Math 635

USC Spring 2026

Cluster Varieties:
Algebra, Topology, Geometry, Duality

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Handwritten Lecture Notes

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Math 635, USC Spring 2026

Cluster varieties: algebra, topology, geometry, duality

Lecture 1

1/12/26

Roughly speaking:

- a cluster variety is a complex algebraic variety obtained by gluing together many copies of $(\mathbb{C}^*)^n$, where the gluing maps take a very particular form
- a cluster algebra is the algebra of regular functions $f: V \longrightarrow \mathbb{C}$ on a cluster variety

Fomin–Zelevinsky, early '00s: introduced cluster algebras

Arise in many parts of math and physics as kind of "universal model" for mutation/wall-crossing phenomena:

- quiver representation theory
- ~~Teichmüll~~ Teichmüller theory
- Poisson geometry
- Grassmannians
- total positivity
- QFT scattering amplitudes (~~amplitudre~~ amplituhedron)
- integrable systems
- string theory (BPS states),

etc

Gross–Hacking–Keel–Kontsevich '18:

- constructed canonical bases for cluster algebras
- established ~~positivity of the~~ Positive Laurent phenomenon
- proof uses mirror symmetry for log Calabi-Yau varieties

can think of as generalization of toric varieties (related to almost toric fibrations in symplectic geometry)

- many strong applications in representation theory, e.g. canonical bases for finite-dimensional irreducible representations of $SL_n(\mathbb{C})$

originally found independently by Lusztig and Kashiwara in early '90s using quantum groups

amazingly, the construction of GHKK uses only general geometry – no rep. thry!

Total positivity

Def: $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ is totally positive (TP) is all of its minors are positive.

Gantmacher–Krein '30's: A TP $\Rightarrow$ eigenvalues are real, positive, and distinct

Binet–Cauchy theorem: The TP matrices in $G = SL_n(\mathbb{C})$ ~~$\mathbb{C}$~~ ~~$\mathbb{R}$~~ are closed under multiplication, and hence form a multiplicative semigroup $G_{>0}$.

Lusztig: Extended definition of $G_{>0}$ for other semisimple Lie groups G.

More generally: If a given complex algebraic variety Z has a distinguished family Δ of regular functions $Z \to \mathbb{C}$, we define the TP variety by

$$Z_{>0} := \{ z \in Z \mid f(z) > 0 \quad \forall f \in \Delta \}$$

Ex: For $Z = \mathrm{Mat}_{n\times n}(\mathbb{C}), GL_n(\mathbb{C}), SL_n(\mathbb{C})$, $\Delta =$ minors, recover above notion of TP

Ex: Grassmannian $Gr_{k,m}(\mathbb{C}) = \{k\text{-dim'l linear subspaces of } \mathbb{C}^m\}$

$\Delta =$ Plücker coordinates ← slight scaling ambiguity

Ex: partial flag manifolds, homogeneous spaces for semisimple complex Lie groups, etc

Lemma: $A \in \mathrm{Mat}_{n\times n}$ has $\binom{2n}{n} - 1$ minors

pf: $\# = \sum_{k=1}^{n} \binom{n}{k}\binom{n}{k}$

Vandermonde's identity: $\binom{m+w}{r} = \sum_{k=0}^{n} \binom{m}{k}\binom{w}{r-k}$ (both sides count: given committee with m men, w women, how many subcommittees with r members?)

Setting $m=w=r=n \rightsquigarrow \binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{k}$

Q: Can we check that $A \in \mathrm{Mat}_{n\times n}$ is TP by only testing a subset of the $\binom{2n}{n}-1$ minors? How many tests are needed? ← i.e. want "efficient TP testing"

Ex: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_{2\times 2}$

$\delta := ad - bc \Rightarrow d = \dfrac{\delta + bc}{a}$.

So if $a, b, c, \delta > 0$, so is d.

Reduce $\binom{4}{2} - 1 = 5$ checks to 4 checks.

Plücker coordinates on Grassmannians:

Given $A \in \mathrm{Mat}_{k\times m}$ of rank k $\rightsquigarrow$ row span $[A] \in Gr_{k,m}$

For $J \subset \{1,\dots,m\}$, $|J| = k$ $\rightsquigarrow$ Plücker coordinates $P_J(A) :=$ $k\times k$ minor of A corresponding to J

Note: For $A, B \in \mathrm{Mat}_{k\times m}$ with $[A] = [B]$ (i.e. same row spans) $(P_J(A))_{|J|=k}$ and $(P_J(B))_{|J|=k}$ agree up to common rescaling, i.e. get

$$Gr_{k,m} \longrightarrow \mathbb{CP}^N \quad \text{for } N = \binom{m}{k} - 1.$$

In fact this is an embedding, the <u>Plücker embedding</u>.

Let $\mathbb{C}[\mathrm{Mat}_{k\times m}]$ = coord. ring of $\mathrm{Mat}_{k\times m}$, i.e. the polynomial algebra in variables x_{ij} for $1 \le i \le k$, $1 \le j \le m$

Def: The Plücker ring $R_{k,m}$ is the subring of $\mathbb{C}[\mathrm{Mat}_{k\times m}]$ generated by P_J over $J \in \{1,\dots,m\}$, $|J| = k$.

Claim: the ideal of relations in $R_{k,m}$ is gen'd by certain quadratic relations called the Grassmann–Plücker relations.

Def: The totally positive Grassmannian $Gr^+_{k,m}$ is the subset of $Gr_{k,m}$ of those pts whose Plücker coords are all positive (up to common scaling).

Note: For $A \in \mathrm{Mat}_{k\times m}(\mathbb{R})$, $[A] \in Gr^+_{k,m}$ iff all $k\times k$ minors of A have the same sign. ← positive WLOG

Q: For $A \in \mathrm{Mat}_{k\times m}(\mathbb{R})$, can we verify that all $k\times k$ minors are positive by only checking a subset of the $\binom{m}{k}$ minors? How many tests are needed?

Positivity testing for $Gr_{2,m}$

Claim: Given $A \in Mat_{2\times m}$, put $P_{ij} := P_{\{i,j\}}$ for $1\le i<j\le m$. To check that all 2×2 minors $P_{ij}(A) > 0$, suffices to check only $2m-3$ special ones.

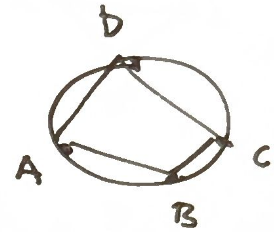
Note: $2m-3 = \dim Gr_{2,m} + 1$

Lemma: For $1\le i<j<k<l\le m$, have three-term Grassmann-Plücker relations:

$$P_{ik}P_{jl} = P_{ij}P_{kl} + P_{il}P_{jk}$$

Rmk: For inscribed quadrilateral Ptolemy's thm (2nd century) gives

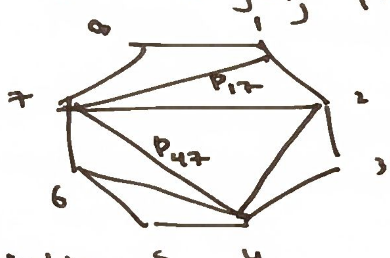
AC · BD = AB · CD + BC · AD



Ex: $A = \begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix}$ wts $P_{13}P_{24} = P_{12}P_{34} + P_{14}P_{23}$, i.e.

$$(ag-ce)(bh-df) = (af-be)(ch-dg) + (ah-de)(bg-cf) \checkmark$$

Put P_m = regular m-gon, T = triangulation.



To each side or diagonal associate P_{ij}, where i,j are the end pts

Cluster variables: P_{ij} ranging over diagonals

frozen variables: P_{ij} ranging over sides

extended cluster: $\{\text{cluster vars}\} \cup \{\text{frozen vars}\} =: \tilde{x}(T)$

Note: extended cluster has $2m-3$ vars, and we claim that these are algebraically independent.

Ex: In above picture, have cluster variables $P_{17}, P_{27}, P_{47}, P_{46}, P_{24}$
frozen variables $P_{12}, P_{23}, \dots, P_{78}, P_{18}$

Thm Each P_{ij} for $1\le i<j\le n$ can be written as a subtraction-free rational expression in the elements of a given extended cluster $\tilde{x}(T)$.

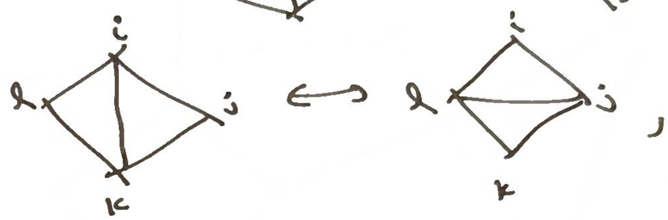
Cor: ~~In particular~~ If each ~~variable in~~ $\underbrace{P_{ij}\in\tilde{x}(T)}_{2m-3 \text{ of these}}$ evaluates positively on given $A \in Mat_{2\times m}$, then all of the ~~...~~ 2×2 minors of A ($\binom{m}{2}$ of these) are positive.

<u>Pf of thm</u>: Follows by combining

(1) each P_{ij} appears as an elt of an extended cluster $\tilde{x}(T)$ for some triangulation T of $\mathbb{P}_m$

(2) any two triangulations of $\mathbb{P}_m$ are related by a sequence of <u>flips</u>

(3) For a flip



replace P_{ik} with P_{lj}.

Using three-term GP relation, have $P_{ik} = \dfrac{P_{ij}P_{lk} + P_{il}P_{jk}}{P_{jl}}$

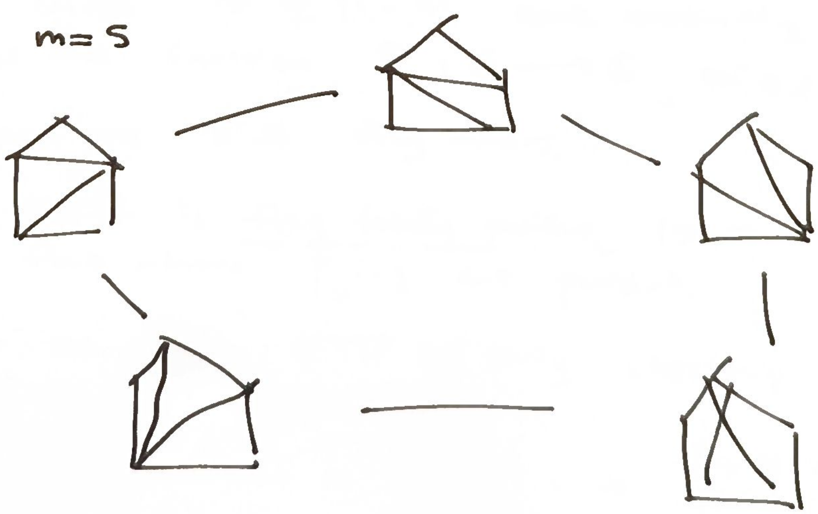
<u>Rmk</u>: In fact, each Plücker coordinate P_{ij} can be written as a Laurent polynomial with positive coefficients in the Plücker coordinates from $\tilde{x}(T)$. Example of ~~positivity~~ positive Laurent phenomenon.

Combinatorics of flips encoded by graph:

- vertices are ~~tri~~ triangulations
- edges are flips

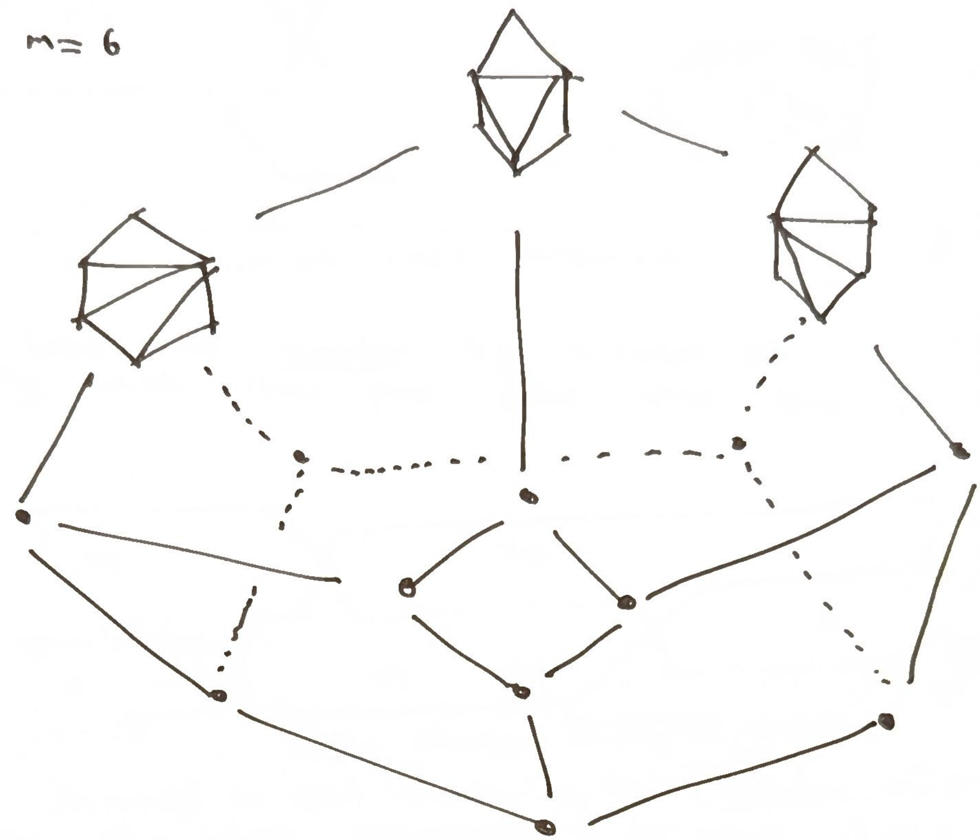
Each vertex has degree $m-3$. In fact, this is the 1-skeleton of an $(m-3)$-dim'l convex polytope called the associahedron (discovered by Stasheff).

<u>Ex</u>: $m=5$



Wiring diagrams :

Ex: $m = 6$



Def: A cluster monomial is a monomial in the variables of a given extended cluster $\tilde{x}(T)$.

Thm (19th century invariant theory): The set of all cluster monomials give a linear basis for the Plücker ring $R_{2,m}$.

Lecture 2

1/14/26

Before moving to TP for $n \times n$ matrices, we discuss an intermediate notion called "flag positivity". Put $G = SL_n$.

Def. Given $J \subsetneq \{1, \dots, n\}$ ~~none~~ nonempty, the flag minor P_J is the function $P_J : G \to \mathbb{C}$, $z = (z_{ij}) \mapsto \det\left(z_{ij} \,\middle|\, \begin{smallmatrix} i \le |J| \\ j \in J \end{smallmatrix}\right)$

$|J| \times |J|$ minor which is "top-justified"

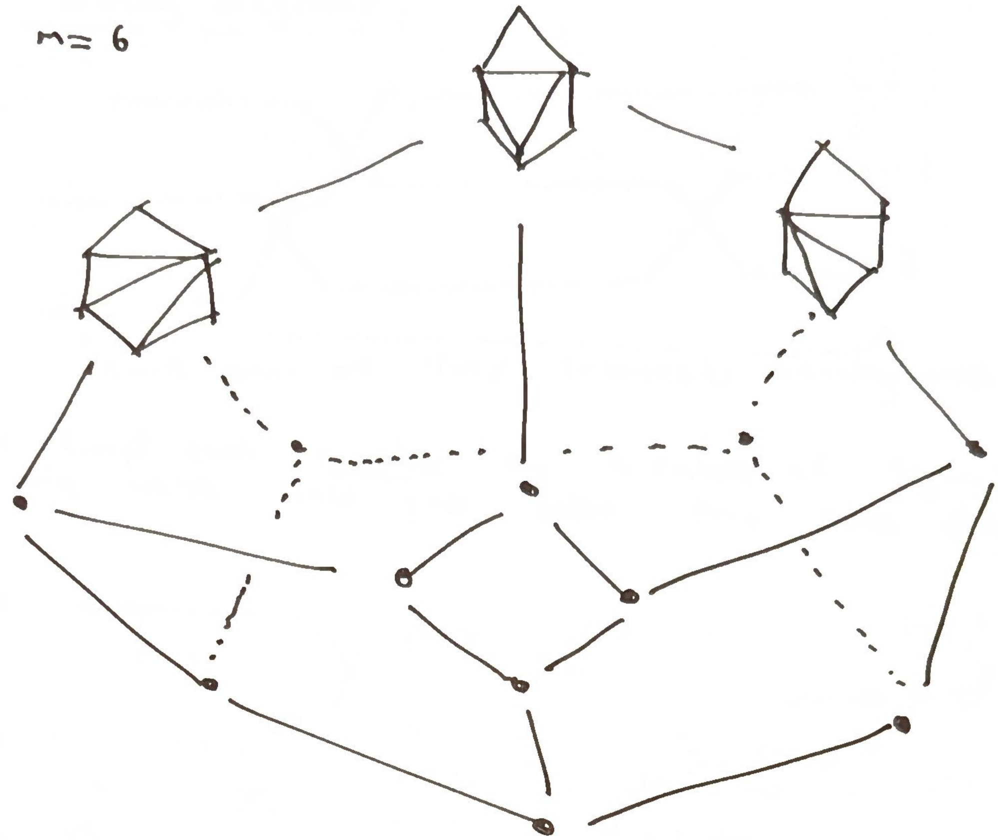
Note: there are $2^n - 2$ flag minors.

Def: $z \in G$ is flag totally positive (FTP) is all flag minors $P_J(z)$ are positive.

Q.: Can we check FTP by only checking a subset of the $2^n - 2$ flag minors.

Claim: It suffices to check only $\frac{(n-1)(n+2)}{2}$ special flag minors.

Ex: $n = 6$



Def: A cluster monomial is a monomial in the variables of a given extended cluster $\tilde{x}(T)$.

Thm (19th century invariant theory): The set of all cluster monomials give a linear basis for the Plücker ring $R_{2,n}$.

Lecture 2

1/14/26

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$|J| \times |J|$ minor which is "top-justified"

Note: there are $2^n - 2$ flag minors.

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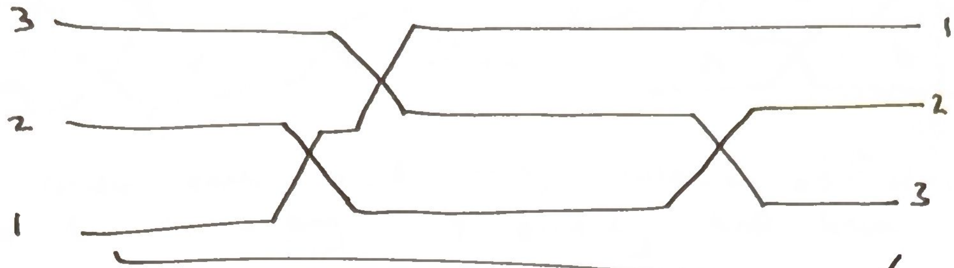
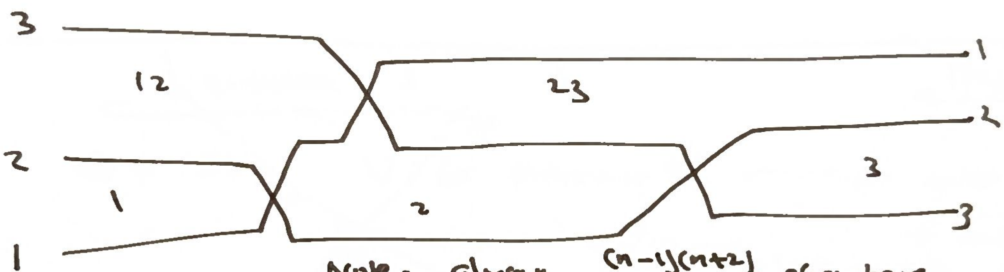
Wiring diagrams:



(case $n=3$)

each pair of lines intersects exactly once

We label each chamber by a subset of $\{1,\dots,n\}$ indicating which lines pass below that ~~some~~ chamber



Note: always $\frac{(n-1)(n+2)}{2}$ chambers

Associated to each chamber is its chamber minor P_J, the flag minor corresponding to its subset $J \subseteq \{1,\dots,n\}$.

extended cluster: all chamber minors of a wiring diagram

~~be~~ cluster variables: the chamber minors for bounded chambers

frozen variables: the chamber minors for unbounded chambers

$2n-2$ of these

$\binom{n-1}{2}$ of these

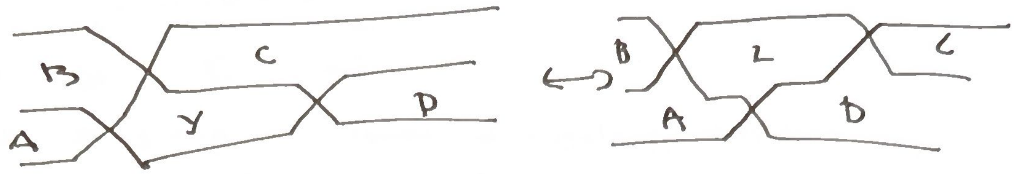
Thm Every flag minor can be written as a subtraction-free rat'l expr in the chamber minors of a given wiring diag.

Cor: If these $\frac{(n-1)(n+2)}{2}$ evaluate positively at a matrix $z \in SL_n$, then z is FTP.

Pf: Follows by

(1) each flag minor appears as a chamber minor in some wiring diagram

(2) any two wiring diagrams can be transformed into each other by a sequence of local braid moves



(3) Under each braid move, collection of chamber minors changes by exchanging $Y \leftrightarrow Z$, and have

$$YZ = AC + BD$$

Rmk: In fact, each flag minor can be written as a Laurent poly with pos. coeffs in the chamber minors of a given ~~wi~~ wiring diagram.

Lecture 3 ~~1/22/26~~

Put $G = SL_n$, $U \subset G$ subgroup of unipotent lower-triangulars
i.e. lower triangular matrices with 1's on diagonal

$U \circlearrowright G$ left multiplication action

$\rightsquigarrow U \circlearrowright \mathbb{C}[G]$ = ring of polynomials in the entries of the matrix $A \in G$

$\mathbb{C}[G]^U$ = ring of U-invariant polynomials
i.e. $P \in \mathbb{C}[G]$ s.t. $P(yz) = P(z)$ $\forall\, y \in U,\ z \in G$

Note: $\begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a + \beta c & \alpha b + \beta d \\ \gamma c & \gamma d \end{pmatrix}$

i.e. $\begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}\begin{pmatrix} - v_1 - \\ - v_2 - \end{pmatrix} = \begin{pmatrix} - \alpha v_1 + \beta v_2 - \\ - \gamma v_2 - \end{pmatrix}$

Similarly, $\begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \delta & \varepsilon \\ 0 & 0 & \eta \end{pmatrix}\begin{pmatrix} - v_1 - \\ - v_2 - \\ - v_3 - \end{pmatrix} = \begin{pmatrix} - \alpha v_1 + \beta v_2 + \gamma v_3 - \\ - \delta v_2 + \varepsilon v_3 - \\ - \eta v_3 - \end{pmatrix}$, etc.

Def: The full flag variety in $\mathbb{C}^n$ is

$$\left\{ \{0\} \subset V_1 \subset V_2 \subset \dots \subset V_{n-1} \subset \mathbb{C}^n \;\middle|\; V_i \text{ is a subspace of dimension } i \text{ for } i = 1, \dots, n-1 \right\}$$

This can be identified with the homogeneous space G/B, where $B \subset G$ is the subgrou

Lecture 3

1/23/26

Put $G = SL_n(\mathbb{C})$

$B \subset G$ subgroup of lower triangular matrices ← Borel subgroup

$U \subset G$ subgroup of $\underbrace{\text{unipotent}}_{\text{i.e. 1's on diagonal}}$ lower triangular matrices

Note: $\begin{pmatrix} \alpha & 0 \\ \beta & \gamma \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha b \\ \beta a + \gamma c & \beta b + \gamma d \end{pmatrix}$, i.e.

$$\begin{pmatrix} \alpha & 0 \\ \beta & \gamma \end{pmatrix}\begin{pmatrix} - v_1 - \\ - v_2 - \end{pmatrix} = \begin{pmatrix} - \alpha v_1 - \\ - \beta v_1 + \gamma v_2 - \end{pmatrix}$$

Similarly,

$$\begin{pmatrix} \alpha & 0 & 0 \\ \beta & \gamma & 0 \\ \delta & \varepsilon & \eta \end{pmatrix}\begin{pmatrix} - v_1 - \\ - v_2 - \\ - v_3 - \end{pmatrix} = \begin{pmatrix} - \alpha v_1 - \\ - \beta v_1 + \gamma v_2 - \\ - \delta v_1 + \varepsilon v_2 + \eta v_3 - \end{pmatrix} \quad \text{etc}$$

Def: The __(full) flag variety__ in $\mathbb{C}^n$ is
$\{ \{0\} \subset V_1 \subset V_2 \subset \dots \subset V_{n-1} \subset \mathbb{C}^n \mid V_i \text{ is an } i\text{-dimensional subspace for } i = 1, \dots, n-1 \}$

Exercise. This is identified with the homogeneous space $B \backslash G$

Def: The __basic affine space__ is $U \backslash G$

Note that we have $(\mathbb{C}^*)^n \hookrightarrow U\backslash G \twoheadrightarrow B\backslash G$, i.e.
the basic affine space is a torus bundle over the flag variety

Here $U \curvearrowright G$ action by left multiplication
$\rightsquigarrow U \curvearrowright \mathbb{C}[G]$ = ring of polynomials in the entries of $A \in SL_n$

$\mathbb{C}[G]^U$ = ring of U-invariant polynomials ← i.e. $P \in \mathbb{C}[G]$ s.t. $P(yz) = P(z)$ $\forall y \in U, z \in G$

Claim: ← by First and Second Fundamental Theorems of invariant theory

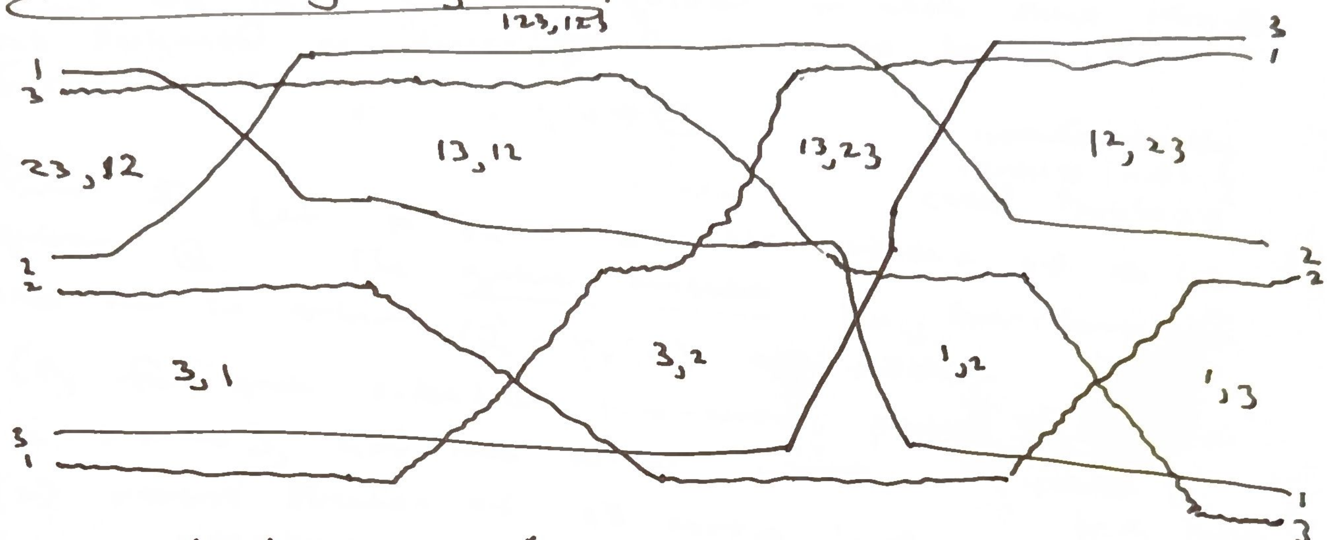
(1) the flag minors generate $\mathbb{C}[G]^U$

(2) the ideal of relations among flag minors is generated by quadratic relations called "generalized Plücker relations"

Checking TP for general $n\times n$ matrices

Given $I, J \subset \{1,\dots,n\}$ of same cardinality, put $\Delta_{I,J}$:= minor determined by rows in I and columns in J.
Thus $z \in \mathrm{Mat}_{n\times n}$ is TP $\iff$ $\Delta_{I,J}(z) > 0$ for all $I, J \subset \{1,\dots,n\}$ with $|I| = |J|$

Double wiring diagrams:



$\rightsquigarrow$ chamber minors $\Delta_{3,1}, \Delta_{3,2}, \Delta_{1,2}, \Delta_{1,3}, \Delta_{23,12}, \Delta_{13,12}$
$\Delta_{13,23}, \Delta_{12,23}, \Delta_{123,123}$

<u>Claim</u>: number of chamber minors for a double wiring diagram is always n^2

<u>Thm</u>: Every minor of an $n\times n$ matrix can be written as a subtraction-free rational expression in the chamber minors of a given double wiring diagram.

<u>Cor</u>: Only need n^2 tests for positivity.

<u>pf idea</u>:

(1) every minor is a chamber minor for some double wiring diagram

(2) any two double wiring diagrams are related by sequence of local moves of three different kind

(3) each local move results in an exchange of minors $Y \leftrightarrow Z$, where ~~$\not\equiv$~~ $YZ = AC + BD$.

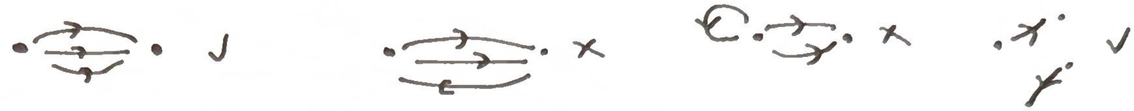
<u>Rmk</u>: In fact in thm we really have Laurent polynomials with positive coefficients.

<u>Rmk</u>: The graph with vertices double wiring diagrams and edges local moves is not regular, but this will be rectified by the theory of cluster algebras.

Quivers and their mutations

Def: A *quiver* is a finite oriented graph with no loops or oriented 2-cycles.

Ex:



Def: An *ice quiver* is a quiver in which some vertices are designated as "frozen", and no arrows between two frozen vertices.

Ex:

non-frozen vertices will be called "mutable"

Def: Let k be a mutable vertex of an ice quiver Q. The *quiver mutation* μ_k transforms Q into new ice quiver $Q' = \mu_k(Q)$ as follows:

(1) for each oriented two-arrow path $i \to k \to j$, add new arrow $i \to j$ (unless i, j both frozen)

(2) reverse direction of all arrows incident to k

(3) repeatedly remove any oriented 2-cycles until none left

Ex:

Exercise:

(1) mutation is an involution, i.e. $\mu_k(\mu_k(Q)) = Q$

(2) mutation commutes with reversing orientations of all arrows

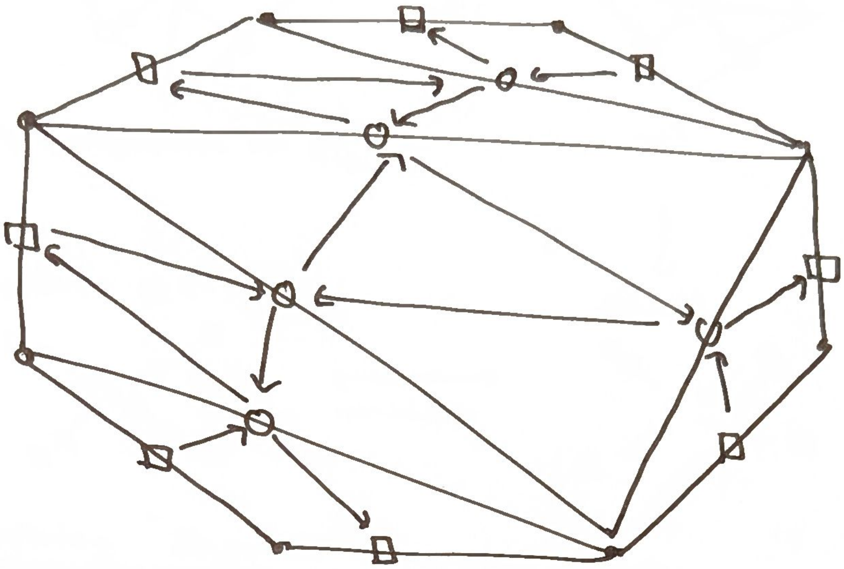
(3) if k, l mutable vertices with no arrows between them, then $\mu_l(\mu_k(Q)) = \mu_k(\mu_l(Q))$

Rmk: If k sink or source, μ_k simply reverses all arrows incident to k.

Exercise: If Q is a tree with no frozens, can get from any orientation to any other by a sequence of mutations at sinks and sources.

Triangulation and quiver

Can define a quiver from a ~~triangul~~ triangulation T of P_m.



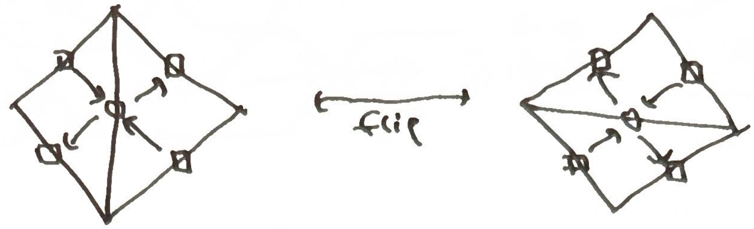
$Q(T)$

Exercise: If T is a triangulation of P_m and T' obtained by flip along diagonal γ, then
$Q(T') = \mu_\gamma(Q(T))$

Lecture #4

1/26/26

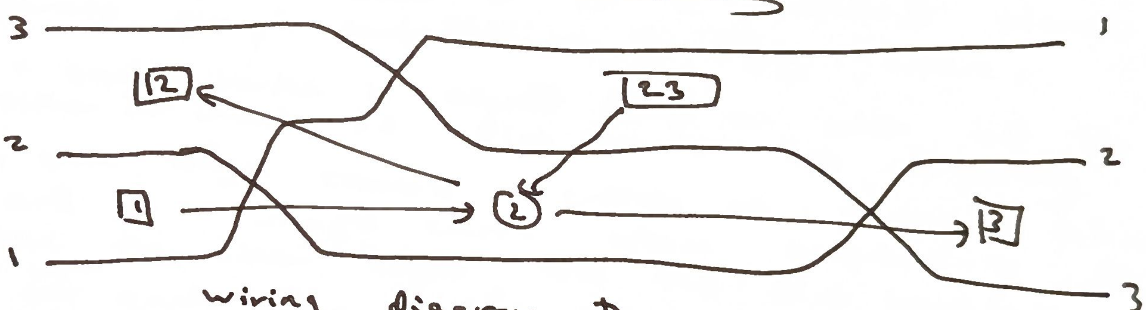
Ex:



T = triangulation of $\mathbb{P}_4$ $\qquad$ T'

quiver $Q(T)$ $\qquad$ $Q(T')$

wiring diagram ⇝ quiver



wiring diagram D

quiver $Q(D)$

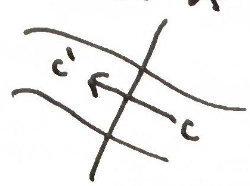
Vertices: chambers of D (mutable if bounded, else frozen)

arrows: for chambers c, c', have $c \to c'$ in $Q(D)$ iff one of following holds.

(i) right end of c = left end of c'

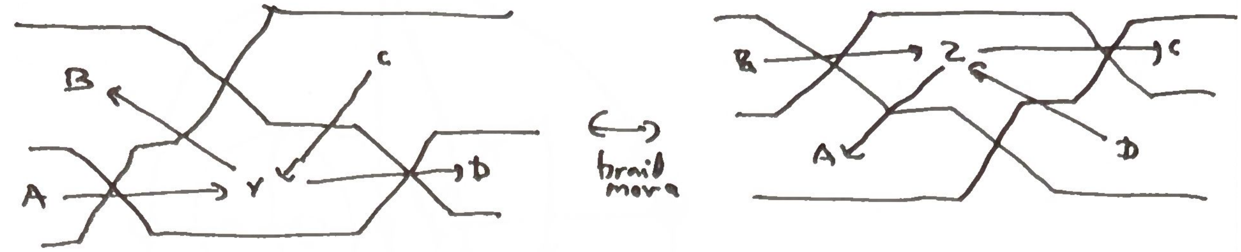
(ii) left end of c is directly above c', right end of c' is directly below c

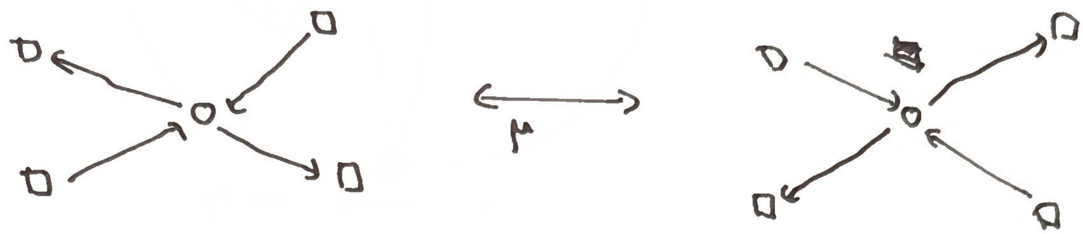
(iii) left end of c is directly below c', right end of c' directly above c,



Exercise: If D, D' wiring diagrams related by a braid move at chamber Y, then $Q(D') = \mu_Y(Q(D))$.

Ex:



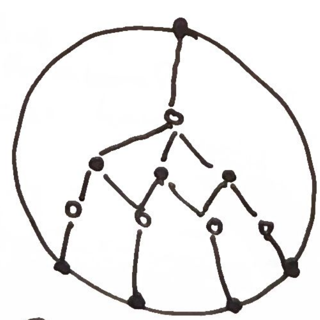


Rmk: Also have double wiring diagram D ⟿ quiver $Q(D)$

Description is more complicated, but special case of quiver associated to a planar bipartite graph.

Def: A plabic graph G is a connected planar bipartite graph embedded in a disk, where:

- each vertex is colored black or white ~~and~~ lies either in interior of disk or on its boundary
- each edge connects vertices of different colors and is a simple curve whose interior is disjoint from the other edges and the disk boundary
- for each face (connected cpnt of complement), the closure is simply connected
- each ~~internal~~ interior vertex has degree ≥ 2
- each boundary vertex has degree 1

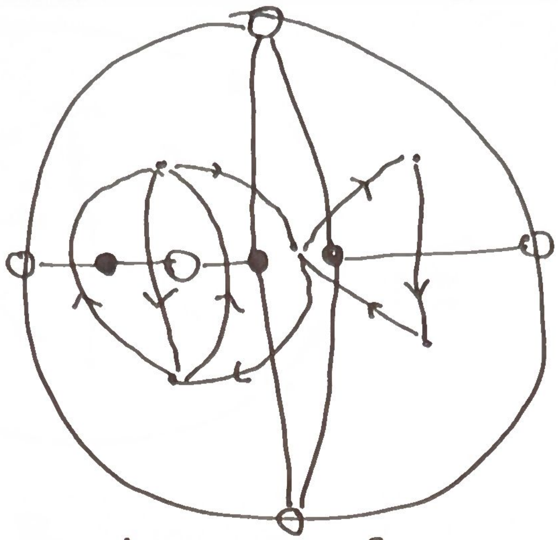


Note: we consider plabic graphs up to isotopy.

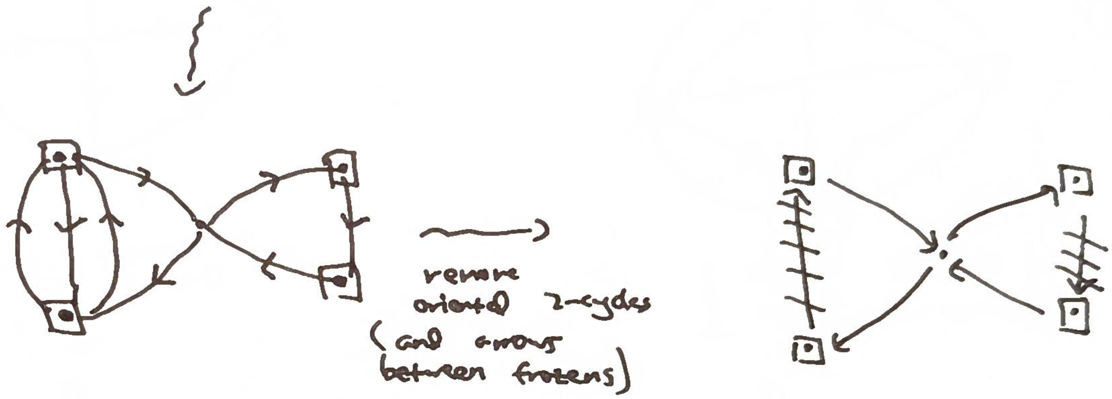
plabic graph G ⟿ quiver $Q(G)$

- vertices are faces of G (frozen if incident to disk boundary, else mutable)
- for each edge of G, have arrow joining the two faces it separates, using rule
- remove oriented 2-cycles

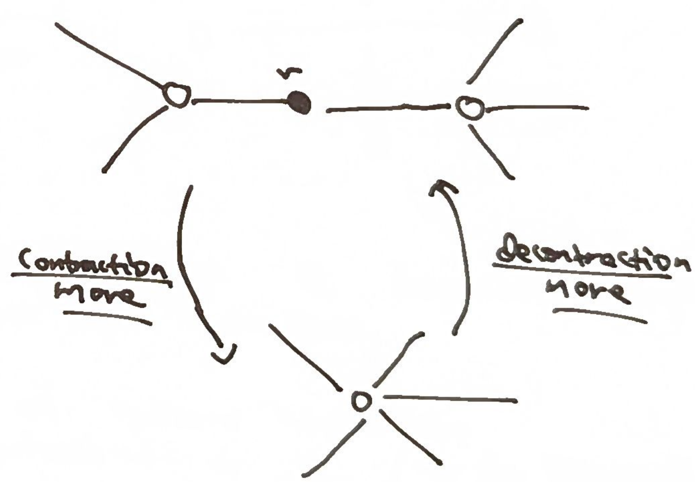
Ex:



plabic graph G

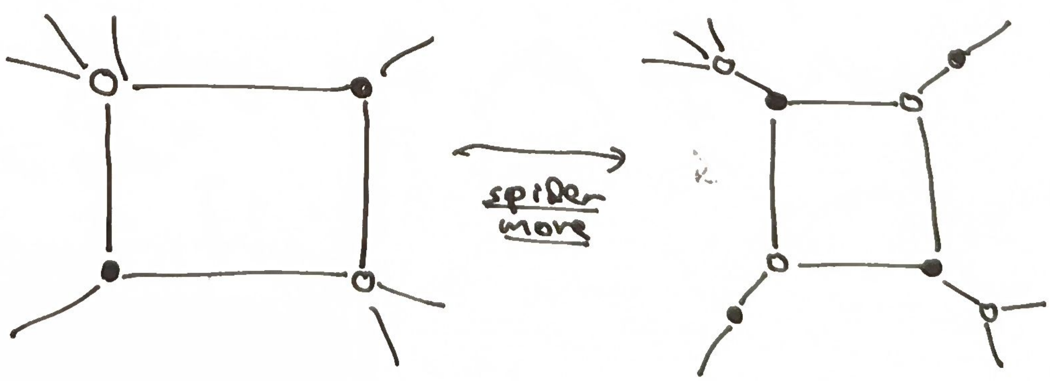


Def: Say v bivalent vertex adjacent to two interior vertices



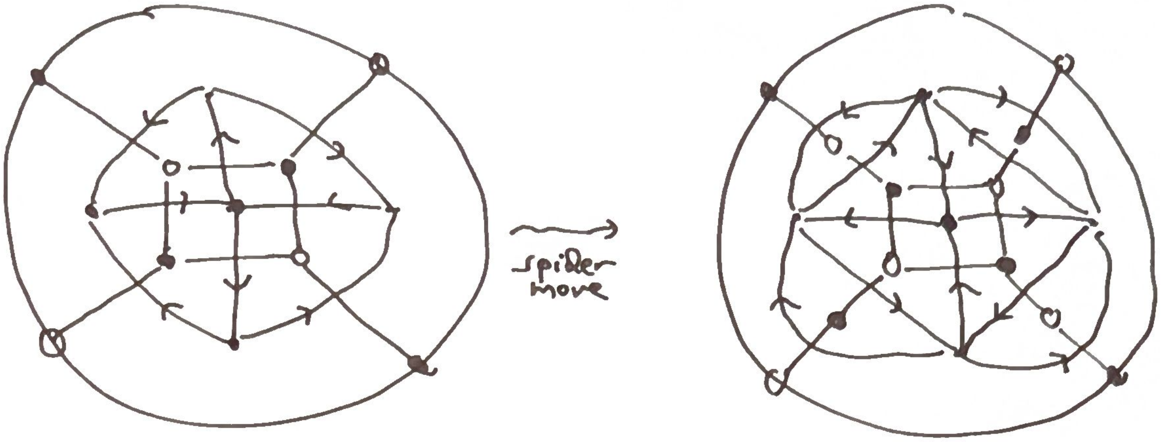
Rmk: does not change associated quiver

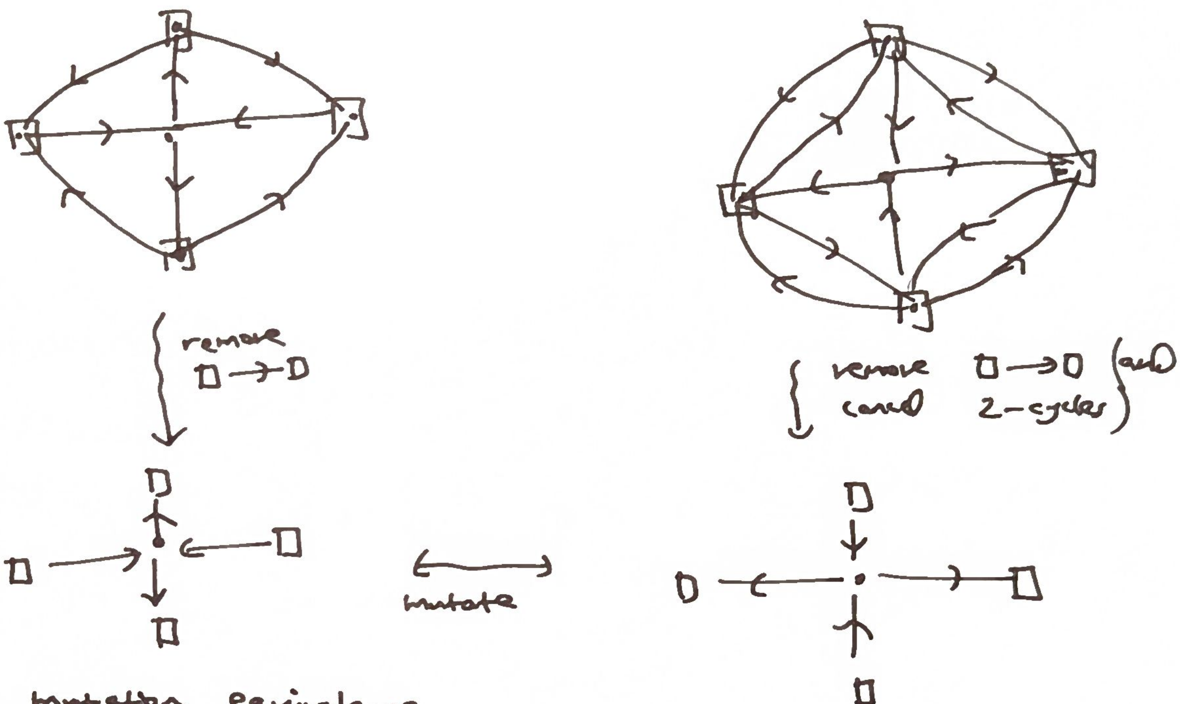
Def: Say G has a ~~quadrilateral~~ quadrilateral face whose vertices have degree ≥ 3.



Exercise: If G, G' related by spider move, then ~~$Q(G)$~~ $Q(G), Q(G')$ related by mutation

Ex



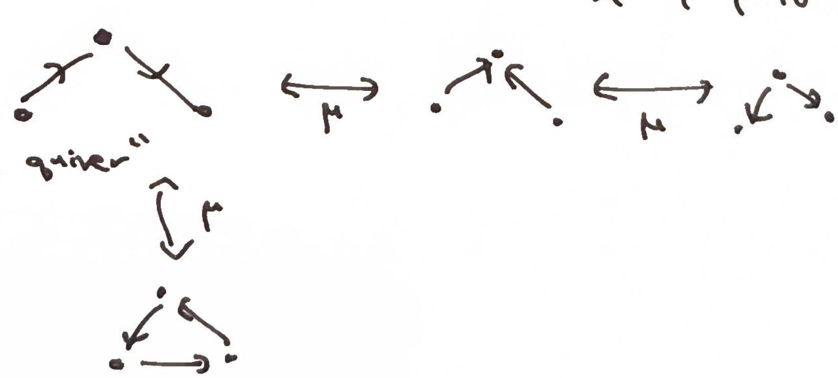


mutation equivalence

Def: Q, Q' mutation equivalent if Q becomes isomorphic to Q' after a sequence of mutations.

Put $[Q] :=$ set of ~~all~~ all quivers which are mutation equivalent to Q (up to isomorphism)

Ex: $Q =$



"A_3 quiver"

Exercise: $[Q]$ has 4 elements

Ex: $Q =$ 

"Markov quiver"

In fact, $[Q]$ is just a single element.

Def. Q has <u>finite mutation type</u> if $[Q]$ is finite.

Rmk. There is a classification theorem for quivers with no frozen vertices and finite mutation type.

Def: Q <u>acyclic</u> if no oriented cycles.

Thm (Caldero–Keller '06): If Q, Q' acyclic and mutation ~~eq~~ equivalent, then we can transform Q into Q' by a sequence of mutations at sources and sinks. In particular, Q, Q' have the same underlying undirected graphs.

Lecture 5

1/28/26

Def: Q quiver with vertices labeled by $1,\dots,m$, such that $1,\dots,n$ are the mutable ones $(n \le m)$.

The extended exchange matrix is

$$\tilde{B}(Q) = (b_{ij})_{\substack{1\le i\le m \\ 1\le j\le n}}, \quad \text{where } b_{ij} = \begin{cases} \ell & \text{if } \ell \text{ arrows } i \to j \\ -\ell & \text{if } \ell \text{ arrows } j \to i \\ 0 & \text{else} \end{cases}$$

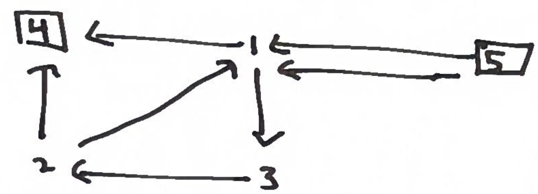
$m\times n$ matrix

The exchange matrix is the submatrix $B(Q) := (b_{ij})_{1\le i,j\le n}$

$n\times n$ skew-symmetric matrix

Ex:

Q



$$\tilde{B}(Q) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \\ -1 & -1 & 0 \\ 2 & 0 & 0 \end{pmatrix} \end{array} \qquad B(Q) = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

Ex: $Q =$

Markov quiver

$$\tilde{B}(Q) = \begin{pmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{pmatrix}$$

$$\tilde{B}(Q) = \begin{pmatrix} 0 & -2 & 2 \\ 2 & 0 & -2 \\ -2 & 2 & 0 \end{pmatrix}$$

Rmk: Reordering the vertices of Q results in simultaneously reordering the rows and columns $1,\dots,n$ and reordering the rows $n+1,\dots,m$.

<u>Lemma</u>: For quiver Q with $\widetilde{B}(Q) = (b_{ij})$ and $Q' = \mu_k(Q)$ for a mutable vertex k of Q, have $\widetilde{B}(Q') = (b'_{ij})$, with

$$(*) \quad b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + b_{ik} b_{kj} & \text{if } b_{ik}, b_{kj} > 0 \\ b_{ij} - b_{ik} b_{kj} & \text{if } b_{ik}, b_{kj} < 0 \\ b_{ij} & \text{else} \end{cases}$$

<u>Note</u>: Can replace middle two cases with
$b'_{ij} = b_{ij} + |b_{ik}| b_{kj}$ if $b_{ik} b_{kj} > 0$

<u>Ex</u>:

$$\begin{tikzcd} 1 \arrow[r, shift left] \arrow[r, shift right] & 2 \arrow[d, shift left] \arrow[d] \arrow[d, shift right] \\ & 3 \end{tikzcd} \quad \overset{\mu_2}{\rightsquigarrow} \quad \begin{tikzcd} 1 \arrow[dr, "6"'] & 2 \arrow[l, shift left] \arrow[l, shift right] \\ & 3 \arrow[u, shift left] \arrow[u] \arrow[u, shift right] \end{tikzcd}$$

$$\begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 3 \\ 0 & -3 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & -2 & 6 \\ 2 & 0 & -3 \\ -6 & 3 & 0 \end{pmatrix}$$

<u>Def</u>: An $n \times n$ matrix $B = (b_{ij}) \in \mathrm{Mat}_{n\times n}(\mathbb{Z})$ is <u>skew-symmetrizable</u> if for some $d_1, \dots, d_n \in \mathbb{Z}_{>0}$ we have $d_i b_{ij} = -d_j b_{ji}$

i.e. becomes skew-symmetric after rescaling the rows by positive integers

<u>Def</u>: An $m \times n$ matrix is <u>extended skew-symmetrizable</u> if the top $n \times n$ submatrix is skew-symmetrizable.

<u>Def</u>. For $\widetilde{B} = (b_{ij})$ extended ~~skew-sym.~~ skew-symmetrizable $m \times n$ matrix, $k \in \{1, \dots, n\}$, we define $\mu_k(\widetilde{B}) = (b'_{ij})$ using same formula $(*)$.

<u>Exercise</u>:

- $\mu_k(\widetilde{B})$ is again extended skew-sym., using same $d_1, \dots, d_n$
- $\mu_k(\mu_k(\widetilde{B})) = \widetilde{B}$
- $\mu_k(-\widetilde{B}) = -\mu_k(\widetilde{B})$
- if $b_{ij} = b_{ji} = 0$, then $\mu_i \mu_j \widetilde{B} = \mu_j \mu_i \widetilde{B}$

Def: For a skew-symmetrizable $n\times n$ matrix $B=(b_{ij})$, its <u>Diagram</u> is the weighted directed graph $\Gamma(G)$ with vertices $1,\dots,n$ and $i\to j$ iff $b_{ij}>0$, with weight $|b_{ij}b_{ji}|$.

Lemma: If the diagram $\Gamma(B)$ of an $n\times n$ skew-symmetrizable matrix B is connected then the skew-symmetrizing vector $(d_1,\dots,d_n)$ is unique up to rescaling.

pf: By connectedness, there is an ~~order~~ ordering $l_1,\dots,l_n$ of $\{1,\dots,n\}$ s.t. for each $j\geq 2$ we have $b_{l_i,l_j}\neq 0$ for some $i<j$.

If $(d_1,\dots,d_n)$ and $(d_1',\dots,d_n')$ skew-symmetrizing vectors, have
$d_i b_{ij} = -d_j b_{ji}$ and $d_i' b_{ij} = -d_j' b_{ji}$ $\forall\, i,j$.

If $b_{ij}\neq 0$, have
$$\frac{b_{ij}}{b_{ji}} = \frac{-d_j}{d_i} = \frac{-d_j'}{d_i'}$$
$$\Rightarrow \frac{d_j}{d_j'} = \frac{d_i}{d_i'}.$$

Def: Two extended skew-symmetrizable matrices $\tilde{B},\tilde{B}'$ are <u>mutation equivalent</u> if can get from B to B' by a sequence of mutations, ~~possibly~~ followed by a reordering of the rows and columns in the sense from before.

Put $[B] :=$ ~~set of~~ mutation equivalence class of B.

Prop: For an $n\times n$ skew-symmetrizable matrix, its rank and determinant are preserved by mutations.

pf: Can write
$$b_{ij}' = \begin{cases} -b_{ij} & \text{if } k\in\{i,j\} \\ b_{ij}+\max(0,-b_{ik})b_{kj}+b_{ik}\max(0,b_{kj}) & \text{otherwise}\end{cases}$$

Have
$$\mu_k(\tilde{B}) = J_{m,k}\tilde{B}J_{n,k} + J_{m,k}\tilde{B}F_k + E_k\tilde{B}J_{n,k}$$
$$= (J_{m,k}+E_k)\tilde{B}(J_{n,k}+F_k)$$

where
- $J_{m,k}$ (resp. $J_{n,k}$) is diagonal $m\times m$ (resp. $n\times n$) and has 1s on diagonal except for -1 in (k,k) entry
- $E_k=(e_{ij})$ is $m\times m$ matrix with $e_{ik}=\max(0,-b_{ik})$ and all other entries 0

• $F_k = (f_{ij})$ is the $n \times n$ matrix with $f_{kj} = \max(0, b_{kj})$ and all other entries 0.

Note: $E_k \tilde{B} F_k$ since $b_{ii} = 0$

Have $\det(J_{n,k} + E_k) = \det(J_{n,k} + F_k) = -1$.

<u>Def</u>: A <u>labeled seed</u> of geometric type in $\mathcal{F} = \mathbb{C}(q_1, \dots, q_m)$ is a pair $(\tilde{x}, \tilde{B})$ where

field of rational functions over $\mathbb{C}$

- $\tilde{x} = (x_1, \dots, x_m)$ is an ~~multiple~~ m-tuple of elts of $\mathcal{F}$ which form a <u>free generating seed</u>

  i.e. $\mathcal{F} = \mathbb{C}(x_1, \dots, x_n)$ and $x_1, \dots, x_n$ alg. indep.

- $\tilde{B} = (b_{ij})$ is an $m \times n$ extended skew-symmetrizable integer matrix

We say:

- $\tilde{x}$ is the (labeled) <u>extended cluster</u> of $(\tilde{x}, \tilde{B})$
- $x = (x_1, \dots, x_n)$ is the (labeled) <u>cluster</u>
- $x_1, \dots, x_n$ are the <u>cluster variables</u>
- $x_{n+1}, \dots, x_m$ are the <u>frozen variables</u>
- $\tilde{B}$ is the <u>extended exchange matrix</u>
- its top $n \times n$ submatrix B is the <u>exchange matrix</u>

<u>Ex</u>:

	Σ	Σ'
extended cluster	$\tilde{x} = (x_1, x_2, x_3)$	$\tilde{x}' = (x_1, \frac{x_1 + x_3}{x_2}, x_3)$
cluster vars	x_1, x_2	$x_1, \frac{x_1 + x_3}{x_2}$
frozen vars	x_3	x_3
extended exchange matrix	$\tilde{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}$	$\tilde{B}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$
exchange matrix	$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$B' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Here $m = 3$, $n = 2$.

Σ
$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & \boxed{3} \arrow[ll, bend left] \end{tikzcd} \qquad \begin{tikzcd} x_1 \arrow[r] & x_2 \arrow[r] & \boxed{x_3} \arrow[ll, bend left] \end{tikzcd}$$

Σ'
$$\begin{tikzcd} 1 & 2 \arrow[l] & \boxed{3} \arrow[l] \end{tikzcd} \qquad \begin{tikzcd} x_1 & \frac{x_1 + x_3}{x_2} \arrow[l] & \boxed{x_3} \arrow[l] \end{tikzcd}$$

Lecture 6

2/4/26

Recall: $\mathcal{F} = \mathbb{C}(q_1,\dots,q_m)$ field of rational functions, $m \geq n$. Say $x_1,\dots,x_m \in \mathcal{F}$ free generating set if algebraically independent and $\mathcal{F} = \mathbb{C}(x_1,\dots,x_m)$.

Def: A labeled seed of geometric type in $\mathcal{F}$ is $(\tilde{x}, \tilde{B})$, where:

- $\tilde{x} = (x_1,\dots,x_m)$ free generating set of $\mathcal{F}$
- $\tilde{B} = (b_{ij})$ $m \times n$ extended skew-symmetrizable integer matrix

Terminology:

- $\tilde{x}$ extended cluster
- $x = (x_1,\dots,x_n)$ cluster, $x_1,\dots,x_n$ cluster variables
- $x_{n+1},\dots,x_m$ frozen variables
- $\tilde{B}$ ~~exchange matrix~~ extended exchange matrix
- top $n \times n$ submatrix B is the exchange matrix

Def: Given $(\tilde{x}, \tilde{B})$ labeled seed, $k \in \{1,\dots,n\}$, define a new labeled seed $\mu_k(\tilde{x}, \tilde{B}) = (\tilde{x}', \tilde{B}')$, where

- $\tilde{B}' = \mu_k(\tilde{B})$
- $\tilde{x}' = (x_1',\dots,x_m')$, where $x_j' = x_j$ for $j \neq k$ and

$$x_k x_k' = \prod_{b_{ik}>0} x_i^{b_{ik}} + \prod_{b_{ik}<0} x_i^{-b_{ik}}$$

exchange relation

Rmk: When $\tilde{B}$ comes from a quiver, the first product is over arrows ending at k and the second product is over arrows starting at k.

Ex:

$$\begin{tikzcd} x_1 \arrow[r] & x_2 \arrow[r] & \boxed{x_3} \arrow[ll, bend left] \end{tikzcd}$$

$$\updownarrow \mu_2$$

$$\begin{tikzcd} x_1 & \frac{x_1+x_3}{x_2} \arrow[l] & \boxed{x_3} \arrow[l] \end{tikzcd}$$

$$\updownarrow \mu_1$$

$$\begin{tikzcd} \frac{x_1+x_2+x_3}{x_1x_2} \arrow[r] & \frac{x_1+x_3}{x_2} & \boxed{x_3} \arrow[l] \end{tikzcd}$$

$$\updownarrow \mu_2$$

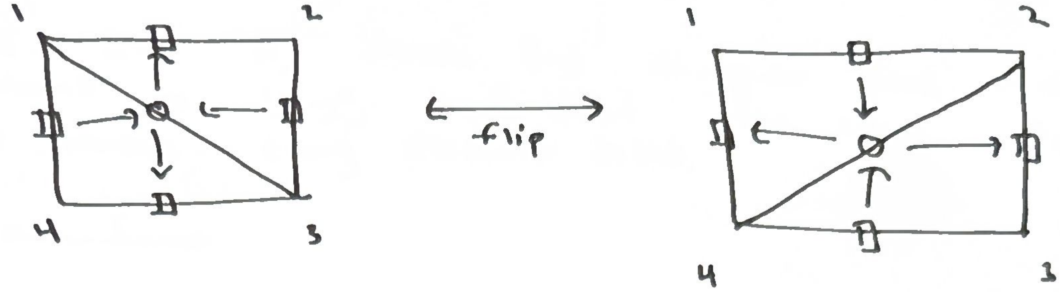
$$\begin{tikzcd} \frac{x_1+x_2+x_3}{x_1x_2} & \frac{x_2+x_3}{x_1} \arrow[l] \arrow[r] & \boxed{x_3} \end{tikzcd}$$

$$\updownarrow \mu_1$$

$$\begin{tikzcd} x_2 \arrow[r] & \frac{x_2+x_3}{x_1} \arrow[r] & \boxed{x_3} \end{tikzcd} \quad \overset{\mu_2}{\longleftrightarrow} \quad \begin{tikzcd} x_2 \arrow[rr, bend right] & x_1 \arrow[l] & \boxed{x_3} \arrow[l] \end{tikzcd}$$

Note: the last seed agrees with the first one up to relabelling.

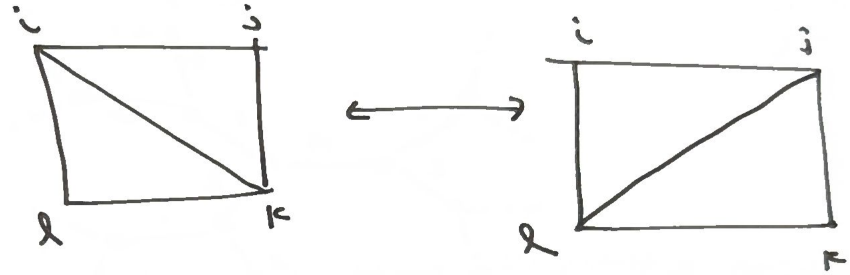
Ex:



$$\begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix} \qquad P_{13} = ag - ce \qquad P_{24} = bh - df$$

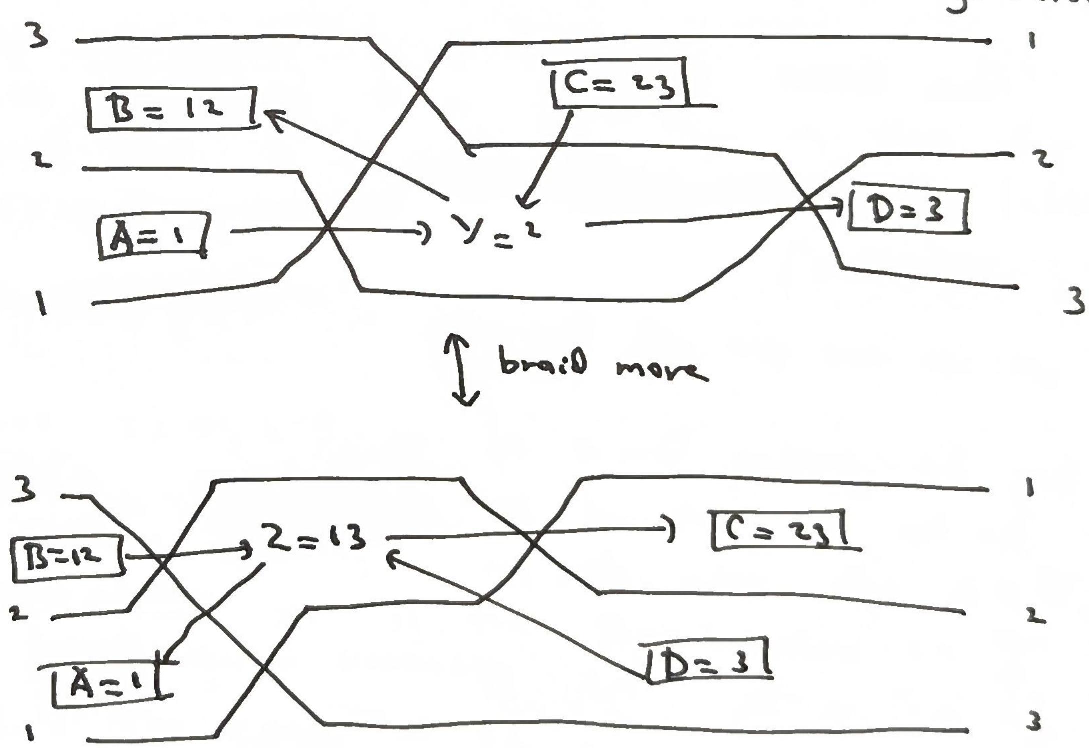
Recall: $P_{13}P_{24} = P_{12}P_{34} + P_{14}P_{23}$

More generally



$\rightsquigarrow$ $P_{ik}P_{jl} = P_{ij}P_{lk} + P_{il}P_{jk}$ special case of the exchange relation

Ex:



$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$A \longleftrightarrow a$
$B \longleftrightarrow ae - bd$
$C \longleftrightarrow bf - ce$) etc

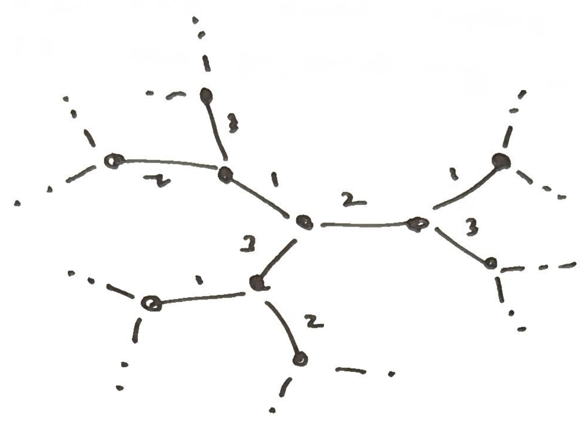
Have $YZ = AC + BD$ special case of the exchange relation

Notation: Let $\mathbb{T}_n$ denote the n-regular tree with edges labeled by $1, \dots, n$, such that the edges incident to each vertex carry distinct labels.

$\mathbb{T}_1$

$\mathbb{T}_2$

$\mathbb{T}_3$



Def.: A <u>seed pattern</u> is a choice of labeled seed $(\tilde{x}(t), \tilde{B}(t))$ for each vertex $t \in \mathbb{T}_n$, so that for each labeled edge $t \overset{k}{\text{———}} t'$ the corresponding labeled seeds $(\tilde{x}(t), \tilde{B}(t))$, $(\tilde{x}(t'), \tilde{B}(t'))$ differ by μ_k.

Note: a seed pattern is determined by any one of its seeds.

Def: Let $(\tilde{x}(t), \tilde{B}(t))_{t \in \mathbb{T}_n}$ be a seed pattern, and put $R := \mathbb{C}[x_{n+1}, \dots, x_m]$. Let $\mathcal{X}$ be the set of all cluster variables appearing in the seeds $x(t)$ for $t \in \mathbb{T}_n$. The <u>cluster algebra</u> $\mathcal{A}$ is the R-subalgebra of $\mathcal{F}$ generated by all cluster variables, i.e. $\mathcal{A} = R[\mathcal{X}]$.

Terminology: The <u>rank</u> n of a cluster algebra is the cardinality of any cluster.

Rmk. Note that there is an isomorphism of $\mathcal{F}$ mapping any free generating set to any other. In particular, up to isomorphism $\mathcal{A}$ depends only on $\tilde{B}_0$ for any initial seed $(\tilde{x}_0, \tilde{B}_0)$, and in fact only on the mutation equivalence class of $\tilde{B}$. In particular, each (ice) quiver Q determines an extended exchange matrix $\tilde{B}$ and hence a cluster algebra.

Ex: For T a triangulation of the regular m-gon P_m, the associated cluster algebra is the Plücker ring $R_{2,m}$.

Ex: For a wiring diagram on k strands, the associated cluster algebra is the algebra generated by flag minors of a $k \times k$ matrix, i.e. the ring of invariants $\mathbb{C}[SL_k]^U$ (here U = group of lower-triangular matrices with 1s on the diagonal)

i.e. functions on the basic affine space

Ex: For a double wiring diagram on k strands, the associated cluster algebra is $\mathbb{C}[GL_n]$, i.e. the polynomial ring in n^2 variables.

Lecture 7

2/6/26

Recall: labeled seed $(\tilde{x}_0, \tilde{B}_0) \longrightarrow$ seed pattern $(\tilde{x}(t), \tilde{B}(t))_{t \in \mathbb{T}_n}$

$\longrightarrow$ cluster algebra $\mathcal{A} \subset \mathcal{F}$ generated by all cluster variables and the frozen variables

Here $\tilde{x}_0 = (x_1, \dots, x_m)$ free generating set of $\mathcal{F} = \mathbb{C}(q_1, \dots, q_m)$, cluster variables $x_1, \dots, x_n$, frozen variables $x_{n+1}, \dots, x_m$.
The rank of $\mathcal{A}$ is n.

Ex: rank $n=1$ $\quad \mathbb{T}_1 = \bullet \overset{1}{\text{———}} \bullet$

$$\tilde{B}_0 = \begin{pmatrix} 0 \\ b_{21} \\ \vdots \\ b_{m1} \end{pmatrix} \quad \text{Exchange relation: } x_1 x_1' = \prod_{b_{i1} > 0} x_i^{b_{i1}} + \prod_{b_{i1} < 0} x_i^{-b_{i1}}$$

$$= M_1 + M_2$$

monomials in the frozen variables $x_2, \dots, x_m$

$$\mathcal{A} = \mathbb{C}[x_1, x_1', x_2, \dots, x_m] \subset \mathcal{F}$$

$$\mathcal{A} \cong \mathbb{C}[z_1, z_1', z_2, \dots, z_m] / (z_1 z_1' = M_1 + M_2), \qquad \mathcal{F} = \mathbb{C}(x_1, x_2, \dots, x_m)$$

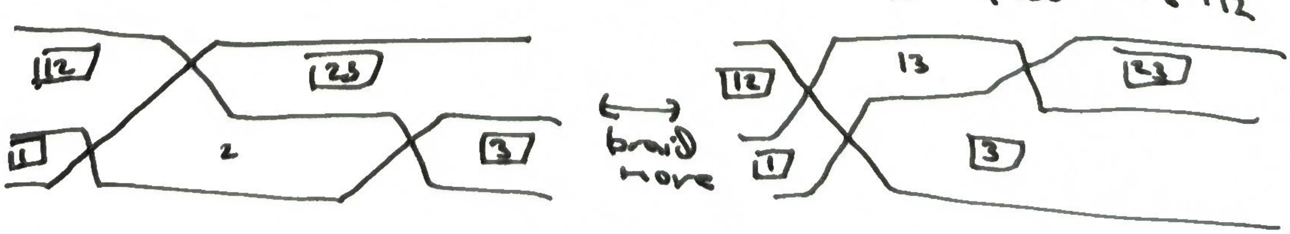
M_1, M_2: monomials in $z_2, \dots, z_m$

Ex: $G = SL_3(\mathbb{C})$, $\quad U =$ subgroup of unipotent lower triangular 3×3 matrices

Then $\mathbb{C}[G]^U$ is a cluster algebra of rank $\underline{1}$.

Recall: $\mathbb{C}[G]^U$ generated by flag minors P_J, $J \subsetneq \{1, 2, 3\}$

Here
- $\mathcal{F} = \mathbb{C}(P_1, P_2, P_3, P_{12}, P_{23})$
- frozen variables: P_1, P_3, P_{12}, P_{23}
- cluster variables P_2, P_{13}
- single exchange relation: $P_2 P_{13} = P_1 P_{23} + P_3 P_{12}$



Ex: rank $n=2$, $\widetilde{B}_0 = \begin{pmatrix} 0 & \pm b \\ \mp c & 0 \\ b_{31} & b_{32} \\ \vdots & \\ b_{m1} & b_{m2} \end{pmatrix}$ either $b, c > 0$ or $b = c = 0$

Suppose no frozens, i.e. $n = m$, $\widetilde{B}_0 = \begin{pmatrix} 0 & \pm b \\ \mp c & 0 \end{pmatrix}$

Then $\mu_1(\widetilde{B}_0) = \mu_2(\widetilde{B}_0) = -\widetilde{B}_0$

Exchange pattern:

$$\cdots \text{---} \underset{(z_1, z_0)}{\begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}} \xrightarrow[]{\ 2\ } \underset{(z_1, z_2)}{\begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix}} \xrightarrow[]{\ 1\ } \underset{(z_3, z_2)}{\begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}} \xrightarrow[]{\ 2\ } \underset{(z_3, z_4)}{\begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix}} \xrightarrow[]{\ 1\ } \cdots$$

where $z_{k-1} z_{k+1} = \begin{cases} z_k^c + 1 & \text{if } k \text{ even} \\ z_k^b + 1 & \text{if } k \text{ odd} \end{cases}$

Ex: $\widetilde{B} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ b_{31} & b_{32} \\ \vdots & \\ b_{m1} & b_{m2} \end{pmatrix}$ μ_k flips sign of kth column for $k = 1, 2$

Exchange relations: $x_1 x_1' = M_1 + M_2$
$x_2 x_2' = M_3 + M_4$

M_1, M_2, M_3, M_4 monomials in frozens

Cluster variables: x_1, x_1', x_2, x_2' (reduces to two rank 1 exchange patterns)

Notation: Let $\mathcal{A}(b,c)$ denote the cluster algebra of rank 2 with exchange matrices $\begin{pmatrix} 0 & \pm b \\ \mp c & 0 \end{pmatrix}$ and no frozens.

Ex: $\mathcal{A}(1,1)$

$$z_{k-1} z_{k+1} = z_k + 1$$

$z_3 = \dfrac{z_2+1}{z_1}$ $\quad z_4 = \dfrac{z_3+1}{z_2} = \dfrac{\frac{z_2+1}{z_1}+1}{z_2} = \dfrac{z_1+z_2+1}{z_1 z_2}$

$z_5 = \dfrac{z_1+1}{z_2}$ $\quad z_6 = z_1 \quad z_7 = z_2$ etc, so 5-periodic.

Ex: $\widetilde{B}_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ p & q \end{pmatrix}$, rank $=2$, 1 frozen variable

$p, q \geq 0$ integers

<u>seed pattern</u>:

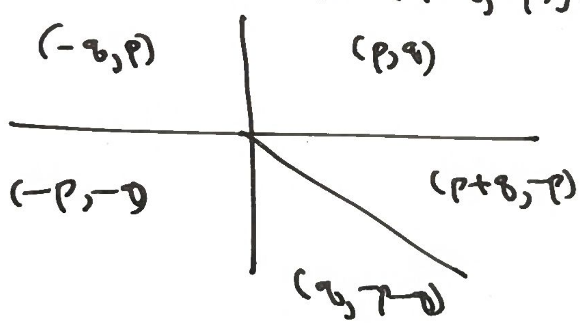
$$\cdots \text{———} \underset{\begin{pmatrix} 0 & 1 \\ -1 & 0 \\ p & q \end{pmatrix}}{(z_1, z_2)} \overset{1}{\text{———}} \underset{\begin{pmatrix} 0 & -1 \\ 1 & 0 \\ -p & p+q \end{pmatrix}}{(z_3, z_2)} \overset{2}{\text{———}} \underset{\begin{pmatrix} 0 & 1 \\ -1 & 0 \\ q & -p-q \end{pmatrix}}{(z_3, z_4)} \overset{1}{\text{———}} \underset{\begin{pmatrix} 0 & -1 \\ 1 & 0 \\ -q & -p \end{pmatrix}}{(z_5, z_4)} \overset{2}{\text{———}} \underset{\begin{pmatrix} 0 & 1 \\ -1 & 0 \\ -q & p \end{pmatrix}}{(z_5, z_6)} \overset{1}{\text{———}} \underset{\begin{pmatrix} 0 & -1 \\ 1 & 0 \\ q & p \end{pmatrix}}{(z_7, z_6)}$$

Have $z_3 = \dfrac{z_2 + y^p}{z_1}$, $z_4 = \dfrac{y^{p+q} z_1 + z_2 + y^p}{z_1 z_2}$,

$z_5 = \dfrac{y^q z_1 + 1}{z_2}$, $z_6 = z_1$, $z_7 = z_2$, etc,

so still 5-periodic.

Although we assumed $p, q \geq 0$, up to mutating and swapping columns every $(i,j) \in \mathbb{Z}^2$ can be written in one of the forms $(p,q), (p+q, -p), (q, -p-q), (-p,-q), (-q, p)$:



Later we will view this as a ~~great~~ simple example of a <u>scattering diagram.</u>

Lecture 8

2/13/26

Ex: $\mathcal{A}(1,2)$ $\qquad z_{k-1}z_{k+1} = \begin{cases} z_k^2+1 & k \text{ even} \\ z_k+1 & k \text{ odd} \end{cases}$

$$z_3 = \frac{z_2^2+1}{z_1} \qquad z_4 = \frac{z_3+1}{z_2} = \frac{\frac{z_2^2+1}{z_1}+1}{z_2} = \frac{z_2^2+z_1+1}{z_1z_2}$$

$$z_5 = \frac{z_1^2+z_2^2+2z_1+1}{z_1z_2^2} \qquad z_6 = \frac{z_1+1}{z_2} \qquad z_7 = z_1 \qquad z_8 = z_2 \quad \text{etc}$$

So it's 6-periodic.

Ex: $\mathcal{A}(1,3)$ $\qquad z_{k-1}z_{k+1} = \begin{cases} z_k^3+1 & k \text{ even} \\ z_k+1 & k \text{ odd} \end{cases}$

Set $z_1 = z_2 = 1$. $\qquad z_3 = \dfrac{z_2^3+1}{z_1} = 2$

$$z_4 = \frac{z_3+1}{z_2} = \frac{2+1}{1} = 3$$

$$z_5 = \frac{z_4^3+1}{z_3} = \frac{28}{2} = 14$$

$$z_6 = \frac{z_5+1}{z_4} = \frac{15}{3} = 5$$

$$z_7 = \frac{z_6^3+1}{z_5} = \frac{126}{14} = 9$$

$$z_8 = \frac{z_7+1}{z_6} = \frac{10}{5} = 2$$

$$z_9 = \frac{z_8^3+1}{z_7} = \frac{9}{9} = 1$$

$$z_{10} = \frac{z_9+1}{z_8} = \frac{2}{2} = 1$$

So it's 8-periodic at least after specializing $z_1 = z_2 = 1$, and we claim that it's 8-periodic even without this specialization.

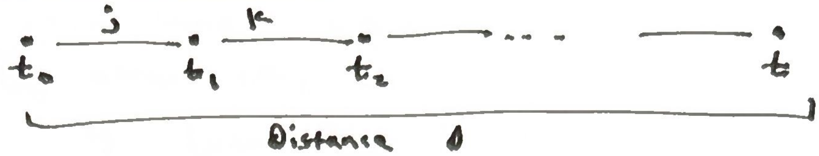
Ex: $\mathcal{A}(1,4)$ $\qquad z_1 = z_2 = 1 \rightsquigarrow 1, 1, 2, 3, 41, 14, 937, 67, 21506, 321, \ldots$

<u>not</u> periodic. However, all <u>integers</u>, and in fact each z_k is a <u>Laurent polynomial</u> in z_1, z_2.

<u>**Thm**</u> Let $(\tilde{x}_0, \tilde{B}_0)$ be a labeled seed with $\tilde{x}_0 = (x_1, \ldots, x_m)$ and associated cluster algebra $\mathcal{A}$. Every cluster variable of $\mathcal{A}$ is a Laurent polynomial with integer coefficients in the variables $x_1, \ldots, x_m$. Moreover, $x_{m+1}, \ldots, x_n$ do not appear in the denominators.

Rmk: Note that we can replace $\tilde{x}_0$ equally with any other extended cluster of $\mathcal{A}$.

proof idea



Say $t_0 \in \mathbb{T}_n$ initial vertex, $(\tilde{x}_0, \tilde{B}_0)$ initial ((labeled) seed), x cluster variable in the seed $x(t)$ for some vertex $t \in \mathbb{T}_n$. For $\tilde{x}_0 = (x_1, \dots, x_m)$ want to show that x is a Laurent polynomial in $x_1, \dots, x_m$. Will use induction on $d = \mathrm{dist}(t, t_0)$.

Base cases: • if $d = 1$, then $x(t) = x(t_1) = (x_1, \dots, x_j', \dots, x_m)$, where

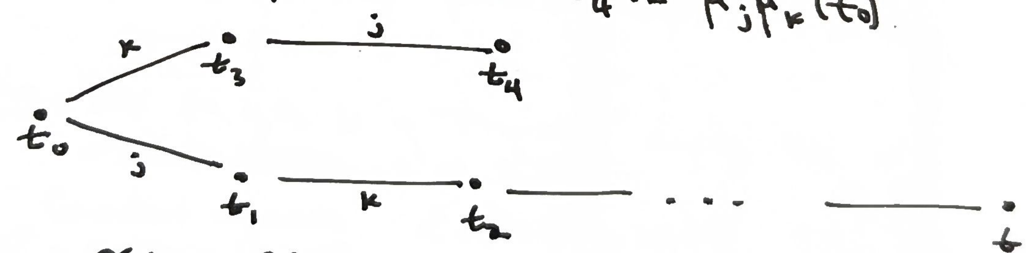
$$x_j' = \frac{\prod_{b_{ij} > 0} x_i^{b_{ij}} + \prod_{b_{ij} < 0} x_i^{-b_{ij}}}{x_j}$$

• if $d = 2$, then $x(t) = \cancel{x(t_2)} = x(t_2) = (x_1, \dots, x_j', \dots, x_k', \dots, x_n)$, (or swap)
where

$$x_k' = \frac{\text{poly in } x_1, \dots, x_j', \dots, x_m}{x_k}$$

$$= \frac{\text{Laurent poly in } x_1, \dots, x_m}{x_k}$$

Inductive step: Now assume $d \geq 3$, and assume for simplicity that $b_{jk}^0 = b_{kj}^0 = 0$ where $\tilde{B}_0 = (b_{ij}^0)$
(the case $b_{jk}^0 b_{kj}^0 < 0$ is more complicated)
Put $t_3 := \mu_k(t_0)$ and $t_4 := \mu_j \mu_k(t_0)$.



Note: $\tilde{x}(t_4) = \tilde{x}(t_2)$, so both t_1, t_3 lie at distance $d-1$ from a seed containing x. By induction:

$$x = \text{Laurent poly in } \underbrace{\tilde{x}(t_1)}_{(x_1, \dots, x_j', \dots, x_m)} = \text{Laurent poly in } \underbrace{\tilde{x}(t_3)}_{(x_1, \dots, x_k', \dots, x_m)}$$

Meanwhile, $x_j' = \dfrac{M_1 + M_2}{x_j}$, $x_k' = \dfrac{M_3 + M_4}{x_k}$, for M_1, M_2, M_3, M_4 monomials in $x_1, \dots, x_m$

Substituting, have:

$$x = \frac{\text{poly in } x_1, \dots, x_m}{(\text{monomial in } x_1, \dots, x_m) \cdot (M_1 + M_2)^a} = \frac{\text{poly in } x_1, \dots, x_m}{(\text{monomial in } x_1, \dots, x_m) \cdot (M_3 + M_4)^b}$$

(after clearing denominators)

It suffices to show that $a \equiv 0$.

Let $\widetilde{B}_0^{aug}$ be $\widetilde{B}_0$ after adding an extra row of the form $(0, \dots, \underset{j\text{th entry}}{1}, \dots, 0)$. Let $\mathcal{A}_{aug}$ be the resulting cluster algebra with coefficient variables $x_{n+1}, \dots, x_{m+1}$.

<u>Observe</u>: expression in $\mathcal{A}_{aug}$ for x in terms of $x_1, \dots, x_{m+1}$ $\xrightarrow[x_{m+1}=1]{\text{specialize}}$ expression in $\mathcal{A}$ for x in terms of $x_1, \dots, x_m$

So x Laurent polynomial in $x_1, \dots, x_{m+1}$ in $\mathcal{A}_{aug}$ $\Longrightarrow$ x Laurent poly in $x_1, \dots, x_m$ in $\mathcal{A}$, hence WLOG can assume $\widetilde{B}_0^{aug}$ instead of $\widetilde{B}_0$.

But then

$$x_j' = \frac{M_1^{aug} + M_2^{aug}}{x_j} = \frac{M_1 x_{m+1} + M_2}{x_j}$$

$$x_k' = \frac{M_3^{aug} + M_4^{aug}}{x_k} = \frac{M_3 + M_4}{x_k}$$

Then $M_1^{aug} + M_2^{aug}$ and $M_3 + M_4$ have no common factor (think about what happens if we specialize $x_1 = \dots = x_m = 1$)

$\Longrightarrow a = 1$. $\square$

<u>Def</u>: A <u>Markov triple</u> is a triple $(a, b, c) \in \mathbb{Z}^3_{\geq 1}$ which satisfies the Markov equation $a^2 + b^2 + c^2 = 3abc$

<u>Ex</u>: $(1, 1, 1)$ is a Markov triple and hence also its permutations. So is $(1, 2, 5)$, $(1, 5, 2)$, $(2, 1, 5)$, $(5, 1, 2)$, $(2, 5, 1)$, $(5, 2, 1)$

<u>Lemma</u>: If (a, b, c) is a Markov triple, then so is (a, b, c') with $c' = \frac{a^2 + b^2}{c}$

<u>pf</u>: Consider equation $a^2 + b^2 + t^2 = 3abt$, i.e. $t^2 - 3abt + (a^2 + b^2) = 0$. If c is one root, the other one c' must satisfy $c + c' = 3ab$, i.e.

$$c' = 3ab - c = \frac{3abc - c^2}{c} = \frac{a^2 + b^2}{c},$$

"Markov mutation"

<u>Lemma</u>: If (a, b, c) is a Markov triple and $a \leq b < c$, then $c' = 3ab - c < c$.

<u>pf</u>: Put $f(t) = t^2 - 3abt + (a^2 + b^2)$.

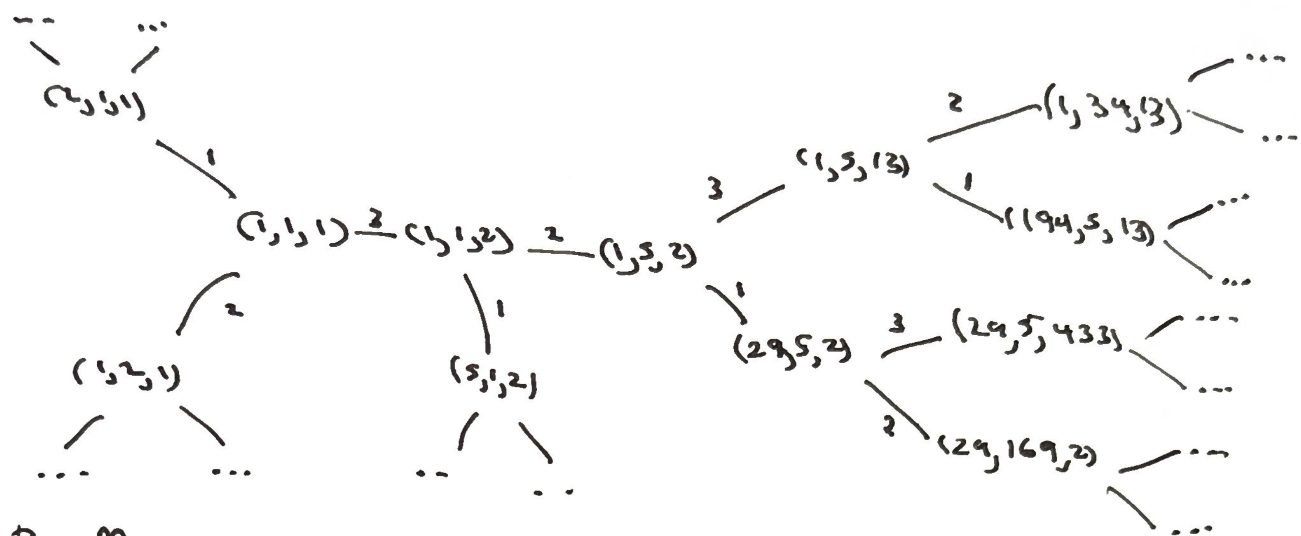
Then

$$f(b) = b^2 - 3ab^2 + a^2 + b^2$$
$$= b^2(2 - 3a) + a^2$$
$$\leq -b^2 + a^2 \leq 0$$

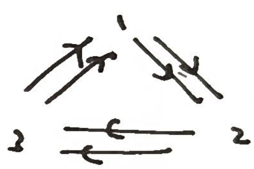
~~Then~~ Then c', the other root of f, must satisfy $c' \leq b < c$.

Cor: Every Markov triple can be connected to $(1,1,1)$ by a sequence of Markov mutations.

The Markov tree:



Recall: The Markov quiver is



Exchange relations:

$$x_1' x_1 = x_2^2 + x_3^2$$
$$x_2' x_2 = x_1^2 + x_3^2$$
$$x_3' x_3 = x_1^2 + x_2^2$$

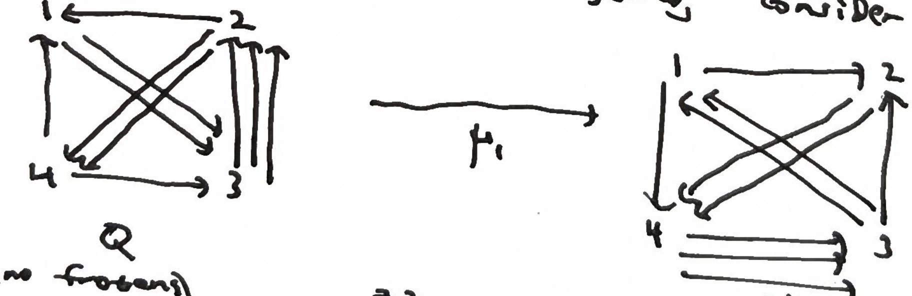
Thus for any cluster $\tilde{x} = (x_1, x_2, x_3)$ specializing initial cluster variables to $(1,1,1)$ turns this into a Markov triple.

Ex: The Somos-4 sequence is $z_0, z_1, z_2, z_3, \ldots$ defined by $z_0 = z_1 = z_2 = z_3 = 1$ and $z_{m+2} z_{m-2} = z_{m+1} z_{m-1} + z_m^2$, i.e.

$1, 1, 1, 1, 2, 3, 7, 23, 59, 314, 1529, \ldots$

Somos '80s: these are all integers!

To explain using cluster algebras, consider quiver



= Q rotated by $\pi/2$

Q (no frozens)

$z_1 z_5 = z_2 z_4 + z_3^2$ $\quad Q' = \mu_1(Q)$

Then μ_2 rotates Q' by $\pi/2$, $z_2 z_6 = z_5 z_3 + z_4^2$

Continue in this way with mutation sequence $\mu_1, \mu_2, \mu_3, \mu_4, \mu_1, \mu_2, \mu_3, \mu_4, \ldots$

gives z_k = Laurent polynomial in z_1, z_2, z_3, z_4 $\xrightarrow[z_1 = z_2 = z_3 = z_4 = 1]{\text{specialize}}$ kth elt of Somos-4, necessarily an integer

Lecture 9

Let $(\tilde{x}, \tilde{B})$ be a labeled seed with $\tilde{x} = (x_1, \dots, x_m)$, $\tilde{B} = (b_{ij})$.
Put $(\tilde{x}', \tilde{B}') = \mu_k(\tilde{x}, \tilde{B})$, with $\tilde{x}' = (x'_1, \dots, x'_m)$, $\tilde{B}' = (b'_{ij})$.
Put $\hat{y} := (\hat{y}_1, \dots, \hat{y}_n)$, where $\hat{y}_j = \prod_{i=1}^{m} x_i^{b_{ij}}$, and similarly $\hat{y}' = (\hat{y}'_1, \dots, \hat{y}'_n)$ with $\hat{y}'_j = \prod_{i=1}^{m} (x'_i)^{b'_{ij}}$.

__Prop:__ We have (for $j = 1, \dots, n$)

$$\hat{y}'_j = \begin{cases} \hat{y}_k^{-1} & \text{if } j = k \\ \hat{y}_j \left(\hat{y}_k^{-\operatorname{sgn}(b_{kj})} + 1 \right)^{-b_{kj}} & \text{else} \end{cases}$$

Here $\operatorname{sgn}(b) = \begin{cases} 1 & \text{if } b > 0 \\ -1 & \text{if } b < 0. \end{cases}$

__Rmk:__

- recall that the exchange relation is

$$x_k x'_k = \underbrace{\prod_{b_{ik} > 0} x_i^{b_{ik}}} + \underbrace{\prod_{b_{ik} < 0} x_i^{-b_{ik}}}$$

$\hat{y}_k$ is the ratio of these

- the above formula for $\hat{y}'_j$ depends only on the top $n \times n$ submatrix of $\tilde{B}$

__Proof:__ • if $j = k$, $\hat{y}'_k = \prod_{i=1}^{m} (x'_i)^{b'_{ik}} = \prod_{i \neq k} x_i^{b'_{ik}} = \prod_{i \neq k} x_i^{-b_{ik}} = \hat{y}_k^{-1}$

$$\left(\text{recall that we have } b'_{ij} = \begin{cases} -b_{ij} & \text{if } k \in \{i, j\} \\ b_{ij} + |b_{ik}| b_{kj} & \text{if } b_{ik} b_{kj} > 0 \\ b_{ij} & \text{else} \end{cases} \right)$$

• if $j \neq k$ and $b_{kj} \leq 0$, have

$$\begin{aligned}
\hat{y}'_j &= (x'_k)^{b'_{kj}} \prod_{i \neq k} x_i^{b'_{ij}} \\
&= (x'_k)^{-b_{kj}} \left(\prod_{i \neq k} x_i^{b_{ij}} \right) \left(\prod_{b_{ik} < 0} x_i^{-b_{ik} b_{kj}} \right) \\
&= x_k^{b_{kj}} \left(\prod_{b_{ik} > 0} x_i^{b_{ik}} + \prod_{b_{ik} < 0} x_i^{-b_{ik}} \right)^{-b_{kj}} \left(\prod_{i \neq k} x_i^{b_{ij}} \right) \left(\prod_{b_{ik} < 0} x_i^{-b_{ik} b_{kj}} \right) \\
&= \left(\prod_i x_i^{b_{ij}} \right) \left(\prod_i x_i^{b_{ik}} + 1 \right)^{-b_{kj}} \\
&= \hat{y}_j (\hat{y}_k + 1)^{-b_{kj}}.
\end{aligned}$$

• case $j \neq k$, $b_{kj} \geq 0$ similar.

Def: A <u>Y-seed</u> of rank n in a field $\mathcal{F}$ is (Y, B), where:

- Y = n-tuple of elts in $\mathcal{F}$
- B = skew-symmetrizable $n\times n$ integer matrix

We mutate Y-seeds as follows:

$$(Y, B) \xrightarrow[\mu_k]{} (Y', B'), \text{ where } B' = \mu_k(B),$$

$$Y' = (Y'_1, \dots, Y'_n) \text{ with } Y'_j = \begin{cases} Y_k^{-1} & \text{if } j = k \\ Y_j\,(Y_k^{-\operatorname{sgn}(b_{kj})} + 1)^{-b_{kj}} & \text{else} \end{cases}$$

Thus labeled seed $(\tilde{x}, \tilde{B}) \longrightarrow$ Y-seed $(\hat{y}, B)$, where

B = top $n\times n$ submatrix of $\tilde{B}$

$\hat{y} = (\hat{y}_1, \dots, \hat{y}_n)$ with $\hat{y}_j = \prod_{i=1}^{m} x_i^{b_{ij}}$

Rmk: The seed mutation at k only changes x_k and leaves x_j alone for $j \neq k$, whereas Y-seed mutation at k potentially changes all of $Y_1, \dots, Y_n$. However, the formula for x'_k involves all of $x_1, \dots, x_m$, whereas Y'_j only involves Y_k and Y_j.

Def: A <u>semifield</u> is an abelian group P endowed with an auxiliary operation $\oplus$ which is commutative, associative, and distributive with respect to the group operation on P (written multiplicatively).

Note that $(P, \oplus)$ is only a semigroup (i.e. not necessarily identity or inverses)

Ex: The multiplicative group $\mathbb{Q}_{>0}$, with $\oplus$ given by ordinary addition.

Def: The <u>tropical semifield</u> $\mathrm{Trop}(q_1, \dots, q_\ell)$ is defined by:

- the multiplicative group of Laurent ~~poly~~ monomials in $q_1, \dots, q_\ell$
- $\prod_{i=1}^{\ell} q_i^{a_i} \oplus \prod_{i=1}^{\ell} q_i^{b_i} = \prod_{i=1}^{\ell} q_i^{\min(a_i, b_i)}$ "tropical addition"

<u>Check</u>:

- commutative: $\min(a_i, b_i) = \min(b_i, a_i)$
- associative: $\min(\min(a_i, b_i), c_i) = \min(a_i, b_i, c_i) = \min(a_i, \min(b_i, c_i))$
- Distributive: (i.e. $(p\oplus q)r = pr \oplus qr$) $\min(a_i, b_i) + c_i = \min(a_i + c_i, b_i + c_i)$

For $(\tilde{x}, \tilde{B})$ labeled seed, $\rightsquigarrow$ <u>coefficient tuple</u>

$\tilde{x} = (x_1, \dots, x_n, \underbrace{x_{n+1}, \dots, x_m}_{\text{frozen variables}})$

$y = (y_1, \dots, y_n)$, where

$$y_j = \prod_{i=n+1}^{m} x_i^{b_{ij}} \in \text{Trop}(x_{n+1}, \dots, x_m)$$

for $j = 1, \dots, n$

<u>Note</u>: $B =$ top $n \times n$ submatrix of $\tilde{B}$ together with coeff. tuple y recover the extended exchange matrix $\tilde{B}$.

<u>Prop</u>: $\tilde{B} = (b_{ij})$ extended skew-symmetrizable $m \times n$ matrix with coef. tuple $y = (y_1, \dots, y_n)$, and $\tilde{B}' = (b_{ij}') = \mu_k(\tilde{B})$ with coef tuple $y' = (y_1', \dots, y_n')$. Then

$$y_j' = \begin{cases} y_k^{-1} & \text{if } j = k \\ y_j \left(y_k^{-\text{sgn}(b_{kj})} \oplus 1 \right)^{-b_{kj}} & \text{else.} \end{cases}$$

"tropical Y-seed mutation"

<u>Def</u>: The <u>universal semifield</u> $\mathbb{Q}_{sf}(x_1, \dots, x_m)$ is

$$\left\{ \frac{P(x_1, \dots, x_m)}{Q(x_1, \dots, x_m)} \in \mathbb{Q}(x_1, \dots, x_m) \;\middle|\; P, Q \text{ have positive coefficients} \right\}$$

with ordinary multiplication and addition.

<u>Lemma</u>: Given any semifield S, and elts $s_1, \dots, s_m \in S$, $\exists!$ semifield homomorphism $\mathbb{Q}_{sf}(x_1, \dots, x_m) \to S$ sending x_i to s_i for $i = 1, \dots, m$.

<u>pf of prop</u>: Let $f: \mathbb{Q}_{sf}(x_1, \dots, x_m) \to \text{Trop}(x_{n+1}, \dots, x_m)$ be semifield homo. sending $f(x_i) = \begin{cases} 1 & \text{if } i \le n \\ x_i & \text{if } i > n. \end{cases}$

Note that f also sends x_k' to 1, since

$x_k x_k' = M_1 + M_2 \rightsquigarrow 1 \cdot f(x_k') = \underbrace{f(M_1) \oplus f(M_2)}_{1 \text{ since } M_1, M_2 \text{ monomials which share no frozen variables}} = 1$

$\Rightarrow f(x_k') = 1.$

Also, $\hat{y}_j = \prod_{i=1}^{m} x_i^{b_{ij}} \rightsquigarrow f(\hat{y}_j) = \prod_{i=n+1}^{m} x_i^{b_{ij}}$

$= y_j$, for $j = 1, \dots, n$,

~~tt~~ and similarly $f(\hat{y}_j') = y_j'$.

Thus $\hat{y}_j' = \begin{cases} \hat{y}_k^{-1} & \text{if } j = k \\ \hat{y}_j \left(\hat{y}_k^{-\text{sgn}(b_{kj})} + 1 \right)^{-b_{kj}} & \text{else} \end{cases} \xrightarrow[f]{\text{apply}} y_j' = \begin{cases} y_k^{-1} & \text{if } j = k \\ y_j \left(y_k^{-\text{sgn}(b_{kj})} \oplus 1 \right)^{-b_{kj}} & \text{else} \end{cases}$ ~~with~~ $\square$

Lecture 10

2/13/26

We can now give an alternative characterization of labeled seeds and their mutations. Fix $\mathcal{F} = \mathbb{C}(q_1, \dots, q_m)$.

A labeled seed is a triple $\Sigma = (x, y, B)$, where

- cluster $x = (x_1, \dots, x_n) \in \mathcal{F}^n$ s.t. $x \cup \{q_{n+1}, \dots, q_m\}$ freely generates $\mathcal{F}$
- exchange matrix B = skew-symmetrizable integer matrix
- coefficient tuple $y = (y_1, \dots, y_n)$ where y_j is a Laurent monomial in $\mathrm{Trop}(q_{n+1}, \dots, q_m)$

For a mutation $(x, y, B) \xrightarrow{\mu_k} (x', y', B')$, have

- $B' = \mu_k(B)$
- y' given by tropical Y-seed mutation rule
- $x' = (x \setminus \{x_k\}) \cup \{x_k'\}$, with

$$x_k x_k' = \frac{y_k}{y_k \oplus 1} \prod_{b_{ik} > 0} x_i^{b_{ik}} + \frac{1}{y_k \oplus 1} \prod_{b_{ik} < 0} x_i^{-b_{ik}}$$

<u>Key point</u>: from this perspective, the complexity of the mutation process does not really grow with the number $m-n$ of frozen variables

<u>Ex</u>: (A_2 revisited) $\quad B_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\rightsquigarrow$ labeled seed pattern

$$\cdots \overset{1}{\text{———}} \Sigma(-1) \overset{2}{\text{———}} \Sigma(0) \overset{1}{\text{———}} \Sigma(1) \overset{2}{\text{———}} \Sigma(2) \overset{1}{\text{———}} \Sigma(3) \overset{2}{\text{———}} \cdots$$

$\Sigma(t) = (x(t), y(t), B(t))$, $\quad B(t) = (-1)^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

t	$x(t)$		$y(t)$	
0	x_1	x_2	y_1	y_2
1	$\frac{y_1 + x_2}{x_1(y_1 \oplus 1)}$	x_2	$\frac{1}{y_1}$	$\frac{y_1 y_2}{y_1 \oplus 1}$
2	$\frac{y_1 + x_2}{x_1(y_1 \oplus 1)}$	$\frac{x_1 y_1 y_2 + y_1 + x_2}{(y_1 y_2 \oplus y_1 \oplus 1) x_1 x_2}$	$\frac{y_2}{y_1 y_2 \oplus y_1 \oplus 1}$	$\frac{y_1 \oplus 1}{y_1 y_2}$
3	$\frac{x_1 y_2 + 1}{x_2(y_2 \oplus 1)}$	$\frac{x_1 y_1 y_2 + y_1 + y_2}{(y_1 y_2 \oplus y_1 \oplus 1) x_1 x_2}$	$\frac{y_1 y_2 \oplus y_1 \oplus 1}{y_2}$	$\frac{1}{y_1(y_2 \oplus 1)}$
4	$\frac{x_1 y_2 + 1}{x_2(y_2 \oplus 1)}$	x_1	$\frac{1}{y_2}$	$y_1(y_2 \oplus 1)$
5	x_2	x_1	y_2	y_1

<u>Thm</u> A seed pattern with initial labeled seed $\Sigma = (x, y, B)$ with $B = \pm\begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix}$, $b, c \in \mathbb{Z}_{\geq 1}$, is of finite type if and only if $bc \leq 3$.

Compare:

<u>Prop</u>: For $b, c \in \mathbb{Z}_{\geq 1}$, the subgroup $W = \langle R_1, R_2 \rangle \subset GL_2$ generated by reflections $R_1 = \begin{pmatrix} -1 & b \\ 0 & 1 \end{pmatrix}$ $R_2 = \begin{pmatrix} 1 & 0 \\ c & -1 \end{pmatrix}$ is finite if and only if $bc \leq 3$.

<u>pf.</u>: $R_1^2 = R_2^2 = \mathbb{1}$, so W finite iff $R_1 R_2$ has finite order.

$$R_1 R_2 = \begin{pmatrix} bc-1 & -b \\ c & -1 \end{pmatrix}$$

characteristic equation: $\lambda^2 - (bc-2)\lambda + 1 = 0$

$$\rightarrow \frac{bc-2 \pm \sqrt{(bc-2)^2 - 4}}{2}$$

For $bc = 1, 2, 3$, roots have order $3, 4, 6$ respectively.

If $bc > 4$, roots are real and not $\pm 1 \Rightarrow$ infinite order.

If $bc = 4$, $(s_1 s_2)^k = \begin{pmatrix} 2k+1 & -kb \\ kc & 2k+1 \end{pmatrix}$ also infinite order.

<u>pf of thm</u>:

Can check that in the case $B = \pm\begin{pmatrix} 0 & 1 \\ -c & 0 \end{pmatrix}$ has 5 seeds if $c=1$, 6 seeds if $c=2$, 8 seeds if $c=3$.

Now assume $bc \geq 4$, seed pattern $(x(t), y(t), B(t))$, $t \in \mathbb{Z}$.

Put $x(0) = (z_1, z_2)$, $x(1) = (z_3, z_2)$, $x(2) = (z_3, z_4)$, etc.

Let $U = \{u^r \mid r \in \mathbb{R}\}$ semifield (u formal variable)

with $u^r \oplus u^s = u^{\max(r,s)}$

$u^r \cdot u^s = u^{r+s}$.

<u>Aim</u>: construct semifield homomorphism $\psi: \mathcal{F} \to U$ such that $\{\psi(z_t) \mid t \in \mathbb{Z}\}$ is infinite.

warning: $\mathcal{F}$ is not right domain... should be $\mathbb{Q}(z_1, z_2)$?

<u>Case $bc > 4$</u>: Let λ be a real number > 1 which is an eigenvalue of $\begin{pmatrix} bc-1 & -b \\ c & -1 \end{pmatrix}$

Put $\psi(z_1) = u^c$, $\psi(z_2) = u^{\lambda+1}$

Exchange relations become:

$$\psi(z_{t-1})\psi(z_{t+1}) = \begin{cases} \psi(z_t)^c \oplus 1 & t \text{ even} \\ \psi(z_t)^b \oplus 1 & t \text{ odd} \end{cases}$$

Claim: $\psi(z_{2k+1}) = u^{\lambda^k c}$, $\psi(z_{2k+2}) = u^{\lambda^k(\lambda+1)}$

Use induction: $\psi(z_{2k+3}) = \dfrac{\psi(z_{2k+2})^c \oplus 1}{\psi(z_{2k+1})} = u^{\lambda^k(\lambda+1)c - \lambda^k c}$

$= u^{\lambda^{k+1} c}$

$\psi(z_{2k+4}) = \dfrac{\psi(z_{2k+3})^b \oplus 1}{\psi(z_{2k+2})} = u^{\lambda^{k+1} cb - \lambda^k(\lambda+1)}$

$= u^{\lambda^k(\lambda cb - \lambda - 1)}$

$= u^{\lambda^{k+1}(\lambda+1)}$

(using $\lambda^2 - (bc-2)\lambda + 1 = 0$)

Case $bc=4$: Instead use $\psi(z_1) = u$, $\psi(z_2) = u^b$.

Claim: $\psi(z_{2k-1}) = u^{2k-1}$, $\psi(z_{2k+2}) = u^{(k+1)b}$

(also by induction)

Def: A skew-symmetrizable matrix $B = (b_{ij})$ is 2-finite if for any $B' = (b'_{ij})$ mutation equivalent to B, we have $|b'_{ij} b'_{ji}| \leq 3$ $\forall\, i,j$.

Cor: Finite type seed pattern $\Longrightarrow$ every exchange matrix is 2-finite

pf: If $B \underset{\text{mutations}}{\sim} B'$ with $|b'_{ij} b'_{ji}| \geq 4$ for some i,j, then by freezing all the cluster variables in that seed except for x_i, x_j, we are reduced to the rank 2 case.

Rmk: Turns out the converse to above corollary is also true!

Lecture 11

2/18/26

Def: A <u>symmetrizable generalized Cartan matrix</u> is a square integer matrix $A = (a_{ij})$ such that:

- all diagonal entries are 2
- all off-diagonal entries are ≤ 0
- DA is symmetric for some diagonal matrix D with positive entries

Def: A <u>Cartan matrix</u> is a symmetrizable generalized Cartan matrix such that DA is positive definite (i.e. has only >0 eigenvalues, or equivalently >0 principal minors).

Note: For a Cartan matrix A, we must have $\det\begin{pmatrix} 2 & a_{ij} \\ a_{ji} & 2 \end{pmatrix} = 4 - a_{ij}a_{ji} > 0$ for all $i \neq j$,

i.e. $a_{ij}a_{ji} \leq 3$. In particular, $|a_{ij}|, |a_{ji}| \in \{0,1,2,3\}$.

Ex: $A = \begin{pmatrix} 2 & -b \\ -c & 2 \end{pmatrix}$ for $b,c \in \mathbb{Z}_{\geq 0}$ is Cartan if and only if one of:

- $b = c = 0$
- $b = c = 1$
- $b = 1, c = 2$ or $b = 2, c = 1$
- $b = 1, c = 3$ or $b = 3, c = 1$

Note that these "match" our classification of rank 2 cluster algebras of finite type

Given an $n \times n$ Cartan matrix A, its <u>Dynkin diagram</u> $\text{Dynk}(A)$ is the graph with vertices $1, \dots, n$, where for each $i \neq j$ we put


if $a_{ij} = -1$, $a_{ji} = -2$


if $a_{ij} = -1$, $a_{ji} = -3$


if $a_{ij} = a_{ji} = -1$

Ex: $A = \begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \rightsquigarrow \text{Dynk}(A) =$ 

Ex: $A = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \rightsquigarrow \text{Dynk}(A) =$ 

Note: this is unrelated to the fact that the quiver  corresponds to the skew-symmetric matrix $\begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}$

Def: A Cartan matrix is <u>indecomposable</u> if its Dynkin diagram is connected. The <u>type</u> of A is its equivalence class up to simultaneous permutations of the rows and columns.

Note: Any Cartan matrix is equivalent to a block-diagonal matrix with indecomposable blocks, which correspond to the connected components of the corresponding Dynkin diagram. The type of A is determined by the multiplicity of each type of connected Dynkin diagram appearing in such a decomposition.

<u>Thm</u> (Cartan-Killing): The Dynkin diagrams of indecomposable Cartan matrices are as follows:

A_n $(n \geq 1)$

n vertices

B_n $(n \geq 2)$

C_n $(n \geq 3)$

D_n $(n \geq 4)$

E_6

E_7

E_8

F_4

G_2

<u>Def</u>: Given an $n \times n$ skew-symmetrizable integer matrix $B = (b_{ij})$, its <u>Cartan counterpart</u> $\mathrm{Cart}(B)$ is the symmetrizable generalized Cartan matrix (a_{ij}), also $n \times n$, defined by

$$a_{ij} = \begin{cases} 2 & i = j \\ -|b_{ij}| & i \neq j. \end{cases}$$

<u>Thm</u>: A cluster algebra is of finite type if and only if its seed pattern contains an exchange matrix B such that $\mathrm{Cart}(B)$ is a Cartan matrix.

<u>Thm</u> Suppose that B_1, B_2 are skew-symmetrizable integer matrices s.t. $\mathrm{Cart}(B_1)$, $\mathrm{Cart}(B_2)$ are Cartan. Then $\mathrm{Cart}(B_1)$, $\mathrm{Cart}(B_2)$ have the same type if and only if B_1 and B_2 are mutation equivalent.

<u>Recall</u>: The classification of simple complex Lie algebras (or equivalently compact simply connected Lie groups) is precisely

- A_n $(n \geq 1)$: $\mathfrak{sl}_{n+1}(\mathbb{C})$ special linear
- B_n $(n \geq 2)$: $\mathfrak{so}_{2n+1}(\mathbb{C})$ odd orthogonal
- C_n $(n \geq 3)$: $\mathfrak{sp}_{2n}(\mathbb{C})$ symplectic
- D_n $(n \geq 4)$: $\mathfrak{so}_{2n}(\mathbb{C})$ even orthogonal
- exceptional algebras : G_2, F_4, E_6, E_7, E_8 sporadic

<u>Note</u>: A Lie algebra is <u>simple</u> if not abelian and no nontrivial ideals.

Lecture 12

2/20/26

Def: A <u>bordered surface with marked points</u> is a pair (S, M), where

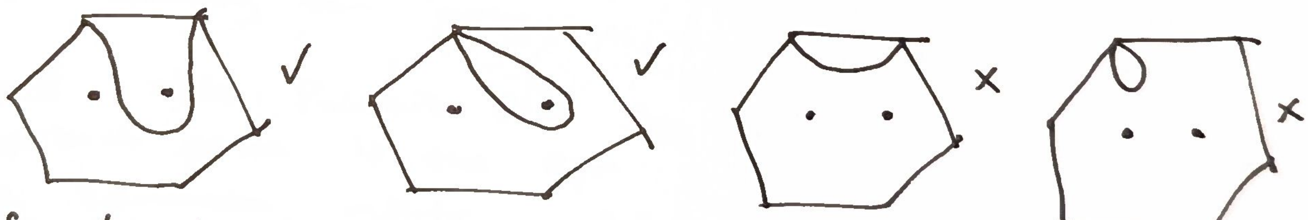
- S = oriented connected surface, possibly with boundary
- $M \subset S$ nonempty subset with at least one point on each boundary cpnt

Will refer to M as "marked points" and those in the interior of S as "punctures"

For technical reasons, will assume (S, M) is not a sphere with 1, 2, 3 punctures, a monogon with 0, 1 punctures, or a bigon or triangle without punctures

Def: An <u>arc</u> γ in (S, M) is a curve in S (up to isotopy) such that

- γ does not cross itself (except that endpoints may coincide)
- apart from its endpoints, γ is disjoint from M and ∂S
- γ does not cut out an unpunctured monogon or unpunctured bigon



Def: A <u>boundary segment</u> is a curve which connects two marked points and lies entirely in ∂S, without passing through a third marked point.

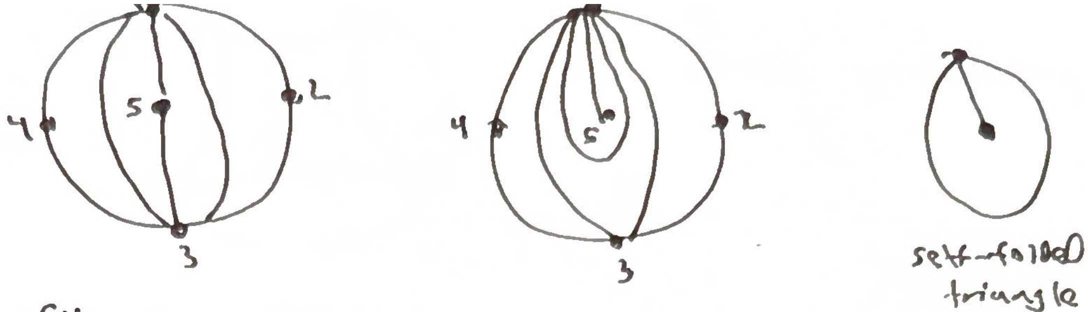
Def: Two arcs are <u>compatible</u> if they have isotopic representatives which do not cross (except possibly at endpoints)

A <u>triangulation</u> is a maximal collection of pairwise compatible arcs, along with all boundary segments.

We refer to the components cut out by the arcs of a triangulation as "triangles".

NB: triangles may have either 3 distinct sides or only 2 ("self-folded")

Ex:



Def: A <u>flip</u> of a triangulation T replaces a single arc γ be another arc $\gamma' \neq \gamma$ (which is unique if it exists) such that $T \setminus \{\gamma\} \cup \{\gamma'\}$ forms a new triangulation.

Ex:

, but we cannot flip along η.

Def: Given a bordered surface S with marked points M, the <u>(cusped) Teichmüller space</u> $\mathcal{T}(S, M)$ is the space of all complete finite-area Riemannian metrics with constant curvature -1 on $S \setminus M$ and with geodesic boundary $\partial S \setminus M$, modulo $\mathrm{Diff}_0(S, M)$.

Here $\mathrm{Diff}_0(S, M)$ is the group of diffeomorphisms of S which fix M and are isotopic to the identity.

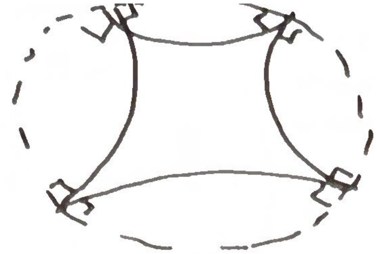
Note: There are cusps at the points M, meaning they are infinitely far away yet the total area is finite.

Recall: The Poincaré disk ~~___~~ model for two-dimensional hyperbolic space is the open unit disk $\mathbb{D}$ with the Riemannian metric
$$ds^2 = \frac{4(dx^2 + dy^2)}{(1 - x^2 - y^2)^2}$$
(constant curvature -1).

The geodesics are of the form $C \cap \mathbb{D}$ for Euclidean circles $C \subset \mathbb{R}^2$ meeting $\partial\mathbb{D}$ orthogonally, and also $L \cap \mathbb{D}$ where $L \subset \mathbb{R}^2$ is a Euclidean line through the origin.

Ex: Let P be a k-sided hyperbolic polygon cut out by geodesics in $\mathbb{D}$, equipped with the restriction of the hyperbolic metric. This defines an element of $\mathcal{T}(S, M)$ with (S, M) having genus zero, one boundary component, and k boundary marked points.

Ex:

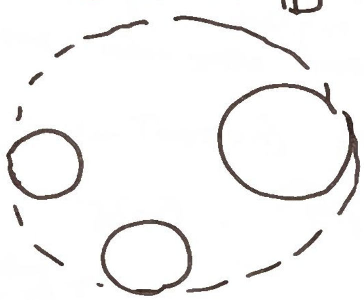


a hyperbolic quadrilateral

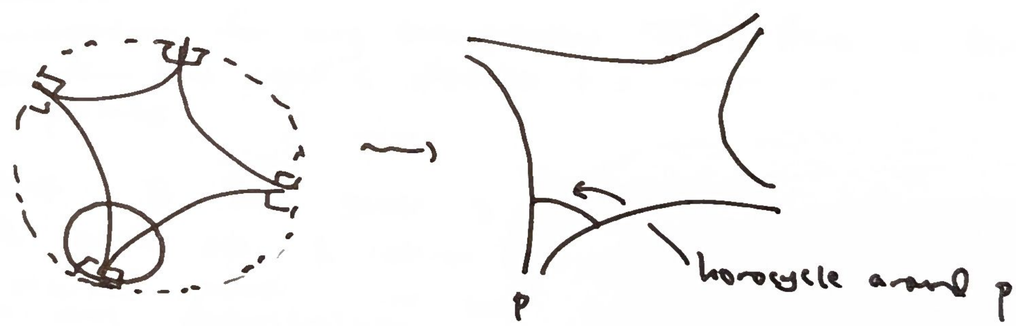
Def: Given $\Sigma \in \mathcal{T}(S,M)$, a horocycle around a puncture p is a closed curve in Σ which is orthogonal to all geodesics asymptotic to p. Similarly, a horocycle around a boundary marked point p is an arc joining two points of $\partial\Sigma$ which is orthogonal to all geodesics asymptote to p.

RmK: Intuitively the horocycle around p is the set of all points of a fixed distance from p, but this distance is infinite.

Ex: The horocycles in $\mathbb{D}$ are circles tangent to the boundary



Ex:

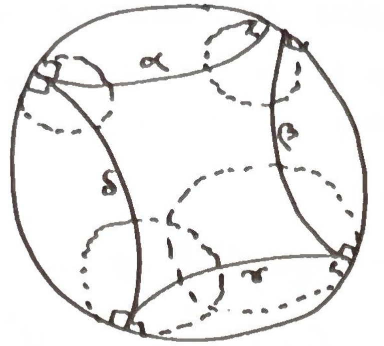


Def: The decorated Teichmüller space $\widetilde{\mathcal{T}}(S,M)$ is defined similarly to $\mathcal{T}(S,M)$ but now we equip $S \setminus M$ with a ~~collecte~~ collection of horocycles, one from each marked point in M.

Def: Fix $\Sigma \in \widetilde{\mathcal{T}}(S,M)$, and let γ be an arc or boundary segment in (S,M). We define the lambda length $\lambda(\gamma)$ as follows. Let γ_Σ be the unique representative of γ which is geodesic with respect to the hyperbolic metric on Σ. Let $\ell(\gamma_\Sigma)$ be the signed distance along γ_Σ between the two horocycles at either end of γ_Σ, where the sign is positive if the two horocycles are disjoint and negative otherwise.

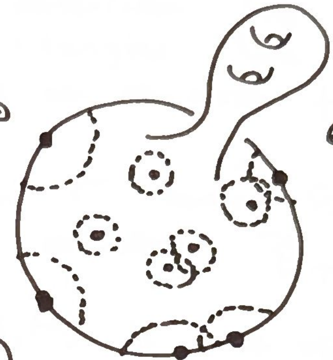
Put $\lambda(\gamma) := \exp\left(\ell(\gamma_\Sigma)/2\right)$.

Ex :



In this example with 4 boundary marked points and no punctures, we have $l(\alpha_\Sigma), l(\beta_\Sigma), l(\delta_\Sigma) > 0$, but $l(\gamma_\Sigma) < 0$.

Note : we can depict a typical element $\Sigma \in \widetilde{T}(S,M)$ by a cartoon such as



In this example S has genus two and one boundary cpnt, and M consists of 5 boundary marked pts and 5 punctures (i.e. interior marked pts). Some of the horocycles intersect.

this picture is not in any way faithful to the hyperbolic metric on Σ

the boundary segments should be geodesic w.r.t. the hyperbolic metric

Thm (Penner, Fomin-Thurston): The map

$$\prod_{\substack{\gamma \\ \text{arc or boundary} \\ \text{segment of } T}} \lambda(\gamma) : \widetilde{T}(S,M) \longrightarrow \mathbb{R}_{>0}^{n+c} \quad \text{is}$$

a homeomorphism for any triangulation T. Here n denotes the number of arcs and c denotes the number of boundary marked points.

Rmk : If S has genus g and b boundary cpnts and M consists of i interior marked points and c boundary marked points, one can compute the number n of arcs in any triangulation to be:

$$n = 6g + 3b + 3i + c - 6$$

So $\dim \widetilde{T}(S,M) = 6g - 6 + 3b + 3i + 2c$

and
$$\dim T(S,M) = \dim \widetilde{T}(S,M) \underbrace{- i - c}_{\text{due to choice of horocycles}}$$
$$= 6g - 6 + 3b + \underbrace{2i + c}_{\text{2 parameters for locations of interior marked pts and 1 parameter for locations of bdy marked pts}}$$

Prop ("hyperbolic Ptolemy"): Let $\alpha, \beta, \gamma, \delta$ be arcs or boundary segments which cut out a quadrilateral with diagonals η, θ. Then we have

$$\lambda(\eta)\lambda(\theta) = \lambda(\alpha)\lambda(\gamma) + \lambda(\beta)\lambda(\delta)$$

(here we assume $\alpha, \beta, \gamma, \delta$ are ordered cyclically).

We next associate an extended exchange matrix to any triangulation.

Def: Let T be a triangulation of (S,M) with arcs $\tau_1,\dots,\tau_n$ and boundary segments $\tau_{n+1},\dots,\tau_{n+c}$.

Put $b_{ij} = \#\{\text{triangles with sides } \tau_i,\tau_j \text{ in clockwise order}\} - \#\{\text{triangles with sides } \tau_i,\tau_j \text{ in counterclockwise order}\}$.

Put $\widetilde{B}_T := (b_{ij})_{\substack{1\le i\le n+c\\ 1\le j\le n}}$, the $(n+c)\times n$ extended exchange matrix of T.

Rmk: Actually the above definition has to be modified if there are any self-folded triangles. We have $|b_{ij}|\le 2$ since every arc is a side of at most two triangles.

Prop: Flipping a triangulation T corresponds to mutating the associated extended exchange matrix $\widetilde{B}_T$.

Facts:

- every arc in (S,M) is part of a triangulation
- any two triangulations differ by a sequence of flips

Now let $\mathcal{A}$ denote the cluster algebra associated to $\widetilde{B}_T$. It follows that:

- each arc γ in (S,M) corresponds to a cluster variable $x_\gamma \in \mathcal{A}$
- each triangulation T of (S,M) gives rise to a seed of $\mathcal{A}$

Rmk: There is an injective map $\left\{\begin{matrix}\text{arcs in}\\ (S,M)\end{matrix}\right\} \hookrightarrow \left\{\begin{matrix}\text{cluster variables}\\ \text{in } \mathcal{A}\end{matrix}\right\}$ but it is not generally surjective if there are any interior marked points, due to the fact that not all arcs can be flipped. There is a more general notion of "tagged arcs" and "tagged triangulations" (due to Fomin-Shapiro-Thurston) which are in bijection with cluster variables and seeds respectively.

Rmk: At least in the absence of punctures (i.e. interior marked pts), we can view every element in $\mathcal{A}$ as a function $\widetilde{T}(S,M) \to \mathbb{R}$. Are these "all of them" in any sense?

Even though not every seed corresponds to a triangulation (but rather a tagged triangulation), we still have:

Lemma: For each seed of $\mathcal{A}$, the corresponding (extended) exchange matrix has all entries equal to $0, \pm 1, \pm 2$.

Cor: For any (S,M), the associated exchange matrix B is mutation-finite, i.e. only finitely many matrices appear in its mutation graph.

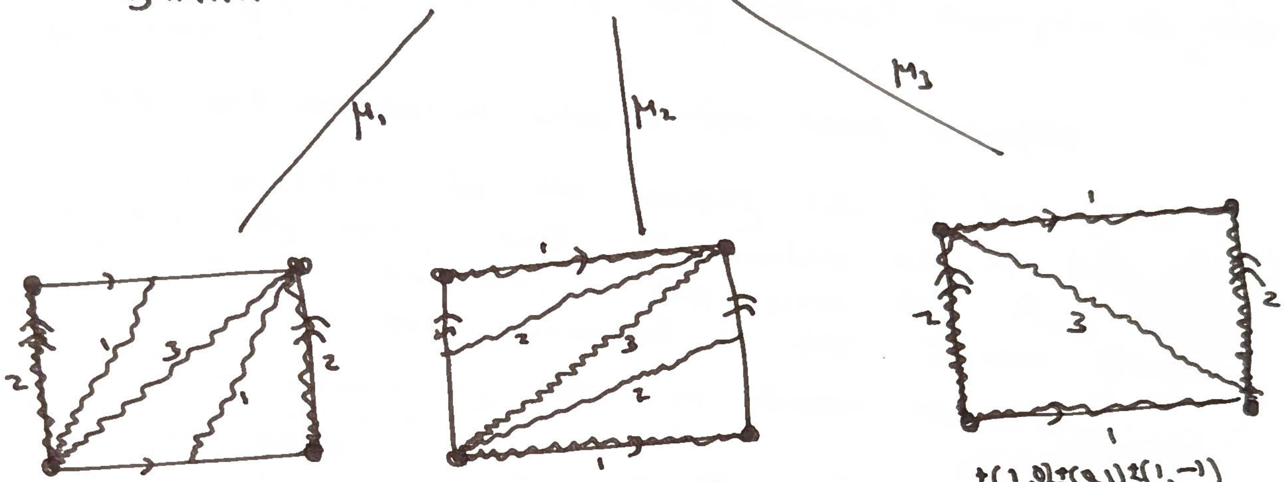
Ex: Consider the once-punctured torus (S, M), i.e. $g=1$, $b=0$, $i=1$, $c=0$, and hence every triangulation has

$$n = 6g+3b+3i+c-6 = 3 \text{ arcs.}$$

Here is the beginning of the exchange graph:

Using $H_1(\mathbb{T}^2) \cong \mathbb{Z}^2$, these represent homology classes

$\pm(1,0)$, $\pm(0,1)$, $\pm(1,1)$



$\pm(1,2)$, $\pm(0,1)$, $\pm(1,1)$ $\qquad$ $\pm(1,0)$, $\pm(2,1)$, $\pm(1,1)$ $\qquad$ $\pm(1,0)$, $\pm(0,1)$, $\pm(1,-1)$

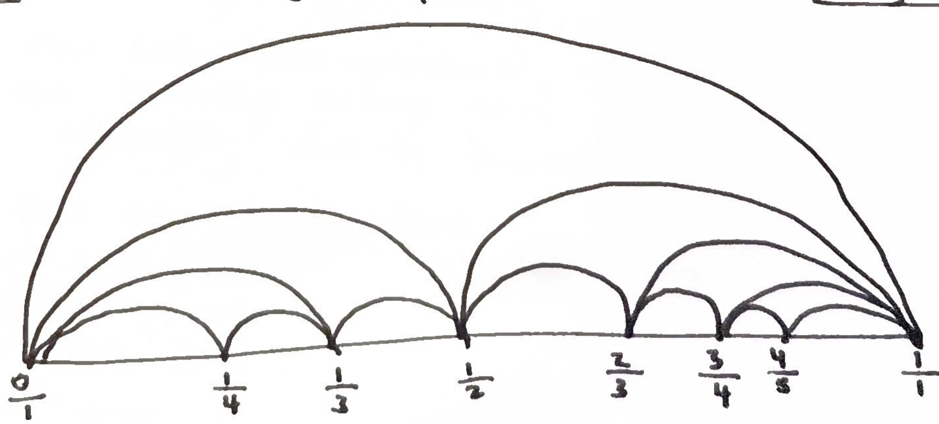
<u>Observe</u>: if a triangulation has arcs with homology classes $\pm(a_1,b_1)$, $\pm(a_2,b_2)$, $\pm(a_3,b_3)$, then must have

$$(a_3,b_3) = \pm(a_1+a_2, b_1+b_2) \quad \text{or} \quad (a_3,b_3) = \pm(a_1-a_2, b_1-b_2),$$

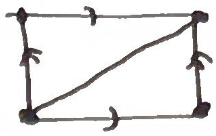
and μ_3 replaces one option with the other (and similarly for μ_1, μ_2).

Also, γ_1 and γ_2 intersect in $\left|\det\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}\right| = |a_1b_2 - a_2b_1|$ points, so we must have $|a_1b_2-a_2b_1| = 1$, and similarly $|a_1b_3 - a_3b_1| = |a_2b_3 - a_3b_2| = 1$.

<u>Upshot</u>: the exchange graph is dual to the <u>Farey tessellation</u>:



Here p/q, p'/q' are connected by an arc iff $|pq'-p'q| = 1$.

Observe: For the triangulation $T =$ 
the corresponding exchange matrix is

$$\widetilde{B}_T = \begin{pmatrix} 0 & -2 & 2 \\ 2 & 0 & -2 \\ -2 & 2 & 0 \end{pmatrix}$$

(reversing the orientation on Π^2 would flip these signs)

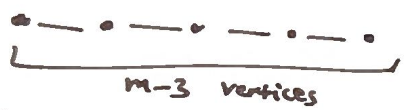
(no frozens)

This is exactly the exchange matrix of the Markov quiver!

Question: It follows that there is a bijective correspondence between triangles in the Farey tessellation and Markov triples. Is there any "direct" description of this bijection?

We end this lecture with a few more examples

Ex: Let (S,M) be the m-gon, i.e. S has genus 0 and one bdy cpnt, and M consists of m bdy marked points and no punctures. This gives the A_{m-3} cluster algebra, i.e. the one associated to the Dynkin diagram

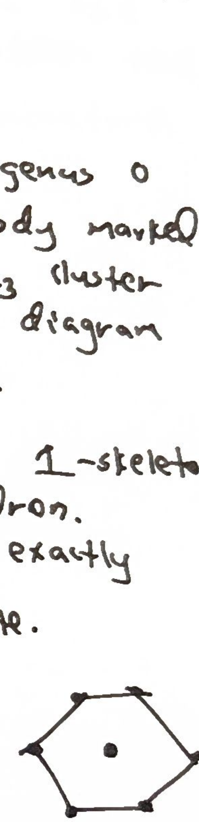


, plus m frozen variables.

Recall that the exchange graph is identified with the 1-skeleton of the $(m-3)$-dimensional Stasheff associahedron.
Note that the number of cluster variables is exactly

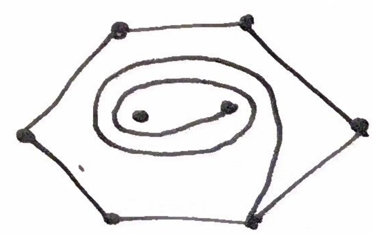
$$\binom{m}{2} - m = \frac{m(m-3)}{2},$$

and in particular finite.

Ex: Let (S,M) be the once-punctured m-gon 
This is the cluster algebra associated with the D_m Dynkin diagram, i.e.

Note that there are only finitely many arcs, but this not a priori imply only finitely many cluster variables.

Ex: For the twice-punctured m-gon, it is easy to see that there are infinitely many arcs, and hence infinitely many cluster variables, due to braiding phenomena:



Rmk: One way to prove that
$\{\text{arcs}\} \longrightarrow \{\text{cluster variables}\}$ is injective is using hyperbolic Ptolemy and the fact that lambda lengths give ~~an isomorphism~~ a homeomorphism $\widetilde{\mathcal{T}}(S,M) \cong \mathbb{R}^{n+c}_{>0}$.
Another way is as follows. For any arc γ, the cluster variable x_γ is a Laurent polynomial ~~polynomial~~ in the initial (extended) cluster variables $x_1, \dots, x_m$.

Writing $x_\gamma = \dfrac{P_\gamma(x_1,\dots,x_m)}{x_1^{d_1}x_2^{d_2}\cdots x_m^{d_m}}$, it turns out that the denominator vector $(d_1,\dots,d_m)$ precisely records the intersection numbers of γ with the curves $\gamma_1,\dots,\gamma_m$ of the initial triangulation. Moreover, for ~~distin~~ distinct arcs γ_1, γ_2, these intersection numbers cannot all be the same, i.e. $x_{\gamma_1} \neq x_{\gamma_2}$.

Rmk: Here intersection number means interior intersections, i.e. γ intersects any boundary segment trivially. This is consistent with the fact that, when writing a cluster variable as a Laurent polynomial in the extended cluster variables of a seed, ~~only the~~ the frozen variables do not appear in denominators.