A TREE FORMULA FOR THE ELLIPSOIDAL SUPERPOTENTIAL OF COMPLEX PROJECTIVE SPACE

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Contents

Let M^{2n} be a closed symplectic manifold and $A \in H_2(M)$ a homology class. Given a tuple $\vec{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n_{>0}$, let

$$E(\vec{a}) = \{\pi \sum_{i=1}^{n} |z_i|^2 / a_i \leqslant\} \subset \mathbb{C}^n$$

denote the symplectic ellipsoid with area factors $a_1, \ldots, a_n$, and let $\mathring{E}(\vec{a})$ denote its interior. We consider the compact symplectic cap $M_{\vec{a}} := M \setminus \iota(\mathring{E}(\varepsilon\vec{a}))$, where ι is a symplectic embedding $E(\varepsilon\vec{a}) \overset{s}{\hookrightarrow} M$ for some $\varepsilon > 0$ sufficiently small, with symplectic completion $\widehat{M}_{\vec{a}} = M_{\vec{a}} \cup (\mathbb{R}_{\leqslant 0} \times \partial M_{\vec{a}})$. The **ellipsoidal superpotential** $\mathbf{T}^{\vec{a}}_{M,A} \in \mathbb{Q}$ is the count of index zero finite energy pseudoholomorphic planes in $\widehat{M}_{\vec{a}}$. These counts play an important role in obstructing stabilized symplectic embeddings of ellipsoids into M, and in the case $\dim M = 4$ they also encode information about singular symplectic curves in M. A central question is to understand when $\mathbf{T}^{\vec{a}}_{M,A}$ is nonzero.

In the article [MS] a recursive formula is given for $\mathbf{T}^{\vec{a}}_{M,A}$ in terms stationary descendant Gromov–Witten invariants of M, which are readily computable e.g. if M is a smooth toric Fano variety. In this note, a companion to [MS], we give an explicit closed formula for $\mathbf{T}^{\vec{a}}_{M,A}$ in the case $M = \mathbb{CP}^2$ as a sum over trees. This is a step towards the goal of finding a positive formula of $\mathbf{T}^{\vec{a}}_{M,A}$.

Fix M^{2n} be a closed symplectic manifold, $A \in H_2(M)$ a homology class, and $a > 1$ an irrational number. We put $M_a := M \setminus \iota(\mathring{E}(\varepsilon, a\varepsilon))$

A broad goal for this paper is to develop purely combinatorial methods for approaching the problems discussed in §**??** and §**??**. Although Theorem **??** and its generalization make it possible to compute any given $\mathbf{T}^{\vec{a}}_{M,A}$ given enough computational power, its recursivity somewhat obscures its enumerative nature, and the

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presence of terms of both positive and negative sign complicates efforts to prove general nonvanishing results. On the other hand, geometric considerations such as automatic transversality imply that $\mathbf{T}_{M,A}^{\vec{a}}$ is frequently nonnegative at least in dimension four, and this raises the possibility of a positive bijective formula underlying Theorem **??**. As a step in this direction, in §**??** we prove a closed formula for $\mathbf{T}_{M,A}^{\vec{a}}$ as a sum over trees, with "relatively few" negative terms.

Before stating the formula, we need some graph theoretic terminology (see also §1 and §2). For $d \in \mathbb{Z}_{\geqslant 1}$, let $\mathfrak{T}_d^{\mathrm{un}}$ denote the set of rooted trees with d (unordered) leaves and no bivalent vertices (i.e. no vertices with $|v| = 2$). For $T \in \mathfrak{T}_d^{\mathrm{un}}$, let $V_{\mathrm{in}}(T)$ denote its set of internal vertices, and let $V_{\mathrm{mov}}(T)$ denote its set of *movable vertices*, i.e. those $v \in V_{\mathrm{in}}(T)$ such that there are no internal vertices lying above v. Let $\ell(v)$ denote the *leaf number* of v, i.e. the number of leaf vertices lying above v. Finally, let $|\mathrm{Aut}(T)|$ denote the number of symmetries of T. See Figure 1 for a picture of $\mathfrak{T}_4^{\mathrm{un}}$.

For simplicity in this introduction we formulate just the special case $\mathbf{T}_d := \mathbf{T}_d^{\infty}$.

Theorem A (= Corollary 2.0.6)**.** *For $d \in \mathbb{Z}_{\geqslant 1}$ we have:*

$$\widetilde{\mathbf{T}}_d = 2^d \sum_{T \in \mathfrak{T}_d^{\mathrm{un}}} \frac{(-1)^{|V_{\mathrm{in}}(T) \setminus V_{\mathrm{mov}}(T)|}}{|\mathrm{Aut}(T)|} \cdot \prod_{v \in V_{\mathrm{in}}(T)} \left(\frac{(3\ell(v)-1)!}{(3\ell(v)-|v|+1)!} \right) \cdot \prod_{v \in V_{\mathrm{mov}}(T)} \left(2^{-\ell(v)} \binom{2\ell(v)}{\ell(v)} - 1 \right). \tag{0.0.1}$$

The proof incorporates techniques homological perturbation theory for $\mathcal{L}_\infty$ algebras, and the formula for general $a \in \mathbb{R}_{>0}$ involves the lattice path Γ^a. Note that the term $2^{-\ell(v)}\binom{2\ell(v)}{\ell(v)} - 1$ is always positive, so all negativity comes from the nonmovable vertices.

Example 0.0.1. In the case $d = 1$, there is just a single tree in $\mathfrak{T}_1^{\mathrm{un}}$, which has no internal vertices. Both products in (0.0.1) are vacuous, so we get $\widetilde{\mathbf{T}}_1 = 1$.

In the case $d = 2$, there is also just one tree in $\mathfrak{T}_2^{\mathrm{un}}$, which has one internal vertex, which is movable and has leaf number is 2. Then (0.0.1) gives

$$\widetilde{\mathbf{T}}_2 = 2^2 \cdot \frac{(-1)^0}{2} \cdot \frac{5!}{4!} \cdot \left(2^{-2} \binom{4}{2} - 1 \right) = 5.$$

$\diamondsuit$

1. Homological perturbation theory

In this subsection we formulate our main algebraic tool for inverting $\mathcal{L}_\infty$ homomorphisms. We include a simple proof for completeness, although the technique is well-known to experts.

Definition 1.0.1.

(1) Let $\mathfrak{T}_k^{\mathrm{or}}$ denote the set of (isomorphism classes of) trees with $k+1$ ordered *univalent vertices, such that all other vertices have valency at least three.*

(2) Similarly, let $\mathfrak{T}_k^{\mathrm{un}}$ denote the set of (isomorphism classes of) rooted trees with $k+1$ unordered *univalent vertices, such that all other vertices have valency at least three.*

For $T \in \mathfrak{T}_k^{\mathrm{or}}$, the first univalent vertex is called the **root vertex** and the remaining univalent vertices are called the **leaf vertices**. We denote the set of leaf vertices by $V_{\mathrm{leaf}}(T)$. Note that the ordering of the leaf vertices amounts to a bijection between $V_{\mathrm{leaf}}(T)$ and $\{1, \ldots, k\}$. We will also refer to the edges connected to the leaf vertices as the **leaf edges**, and the edge connected to the root vertex as the **root edge**. The set $\mathfrak{T}_k^{\mathrm{un}}$ is the quotient $\mathfrak{T}_k^{\mathrm{or}}/\Sigma_k$ by the natural symmetric group action on $\mathfrak{T}_k^{\mathrm{or}}$ which reorders the leaves. The stabilizer of a tree $T \in \mathfrak{T}_k^{\mathrm{or}}$ is the automorphism group $\mathrm{Aut}(T)$ of T.

We orient all edges of T towards the root, and we will say that v is "above" w if w lies on the oriented path from v to the root. We denote by $V_{\mathrm{in}}(T)$ the set of internal vertices of T, i.e. those having valency at least three. For $v \in V_{\mathrm{in}}(T)$, the **incoming** (resp. **outgoing**) edges of v are those edges of T which have v as an endpoint and are oriented towards (resp. away from) v.

An $\mathcal{L}_\infty$ algebra is called **abelian** if all of the $\mathcal{L}_\infty$ operations $\ell^1, \ell^2, \ell^3, \ldots$ vanish identically. Note that, with our degree conventions, any $\mathcal{L}_\infty$ algebra supported entirely in even degrees is automatically abelian. The following proposition is proved using standard techniques from homological perturbation theory. One can formulate a more general version without any grading restrictions, but here we state a simplified version which suffices for our purposes. Note that for any evenly graded $\mathcal{L}_\infty$ algebra V all of the signs $\diamondsuit(V, \sigma; v_1, \ldots, v_k)$ are automatically positive, and hence can be ignored.

Proposition 1.0.2. *Let V and W be evenly graded $\mathcal{L}_\infty$ algebras over $\mathbb{Q}$, and let $\Phi : V \to W$ be an $\mathcal{L}_\infty$ homomorphism such that the linear map $\Phi^1 : V \to W$ is invertible. Then there exists an $\mathcal{L}_\infty$ homomorphism $\Psi : W \to V$ such that*

$$\Psi \circ \Phi = \mathbb{1}_V \quad \textit{and} \quad \Phi \circ \Psi = \mathbb{1}_W.$$

Moreover, Ψ is given explicitly as follows. We first set $\Psi^1 : W \to V$ to be the inverse of $\Phi^1 : V \to W$. For $k \geqslant 2$, $T \in \mathfrak{T}_k^{\mathrm{or}}$, and $w_1, \ldots, w_k \in W$, define $\Psi^T(w_1, \ldots, w_k)$ as follows. Start by labeling the ith leaf edge of T by $\Psi^1(w_i)$ for $i = 1, \ldots, k$. Recursively, for each internal vertex $v \in V_{\mathrm{in}}(T)$, say with j incoming edges, label the outgoing edge by the result after applying $\Psi^1 \circ \Phi^j$ to the corresponding labels of its incoming edges. We define $\Psi^T(w_1, \ldots, w_k)$ to be the resulting label on the root edge, and finally put

$$\Psi^k(w_1, \ldots, w_k) = \sum_{T \in \mathfrak{T}_k^{\mathrm{or}}} (-1)^{|V_{\mathrm{in}}(T)|} \Psi^T(w_1, \ldots, w_k).$$

Example 1.0.3. We have:

- $\Psi^2(x, y) = -\Psi^1\Phi^2(\Psi^1(x), \Psi^1(y))$
- $\Psi^3(x, y, z) = -\Psi^1\Phi^3(\Psi^1(x), \Psi^1(y), \Psi^1(z)) + \Psi^1\Phi^2(\Psi^1(x), \Psi^1\Phi^2(\Psi^1(y), \Psi^1(z))) + \Psi^1\Phi^2(\Psi^1(y), \Psi^1\Phi^2(\Psi^1(x), \Psi^1(z))) + \Psi^1\Phi^2(\Psi^1(z), \Psi^1\Phi^2(\Psi^1(x), \Psi^1(y)))$.

$\diamondsuit$

Proof. The relation $\Psi \circ \Phi = \mathbb{1}$ means that for $k \geqslant 2$ we must have $(\Psi \circ \Phi)^k(v_1, \ldots, v_k) = 0$ for any $v_1, \ldots, v_k \in V$, and this amounts to

$$\sum_{\substack{s \geqslant 1 \\ 1 \leqslant k_1 \leqslant \cdots \leqslant k_s \\ k_1 + \cdots + k_s = k}} \sum_{\sigma \in \overline{\mathrm{Sh}}(k_1, \ldots, k_s)} \Psi^s \circ (\Phi^{k_1} \odot \cdots \odot \Phi^{k_s})(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = 0,$$

or equivalently

$$\Psi^k(\Phi^1(v_1), \ldots, \Phi^1(v_k)) = -\sum_{\substack{1 \leqslant s \leqslant k-1 \\ 1 \leqslant k_1 \leqslant \cdots \leqslant k_s \\ k_1 + \cdots + k_s = k}} \sum_{\sigma \in \overline{\mathrm{Sh}}(k_1, \ldots, k_s)} \Psi^s \circ (\Phi^{k_1} \odot \cdots \odot \Phi^{k_s})(v_{\sigma(1)}, \ldots, v_{\sigma(k)}),$$

i.e. for any $w_1, \ldots, w_k \in W$ we must have

$$\Psi^k(w_1, \ldots, w_k) = -\sum_{\substack{1 \leqslant s \leqslant k-1 \\ 1 \leqslant k_1 \leqslant \cdots \leqslant k_s \\ k_1 + \cdots + k_s = k}} \sum_{\sigma \in \overline{\mathrm{Sh}}(k_1, \ldots, k_s)} \Psi^s \circ (\Phi^{k_1} \odot \cdots \odot \Phi^{k_s})(\Psi^1(w_{\sigma(1)}), \ldots, \Psi^1(w_{\sigma(k)})).$$

This is easily seen to agree with the definition of Ψ given in the statement of the proposition, which therefore necessarily satisfies $\Psi \circ \Phi = \mathbb{1}$.

As for the relation $\Phi \circ \Psi = \mathbb{1}$, we need

$$\sum_{\substack{s \geqslant 1 \\ 1 \leqslant k_1 \leqslant \cdots \leqslant k_s \\ k_1 + \cdots + k_s = k}} \sum_{\sigma \in \overline{\mathrm{Sh}}(k_1, \ldots, k_s)} \Phi^s \circ (\Psi^{k_1} \odot \cdots \odot \Psi^{k_s})(w_{\sigma(1)}, \ldots, w_{\sigma(k)}) = 0$$

for any $w_1, \ldots, w_k \in W$, or equivalently

$$\Phi^1(\Psi^k(w_1, \ldots, w_k)) = -\sum_{\substack{s \geqslant 2 \\ 1 \leqslant k_1 \leqslant \cdots \leqslant k_s \\ k_1 + \cdots + k_s = k}} \sum_{\sigma \in \overline{\mathrm{Sh}}(k_1, \ldots, k_s)} \Phi^s \circ (\Psi^{k_1} \odot \cdots \odot \Psi^{k_s})(w_{\sigma(1)}, \ldots, w_{\sigma(k)}),$$

i.e.

$$\Psi^k(w_1, \ldots, w_k) = -\sum_{\substack{s \geqslant 2 \\ 1 \leqslant k_1 \leqslant \cdots \leqslant k_s \\ k_1 + \cdots + k_s = k}} \sum_{\sigma \in \overline{\mathrm{Sh}}(k_1, \ldots, k_s)} \Psi^1 \circ \Phi^s \circ (\Psi^{k_1} \odot \cdots \odot \Psi^{k_s})(w_{\sigma(1)}, \ldots, w_{\sigma(k)}),$$

which is equivalent to our definition of Ψ^k. □

2. Tree formula

We now prove an explicit formula for $\widetilde{\mathbf{T}}_d^a$ as a sum over trees. It is likely that similar techniques could be used to study curves in more general closed symplectic manifolds M, but for concreteness we restrict the discussion to $\mathbb{CP}^2$.

We first need some definitions. Recall that the sets of trees $\mathfrak{T}_k^{\mathrm{or}}$ and $\mathfrak{T}_k^{\mathrm{un}}$ with k ordered and unordered leaves respectively were defined in Definition 1.0.1.

Definition 2.0.1. *For a tree T in $\mathfrak{T}_k^{\mathrm{or}}$ or $\mathfrak{T}_k^{\mathrm{un}}$, an interval vertex $v \in V_{\mathrm{in}}(T)$ is* **movable** *if there are no internal vertices above it. We denote the set of movable vertices of T by $V_{\mathrm{mov}}(T) \subset V_{\mathrm{in}}(T)$.*

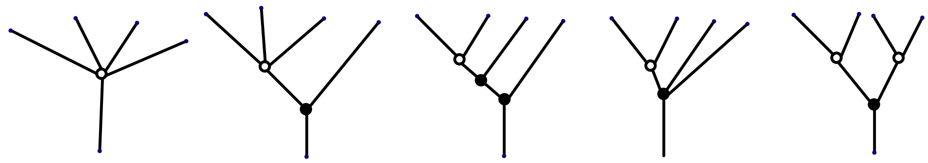

FIGURE 1. The trees comprising $\mathfrak{T}_4^{\mathrm{un}}$. The top vertices are the leaves and the bottom vertex is the root. The internal vertices are denoted with a large circle, which is open for a movable vertex and solid otherwise.

Definition 2.0.2. *For a vertex v of a tree T in $\mathfrak{T}_d^{\mathrm{or}}$ or $\mathfrak{T}_d^{\mathrm{un}}$, the* **leaf number** $\ell(v)$ *is the number of leaf vertices lying above v (including possibly v itself). In particular, $\ell(v) = 1$ if v is a leaf vertex.*

Example 2.0.3. Figure 1 shows the elements of $\mathfrak{T}_4^{\mathrm{un}}$ with the movable vertices labeled by open circles. ◊

Theorem 2.0.4. *For any $d \in \mathbb{Z}_{\geqslant 1}$, we have:*

$$\widetilde{\mathbf{T}}_d^a = (\Gamma_2^a!)^d \sum_{T \in \mathfrak{T}_d^{\mathrm{un}}} \frac{(-1)^{|V_{\mathrm{in}}(T)|}}{|\mathrm{Aut}(T)|} \cdot \prod_{v \in V_{\mathrm{in}}(T)} \left(\frac{\Gamma_{3\ell(v)-1}^a!}{\left(\sum\limits_{v' \to v} \Gamma_{3\ell(v')-1}^a\right)!} \right) \cdot \prod_{v \in V_{\mathrm{mov}}(T)} \left(1 - (\ell(v)!)^{-2} \frac{(\ell(v)\Gamma_2^a)!}{(\Gamma_2^a!)^{\ell(v)}} \right).$$

Here the sum $\sum\limits_{v' \to v}$ is over all vertices v' which are adjacent to v and lie strictly above it.

Remark 2.0.5. More explicitly, for $1 < a < 2$ we have

$$\widetilde{\mathbf{T}}_d^a = \sum_{T \in \mathfrak{T}_d^{\mathrm{un}}} \frac{(-1)^{|V_{\mathrm{in}}(T)|}}{|\mathrm{Aut}(T)|} \cdot \prod_{v \in V_{\mathrm{in}}(T)} \left(\frac{\Gamma_{3\ell(v)-1}^a!}{\left(\sum\limits_{v' \to v} \Gamma_{3\ell(v')-1}^a\right)!} \right) \cdot \prod_{v \in V_{\mathrm{mov}}(T)} \left(1 - \frac{1}{\ell(v)!} \right),$$

while for $a \geqslant 2$ we have

$$\begin{aligned} \widetilde{\mathbf{T}}_d^a &= 2^d \sum_{T \in \mathfrak{T}_d^{\mathrm{un}}} \frac{(-1)^{|V_{\mathrm{in}}(T)|}}{|\mathrm{Aut}(T)|} \cdot \prod_{v \in V_{\mathrm{in}}(T)} \left(\frac{\Gamma_{3\ell(v)-1}^a!}{\left(\sum\limits_{v' \to v} \Gamma_{3\ell(v')-1}^a\right)!} \right) \cdot \prod_{v \in V_{\mathrm{mov}}(T)} \left(1 - 2^{-\ell(v)} \binom{2\ell(v)}{\ell(v)} \right) \\ &= 2^d \sum_{T \in \mathfrak{T}_d^{\mathrm{un}}} \frac{(-1)^{|V_{\mathrm{in}}(T) \setminus V_{\mathrm{mov}}(T)|}}{|\mathrm{Aut}(T)|} \cdot \prod_{v \in V_{\mathrm{in}}(T)} \left(\frac{\Gamma_{3\ell(v)-1}^a!}{\left(\sum\limits_{v' \to v} \Gamma_{3\ell(v')-1}^a\right)!} \right) \cdot \prod_{v \in V_{\mathrm{mov}}(T)} \left(2^{-\ell(v)} \binom{2\ell(v)}{\ell(v)} - 1 \right). \end{aligned}$$

◊

Note that the term $2^{-\ell(v)}\binom{2\ell(v)}{\ell(v)} - 1$ is always positive, so all negativity in the above expression comes from the "unmovable" vertices $v \in V_{\text{in}}(T) \setminus V_{\text{mov}}(T)$. In the special case $a \gg 1$, we have $\Gamma_k^\infty = (k, 0)$, so we get the following formula for $\widetilde{\mathbf{T}}_d = (3d-1)\mathbf{T}_d$:

Corollary 2.0.6. *For $d \in \mathbb{Z}_{\geqslant 1}$ we have:*

$$\widetilde{\mathbf{T}}_d = 2^d \sum_{T \in \mathfrak{T}_d^{\text{un}}} \frac{(-1)^{|V_{\text{in}}(T) \setminus V_{\text{mov}}(T)|}}{|\text{Aut}(T)|} \cdot \prod_{v \in V_{\text{in}}(T)} \left(\frac{(3\ell(v)-1)!}{(3\ell(v)-|v|+1)!} \right) \cdot \prod_{v \in V_{\text{mov}}(T)} \left(2^{-\ell(v)} \binom{2\ell(v)}{\ell(v)} - 1 \right). \tag{2.0.1}$$

Here $|v|$ denotes the valency of v, i.e. the number of incoming edges plus 1.

Example 2.0.7. To compute $\widetilde{\mathbf{T}}_3$ using Corollary 2.0.6, we put $\mathfrak{T}_3^{\text{un}} = \{T_1, T_2\}$, where T_1 has a single internal vertex of valency 4 and T_2 has two internal vertices each of valency 3. We have $|\text{Aut}(T_1)| = 3!$ and $|\text{Aut}(T_2)| = 2!$. The internal vertex of T_1 is movable and has leaf number 3. The internal vertices of T_2 have leaf numbers $2, 3$, and the first is movable while the second is not. Plugging these into (2.0.1) gives $\widetilde{\mathbf{T}}_3 = 32$. $\Diamond$

Remark 2.0.8. Curiously, we have precisely $\mathbf{T}_d = |\mathfrak{T}_d^{\text{or}}|$ for $d = 1, 2, 3, 4$, while experimentally we have $\mathbf{T}_d < |\mathfrak{T}_d^{\text{or}}|$ for $d \geqslant 5$. This perhaps suggests that there could be a positive formula for $\mathbf{T}_d$ which counts elements of $\mathfrak{T}_d$ with some additional conditions which only become relevant starting in degree 5. $\Diamond$

Given $\vec{a} \in \mathbb{R}^n_{>0}$, recall that $\eta_{\vec{a}} : C_o \to C_{\vec{a}}$ is the $\mathcal{L}_\infty$ homomorphism inverse to the stationary descendant map $\epsilon_{\vec{a}} : C_{\vec{a}} \to C_o$, and the explicit construction of $\eta_{\vec{a}}$ is provided by Proposition 1.0.2. In particular, $\eta^1_{\vec{a}}$ is the linear inverse of $\epsilon^1_{\vec{a}}$, i.e. for $k \in \mathbb{Z}_{\geqslant 1}$ we have

$$\epsilon^1_{\vec{a}}(\mathfrak{o}_k^{\vec{a}}) = \frac{\mathfrak{q}_k}{(\Gamma_k^{\vec{a}})!} \quad \text{and} \quad \eta^1_{\vec{a}}(\mathfrak{q}_k) = (\Gamma_k^{\vec{a}})!\,\mathfrak{o}_k^{\vec{a}}.$$

Applying $\pi_1 \circ \widehat{\eta}_{\vec{a}}$ to both sides of (**??**) gives

$$\pi_1(\exp_A(\mathfrak{m}_M^{\vec{a}})) = (\pi_1 \circ \widehat{\eta}_{\vec{a}})(\exp_A(\mathfrak{m}_M^o)),$$

i.e.

$$\begin{aligned}
\mathfrak{m}^{\vec{a}}_{M,A} &= \sum_{\substack{k \geqslant 1 \\ A_1,\ldots,A_k \in H_2(M) \\ A_1+\cdots+A_k = A}} \tfrac{1}{k!} \eta^k_{\vec{a}}(\mathfrak{m}^o_{M,A_1}, \ldots, \mathfrak{m}^o_{M,A_k}) \\
&= \sum_{\substack{k \geqslant 1 \\ A_1,\ldots,A_k \in H_2(M) \\ A_1+\cdots+A_k = A}} \tfrac{1}{k!} \sum_{T \in \mathfrak{T}_k^{\text{or}}} (-1)^{|V_{\text{in}}(T)|} \eta^T_{\vec{a}}(\mathfrak{m}^o_{M,A_1}, \ldots, \mathfrak{m}^o_{M,A_k}).
\end{aligned}$$

Using $\mathfrak{m}^{\vec{a}}_{M,A} = \widetilde{\mathbf{T}}^{\vec{a}}_{M,A} \cdot \mathfrak{o}^{\vec{a}}_{c_1(A)-1}$ and $\mathfrak{m}^o_{M,A} = N_{M,A} \langle\!\langle \psi^{c_1(A)-2} pt \rangle\!\rangle \cdot \mathfrak{q}_{c_1(A)-1}$, we can rewrite the above as

$$\widetilde{\mathbf{T}}^{\vec{a}}_{M,A} \cdot \mathfrak{o}^{\vec{a}}_{c_1(A)-1}$$

$$= \sum_{\substack{k \geqslant 1 \\ A_1,\ldots,A_k \in H_2(M) \\ A_1+\cdots+A_k = A}} \tfrac{1}{k!} \left(\prod_{s=1}^{k} N_{M,A_s} \langle\!\langle \psi^{c_1(A_s)-2} pt \rangle\!\rangle \right) \sum_{T \in \mathfrak{T}^{\mathrm{or}}_k} (-1)^{|V_{\mathrm{in}}(T)|} \eta^T_{\vec{a}}(\mathfrak{q}_{c_1(A_1)-1}, \ldots, \mathfrak{q}_{c_1(A_k)-1}). \tag{2.0.2}$$

We now specialize to the case $M = \mathbb{CP}^2$ and $A = d[L]$, so that equation (2.0.2) becomes

$$\widetilde{\mathbf{T}}^{a}_{d} \cdot \mathfrak{o}^{a}_{3d-1} = \sum_{\substack{k \geqslant 1 \\ d_1,\ldots,d_k \in \mathbb{Z}_{\geqslant 1} \\ d_1+\cdots+d_k = d}} \tfrac{1}{k!(d_1!)^3\cdots(d_k!)^3} \sum_{T \in \mathfrak{T}^{\mathrm{or}}_k} (-1)^{|V_{\mathrm{in}}(T)|} \eta^T_{a}(\mathfrak{q}_{3d_1-1}, \ldots, \mathfrak{q}_{3d_k-1}). \tag{2.0.3}$$

Our goal is to rewrite the above as a sum over trees with exactly d leaves. We will introduce relevant notation as we need it.

Notation 2.0.9. *Given $d_1, \ldots, d_k \in \mathbb{Z}_{\geqslant 1}$ with $d_1 + \cdots + d_k = d$, let $\mathcal{P}(d_1, \ldots, d_k)$ denote the set of surjections $h : \{1, \ldots, d\} \twoheadrightarrow \{1, \ldots, k\}$ such that $|h^{-1}(i)| = d_i$ for $i = 1, \ldots, k$.*

Then we can rewrite (2.0.3) temporarily more redundantly as

$$\widetilde{\mathbf{T}}^{a}_{d} \cdot \mathfrak{o}^{a}_{3d-1} = \sum_{\substack{k \geqslant 1 \\ d_1,\ldots,d_k \in \mathbb{Z}_{\geqslant 1} \\ d_1+\cdots+d_k = d}} \tfrac{1}{k!(d_1!)^3\cdots(d_k!)^3} \binom{d}{d_1, \ldots, d_k}^{-1} \sum_{h \in \mathcal{P}(d_1,\ldots,d_k)} \sum_{T \in \mathfrak{T}^{\mathrm{or}}_k} (-1)^{|V_{\mathrm{in}}(T)|} \eta^T_{a}(\mathfrak{q}_{3d_1-1}, \ldots, \mathfrak{q}_{3d_k-1}).$$

Notation 2.0.10.

(1) *Put*

$$\mathfrak{T}^{\mathrm{or}}_k(\{1, \ldots, d\}) := \left\{(T, h) \mid T \in \mathfrak{T}^{\mathrm{or}}_k,\ h : \{1, \ldots, d\} \twoheadrightarrow \{1, \ldots, k\}\right\},$$

i.e. $\mathfrak{T}^{\mathrm{or}}_k(\{1, \ldots, d\})$ is the set of trees $T \in \mathfrak{T}^{\mathrm{or}}_k$ equipped with a partition of $\{1, \ldots, d\}$ into k parts labeled by the leaves of T.

(2) *Given $(T, h) \in \mathfrak{T}^{\mathrm{or}}_k(\{1, \ldots, d\})$, put*

$$C_{(T,h)} := \frac{1}{(d_1!)^2 \cdots (d_k!)^2} \quad \text{and} \quad \epsilon^{(T,h)}_a(\mathfrak{q}_\bullet) := \epsilon^T_a(\mathfrak{q}_{3d_1-1}, \ldots, \mathfrak{q}_{3d_k-1}),$$

where $d_i := |h^{-1}(i)|$ for $i = 1, \ldots, k$.

We then have

$$\widetilde{\mathbf{T}}^{a}_{d} \cdot \mathfrak{o}^{a}_{3d-1} = \sum_{k \geqslant 1} \sum_{(T,h) \in \mathfrak{T}^{\mathrm{or}}_k(\{1,\ldots,d\})} \tfrac{1}{k!d!} \cdot C_{(T,h)} \cdot (-1)^{|V_{\mathrm{in}}(T)|} \eta^{(T,h)}_a(\mathfrak{q}_\bullet).$$

Notation 2.0.11. *Put*

$$\mathfrak{T}^{\mathrm{un}}_k(\{1, \ldots, d\}) = \{(T, h) \mid T \in \mathfrak{T}^{\mathrm{un}}_k,\ h : \{1, \ldots, d\} \twoheadrightarrow V_{\mathrm{leaf}}(T)\},$$

i.e. an element of $\mathfrak{T}^{\mathrm{un}}_k(\{1,\ldots,d\})$ *is a tree* $T \in \mathfrak{T}^{\mathrm{un}}_k$ *with* k *unordered leaves equipped with a partition of* $\{1,\ldots,d\}$ *into* k *parts labeled by the leaves of* T*. Put also*

$$\mathfrak{T}^{\mathrm{un}}(\{1,\ldots,d\}) := \bigcup_{1\leqslant k\leqslant d} \mathfrak{T}^{\mathrm{un}}_k(\{1,\ldots,d\}).$$

Note that the symmetric group Σ_k acts freely on $\mathfrak{T}^{\mathrm{or}}_k(\{1,\ldots,d\})$ via $\sigma\cdot(T,h) = (T',\sigma\circ h)$, where T' is given by reordering the leaves of T according to the permutation $\sigma\in\Sigma_k$, and the quotient is identified with $\mathfrak{T}^{\mathrm{un}}_k(\{1,\ldots,d\})$. We then have

$$\widetilde{\mathbf{T}}^a_d \cdot \mathfrak{o}^a_{3d-1} = \sum_{(T,h)\in\mathfrak{T}^{\mathrm{un}}(\{1,\ldots,d\})} \tfrac{1}{d!}\cdot C_{(T,h)}\cdot(-1)^{|V_{\mathrm{in}}(T)|}\eta_a^{(T,h)}(\mathfrak{q}_\bullet). \tag{2.0.4}$$

Observe that there is a natural map

$$\zeta : \{(T,\mathcal{S}) \mid T\in\mathfrak{T}^{\mathrm{or}}_d, \mathcal{S}\subset V_{\mathrm{mov}}(T)\} \to \mathfrak{T}^{\mathrm{un}}(\{1,\ldots,d\}),$$

defined as follows. Given $(T,\mathcal{S})$, let T' be the tree obtained by removing all edges and vertices strictly above v, for each $v\in\mathcal{S}$. Note that each $v\in\mathcal{S}$ becomes a leaf in T'. We define a surjection $h : \{1,\ldots,d\}\twoheadrightarrow V_{\mathrm{leaf}}(T')$ in such a way that

- if $v\in V_{\mathrm{leaf}}(T')$ corresponds to a leaf of T, then $h^{-1}(v)$ is the original label of v
- for $v\in\mathcal{S}$, $h^{-1}(v)$ is the set of labels of leaves lying above v in T.

We put $\zeta(T,\mathcal{S}) = (T',h)$.

In fact, ζ is a bijection, with inverse map

$$\zeta^{-1} : \mathfrak{T}^{\mathrm{un}}(\{1,\ldots,d\}) \to \{(T,\mathcal{S}) \mid T\in\mathfrak{T}^{\mathrm{or}}_d, \mathcal{S}\subset V_{\mathrm{mov}}(T)\}$$

described as follows. Given $(T,h)\in\mathfrak{T}^{\mathrm{un}}(\{1,\ldots,d\})$, for each $v\in V_{\mathrm{leaf}}(T)$ such that $|h^{-1}(v)|\geqslant 2$ we add $|h^{-1}(v)|$ new leaf vertices, each joined to v by a new leaf edge. By construction the resulting tree T' comes with a natural bijection $\{1,\ldots,d\}\xrightarrow{\sim} V_{\mathrm{leaf}}(T')$, i.e. $T'\in\mathfrak{T}^{\mathrm{or}}_d$. Also, each $v\in V_{\mathrm{leaf}}(T)$ satisfying $|h^{-1}(v)|\geqslant 2$ naturally corresponds to a movable vertex in T', and we denote the set of these by $\mathcal{S}\subset V_{\mathrm{mov}}(T')$. Then we have $\zeta^{-1}(T,h) = (T',\mathcal{S})$.

Using this bijection, we rewrite (2.0.4) as:

$$\widetilde{\mathbf{T}}^a_d \cdot \mathfrak{o}^a_{3d-1} = \tfrac{1}{d!}\sum_{T\in\mathfrak{T}^{\mathrm{or}}_d}\sum_{\mathcal{S}\subset V_{\mathrm{mov}}(T)} C_{\zeta(T,\mathcal{S})}\cdot(-1)^{|V_{\mathrm{in}}(\zeta(T,\mathcal{S}))|}\eta_a^{\zeta(T,\mathcal{S})}(\mathfrak{q}_\bullet). \tag{2.0.5}$$

It will be convenient to extend the notion of leaf number to elements of $\mathfrak{T}^{\mathrm{un}}(\{1,\ldots,d\})$ as follows. For $(T,h)\in\mathfrak{T}^{\mathrm{un}}(\{1,\ldots,d\})$, we define the leaf number $\ell(v)$ to be the cardinality of $\bigcup\limits_w |h^{-1}(w)|$, where the union is over all leaf vertices w lying above v (including possibly v itself).

Proof of Theorem 2.0.4. According to Proposition 1.0.2, $\eta_a^T(\mathfrak{q}_{3d_1-1},\ldots,\mathfrak{q}_{3d_k-1})$ is computed as follows. Recall that $T\in\mathfrak{T}^{\mathrm{or}}_k$ has k ordered leaves. For $i=1,\ldots,k$, we label the ith leaf edge by $\eta^1_a(\mathfrak{q}_{3d_i-1}) = (\Gamma^a_{3d_i-1})!\,\mathfrak{o}^a_{3d_i-1}$. We then iteratively label each edge of T, say with source vertex v, by the result of applying $\eta^1_a\circ\epsilon^j_a$ to the labels of the incoming edges of v (here the valency of v is $j+1$).

Then for $T \in \mathfrak{T}_d^{\mathrm{or}}$ and $\mathcal{S} \subset V_{\mathrm{mov}}(T)$, with $\zeta(T,\mathcal{S}) \in \mathfrak{T}^{\mathrm{un}}(\{1,\dots,d\})$ as defined above we have:

$$\eta_a^{\zeta(T,\mathcal{S})}(\mathfrak{q}_\bullet) = \prod_{v \in V_{\mathrm{leaf}}(\zeta(T,\mathcal{S}))} \left(\Gamma^a_{3\ell(v)-1}\right)! \cdot \frac{\prod\limits_{v \in V_{\mathrm{in}}(\zeta(T,\mathcal{S}))} \left(\Gamma^a_{3\ell(v)-1}\right)!}{\prod\limits_{v \in V_{\mathrm{in}}(\zeta(T,\mathcal{S}))} \left(\sum\limits_{v' \to v} \Gamma^a_{3\ell(v')-1}\right)!} \cdot \mathfrak{o}^a_{3d-1}. \tag{2.0.6}$$

Observe that in the case $\mathcal{S} = \varnothing$, $\zeta(T,\varnothing)$ is simply T itself but viewed as an element of $\mathfrak{T}^{\mathrm{un}}(\{1,\dots,d\})$, and we have

$$\eta_a^{\zeta(T,\varnothing)}(\mathfrak{q}_\bullet) = \eta_a^T(\underbrace{\mathfrak{q}_2,\dots,\mathfrak{q}_2}_{d}) = \prod_{v \in V_{\mathrm{leaf}}(T)} \left(\Gamma^a_{3\ell(v)-1}\right)! \cdot \frac{\prod\limits_{v \in V_{\mathrm{in}}(T)} \left(\Gamma^a_{3\ell(v)-1}\right)!}{\prod\limits_{v \in V_{\mathrm{in}}(T)} \left(\sum\limits_{v' \to v} \Gamma^a_{3\ell(v')-1}\right)!} \cdot \mathfrak{o}^a_{3d-1}, \tag{2.0.7}$$

where

$$\Gamma^a_2! = \begin{cases} 1 & \text{if } \; 1 < a < 2 \\ 2 & \text{if } \; a \geqslant 2. \end{cases}$$

We have natural a identification $V_{\mathrm{in}}(T) \approx V_{\mathrm{in}}(\zeta(T,\mathcal{S})) \cup \mathcal{S}$, and the leaves of $\zeta(T,\mathcal{S})$ correspond to (a) leaves of T not lying above any $v \in \mathcal{S}$ and (b) elements of $\mathcal{S}$. Then, comparing (2.0.6) and (2.0.7), we have

$$\begin{aligned} \eta_a^{\zeta(T,\mathcal{S})}(\mathfrak{q}_\bullet) &= \eta_a^{\zeta(T,\varnothing)}(\mathfrak{q}_\bullet) \cdot \frac{\prod\limits_{v \in \mathcal{S}} \left(\Gamma^a_{3\ell(v)-1}\right)!}{\prod\limits_{v \in \mathcal{S}} (\Gamma^a_2!)^{\ell(v)}} \cdot \frac{\prod\limits_{v \in \mathcal{S}} \left(\sum\limits_{v' \to v} \Gamma^a_{3\ell(v')-1}\right)!}{\prod\limits_{v \in \mathcal{S}} \left(\Gamma^a_{3\ell(v)-1}\right)!} \\ &= \eta_a^{\zeta(T,\varnothing)}(\mathfrak{q}_\bullet) \cdot \prod_{v \in \mathcal{S}} \frac{(\ell(v)\Gamma^a_2)!}{(\Gamma^a_2!)^{\ell(v)}}. \end{aligned}$$

Note that we have $C_{\zeta(T,\mathcal{S})} = \left(\prod\limits_{v \in \mathcal{S}} \ell(v)!\right)^{-2}$, and hence

$$\begin{aligned} \widetilde{\mathbf{T}}^a_d \cdot \mathfrak{o}^a_{3d-1} &= \tfrac{1}{d!} \sum_{T \in \mathfrak{T}_d^{\mathrm{or}}} (-1)^{|V_{\mathrm{in}}(T)|} \sum_{\mathcal{S} \subset V_{\mathrm{mov}}(T)} C_{\zeta(T,\mathcal{S})} \cdot (-1)^{|\mathcal{S}|} \cdot \eta_a^{\zeta(T,\mathcal{S})}(\mathfrak{q}_\bullet) \\ &= \tfrac{1}{d!} \sum_{T \in \mathfrak{T}_d^{\mathrm{or}}} (-1)^{|V_{\mathrm{in}}(T)|} \cdot \eta_a^{\zeta(T,\varnothing)}(\mathfrak{q}_\bullet) \sum_{\mathcal{S} \subset V_{\mathrm{mov}}(T)} \left(\prod_{v \in \mathcal{S}} \ell(v)!\right)^{-2} \cdot (-1)^{|\mathcal{S}|} \cdot \prod_{v \in \mathcal{S}} \frac{(\ell(v)\Gamma^a_2)!}{(\Gamma^a_2!)^{\ell(v)}}. \end{aligned}$$

We also have

$$\eta_a^{\zeta(T,\varnothing)}(\mathfrak{q}_\bullet) = (\Gamma^a_2!)^d \cdot \prod_{v \in V_{\mathrm{in}}(T)} \frac{\left(\Gamma^a_{3\ell(v)-1}\right)!}{\left(\sum\limits_{v' \to v} \Gamma^a_{3\ell(v')-1}\right)!} \cdot \mathfrak{o}^a_{3d-1},$$

and hence

$$\begin{aligned}\widetilde{\mathbf{T}}^a_d &= \tfrac{1}{d!}(\Gamma^a_2!)^d \sum_{T\in\mathfrak{T}^{\mathrm{or}}_d} (-1)^{|V_{\mathrm{in}}(T)|}\cdot \prod_{v\in V_{\mathrm{in}}(T)} \frac{\left(\Gamma^a_{3\ell(v)-1}\right)!}{\left(\sum\limits_{v'\to v}\Gamma^a_{3\ell(v')-1}\right)!}\cdot \sum_{\mathcal{S}\subset V_{\mathrm{mov}}(T)} (-1)^{|\mathcal{S}|}\cdot \prod_{v\in\mathcal{S}} (\ell(v)!)^{-2}\frac{(\ell(v)\Gamma^a_2)!}{(\Gamma^a_2!)^{\ell(v)}}\\ &= \tfrac{1}{d!}(\Gamma^a_2!)^d \sum_{T\in\mathfrak{T}^{\mathrm{or}}_d} (-1)^{|V_{\mathrm{in}}(T)|}\cdot \prod_{v\in V_{\mathrm{in}}(T)} \frac{\left(\Gamma^a_{3\ell(v)-1}\right)!}{\left(\sum\limits_{v'\to v}\Gamma^a_{3\ell(v')-1}\right)!}\cdot \prod_{v\in V_{\mathrm{mov}}(T)} \left(1-(\ell(v)!)^{-2}\frac{(\ell(v)\Gamma^a_2)!}{(\Gamma^a_2!)^{\ell(v)}}\right).\end{aligned}$$

Finally, we can write this as a sum over $\mathfrak{T}^{\mathrm{un}}_d$ by recalling that there is a natural forgetful map $\mathfrak{T}^{\mathrm{or}}_d \to \mathfrak{T}^{\mathrm{un}}_d$ and the fiber of any $T \in \mathfrak{T}^{\mathrm{un}}_d$ has cardinality $\frac{d!}{|\mathrm{Aut}(T)|}$. $\square$

References

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