Problem Set 3

2.1

33)

(1) when $a = \lambda$

$y' + ay = be^{-at}$

$e^{at}y' + ae^{at}y = b$

$(e^{at}y)' = b$

$e^{at}y = bt + C$

$y = bte^{-at} + Ce^{-at}$ (by L'Hospital's Rule)

$\lim_{t\to\infty}\left(b\frac{t}{e^{at}} + Ce^{-at}\right) = \lim_{t\to\infty}\frac{bt}{e^{at}} = \lim_{t\to\infty}\frac{b}{ae^{at}} = 0$

Therefore, $y \to 0$ as $t \to \infty$

(2) When $a \neq \lambda$

$y' + ay = be^{-\lambda t}$ (multiply both sides by e^{at})

$e^{at}y' + ae^{at}y = be^{(a-\lambda)t}$

$(e^{at}y)' = be^{(a-\lambda)t}$

$e^{at}y = \frac{b}{a-\lambda}e^{(a-\lambda)t} + C$

$y = \frac{b}{a-\lambda}e^{-\lambda t} + Ce^{-at}$

$\lim_{t\to\infty}\left(\frac{b}{a-\lambda}e^{-\lambda t} + Ce^{-at}\right) = 0$

Therefore, $y \to 0$ as $t \to \infty$

(We need to separate $a = \lambda$ and $a \neq \lambda$, as $\frac{1}{a-\lambda}$ doesn't exist when $a = \lambda$)

Thus, $y \to 0$ as $t \to \infty$

2.1

38) $y' + p(t)y = g(t)$

(a) If $g(t) = 0$ for all t,

$$\frac{dy}{dt} + p(t)y = 0$$

$$\frac{dy}{dt} = -p(t)y$$

$$\frac{1}{y}\,dy = -p(t)\,dt$$

$$\ln|y| = -\int p(t)\,dt + C$$

$$y = \pm e^{-\int p(t)\,dt + C} = Ae^{-\int p(t)\,dt} \quad (\text{let } A = \pm e^{c})$$

Therefore, if $g(t) = 0$ for all t,

$$y = A\exp\left[-\int p(t)\,dt\right]$$

(b) Suppose $y = A(t)\,e^{-\int p(t)dt}$

Then, $y' = A'(t)e^{-\int p(t)\,dt} - p(t)A(t)\,e^{-\int p(t)\,dt}$

$$y' + p(t)\,y = A'(t)e^{-\int p(t)dt} - p(t)\,A(t)e^{-\int p(t)\,dt} + p(t)\,A(t)\,e^{-\int p(t)\,dt}$$

$$= A'(t)\,e^{-\int p(t)dt} = g(t)$$

Therefore, if $g(t)$ is not everywhere zero,

$$A'(t) = g(t)\exp\left[\int p(t)\,dt\right]$$

(c) Integrate both sides of (b)

$$\int A'(t)\,dt = \int g(t)\,e^{\int p(t)dt}\,dt$$

$\mu(t) = e^{\int p(t)\,dt}$, so

$$A(t) = \int g(t)\,\mu(t)\,dt + k$$

Since $y = A(t)\exp\left[-\int p(t)\,dt\right]$ (this is given in 38(b)),

$$y = \left[\int g(t)\,\mu(t)\,dt + k\right]e^{-\int p(t)\,dt}$$

$= \frac{1}{\mu(t)}\left[\int_{t_0}^{t} \mu(s) g(s)\, ds + c\right]$

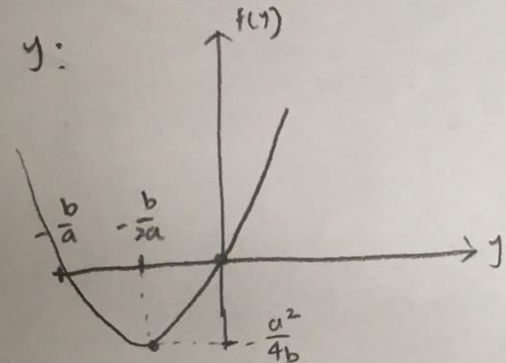
This solution agrees with the solution given in (33)

$b \times \frac{a^2}{4b^2} = \frac{a^2}{4b}$

2.5

2) $\frac{dy}{dt} = f(y) \Rightarrow f(y) = ay + by^2 = b\left(y^2 + \frac{a}{b}y\right) = b\left(y + \frac{a}{2b}\right)^2 - \frac{a^2}{4b}$

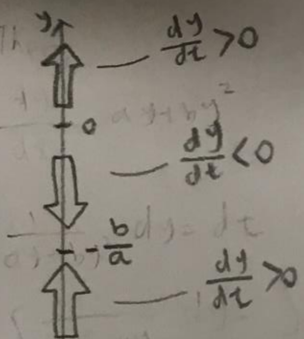
Graph of $f(y)$ versus y:



Critical Points: Points where $\frac{dy}{dt} = 0$ ⇒ It was covered in Calc I

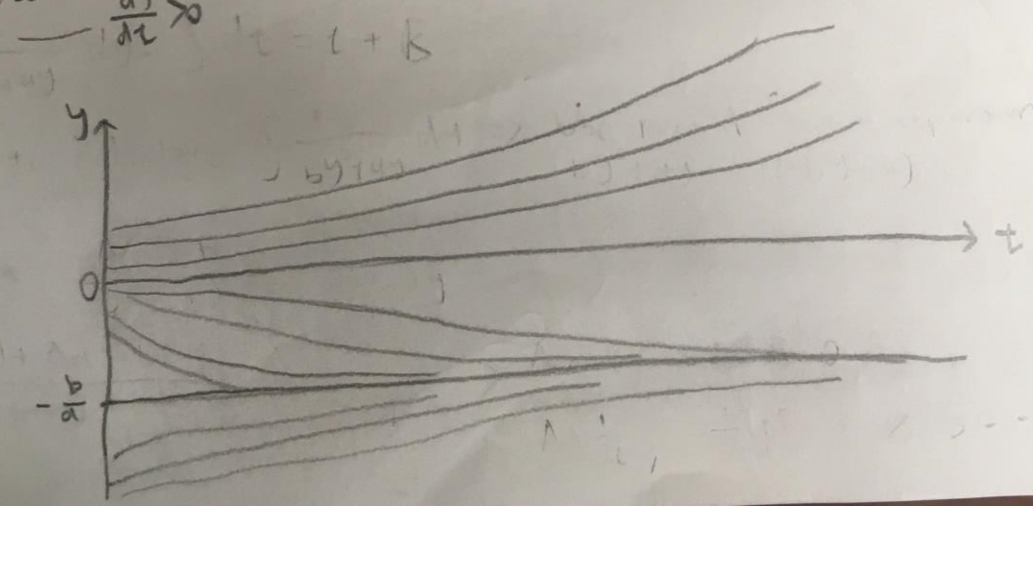
$\frac{dy}{dt} = ay + by^2 = y(a+by) = 0 \Rightarrow \boxed{y=0 \text{ and } y=-\frac{a}{b}}$ are critical points

Draw phase diagram based on graph of $f(y)$ versus y



From this, we can understand that $y=0$ is asymptotically unstable, while $y=-\frac{b}{a}$ is asymptotically stable

(At very close points to $y=0$, the solution doesn't approach to $y=0$, but at very close points to $y=-\frac{b}{a}$, the solution approaches $y=-\frac{a}{b}$)



⇒ these lines can be drawn just by using asymptotically stable vs unstable. We don't need to find the exact solution.

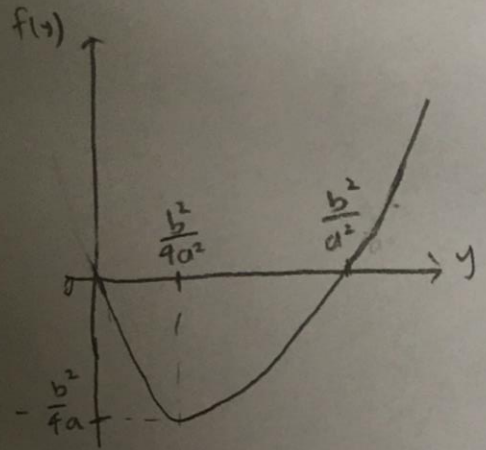
2.5

11) $\frac{dy}{dt} = ay - b\sqrt{y} = f(y)$

$\frac{d}{dy}(ay - b\sqrt{y}) = a - \frac{b}{2\sqrt{y}} = 0 \Rightarrow a = \frac{b}{2\sqrt{y}} \Rightarrow 2\sqrt{y} = \frac{b}{a} \Rightarrow 4y = \frac{b^2}{a^2}$

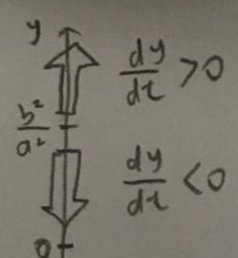
$\Rightarrow y = \frac{b^2}{4a^2}$ (this will be local min point)

$ay - b\sqrt{y} = 0 \Rightarrow a^2y^2 = b^2 y \Rightarrow y(a^2y - b^2) = 0 \Rightarrow y = 0$ or $y = \frac{b^2}{a^2}$

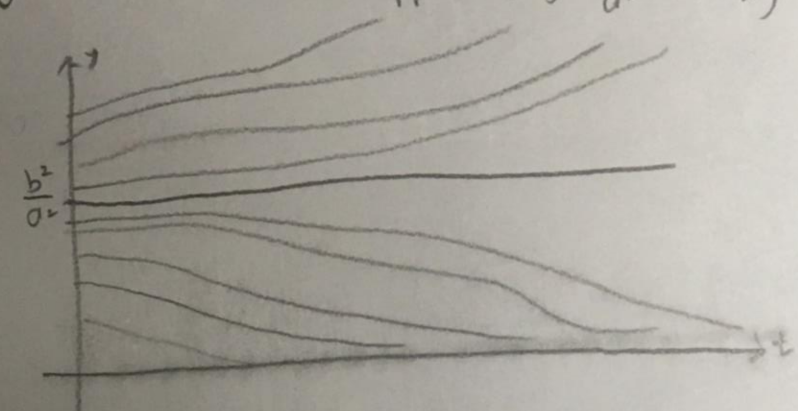


$a \times \frac{b^2}{4a^2} - b \times \sqrt{\frac{b^2}{4a^2}} = \frac{b^2}{4a} - \frac{b^2}{2a} = -\frac{b^2}{4a}$

Draw phase line from this graph



From this, $y=0$ will be asymptotically stable (solution will approach $y=0$), and $y = \frac{b^2}{a^2}$ will be asymptotically unstable (solution will not approach $y = \frac{b^2}{a^2}$ at all)



$\Rightarrow$ we were able to find this just by the phase line

2.5

15) $\frac{dy}{dt} = r\left[y - \frac{y^2}{k}\right] = \frac{r}{k} \cdot (ky - y^2)$

$\frac{k}{ky - y^2}\, dy = r\, dt$

$\int \frac{k}{ky - y^2}\, dy = rt + c$

$\int \frac{k}{ky - y^2}\, dy$ can be solved by splitting into partial fractions

$k \cdot \frac{1}{y(k-y)} = k \cdot \frac{A}{y} + \frac{B}{k-y} = k \cdot \frac{Ak - Ay + By}{y(k-y)}$

Therefore $\begin{cases} Ak = 1 \\ -A + B = 0 \end{cases} \Rightarrow A = \frac{1}{k},\ B = \frac{1}{k}$

$\int \frac{k}{ky - y^2}\, dy = \int k \cdot \left(\frac{1}{ky} + \frac{1}{k(k-y)}\right) dy = \int \frac{1}{y} + \frac{1}{k-y}\, dy$

$= \ln|y| - \ln|k-y| = \ln\left|\frac{y}{k-y}\right| = rt + c$

$\frac{y}{k-y} = \pm e^{rt+c} = Ce^{rt}$

$y = (k-y)Ce^{rt} = kCe^{rt} - Cye^{rt}$

$y(1 + Ce^{rt}) = kCe^{rt} \Rightarrow y = \frac{kCe^{rt}}{1 + Ce^{rt}}$

(a) We know that $y(0) = y_0 = \frac{k}{3}$

$\frac{kC}{1+C} = \frac{k}{3} \Rightarrow 3kC = k + Ck \Rightarrow 2kC = k \Rightarrow C = \frac{1}{2}$

$y = \frac{\frac{k}{2}e^{rt}}{1 + \frac{1}{2}e^{rt}} = \frac{ke^{rt}}{2 + e^{rt}}$

So at time τ,

$\frac{ke^{r\tau}}{2 + e^{r\tau}} = 2y_0 = \frac{2k}{3}$

$3ke^{r\tau} = 4k + 2ke^{r\tau}$

$e^{r\tau} = 4$

$r\tau = \ln 4 \Rightarrow \boxed{\tau = \frac{\ln 4}{r}}$

at $r = 0.025$, $\quad \tau = \frac{1000}{25}\ln 4 = 40\ln 4 \qquad \boxed{\tau = 40\ln 4}$

(b) From the previous section, we know $y = \frac{kCe^{rt}}{1+Ce^{rt}}$

$y_0 = k\alpha \Rightarrow y(0) = K\alpha$

$\frac{kC}{1+C} = k\alpha \Rightarrow \frac{C}{1+C} = \alpha \Rightarrow C = \alpha + C\alpha \Rightarrow (1-\alpha)C = \alpha$

$\Rightarrow C = \frac{\alpha}{1-\alpha}$

$y = \frac{\alpha}{1-\alpha} \cdot \frac{ke^{rt}}{1+\frac{\alpha}{1-\alpha}e^{rt}} = \frac{k\alpha e^{rt}}{1-\alpha+\alpha e^{rt}}$

$\frac{y(T)}{k} = \frac{\alpha e^{rT}}{1-\alpha+\alpha e^{rT}} = \beta$

$\alpha e^{rT} = (1-\alpha)\beta + \alpha\beta e^{rT} \Rightarrow \alpha(1-\beta)e^{rT} = (1-\alpha)\beta$

$\Rightarrow e^{rT} = \frac{(1-\alpha)\beta}{\alpha(1-\beta)} \Rightarrow rT = \ln\left(\frac{(1-\alpha)\beta}{(1-\beta)\alpha}\right) \Rightarrow \boxed{T = \frac{1}{r}\ln\left(\frac{(1-\alpha)\beta}{(1-\beta)\alpha}\right)}$

$\lim_{\alpha\to 0} T = \lim_{\alpha\to 0} \frac{1}{r}\ln\left(\frac{1-\alpha}{1-\beta}\cdot\frac{\beta}{\alpha}\right) = \frac{1}{r}\lim_{\alpha\to 0}\left(\ln\left[\left(\frac{1-\alpha}{\alpha}\right)\cdot\left(\frac{\beta}{1-\beta}\right)\right]\right] = \infty$

This holds true, as $\lim_{\alpha\to 0}\frac{1-\alpha}{\alpha} = \frac{1}{0} = \infty$

$\lim_{\beta\to 1} T = \lim_{\beta\to 1}\frac{1}{r}\ln\left[\left(\frac{1-\alpha}{\alpha}\right)\cdot\left(\frac{\beta}{1-\beta}\right)\right] = \infty$

This holds true as $\lim_{\beta\to 1}\frac{\beta}{1-\beta} = \frac{1}{0} = \infty$

So, $T\to\infty$ as $\alpha\to 0$ or as $\beta\to 1$

$$T = \frac{1}{r}\ln\left(\frac{1-\alpha}{\alpha}\cdot\frac{\beta}{1-\beta}\right)$$

$$= \frac{1000}{25}\times\ln\left(\frac{0.9}{0.1}\cdot\frac{0.9}{0.1}\right) = 40\ln 81 \Rightarrow \boxed{T = 160\ln 3}$$

23)

(a) $\frac{dy}{dt} = -\beta y$

$$\frac{1}{y}\,dy = -\beta\,dt$$

$$\int\frac{1}{y}\,dy = -\beta\int dt$$

$$\ln|y| = -\beta t + k$$

$$y = e^{-\beta t + k} = Ce^{-\beta t}$$

Since $y(0) = y_0$, $\underline{y = y_0 e^{-\beta t}}$ //

(b) $\frac{dx}{dt} = -\alpha x y = -\alpha x y_0 e^{-\beta t}$

$$\frac{1}{x}\,dx = -\alpha y_0 e^{-\beta t}\,dt$$

$$\int\frac{1}{x}\,dx = \int -\alpha y_0 e^{-\beta t}\,dt \quad (\alpha \text{ and } y_0 \text{ are constants})$$

$$\ln|x| = \frac{\alpha y_0}{\beta}e^{-\beta t} + k$$

$$x = \exp\left[\frac{\alpha y_0}{\beta}e^{-\beta t} + k\right] = C\exp\left[\frac{\alpha y_0}{\beta}e^{-\beta t}\right]$$

$(\exp(k) = C)$

Since $x(0) = x_0$,

$x(0) = C\exp\left[\frac{\alpha y_0}{\beta}\right] = x_0 \Rightarrow C = x_0 e^{-\frac{\alpha y_0}{\beta}}$

$$x = x_0\exp\left[-\frac{\alpha y_0}{\beta}\right]\times\exp\left[\frac{\alpha y_0}{\beta}e^{-\beta t}\right]$$

$$\Rightarrow \boxed{x = x_0\exp\left[\frac{\alpha y_0}{\beta}\left(e^{-\beta t} - 1\right)\right]}$$

(c) From (b),

$$x = x_0 \exp\left[\frac{\alpha y_0}{\beta} e^{-\beta t} - \frac{\alpha y_0}{\beta}\right]$$

Since $\lim_{t\to\infty} e^{-\beta t} = 0$,

$$\lim_{t\to\infty} x = \boxed{x_0 \exp\left[-\frac{\alpha y_0}{\beta}\right]}$$

2.6

24) $M + N y' = 0$

Multiply both sides by $\mu(xy)$

$\mu(xy) M + \mu(xy) N y' = 0$

This is exact if and only if

$(\mu(xy) M)_y = (\mu(xy) N)_x$

$(\mu(xy))_x = y \mu_x(xy)$, $(\mu(xy))_y = x \mu_y(xy)$ (from product rule)

So, $x\mu_y(xy)M + \mu(xy)M_y = x\mu_x(xy)N + \mu(xy)N_x$

$x\mu_y(xy)M - y\mu_x(xy)N = \mu(N_x - M_y)$

Let $xy = t$

Then, $\mu_x = \mu_y = \mu_t$

The equation will be

$\mu_t x M - \mu_t y N = \mu(N_x - M_y)$

$\Rightarrow \dfrac{\mu_t}{\mu(t)} = \dfrac{N_x - M_y}{Mx - Ny} = R$ (This is given in the problem)

Therefore,

$\ln \mu(t) = \int R\,dt$

$\mu(t) = e^{\int R\,dt} \Rightarrow \boxed{\mu(t) = \exp \int R(t)\,dt}$, where $t = xy$

31. Use problem 24 as a hint

$$\left(3x + \frac{6}{y}\right) + \left(\frac{x^2}{y} + 3\frac{y}{x}\right)\frac{dy}{dx} = 0$$

this is the same as y'

Let $M = 3x + \frac{6}{y}$, $N = \frac{x^2}{y} + 3\frac{y}{x}$

$N_x = \frac{2x}{y} - 3\frac{y}{x^2}$, $M_y = -\frac{6}{y^2}$

$$N_x - M_y = \frac{2x}{y} - 3\frac{y}{x^2} + \frac{6}{y^2} = \frac{2x^3y^2 - 3y^3 + 6x^2}{x^2y^2}$$

$$xM - yN = 3x^2 + \frac{6x}{y} - x^2 - 3\frac{y^2}{x} = 2x^2 + \frac{6x}{y} - 3\frac{y^2}{x} = \frac{2x^3y + 6x^2 - 3y^3}{xy}$$

$$\frac{N_x - M_y}{xM - yN} = \frac{2x^3y^2 - 3y^3 + 6x^2}{x^2y^2} \times \frac{xy}{2x^3y^2 - 3y^3 + 6x^2} = \frac{1}{xy}$$

Since this depends on the quantity xy only, we can use the same technique as 24).

Here, we showed that the general formula for the integrating factor is:

$$\mu(t) = \exp\int R(t)\,dt = \exp\int \frac{1}{t}\,dt = \exp\left[\ln(t)\right] = t$$

Therefore, the integrating factor is $\mu(xy) = xy$

$\Rightarrow$ Multiply the original function by xy

$$(3x^2y + 6x) + (x^3 + 3y^2)\frac{dy}{dx} = 0$$

Let $M = 3x^2y + 6x$, $N = x^3 + 3y^2$

Since $M_y = N_x = 3x^2$, this equation is exact

Therefore, there exists a function $\psi(x,y)$ such that

$\psi_x(x,y) = M$, $\psi_y(x,y) = N$

$\psi_x(x,y) = M = 3x^2y + 6x$

$\Rightarrow \psi(x,y) = x^3y + 3x^2 + f(y)$

$\psi_y(x,y) = N = x^3 + 3y^2$

Substitute $\psi(x,y) = x^3y + 3x^2 + f(y)$

$\psi_y = x^3 + f'(y) = x^3 + 3y^2 \Rightarrow f'(y) = 3y^2 \Rightarrow f(y) = y^3$

This implies that $\psi(x,y) = x^3y + 3x^2 + y^3$, so the solution is

$\boxed{x^3y + 3x^2 + y^3 = C}$

Chapter 2.7

4) Euler's method: $y_{n+1} = y_n + f_n \cdot (t_{n+1} - t_n)$

(a) Since $h = 0.1$, $y_{n+1} = y_n + y'_n \cdot 0.1$ / $y' = 3\cos t - 2y$

t	y	y'
0	0	$3 - 0 = 3$
0.1	$0 + 3 \times 0.1 = 0.3$	$3\cos(0.1) - 2 \times 0.3 = 2.385$
0.2	$0.3 + 0.1 \times 2.385 = 0.539$	$3\cos(0.2) - 2 \times 0.539 = 1.862$
0.3	$0.539 + 0.1 \times 1.862 = 0.725$	$3\cos(0.3) - 2 \times 0.725 = 1.416$
0.4	$0.725 + 0.1 \times 1.416 = 0.867$	$3\cos(0.4) - 2 \times 0.867 = 1.029$
0.5	$0.867 + 0.1 \times 1.029 = 0.970$	

$\boxed{y(0.1) = 0.3,\ y(0.2) = 0.539,\ y(0.3) = 0.725,\ y(0.4) = 0.867}$

(b) $y' = 3\cos t - 2y$

$h = 0.05 \Rightarrow y_{n+1} = y_n + 0.05 \times y'_n$

t	y	y'
0	0	$3 - 0 = 3$
0.05	$0 + 0.05 \times 3 = 0.15$	$3\cos(0.05) - 2 \times 0.15 = 2.696$
0.1	$0.15 + 0.05 \times 2.696 = 0.285$	$3\cos(0.1) - 2 \times 0.285 = 2.415$
0.15	$0.285 + 0.05 \times 2.415 = 0.406$	$3\cos(0.15) - 2 \times 0.406 = 2.154$
0.2	$0.406 + 0.05 \times 2.154 = 0.514$	$3\cos(0.2) - 2 \times 0.514 = 1.912$
0.25	$0.514 + 0.05 \times 1.912 = 0.610$	$3\cos(0.25) - 2 \times 0.610 = 1.687$
0.3	$0.61 + 0.05 \times 1.687 = 0.694$	$3\cos(0.3) - 2 \times 0.694 = 1.478$
0.35	$0.694 + 0.05 \times 1.478 = 0.768$	$3\cos(0.35) - 2 \times 0.768 = 1.282$
0.4	$0.768 + 0.05 \times 1.282 = 0.832$	

So, $\boxed{y(0.1) = 0.285,\ y(0.2) = 0.514,\ y(0.3) = 0.694,\ y(0.4) = 0.832}$

(c) I computed this by using Python (code is attached later)

$\boxed{y(0.1) = 0.278,\ y(0.2) = 0.502,\ y(0.3) = 0.699,\ y(0.4) = 0.815}$

(d) $y' = 3\cos t - 2y$

$y' + 2y = 3\cos t$

$e^{2t} y' + 2e^{2t} y = 3e^{2t}\cos t$

$(e^{2t} y)' = 3e^{2t}\cos t$

$e^{2t} y = \int 3e^{2t}\cos t \, dt + C$ ⇒ solve this by integration by parts

$\int 3e^{2t}\cos t \, dt = \int 3e^{2t}(\sin t)' dt = 3e^{2t}\sin t - \int 6e^{2t}\sin t \, dt$

$= 3e^{2t}\sin t + \int 6e^{2t}(\cos t)' dt = 3e^{2t}\sin t + 6e^{2t}\cos t - \int 12e^{2t}\cos t \, dt$

$\Rightarrow 15 \int e^{2t}\cos t \, dt = 3e^{2t}\sin t + 6e^{2t}\cos t$

$\Rightarrow \int 3e^{2t}\cos t \, dt = \frac{3}{5} e^{2t}\sin t + \frac{6}{5} e^{2t}\cos t$

$$e^{2t} y = \frac{3}{5} e^{2t} \sin t + \frac{6}{5} e^{2t} \cos t + C$$

$$y = \frac{3}{5} \sin t + \frac{6}{5} \cos t + C e^{-2t}$$

From the initial condition, $y(0) = 0$

$$\Rightarrow y(0) = \frac{6}{5} + C = 0 \Rightarrow C = -\frac{6}{5}$$

$$y(t) = \frac{3}{5} \sin t + \frac{6}{5} \cos t - \frac{6}{5} e^{-2t}$$

From this, we can understand that

$$\boxed{y(0.1) = 0.271, \quad y(0.2) = 0.491, \quad y(0.3) = 0.665, \quad y(0.4) = 0.800}$$

When we compare the results, $h = 0.025$ gives the most accurate estimate.

11) I completely used Python for this. See the solutions below

For 2.7 (4)

```
import math
# Defining function to find dy/dt
def func( t, y ):
    return (3 * math.cos(t) - 2 * y)

# Function for euler formula
def euler( t0, y, h, t ):
    temp = 0

    # Iterating till the point at which we need approximation
    while t0 < t:
        temp = y
        y = y + h * func(t0, y)
        t0 = t0 + h

    # Printing approximation
    print("Approximate solution at t = ", t, " is ", "%.6f"% y)

# Initial Values
t0 = 0
y0 = 0
h = 0.025

# Value of t at which we need approximation
t = 0.1
euler(t0, y0, h, t)

t = 0.2
euler(t0, y0, h, t)

t = 0.3
euler(t0, y0, h, t)

t = 0.4
euler(t0, y0, h, t)
```

For 2.7 (11)

Defining Function:

```
In [35]: import math
         # Defining function to find dy/dt
         def func( t, y ):
             return (5 - 3 * math.sqrt(y))

         # Function for euler formula
         def euler( t0, y, h, t ):
             temp = 0

             # Iterating till the point at which we need approximation
             while t0 < t:
                 temp = y
                 y = y + h * func(t0, y)
                 t0 = t0 + h

             # Printing approximation
             print("Approximate solution at t = ", t, " is ", "%.6f"% y)
```

(a) h = 0.1

```
In [36]: # Initial Values
         t0 = 0
         y0 = 2
         h = 0.1

         # Value of t at which we need approximation
         t = 0.5
         euler(t0, y0, h, t)
         t = 1
         euler(t0, y0, h, t)
         t = 1.5
         euler(t0, y0, h, t)
         t = 2
         euler(t0, y0, h, t)
         t = 2.5
         euler(t0, y0, h, t)
         t = 3
         euler(t0, y0, h, t)

         Approximate solution at t =  0.5  is  2.307998
         Approximate solution at t =  1  is  2.516664
         Approximate solution at t =  1.5  is  2.600226
         Approximate solution at t =  2  is  2.667728
         Approximate solution at t =  2.5  is  2.709388
         Approximate solution at t =  3  is  2.735210
```

Be Careful: The true value for y(1) = 2.49006 / It seems that this only happens for the second one, and I was unable to fix this

(b) h = 0.05

```
In [39]: # Initial Values
t0 = 0
y0 = 2
h = 0.05

# Value of t at which we need approximation
t = 0.5
euler(t0, y0, h, t)
t = 1
euler(t0, y0, h, t)
t = 1.5
euler(t0, y0, h, t)
t = 2
euler(t0, y0, h, t)
t = 2.5
euler(t0, y0, h, t)
t = 3
euler(t0, y0, h, t)
```

```
Approximate solution at t =  0.5  is  2.324097
Approximate solution at t =  1  is  2.482626
Approximate solution at t =  1.5  is  2.593517
Approximate solution at t =  2  is  2.662270
Approximate solution at t =  2.5  is  2.708477
Approximate solution at t =  3  is  2.734155
```

Be Careful: The true value for y(0.5) = 2.30167 / It seems that this only happens for the first one, and I was unable to fix this

(c) h = 0.025

```
In [40]: # Initial Values
t0 = 0
y0 = 2
h = 0.025

# Value of t at which we need approximation
t = 0.5
euler(t0, y0, h, t)
t = 1
euler(t0, y0, h, t)
t = 1.5
euler(t0, y0, h, t)
t = 2
euler(t0, y0, h, t)
t = 2.5
euler(t0, y0, h, t)
t = 3
euler(t0, y0, h, t)
```

```
Approximate solution at t =  0.5  is  2.298638
Approximate solution at t =  1  is  2.479030
Approximate solution at t =  1.5  is  2.594533
Approximate solution at t =  2  is  2.662268
Approximate solution at t =  2.5  is  2.704792
Approximate solution at t =  3  is  2.731593
```

(d) h = 0.01

```
In [41]: # Initial Values
t0 = 0
y0 = 2
h = 0.01

# Value of t at which we need approximation
t = 0.5
euler(t0, y0, h, t)
t = 1
euler(t0, y0, h, t)
t = 1.5
euler(t0, y0, h, t)
t = 2
euler(t0, y0, h, t)
t = 2.5
euler(t0, y0, h, t)
t = 3
euler(t0, y0, h, t)
```

```
Approximate solution at t =  0.5  is  2.296863
Approximate solution at t =  1  is  2.476909
Approximate solution at t =  1.5  is  2.588297
Approximate solution at t =  2  is  2.657977
Approximate solution at t =  2.5  is  2.702539
Approximate solution at t =  3  is  2.730021
```