Math 2030 PSet 8 Solutions

Q2(a) $y'' - 2xy' + \lambda y = 0$, the point $x=0$ is an ordinary point.

Let $y = \sum_{n=0}^{\infty} a_n x^n$, then $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

So $y'' - 2xy' + \lambda y = 0$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} 2n a_n x^n + \lambda \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} 2n a_n x^n + \lambda \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow 2a_2 + \lambda a_0 + \sum_{n=1}^{\infty} \left((n+2)(n+1)a_{n+2} + (\lambda - 2n) a_n\right) x^n = 0$$

So $2a_2 + \lambda a_0 = 0$ and $a_{n+2} = \frac{2n-\lambda}{(n+2)(n+1)} a_n$

So $a_0 = a_0$, $a_1 = a_1$, $a_2 = -\frac{\lambda a_0}{2}$, $a_3 = \frac{2-\lambda}{6} a_1$

$a_4 = \frac{(\lambda-4)\lambda}{24}$, $a_5 = \frac{(\lambda-2)(\lambda-6)}{120} a_1$

$a_6 = \frac{(\lambda-4)\lambda(8-\lambda)}{720}$, $a_7 = \frac{(10-\lambda)(\lambda-2)(\lambda-6)a_1}{120}$

By letting $a_0 = 1$, $a_1 = 0$ the first solution is:

$$y_1(x) = 1 - \frac{\lambda}{2}x^2 + \frac{(\lambda-4)\lambda}{4!}x^4 - \frac{(\lambda-4)(\lambda-8)\lambda}{6!}x^6$$

By letting $a_0 = 0$, $a_1 = 1$ the second solution is:

$$y_2(x) = x - \frac{(\lambda-2)x^3}{3!} + \frac{(\lambda-2)(\lambda-6)}{5!}x^5 - \frac{(\lambda-10)(\lambda-2)(\lambda-6)}{7!}x^7$$

We have that $W(y_1, y_2)(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix}$

$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$

So indeed they form a fundamental set of solutions.

b) if $\lambda = 0$, then $y_1(x) = 1$
if $\lambda = 1$, then $y_2(x) = x$
if $\lambda = 4$, then $y_1(x) = 1 - 2x^2$
if $\lambda = 6$, then $y_2(x) = x - \frac{2}{3}x^3$
if $\lambda = 8$, then $y_1(x) = 1 - 4x^2 + \frac{4}{3}x^4$

c) $H_0(x) = 1$, $H_1(x) = 2x$, $H_2(x) = 4x^2 - 2$

$H_3(x) = -12 + 8x^3$, $H_4(x) = 12 - 48x^2 + 16x^4$

$H_5(x) = 120 - 160x^3 + 32x^5$

Q2) $y'' + (\sin x)y' + (\cos x)y = 0 \Rightarrow y''(0) = 0$

$y''' + (\cos x)y' + (\sin x)y'' - (\sin x)y + (\cos x)y' = 0 \Rightarrow y'''(0) = 2$

$y^{(4)} + (\sin x)y''' + (2\cos x)y'' - (3\sin x)y' - (\cos x)y = 0$
so $y^{(4)}(0) = 0$

Q7) $(1+x^3)y'' + 4xy' + y = 0$

$\Rightarrow y'' + \frac{4x}{1+x^3}y' + y = 0$

we have that $1+x^3 = (1+x)(x^2-x+1)$

The roots of x^2-x+1 are $\frac{1 \pm \sqrt{3} \cdot i}{2}$

So the roots of $1+x^3$ are; -1, $\frac{1 \pm \sqrt{3} \cdot i}{2}$

For $x_0 = 0$ we have that the distance from 0 to the roots is 1 is the radius of convergence is atleast $\underline{\underline{1}}$

For $x_0 = 2$, we have that the distance from 0 to the roots is: $3, \sqrt{3}$

So the radius of convergence is atleast $\sqrt{3}$

Q11) $y'' + (\sin x)y = 0$ Since 0 is an ordinary point.

$\Rightarrow y^{(3)} + (\sin x)y' + (\cos x)y = 0$

$\Rightarrow y^{(4)} + (\sin x)y'' + (2\cos x)y' - (\sin x)y = 0$

$\Rightarrow y^{(5)} + (\sin x)y''' + (3\cos x)y'' - (3\sin x)y' - (\cos x)y = 0$

$\Rightarrow y^{(6)} + (\sin x)y^{(4)} + (4\cos x)y^{(3)} - (6\sin x)y'' - (4\cos x)y'$
$+ (\sin x)y = 0$

$\Rightarrow y^{(7)} + (\sin x)y^{(5)} + (5\cos x)y^{(4)} - (10\sin x)y^{(3)} - (10\cos x)y''$
$+ (5\sin x)y' + (\cos x)y = 0$

By setting $y(0) = 1,\ y'(0) = 0$

we get that: $y(0) = 1,\ y'(0) = 0,\ y''(0) = 0$
$y^{(3)}(0) = -1,\ y^{(4)}(0) = 0,\ y^{5}(0) = 1$
$y^{(6)}(0) = 4,\ y^{(7)}(0) = -1$

So $y_1(x) = 1 - \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{4x^6}{6!}$

(By writing $y = \sum_{n=0}^{\infty} a_n x^n$ and noting that $y^n(0) = a_n n!$)

By setting $y(0)=0$, $y'(0)=1$ we get:

$y(0)=0$, $y'(0)=1$, $y''(0)=0$, $y^{(3)}(0)=0$

$y^{(4)}(0)=-2$, $y^{5}(0)=0$, $y^{(6)}(0)=4$

$y^{7}(0)=10$

So $y_2(x) = x - \frac{x^4}{12} + \frac{x^6}{180} + \frac{x^7}{504}$

We have that $W(y_1, y_2)(0) = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 \neq 0$

So indeed they form a fundamental set of solutions

There are no zeroes of 1, so the radius of convergence is ∞

Q13) $(\cos x)y'' + xy' - 2y = 0$

Let $y(0) = 1$, $y'(0) = 0$

$$(\cos x)y'' + xy' - 2y = 0 \Rightarrow y''(0) - 2 = 0$$
$$\Rightarrow y''(0) = 2$$

We have after differentiating:

$$y^{(3)}\cos x - (\sin x)y'' + y' + xy'' - 2y' = 0$$
$$= y^{(3)}\cos x + y''(x - \sin x) - y' = 0$$
$$y^{(3)}(0) = 0$$

We have after differentiating that:

$$y^{(4)}\cos x + y^{(3)}(x - 2\sin x) + y''(-\cos x) = 0$$

So $y^{(4)}(0) = 2$

Using similar steps to the above we find that $y^{5}(0) = 0$ and $y^{(6)}(0) = 6$

So the first solution is: $1 + x^2 + \frac{1}{12}x^4 + \frac{6}{6!}x^6$

(By noting that $y^n(0) = n!\, a_n$)

Now by Letting $y(0)=0, y'(0)=1$
we have:

$(\cos x)y'' + xy' - 2y = 0 \Rightarrow y''(0) = 0$

$\Rightarrow y^{(3)} \cos x + y''(x - \sin x) - y' \Rightarrow y^{(3)}(0) = 1$

We have after differentiating above:

$y^{(4)} \cos x + y^{(3)}(x - 2\sin x) + y''(-\cos x) = 0$
so $y^{(4)}(0) = 0$

$y^{(5)} \cos x + y^{(4)}(x - 3\sin x) + y^{(3)}(1 - 3\cos x) + y'' \sin x = 0$
so $y^{(5)}(0) = 2$

Using a similar method we find that $y^{7}(0) = 9$

So the second solution is:

$$y_2(x) = x + \frac{1}{6}x^3 + \frac{2}{5!}x^5 + \frac{9}{7!}x^7$$

We have that $W(y_1, y_2)(0) = \det \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

So indeed they form a fundamental set of solutions

The zeroes of $\cos x$ are $\frac{(2k+1)\pi}{2}$ so the radius of convergence is atleast $\frac{\pi}{2}$

Section 5.4

Q1) $x^2y'' + 4xy' + 2y = 0$, Let $y = x^r$

So $(r(r-1) + 4r + 2)x^r = 0$

$\Rightarrow (r^2 + 3r + 2)x^r = 0 \Rightarrow r = -1,\ r = -2$

So the general solution is $y = C_1 x^{-1} + C_2 x^{-2}$

Q2) $(x+1)^2 y'' + 3(x+1)y' + 0.75y = 0$, Let $y = (x+1)^r$

So $(x+1)^r(r(r-1) + 3r + 0.75) = 0$

$= (x+1)^r(r^2 + 2r + 0.75) = 0$

$(x+1)^r((r+0.5)(r+1.5)) = 0 \Rightarrow r = -0.5,\ r = -1.5$

So the general solution is: $y = C_1|x+1|^{-0.5} + C_2|x+1|^{-1.5}$

Q8) Let $y = x^r$, so $x^r(2(r)(r-1) - 4r + 6) = 0$

$x^r(2r^2 - 6r + 6) = 0$, $\dfrac{6 \pm \sqrt{36 - 4(2)(6)}}{4}$

$\dfrac{6 \pm \sqrt{12}\, i}{4} = \dfrac{6 \pm 2\sqrt{3}\, i}{4} = r$

So the general solution is

$y = |x|^{\frac{3}{2}}\left(C_1 \cos\left(\frac{\sqrt{3}}{2}\ln(x)\right) + C_2 \sin\left(\frac{\sqrt{3}}{2}\ln(x)\right)\right)$

Q9) Let $y=x^r$, so $x^r\left(r(r-1)-5r+9\right)=0$

so $x^r\left(r^2-6r+9\right)=0 \Rightarrow (r-3)^2x^r=0$

So the general solution is $y=(c_1+c_2\ln x)x^3$

Q19) $x^2(1-x)y''+(x-2)y'-3xy=0$

$$\Rightarrow y''+\frac{(x-2)}{x^2(1-x)}y'-\frac{3x}{x^2(1-x)}y=0$$

the zeroes of $x^2(1-x)$ are $0, 1$ so the singular points are $0, 1$.

For 0: $\lim_{x\to 0} x\left(\frac{x-2}{x^2(1-x)}\right)=\lim_{x\to 0}\left(\frac{(x-2)}{x(1-x)}\right)$

but this limit does not exist so zero is not a regular singular point

For 1: $\lim_{x\to 1}\frac{(x-1)(x-2)}{x^2(1-x)}=\lim_{x\to 1}\frac{x-2}{x^2}=-1$

$$\lim_{x\to 1}\frac{(x-1)^2(x-2)}{x(1-x)}=\lim_{x\to 1}\frac{(x-2)(x-1)}{x}=0$$

So 1 is a regular singular point.

Q27) $(x^2+x-2) = (x-1)(x+2)$

So $y'' + \frac{x+1}{(x-1)(x+2)} y' + \frac{2}{(x-1)(x+2)} y = 0$

The singular points are $1, -2$

For 1:

$$\lim_{x \to 1} \frac{(x-1)(x+1)}{(x-1)(x+2)} = \frac{2}{3}$$

$$\lim_{x \to 1} \frac{(x-1)^2 \, 2}{(x-1)(x+2)} = 0$$ so 1 is regular singular

For 2: $$\lim_{x \to -2} \frac{(x+2)(x+1)}{(x-1)(x+2)} = \frac{-1}{-3} = \frac{1}{3}$$

$$\lim_{x \to -2} \frac{(x+2)^2 \cdot 2}{(x-1)(x+2)} = 0$$ so 2 is regular singular

Q28) $y'' + \frac{e^x}{x} y' + \frac{(3\cos x)}{x} y = 0$

the singular point is 0

$\lim_{x \to 0} \frac{x \cdot e^x}{x} = 1 \quad \lim_{x \to 0} x^2 \cdot \frac{e^x}{x} = 0$

so 0 is regular singular

Q31) $y'' - \frac{3\sin x}{x^2} y' + \frac{(1+x^2)}{x^2} y = 0$

the singular point is 0

$\lim_{x \to 0} -\frac{3\sin x}{x^2} \cdot x = \lim_{x \to 0} \frac{-3\sin x}{x}$

by apply L'Hospital's rule

$\lim_{x \to 0} \frac{-3\sin x}{x} = \lim_{x \to 0} \frac{-3\cos x}{1} = -3$

$\lim_{x \to 0} \frac{x^2(1+x^2)}{x^2} = 1$

so 0 is regular singular

Q37) $x^2y'' - 2y = 0$, Let $y = x^r$

so $x^r(r(r-1)-2) = 0$

$x^r(r^2-r-2) = 0$, $x^r(r-2)(r+1) = 0$

$\Rightarrow r=2,\ r=-1$

So $y = C_1x^2 + C_2x^{-1}$

$y(1) = C_1 + C_2 = 1$

$y'(1) = 2C_1 - \underline{C_2} = \gamma$

In order for y to be bonded we need C_2 to be 0.

So $C_1 = 1$, and so $\gamma = 2$