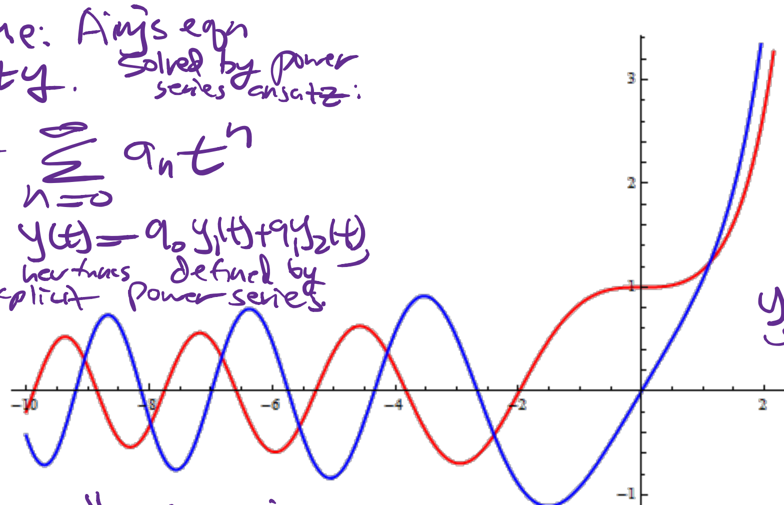
Lecture 16

Last time: Airy's eqn $y'' = ty$. Solved by power series ansatz:

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Found $y(t) = a_0 y_1(t) + a_1 y_2(t)$

y_1, y_2 two new functions defined by explicit power series.



$y_1(0) = 1$
$y_1'(0) = 0$
$y_2(0) = 0$
$y_2'(0) = 1$

Ex: Solve $y'' = ty$ using ansatz $y(t) = \sum_{n=0}^{\infty} a_n (t-1)^n$

$y'(t) = \sum_{n=1}^{\infty} a_n n (t-1)^{n-1}$

$y''(t) = \sum_{n=2}^{\infty} a_n (n)(n-1)(t-1)^{n-2}$

Plug into Airy's eqn:

$$\underbrace{\sum_{n=2}^{\infty} a_n (n)(n-1)(t-1)^{n-2} - \sum_{n=0}^{\infty} a_n t (t-1)^n = 0}_{\text{write as a single power series (centered at } t=1\text{) and set coeffs to } 0}$$

Note: have some terms involving t, and some involving $(t-1)$

Write $t = (t-1) + 1$

Then have

$$\sum_{n=2}^{\infty} a_n n(n-1)(t-1)^{n-2} - \sum_{n=0}^{\infty} a_n (t-1)^n - \sum_{n=0}^{\infty} a_n (t-1)(t-1)^n = 0$$

$2\cdot 1 a_2 + 3\cdot 2 a_3 (t-1) + 4\cdot 3 a_4 (t-1)^2 + \dots$
$- a_0 - a_1(t-1) - a_2(t-1)^2 - a_3(t-1)^3 - \dots$
$- a_0(t-1) - a_1(t-1)^2 - a_2(t-1)^3 - \dots$
$= 0$

$(2a_2 - a_0) + (3\cdot 2 a_3 - a_1 - a_0)(t-1) + \left(\begin{matrix}4\cdot 3 a_4 \\ -a_2 - a_1\end{matrix}\right)(t-1)^2 + \left(\begin{matrix}5\cdot 4 a_5 \\ -a_3 - a_2\end{matrix}\right)(t-1)^3 + \dots = 0$

Gives eqns: $2a_2 = a_0$, $3\cdot 2 a_3 = a_0 + a_1$, $4\cdot 3 a_4 = a_1 + a_2$, $5\cdot 4 a_5 = a_2 + a_3$ etc

$a_2 = a_0/2$, $a_3 = \frac{a_0 + a_1}{3\cdot 2}$, $a_4 = \frac{a_1 + a_2}{4\cdot 3} = \frac{a_1 + a_0/2}{12} = a_1/12 + \frac{a_0}{24}$, etc

B/c harder to write out general term, but get

$$y(t) = a_0 y_3(t) + a_1 y_4(t)$$
$$= a_0 + a_1(t-1) + a_2(t-1)^2 + a_3(t-1)^3 + \dots$$

So $y_4(t) = 0 + (1)(t-1) + (0)(t-1)^2 + \frac{1}{6}(t-1)^3 + \frac{1}{12}(t-1)^4 + \dots$

$$y(t) = a_0 + a_1(t-1) + \frac{a_0}{2}(t-1)^2 + \left(\frac{a_0+a_1}{6}\right)(t-1)^3 + \left(\frac{a_0}{24} + \frac{a_1}{12}\right)(t-1)^4 + \dots$$

Pendulum equation:

$$\theta'' = -\sin(\theta)$$

(small angle approx. $\sin(\theta) \approx \theta$; approx have $\theta'' = -\theta$)

Ex: $t^2 y'' + t y' + y = 0$

Ansatz $y(t) = \sum_{n=0}^{\infty} a_n t^n$

$y' = \sum_{n=1} a_n n t^{n-1}$ $y'' = \sum_{n=2} a_n n(n-1) t^{n-2}$

$$\sum_{n=2}^{\infty} \left(a_n n(n-1) + n a_n + a_n\right) t^n + \underbrace{a_0 + a_1 t}_{} + \underbrace{a_1 \cdot 1 \cdot t}_{} = 0$$

Conclusion: $a_0 = a_1 = a_2 = a_3 = \dots = 0$.
So $y(t) = 0$. This is a soln, but certainly not the general solution. Out of luck...

Recall: $P(t)y'' + Q(t)y' + R(t)y = 0$, or write as $y'' + p(t)y' + q(t)y = 0$, for $p(t) = Q(t)/P(t)$, $q(t) = R(t)/P(t)$.

Thm: Consider the ODE $y'' + p(t)y' + q(t)y = 0$. If t_0 is an ordinary pt, then the ODE has general soln $y(t) = \sum_{n=0}^{\infty} a_n (t-t_0)^n$, and have $y(t) = a_0 y_1(t) + a_1 y_2(t)$, and the r.o.c. of $y(t)$ is at least $\min\left(\begin{matrix}\text{r.o.c. of } p(t) \\ \text{at } t_0\end{matrix}, \begin{matrix}\text{r.o.c. of } q(t) \\ \text{at } t_0\end{matrix}\right)$.

$$\left(p(t) = 0,\ q(t) = \frac{t^2 - 5t + 6}{t-3} = t-2\right)$$

Note: $y_3(0) =$ sum of the coeffs if we set $a_0 = 1$, $a_1 = 0$.

So $y_3(0) = \underbrace{1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} + \dots}_{\text{neither } 0 \text{ nor } 1}$

$y_4(t) = \sum_{n=0}^{\infty} a_n (t-1)^n$ after setting $a_0 = 0$, $a_1 = 1$

$y_4(0) = 1 - \frac{1}{6} + \frac{1}{12} + \dots$

$$= a_0 \underbrace{\left(1 + \frac{1}{2}(t-1)^2 + \frac{1}{6}(t-1)^3 + \frac{1}{24}(t-1)^4 + \dots\right)}_{y_3}$$
$$+ a_1 \underbrace{\left((t-1) + \frac{1}{6}(t-1)^3 + \frac{1}{12}(t-1)^4 + \dots\right)}_{y_4}$$

Power series ansatz: $\theta(t) = \sum_{n=0}^{\infty} a_n t^n$

also write $\sin(\theta) = \theta - \theta^3/3! + \theta^5/5! - \dots$

then can plug in ansatz and write everything as a power series ~~centered in t~~ centered at 0.

Plug in: $\sum_{n=2}^{\infty} a_n n(n-1) t^n + \sum_{n=1}^{\infty} a_n n t^n + \sum_{n=0}^{\infty} a_n t^n = 0$

We get: $a_0 = 0$, $2a_1 = 0$,
$a_n(n)(n-1) + n a_n + a_n = 0$ for all $n \geq 2$,
i.e. $n^2 a_n + a_n = 0$, i.e.
$(n^2+1) a_n = 0$

Possible issue: What if $p(t), q(t)$ are ill-defined at $t = t_0$?

Def: $t = t_0$ is an ordinary pt if $P(t_0) \neq 0$ or more generally if Q/P and R/P are analytic at t_0.
Otherwise, $t = t_0$ is called a singular point.

Ex: $y'' - ty = 0$ sing pts? none
($p(t) = 0$, $q(t) = -t$)

Ex: $(t-1)y'' + y' + ty = 0$ sing pts? $t = 1$
($p(t) = \frac{1}{t-1} = q(t)$)

Ex: $(t-3)y'' + (t^2 - 5t + 6)y = 0$
sing pts? none.

Ex: $(t-1)ty'' + t^3y' + \sin(t)y = 0$.

sing pts? $t = 1$

$$p(t) = \frac{t^2}{t-1} \qquad q(t) = \frac{\sin(t)}{t(t-1)}$$

recall: $\sin(t)/t = \frac{1}{t}\left(t - t^3/3! + t^5/5! - \cdots\right)$

$= 1 - t^2/3! + t^4/5! - \cdots$

Ex: Consider $(t^2+t+3)y'' + ty' + y = 0$.

Suppose have a solution $y(t) = \sum_{n=0}^{\infty} a_n (t-\pi)^n$.

What can say about r.o.c. of $y(t)$?

Write ODE as $y'' + \underbrace{\frac{t}{t^2+t+3}}_{p(t)} y' + \underbrace{\frac{1}{t^2+t+3}}_{q(t)} y = 0$

By thm, r.o.c. is at least

$$\min\Big(\underbrace{\text{r.o.c. of } p \text{ at } \pi}_{\text{distance from } \pi \text{ to nearest root of } t^2+t+3},\ \underbrace{\text{r.o.c. of } q \text{ at } \pi}_{\text{distance from } \pi \text{ to nearest root of } t^2+t+3}\Big).$$

roots of t^2+t+3: $\dfrac{-1 \pm \sqrt{1-4}}{2}$

$= -1/2 \pm i\sqrt{3}/2$