

# Math 635

USC Spring 2026

*Cluster Varieties:  
Algebra, Topology, Geometry, Duality*

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Handwritten Lecture Notes

## Table of Contents

<b>Lecture 1</b> .....	<b>2</b>
<b>Lecture 2</b> .....	<b>8</b>
<b>Lecture 3</b> .....	<b>11</b>
<b>Lecture 4</b> .....	<b>15</b>
<b>Lecture 5</b> .....	<b>20</b>
<b>Lecture 6</b> .....	<b>24</b>
<b>Lecture 7</b> .....	<b>28</b>
<b>Lecture 8</b> .....	<b>31</b>
<b>Lecture 9</b> .....	<b>35</b>

Cluster varieties: algebra, topology, geometry, duality

Lecture 1

1/12/26

Roughly speaking:

- a cluster variety is a complex algebraic variety obtained by gluing together many copies of  $(\mathbb{C}^*)^n$  where the gluing maps take a very particular form
- a cluster algebra is the algebra of regular functions  $f: V \rightarrow \mathbb{C}$  on a cluster variety

Fomin-Zelevinsky, early '00s: introduced cluster algebras  
Arise in many parts of math and physics as kind of "universal model" for mutation/wall-crossing phenomena:

- quiver representation theory
- ~~Dehn~~ Teichmüller theory
- Poisson geometry
- Grassmannians
- total positivity
- QFT scattering amplitudes (amplitude amplituhedron)
- integrable systems
- string theory (BPS states), etc

Gross-Hacking-Kontsevich 1/9:

- constructed canonical bases for cluster algebras
- established ~~positivity of the Laurent phenomenon~~ positive Laurent phenomenon
- proof uses mirror symmetry for log Calabi-Yau varieties

many strong applications  
in representation theory, e.g.  
canonical bases for  
finite-dimensional irreducible  
representations of  $SL_n(\mathbb{C})$

can think of as generalization  
of toric varieties

(related to almost toric  
fibrations in symplectic geometry)

originally found independently  
by Lusztig and  
Kashiwara in early 90s  
using quantum groups

amazingly, the construction  
of GTKK uses only  
general geometry - no  
rep. theory!

## Total positivity

Def :  $A \in \text{Mat}_{n \times n}(\mathbb{R})$  is totally positive (TP) if all of its minors are positive.

Gantmacher-Krein '30's :  $A \text{ TP} \Rightarrow$  eigenvalues are real, positive, and distinct

Binet-Cauchy theorem : The TP matrices in  $G = \text{SL}_n(\mathbb{C}) \times \text{GL}_n(\mathbb{C})$  are closed under multiplication, and hence form a multiplicative semigroup  $G_{>0}$ .

Lusztig : Extended definition of  $G_{>0}$  for other semisimple Lie groups  $G$ .

More generally : If a given complex algebraic variety  $Z$  ~~comes~~ has a distinguished family  $\Delta$  of regular functions  $Z \rightarrow \mathbb{C}$ , we define the TP variety by

$$Z_{>0} := \{ z \in Z \mid \begin{matrix} \text{for } z \in Z \\ f(z) > 0 \end{matrix} \forall f \in \Delta \}$$

Ex : For  $Z = \text{Mat}_{n \times n}(\mathbb{C})$ ,  $\text{GL}_n(\mathbb{C})$ ,  $\text{SL}_n(\mathbb{C})$ , we recover above notion of TP,  $\Delta = \text{minors}$ ,

Ex : Grassmannian  $\text{Gr}_{k \times m}(\mathbb{C}) = \{ k\text{-dim linear subspaces of } \mathbb{C}^m \}$   
 $\Delta = \text{Plücker coordinates}$

Ex : partial flag manifolds, homogeneous spaces for semisimple complex Lie groups, etc. Slight scaling ambiguity

Lemma :  $A \in \text{Mat}_{n \times n}$  has  $\binom{2n}{n} - 1$  minors

$$\text{pf} : \# = \sum_{k=1}^n \binom{n}{k} \binom{n}{k}$$

Vandermonde's identity :  $\binom{m+w}{r} = \sum_{k=0}^n \binom{m}{k} \binom{w}{r-k}$

$$\text{Setting } m=w=r=n \Rightarrow \binom{2n}{n} = \sum_{k=0}^n \binom{n}{k} \binom{n}{k}$$

(both sides count:  
 Given committee with  $n$  men  
 $\sim$  women,  
 how many subcommittees with  $r$  members?)

Q : Can we check that  $A \in \text{Mat}_{n \times n}$  is TP testing a subset of the  $\binom{2n}{n} - 1$  minors? How many tests are needed?

by only

i.e. want  
 "efficient  
 TP  
 testing"

$$\text{Ex} : A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Mat}_{2 \times 2}$$

$$\delta := ad - bc \Rightarrow d = \frac{a + bc}{a}.$$

So if  $a, b, c, \delta > 0$ , so is  $\delta$ .

Reduce  $\binom{4}{2} - 1 = 5$  checks to 4 checks.

## Plücker coordinates on Grassmannians:

Given  $A \in \text{Mat}_{k \times m}$   $\rightarrow \text{row span } [A] \in \text{Gr}_{k, m}$   
 If rank  $k$

For  $J \subset \{1, \dots, m\}$   $\rightarrow$  Plücker coordinates  
 $|J|=k$   $P_J(A) := k \times k$  minor of  $A$  corresponding  
 to  $J$

Note: For  $A, B \in \text{Mat}_{k \times m}$  with  $[A] = [B]$  (i.e. same row spans)  
 $(P_J(A))_{|J|=k}$  and  $(P_J(B))_{|J|=k}$  agree up to common rescaling, i.e. get  
 $\text{Gr}_{k, m} \rightarrow \mathbb{CP}^N$  for  $N = \binom{m}{k} - 1$ .

In fact this is an embedding, the Plücker embedding.

Let  $\mathbb{C}[\text{Mat}_{k \times m}] = \text{word. ring of } \text{Mat}_{k \times m}$ , i.e. the polynomial algebra in variables  ~~$x_{ij}$~~   $x_{ij}$  for  $1 \leq i \leq k$   
 $1 \leq j \leq m$

Def: The Plücker ring  $R_{k, m}$  is the subring of  $\mathbb{C}[\text{Mat}_{k \times m}]$  generated by  $P_J$  over  $J \in \{1, \dots, m\}, |J|=k$ .

Claim: the ideal of relations in  $R_{k, m}$  is generated by certain quadratic relations called the Grassmann-Plücker relations.

Def: The totally positive Grassmannian  $\text{Gr}_{k, m}^+$  is the subset of  $\text{Gr}_{k, m}$  of those pts whose Plücker coords are all positive (up to common scaling).

Note: For  $A \in \text{Mat}_{k \times m}(\mathbb{R})$ ,  $[A] \in \text{Gr}_{k, m}^+$  iff all  $k \times k$  minors of  $A$  have the same sign.

Q: For  $A \in \text{Mat}_{k \times m}(\mathbb{R})$ , can we verify that all  $k \times k$  minors are positive by only checking a subset of the  $\binom{m}{k}$  minors?  
 How many tests are needed? positive wlog

## Positivity testing for $\text{Gr}_{2,m}$

Claim: Given  $A \in \text{Mat}_{2 \times m}$ , put  $P_{ij} := P_{\{i,j\}}$  for  $1 \leq i, j \leq m$ .  
 To check that all  $2 \times 2$  minors  $P_{ij}(A) \geq 0$ , suffices to check only  $2m-3$  special ones.

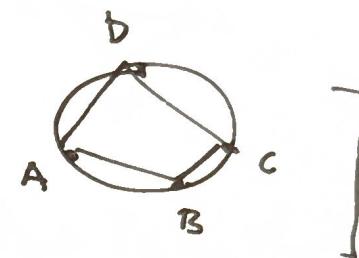
Note:  $2m-3 = \dim \text{Gr}_{2,m} + 1$

Lemma: For  $1 \leq i_1 < i_2 < i_3 \leq m$ , have three-term Grassmann-Plücker relations:

$$P_{i_1 k} P_{i_2 l} = P_{i_1 i_2} P_{k l} + P_{i_1 l} P_{i_2 k}$$

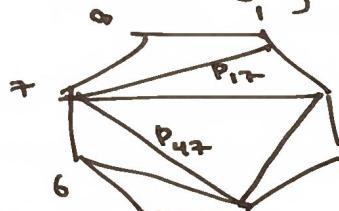
Rmk: For inscribed quadrilateral Ptolemy's thm (2nd century) gives

$$AC \cdot BD = AB \cdot CD + BC \cdot AD$$



Ex:  $A = \begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix}$  v/s  $P_{13} P_{24} = P_{12} P_{34} + P_{14} P_{23}$ , i.e.  
 $(ag-ce)(bh-df) = (af-be)(ch-dg) + (ah-de)(bg-cf)$  ✓

Put  $P_m = \text{regular } m\text{-gon}$ ,  $T = \text{triangulation}$ .



To each side or diagonal associate  $P_{ij}$ , where  $i, j$  are the end pts

Cluster variables:  $P_{ij}$  ranging over diagonals  
frozen variables:  $P_{ij}$  ranging over sides  
extended cluster:  $\{ \text{cluster vars} \} \cup \{ \text{frozen vars} \} =: \tilde{x}(T)$

Note: extended cluster has  $2m-3$  vars, and we claim that these are algebraically independent.

Ex: In above picture, have cluster variables  $P_{13}, P_{14}, P_{15}, P_{23}, P_{24}$   
 frozen variables  $P_{12}, P_{34}, \dots, P_{28}, P_{19}$

Thm: Each  $P_{ij}$  for  $1 \leq i < j \leq n$  subtraction-free rational expression can be written as a rational expression in the elements of a given extended cluster  $\tilde{x}(T)$ .

Cor: For  $A \in \text{Mat}_{2 \times m}$ , positively evaluating  $P_{ij} \in \tilde{x}(T)$  on give  $A \in \text{Mat}_{2 \times m}$ , then all  $\binom{m}{2}$  of the  $2 \times 2$  minors of  $A$  are positive.

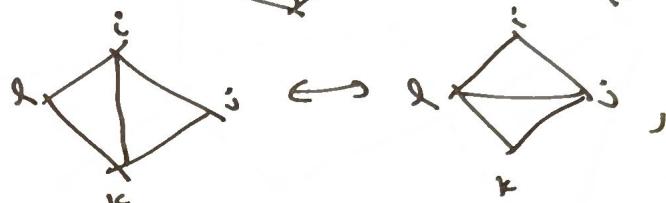
$\frac{2m-3}{2}$  of these

Pf of thm: Follows by combining

- (1) each  $P_{ij}$  appears as an elt of an extended cluster  $\tilde{x}(T)$  for some triangulation  $T$  of  $\mathbb{P}_m$
- (2) any two triangulations of  $\mathbb{P}_m$  are related by a sequence of flips



(3) For a flip



replace  $P_{ik}$  with  $P_{li}$ .

Using three-term GP relation, have  $P_{ik} = \frac{P_{ij}P_{lk} + P_{il}P_{jk}}{P_{il}}$

Rank: In fact, each Plücker coordinate  $P_{ij}$  can be written as a Laurent polynomial with positive coefficients in the Plücker coordinates from  $\tilde{x}(T)$ .

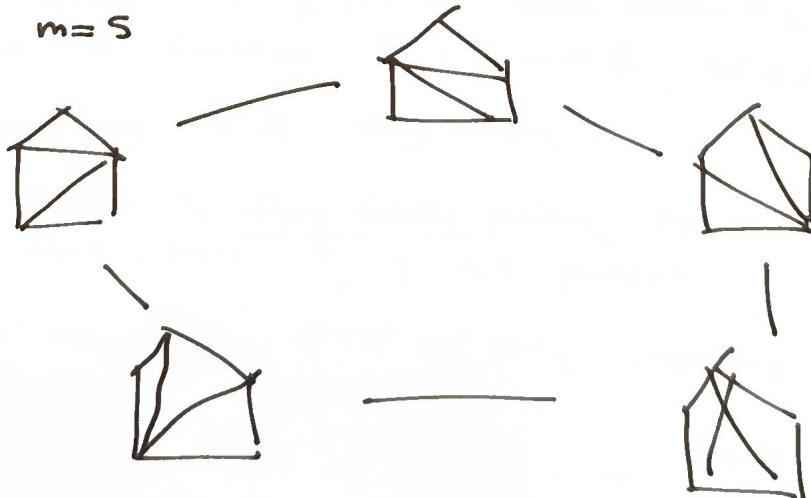
Example of ~~possibly~~ positive Laurent phenomenon.

Combinatorics of flips encoded by graph:

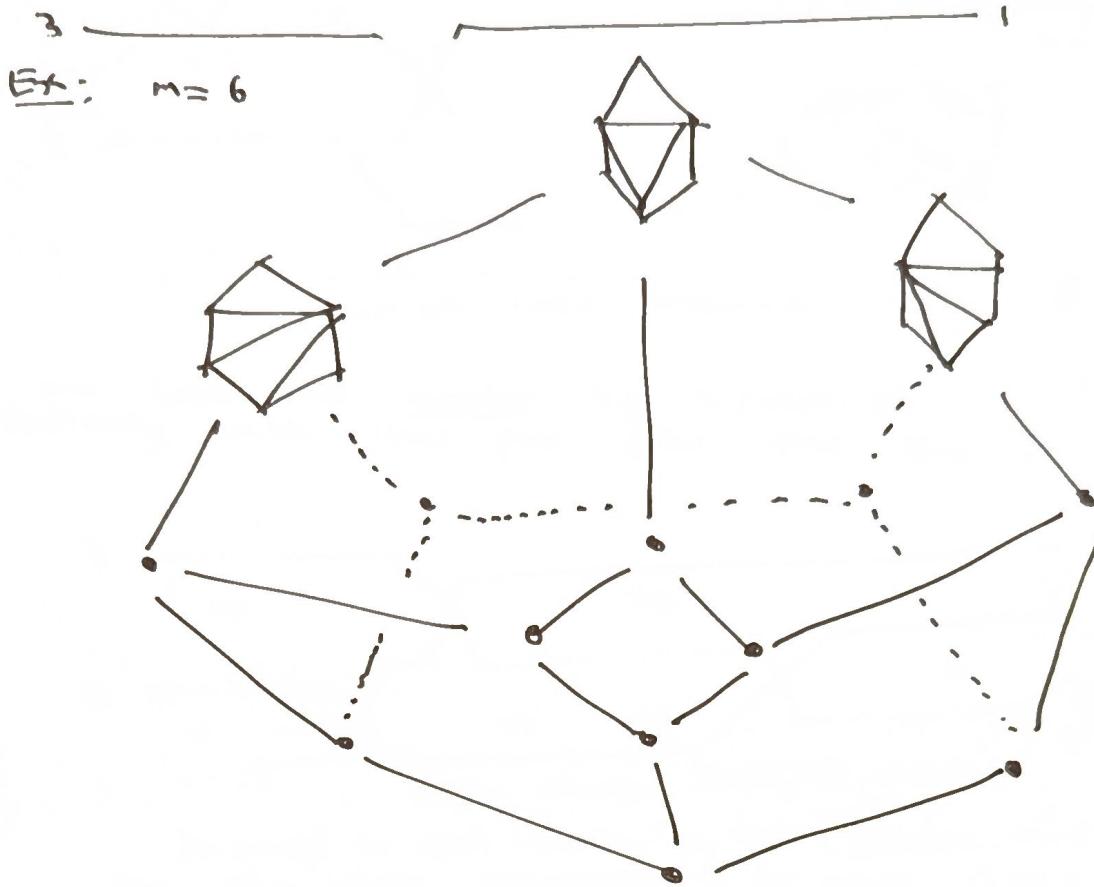
- vertices are ~~triangulations~~ triangulations
- edges are flips

Each vertex has degree  $m-3$ . In fact, this is the 1-skeleton of an  $(m-3)$ -diml convex polytope called the associahedron (discovered by Stasheff).

Ex:  $m=5$



## Wiring diagrams:



Def: A cluster monomial is a monomial in the variables of a given extended cluster  $\tilde{x}(\tau)$ .

Thm (19th century invariant theory): The set of all cluster monomials give a linear basis for the Plücker ring  $R_{2,n}$ .

## Lecture 2

11/11/25

Before moving to TP for non matrices, we discuss an intermediate notion called "flag positivity". Put  $G = SL_n$ .

Def. Given  $J \subsetneq \{1, \dots, n\}$  non empty, the flag minor  $P_J$  is the function  $P_J: G \rightarrow \mathbb{Q}$ ,  $z = (z_{ij}) \mapsto \det(z_{ij}) \mid i \in |J|, j \in J$

Note: there are  $2^n - 2$  flag minors.

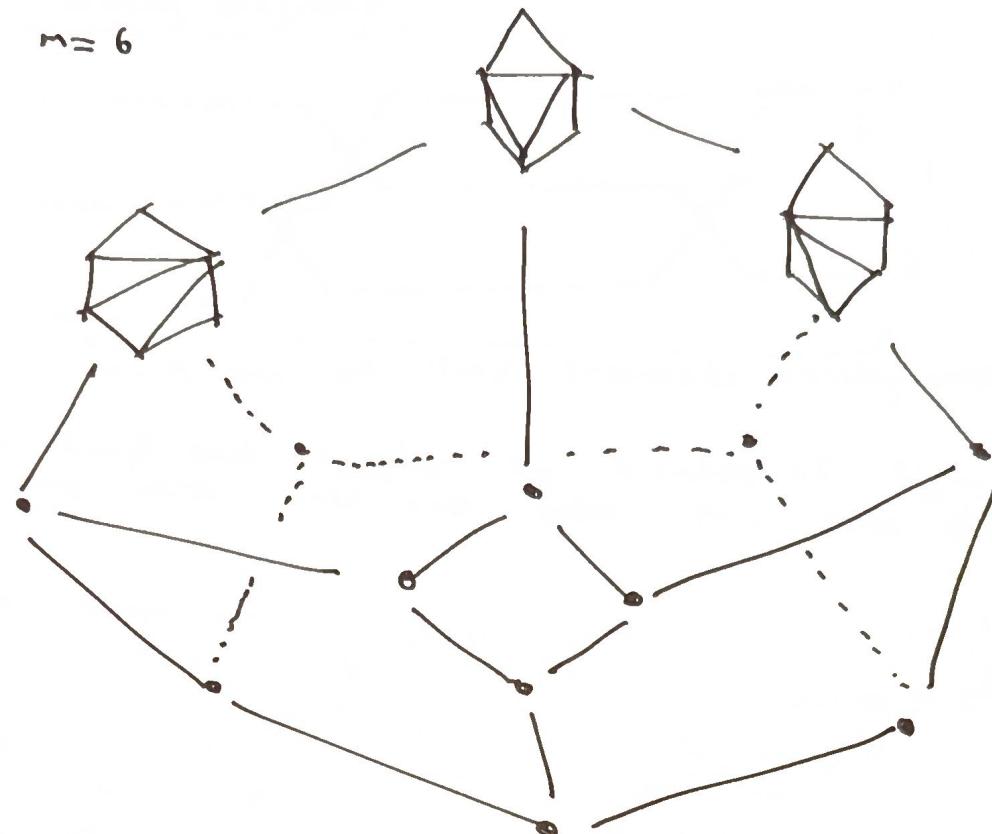
Def:  $z \in G$  is flag totally positive (FTP) if all flag minors  $P_J(z)$  are positive.

$|J| \times |J|$  minor which is "top-justified"

Q: Can we check FTP by only checking a subset of the  $2^n - 2$  flag minors.

Claim: It suffices to check only  $\frac{(n-1)(n+2)}{2}$  special flag minors.

Ex:  $n=6$



Def: A cluster monomial is a monomial in the variables of a given extended cluster  $\tilde{x}(T)$ .

Thm (19th century invariant theory): The set of all cluster monomials give a linear basis for the Plücker ring  $R_{2,n}$ .

## Lecture 2

11/11/26

Before moving to TP for  $n \times n$  matrices, we discuss an intermediate notion called "flag positivity". Put  $G = SL_n$ .

Def: Given  $J \subseteq \{1, \dots, n\}$  nonempty, the flag minor  $P_J$  is the function  $P_J: G \rightarrow \mathbb{Q}$ ,  $z = (z_{ij}) \mapsto \det(z_{ij})$  for  $i \in |J|$ ,  $j \in J$ .

Note: there are  $2^n - 2$  flag minors.

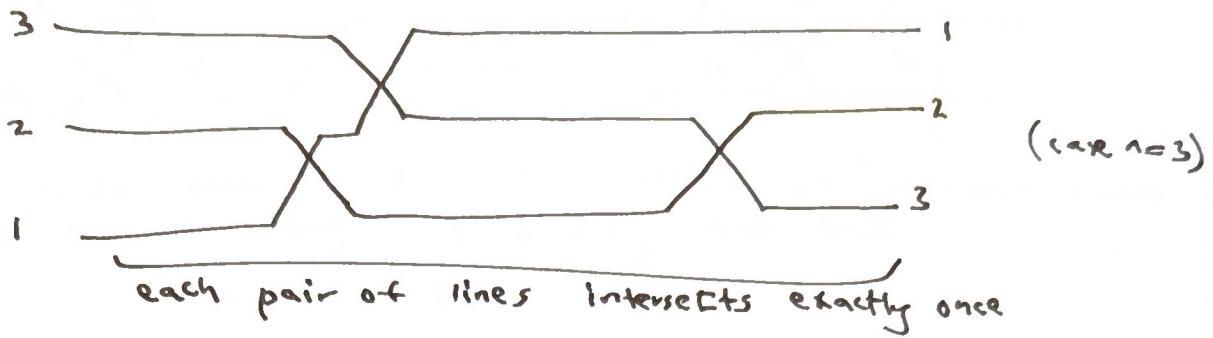
Def:  $z \in G$  is flag totally positive (FTP) if all flag minors  $P_J(z)$  are positive.

$|J| \times |J|$  minor  
which is  
"top-justified"

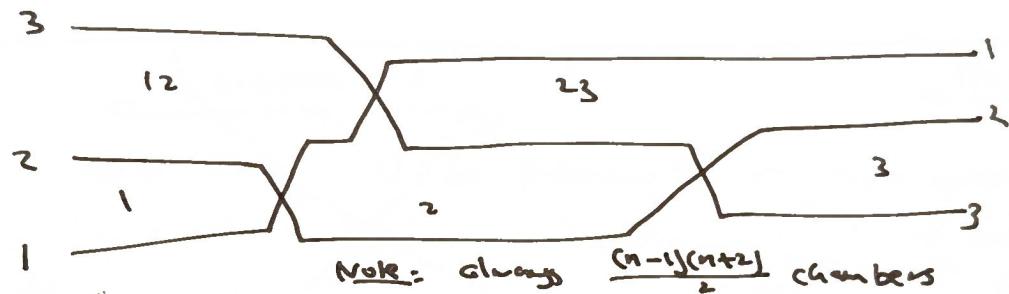
Q: Can we check FTP by only checking a subset of the  $2^n - 2$  flag minors.

Claim: It suffices to check only  $\frac{(n-1)(n+2)}{2}$  special

### Wiring diagrams:



We label each chamber indicating which lines pass by a subset of  $\{1, \dots, n\}$  below that ~~one~~ chamber



Associated to each chamber is its chamber minor  $P_J$  the flag minor corresponding to its subset  $J \subseteq \{1, \dots, n\}$ .

extended cluster: all chamber minors of a wiring diagram  
cluster variables: the chamber minors for bounded chambers  
frozen variables: the chamber minors for unbounded chambers

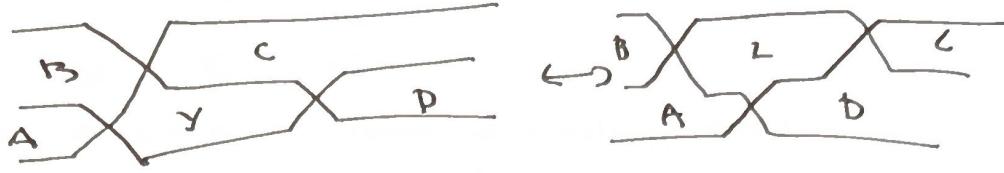
$\frac{n-1}{2}$  of these

$\binom{n-1}{2}$

Thm Every flag minor can be written as a subtraction-free rat'l expr in the chamber minors of a given wiring diag.  
Car: If there  $\frac{(n-1)(n+2)}{2}$  evaluate positively at a matrix  $z \in \text{SL}_n$ , then  $z$  is FTFP.

Prf: Follows by

- (1) each flag minor appears as a chamber minor in some wiring diagram
- (2) any two wiring diagrams can be transformed into each other by a sequence of local braid moves



(3) Under each braid move, collection of chamber minors changes by exchanging  $Y \leftrightarrow Z$ , and have  
 $YZ = AC + BD$

Point: In fact, each flag minor can be written as a Laurent poly with pos. coeffs in the chamber minors of a given ~~wire~~ wiring diagram.

~~Lecture 3~~ ~~1722126~~

Put  $G = SL_n$ ,  $U \times G$  subgroup of unipotent lower-triangulars  
i.e. lower triangular matrices with 1s on diagonal

$U \times G$  left multiplication action

$\rightarrow U \times G[G] =$  ring of polynomials in the matrix entries of  $A \in G$

$\mathbb{C}[G]^U =$  ring of  $U$ -invariant polynomials

Note:  $\begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a + \beta c & \alpha b + \beta d \\ \gamma a & \gamma b \end{pmatrix}$

...  $\begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} -v_1 \\ -v_2 \end{pmatrix} = \begin{pmatrix} -\alpha v_1 + \beta v_2 \\ \gamma v_1 \end{pmatrix}$

Similarly,  $\begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \delta & \varepsilon \\ 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} -v_1 \\ -v_2 \\ -v_3 \end{pmatrix} = \begin{pmatrix} -\alpha v_1 + \beta v_2 + \gamma v_3 \\ -\delta v_1 + \varepsilon v_3 \\ \gamma v_1 \end{pmatrix}$

i.e.  $P \in \mathbb{C}[G]$   
s.t.  $P(yz) = P(z)$   
 $y \in U, z \in G$

Def: The full flag variety in  $\mathbb{C}^n$  is  
 $\{ \sum c_i v_i \in V_1 \subset \mathbb{C}^n \subset V_2 \subset \dots \subset V_{n-1} \subset \mathbb{C}^n \}$   
 $v_i$  is a subspace of dimension  $i$   
for  $i=1, \dots, n-1$

This can be identified with the homogeneous space  $G/B$ , where  $B \subset G$  is the subgroup

## Lecture 3

1/23/26

Put  $G = \mathrm{SL}_n(\mathbb{C})$

$B \subset G$  subgroup of lower triangular matrices

$V \subset G$  subgroup of unipotent lower triangular matrices

i.e. 1's on  
diagonal

Borel  
subgroup

Note:

$$\begin{pmatrix} \alpha & 0 \\ \beta & \gamma \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha b \\ \beta a + \gamma c & \beta b + \gamma d \end{pmatrix}, \quad \text{i.e.}$$

$$\begin{pmatrix} \alpha & 0 \\ \beta & \gamma \end{pmatrix} \begin{pmatrix} -r_1 \\ -r_2 \end{pmatrix} = \begin{pmatrix} -\alpha r_1 \\ -\beta r_1 + \gamma r_2 \end{pmatrix}$$

Similarly,

$$\begin{pmatrix} \alpha & 0 & 0 \\ \beta & \gamma & 0 \\ 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} -r_1 \\ -r_2 \\ -r_3 \end{pmatrix} = \begin{pmatrix} -\alpha r_1 \\ -\beta r_1 + \gamma r_2 \\ -\gamma r_1 + \gamma r_2 + \gamma r_3 \end{pmatrix} \quad \text{etc}$$

Def: The (full) flag variety

$\{ \{v_i\} \subset V, v_i \subset \dots \subset V_{n-1} \subset \mathbb{C}^n \mid v_i \text{ is an } i\text{-dimensional subspace for } i=1, \dots, n-1 \}$

Exercise: This is identified with the homogeneous space

Def: The basic affine space is  ~~$\mathbb{C}^n$~~   $\mathbb{C}/G$

Note that we have

the basic affine space  $\mathbb{C}^n \hookrightarrow \mathbb{C}/G \rightarrow B/G$ , i.e.

Here  $V \times G$  action by left multiplication

$\rightsquigarrow V \times G[G] = \text{ring of polynomials in the entries of } A \in \mathrm{SL}_n$

$\mathbb{C}[G]^V = \text{ring of } V\text{-invariant polynomials}$

Claim:

by First and Second Fundamental Theorems of invariant theory

(1) the flag

minors generate  $\mathbb{C}[G]^V$

i.e.  $P \in \mathbb{C}[G]$   
s.t.  $P(yz) = P(xz)$   
 $y \in V, z \in G$

(2) the ideal

of relations among flag minors is generated by quadratic relations called "generalized Plücker relations"

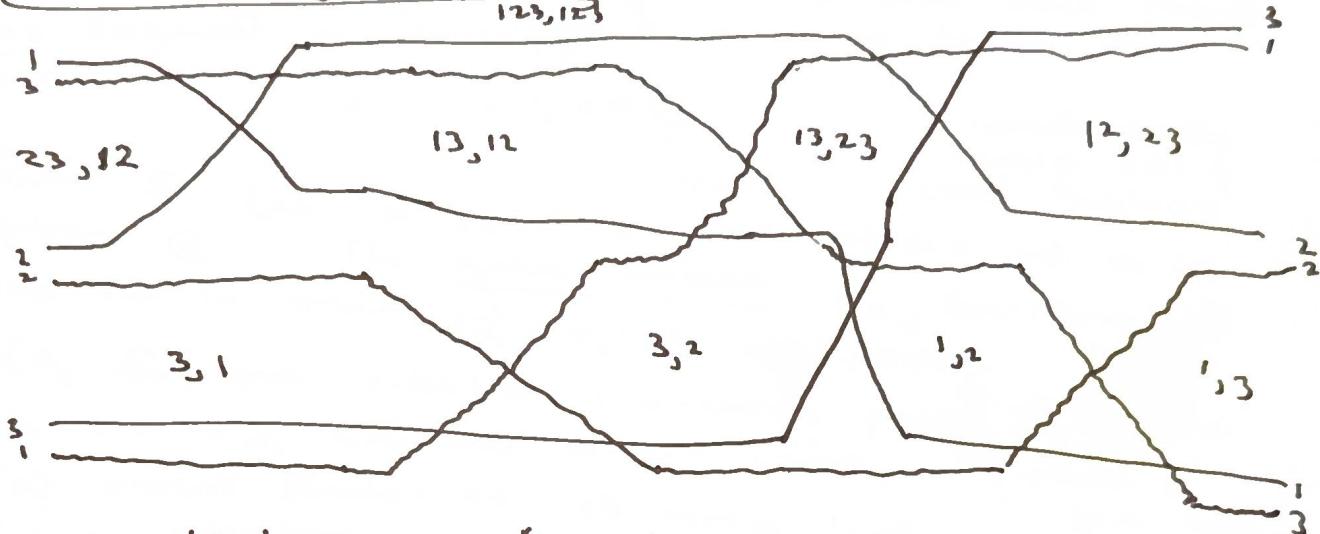
by

### Checking TP for general nn matrices

Given  $I, J \subset \{1, \dots, n\}$  of same cardinality, put  
 $\Delta_{I,J} :=$  minor determined by rows in  $I$  and columns in  $J$

Thus  $\mathbf{z} \in \text{Mat}_{nn}$  is TP  $\iff \Delta_{I,J} (\pm) > 0$  for all  
 $I, J \subset \{1, \dots, n\}$  with  $|I| = |J|$

### Double wiring diagrams:



$\rightarrow$  chamber minors  $\Delta_{3,1}, \Delta_{3,2}, \Delta_{1,2}, \Delta_{1,3}, \Delta_{23,12}, \Delta_{13,12}, \Delta_{13,23}, \Delta_{12,23}, \Delta_{123,123}$

Claim: number of chamber minors for a double wiring diagram is always  $n^2$  minors

Thm: Every minor of an  $nn$  matrix can be written as a subtraction-free rational expression in the chamber minors of a given double wiring diagram.

Cor: Only need  $n^2$  tests for positivity.

pt idea:

(1) every minor is a chamber minor for some double wiring diagram

(2) any two double wiring diagrams are related by sequence of local moves of three different kinds

(3) each local move results in an exchange of minors  $\gamma \leftrightarrow \gamma'$ , where  $\gamma' = \gamma - A\gamma + B\gamma$ .

Rank: In fact in this we really have Laurent polynomials with positive coefficients.

Rank: The graph with vertices double wiring diagrams and edges local moves is not regular, but this will be rectified by the theory of cluster algebras.

## Quivers and their mutations

Def: A quiver is a finite oriented graph with no loops or oriented 2-cycles.

Ex:



Def: An ice quiver is a quiver in which some vertices are designated as "frozen", and no arrows between two frozen vertices.



Def:

Let  $\mathbb{K}$  be a quiver  $Q$ . The quiver mutation into new ice quiver

$$Q' = \mu_k(Q)$$

- (1) for each oriented two-arrow path  $i \rightarrow k \rightarrow j$ , add new arrow  $i \rightarrow j$  (unless  $i, j$  both frozen)
- (2) reverse direction of all arrows incident to  $k$
- (3) repeatedly reverse any oriented 2-cycles until none left

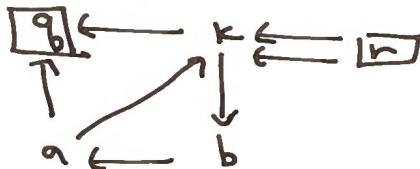
non-frozen vertices will be called "mutable"

vertex of an ice  $\mathbb{K}$  transforms  $Q$  follows:

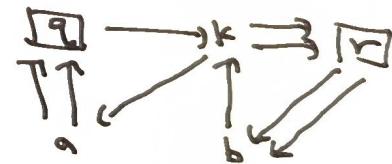
path  ~~$i \rightarrow k \rightarrow j$~~

(unless  $i, j$  both frozen)

Ex:



$$\mu_k$$



Exercise:

- (1) mutation is an involution i.e.  $\mu_k(\mu_k(Q)) = Q$
- (2) mutation commutes with reversing orientation of all arrows

(3) if  $k, l$  are

mutable vertices with no arrows between them, then

$$\mu_l(\mu_k(Q)) = \mu_k(\mu_l(Q))$$

Rank: If  $\mathbb{K}$  has

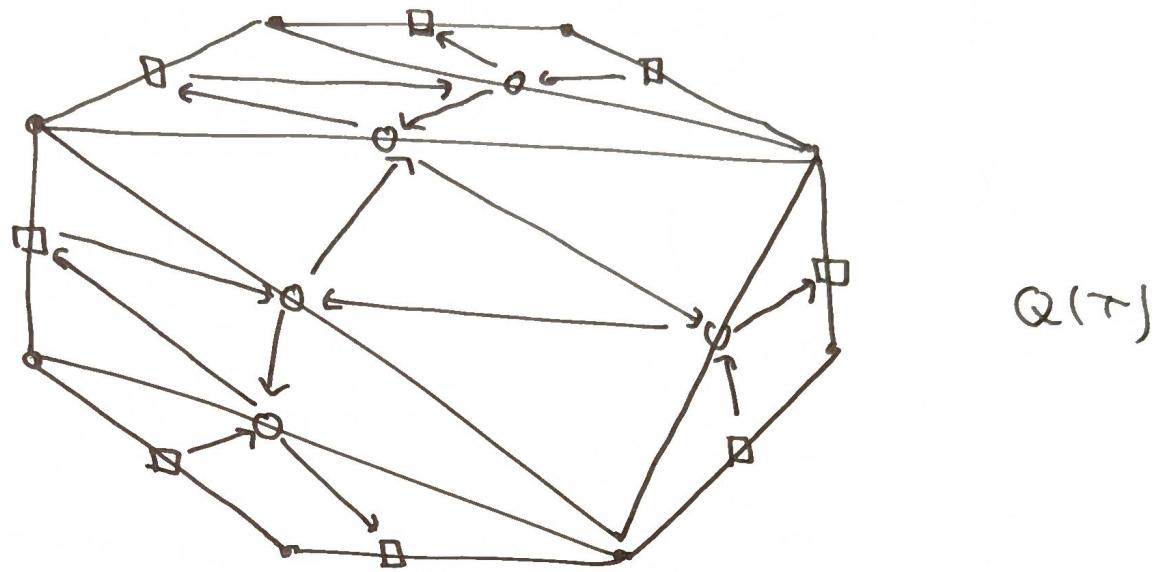
arrows incident to  $k$ ,  $\mu_k$  simply reverses all

Exercise: If  $Q$  is a tree with no frozen, can get from any orientation to any other by a sequence of mutations.

at sinks and sources.

Triangulation and quiver

Can define a quiver from a ~~triangulation~~ triangulation  $T$  of  $P_m$ .



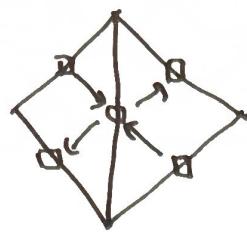
Exercise: If  $T$  is a triangulation of  $P_m$  and  $T'$  obtained by flip along diagonal  $\gamma$ , then

$$Q(T') = \mu_\gamma(Q(T))$$

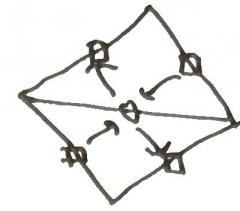
# Lecture 34

1/25/26

Ex:

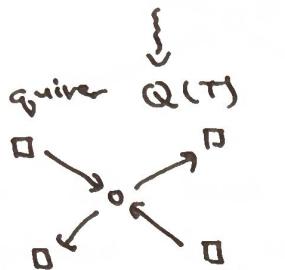


flip

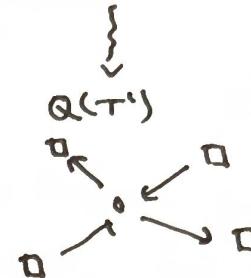


$T'$

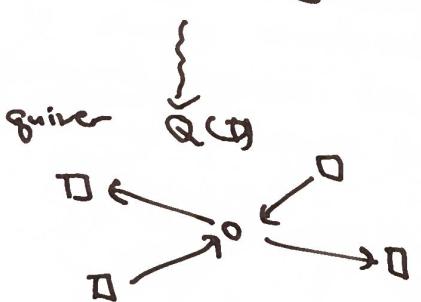
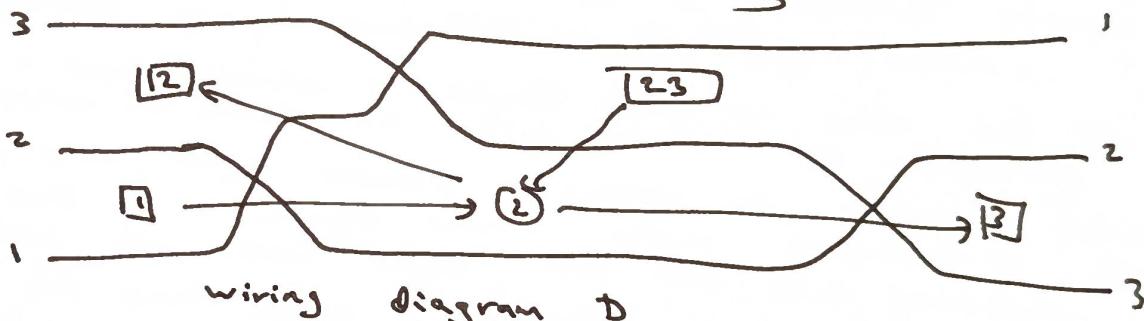
$T = \text{triangulation of } RP_4$



mutation



wiring diagram  $\rightsquigarrow$  quiver



vertices: chambers of  $\mathfrak{p}$   
(mutable if bounded,  
else frozen)

arrows: for chambers  $c, c'$   
here  $c \rightarrow c'$  in  $Q(D)$  if  
one of following holds.

- (i) right end of  $c$  = left end of  $c'$
- (ii) left end of  $c$  is directly above  $c'$ ,  
right end of  $c$  is directly below  $c'$

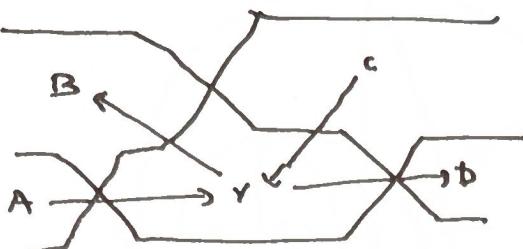


- (iii) left end of right end of  $c'$  is directly below  $c'$  directly above  $c$

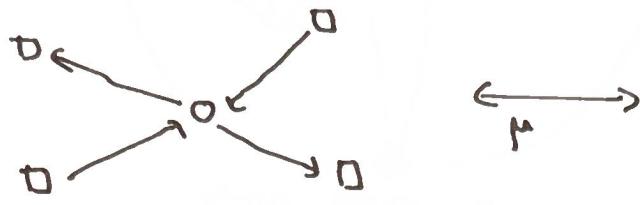
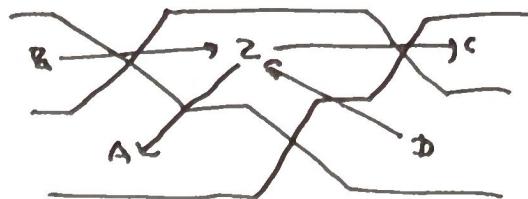


Exercise: If  $D, D'$  wiring diagrams related by a braid move at chamber  $Y$ , then  $(Q(D))' = \mu_Y(Q(D))$ .

Ex:



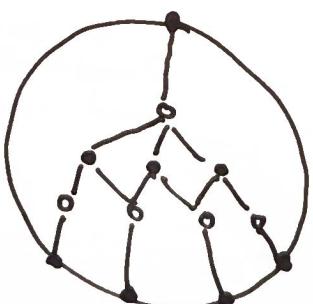
brail  
move



Rmk: Also have double wiring diagram  $\xrightarrow{\text{?}}$  quiver  $Q(G)$   
 Description is more complicated, but special case of  
 quiver associated to a planar bipartite graph.

Def: A plabic graph  $G$  is a connected planar bipartite graph embedded in a disk, where:

- each vertex is colored black or white and lies either in interior of disk or on its boundary
- each edge connects vertices of different colors and is a simple curve whose interior is disjoint from the other edges and the disk boundary
- for each face closure is simply connected (connected comp of complement), the closure is connected
- each internal vertex has degree  $\geq 2$
- each boundary vertex has degree 1



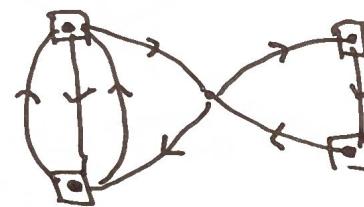
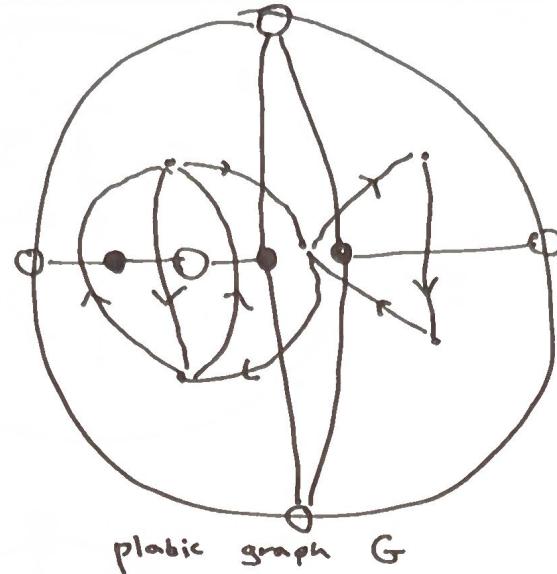
Note: we consider plabic graphs up to isotopy.

plabic graph  $G \xrightarrow{\text{?}} \text{quiver } Q(G)$

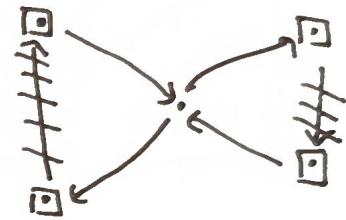
- vertices are faces of  $G$  (frozen if incident to disk boundary, else mutable)
- for each edge of  $G$ , have arrow joining the two faces it separates, using rule
- remove oriented 2-cycles



Ex:



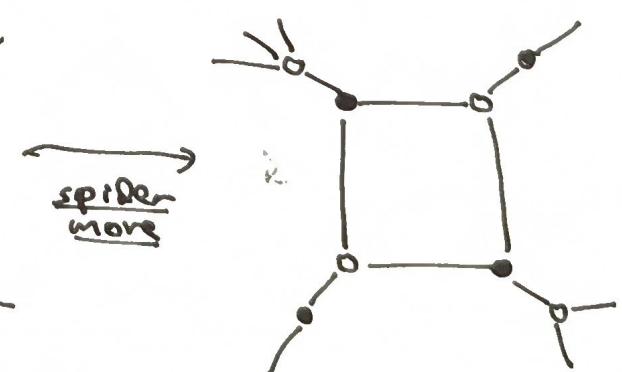
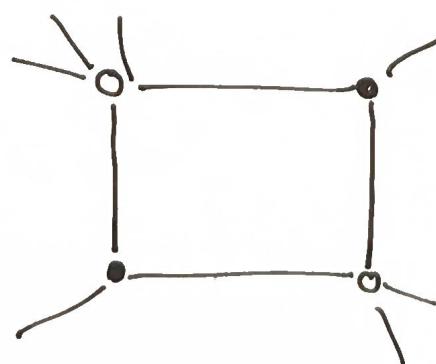
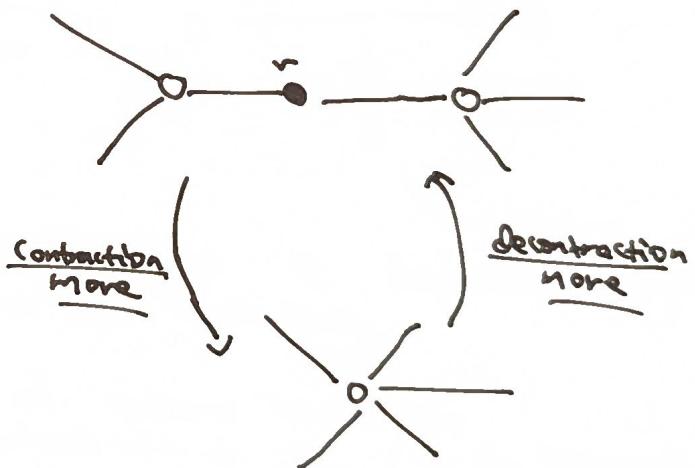
remove  
oriented 2-cycles  
(and arrows  
between frozen)



Def: Say  $v$  bivalent vertex  
adjacent to two interior  
vertices

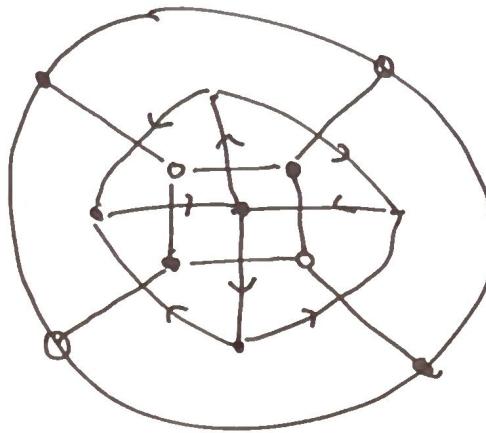
Rule: does not change  
associated quiver

Def: Say  
~~quadrilater~~  
face whose  
degree  $\geq 3$ .  
G has a  
quadrilateral  
vertices have

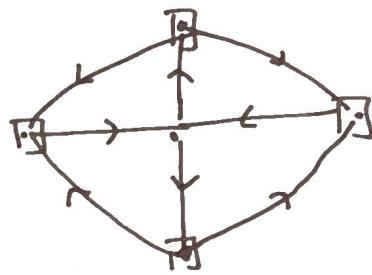
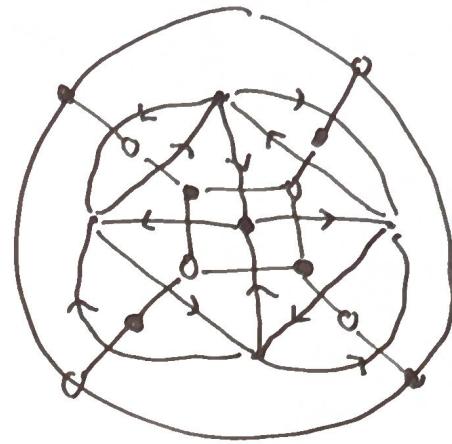


Exercise: If  $G, G'$  related by  
~~Q(G), Q(G')~~ related by spider move, then  
mutation

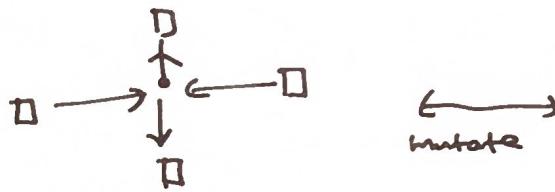
Ex



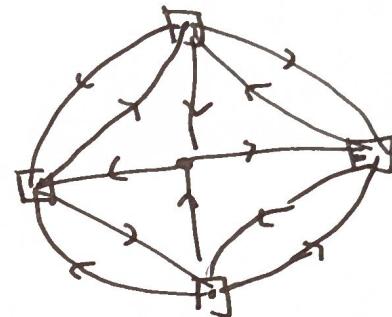
spider move



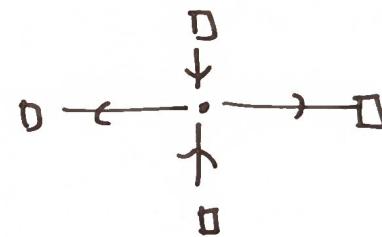
remove  $D \rightarrow D$



mutate



remove  $D \rightarrow D$  (and  
cancel 2-cycles)



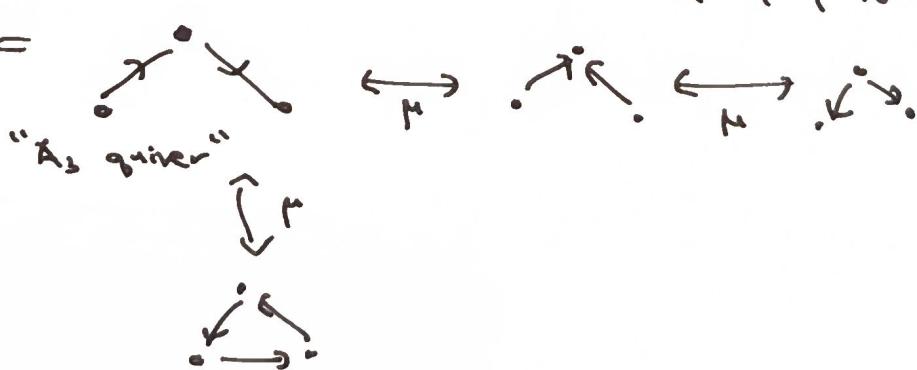
mutation equivalence

Def:  $Q, Q', Q''$  are mutation equivalent if  $Q$  becomes isomorphic after a sequence of mutations.

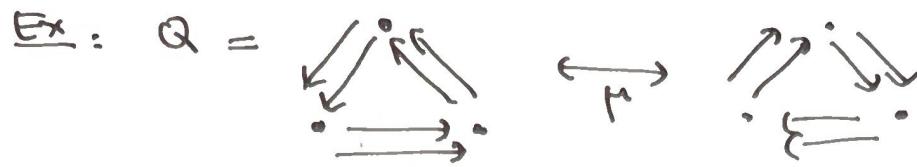
Put  $[Q] :=$  set of all quivers which are mutation equivalent to  $Q$  (up to isomorphism)

Ex:

$Q =$



Exercise:  $[Q]$  has 4 elements



"Markov quiver"

In fact,  $[Q]$  is just a single element.

Def.  $Q$  has finite mutation type if  $[Q]$  is finite.

Rmk: there is a classification theorem for quivers with no frozen vertices and finite mutation type.

Def:  $Q$  acyclic if no oriented cycles.

Thm (Caldero-Keller '06): If  $Q, Q'$  acyclic and mutation ~~equivalent~~ equivalent, then we can transform  $Q$  into  $Q'$  by a sequence of mutations at sources and sinks. In particular,  $Q, Q'$  have the same underlying undirected graphs.

## Lecture 5

11/28/26

Def :  $Q$  quiver with vertices labeled by  $1, \dots, m$ , such that  $1, \dots, n$  are the mutable ones ( $n \leq m$ ).

The extended exchange matrix is

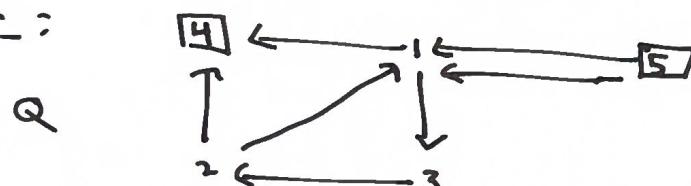
$\tilde{B}(Q) = (b_{ij})_{\substack{1 \leq i, m \\ 1 \leq j \leq n}}$ , where  $b_{ij} = \begin{cases} 1 & \text{if } \text{arrows } i \rightarrow j \\ -1 & \text{if } \text{arrows } j \rightarrow i \\ 0 & \text{else} \end{cases}$

$m \times n$  matrix

The exchange matrix is the submatrix  $B(Q) := (b_{ij})_{1 \leq i, j \leq n}$

$n \times n$  skew-symmetric matrix

Ex :

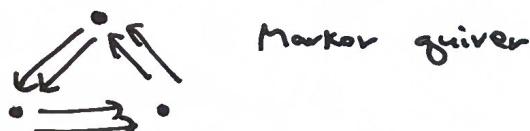


$\tilde{B}(Q) =$

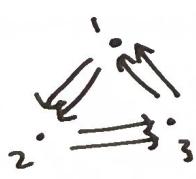
$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \\ -1 & -1 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

$$B(Q) = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

Ex :  $Q =$



Marker quiver



$$\tilde{B}(Q) = \begin{pmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{pmatrix}$$



$$\tilde{B}(Q) = \begin{pmatrix} 0 & -2 & 2 \\ 2 & 0 & -2 \\ -2 & 2 & 0 \end{pmatrix}$$

Rank : Rearranging the vertices of  $Q$  results in simultaneously rearranging the rows and columns  $1, \dots, n$  and reordering the rows  $1, \dots, m$ .

Lemma : For quiver  $Q$  with  $\tilde{B}(Q) = (b_{ij})$  and  $Q' = \mu_k(Q)$  for a mutable vertex  $k$  of  $Q$ , have  $\tilde{B}(Q') = (b'_{ij})$ , with  $b'_{ij} = \begin{cases} -b_{ij} & \text{if } i=k \text{ or } j=k \\ b_{ij} + b_{ik}b_{kj} & \text{if } b_{ik}b_{kj} > 0 \\ b_{ij} - b_{ik}b_{kj} & \text{if } b_{ik}b_{kj} < 0 \\ b_{ij} & \text{else} \end{cases}$

Note : Can replace middle two cases with  $b'_{ij} = b_{ij} + |b_{ik}|b_{kj}$  if  $b_{ik}b_{kj} > 0$

Ex : 

$$\begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 3 \\ 0 & -3 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -2 & 6 \\ 2 & 0 & -3 \\ -6 & 3 & 0 \end{pmatrix}$$

Def : An  $n \times n$  matrix  $B = (b_{ij}) \in \text{Mat}_{n \times n}(\mathbb{Z})$  is skew-symmetrizable if for some  $\alpha_1, \dots, \alpha_n \in \mathbb{Z}_{>0}$  we have  $\alpha_i b_{ij} = -\alpha_j b_{ji}$

Def : An  $n \times n$  matrix is extended skew-symmetrizable if the top  $n-1$  submatrix is skew-symmetrizable.

i.e. becomes skew-symmetric after rescaling the rows by positive integers

Def : For  $\tilde{B} = (b_{ij})$  extended skew-sym.  $n \times n$  matrix,  $k \in \{1, \dots, n\}$ , we define  $\mu_k(\tilde{B}) = (b'_{ij})$  using same formula  $(*)$ .

Exercise :

- $\mu_k(\tilde{B})$  is again extended skew-sym., using same  $\alpha_1, \dots, \alpha_n$
- $\mu_k(\mu_k(\tilde{B})) = \tilde{B}$
- $\mu_k(-\tilde{B}) = -\mu_k(\tilde{B})$
- ~~$\mu_k(\tilde{B}) = \tilde{B}$~~
- if  $b_{ij} = b_{ji} = 0$ , then  $\mu_i \mu_j \tilde{B} = \mu_j \mu_i \tilde{B}$

Def: For a skew-symmetrizable  $n \times n$  matrix  $B = (b_{ij})$ , its Diagram is the weighted directed graph  $\Gamma(B)$  with vertices  $1, \dots, n$  and  $i \rightarrow j$  iff  $b_{ij} > 0$ , with weight  $|b_{ij}|b_{ji}|$ .

Lemma: If the diagram  $\Gamma(B)$  of an  $n \times n$  skew-symmetrizable matrix  $B$  is connected then the skew-symmetrizing vector  $(\theta_1, \dots, \theta_n)$  is unique up to rescaling.

pf: By connectedness, there is an ~~strict~~ ordering  $i_1, \dots, i_n$  of  $\{1, \dots, n\}$  s.t. ~~not~~ for each  $j \geq 2$  we have  $b_{i_1, i_j} \neq 0$  for some  $i < j$ .

If  $(\theta_1, \dots, \theta_n)$  and  $(\theta'_1, \dots, \theta'_n)$  skew-symmetrizing vectors, have  $\theta_i \theta_j = -\theta_j \theta_i$  and  $\theta'_i \theta_j = -\theta'_j \theta_i \neq \theta_i \theta'_j$ .

If  $b_{i_1, i_2} \neq 0$ , have  $\frac{\theta_{i_1}}{\theta_{i_2}} = -\frac{\theta'_1}{\theta'_2} = -\frac{\theta'_j}{\theta'_i}$

$$\Rightarrow \frac{\theta_1}{\theta_2} = \frac{\theta'_1}{\theta'_2}.$$

Def: Two extended

are mutation equivalent if can get from  $B$  to  $B'$  by a sequence of mutations, followed by a reordering of the rows and columns in the sense from before.

Put  $[B] := \text{mutation equivalence class of } B$ .

Prop: For an  $n \times n$  skew-symmetrizable matrix, its rank and determinant are preserved by mutations.

pf: Can write  $b_{ij} = \begin{cases} -b_{ji} & \text{if } k \in \{i, j\} \\ b_{ij} + \max(0, -b_{ik})b_{ki} + b_{ik} \max(0, b_{kj}) & \text{otherwise} \end{cases}$

$$\begin{aligned} \text{Have } f_K(\tilde{B}) &= J_{m, k} \tilde{B} J_{n, k} + J_{m, k} \tilde{B} F_k + E_k \tilde{B} J_{n, k} \\ &= (J_{m, k} + E_k) \tilde{B} (J_{n, k} + F_k) \end{aligned}$$

where  $\bullet J_{m, k}$  (resp.  $J_{n, k}$ ) is diagonal  $m \times m$  (resp.  $n \times n$ ) and has  $1s$  on diagonal except for  $-1$  in  $(k, k)$  entry  
 $\bullet E_k = (e_{ij})$  is  $m \times m$  matrix with  $e_{ik} = \max(0, -b_{ik})$  and all other entries  $0$

$F_{1, \kappa} = (f_{ij})$  is the  $n \times n$  matrix with  $f_{kj} = \max(0, b_{kj})$  and all other entries 0.

Note:  $E_{1, \kappa} \tilde{B} F_{\kappa}$  since  $b_{ii} = 0$

Have  $\det(I_{n, \kappa} + E_{1, \kappa}) = \det(I_{n, \kappa} + F_{\kappa}) = -1$ .

Def: A labeled seed of geometric type in  $\mathcal{G} = \mathbb{C}(x_1, \dots, x_n)$  is a pair  $(\tilde{x}, \tilde{B})$  where

- $\tilde{x} = (x_1, \dots, x_n)$  is an adapted  $n$ -tuple of elts of  $\mathcal{G}$  which form a free generating seed
  - $\tilde{B} = (b_{ij})$  is an  $n \times n$  extended matrix
- ie  $\mathcal{G} = \mathbb{C}(x_1, \dots, x_n)$   
and  $x_1, \dots, x_n$  alg. indep.  
skew-symmetrizable integer

We say:

- $\tilde{x}$  is the labeled extended cluster of  $(\tilde{x}, \tilde{B})$
- $x = (x_1, \dots, x_n)$  is the labeled cluster
- $x_1, \dots, x_n$  are the cluster variables
- $x_{n+1}, \dots, x_m$  are the frozen variables
- $\tilde{B}$  is the extended exchange matrix
- its top  $n \times n$  submatrix  $B$  is the exchange matrix

	$\Sigma$	$\Sigma'$
extended cluster	$\tilde{x} = (x_1, x_2, x_3)$	$\tilde{x}' = (x_1, \frac{x_1+x_2}{x_2}, x_3)$
cluster vars	$x_1, x_2$	$x_1, \frac{x_1+x_3}{x_2}$
frozen vars	$x_3$	$x_3$
extended exchange matrix	$\tilde{B} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$	$\tilde{B}' = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
exchange matrix	$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$B' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Here  $n=3$ ,  $\kappa=2$ .



## Lecture 6

Recall:  $\mathcal{L} = \mathbb{Q}(q_1, \dots, q_m)$  field of rational functions,  $m \geq n$ . Say  $x_1, \dots, x_m \in \mathcal{L}$  a free generating set if algebraically independent and  $\mathcal{L} = \mathbb{Q}(x_1, \dots, x_m)$ .

Def: A labeled seed of geometric type in  $\mathcal{L}$  is  $(\tilde{x}, \tilde{B})$ , where:

- $\tilde{x} = (x_1, \dots, x_m)$  free generating set of  $\mathcal{L}$
- $\tilde{B} = (b_{ij})$   $m \times n$  extended skew-symmetrizable integer matrix

### Terminology:

- $\tilde{x}$  extended cluster
- $x = (x_1, \dots, x_n)$  cluster,  $x_1, \dots, x_n$  cluster variables
- $x_{n+1}, \dots, x_m$  frozen variables
- $\tilde{B} \leftrightarrow$  ~~exchange matrix~~ extended exchange matrix
- top  $n \times n$  submatrix  $B$  is the exchange matrix

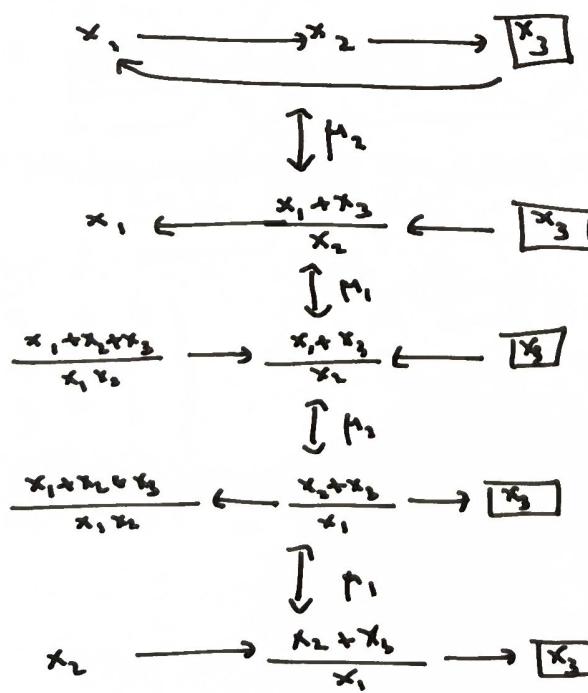
Def: Given  $(\tilde{x}, \tilde{B})$  labeled seed,  $k \in \{1, \dots, n\}$ , define a new labeled seed  $\mu_k(\tilde{x}, \tilde{B}) = (\tilde{x}', \tilde{B}')$ , where

- $\tilde{B}' = \mu_k(\tilde{B})$
- $\tilde{x}' = (x'_1, \dots, x'_m)$ , where  $x'_j = x_j$  for  $j \neq k$  and

$$x_k x'_k = \prod_{b_{ik} > 0} x_i^{b_{ik}} + \prod_{b_{ik} < 0} x_i^{-b_{ik}} \quad \text{exchange relation}$$

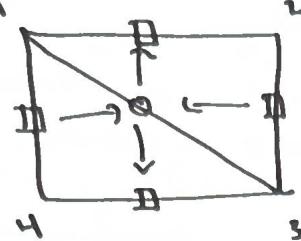
Rmk: When  $\tilde{B}$  comes from a quiver, the first product is over arrows ending at  $k$  and the second product is over arrows starting at  $k$ .

Ex:

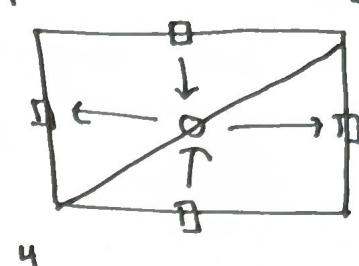


Note: the last seed agrees with the first one up to relabelling.

Ex:



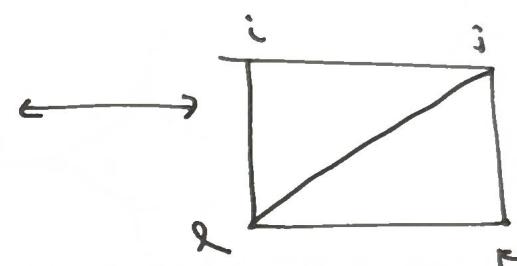
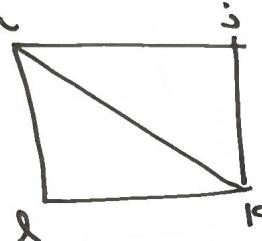
↔ flip



$$\begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix} \quad P_{13} = ag - ce \quad P_{24} = bh - df$$

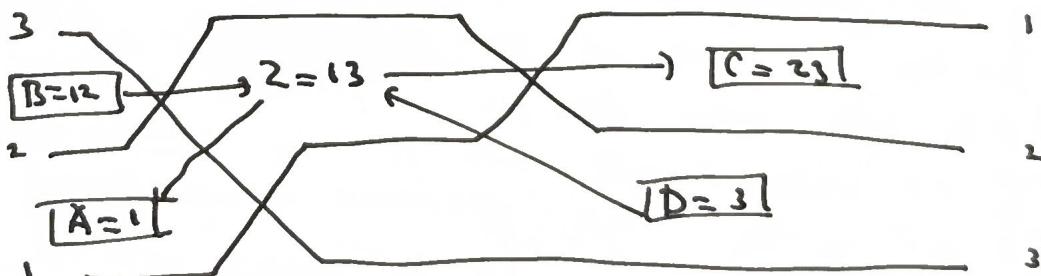
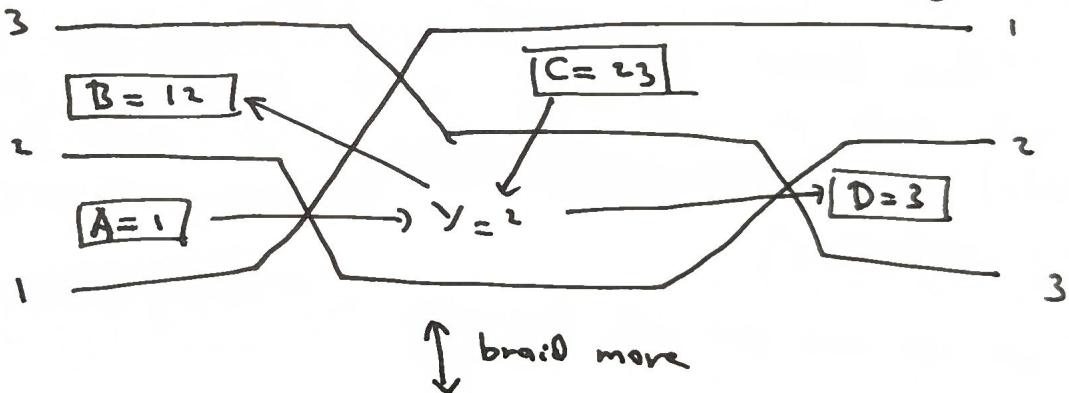
$$\text{Recall: } P_{13} P_{24} = P_{12} P_{34} + P_{14} P_{23}$$

More generally,



$$\rightsquigarrow P_{ik} P_{jl} = P_{ij} P_{lk} + P_{il} P_{jk} \quad \text{special case of the exchange relation}$$

Ex:



$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$\begin{aligned} A &\leftrightarrow a \\ B &\leftrightarrow ae - bf \\ C &\leftrightarrow bf - ce \end{aligned} \quad )$$

etc

$$\text{Have } y_2 = Ac + Bd$$

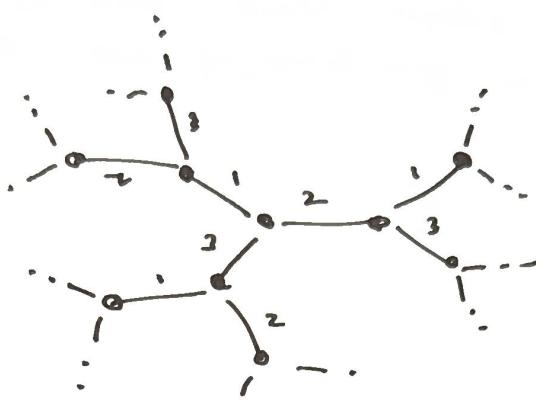
special case of the exchange relation

Notation: Let  $\Pi_n$  denote the  $n$ -regular tree with edges labeled by  $l_1, \dots, l_n$  such that the edges incident to each vertex carry distinct labels.

$$\Pi_1 \quad \bullet - 1 -$$

$$\Pi_2 \quad \dots - 2 - \bullet - 1 - \bullet - 2 - \bullet - 1 - \bullet - 2 - \bullet - \dots$$

$$\Pi_3$$



Def.: A seed pattern is a choice of labeled seeds  $(\tilde{x}(t), \tilde{B}(t))$  for each vertex  $t \in \Pi_n$ , so that for each labeled edge  $t \xrightarrow{\kappa} t'$  the corresponding labeled seeds  $(\tilde{x}(t), \tilde{B}(t)), (\tilde{x}(t'), \tilde{B}(t'))$  differ by  $\mu_\kappa$ .

Note: a seed pattern is determined by any one of its seeds.

Def: Let  $(\tilde{x}(t), \tilde{B}(t))_{t \in \Pi_n}$  be a seed pattern, and put  $R := \mathbb{Q}[x_{n_1}, \dots, x_n]$ . Let  $\mathcal{X}$  be the set of all cluster variables appearing in the seeds  $x(t)$  for  $t \in \Pi_n$ . The cluster algebra  $A$  is the  $R$ -subalgebra of  $\mathcal{L}$  generated by all cluster variables i.e.  $A = R[\mathcal{X}]$ .

Terminology: The rank  $n$  of a cluster algebra is the cardinality of any cluster.

Rank: Note that there is an isomorphism of any free generating set to any other. In particular, up to isomorphism  $A$  depends only on  $\tilde{B}_0$  for any initial seed  $(\tilde{x}_0, \tilde{B}_0)$ , and in fact only on the mutation equivalence class of  $\tilde{B}$ . In particular, each (i.e) give  $Q$  defines an extended exchange matrix  $\tilde{B}$  and hence a cluster algebra.

Ex: For  $T$  a triangulation of the regular  $n$ -gon  $P_m$ , the associated cluster algebra is the Plücker ring  $P_{2,m}$ .

Ex: For a wiring diagram on  $k$  strands, the associated cluster algebra is the algebra generated by flag minors of a  $k \times k$  matrix, i.e. the ring of invariants  $\mathbb{C}[\mathrm{SL}_k]^U$  (here  $U = \text{group of lower-triangular matrices with } 1s \text{ on the diagonal}$ )

Ex: For a double wiring diagram on  $k$  strands, the associated cluster algebra is  $\mathbb{C}[\mathrm{GL}_n]$ , i.e. the polynomial ring in  $k^2$  variables. i.e. functions on the basic affine space

## Lecture 7

2/6/26

Recall: Labeled seed  $(\tilde{x}_0, \tilde{B}_0) \rightsquigarrow$  seed pattern  $(\tilde{x}(t), \tilde{B}(t))_{t \in \mathbb{T}_n}$

→ cluster algebra  $\mathbb{A}$  of  $\mathcal{L}$ ,  
generated by all cluster  
variables and the frozen  
variables

Here  $\tilde{x}_0 = (x_1, \dots, x_m)$  free generating set of  $\mathcal{L} = \mathbb{C}(x_1, \dots, x_m)$ ,  
cluster variables  $x_1, \dots, x_n$ , frozen variables  $x_{n+1}, \dots, x_m$ .  
The rank of  $\mathbb{A}$  is  $n$ .

Ex: rank  $n=1$   $\mathbb{T}_1 = \{1\}$ .

$$\tilde{B}_0 = \begin{pmatrix} 0 \\ b_{11} \\ \vdots \\ b_{m1} \end{pmatrix}$$

$$\text{Exchange relation: } x_i x_i' = \prod_{b_{ii} > 0}^{b_{ii}} x_i + \prod_{b_{ii} < 0}^{-b_{ii}} x_i$$

$$= M_1 + M_2$$

monomials in the  
frozen variables  $x_{n+1}, \dots, x_m$

$$\mathbb{A} = \mathbb{C}[x_1, x_1', x_2, \dots, x_m] \subset \mathcal{L}$$

||  
 $\mathbb{C}(x_1, x_2, \dots, x_m)$

$$\mathbb{C}[z_1, z_1', z_2, \dots, z_m] / (z_1 z_1' = M_1 + M_2)$$

monomials in  $z_3, \dots, z_m$

Ex:  $G = \mathrm{SL}_3(\mathbb{C})$ ,  $U =$  subgroup of unipotent lower-triangular  $3 \times 3$  matrices

Then  $\mathbb{C}[G]^U$  is a cluster algebra of rank 1.

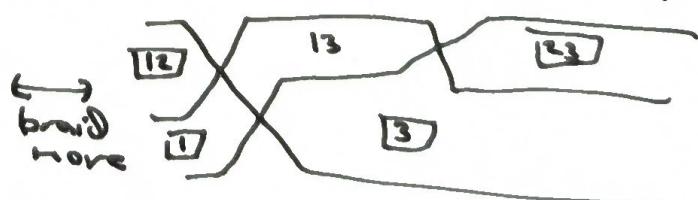
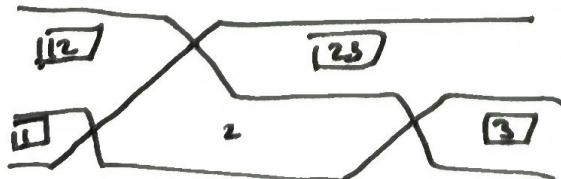
Recall:  $\mathbb{C}[G]^U$  generated by flag minors  $P_J$ ,  $J \subset \{1, 2, 3\}$

Here •  $\mathcal{L} = \mathbb{C}(P_1, P_2, P_3, P_{12}, P_{23})$

• frozen variables:  $P_{11}, P_{33}, P_{12}, P_{23}$

• cluster variables  $P_{22}, P_{13}$

• single exchange relation:  $P_2 P_{13} = P_1 P_{23} + P_2 P_{12}$



$$\underline{\text{Ex:}} \text{ rank } n=2, \quad \widetilde{B}_0 = \begin{pmatrix} 0 & \pm b \\ \mp c & 0 \\ b_{31} & b_{32} \\ \vdots & \vdots \\ b_{m1} & b_{m2} \end{pmatrix} \quad \begin{array}{l} \text{either } b, c > 0 \\ \text{or} \\ b = c = 0 \end{array}$$

$$\text{Suppose no frozen, i.e. } n=m, \quad \widetilde{B}_0 = \begin{pmatrix} 0 & \pm b \\ \mp c & 0 \end{pmatrix}$$

$$\text{Then } \mu_1(\widetilde{B}_0) = \mu_2(\widetilde{B}_0) = -\widetilde{B}_0$$

Exchange pattern:

$$\dots - \begin{pmatrix} (z_1, z_0) & \\ (0 & -b) \\ c & 0 \end{pmatrix} \xrightarrow{2} \begin{pmatrix} (z_1, z_2) & \\ (0 & b) \\ -c & 0 \end{pmatrix} \xrightarrow{1} \begin{pmatrix} (z_3, z_2) & \\ (0 & -b) \\ c & 0 \end{pmatrix} \xrightarrow{2} \begin{pmatrix} (z_3, z_4) & \\ (0 & b) \\ -c & 0 \end{pmatrix} \xrightarrow{1} \dots$$

where

$$z_{k-1}, z_{k+1} = \begin{cases} z_k^c + 1 & \text{if } k \text{ even} \\ z_k^b + 1 & \text{if } k \text{ odd} \end{cases}$$

$$\underline{\text{Ex:}} \quad \widetilde{B} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ b_{31} & b_{32} \\ \vdots & \vdots \\ b_{m1} & b_{m2} \end{pmatrix}$$

$\mu_1$  flips sign of  $k$ th column  
for  $k = l_j$

Exchange relations:

$$x_1 x_1' = M_1 + M_2$$

$$x_2 x_2' = M_3 + M_4$$

Cluster variables:

$x_1, x_2, M_1, M_2, M_3, M_4$  monomials in frozen  
(reduces to two rank 1)  
exchange patterns

Notation: Let  $A(b, c)$  denote the  
of rank 2 with exchange matrices

$$\underline{\text{Ex:}} \quad A(1, 1)$$

$$z_{k-1}, z_{k+1} = z_k + 1$$

$$z_3 = \frac{z_2 + 1}{z_1}$$

$$z_4 = \frac{z_3 + 1}{z_2} = \frac{z_2 + 1}{z_1} + 1 = \frac{z_1 + z_2 + 1}{z_1 z_2}$$

$$z_5 = \frac{z_4 + 1}{z_3}$$

$$z_6 = z_1, \quad z_7 = z_2 \quad (\text{etc.}) \quad \text{so 5-periodic.}$$

cluster algebra  
 $(0 \pm b)$  and no frozens.

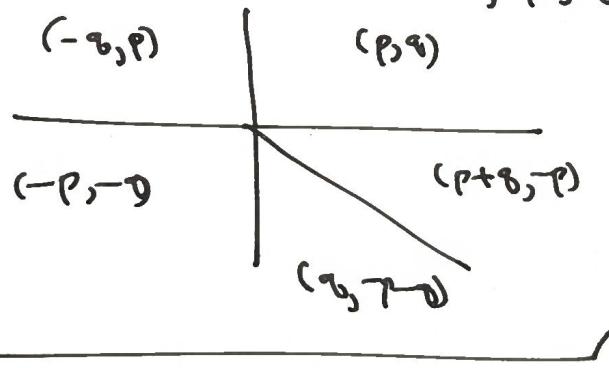
$\text{Ex: } \tilde{B}_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ p & q \end{pmatrix}, \text{ rank} = 2, 1 \text{ frozen variable}$   
 $p, q \geq 0 \text{ integers}$

seed pattern:

$$\dots - \begin{pmatrix} z_1, z_2 \\ 0 & 1 \\ -1 & 0 \\ p & q \end{pmatrix} \xrightarrow{1} \begin{pmatrix} z_3, z_4 \\ 0 & -1 \\ 1 & 0 \\ -p & p+q \end{pmatrix} \xrightarrow{2} \begin{pmatrix} z_5, z_6 \\ 0 & 1 \\ -1 & 0 \\ q-p & q \end{pmatrix} \xrightarrow{1} \begin{pmatrix} z_7, z_8 \\ 0 & -1 \\ 1 & 0 \\ -q & -p \end{pmatrix} \xrightarrow{2} \begin{pmatrix} z_9, z_{10} \\ 0 & 1 \\ -1 & 0 \\ -q & p \end{pmatrix} \dots$$

Have  $z_3 = \frac{z_2 + y^p}{z_1}, z_4 = \frac{y^{p+q} z_1 + z_2 + y^p}{z_1 z_2},$   
 $z_5 = \frac{y^q z_1 + 1}{z_2}, z_6 = z_1, z_7 = z_2, \text{ etc,}$   
so still 5-periodic.

Although we assumed  $p, q \geq 0$ , up to unitating and swapping columns every  $(i, j) \in \mathbb{Z}^2$  can be written in one of the forms  $(p, q), (p+q, -p), (q, -p-q), (-p, -q), (-q, p)$ :



Later we will view this as a ~~scattering~~ simple example of a scattering Diagram.

## Lecture 8

2/8/26

Ex:  $A(1,2)$

$$z_{k+1} = \begin{cases} z_k^2 + 1 & k \text{ even} \\ z_k + 1 & k \text{ odd} \end{cases}$$

$$z_3 = \frac{z_2^2 + 1}{z_1} \quad z_4 = \frac{z_3^2 + 1}{z_2} = \frac{z_2^2 + 1}{z_1} + 1 = \frac{z_1^2 + z_1 + 1}{z_1 z_2}$$

$$z_5 = \frac{z_1^2 + z_2^2 + 2z_1 + 1}{z_1 z_2} \quad z_6 = \frac{z_1 + 1}{z_2} \quad z_7 = z_1 \quad z_8 = z_2 \quad \text{etc}$$

So it's 6-periodic.

Ex:  $A(1,3)$

$$z_{k+1} = \begin{cases} z_k^3 + 1 & k \text{ even} \\ z_k + 1 & k \text{ odd} \end{cases}$$

$$\text{Set } z_1 = z_2 = 1. \quad z_3 = \frac{z_2^3 + 1}{z_1} = 2$$

$$z_4 = \frac{z_3 + 1}{z_2} = \frac{2+1}{1} = 3$$

$$z_5 = \frac{z_4^3 + 1}{z_3} = \cancel{z_4^3 + 1} \cancel{z_3} \cancel{z_2} \cancel{z_1} = \frac{28}{2} = 14$$

$$z_6 = \frac{z_5 + 1}{z_4} = \cancel{z_5 + 1} \cancel{z_4} \cancel{z_3} \cancel{z_2} \cancel{z_1} = \frac{15}{3} = 5$$

$$z_7 = \frac{z_6^3 + 1}{z_5} = \frac{126}{14} = 9$$

$$z_8 = \frac{z_7 + 1}{z_6} = \frac{10}{5} = 2$$

$$z_9 = \frac{z_8^3 + 1}{z_7} = \frac{9}{9} = 1$$

$$z_{10} = \frac{z_9 + 1}{z_8} = \frac{2}{2} = 1$$

So it's 8-periodic at least after specializing  $z_1 = z_2 = 1$  and we claim that it's 8-periodic even without this specialization.

Ex:  $A(1,4)$

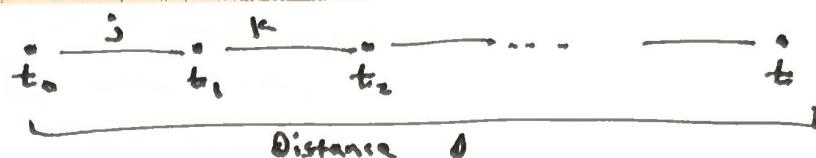
$$z_1 = z_2 = 1 \implies 1, 1, 2, 3, 41, 14, 937, 67, 21505, 321 \dots$$

not periodic. However, all integers and in fact each  $z_k$  is a Laurent polynomial in  $z_1, z_2$ .

Thm Let  $(\tilde{X}_0, \tilde{B}_0)$  be a labeled seed, with  $\tilde{X}_0 = (x_1, \dots, x_m)$  and associated cluster algebra  $A$ . Every cluster variable of  $A$  is a Laurent polynomial with integer coefficients in the variables  $x_1, \dots, x_m$ . Moreover,  $x_{m+1}, \dots, x_n$  do not appear in the denominators.

Rank: Note that we can replace  $\tilde{X}_0$  equivalently with any other extended cluster of  $A$ .

proof idea



Say  $t_0 \in T_n$  initial vertex,  $(\tilde{x}_0, \tilde{B}_0)$  initial ((labeled) seed),  
 $x$  cluster variable in the seed

For  $\tilde{x}_0 = (x_1, \dots, x_n)$ , want to show that  $x$  is a  
 Laurent polynomial in  $x_1, \dots, x_n$ . Will use induction on

$$d = \text{dist}(t_0, t_1).$$

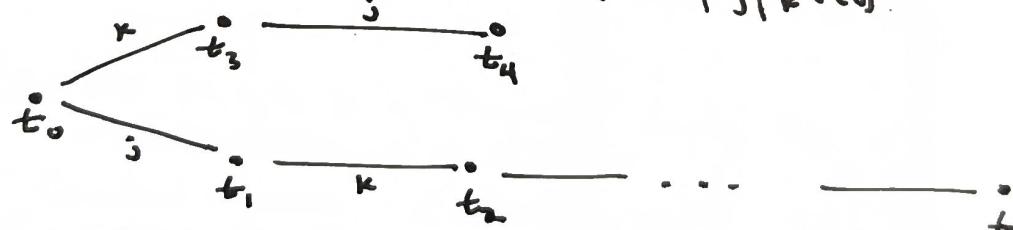
Base cases : if  $d=1$ , then  $x(t) = x(t_1) = (x_1, \dots, x_{j-1}^1, x_j, \dots, x_n)$ ,  
 where  $x_j^1 = \frac{\prod_{b_{ij} > 0} x_i^{b_{ij}} + \prod_{b_{ij} < 0} x_i^{-b_{ij}}}{x_j}$

if  $d=2$ , then  $x(t) = x(t_1) = x(t_2) = (x_1, \dots, x_{j-1}^1, x_j^1, x_{j+1}^1, \dots, x_n)$ ,  
 where  $x_k^1 = \frac{\text{poly in } x_1, \dots, x_{j-1}^1, x_j, \dots, x_n}{x_k}$   
 $\quad \quad \quad = \frac{\text{Laurent poly in } x_1, \dots, x_n}{x_k}$

(or swap)

Inductive step : Now assume  $d \geq 3$ , and assume for simplicity  
 that  $b_{j,k}^0 = b_{k,j}^0 = 0$  where  $\tilde{B}_0 = (b_{i,j}^0)$   
 (the case  $b_{j,k}^0, b_{k,j}^0 < 0$  is more complicated)

Put  $t_3 := \mu_k(t_0)$  and  $t_4 := \mu_j \mu_k(t_0)$



Note :  $\tilde{x}(t_0) = \tilde{x}(t_3)$ , so both  $t_1, t_3$  lie at distance  $d-1$  from a seed containing  $x$ . By induction:

$$x = \text{Laurent poly in } \tilde{x}(t_0) = \text{Laurent poly in } \tilde{x}(t_3)$$

$\underbrace{(x_1, \dots, x_{j-1}^1, x_j, \dots, x_n)}$        $\underbrace{(x_1, \dots, x_{j-1}^1, x_{j+1}^1, \dots, x_n)}$

Meanwhile,  $x_j^1 = \frac{M_1 + M_2}{x_j}$ ,  $x_k^1 = \frac{M_3 + M_4}{x_k}$  for  $M_1, M_2, M_3, M_4$   
 monomials in  $x_1, \dots, x_n$

$$x = \frac{\text{poly in } x_1, \dots, x_n}{(\text{monomial in } x_1, \dots, x_n) \cdot (M_1 + M_2)^a} = \frac{\text{poly in } x_1, \dots, x_n}{(\text{monomial in } x_3, \dots, x_n) \cdot (M_3 + M_4)^b}$$

(after clearing denominators)

It suffices to show that  $a=0$ .  
 Let  $\tilde{B}_0^{\text{aug}}$  be  $\tilde{B}_0$  after adding an extra row of the form  $(0, \dots, \underbrace{1, \dots, 0}_{i\text{th entry}})$ . Let  $A^{\text{aug}}$  be the resulting cluster algebra with coefficient variables  $x_{n+1}, \dots, x_m$ .  
Observe: expression in  $A^{\text{aug}}$  for  $x$  in terms of  $x_{n+1}, \dots, x_m$  (Specialize  $x_{n+1} = 1$ ) expression in  $A^{\text{aug}}$  for  $x$  in terms of  $x_1, \dots, x_m$

So  $x$  Laurent polynomial in  $x_1, \dots, x_m$  and in  $A^{\text{aug}}$   $\Rightarrow x$  Laurent poly in  $x_1, \dots, x_m$  in  $A$ , hence

WLOG can assume  $\tilde{B}_0^{\text{aug}}$  instead of  $\tilde{B}_0$ .

But then  $x'_j = \frac{M_1^{\text{aug}} + M_2^{\text{aug}}}{x_j} = \frac{M_1 x_{n+1} + M_2}{x_j}$

$$x'_{j'} = \frac{M_3^{\text{aug}} + M_4^{\text{aug}}}{x_{j'}} = \frac{M_3 + M_4}{x_{j'}}$$

Then  $M_1^{\text{aug}} + M_2^{\text{aug}}$  and  $M_3 + M_4$  have no common factor  
 (think about what happens if we specialize  $x_1 = \dots = x_m = 1$ )  
 $\Rightarrow a = 1$

□

Def: A Markov triple is a triple  $(a, b, c) \in \mathbb{Z}_{\geq 1}^3$  which satisfies the Markov equation  $a^2 + b^2 + c^2 = 3abc$

Ex:  $(1, 1, 1)$  is a Markov triple and hence also its permutations

So is  $(1, 2, 5)$ ,  
 $(1, 5, 2)$ ,  $(2, 1, 5)$ ,  $(5, 1, 2)$ ,  
 $(2, 5, 1)$ ,  $(5, 2, 1)$

Lemma: If  $(a, b, c)$  is a Markov triple, then so is  $(a, b, c')$  with  $c' = \frac{a^2 + b^2}{c}$

Pf: Consider equation  $a^2 + b^2 + c^2 = 3abc$ , i.e.  $t^2 - 3abt + (a^2 + b^2) = 0$ . If  $c$  is one root, the other root  $c'$  must satisfy  $c + c' = 3ab$ , i.e.  $c' = 3ab - c = \frac{3abc - c^2}{c} = \frac{a^2 + b^2}{c}$  "Markov mutation"

Lemma: If  $(a, b, c)$  is a Markov triple and  $a \leq b \leq c$ , then  $c' = 3abc - c < c$ .

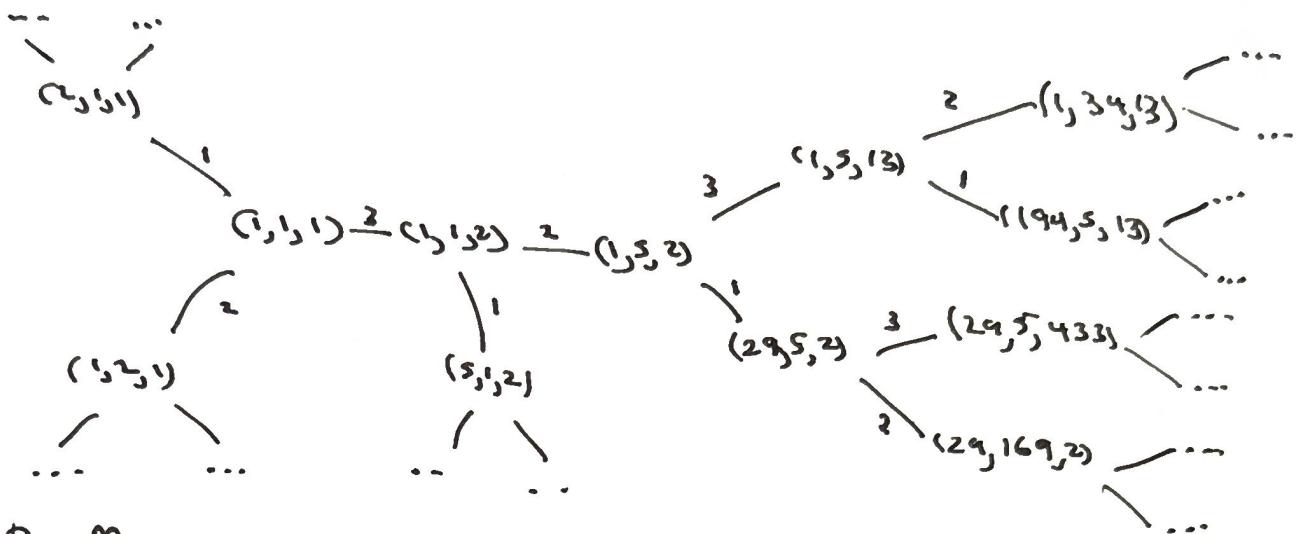
Pf: Put  $f(t) = t^2 - 3abt + (a^2 + b^2)$ .

$$\begin{aligned} \text{Then } f(b) &= b^2 - 3ab^2 + a^2 + b^2 \\ &= b^2(2 - 3a) + a^2 \\ &\leq -b^2 + a^2 \leq 0 \end{aligned}$$

~~Then  $c'$  is the other root of  $f$ , must satisfy  $c' \leq b < c$ .~~

Cor by : Every Markov triple can be connected to  $(1, 1, 1)$  by a sequence of Markov mutations.

## The Maroon tree :



Recall : The Marian guiver is

Exchange relations:

$$x_1 - x_2 + x_3$$

$$x_1 x_2 = x_1^2 + x_2^2$$

$$x_3^1 x_3 = x_1^2 + x_2^2$$

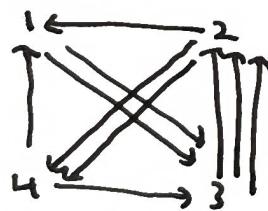
Ex: The Somos-4 sequence  $z_0 = z_1 = z_2 = \dots$  is formed by the recurrence relation  $z_{n+2} = z_{n+1} z_n + z_{n-1}^2$ . This sequence is known as a **Markov triple**.

$z_0 = z_1 = z_2 = z_3 = 1$  en 0 sequence  $z_0, z_1, z_2, z_3, \dots$  defined by

Some 180s: these are all integers. To establish, we note that  $z_{m+2} z_{m-1} = z_{m+1} z_{m-1} + z_m^2$  i.e.  $z_{m+2} z_{m-1} - z_{m+1} z_{m-1} = z_m^2$  defined by  $z_1, z_2, z_3, z_4, z_5, z_6, \dots$

To explain using cluster algebras!

1 ← 2      [ ]      algebra consider given



(no <sup>Q</sup> frosen)

$$z_1 z_3 = z_2 z_4 + z_3^2 \quad \overrightarrow{Q' = p_1(Q)}$$

The  $\mu_2$  rotates  $Q'$  by  $\pi_{\mu_2}$ ,  
 Continue in this way.

Continue in this way with  $\frac{2}{5}, \frac{2}{3}, \frac{2}{4}$

gives a  $\beta$  - mutation

gives  $\bar{z}_n = \text{ Laurent polynomial}$   
 $\text{in } z_1, z_2, z_3, z_4, z_5$

Sequence  $\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7, \mu_8, \mu_9$

Specialize  
 $t_1 = t_2 = t_3 = t_4 = 1$

1<sup>st</sup> to 8<sup>th</sup> off of  $S_{0,0,0,0,0,0,0,0}$  - 4 necessarily an integer

# Lecture 9

2/11/26

Let  $(\tilde{x}, \tilde{B})$  be a labeled seed, with  $\tilde{x} = (x_1, \dots, x_m)$ ,  $\tilde{B} = (b_{ij})$ .  
 Put  $(\tilde{x}', \tilde{B}') = \mu_k(\tilde{x}, \tilde{B})$ , with  $\tilde{x}' = (x'_1, \dots, x'_m)$ ,  $\tilde{B}' = (b'_{ij})$ .

Put  $\hat{y} := (\hat{y}_1, \dots, \hat{y}_n)$ , where  $\hat{y}_{j,i} = \prod_{i=1}^m x_i^{b_{ij}}$  and  
 similarly  $\hat{y}' = (\hat{y}'_1, \dots, \hat{y}'_n)$  with  $\hat{y}'_{j,i} = \prod_{i=1}^m (x'_i)^{b'_{ij}}$ .

Prop: We have  $\hat{y}'_j = \begin{cases} \hat{y}'_k^{-1} & \text{if } j = k \\ \hat{y}'_j (\hat{y}'_k^{-\text{sgn}(b_{kj})} + 1)^{-b_{kj}} & \text{else} \end{cases}$   
 (for  $j = 1, \dots, n$ )

Here  $\text{sgn}(b) = \begin{cases} 1 & \text{if } b > 0 \\ -1 & \text{if } b < 0. \end{cases}$

Rank: • recall that the exchange relation is

$$x_k x_{i^*}^{-1} = \underbrace{\prod_{b_{ik} > 0} x_i^{b_{ik}}}_{\text{top non submatrix of } \tilde{B}} + \underbrace{\prod_{b_{ik} < 0} x_i^{-b_{ik}}}_{\text{bottom non submatrix of } \tilde{B}}$$

$\hat{y}_k$  is the ratio of these

• the above formula for  $\hat{y}'_j$  depends only on the

Proof: • if  $j = k$ ,  $\hat{y}'_k = \prod_{i=1}^m (x'_i)^{b'_{ik}} = \prod_{i \neq k} x_i^{b'_{ik}}$

(recall that we have

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } k \in \{i, j\} \\ b_{ij} + b_{ik} b_{kj} & \text{if } b_{ik} b_{kj} > 0 \\ b_{ij} & \text{else} \end{cases}$$

$$= \prod_{i \neq k} x_i^{-b_{ik}} = \hat{y}_k^{-1}$$

• if  $j \neq k$  and  $b_{kj} \leq 0$ , have

$$\hat{y}'_j = (x'_{i^*})^{b'_{ij}} \prod_{i \neq k} x_i^{b'_{ij}}$$

$$= (x'_{i^*})^{-b_{kj}} \left( \prod_{i \neq k} x_i^{b_{ij}} \right) \left( \prod_{b_{ik} < 0} x_i^{-b_{ik} b_{kj}} \right)$$

$$= x_{i^*}^{b_{ij}} \left( \prod_{b_{ik} > 0} x_i^{b_{ik}} + \prod_{b_{ik} < 0} x_i^{-b_{ik}} \right)^{-b_{kj}} \left( \prod_{i \neq k} x_i^{b_{ij}} \right) \left( \prod_{b_{ik} < 0} x_i^{-b_{ik} b_{kj}} \right)$$

$$= \left( \prod_i x_i^{b_{ij}} \right) \left( \prod_i x_i^{b_{ik}} + 1 \right)^{-b_{kj}}$$

$$= \hat{y}'_j (\hat{y}_k^{-1} + 1)^{-b_{kj}}.$$

• case  $j \neq k$ ,  $b_{ik} \geq 0$  similar.

Def : A  $\mathbb{Y}$ -seed of rank  $n$  in a field  $\mathbb{L}$  is  $(\mathbb{Y}, \mathbb{B})$ , where

- $\mathbb{Y}$  =  $n$ -tuple of elts in  $\mathbb{L}$
- $\mathbb{B}$  = skew-symmetrizable  $n \times n$  integer matrix

We mutate  $\mathbb{Y}$ -seeds as follows:

$$(\mathbb{Y}, \mathbb{B}) \xrightarrow{\mu_k} (\mathbb{Y}', \mathbb{B}'), \text{ where } \mathbb{B}' = \mu_k(\mathbb{B}),$$

$$\mathbb{Y}' = (\mathbb{Y}_1, \dots, \mathbb{Y}_n) \text{ with } \mathbb{Y}'_j = \begin{cases} \mathbb{Y}_k & \text{if } j = k \\ \mathbb{Y}_j (\gamma^{-\text{sgn}(b_{jk})} + 1)^{-b_{jk}} & \text{else} \end{cases}$$

Thus labeled seed  $(\tilde{\mathbb{Y}}, \tilde{\mathbb{B}})$   $\longrightarrow$   $\mathbb{Y}$ -seed  $(\hat{\mathbb{Y}}, \mathbb{B})$ , where

$$\mathbb{B} = \text{top row submatrix of } \tilde{\mathbb{B}}$$

$$\hat{\mathbb{Y}} = (\hat{\mathbb{Y}}_1, \dots, \hat{\mathbb{Y}}_n) \text{ with } \hat{\mathbb{Y}}_i = \prod_{j=1}^n x_i^{b_{ij}}$$

Part : The seed mutation leaves  $x_j$  alone for  $j \neq k$  whereas  $\mathbb{Y}$ -seed mutation at  $k$  only changes  $\mathbb{Y}_k$  and  $\mathbb{Y}_j$ . Potentially changes all of  $\mathbb{Y}_1, \dots, \mathbb{Y}_n$ . However, the formula for  $x_k$  involves all of  $x_1, \dots, x_n$ , whereas  $\mathbb{Y}_j$  only involves  $\mathbb{Y}_k$  and  $\mathbb{Y}_j$ .

Def : A semifield is an abelian group  $P$  endowed with an auxiliary operation  $\oplus$  which is commutative, associative, and distributive with respect to the group operation on  $P$  (written multiplicatively). Note that  $(P, \oplus)$  is only a semigroup (i.e. not necessarily identity or inverses)

Ex : The multiplicative group  $\mathbb{Q}_{>0}$  with  $\oplus$  given by ordinary addition.

Def : The tropical semifield  $\text{Trop}(\mathbb{Q}_1, \dots, \mathbb{Q}_l)$  is defined by :

- the multiplicative group of Laurent monomials in  $\mathbb{Q}_1, \dots, \mathbb{Q}_l$
- $\prod_{i=1}^l q_i^{a_i} \oplus \prod_{i=1}^l q_i^{b_i} = \prod_{i=1}^l q_i^{\min(a_i, b_i)}$  ("tropical addition")

Check :

- commutative :  $\min(a_i, b_i) = \min(b_i, a_i)$
- associative :  $\min(\min(a_i, b_i), c_i) = \min(a_i, \min(b_i, c_i)) = \min(a_i, b_i, c_i)$  (i.e.  $(p \oplus q) \oplus r = p \oplus (q \oplus r)$ )
- Distributive :  $\min(a_i, b_i) + c_i = \min(a_i + c_i, b_i + c_i)$

For  $(\tilde{x}, \tilde{B})$  labeled seed,  $\rightarrow$  coefficient tuple

$$\tilde{x} = (x_1, \dots, \underbrace{x_n, \dots, x_m}_{\text{frozen variables}})$$

$$y = (y_1, \dots, y_m), \text{ where}$$

$$y_j = \prod_{i=n+1}^m x_i^{b_{ij}} \in \text{Trop}(x_{n+1}, \dots, x_m)$$

for  $j=1, \dots, n$

Note:  $B = \text{top non submatrix of } \tilde{B}$  together with coeff. tuple  $y$  recover the extended exchange matrix  $\tilde{B}$ .

Prop:  $\tilde{B} = (b_{ij})$  extended skew-symmetrizable max matrix with coeff. tuple  $y = (y_1, \dots, y_n)$ , and  $\tilde{B}' = (b'_{ij}) = \mu_k(\tilde{B})$  with coeff. tuple  $y' = (y'_1, \dots, y'_n)$ . Then

$$y'_j = \begin{cases} y_k^{-1} & \text{if } j=k \\ y_j \left( y_k^{-\text{sgn}(b_{kj})} \oplus 1 \right)^{-b_{kj}} & \text{else.} \end{cases}$$

"tropical Y-seed mutation"

Def: The universal semifield  $\mathbb{Q}_{sf}(x_1, \dots, x_m)$  is

$$\left\{ \frac{P(x_1, \dots, x_m)}{Q(x_1, \dots, x_m)} \in \mathbb{Q}(x_1, \dots, x_m) \mid P, Q \text{ have positive coefficients} \right\}$$

with ordinary multiplication and addition.

Lemma: Given any semifield  $\mathbb{S}$ , and its  $s_1, \dots, s_m \in \mathbb{S}$ ,  $x_i \mapsto s_i$  for  $i=1, \dots, m$ ,  $\mathbb{Q}_{sf}(x_1, \dots, x_m) \rightarrow \mathbb{S}$  sending

pf of prop: Let  $f: \mathbb{Q}_{sf}(x_1, \dots, x_m) \rightarrow \text{Trop}(x_{n+1}, \dots, x_m)$  be semifield homo. sending  $f(x_i) = \begin{cases} 1 & \text{if } i \in n \\ x_i & \text{if } i \in n. \end{cases}$

Note that  $f$  also sends  $x_k^1$  to 1, since

$$x_k x_k^1 = M_1 + M_2 \implies 1 \cdot f(x_k^1) = f(M_1) \oplus f(M_2) = 1$$

$$\text{Also, } \hat{y}_j = \prod_{i=1}^n x_i^{b_{ij}} \implies f(\hat{y}_j) = \prod_{i=n+1}^m x_i^{b_{ij}} = y_j \text{ for } j=1, \dots, n,$$

1 since  $M_1, M_2$  monomials which share no frozen variables

$\therefore$  and similarly  $f(\hat{y}_j) = y_j$ .

$$\text{Thus } \hat{y}'_j = \begin{cases} \hat{y}_k^{-1} & \text{if } j=k \\ \hat{y}_j \left( \hat{y}_k^{-\text{sgn}(b_{kj})} \oplus 1 \right)^{-b_{kj}} & \text{else} \end{cases}$$

$$\hat{y}'_j = \begin{cases} y_k^{-1} & \text{if } j=k \\ y_j \left( y_k^{-\text{sgn}(b_{kj})} \oplus 1 \right)^{-b_{kj}} & \text{else} \end{cases}$$