HIGHER SYMPLECTIC CAPACITIES AND THE STABILIZED ELLIPSOID INTO POLYDISC PROBLEM

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1. Introduction

The paper [12] defines a new sequence of symplectic capacities $\mathfrak{g}_k$. The capacities $\mathfrak{g}_k$ are invariant under taking products with $\mathbb{C}$, and so are well-suited for exploring the "stabilized" embedding problems that have recently been studied in order to better understand symplectic embedding problems in dimension greater than 4, see e.g. [6, 7, 2, 3, 9] and the discussion below.

It is interesting to try to understand how strong the capacities $\mathfrak{g}_k$ are. The purpose of this note is to show that for ellipsoids into integral polydiscs, they often given sharp obstructions to the stabilized problem. In particular, they are strong enough to prove a conjecture from [1], Conjecture 1.4.

To explain Conjecture 1.4, define $c_b(a)$ to be the infimum, over λ, such that an embedding

$$E(1,a) \overset{s}{\hookrightarrow} \lambda \cdot P(1,b). \tag{1}$$

exists. This is a nondecreasing, continuous, function of a. When $b \geq 2$ is an integer, it is shown in [1] that the function $c_b(a)$ is given by the volume constraint $\frac{a}{2b}$, except on finitely many intervals. On all but one of these intervals, the function $c_b(a)$ is given by a "linear step": it is piecewise linear, with a single nonsmooth point, called its corner, where its graph changes from lying on a line through the origin to being horizontal. On the remaining interval, it is also piecewise linear with a single nonsmooth point, but the linear piece does not lie on a line through the origin – it has an intercept, and so we call it the "affine step". For more detail, see [1].

Conjecture 1.4 asserts that the linear steps from above are "stable". Namely, define $c_b^N(a)$ to be the infimum, over λ, such that an embedding

$$E(1,a) \times \mathbb{C}^N \overset{s}{\hookrightarrow} \lambda \cdot P(1,b) \times \mathbb{C}^N \tag{2}$$

exists. This is a stabilized version of (1), and we always have $c_b^N(a) \leq c_b(a)$, by taking the product with the identity mapping. The conjecture, then, is that for a in the domain of the linear steps, we in fact have $c_b^N(a) = c_b(a)$. To make this precise, we define, for $k \in \{0, 1, 2, \ldots, \lfloor\sqrt{2b}\rfloor\}$, the numbers

$$u_b(k) = \frac{(2b+k)^2}{2b}, \qquad v_b(k) = 2b\left(\frac{2b+2k+1}{2b+k}\right)^2$$

We always have $u_b(k) < v_b(k)$; for $u_b(k) < a < v_b(k)$, the graph of $c_b(a)$ is precisely the linear steps mentioned above.

In this note, we will show:

Theorem 1.1. *Assume that*
$$u_b(k) \leq a \leq v_b(k).$$
Then
$$c_b(a) = c_b^N(a).$$

It remains an interesting question to understand the stabilized embedding problem for other values of a. For a sequence of values of a for which we can understand this problem, see Lemma 4.1 below. It is known from [5] that for $a > (\sqrt{2b}+1)^2$, there are always embeddings (2) for which the corresponding embedding (1) does not exist.

For $b = 1$, the graph of $c_b(a)$ still starts with a staircase, see [4], but it is an infinite staircase determined by the Pell numbers. It seems likely that an analogue of Theorem 1.1 holds in the $b = 1$ case as well, but this is beyond the scope of this note.

2. New capacities

We first briefly review the capacities $\mathfrak{g}_k$ defined for $k \in \mathbb{Z}_{>0}$ in [12]. These are part of a more general family of capacities $\mathfrak{g}_\mathfrak{b}$ indexed by elements in the symmetric tensor algebra $\overline{S}\mathbb{K}[t] = \bigoplus_{k=1}^{\infty}(\otimes^k \mathbb{K}[t])/\Sigma_k$. We give here only an impressionistic sketch, omitting some of the more technical details.

2.1. **The first approximation.** Suppose that X is a Liouville domain. We work with almost complex structures J on the symplectic completion $\widehat{X}$ which are admissible in the sense of symplectic field theory (SFT). Fix a point $p \in X$ along with a local J-holomorphic divisor D passing through p. To first approximation, $\mathfrak{g}_k(X)$ is simply the minimal energy of a punctured J-holomorphic sphere $u : \Sigma \to \widehat{X}$ with some number $l \geq 1$ of positive ends asymptotic to Reeb orbits in ∂X, such that u passes through p and is tangent to D to order $k-1$. We denote this tangency constraint by $\langle\!\langle \mathcal{T}^{k-1}p \rangle\!\rangle$.

To see why this should be monotone with respect to symplectic embeddings, the basic point is that given such a curve u in $\widehat{X}$ and an embedding $X' \overset{s}{\hookrightarrow} X$, we can neck-stretch along $\partial X'$. This forces u to break into a pseudoholomorphic building consisting of

- a curve u_{top} (possibly disconnected) in the completed symplectic cobordism $\widehat{X \setminus X'}$ with the same positive asymptotics as u
- a curve u_{bot} in $\widehat{X'}$ which inherits the tangency constraint $\langle\!\langle \mathcal{T}^{k-1}p \rangle\!\rangle$.

Since u_{bot} is a candidate minimizer for $\mathfrak{g}_k(X')$ and it has energy at most that of u, this shows that $\mathfrak{g}_k(X') \leq \mathfrak{g}_k(X)$.

2.2. **Behavior under stabilization.** One role of the local tangency constraint in the definition of $\mathfrak{g}_k$ is to cut down the dimension of familes of curves, thereby giving access to curves of higher Fredholm index. There are certainly other natural geometric constraints which lower the index, the most obvious being to impose k distinct point constraints. In fact, doing so leads to the "rational symplectic field theory capacities" (RSFT) first considered in [8].

However, point constraints behave in a rather complicated way under dimensional stabilization. The RSFT capacities are therefore perhaps not well-suited for stabilized problems (although they may have other applications yet to be discovered). For example, note that each point constraint is codimension 2 when $\dim X = 4$, but is generally codimension $2n-2$ when $\dim X = 2n$. This means that the same curve with the same point constraints has negative total index after stabilizing.

By contrast, local tangency constraints behave quite well with respect to stabilization, and this is easy to check at least on the level of Fredholm indices. This is closely related to the observation of Hind and Kerman from [6] (see also [2, 3, 9]) that punctured rational curves with exactly one negative end have stable Fredholm index. More precisely, let X be any Liouville domain and let $B^2(S)$ denote the two-ball of area S. We have the following stabilization theorem:

Theorem 2.1. [12, §6.2] *For all $k \in \mathbb{Z}_{>0}$ we have $\mathfrak{g}_k(X \times B^2(S)) = \mathfrak{g}_k(X)$, provided that $S > \mathfrak{g}_k(X)$. The same holds for the capacities $\mathfrak{g}_\mathfrak{b}$.*

2.3. **The naive chain complex.** Unfortunately, the definition given in §2.1 is not particularly robust, since it might depend on the choice of almost complex structure J. Indeed, if we try to deform J to some other almost complex structure J', somewhere along the way the curve u might degenerate into a pseudoholomorphic building and then disappear. Therefore, in order to get something which is truly a symplectomorphism invariant, we have to be a bit more "homological". This is where the chain complexes coming from Floer theory or symplectic field theory become essential.

The idea is to associate to X a filtered chain complex $C(X)$, where

- as a vector space, $C(X)$ is the (graded) polynomial algebra on the (not necessarily primitive) Reeb orbits of ∂X
- the differential is defined by counting rigid-up-to-translation connected rational curves in $\mathbb{R} \times \partial X$ with several positive ends and one negative end
- the filtration is by the symplectic action functional, or equivalently by the periods of Reeb orbits.

Similarly, given an exact[1] symplectic cobordism W with positive end $\partial^+ W = \partial X$ and negative end $\partial^- W = \partial X'$, we define a chain map from $C(X)$ to $C(X')$ by counting rigid possibly disconnected rational curves in W, such that each component has several positive ends and one negative end.

By Stokes' theorem, both the differential and the cobordism map are action-nondecreasing and hence preserve the filtrations.

However, the above prescription does not work on face value due to transversality issues. Namely, in order to show that the differential squares to zero and that the cobordism map is a chain map, the typical strategy is to analyze analogous moduli spaces of dimension one and show that (after compactifying) their boundaries give precisely the desired relations. But it is well-known that the relevant SFT moduli spaces are rarely transversely cut out for any choice of generic J. Multiply covered curves tend to appear with higher-than-expected dimension, and this spoils our strategy.

2.4. **Using SFT or Floer theory.** One way is get around this issue is to count curves in a "virtual" sense, by introducing suitable abstract perturbations which allow more room to achieve transversality. This is the basic strategy being pursued to define SFT in full generality by various groups, with much recent progress but consensus not yet achieved.

[1]There is also a nice story extending the theory to non-exact symplectic cobordisms, but we will ignore this for simplicity.

Another approach is to use Floer theory. In Floer theory we introduce Hamiltonian perturbations by adding inhomogeneous terms to all Cauchy–Riemann equations. The geometry of Floer theory is somewhat different from SFT, since the curves of the latter are always unbounded in the target space whereas the curves of the former lie in compact regions of the target space. In particular, the relevant compactness theorem in Floer theory is simpler than that of SFT, and it turns out that we can achieve Floer-theoretic transversality (at least in the exact setting) using only "classical" perturbation methods.

In the setting of SFT, the desired invariant $C(X)$ can be written as $\mathcal{B}CH_{\text{lin}}(X)$. Here $CH_{\text{lin}}(X)$ is the linearized contact homology of X, which is roughly the chain complex generated by Reeb orbits of ∂X with differential counting cylinders in the symplectization $\mathbb{R} \times \partial X$.[2] Linearized contact homology only involves curves with one positive end, but by incorporating curves with several positive ends we get an $\mathcal{L}_\infty$ structure, consisting of l-to-1 operations for all $l \geq 1$ satisfying various compatibility conditions. We can conveniently package this $\mathcal{L}_\infty$ structure into one large chain complex $\mathcal{B}CH_{\text{lin}}(X)$, the *bar complex*.

Alternatively, the counterpart of $CH_{\text{lin}}(X)$ in Floer theory is the positive S^1-equivariant symplectic cochain complex $\text{SC}_{S^1,+}(X)$. In short, $\text{SC}(X)$ is the Hamiltonian Floer homology of X for an appropriate (sequence of) Hamiltonians, $\text{SC}_+(X)$ is the result after modding out the constant orbits, and $\text{SC}_{S^1,+}(X)$ is the quotient by the S^1-action induced by loop rotation. Again, $\text{SC}_{S^1,+}(X)$ only involves curves with one positive end, but curves with several positive ends define its $\mathcal{L}_\infty$ structure. We can then rigorously define $C(X)$ to be the bar complex $\mathcal{B}\text{SC}_{S^1,+}(X)$. At least in characteristic zero, the complexes $\mathcal{B}CH_{\text{lin}}(X)$ and $\text{SC}_{S^1,+}(X)$ are equivalent.

2.5. **From spectral invariants to capacities.** Getting back to the high level viewpoint, we have a filtered chain complex $C(X)$ for each Liouville domain X, and filtration-preserving chain maps $\Phi : C(X) \to C(X')$ for any (exact) symplectic embedding $X' \overset{s}{\hookrightarrow} X$. Now for any class α in the homology of $C(X)$, define $c_\alpha(X)$ to be the minimal action of any closed element of $C(X)$ which represents α. By a simple diagram chase, we have $c_{[\Phi](\alpha)}(X') \leq c_\alpha(X)$, where $[\Phi]$ denotes the homology-level map induced by Φ.

At first glance, this construction appears to give a new family of symplectic capacities indexed by homology classes of $C(X)$. But there is still one issue, which is that we need a canonical way to reference these homology classes. Indeed, in principle the homology level map $[\Phi]$ might be quite nontrivial, so how do we know when two numbers $c_\alpha(X)$ and $c_\beta(X')$ can be compared to each other?

This is where the tangency constraints come in. The claim is that by counting possibly disconnected curves in $\widehat{X}$ with each component u_i satisfying a $\lll \mathcal{T}^{k_i-1}p \ggg$ constraint for some $k_i \in \mathbb{Z}_{>0}$, we get a chain map

$$\epsilon_X \lll \mathcal{T}^\bullet \ggg : C(X) \to \overline{S}\mathbb{K}[t].$$

For example, a term $t^3 \otimes t^2 \otimes t^5$ in $\overline{S}\mathbb{K}[t]$ corresponds to counting curves with three components which satisfy constraints $\lll \mathcal{T}^3 p \ggg$, $\lll \mathcal{T}^2 p \ggg$, and $\lll \mathcal{T}^5 p \ggg$ respectively. Moreover, these maps are natural in the sense that the composition $\epsilon_{X'} \lll \mathcal{T}^\bullet \ggg \circ \Phi$ agrees with $\epsilon_X \lll \mathcal{T}^\bullet \ggg$ up to filtered chain homotopy.

[2]More precisely, we only allow "good" Reeb orbits, and we count cylinders which are additionally "anchored" in X.

Now for any $\mathfrak{b} \in \overline{S}\mathbb{K}[t]$, we define the capacity $\mathfrak{g}_{\mathfrak{b}}(X) \in \mathbb{R}_{>0}$ by

$$\mathfrak{g}_{\mathfrak{b}}(X) := \inf\{c_\alpha(X) \;:\; [\epsilon_X \lll \mathcal{T}^\bullet \ggg](\alpha) = \mathfrak{b}\}.$$

This defines a symplectomorphism invariant which scales like symplectic area, and for any symplectic embedding $X' \overset{s}{\hookrightarrow} X$ we have $\mathfrak{g}_{\mathfrak{b}}(X') \leq \mathfrak{g}_{\mathfrak{b}}(X)$. In the case that X is Liouville deformation equivalent to a ball, one can show that $\epsilon_X \lll \mathcal{T}^\bullet \ggg$ is actually a chain homotopy equivalence, so every spectral invariant of $C(X)$ corresponds to some choice of $\mathfrak{b}$.

Finally, to define the simplified capacities $\mathfrak{g}_k$, let $\pi_1 : \overline{S}\mathbb{K}[t] \to \mathbb{K}[t]$ denote the projection to tensors of length 1 (e.g. $t^2 + t^3 \otimes t^2 \otimes t^5$ maps to t^2). We define

$$\mathfrak{g}_k(X) := \inf_{\mathfrak{b}\,:\,\pi_1(\mathfrak{b}) = t^{k-1}} \mathfrak{g}_{\mathfrak{b}}(X).$$

In essence, this means we look for the collection of Reeb orbits in ∂X of minimal action which is closed with respect to the differential of $C(X)$, and which bounds a connected rational curve in $\widehat{X}$ satisfying a $\lll \mathcal{T}^{k-1}p \ggg$ constraint (but disregarding any disconnected curves bounded by the same collection).

2.6. **The case of ellipsoids.** To get some intuition for $\mathfrak{g}_{\mathfrak{b}}(X)$, we note that when X is an irrational ellipsoid $E(a_1, \ldots, a_n)$, the differential on $C(X)$ vanishes for degree parity reasons. This means that $C(X)$ already agrees with its homology, and the map

$$\epsilon_X \lll \mathcal{T}^\bullet \ggg : C(X) \to \overline{S}\mathbb{K}[t]$$

is in fact an isomorphism. Then $\mathfrak{g}_{\mathfrak{b}}(X)$ is simply the action of the unique element $(\epsilon_X \lll \mathcal{T}^\bullet \ggg)^{-1}(\mathfrak{b}) \in C(X)$ which corresponds to $\mathfrak{b}$. However, recall that the map $\epsilon_X \lll \mathcal{T}^\bullet \ggg$ is defined by counting curves in $E(a_1, \ldots, a_n)$ satisfying local tangency constraints, so it could be quite nontrivial even in the case $n = 2$! Indeed, in the very special case of the nearly round ball $E(1, 1 + \epsilon)$, a closely related problem is to count rational curves in $\mathbb{CP}^2$ satisfying local tangency constraints, which was recently solved in [10]. For other ellipsoids, including those in higher dimensions, and for more general Liouville domains, computing $\mathfrak{g}_{\mathfrak{b}}$ seems to involve some very interesting and challenging enumerative problems.

3. Some computations

The following computations for the capacities of ellipsoids and polydisks are given in [12, §6.3]:

$$\mathfrak{g}_k(P(1,a)) = \min(k, a + \lceil \tfrac{k-1}{2} \rceil) \qquad \text{for } a \geq 1,\ k \geq 1 \text{ odd} \tag{3}$$

$$\mathfrak{g}_k(E(1,a)) = k \qquad \text{for } a \geq 1,\ 1 \leq k \leq a. \tag{4}$$

We will see below that this is enough to deduce Theorem 1.1.

It seems plausible that the computation for $P(1,a)$ is also valid for k even. This would follow if we knew that the capacities $\mathfrak{g}_k$ are nondecreasing with k, although this is not yet clear. While it is not needed for the proof below, let us also mention the following more general expected formula for ellipsoids, which will be treated in [11]. For $1 \leq a \leq 3/2$, we have

$$g_k(E(1,a)) = \begin{cases} 1+ia & \text{for } k = 1+3i \text{ with } i \geq 0 \\ a+ia & \text{for } k = 2+3i \text{ with } i \geq 0 \\ 2+ia & \text{for } k = 3+3i \text{ with } i \geq 0. \end{cases} \tag{5}$$

For $a > 3/2$, we have

$$g_k(E(1,a)) = \begin{cases} k & \text{for } 1 \leq k \leq \lfloor a \rfloor \\ a+i & \text{for } k = \lceil a \rceil + 2i \text{ with } i \geq 0 \\ \lceil a \rceil + i & \text{for } k = \lceil a \rceil + 2i + 1 \text{ with } i \geq 0. \end{cases} \tag{6}$$

4. Proof of the main theorem

The key to proving Theorem 1.1 is the following lemma.

Lemma 4.1. *Let b and N be any positive integers, and let $a = 2b + 2i + 1$, where i is any nonnegative integer. Then there is an embedding*

$$E(1,a) \times \mathbb{C}^N \overset{s}{\hookrightarrow} \lambda \cdot P(1,b) \times \mathbb{C}^N \tag{7}$$

if and only if

$$\lambda \geq \frac{2b+2i+1}{2b+i}. \tag{8}$$

Proof. The existence of an embedding as guaranteed by the lemma is shown by Hind's folding construction [5], as explained in [1].

To show that no better embedding exists, we use the above capacities. Namely, let $k = a$. Then, by (3) and (4) above, we have

$$\mathfrak{g}_k(E(1,a) \times \mathbb{C}^n) = 2b + 2i + 1, \qquad \mathfrak{g}_k(P(1,b) \times \mathbb{C}^n) = 2b + i.$$

Thus, if an embedding (7) exists, then we must have

$$2b + 2i + 1 \leq \lambda(2b + i).$$

Rearranging this gives (8). □

We can now give the proof of Theorem 1.1

Proof. As mentioned above, $c_b^N(a)$ is nonincreasing in N. We want to show that it is in fact constant in N for a in the intervals given by the theorem. The computation of $c_b^0(a)$ from [1], together with Lemma 4.1 from above, shows that it does not depend on N for the corners of each linear step.

Now note that if an embedding

$$E(1,a) \times \mathbb{C}^n \overset{s}{\hookrightarrow} \lambda P(1,b) \times \mathbb{C}^n$$

exists, then for any $a' > a$, by scaling there is an embedding

$$E(1,a') \times \mathbb{C}^n \overset{s}{\hookrightarrow} \frac{a'}{a} \lambda P(1,b) \times \mathbb{C}^n.$$

Thus, $c_b^N(a') \leq \frac{a'}{a} c_b^N(a)$. So, given $y_0 = c^N(a)$, the graph of $c^N(a')$ for $a' > a$ can not lie above the line through (a, y_0) and the origin.

We can now prove the theorem.

Consider any linear step for $c_b^0(a)$. Recall that this consists of a linear part, and then a horizontal part. Consider the linear part. We want to show that this stabilizes. We know that $c_b^N(a) \leq c_b(a)$. If there were any a value corresponding to the first step for which strict inequality held, then by the linearity property above, at the corner a_0 of the step, we would have $c_b^N(a_0) < c_b(a_0)$. However, above we saw that the corner is stable. Hence, the whole linear part must stabilize. As for the horizontal part, we know that we must have $c_b^N \leq c_b$, but on the other hand the function c^N is nondecreasing, and so must be constant here. Thus, the whole step stabilizes, so all the linear steps do.

□

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