# Lecture 19

**Ex**: $(t-1)^2 y'' + 5(t-1)y' + 4y = 0.$ ← (shifted) Euler eqn

**Ansatz**: $y(t) = (t-1)^r$

$\longrightarrow r(r-1)+5r+4=0$

$r^2+4r+4=0$

$(r+2)^2=0 \Rightarrow r_1=r_2=-2$

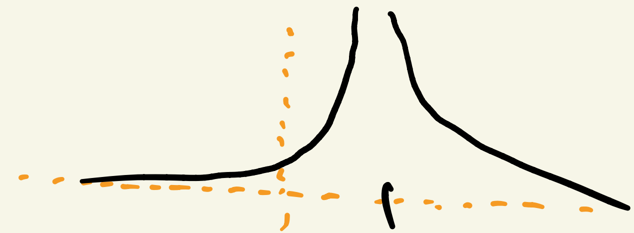
**Solns**: $y_1(t)=|t-1|^{-2}$

$y_2(t)=|t-1|^{-2}\ln|t-1|$

**Note**: $y(t)=\frac{1}{(t-1)^2}$ is also a soln for all $t\neq 1$

( $t=1$ is a regular singular pt )

Consider $\frac{1}{(t-1)^2}$




( similar to $\frac{1}{|t-1|}$ but "blows up faster" )

**Note**: $t=0$ is an ordinary pt, so could seek a sol'n of form
$$\sum_{n=0}^{\infty} a_n t^n.$$
could plug in, solve for $a_0, a_1, a_2$ recursively.

$$\frac{1}{1-x} = 1+x+x^2+x^3+x^4+\ldots$$
take derivative of both sides:
$$\frac{1}{(1-x)^2} = 1+2x+3x^2+4x^3+\ldots$$
so
$$\frac{1}{(t-1)^2} = 1+2t+3t^2+4t^3+\ldots$$

Let's write
$$\frac{1}{(t-1)^2} = \sum_{n=0}^{\infty} a_n t^n$$

**Note**: for $y(t) = \dfrac{1}{(t-1)^2}$,

have $y(0) = 1$
$$y'(t) = \frac{-2}{(t-1)^3}$$
$$y'(0) = 2$$

So $\dfrac{1}{(t-1)^2}$ is the unique sol'n to our ODE w/ these 2 IC's.

**Ex**: $\underbrace{(t-1)^2 y'' + 5(t-1)y' + (4-t)y = 0}_{t=1 \text{ is a regular sing pt.}}$

perturbation of Euler eq'n, w/ extra $-t$ term added

**Recall**: $y''(t) + p(t)y'(t) + q(t) = 0$, $t = t_0$ is a regular sing pt if $(t-t_0)p(t)$, $(t-t_0)^2 q(t)$ are analytic at $t = t_0$.

**Ex**: $2t^2y'' - ty' + (1+t)y = 0$ $\quad t > 0$.

regular sing pt at $t = 0$

**Ansatz**: $y(t) = \underbrace{t^r\left(\sum_{n=0}^{\infty} a_n t^n\right)}_{\text{"Frobenius series"}}$.

note: $r$ not necessarily $\geq 0$, or even an integer

$$y(t) = \sum_{n=0}^{\infty} a_n t^{n+r}$$

$$y'(t) = \sum_{n=0}^{\infty} a_n (n+r) t^{n+r-1}$$

$$y''(t) = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) t^{n+r-2}$$

**Plug in**: $2t^2\left(\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)t^{n+r-2}\right) - t\sum_{n=0}^{\infty} a_n(n+r)t^{n+r-1} + (1+t)\sum_{n=0}^{\infty} a_n t^{n+r} = 0$

$$y(t) = a_0 t^r + a_1 t^{r+1} + a_2 t^{r+2} + \dots$$

so $$y'(t) = a_0 r t^{r-1} + a_1 (r+1) t^r + \dots$$

<u>Want</u>:

$$\sum_{n=0}^{\infty} 2a_n (n+r)(n+r-1) t^{n+r} - \sum_{n=0}^{\infty} a_n (n+r) t^{n+r} + \sum_{n=0}^{\infty} a_n t^{n+r} + \sum_{n=0}^{\infty} a_n t^{n+r+1} = 0.$$

$$j = n+1 \rightsquigarrow \sum_{j=1}^{\infty} a_{j-1} t^{j+r} = \sum_{n=1}^{\infty} a_{n-1} t^{n+r}$$

<u>So get</u>:

$$\sum_{n=1}^{\infty} \left( \begin{array}{l} 2a_n (n+r)(n+r-1) \\ -a_n (n+r) \quad + a_n \\ + a_{n-1} \end{array} \right) t^{n+r} + \underbrace{\left( \begin{array}{l} 2a_0 r(r-1) \\ -a_0 r + a_0 \end{array} \right) t^r}_{t^r a_0 (2r(r-1) - r + 1)}$$

$t^r a_0 (2r(r-1) - r + 1)$ indicial polynomial for Euler eqn part

This Prob. series is identically 0 iff all coeffs are 0.

We must have:

(1) • $t^r$ coeff is zero $\Longrightarrow$ $2r(r-1) - r + 1 = 0$

(2) • $a_n\left(2(n+r)(n+r-1) - (n+r) + 1\right) = -a_{n-1}$ for all $n \geq 1$.

From (1), $2r^2 - 3r + 1 = 0$ $\qquad r = \dfrac{3 \pm \sqrt{9-8}}{4} = \dfrac{3 \pm 1}{4}$

$r_1 = 1, \quad r_2 = 1/2$

[ From now on, always take $r_1 \geq r_2$ ]

Case a: $r = 1$. Then (2) says:

$$a_n\Big(\underbrace{2(n+1)(n) - (n+1) + 1}_{2n^2 + 2n - n - 1 + 1}\Big) = -a_{n-1} \quad \text{for all } n \geq 1.$$

$$\Longrightarrow \quad 2n^2 + 2n - n - 1 + 1 = 2n^2 + n = n(2n+1)$$

So $a_n = \dfrac{-a_{n-1}}{n(2n+1)}$ $\qquad a_0$

$$a_1 = \frac{-a_0}{1\cdot 3}$$

$$a_2 = \frac{-a_1}{2\cdot 5} = \frac{a_0}{1\cdot3\cdot2\cdot5}$$

$$a_3 = \frac{-a_2}{3\cdot 7} = \frac{-a_0}{1\cdot3\cdot2\cdot5\cdot3\cdot7}$$

$$a_4 = \frac{-a_3}{4\cdot 9} = \frac{a_0}{1\cdot3\cdot2\cdot5\cdot3\cdot7\cdot4\cdot9} \quad \text{etc.}$$

So this gives a soln to ODE: (set $a_0=1$)

$$y_1(t) = \underbrace{t^1\left(1 - \frac{1}{1\cdot3}t + \frac{1}{1\cdot3\cdot2\cdot5}t^2 - \frac{1}{1\cdot3\cdot2\cdot5\cdot3\cdot7}t^3 + \dots\right)}_{\text{Frob. series, happens to be an ordinary power series}}$$

Since 2nd order ODE, need another indep soln.

Case 6) : $r = 1/2$.

Then (2) says $a_n\left(\underbrace{\underbrace{2(n+1/2)(n+1/2-1)-n+1/2}_{2(n+1/2)(n-1/2)-n+1/2}}_{2(n^2-1/4)-n+1/2}\right) = -a_{n-1}$

$$2n^2 - 1/2 - n + 1/2$$

$$= 2n^2 - n = n(2n-1)$$

→ so

$$a_n = \frac{-a_{n-1}}{n(2n-1)}$$

So

$a_0$

$$a_1 = \frac{-a_0}{1\cdot 1}$$

$$a_2 = \frac{-a_1}{2\cdot 3} = \frac{a_0}{1\cdot 1\cdot 2\cdot 3}$$

$$a_3 = \frac{-a_2}{3\cdot 5} = \frac{-a_0}{1\cdot 1\cdot 2\cdot 3\cdot 3\cdot 5} \quad \text{etc.}$$

So $$y_2(t) = \underbrace{t^{1/2}\left(1 - \frac{1}{1\cdot 1}t + \frac{1}{1\cdot 1\cdot 2\cdot 3}t^2 - \frac{1}{1\cdot 1\cdot 2\cdot 3\cdot 3\cdot 5}t^3 + \dots\right)}_{\text{Frob. series, not a power series}}$$

Note: Should show that $y_1, y_2$ have pos. r.o.c.

So far: Given $y'' + p(t)y' + q(t)y = 0$, assuming $t = t_0$ is a regular singular pt, can seek soln of form $y(t) = (t-t_0)^r \sum_{n=0}^{\infty} a_n (t-t_0)^n$.

In preceeding example, found 2 solns of this form, which gave fund set of solns.

In general could encounter a few issues:

- repeated roots

- complex roots
- if $r_1 - r_2$ is an integer also have trouble!

  (in ex, $r_1 - r_2 = 1/2$ so no prob)