Math 635: Cluster Varieties

Algebra, Topology, Geometry, Duality

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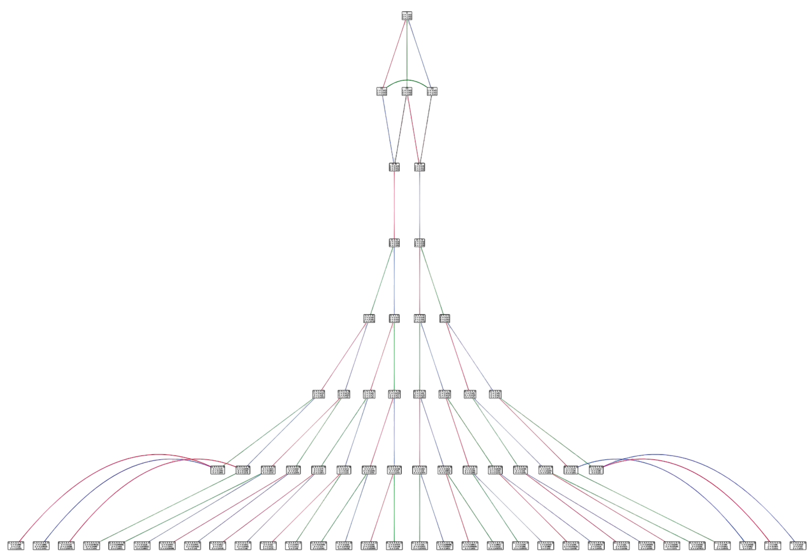

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1 Lecture 1

Date: January 12, 2026
Main reference: [FWZ21], Chapters 1 and 2.

1.1 Introduction

Roughly speaking:

- A **cluster variety** is a complex algebraic variety obtained by gluing together many copies of $(\mathbb{C}^*)^n$, where the gluing maps take a very particular form.
- A **cluster algebra** is the algebra of regular functions $f\colon V \to \mathbb{C}$ on a cluster variety.

Fomin–Zelevinsky, early 2000s: Introduced cluster algebras. They arise in many parts of mathematics and physics as a kind of "universal model" for mutation/wall-crossing phenomena:

- Quiver representation theory
- Teichmüller theory
- Poisson geometry
- Grassmannians
- Total positivity
- QFT scattering amplitudes (amplituhedron)
- Integrable systems
- String theory (BPS states)
- etc.

Gross–Hacking–Keel–Kontsevich (GHKK) [Gro+18]:

- Constructed canonical bases for cluster algebras.
- Established positivity of the Laurent phenomenon.
- Proof uses mirror symmetry for log Calabi–Yau varieties (which can be thought of as a generalization of toric varieties, related to almost toric fibrations in symplectic geometry).
- Many strong applications in representation theory, e.g., canonical bases for finite-dimensional irreducible representations of $\mathrm{SL}_n(\mathbb{C})$.

Remark 1.1. The canonical bases were originally found independently by Lusztig and Kashiwara in the early 1990s using quantum groups. Amazingly, the construction of GHKK uses only general geometry—no representation theory!

1.2 Total Positivity

Definition 1.2. A matrix $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ is **totally positive** (TP) if all of its minors are positive.

Gantmacher–Krein (1930s): If A is TP, then the eigenvalues of A are real, positive, and distinct.

Binet–Cauchy theorem: The TP matrices are closed under multiplication, and hence form a multiplicative semigroup $G_{>0}$.

Remark 1.3. We have $G_{>0} \subset G = \mathrm{SL}(5) \cap \cdots$ (the specific group depends on context).

Lusztig: Extended the definition of $G_{>0}$ to other semisimple Lie groups G.

More generally: If a given complex algebraic variety Z has a distinguished family Δ of regular functions $Z \to \mathbb{C}$, we define the **TP variety** by

$$Z_{>0} := \{z \in Z \mid f(z) > 0 \text{ for all } f \in \Delta\}.$$

Example 1.4. For $Z = \mathrm{Mat}_{n\times n}(\mathbb{C})$, $\mathrm{GL}_n(\mathbb{C})$, or $\mathrm{SL}_n(\mathbb{C})$, we recover the above notion of TP, where $\Delta = \{\text{minors}\}$.

Example 1.5. The **Grassmannian** $\mathrm{Gr}_{k,m}(\mathbb{C}) = \{k\text{-dimensional linear subspaces of } \mathbb{C}^m\}$, with $\Delta = \{\text{Plücker coordinates}\}$.

Example 1.6. Partial flag manifolds, homogeneous spaces for semisimple complex Lie groups, etc. (slight scaling ambiguity).

Lemma 1.7. *A matrix $A \in \mathrm{Mat}_{n\times n}$ has $\binom{2n}{n} - 1$ minors.*

Proof. The number of minors is

$$\# = \sum_{k=1}^{n} \binom{n}{k}^2.$$

By Vandermonde's identity:

$$\binom{m+w}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{w}{r-k}.$$

Setting $m = w = r = n$ gives

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2,$$

from which the result follows. □

Remark 1.8. Vandermonde's identity counts: both sides count the number of ways to choose a committee with r members from m men and w women. How many subcommittees have exactly k members?

Question 1.9. Can we check that $A \in \mathrm{Mat}_{n\times n}$ is TP by only testing a subset of the $\binom{2n}{n} - 1$ minors? How many tests are needed?

Example 1.10. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_{2\times 2}$. Define $\delta := ad - bc$, so $d = \frac{\delta + bc}{a}$. Thus, if $a, b, c, \delta > 0$, then δ is automatically positive. This reduces $\binom{4}{2} - 1 = 5$ checks to 4 checks.

The goal is "efficient TP testing."

1.3 Plücker Coordinates on Grassmannians

Given $A \in \mathrm{Mat}_{k\times m}$ of rank k, we have $\mathrm{rowspan}(A) =: [A] \in \mathrm{Gr}_{k,m}$.

For $J \subseteq \{1,\dots,m\}$ with $|J| = k$, the **Plücker coordinate** is

$$P_J(A) := k \times k \text{ minor of } A \text{ corresponding to columns } J.$$

Note 1.11. For $A, B \in \mathrm{Mat}_{k\times m}$ with $[A] = [B]$ (i.e., same row spans), the tuples $(P_J(A))_{|J|=k}$ and $(P_J(B))_{|J|=k}$ agree up to common rescaling. We thus get a map

$$\mathrm{Gr}_{k,m} \longrightarrow \mathbb{CP}^{N-1}, \quad N = \binom{m}{k}.$$

In fact, this is an embedding, called the **Plücker embedding**.

Let $\mathbb{C}[\mathrm{Mat}_{k\times m}]$ denote the coordinate ring of $\mathrm{Mat}_{k\times m}$, i.e., the polynomial algebra in variables x_{ij} for $1 \le i \le k$, $1 \le j \le m$.

Definition 1.12. The **Plücker ring** $R_{k,m}$ is the subring of $\mathbb{C}[\mathrm{Mat}_{k\times m}]$ generated by P_J over all $J \in \{1,\dots,m\}$ with $|J| = k$.

Claim 1.13. *The ideal of relations in $R_{k,m}$ is generated by certain quadratic relations called the **Grassmann–Plücker relations**.*

Definition 1.14. The **totally positive Grassmannian** $\mathrm{Gr}^+_{k,m}$ is the subset of $\mathrm{Gr}_{k,m}$ consisting of those points whose Plücker coordinates are all positive (up to common scaling).

Note 1.15. For $A \in \mathrm{Mat}_{k\times m}(\mathbb{R})$, we have $[A] \in \mathrm{Gr}^+_{k,m}$ if and only if all $k \times k$ minors of A have the same sign.

Question 1.16. For $A \in \mathrm{Mat}_{k\times m}(\mathbb{R})$, can we verify that all $k \times k$ minors are positive by only checking a subset of the $\binom{m}{k}$ minors? How many tests are needed?

(We may assume positive WLOG by rescaling.)

1.4 Positivity Testing for $\mathrm{Gr}_{2,m}$

Claim 1.17. *Given $A \in \mathrm{Mat}_{2\times m}$, put $P_{ij} := P_{\{i,j\}}$ for $1 \le i < j \le m$. To check that all 2×2 minors $P_{ij}(A) > 0$, it suffices to check only the $2m-3$ special ones.*

Note 1.18. $2m - 3 = \dim \mathrm{Gr}_{2,m} + 1$.

Lemma 1.19. *For $1 \le i < j < k < \ell \le m$, we have the three-term Grassmann–Plücker relation:*

$$P_{ik}P_{j\ell} = P_{ij}P_{k\ell} + P_{i\ell}P_{jk}.$$

Remark 1.20. For an inscribed quadrilateral, Ptolemy's theorem (2nd century) gives

$$AC \cdot BD = AB \cdot CD + BC \cdot AD.$$

Example 1.21. Let $A = \begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix}$. We verify $P_{13}P_{24} = P_{12}P_{34} + P_{14}P_{23}$, i.e.,

$$(ag - ce)(bh - df) = (af - be)(ch - dg) + (ah - de)(bg - cf). \quad \checkmark$$

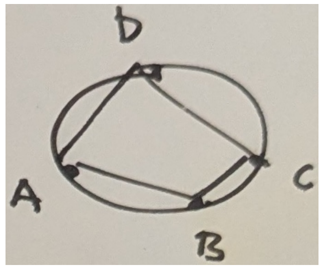

Figure 1: Inscribed quadrilateral for Ptolemy's theorem.

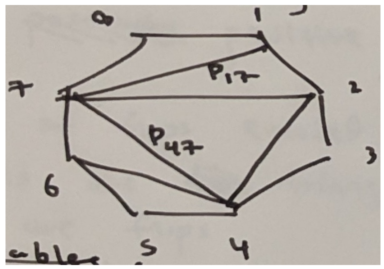

Figure 2: A triangulated polygon $\mathbb{P}_m$ with vertices labeled $1, \ldots, m$.

Put $\mathbb{P}_m$ = regular m-gon, and let T be a triangulation.
To each side or diagonal, associate P_{ij}, where i, j are the endpoints.

- **Cluster variables**: P_{ij} ranging over diagonals.
- **Frozen variables**: P_{ij} ranging over sides.
- **Extended cluster**: $\{\text{cluster vars}\} \cup \{\text{frozen vars}\} =: \widetilde{x}(T)$.

Note 1.22. The extended cluster has $2m-3$ variables, and we claim that these are algebraically independent.

Example 1.23. In the above picture, we have cluster variables $P_{17}, P_{27}, P_{47}, P_{24}$ and frozen variables $P_{12}, P_{23}, \ldots, P_{78}, P_{18}$.

Theorem 1.24. *Each P_{ij} for $1 \leq i < j \leq n$ can be written as a subtraction-free rational expression in the elements of a given extended cluster $\widetilde{x}(T)$.*

Corollary 1.25. *If each $P_{ij} \in \widetilde{x}(T)$ evaluates positively on a given $A \in \mathrm{Mat}_{2\times m}$, then all of the $2m-3$ of the $\binom{m}{2}$ minors of A are positive.*

Proof of Theorem. Follows by combining:

1. Each P_{ij} appears as an element of an extended cluster $\widetilde{x}(T)$ for some triangulation T of $\mathbb{P}_m$.
2. Any two triangulations of $\mathbb{P}_m$ are related by a sequence of **flips**.
3. For a flip, replace P_{ik} with $P_{j\ell}$. Using the three-term GP relation, we have

$$P_{ik} = \frac{P_{ij}P_{k\ell} + P_{i\ell}P_{jk}}{P_{j\ell}}.$$

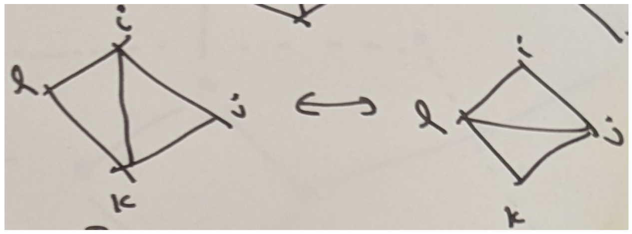

Figure 3: A flip replaces one diagonal with another in a quadrilateral.

Remark 1.26. In fact, each Plücker coordinate P_{ij} can be written as a Laurent polynomial with positive coefficients in the Plücker coordinates from $\widetilde{x}(T)$. This is an example of the **positive Laurent phenomenon**.

The combinatorics of flips is encoded by a graph:

- Vertices are triangulations.
- Edges are flips.

Each vertex has degree $m - 3$. In fact, this is the 1-skeleton of an $(m - 3)$-dimensional convex polytope called the **associahedron** (discovered by Stasheff).

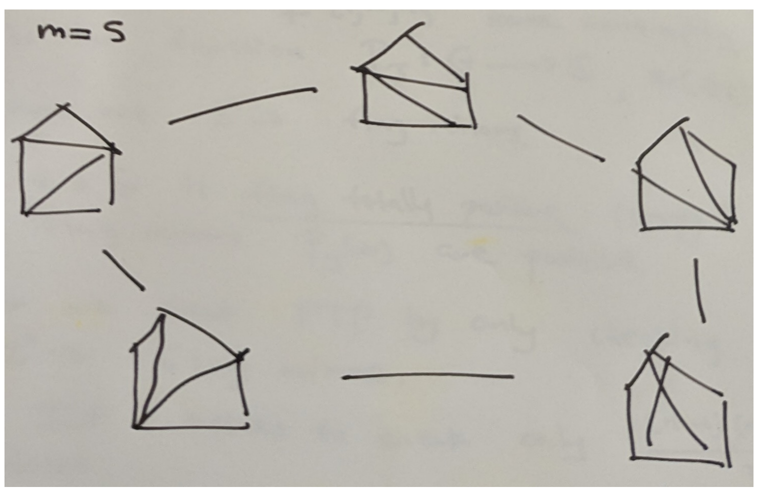

Figure 4: The associahedron for $m = 5$ (a pentagon).

Definition 1.27. A **cluster monomial** is a monomial in the variables of a given extended cluster $\widetilde{x}(T)$.

Theorem 1.28 (19th century invariant theory)**.** *The set of all cluster monomials gives a linear basis for the Plücker ring* $R_{2,m}$.

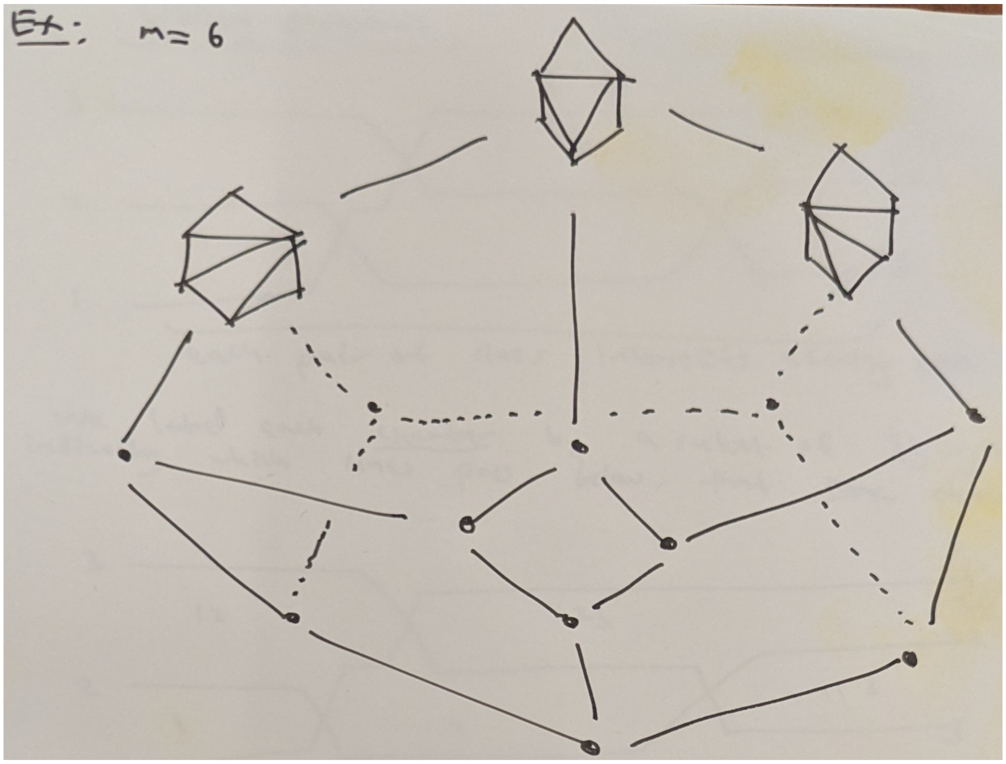

Figure 5: The associahedron for $m = 6$ (a 3-dimensional polytope).

2 Lecture 2

Date: January 14, 2026
Main reference: [FWZ21], Chapters 2 and 3.

2.1 Flag Positivity

Before moving to TP for $n \times n$ matrices, we discuss an intermediate notion called "flag positivity." Put $G = \mathrm{SL}_n$.

Definition 2.1. Given $J \subsetneq \{1, \ldots, n\}$ nonempty, the **flag minor** P_J is the function $P_J \colon G \to \mathbb{C}$ defined by

$$P_J(z) := z(\vec{e}_J) \mapsto \det\big(z_{\alpha\beta} \mid \alpha \leq |J|,\, \beta \in J\big),$$

i.e., the $|J| \times |J|$ minor which is "top-justified."

Note 2.2. There are $2^n - 2$ flag minors.

Definition 2.3. An element $z \in G$ is **flag totally positive** (FTP) if all flag minors $P_J(z)$ are positive.

Question 2.4. Can we check FTP by only checking a subset of the $2^n - 2$ flag minors?

Claim 2.5. *It suffices to check only $\frac{(n-1)(n+2)}{2}$ special flag minors.*

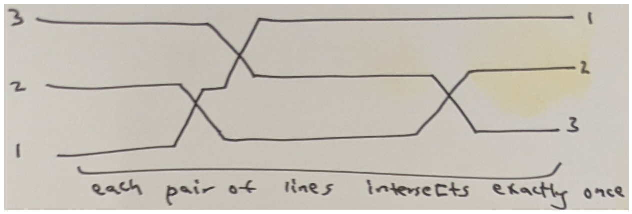

Figure 6: A wiring diagram for $n = 3$: each pair of lines intersects exactly once.

2.2 Wiring Diagrams

Each pair of lines intersects exactly once.

We label each **chamber** by a subset of $\{1, \dots, n\}$ indicating which lines pass below that chamber.

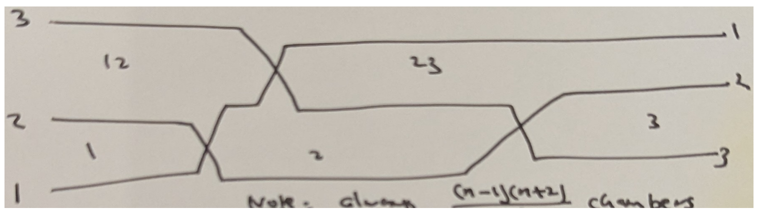

Figure 7: A wiring diagram with chamber labels.

Note 2.6. There are always $\frac{(n-1)(n+2)}{2}$ chambers.

Associated to each chamber is its **chamber minor** P_J, the flag minor corresponding to its subset $J \subsetneq \{1, \dots, n\}$.

Extended cluster: All chamber minors of a wiring diagram.

- **Cluster variables**: the chamber minors for bounded chambers.
- **Frozen variables**: the chamber minors for unbounded chambers.

There are $\frac{(n-1)n}{2}$ of these (the bounded chambers).

Theorem 2.7. *Every flag minor can be written as a subtraction-free rational expression in the chamber minors of a given wiring diagram.*

Corollary 2.8. *If the $\frac{(n-1)(n+2)}{2}$ chamber minors evaluate positively at a matrix $z \in \mathrm{SL}_n$, then z is FTP.*

Proof outline. Follows by:

1. Each flag minor appears as a chamber minor in some wiring diagram.
2. Any two wiring diagrams can be transformed into each other by a sequence of local **braid moves**.
3. Under each braid move, the collection of chamber minors changes by exchanging $Y \leftrightarrow Z$, and we have
$$YZ = AC + BD.$$

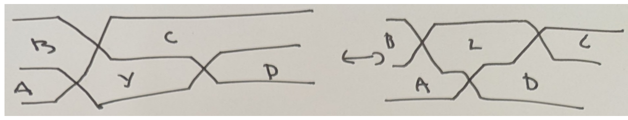

Figure 8: A braid move exchanges two adjacent crossings.

□

Remark 2.9. In fact, each flag minor can be written as a Laurent polynomial with positive coefficients in the chamber minors of a given wiring diagram.

References

[FWZ21] Sergey Fomin, Lauren Williams, and Andrei Zelevinsky. *Introduction to Cluster Algebras*. Chapters 1–6, arXiv:1608.05735. 2021.

[Gro+18] Mark Gross, Paul Hacking, Sean Keel, and Maxim Kontsevich. “Canonical bases for cluster algebras”. In: *J. Amer. Math. Soc.* 31.2 (2018), pp. 497–608.