475 SPRING 2025 PROBLEM SET #9

Problem 1. Let C denote the circle $\{|z| = 2\}$ with its counterclockwise orientation. Compute the following contour integrals.

(1) $\int_C \tan z \, dz$

(2) $\int_C \dfrac{dz}{\sinh 2z}$.

Problem 2. Find the singularities of the function $f(z) = z \sec z$ and determine their residues and singularity types.

Problem 3. Consider the function $f(z) = \dfrac{1}{q(z)^2}$, where q is analytic at z_0, $q(z_0) = 0$, and $q'(z_0) \neq 0$. Show that z_0 is a pole of order 2 of f, with residue $B_0 = -\dfrac{q''(z_0)}{q'(z_0)^3}$.

Problem 4. Let C be the circle $\{|z| = 2\}$ with its counterclockwise orientation, and put $f(z) = \dfrac{z^5}{1 - z^3}$. Compute the contour integral $\int_C f(z)$ by:

(a) using the residue theorem

(b) using a residue at infinity

and show that these give the same answer.

Problem 5. Let $P(z) = a_n z^n + \cdots + a_1 z + a_0$ be a complex polynomial of degree n, and let $Q(z) = b_m z^m + \cdots + b_1 z + b_0$ be a complex polynomial of degree m, where $m \geq n + 2$. Assume that all of the zeros of $Q(z)$ lie in the interior of a simple closed contour C. Prove that

$$\int_C \frac{P(z)}{Q(z)} = 0.$$

Hint: think about the residue at infinity.

Problem 6. Compute the following integral using residues:

$$\int_0^\infty \frac{x^2 dx}{x^6 + 1}.$$

Problem 7. Compute the following integral using residues:

$$\int_0^\infty \frac{\cos(17x)\, dx}{x^2 + 1}.$$