ODIE PSet # 5 Solutions

Section 3.2

Q29) We have that $W(y_1, y_2)(t) = Ce^{\int -p(t)\,dt}$
(Theorem 3.2.7)

$t^2 y'' - t(t+2)y' + (t+2)y = 0$

$\Rightarrow y'' - \frac{t(t+2)}{t^2} y' + \frac{(t+2)}{t^2} y = 0$ so $p(t) = -\frac{t(t+2)}{t^2}$

so $W(y_1, y_2)(t) = Ce^{\int \frac{t(t+2)}{t^2}\,dt}$

we note that $\int \frac{t(t+2)}{t^2}\,dt = \int 1 + \frac{2}{t}\,dt$

$= t + 2\ln t + C$

so $W(y_1, y_2)(t) = Ct^2 e^t$

Section 3.4

Q23) $t^2y'' - 4ty' + 6y = 0 \implies y'' - \frac{4}{t}y' + \frac{6y}{t^2} = 0$

We have that a solution of this ODE is:

$$y_1(t) = t^2$$

Let $y_2(t) = v(t)y_1(t) = v(t)t^2$

We have that $y_2'(t) = v'(t)t^2 + 2tv(t)$

$$y_2''(t) = v''(t)t^2 + 4tv'(t) + 2v(t)$$

We have that: $y_2''(t) - \frac{4}{t}y_2'(t) + \frac{6}{t^2}y_2(t)$

$= v''(t)t^2 + v'(t)\left(4t - 4t\right) + v(t)\left(2 - 8 + 6\right) = 0$

$= v''(t)t^2 = 0 \implies v''(t) = 0$

$\implies v'(t) = C_1$

$\implies v(t) = C_1t + C_2$

So $y_2(t) = C_1t^3 + C_2t^2$

Q31) $y_1(x) = e^{\frac{-\delta x^2}{2}}$

$y_1'(x) = -\delta x e^{\frac{-\delta x^2}{2}}$

$y_1''(x) = \delta^2 x^2 e^{\frac{-\delta x^2}{2}} - \delta e^{\frac{-\delta x^2}{2}}$

So $y_1'' + \delta(xy_1' + y_1)$ is equal to:

$$\delta^2 x^2 e^{\frac{-\delta x^2}{2}} - \delta e^{\frac{-\delta x^2}{2}} - \delta\left(\delta x^2 e^{\frac{-\delta x^2}{2}} + e^{\frac{-\delta x^2}{2}}\right)$$

$$= \underbrace{\delta^2 x^2 e^{\frac{-\delta x^2}{2}} - \delta^2 x^2 e^{\frac{-\delta x^2}{2}}}_{=0} \underbrace{- \delta e^{\frac{-\delta x^2}{2}} + \delta e^{\frac{-\delta x^2}{2}}}_{=0}$$

$$= 0$$

So $y_1(x) = e^{\frac{-\delta x^2}{2}}$ is a solution.

Let $y_2(x) = v(x)\, y_1(x)$ plugging this into Initial problem gives:

$$e^{\frac{-\delta x^2}{2}} v'' + \left(-2\delta x e^{\frac{-\delta x^2}{2}} + \delta x e^{\frac{-\delta x^2}{2}}\right) v' = 0$$

$$\Rightarrow v'' - \delta x v' = 0$$

Let $v' = w$, we have that:

$w' - \delta x w = 0$ this can be solved using the method of integrating factors

$w = Ce^{\delta x^2/2}$, but $v' = w$ so $v = \int Ce^{\frac{\delta x^2}{2}}\,dx$

$$y_2(x) = \left(\int c e^{\frac{\delta x^2}{2}} dx\right) e^{-\frac{\delta x^2}{2}}$$

The general solution is:

$$y(x) = C_1 e^{-\frac{\delta x^2}{2}} + C_2 \int c e^{\frac{\delta x^2}{2}} dx \, e^{-\frac{\delta x^2}{2}}$$

Q37) $ay'' + by' + C = 0$

the characterystic equation is:

$$ar^2 + br + c = 0$$

$$\Longrightarrow r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Case 1: $b^2 - 4ac > 0$

In this the general solution has form

$$y(t) = C_1 e^{\frac{-b + \sqrt{b^2 - 4ac}}{2a} t} + C_2 e^{\frac{-b - \sqrt{b^2 - 4ac}}{2a}}$$

we note that $b^2 - 4ac < b^2$ (since $4ac > 0$)

and so $\sqrt{b^2 - 4ac} < b$.

So $\frac{-b + \sqrt{b^2 - 4ac}}{2a} < 0$, $\frac{-b - \sqrt{b^2 - 4ac}}{2a} < 0$

So $\lim_{t \to \infty} y(t) = 0$

Case 2: $b^2 - 4ac = 0$

The general solution $y(t)$ is of the form:

$$y(t) = C_1 e^{\frac{-b}{2a}t} + C_2 t e^{-\frac{b}{2a}t}$$

Since $\frac{-b}{2a} < 0$ we have that $\lim_{t \to \infty} e^{-\frac{b}{2a}t} = 0$

$\lim_{t \to \infty} t e^{-\frac{b}{2a}t} = 0$

Case 3: $b^2 - 4ac < 0$

The general solution has form:

$$y(t) = e^{-bt}\left(C_1 \cos\left(\sqrt{4ac - b^2}\, t\right) + C_2 \sin\left(\sqrt{4ac - b^2}\, t\right)\right)$$

Since $e^{-bt} \longrightarrow 0$ as $t \longrightarrow \infty$ and

$0 \leq \cos\left(\sqrt{4ac - b^2}\, t\right), \sin\left(\sqrt{4ac - b^2}\, t\right) \leq 1$

we have that $\lim_{t \to \infty} y(t) = 0$

Section 3.5

Q5) Consider the homogenous equation:

$$y'' - 2y' - 3y = 0$$

the characteristic equation is $r^2 - 2r - 3 = 0$

the roots of this equation are $r = 3$, $r = -1$

So the solution to the homogenous equation has the form $y(t) = C_1 e^{3t} + C_2 e^{-t}$

We need to find a solution for the non-homogenous equation.

Let $y_1(t) = u(t) e^{-t}$ where $u(t)$ is a polynomial

we have that:

$$e^{-t}\left(u''(t) - 4u'(t)\right) = -3t e^{-t}$$

$$\Rightarrow u''(t) - 4u'(t) = -3t$$

Let $u(t) = At^2 + bt + c$

then $u'(t) = 2At + b$

$u''(t) = 2A$

So $2A - 8At - 4b = -3t \Longrightarrow A = \frac{3}{8}, B = \frac{3}{16}$

General Sol: $y(t) = e^{-t}\left(\frac{3}{8}t^2 + \frac{3}{16}t\right) + C_1 e^{3t} + C_2 e^{-t}$

Q9) Consider the homogenous equation: $2y''+3y'+y=0$

the characteristic equation is $2r^2+3r+1=0$

the roots of this equation are $r=-1,\ r=-\frac{1}{2}$

So the homogenous has general Solution:

$y(t)=C_1e^{-t}+C_2e^{-\frac{1}{2}t}$

Consider the equation $2y''+3y'+y=t^2$

Let $y(t)=At^2+Bt+C$

then $y'(t)=2At+B$

$y''(t)=2A$

So $4A+6At+3B+At^2+Bt+C$

$\Rightarrow A=1,\ B=-6,\ C=14$

Consider the equation $2y''+3y'+y=3\sin t$

Let $y(t)=E\sin t+F\cos t$

then $y'(t)=E\cos t-F\sin t$

$y''(t)=-E\sin t-F\cos t$

So $2y''+3y'+y=(-E-3F)\sin t+(3E-F)\cos t$

$\Rightarrow E=\frac{-3}{10},\ F=\frac{-9}{10}$

So the general solution is:

$$y(t) = C_1 e^{-t} + C_2 e^{-\frac{1}{2}t} - \frac{3}{10}\sin t - \frac{9}{10}\cos t + t^2 - 6t + 14$$

Q10) Consider the homogenous equation $y'' + y = 0$
the characteristic equation is $r^2 + 1 = 0$

So the general solution of the equation is $C_1 \cos t + C_2 \sin t$

Consider the equation $y'' + y = 3\sin 2t + t\cos 2t$

Let $y(t) = (at+b)\sin 2t + (ct+d)\cos 2t$

We have that $y'' + y = -3(ct+d)\cos 2t - 4c\sin 2t$
$$-3(at+b)\sin 2t + 4a\cos 2t$$
$$= 3\sin 2t + t\cos 2t$$

$\Rightarrow (4a - 3d)\cos 2t - 3ct\cos 2t - (3b+4c)\sin 2t$
$- 3at\sin 2t = 3\sin 2t + t\cos 2t$

$\Rightarrow$ $4a - 3d = 0$
$-3c = 1$
$-3a = 0$
$-(3b+4c) = 3$

$\Longrightarrow a = 0,\ d = 0,\ c = -\frac{1}{3},\ b = -\frac{5}{9}$

So the general solution is:

$$y(t) = C_1 \cos t + C_2 \sin t - \frac{5}{9}\sin 2t - \frac{1}{3}t\cos 2t$$

Q20) Consider the homogenous equation $y'' + 2y' + 5y = 0$

the characteristic equation is $r^2 + 2r + 5 = 0$
the roots of the equation are $r = -1 \pm 2i$

So the solution to the homogenous equation is:
$y(t) = e^{-t}(C_1 \cos 2t + C_2 \sin 2t)$

Consider the equation $y'' + 2y' + 5y = 4e^{-t}\cos 2t$
Let $y(t) = Ate^{-t}\cos 2t + Bte^{-t}\sin 2t$

by substituting we find that $a = 0$, $b = 1$

so the solution to this equation is of the form

$y(t) = C_1 e^{-t}\cos 2t + C_2 e^{-t}\sin 2t + te^{-t}\sin 2t$

Since $y(0) = 1$, $y'(0) = 1$

we have that $y(t) = e^{-t}\cos 2t + \frac{1}{2}e^{-t}\sin 2t$
$+ te^{-t}\sin 2t$

34) The characteristic equation is

$$ar^2 + br + c = 0$$

It has solutions

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow r \neq 0$$

We showed in a previous problem that every solution to the homogenous equation tends to zero as $t \longrightarrow \infty$.

A particular solution of the non-homogenous equation can be found by letting $y(t) = A$

we have that $ay'' + by' + cy = d$

$$\Rightarrow Ac = d \iff A = \frac{d}{c}$$

So a particular solution is $y(t) = \frac{d}{c}$

The general solution is:

$y(t) = Y(t) + \frac{d}{c}$ where $Y(t)$ is the solution to the homogenous equation

$$\lim_{t \to \infty} y(t) = \lim_{t \to \infty} Y(t) + \frac{d}{c} = \frac{d}{c}$$

If $C=0$, we have that the homogenous equation is:

$ay''+by'=0$, and the Characteristic equation is:

$ar^2+br=0 \Rightarrow r=0, r=-\frac{b}{a}$

The solution to the homogenous equation is

$y(t) = A + Be^{-\frac{b}{a}t}$

A particular solution to the non-homogenous equation can be found by letting $y = mt$

We have that $ay''+by' = bm = d \Rightarrow m=\frac{d}{b}$

So the solution to the ODE is:

$y(t) = A + Be^{-\frac{b}{a}t} + \frac{d}{b}t$

$\lim_{t\to\infty} y(t) = \infty$, asymptomatic to $y(t) = \frac{d}{b}t + A$

- If $b=0$, $C=0$

Then a particular solution is: $y = \frac{d}{2a}t^2$

So general solution approaches ∞ as $t \to \infty$

Section 3.6

Q3) $y''+2y'+y=3e^{-t}$

Consider the homogenous equation $y''+2y'+y=0$
the solution to the homogenous equation is:

$y=C_1e^{-t}+C_2te^{-t}$

Now to solve $y''+2y'+y=3e^{-t}$

Let $y=u_1(t)e^{-t}+u_2(t)te^{-t}$

We have that:

$$u_1(t)=-\int\frac{te^{-t}\cdot 3e^{-t}}{W(y_1,y_2)(t)}dt+C_1$$

$$u_2(t)=\int\frac{e^{-t}\cdot 3e^{-t}}{W(y_1,y_2)(t)}dt+C_2$$

We have that $W(y_1,y_2)(t)=\begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & e^{-t}-te^{-t}\end{vmatrix}$

$W(y_1,y_2)(t)=e^{-2t}-te^{-2t}+te^{-2t}=e^{-2t}$

So $u_1(t)=-\int\frac{3te^{-2t}}{e^{-2t}}dt+C_1=-\frac{3}{2}t^2+C_1$

$$u_2(t) = \int \frac{3e^{-2t}}{e^{-2t}}\,dt + C_2 = 3t + C_2$$

So the general solution has the form

$$y(t) = 3t^2 e^{-t} - \frac{3}{2}t^2 e^{-t} + Ae^{-t} + Bte^{-t}$$

$$y(t) = \frac{3}{2}t^2 e^{-t} + Ae^{-t} + Bte^{-t}$$

By letting $y(t) = (at^2 + bt + c)e^{-t}$,

we obtain that a particular solution is:

$y(t) = \frac{3}{2}t^2 e^{-t}$ so, the two methods agree.

Q5) The solution to the homogenous equation is

$y(t) = A\cos t + B\sin t$

To solve the non-homogenous equation let

$y(t) = u_1(t)\cos t + u_2(t)\sin t$

we have that:

$$W(y_1, y_2)(t) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = 1$$

$$\text{so } u_1(t) = -\int \frac{\tan t \cdot \sin t}{1} dt = -\int \sec t - \cos t \, dt$$

$$u_1(t) = -\ln(\sec t + \tan t) + \sin t + C_1$$

$$u_2(t) = \int \frac{\cos t \cdot \tan t}{1} dt = \int \sin t \, dt = -\cos t + C$$

So the general solution is:

$$y(t) = \cos t\left(\sin t - \ln(\sec t + \tan t)\right) - \sin t \cos t$$
$$+ A\cos t + B\sin t$$

$$y(t) = -\ln(\sec t + \tan t)\cos t + A\cos t + B\sin t$$

Q6) The solution to the homogenous equation

$y(t) = A\cos 3t + B\sin 3t$

To solve the non-homogenous equation let:

$y(t) = u_1(t)\cos 3t + u_2(t)\sin 3t$

$$W(y_1, y_2)(t) = \begin{vmatrix} \cos 3t & \sin 3t \\ -3\sin 3t & 3\cos 3t \end{vmatrix} = 3$$

$$u_1(t) = -\int \frac{3\sin 3t}{\cos^2 3t}\,dt = -\frac{1}{?}\sec 3t + C_1$$

$$u_2(t) = \int \frac{3\cos 3t}{\cos^2 3t}\,dt = \ln|\sec 3t + \tan 3t| + C_2$$

So the general solution is:

$y(t) = A\cos 3t + B\sin 3t - 1 + \sin 3t \cdot \ln|\sec 3t + \tan 3t|$

Q17) $y_1 = x^2,\ y_1' = 2x,\ y_1'' = 2$

So $x^2 y_1'' - 3x y_1' + 4y_1 = 2x^2 - 6x^2 + 4x^2 = 0$

$y_2 = x^2 \ln x,\ y_2' = 2x\ln x + x,\ y_2'' = 1 + 2\ln x + 2$

So $x^2 y_2'' - 3x y_2' + 4y_2 = 3x^2 + 2x^2\ln x - 6x^2 \ln x$
$- 3x^2 + 8x^2 \ln x = 0$

To solve the non-homogenous equation det

$y(t) = u_1(x)\,x^2 + u_2(x)\,x^2 \ln x$

$$W(y_1, y_2)(x) = \begin{vmatrix} x^2 & x^2 \ln x \\ 2x & 2x\ln x + x \end{vmatrix}$$

$= 2x^3 \ln x + x^3 - 2x^3 \ln x = x^3$

$$u_1(x) = -\int \frac{x^2 \ln x \cdot \ln x}{x^3}\,dx = -\int \frac{(\ln x)^2}{x}\,dx = -\frac{1}{3}(\ln x)^3 + C_1$$

$$u_2(x) = \int \frac{x^2 \cdot \ln x}{x^3}\,dx = \int \frac{\ln x}{x}\,dx = \frac{1}{2}(\ln x)^2 + C_2$$

So a particular solution is:

$y(t) = \frac{1}{6} x^2 (\ln x)^3$