# Lecture 20

**Goal:**
- finish up series solutions
- systems of ODEs

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**Consider:** $y''(t) + p(t)y'(t) + q(t)y(t) = 0$ $(*)$

suppose $t=0$ is a regular singular pt. Assume $t>0$.

Means: $tp(t) = \sum_{n=0}^{\infty} p_n t^n$

$t^2 q(t) = \sum_{n=0}^{\infty} q_n t^n$

**Strategy:** make ansatz $y(t) = t^r \sum_{n=0}^{\infty} a_n t^n$

plug into $(*)$, solve for $r, a_0, a_1, a_2, \ldots$ recursively.

(ideally, get a fund. set $y_1, y_2$)

(note: could have e.g. $tp(t) = \sin(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \dots$)

Write (*) as $\boxed{t^2 y'' + t(tp(t))y' + (t^2 q(t))y = 0.}$

Plug in above expressions:

$$\begin{aligned}
y(t) &= \sum_{n=0}^{\infty} a_n t^{n+r} = a_0 t^r + a_1 t^{r+1} + a_2 t^{r+2} + \dots \\
y'(t) &= \sum_{n=0}^{\infty} a_n (n+r) t^{n+r-1} = a_0 r t^{r-1} + a_1 (r+1) t^r + \dots \\
y''(t) &= \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) t^{n+r-2} = a_0 (r)(r-1) t^{r-2} + a_1 (r+1) r t^{r-1} + \dots
\end{aligned}$$

$$t^2 \left( \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) t^{n+r-2} \right) + t \underbrace{\left( \sum_{n=0}^{\infty} p_n t^n \right) \left( \sum_{n=0}^{\infty} a_n (n+r) t^{n+r-1} \right)} + \underbrace{\left( \sum_{n=0}^{\infty} q_n t^n \right) \left( \sum_{n=0}^{\infty} a_n t^{n+r} \right)} = 0$$

$$\rightsquigarrow t^2 \left( a_0 r(r-1) t^{r-2} + a_1 (r+1) r t^{r-1} + a_2 (r+2)(r+1) t^r + \dots \right) + t \left( p_0 + p_1 t + p_2 t^2 + \dots \right) \left( a_0 r t^{r-1} + a_1 (r+1) t^r + \dots \right) + \left( q_0 + q_1 t + q_2 t^2 + \dots \right) \left( a_0 t^r + a_1 t^{r+1} + \dots \right) = 0$$

$$\longrightarrow (t^r)\left(\underbrace{q_0(r)(r-1) + p_0 q_0 r + q_0 q_0}\right)$$

$$+ (t^{r+1})\left(q_1(r+1)r + p_0 q_1 (r+1) + p_1 q_0 r + q_0 q_1 + q_1 q_0\right)$$

$$+ (t^{r+2})\left( \cdots \right)$$

$$+ \cdots = 0$$

setting $t^r$ term to zero, get

$$\underbrace{r(r-1) + p_0 r + q_0 = 0}_{\text{"indicial equation"}}$$

Note: If $p_1 = p_2 = \cdots = 0$ and $q_1 = q_2 = q_3 = \cdots = 0$,

then (*) would be $\underbrace{t^2 y'' + t p_0 y' + q_0 y = 0}$

Euler eqn, whose indicial eqn is $r(r-1) + p_0 r + q_0 = 0$.

Set $F(r) := r(r-1) + p_0 r + q_0$

(So indicial eqn $\iff F(r) = 0$).

**Claim:** Have $(t^r)(F(r)) + (t^{r+1})\big(a_1 F(r+1) + p_1 a_0 r + q_1 a_0\big)$

$+ (t^{r+2})(\cdots)$

$$= (t^r)(F(r)) + \sum_{n=1}^{\infty} \left( F(r+n)\, a_n + \underbrace{\sum_{k=0}^{n-1} a_k \big[ (r+k) p_{n-k} + q_{n-k} \big]}_{\text{linear combination of } a_1, \dots, a_{n-1}} \right) t^{\text{nth}}$$

So we following eqns:

(1) $F(r) = 0 \implies r$ has to be a root of ind. eqn.

(2) $$a_n = \frac{-\left(\sum_{k=0}^{n-1} a_k\left[(r+k)p_{n-k} + q_{n-k}\right]\right)}{F(r+n)}$$

Let $r_1 \geq r_2$ be "exponts of the singularity".

<u>Potential issue</u>: $F(r+n) = 0$ ← i.e. $r+n = r_1$ or $r+n = r_2$.

- If $r = r_1$, then $r+n$ cannot be $r_1$ or $r_2$ for $n \geq 1$.
  So no problem.
- If $r = r_2$, $r+n$ could be $r_1$. trouble!

<u>Situation</u>:

- If $r_1$ is real (w/ $r_1 \geq r_2$) then can always find a soln of form $y_1(t) = t^{r_1}\left(1 + \sum_{n=1}^{\infty} a_n t^n\right)$
- If $r_1 - r_2$ is not an integer, then can find a 2nd soln of form $y_2(t) = t^{r_2}\left(1 + \sum_{n=1}^{\infty} b_n t^n\right)$.
- If $r_1, r_2$ are cpx or $r_1 = r_2$ or $r_1 - r_2$ is an integer, then $y_2(t)$ is slightly more complicated.

<u>Rmk</u>: Can alternatively apply red. of order to find a second soln.

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# Systems of (first order, ~~linear~~ constant coefficient) ODEs

Ex: ("SIS system")

$S(t) =$ # of individuals who are suspectible but not get infected

$I(t) =$ # of individual who are infected

$$\begin{cases} S'(t) = -\alpha I(t)S(t) + \beta I(t) \\ I'(t) = \alpha I(t) S(t) - \beta I(t) \end{cases}$$

nonlinear system due to $IS$ term

Goal: Find $S(t)$ and $I(t)$ given $I(0), S(0)$.

Solution: $S'(t) + I'(t) = 0 \Rightarrow S(t) + I(t) = N$

a constant, the total population

So $S(t) = N - I(t)$.

$$\rightsquigarrow \underbrace{I'(t) = \alpha I(t)\big(N - I(t)\big) - \rho I(t)}_{\text{first order separable ODE.}}$$

Can solve for $I(t)$. Then $S(t) = N - I(t)$,
and $N = S(0) + I(0)$.

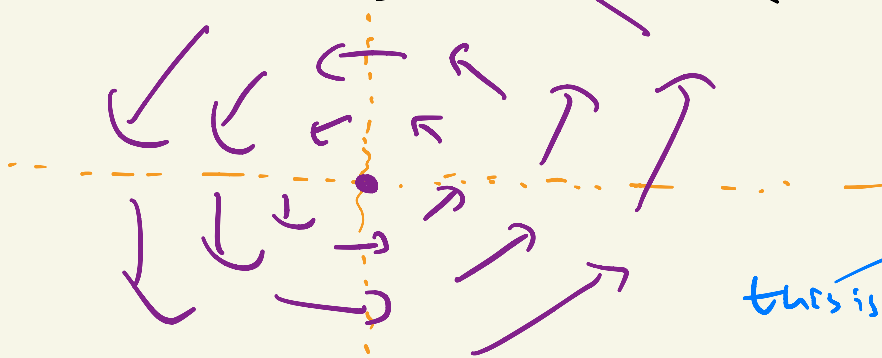
**Ex:**
$$\begin{cases} x'(t) = -y(t) \\ y'(t) = x(t). \end{cases}$$

Equilibrium solutions?

$x(t) \equiv 0$, $y(t) \equiv 0$

( $(x(t), y(t))$ traces out a curve, $(x'(t), y'(t))$ is the velocity )

Goal: find $(x(t), y(t))$.



Let's plot the vector field $(-y, x)$.

this is perpendicular to $(x,y)$.

**Solution:** $y''(t) = x'(t) = -y(t)$

So $y''(t) = -y(t) \implies y(t) = c_1\sin(t) + c_2\cos(t)$

$$x(t) = y'(t) = c_1\cos(t) - c_2\sin(t).$$

$$(x(t), y(t)) = \big(c_1\cos(t) - c_2\sin(t),\ c_1\sin(t) + c_2\cos(t)\big)$$

**Check:** $x(t)^2 + y(t)^2 = c_1^2\cos^2(t) + c_2^2\sin^2(t) \cancel{- 2c_1c_2\cos(t)\sin(t)}$
$\qquad + c_1^2\sin(t) + c_2^2\cos^2(t) \cancel{+ 2c_1c_2\sin(t)\cos(t)}$

$$= c_1^2 + c_2^2$$

So $(x(t), y(t))$ lies on a circle of radius $\sqrt{c_1^2 + c_2^2}$

**Ex:** $x''(t) + \sin(x(t)) = 0$ $(*)$ ("pendulum eq$^n$")

nonlinear

**Claim:** Can convert into a system of 1st order ODEs.

Put $y(t) = x'(t)$. Then $(*)$ is equivalent to

$$\begin{cases} x'(t) = y(t) \\ y'(t) = -\sin(x(t)). \end{cases}$$

In fact, similarly, can convert ~~any~~ $n$th order ODE into a system of $n$ 1st order ODEs.

<u>Ex</u>:

$$\begin{cases} x'(t) = 2x(t) \\ y'(t) = 3y(t) \end{cases}$$

"uncoupled" or "diagonal" system

<u>Solution</u>: $x(t) = C_1 e^{2t}$, $y(t) = C_2 e^{3t}$.

<u>Ex</u>:

$$\begin{cases} x'(t) = 5x(t) - y(t) \\ y'(t) = -x(t) + 5y(t). \end{cases}$$

$$x' + y' = 4x + 4y$$

$$\frac{d}{dt}(x+y) = 4(x+y).$$

$$x'' = 5x' - y'$$

$$x'' - 5x' - (-x + 5y)$$