})$. (The case $b^0_{jk} b^0_{kj} < 0$ is more complicated.)

Put $t_3 := \mu_k(t_0)$ and $t_4 := \mu_j \mu_k(t_0)$. Consider the following portion of $\mathbb{T}_n$:

$$\begin{tikzcd}[arrows=dash]
t_0 \arrow[r, "k"] \arrow[d, "j"'] & t_3 \arrow[r, "j"] & t_4 & & \\
t_1 \arrow[r, "k"] & t_2 \arrow[r] & \cdots \arrow[r] & t
\end{tikzcd}$$

Note: $\widetilde{x}(t_4) = \widetilde{x}(t_2)$, so both t_1, t_3 lie at distance $d-1$ from a seed containing x. By induction:

$$x = \text{Laurent poly in } \widetilde{x}(t_1) = \text{Laurent poly in } \widetilde{x}(t_3).$$

Meanwhile, $x_j' = \frac{M_1+M_2}{x_j}$ and $x_k' = \frac{M_3+M_4}{x_k}$, for monomials M_1, M_2, M_3, M_4 in $x_1, \ldots, x_m$.

Substituting:

$$x = \frac{\text{poly in } x_1, \ldots, x_m}{(\text{monomial in } x_1, \ldots, x_m) \cdot (M_1 + M_2)^a} = \frac{\text{poly in } x_1, \ldots, x_m}{(\text{monomial in } x_1, \ldots, x_m) \cdot (M_3 + M_4)^b}.$$

It suffices to show that $a = 0$.

Let $\widetilde{B}_0^{\text{aug}}$ be $\widetilde{B}_0$ after adding an extra row of the form $(0, \ldots, 1, \ldots, 0)$ (with 1 in the jth entry). Let $\mathcal{A}_{\text{aug}}$ be the resulting cluster algebra with coefficient variables $x_{n+1}, \ldots, x_{m+1}$.

Observe: an expression in $\mathcal{A}_{\text{aug}}$ for x in terms of $x_1, \ldots, x_{m+1}$ specializes (setting $x_{m+1} = 1$) to an expression in $\mathcal{A}$ for x in terms of $x_1, \ldots, x_m$.

So x being a Laurent polynomial in $x_1, \ldots, x_m$ in $\mathcal{A}_{\text{aug}}$ implies x is a Laurent polynomial in $x_1, \ldots, x_m$ in $\mathcal{A}$, hence WLOG we can assume $\widetilde{B}_0^{\text{aug}}$ instead of $\widetilde{B}_0$.

But then

$$x_j' = \frac{M_1^{\text{aug}} + M_2^{\text{aug}}}{x_j} = \frac{M_1 x_{m+1} + M_2}{x_j}, \qquad x_k' = \frac{M_3^{\text{aug}} + M_4^{\text{aug}}}{x_k} = \frac{M_3 + M_4}{x_k}.$$

Then $M_1^{\text{aug}} + M_2^{\text{aug}}$ and $M_3 + M_4$ have no common factors (think about what happens if we specialize $x_1 = \cdots = x_m = 1$), which implies $a = 0$. □

8.3 Markov triples

Definition 8.6. A **Markov triple** is a triple $(a, b, c) \in \mathbb{Z}^3_{\geq 1}$ which satisfies the **Markov equation**:

$$a^2 + b^2 + c^2 = 3abc.$$

Example 8.7. $(1, 1, 1)$ is a Markov triple, and hence also its permutations. So is $(1, 2, 5)$ and its permutations: $(1, 5, 2)$, $(2, 1, 5)$, $(2, 5, 1)$, $(5, 1, 2)$, $(5, 2, 1)$.

Lemma 8.8. *If (a, b, c) is a Markov triple, then so is (a, b, c') with $c' = \frac{a^2+b^2}{c} = 3ab - c$.*

Proof. Consider the equation $a^2 + b^2 + t^2 = 3abt$, i.e., $t^2 - 3abt + (a^2 + b^2) = 0$. If c is one root, the other one c' must satisfy $c + c' = 3ab$, i.e.,

$$c' = 3ab - c = \frac{3abc - c^2}{c} = \frac{a^2 + b^2}{c}.$$

This operation is called **Markov mutation**. □

Lemma 8.9. *If (a, b, c) is a Markov triple and $a \leq b < c$, then $c' = 3ab - c < c$.*

Proof. Put $f(t) = t^2 - 3abt + (a^2 + b^2)$. Then

$$f(b) = b^2 - 3ab^2 + a^2 + b^2 = b^2(2 - 3a) + a^2 \leq -b^2 + a^2 \leq 0.$$

Then c', the other root of f, must satisfy $c' \leq b < c$. □

Corollary 8.10. *Every Markov triple can be connected to $(1, 1, 1)$ by a sequence of Markov mutations.*

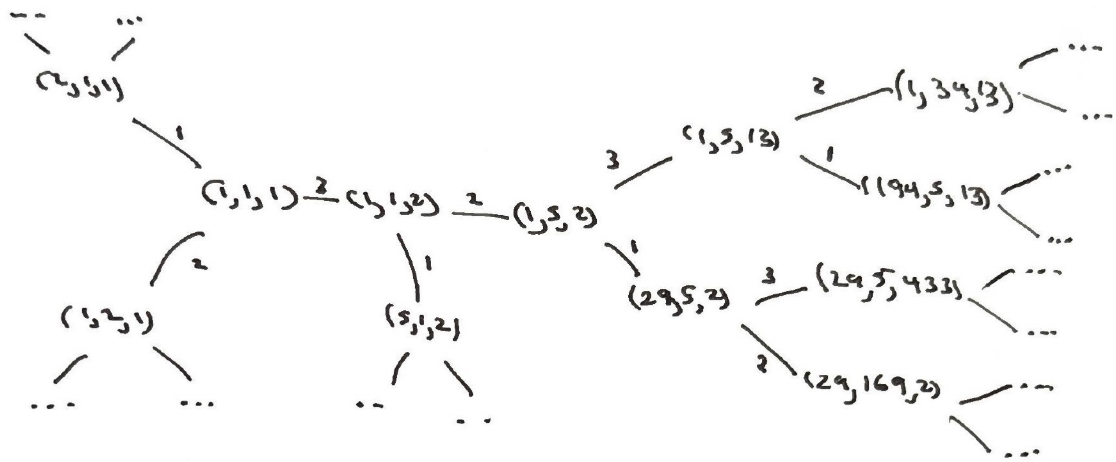

Figure 36: The Markov tree: nodes are Markov triples and edges correspond to Markov mutations.

8.4 The Markov tree

The Markov triples are organized into the **Markov tree**, where the edges correspond to Markov mutations (see Figure 36). Each edge is labeled by the index (1, 2, or 3) of the entry being mutated. For example:

$$(1,1,1) \overset{3}{—} (1,1,2) \overset{2}{—} (1,5,2) \overset{3}{—} (1,5,13) \overset{1}{—} (194,5,13), \quad \ldots$$

Recall: the **Markov quiver** is the quiver on three vertices with exchange relations (see Figure 37):

$$\begin{aligned} x_1' x_1 &= x_2^2 + x_3^2, \\ x_2' x_2 &= x_1^2 + x_3^2, \\ x_3' x_3 &= x_1^2 + x_2^2. \end{aligned}$$

Thus for any cluster $\widetilde{x} = (x_1, x_2, x_3)$, specializing the initial cluster variables to $(1,1,1)$ turns the trio into a Markov triple.

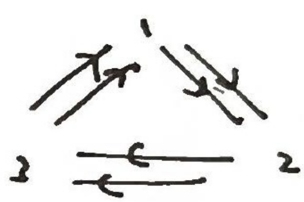

Figure 37: The Markov quiver: three vertices with double arrows forming a cycle.

8.5 The Somos-4 sequence

Example 8.11. The **Somos-4 sequence** is defined by $z_0 = z_1 = z_2 = z_3 = 1$ and the recurrence

$$\tilde{z}_{m+2}\tilde{z}_{m-2} = \tilde{z}_{m+1}\tilde{z}_{m-1} + \tilde{z}_m^2,$$

i.e., the sequence

$$1, 1, 1, 1, 2, 3, 7, 23, 59, 314, 1529, \ldots$$

Somos (1980s): these are all integers!

To explain this using cluster algebras, consider the quiver Q on four vertices with no frozen variables shown in Figure 38. The exchange relation at vertex 1 is

$$z_1 z_5 = z_2 z_4 + z_3^2, \qquad Q' = \mu_1(Q).$$

Then μ_1 rotates Q by $\pi/2$. Applying μ_2 to Q' gives

$$z_2 z_6 = z_3 z_5 + z_4^2.$$

Continuing in this way with the mutation sequence $\mu_1, \mu_2, \mu_3, \mu_4, \mu_1, \mu_2, \mu_3, \mu_4, \ldots$ gives

$$\tilde{z}_n = \text{Laurent polynomial in } z_1, z_2, z_3, z_4.$$

Specializing $z_1 = z_2 = z_3 = z_4 = 1$, the kth element of Somos-4 is necessarily an integer by the Laurent phenomenon.

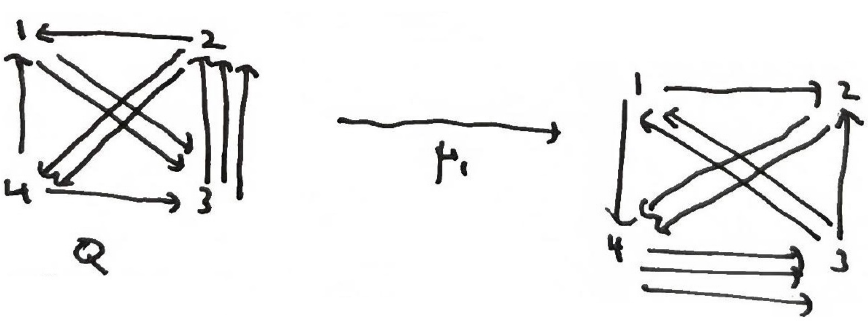

Figure 38: The quiver Q for the Somos-4 sequence (left) and its mutation $\mu_1(Q)$ (right), which is Q rotated by $\pi/2$.

9 Lecture 9

Date: February 11, 2026
Main reference: [FWZ21], §3.4, §3.5, §3.6.

9.1 The $\hat{y}$-variables

Let $(\widetilde{x}, \widetilde{B})$ be a labeled seed, with $\widetilde{x} = (x_1, \ldots, x_m)$, $\widetilde{B} = (b_{ij})$. Put $(\widetilde{x}', \widetilde{B}') = \mu_k(\widetilde{x}, \widetilde{B})$, with $\widetilde{x}' = (x'_1, \ldots, x'_m)$, $\widetilde{B}' = (b'_{ij})$.

Put $\hat{y} := (\hat{y}_1, \ldots, \hat{y}_n)$, where

$$\hat{y}_j = \prod_{i=1}^{m} x_i^{b_{ij}},$$

and similarly $\hat{y}' = (\hat{y}'_1, \ldots, \hat{y}'_n)$ with $\hat{y}'_j = \prod_{i=1}^{m} (x'_i)^{b'_{ij}}$.

Proposition 9.1. *We have (for $j = 1, \ldots, n$):*

$$\hat{y}'_j = \begin{cases} \hat{y}_k^{-1} & \text{if } j = k, \\ \hat{y}_j \left(\hat{y}_k^{-\operatorname{sgn}(b_{kj})} + 1 \right)^{-b_{kj}} & \text{else.} \end{cases}$$

Here $\operatorname{sgn}(b) = \begin{cases} 1 & \text{if } b > 0, \\ -1 & \text{if } b < 0. \end{cases}$

Remark 9.2.

- Recall that the exchange relation is

$$x_k x'_k = \prod_{b_{ik} > 0} x_i^{b_{ik}} + \prod_{b_{ik} < 0} x_i^{-b_{ik}}.$$

  Then $\hat{y}_k$ is the ratio of these two monomials.

- The above formula for $\hat{y}'_j$ depends only on the top $n \times n$ submatrix of $\widetilde{B}$.

Proof of Proposition 9.1. **Case $j = k$:** We have

$$\hat{y}'_k = \prod_{i=1}^{m} (x'_i)^{b'_{ik}} = \prod_{i \neq k} x_i^{b'_{ik}}.$$

Recall the mutation formula:

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } k \in \{i, j\}, \\ b_{ij} + |b_{ik}| b_{kj} & \text{if } b_{ik} b_{kj} > 0, \\ b_{ij} & \text{else.} \end{cases}$$

Since $k \in \{i, k\}$, we have $b'_{ik} = -b_{ik}$, so

$$\hat{y}'_k = \prod_{i \neq k} x_i^{-b_{ik}} = \hat{y}_k^{-1}.$$

Case $j \neq k$ and $b_{kj} \leq 0$:

$$\begin{aligned}
\hat{y}'_j &= (x'_k)^{b'_{kj}} \prod_{i \neq k} x_i^{b'_{ij}} \\
&= (x'_k)^{-b_{kj}} \left(\prod_{i \neq k} x_i^{b_{ij}} \right) \left(\prod_{b_{ik}<0} x_i^{-b_{ik} b_{kj}} \right) \\
&= x_k^{b_{kj}} \left(\prod_{b_{ik}>0} x_i^{b_{ik}} + \prod_{b_{ik}<0} x_i^{-b_{ik}} \right)^{-b_{kj}} \left(\prod_{i \neq k} x_i^{b_{ij}} \right) \left(\prod_{b_{ik}<0} x_i^{-b_{ik} b_{kj}} \right) \\
&= \left(\prod_i x_i^{b_{ij}} \right) (\hat{y}_k + 1)^{-b_{kj}} \\
&= \hat{y}_j \, (\hat{y}_k + 1)^{-b_{kj}} \, .
\end{aligned}$$

The case $j \neq k$, $b_{kj} \geq 0$ is similar. □

9.2 Y-seeds

Definition 9.3. A **Y-seed** of rank n in a field $\mathcal{F}$ is a pair (Y, B), where:

- $Y = (Y_1, \ldots, Y_n)$ is an n-tuple of elements in $\mathcal{F}$,
- B is a skew-symmetrizable $n \times n$ integer matrix.

We mutate Y-seeds as follows:

$$(Y, B) \xrightarrow{\mu_k} (Y', B'), \quad \text{where } B' = \mu_k(B),$$

$Y' = (Y'_1, \ldots, Y'_n)$ with

$$Y'_j = \begin{cases} Y_k^{-1} & \text{if } j = k, \\ Y_j \left(Y_k^{-\operatorname{sgn}(b_{kj})} + 1 \right)^{-b_{kj}} & \text{else.} \end{cases}$$

Thus a labeled seed $(\widetilde{x}, \widetilde{B})$ gives rise to a Y-seed $(\hat{y}, B)$, where B is the top $n \times n$ submatrix of $\widetilde{B}$ and $\hat{y} = (\hat{y}_1, \ldots, \hat{y}_n)$ with $\hat{y}_i = \prod_{i=1}^m x_i^{b_{ij}}$.

Remark 9.4. The seed mutation at k leaves x_j alone for $j \neq k$ but potentially changes all of $Y_1, \ldots, Y_n$. However, the formula for x'_k involves all of $x_1, \ldots, x_m$, whereas the formula for Y'_j only involves Y_k and Y_j.

9.3 Semifields

Definition 9.5. A **semifield** is an abelian group $\mathbb{P}$ (written multiplicatively) endowed with an auxiliary operation $\oplus$ which is commutative, associative, and distributive with respect to the group operation on $\mathbb{P}$. Note that $(\mathbb{P}, \oplus)$ is only a semigroup (i.e., not necessarily having an identity or inverses).

Example 9.6. The multiplicative group $\mathbb{Q}_{>0}$, with $\oplus$ given by ordinary addition.

Definition 9.7. The **tropical semifield** $\operatorname{Trop}(q_1, \ldots, q_\ell)$ is defined by:

- the multiplicative group of Laurent monomials in $q_1, \ldots, q_\ell$,
- the auxiliary addition (**tropical addition**):

$$\prod_{i=1}^{\ell} q_i^{a_i} \oplus \prod_{i=1}^{\ell} q_i^{b_i} = \prod_{i=1}^{\ell} q_i^{\min(a_i,b_i)}.$$

Check:

- Commutative: $\min(a_i, b_i) = \min(b_i, a_i)$.
- Associative: $\min(\min(a_i, b_i), c_i) = \min(a_i, \min(b_i, c_i))$.
- Distributive (i.e., $p(q \oplus r) = pq \oplus pr$): $\min(a_i, b_i) + c_i = \min(a_i + c_i, b_i + c_i)$.

9.4 Coefficient tuples and tropical Y-seed mutation

For a labeled seed $(\widetilde{x}, \widetilde{B})$ with $\widetilde{x} = (x_1, \ldots, x_n, x_{n+1}, \ldots, x_m)$ (where $x_{n+1}, \ldots, x_m$ are the frozen variables), we associate the **coefficient tuple**

$$y = (y_1, \ldots, y_n), \quad \text{where } y_j = \prod_{i=n+1}^{m} x_i^{b_{ij}} \in \mathrm{Trop}(x_{n+1}, \ldots, x_m)$$

for $j = 1, \ldots, n$.

Note 9.8. $B =$ top $n \times n$ submatrix of $\widetilde{B}$. Together with the coefficient tuple y, we recover the extended exchange matrix $\widetilde{B}$.

Proposition 9.9. *Let $\widetilde{B} = (b_{ij})$ be an extended skew-symmetrizable $m \times n$ matrix with coefficient tuple $y = (y_1, \ldots, y_n)$, and $\widetilde{B}' = (b'_{ij}) = \mu_k(\widetilde{B})$ with coefficient tuple $y' = (y'_1, \ldots, y'_n)$. Then*

$$y'_j = \begin{cases} y_k^{-1} & \text{if } j = k, \\ y_j \left(y_k^{-\operatorname{sgn}(b_{kj})} \oplus 1 \right)^{-b_{kj}} & \text{else.} \end{cases}$$

*This is called **tropical Y-seed mutation**.*

Definition 9.10. The **universal semifield** $\mathbb{Q}_{\mathrm{sf}}(x_1, \ldots, x_m)$ is

$$\left\{ \frac{P(x_1, \ldots, x_m)}{Q(x_1, \ldots, x_m)} \in \mathbb{Q}(x_1, \ldots, x_m) \;\middle|\; P, Q \text{ have positive coefficients} \right\}$$

with ordinary multiplication and addition.

Lemma 9.11. *Given any semifield $\mathbb{S}$ and elements $s_1, \ldots, s_m \in \mathbb{S}$, there exists a unique semifield homomorphism $\mathbb{Q}_{\mathrm{sf}}(x_1, \ldots, x_m) \to \mathbb{S}$ sending $x_i \mapsto s_i$ for $i = 1, \ldots, m$.*

Proof of Proposition 9.9. Let $f\colon \mathbb{Q}_{\mathrm{sf}}(x_1, \ldots, x_m) \to \mathrm{Trop}(x_{n+1}, \ldots, x_m)$ be the semifield homomorphism sending

$$f(x_i) = \begin{cases} 1 & \text{if } i \le n, \\ x_i & \text{if } i > n. \end{cases}$$

Note that f also sends x'_k to 1, since $x_k x'_k = M_1 + M_2$ implies

$$1 \cdot f(x'_k) = f(M_1) \oplus f(M_2) = 1$$

(since M_1, M_2 are monomials which share no frozen variables), so $f(x'_k) = 1$.

Also, $\hat{y}_j = \prod_{i=1}^{m} x_i^{b_{ij}}$ implies

$$f(\hat{y}_j) = \prod_{i=n+1}^{m} x_i^{b_{ij}} = y_j, \quad \text{for } j = 1, \ldots, n,$$

and similarly $f(\hat{y}'_j) = y'_j$.

Thus, applying f to the formula from Proposition 9.1:

$$\hat{y}'_j = \begin{cases} \hat{y}_k^{-1} & \text{if } j = k, \\ \hat{y}_j \left(\hat{y}_k^{-\operatorname{sgn}(b_{kj})} + 1 \right)^{-b_{kj}} & \text{else,} \end{cases}$$

gives

$$y'_j = \begin{cases} y_k^{-1} & \text{if } j = k, \\ y_j \left(y_k^{-\operatorname{sgn}(b_{kj})} \oplus 1 \right)^{-b_{kj}} & \text{else.} \end{cases}$$

□

10 Lecture 10

Date: February 13, 2026
Main reference: [FWZ21], §3.5, §3.6, §5.1.

10.1 Alternative characterization of labeled seeds

We can now give an alternative characterization of labeled seeds and their mutations. Fix $\mathcal{F} = \mathbb{C}(b_1, \ldots, b_m)$. A **labeled seed** is a triple $\Sigma = (x, y, B)$, where:

- **cluster** $x = (x_1, \ldots, x_n) \in \mathcal{F}^n$ such that $x \cup \{b_{n+1}, \ldots, b_m\}$ freely generates $\mathcal{F}$,
- **exchange matrix** B = skew-symmetrizable integer matrix,
- **coefficient tuple** $y = (y_1, \ldots, y_n)$, where y_j is a Laurent monomial in $\mathrm{Trop}(b_{n+1}, \ldots, b_m)$.

For a mutation

$$(x, y, B) \xrightarrow{\mu_k} (x', y', B'),$$

we have:

- $B' = \mu_k(B)$,
- y' is given by the tropical Y-seed mutation rule,
- $x' = (x \setminus \{x_k\}) \cup \{x'_k\}$, with

$$x_k x'_k = \frac{y_k}{y_k \oplus 1} \prod_{b_{ik} > 0} x_i^{b_{ik}} + \frac{1}{y_k \oplus 1} \prod_{b_{ik} < 0} x_i^{-b_{ik}}.$$

Key point: From this perspective, the complexity of the mutation process does not really grow with the number $m - n$ of frozen variables.

10.2 Example: A_2 revisited

Consider $B_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. This gives a labeled seed pattern on the line:

$$\cdots \overset{1}{\text{—}} \Sigma(-1) \overset{2}{\text{—}} \Sigma(0) \overset{1}{\text{—}} \Sigma(1) \overset{2}{\text{—}} \Sigma(2) \overset{1}{\text{—}} \Sigma(3) \overset{2}{\text{—}} \cdots$$

Write $\Sigma(t) = (x(t), y(t), B(t))$. Then $B(t) = (-1)^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

The cluster variables $x(t) = (x_1(t), x_2(t))$ and coefficient tuples $y(t) = (y_1(t), y_2(t))$ are given by:

t	$x_1(t)$	$x_2(t)$	$y_1(t)$	$y_2(t)$
0	x_1	x_2	y_1	y_2
1	$\dfrac{y_1+x_2}{x_1(y_1\oplus 1)}$	x_2	$\dfrac{1}{y_1}$	$\dfrac{y_1y_2}{y_1\oplus 1}$
2	$\dfrac{y_1+x_2}{x_1(y_1\oplus 1)}$	$\dfrac{x_1y_1y_2+y_1+x_2}{(y_1y_2\oplus y_1\oplus 1)\,x_1x_2}$	$\dfrac{y_2}{y_1y_2\oplus y_1\oplus 1}$	$\dfrac{y_1\oplus 1}{y_1y_2}$
3	$\dfrac{x_1y_2+1}{x_2(y_2\oplus 1)}$	$\dfrac{x_1y_1y_2+y_1+x_2}{(y_1y_2\oplus y_1\oplus 1)\,x_1x_2}$	$\dfrac{y_1y_2\oplus y_1\oplus 1}{y_2}$	$\dfrac{1}{y_1(y_2\oplus 1)}$
4	$\dfrac{x_1y_2+1}{x_2(y_2\oplus 1)}$	x_1	$\dfrac{1}{y_2}$	$y_1(y_2\oplus 1)$
5	x_2	x_1	y_2	y_1

10.3 Finite type classification in rank 2

Theorem 10.1. *A seed pattern with initial labeled seed $\Sigma = (x, y, B)$ with $B = \pm\begin{pmatrix} 0 & b \\ -c & 0\end{pmatrix}$, $b, c \in \mathbb{Z}_{\geq 1}$, is of finite type if and only if $bc \leq 3$.*

Compare:

Proposition 10.2. *For $b, c \in \mathbb{Z}_{\geq 1}$, the subgroup $W = \langle R_1, R_2\rangle \subset \mathrm{GL}_2$ generated by the reflections*

$$R_1 = \begin{pmatrix} -1 & b \\ 0 & 1\end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 0 \\ c & -1\end{pmatrix}$$

is finite if and only if $bc \leq 3$.

Proof. Since $R_1^2 = R_2^2 = \mathbb{I}$, W is finite if and only if R_1R_2 has finite order. We compute:

$$R_1R_2 = \begin{pmatrix} bc-1 & -b \\ c & -1\end{pmatrix}.$$

The characteristic equation is

$$\lambda^2 - (bc-2)\lambda + 1 = 0,$$

giving

$$\lambda = \frac{bc-2 \pm \sqrt{(bc-2)^2-4}}{2}.$$

For $bc = 1, 2, 3$: the roots have order $3, 4, 6$ respectively.
If $bc > 4$: the roots are real and not ± 1, hence R_1R_2 has infinite order.
If $bc = 4$:

$$(R_1R_2)^k = \begin{pmatrix} 2k+1 & -kb \\ kc & 1-2k\end{pmatrix},$$

which also has infinite order. □

Proof of Theorem 10.1. One can check that in the case $B = \pm\begin{pmatrix} 0 & 1 \\ -c & 0 \end{pmatrix}$, the seed pattern has 5 seeds if $c = 1$, 6 seeds if $c = 2$, and 8 seeds if $c = 3$.

Now assume $bc \geq 4$. Consider the seed pattern $(\Sigma(t))_{t\in\mathbb{Z}}$ with $\Sigma(t) = (x(t), y(t), B(t))$. Label the cluster variables as a sequence z_t ($t \in \mathbb{Z}$), where at each mutation step the new cluster variable gets the next index. The exchange relations become:

$$z_{t-1}z_{t+1} = \begin{cases} z_t^c + 1 & t \text{ even,} \\ z_t^b + 1 & t \text{ odd.} \end{cases}$$

Let $\mathbb{U} = \{u^r \mid r \in \mathbb{R}\}$ be the semifield with

$$u^r \oplus u^s = u^{\max(r,s)}, \quad u^r \cdot u^s = u^{r+s}$$

(u a formal variable).

Aim: Construct a semifield homomorphism $\Psi\colon \mathcal{F} \to \mathbb{U}$[1] such that $\{\Psi(z_t) \mid t \in \mathbb{Z}\}$ is infinite.

Case $bc > 4$**:** Let $\lambda > 1$ be a real eigenvalue of $\begin{pmatrix} bc-1 & -b \\ c & -1 \end{pmatrix}$.

Put $\Psi(z_1) = u^c$, $\Psi(z_2) = u^{\lambda+1}$.

The exchange relations become:

$$\Psi(z_{t-1})\Psi(z_{t+1}) = \begin{cases} \Psi(z_t)^c \oplus 1 & t \text{ even,} \\ \Psi(z_t)^b \oplus 1 & t \text{ odd.} \end{cases}$$

Claim 10.3. *$\Psi(z_{2k+1}) = u^{\lambda^k c}$ and $\Psi(z_{2k+2}) = u^{\lambda^k(\lambda+1)}$.*

Use induction. Writing $\Psi(z_t) = u^{\alpha_t}$:

$$\alpha_{2k+3} = c\,\alpha_{2k+2} - \alpha_{2k+1} = c \cdot \lambda^k(\lambda+1) - \lambda^k c = \lambda^k c(\lambda + 1 - 1) = \lambda^{k+1}c.$$

$$\begin{aligned} \alpha_{2k+4} &= b\,\alpha_{2k+3} - \alpha_{2k+2} = b \cdot \lambda^{k+1}c - \lambda^k(\lambda+1) \\ &= \lambda^k(bc\lambda - \lambda - 1) \\ &= \lambda^k(\lambda^2 + \lambda) \quad (\text{using } \lambda^2 - (bc-2)\lambda + 1 = 0) \\ &= \lambda^{k+1}(\lambda+1). \end{aligned}$$

Since $\lambda > 1$, $\alpha_t \to \infty$, so $\{\Psi(z_t)\}$ is infinite.

Case $bc = 4$**:** Instead use $\Psi(z_1) = u$, $\Psi(z_2) = u^b$.

Claim 10.4. *$\Psi(z_{2k+1}) = u^{2k+1}$ and $\Psi(z_{2k+2}) = u^{(k+1)b}$.*

This also follows by induction. Since $\alpha_t \to \infty$, the seed pattern has infinitely many cluster variables. □

[1] Warning: $\mathcal{F}$ is not quite the right domain - it should be $\mathbb{Q}_{\mathrm{sf}}(z_1, z_2)$.

10.4 2-finiteness

Definition 10.5. A skew-symmetrizable matrix $B = (b_{ij})$ is **2-finite** if for any $B' = (b'_{ij})$ mutation equivalent to B, we have $|b'_{ij}b'_{ji}| \leq 3$ for all i, j.

Corollary 10.6. *Finite type seed pattern* $\implies$ *every exchange matrix is* 2*-finite.*

Proof. If $B \sim B'$ with $|b'_{ij}b'_{ji}| \geq 4$ for some i, j, then by freezing all the cluster variables in that seed except for x_i, x_j, we are reduced to the rank 2 case. □

Remark 10.7. It turns out the converse to the above corollary is also true!

11 Lecture 11

Date: February 18, 2026
Main reference: [FWZ21], §5.2.

11.1 Cartan matrices and Dynkin diagrams

Definition 11.1. A **symmetrizable generalized Cartan matrix** is a square integer matrix $A = (a_{ij})$ such that:

- all diagonal entries are 2,
- all off-diagonal entries are ≤ 0,
- DA is symmetric for some diagonal matrix D with positive entries.

Definition 11.2. A **Cartan matrix** is a symmetrizable generalized Cartan matrix such that DA is positive definite (i.e. has only > 0 eigenvalues, or equivalently > 0 principal minors).

Note 11.3. For a Cartan matrix A, we must have

$$\det \begin{pmatrix} 2 & a_{ij} \\ a_{ji} & 2 \end{pmatrix} = 4 - a_{ij}a_{ji} > 0 \quad \text{for all } i \neq j,$$

i.e. $a_{ij}a_{ji} \leq 3$. In particular, $|a_{ij}|, |a_{ji}| \in \{0, 1, 2, 3\}$.

Example 11.4. $A = \begin{pmatrix} 2 & -b \\ -c & 2 \end{pmatrix}$ for $b, c \in \mathbb{Z}_{\geq 0}$ is Cartan if and only if it is one of:

- $b = c = 0$,
- $b = c = 1$,
- $b = 1, c = 2 \quad$ or $\quad b = 2, c = 1$,
- $b = 1, c = 3 \quad$ or $\quad b = 3, c = 1$.

Note that these "match" our classification of rank 2 cluster algebras of finite type.

Definition 11.5. Given an $n \times n$ Cartan matrix A, its **Dynkin diagram** $\mathrm{Dynk}(A)$ is the graph with vertices $1, \ldots, n$, where for each $i \neq j$ we put:

- a double edge with an arrow from i to j if $a_{ij} = -1$, $a_{ji} = -2$,
- a triple edge with an arrow from i to j if $a_{ij} = -1$, $a_{ji} = -3$,
- a single edge between i and j if $a_{ij} = a_{ji} = -1$.

Example 11.6. $A = \begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$. The Dynkin diagram $\mathrm{Dynk}(A)$ is of type B_3 (see Figure 39).

Example 11.7. $A = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$. The Dynkin diagram $\mathrm{Dynk}(A)$ is of type G_2 (see Figure 40).

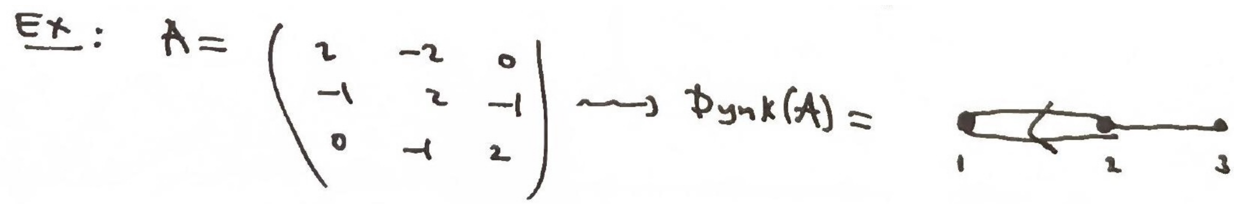

Figure 39: The Dynkin diagram of type B_3.

Figure 40: The Dynkin diagram of type G_2.

Note 11.8. The above Dynkin diagram conventions are unrelated to the fact that the quiver with 3 arrows between two vertices corresponds to the skew-symmetric matrix $\begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}$.

Definition 11.9. A Cartan matrix is **indecomposable** if its Dynkin diagram is connected. The **type** of A is its equivalence class up to simultaneous permutations of the rows and columns.

Note 11.10. Any Cartan matrix is equivalent to a block-diagonal matrix with indecomposable blocks which correspond to the connected components of the corresponding Dynkin diagram. The type of A is determined by the multiplicity of each type of connected Dynkin diagram appearing in such a decomposition.

Theorem 11.11 (Cartan–Killing)**.** *The Dynkin diagrams of indecomposable Cartan matrices are as follows (see Figure 41):*

$$A_n\ (n \geq 1), \quad B_n\ (n \geq 2), \quad C_n\ (n \geq 3), \quad D_n\ (n \geq 4), \quad E_6,\ E_7,\ E_8, \quad F_4, \quad G_2.$$

11.2 Finite type classification

Definition 11.12. Given an $n \times n$ skew-symmetrizable integer matrix $B = (b_{ij})$, its **Cartan counterpart** $\mathrm{Cart}(B)$ is the symmetrizable generalized Cartan matrix (a_{ij}), also $n \times n$, defined by

$$a_{ij} = \begin{cases} 2 & \text{if } i = j, \\ -|b_{ij}| & \text{if } i \neq j. \end{cases}$$

Theorem 11.13. *A cluster algebra is of finite type if and only if its seed pattern contains an exchange matrix B such that* $\mathrm{Cart}(B)$ *is a Cartan matrix.*

Theorem 11.14. *Suppose that B_1, B_2 are skew-symmetrizable integer matrices such that* $\mathrm{Cart}(B_1)$, $\mathrm{Cart}(B_2)$ *are Cartan. Then* $\mathrm{Cart}(B_1)$, $\mathrm{Cart}(B_2)$ *have the same type if and only if B_1 and B_2 are mutation equivalent.*

Recall: The classification of simple complex Lie algebras (or equivalently, compact simply connected Lie groups) is precisely:

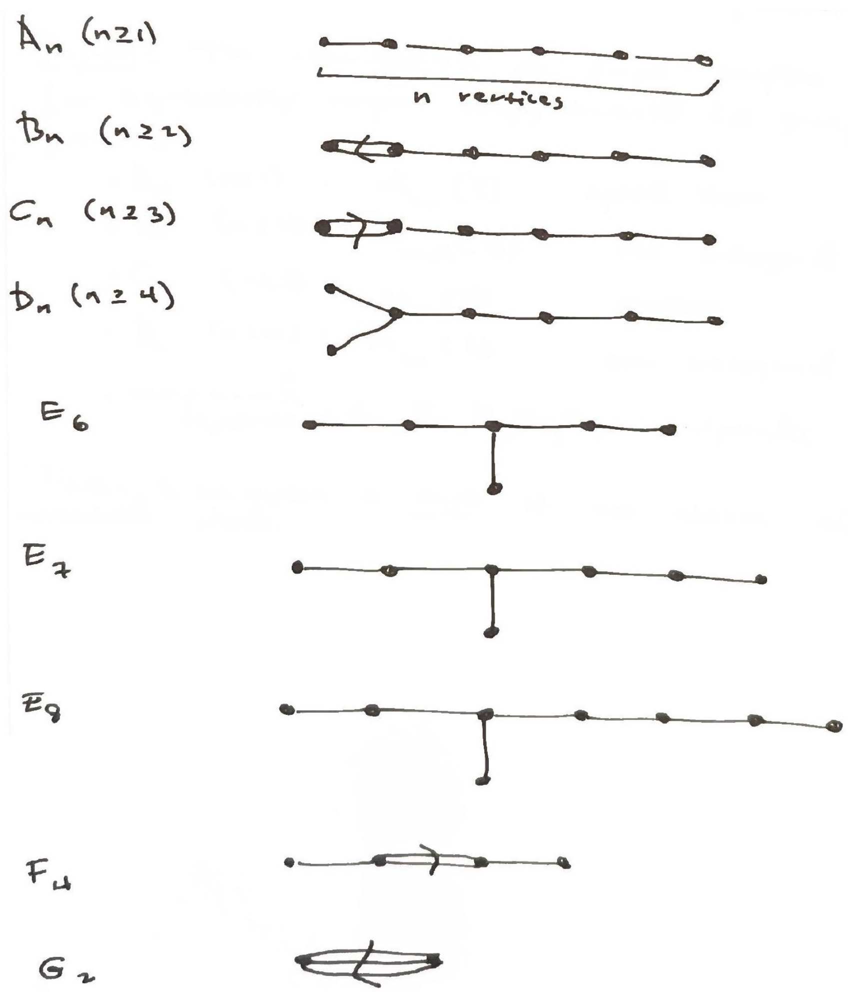

Figure 41: The Dynkin diagrams of indecomposable Cartan matrices (Theorem 11.11).

- A_n $(n \geq 1)$: $\mathfrak{sl}_{n+1}(\mathbb{C})$ (special linear)
- B_n $(n \geq 2)$: $\mathfrak{so}_{2n+1}(\mathbb{C})$ (odd orthogonal)
- C_n $(n \geq 3)$: $\mathfrak{sp}_{2n}(\mathbb{C})$ (symplectic)
- D_n $(n \geq 4)$: $\mathfrak{so}_{2n}(\mathbb{C})$ (even orthogonal)
- exceptional algebras: G_2, F_4, E_6, E_7, E_8 (sporadic)

Note 11.15. A Lie algebra is **simple** if it is not abelian and has no nontrivial ideals.

References

[FWZ21] Sergey Fomin, Lauren Williams, and Andrei Zelevinsky. *Introduction to Cluster Algebras*. Chapters 1–6, arXiv:1608.05735. 2021.

[Gro+18] Mark Gross, Paul Hacking, Sean Keel, and Maxim Kontsevich. “Canonical bases for cluster algebras”. In: *J. Amer. Math. Soc.* 31.2 (2018), pp. 497–608.