635 FALL 2024 PROBLEM SET #1

Problem 1. Describe the fan Σ for $\mathbb{CP}^n$ explicitly by describing all of its cones. Explain carefully why X_Σ is isomorphic to $\mathbb{CP}^n$ by identifying each open subset $V_\sigma \subset X_\Sigma$ associated to a cone $\sigma \in \Sigma$ with an open subset of $\mathbb{CP}^n$, and that the gluing maps for X_Σ match those of $\mathbb{CP}^n$.

Problem 2. Consider the curve $C := \{y^2 = x^3\} \subset \mathbb{C}^2$.

(a) Explain how C is a toric variety.

(b) Consider the subset $S := \{0, 2, 3, 4, \dots\} = \mathbb{Z}_{\geq 0} \setminus \{1\}$ of $\mathbb{Z}_{\geq 0}$. Prove that C is isomorphic to $\operatorname{Spec} \mathbb{C}[S]$.

(c) A submonoid $T \subset \mathbb{Z}^n$ is said to be *saturated* if $km \in T$ implies $m \in T$ for any $k \in \mathbb{Z}_{\geq 1}$ and $m \in \mathbb{Z}^n$. Prove that $\sigma^\vee \cap \mathbb{Z}^n$ is saturated for any rational polyhedral cone $\sigma \subset \mathbb{R}^n$.

(d) Show that S is not saturated.

(e) For any finitely generated submonoid $T \subset \mathbb{Z}^n$, consider the following statements.
   (i) $\operatorname{Spec} \mathbb{C}[T]$ is a normal variety
   (ii) T is saturated
   (iii) $T = \sigma^\vee \cap \mathbb{Z}^n$ for some strongly convex rational polyhedral cone $\sigma \subset \mathbb{R}^n$.

   Prove that (i) implies (ii), and that (ii) implies (iii).[1]

(f) Use the above to conclude C is not normal.

[1]In infact (iii) also implies (i), but you don't have to prove that here.