# Lecture 17 section 1

# Lecture 17

**Recall**: $t^2y'' - 6y = 0$

Try: ansatz $y(t) = \sum_{n=0}^{\infty} a_n t^n$

$y' = \sum_{n=0}^{\infty} a_n n t^{n-1}$, $y'' = \sum_{n=0}^{\infty} a_n n(n-1) t^{n-2}$

$\rightsquigarrow \sum_{n=0}^{\infty} a_n n(n-1) t^n - 6\sum_{n=0}^{\infty} a_n t^n = 0$

$\Rightarrow \sum_{n=0}^{\infty} (a_n n^2 - n a_n - 6a_n) t^n = 0$

$\Rightarrow a_n(n^2 - n - 6) = 0$ for all $n \geq 0$.

$n=0 \Rightarrow a_0 \cdot (-6) = 0 \Rightarrow a_0 = 0$

$n=1 \Rightarrow a_1(1^2 - 1 - 6) = 0 \Rightarrow a_1 = 0$

$\Rightarrow a_2 = a_3 = a_4 = a_5 = \ldots = 0$.

Problem: only power series solution centered at 0 is the zero function...

We noted: in this example, $t=0$ is a singular pt.

**Today**: A class of examples, called Euler's eqns, which involve a singular pt.

**Note**: $t^2y'' - 3\cdot 2y = 0$

$t^3$ is a solution!

Claim: $y(t) = t^{-2}$ is also a soln!

$y' = -2t^{-3}$, $y'' = (-2)(-3)t^{-4}$

$\rightsquigarrow t^2(6)t^{-4} - 6t^{-2} = 0$ ✓

**Note**: ansatz $y(t) = t^r$ encompasses both of these!

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**Ex**: $t^2y'' + 6ty' + 4y = 0$.

Ansatz: $y(t) = t^r$ (r fixed constant)

$y' = rt^{r-1}$, $y'' = r(r-1)t^{r-2}$

$\rightsquigarrow t^2 r(r-1)t^{r-2} + 6t\,rt^{r-1} + 4t^r = 0$

$\rightsquigarrow (r(r-1) + 6r + 4)t^r = 0$

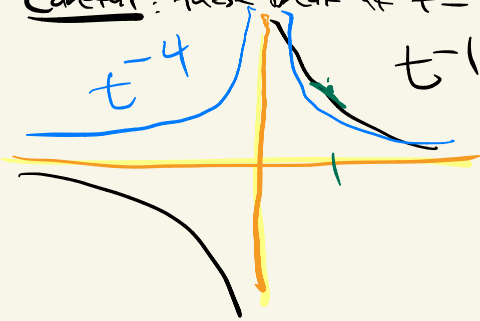
$\Rightarrow$ need $r^2 + 5r + 4 = 0$

$\Rightarrow (r+1)(r+4) = 0$

$\Rightarrow r = -1$ or $r = -4$

$\Rightarrow t^{-1}$ and $t^{-4}$ are solutions.

**Careful**: these break at $t=0$.



Ex: $t^2 y'' + 7ty' + 4y = 0$

ansatz: $y(t) = t^r$

$(r(r-1) + 7r + 4)t^r = 0$

$\rightsquigarrow r^2 + 6r + 4 = 0$

$\Rightarrow r = \frac{-6 \pm \sqrt{36-16}}{2}$

$= \frac{-6 \pm \sqrt{20}}{2} = -3 \pm \sqrt{5}$

$r_1 = -3+\sqrt{5}, \quad r_2 = -3-\sqrt{5}$

For $t > 0$, general soln is

$$\boxed{y(t) = C_1 t^{-3+\sqrt{5}} + C_2 t^{-3-\sqrt{5}}}$$

Define: $t^2 y'' + \alpha t y' + \beta y = 0$

for $\alpha, \beta \in \mathbb{R}$ is called an Euler equation.

Naively: roots $r_1$ and $r_2$

general soln $y(t) = C_1 t^{r_1} + C_2 t^{r_2}$.

Issues:

- roots could be cpx
- root could be repeated
- for $t<0$, $t^r$ not defined

Get $y_1(t) = t^{-2}$.

How to find $y_2(t)$? Use red. of order!

Ansatz: $y(t) = u(t)\, y_1(t)$

Plug in: $t^2(u'' y_1 + 2u' y_1' + \cancel{u y_1''})$

$+ 5t(u' y_1 + \cancel{u y_1'}) + \cancel{4 u y_1} = 0$

(cancel since $y_1$ solves the ODE)

$y_1 = t^{-2}, \quad y_1' = -2t^{-3}, \quad y_1'' = 6t^{-4}$

$t^2(u'' t^{-2} + 2u'[-2t^{-3}]) + 5tu' t^{-2} = 0$

$u'' + t^{-1} u' = 0.$

$\rightsquigarrow$ Gives two solns

$y_1(t) = t^{-3+\sqrt{5}}, \quad y_2(t) = t^{-3-\sqrt{5}}$

[recall: $e^t := 1 + t + t^2/2! + t^3/3! + t^4/4! + \ldots$]

To define $t^r$ for $r$ not an integer

$t^r = (e^{\ln(t)})^r = e^{r \ln(t)}$

so then plug in $r\ln(t)$ to above definition of exp.

Ex: $r = -3+\sqrt{5}$ $\quad 17^r := e^{(-3+\sqrt{5})\ln(17)}$.

Careful: ln(negative) = not defined.

$\Rightarrow t^r$ not really defined for $t < 0$.

For Euler equations:

Ansatz: $y(t) = t^r$. $\rightsquigarrow y' = rt^{r-1}, \; y'' = r(r-1)t^{r-2}$

Plug into Euler eqn: $r(r-1)t^r + r\alpha t^r + \beta t^r = 0$

$\rightarrow (r(r-1) + r\alpha + \beta)t^r = 0$

$\rightarrow \underbrace{r(r-1) + r\alpha + \beta = 0}_{\text{indicial equation}}$

Ex $t^2 y'' + 5ty' + 4y = 0$, assume $t > 0$

$\rightsquigarrow r(r-1) + 5r + 4 = 0$.

$r^2 + 4r + 4 = 0$.

$(r+2)^2 = 0$.

$\rightsquigarrow r_1 = r_2 = -2$ repeated roots.

$y' = u' y_1 + u y_1'$

$y'' = u'' y_1 + 2u' y_1' + u y_1''$

Put $v(t) := u'(t)$.

$v'(t) + t^{-1} v(t) = 0$.

Separable: $\frac{v'}{v} = -t^{-1}$

$\ln(v) = -\ln(t)$

$\Rightarrow v = t^{-1}$.

$\Rightarrow u = \ln(t)$

Conclusion: $t^{-2} \ln(t)$ is also a solution!

**Ex:** $t^2y'' + 2ty' + y = 0$.

indicial eqn: $(r)(r-1) + 2r + 1 = 0$

$r^2 - r + 2r + 1 = 0$

$r^2 + r + 1 = 0$

$r = \frac{-1 \pm \sqrt{1-4}}{2} = -1/2 \pm i\sqrt{3}/2$

$r_1 = -1/2 + \frac{i\sqrt{3}}{2}$, $r_2 = -1/2 - \frac{i\sqrt{3}}{2}$

Could say $c_1 t^{-1/2 + i\sqrt{3}/2} + c_2 t^{-1/2 - i\sqrt{3}/2}$ is the general solution...

Put $\alpha = -1/2$ $\beta = \sqrt{3}/2$.

How to go from $t^{\alpha + i\beta}$, $t^{\alpha - i\beta}$ to two real-valued solutions?

Write $t^{\alpha + i\beta} = t^\alpha t^{i\beta}$

$= t^\alpha e^{i \ln(t) \beta}$

$= t^\alpha [\cos(\ln(t)\beta) + i \sin(\ln(t)\beta)]$

$\Rightarrow$ real and imag. parts are both also solutions.

**Upshot:** $t^\alpha \cos(\ln(t)\beta)$, $t^\alpha \sin(\ln(t)\beta)$ are both solutions to ODE, and they form a fund. set.

**Claim:** If $r_1 = r_2$ is a repeated root to indicial eqn for $t^2y'' + \alpha t y' + \beta y = 0$, then $t^{r_1}\ln(t)$ is also a soln.

**Pf:** Apply red. of order as above.

Or, use trick: Put

$L[y] = t^2y'' + \alpha t y' + \beta y$

Have $L[t^r] = t^r((r-r_1)^2)$

$\frac{\partial}{\partial r} L[t^r] = r t^{r-1}(r-r_1)^2 + t^r 2(r-r_1)$

note: RHS is zero for $r = r_1$.

$$\Rightarrow \underbrace{\frac{\partial}{\partial r}\Big|_{r=r_1} L[t^r] = 0}$$

What is $\frac{d}{dr}(t^r)$?

$\frac{d}{dr}\left(e^{r\ln(t)}\right) = \ln(t) e^{r\ln(t)}$

$= t^r \ln(t)$.

So $\frac{\partial}{\partial r}\Big|_{r=r_1} L[t^r] = 0$

$L\left[\frac{\partial}{\partial r}\Big|_{r=r_1} t^r\right]$ $\|$

$L[t^r \ln(t)] = 0$. $\|$

**Conclusion:** $t^r \ln(t)$ is also a solution!

**Next:**

- what happens for $t < 0$?
- how do these solns look?

specifically, what happens as $t \to 0$?