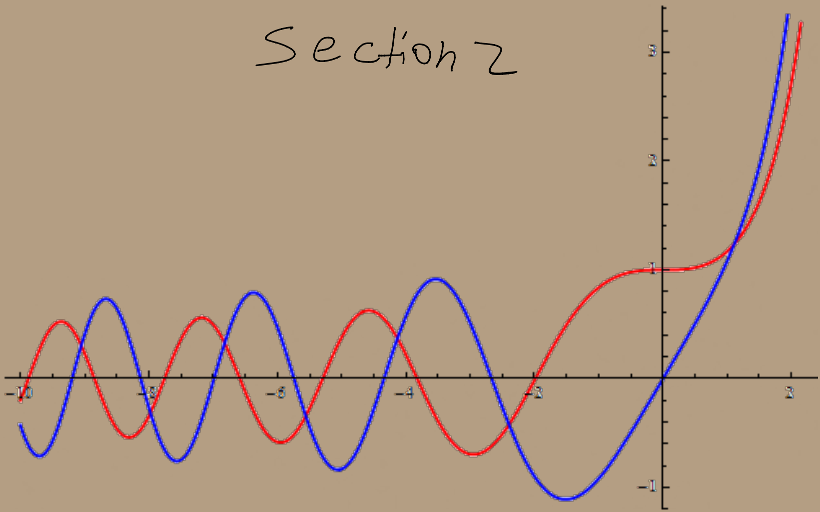
Lecture 16

Section 2



Last lecture: Airy's eqn: $y'' = ty$ via power series ansatz $y(t) = \sum_{n=0}^{\infty} a_n t^n$

Last time: found $y(t) = a_0 y_1(t) + a_1 y_2(t)$ where $y_1(t), y_2(t)$ were explicit power series. Had $\boxed{y_1(0)=1,\ y_1'(0)=0 \quad y_2(0)=0,\ y_2'(0)=1}$

Plug into Airy's eqn:

$$\underbrace{\sum_{n=2}^{\infty} a_n n(n-1)(t-1)^{n-2} - t\sum_{n=0}^{\infty} a_n (t-1)^n = 0.}$$

try to write as a single power series centered at $t=1$

$$\sum_{n=2}^{\infty} a_n n(n-1)(t-1)^{n-2} - \sum_{n=0}^{\infty} a_n (t-1)^n - \sum_{n=0}^{\infty} a_n (t-1)^{n+1} = 0.$$

$$(2\cdot 1 a_2 - a_0) + (3\cdot 2 a_3 - a_1 - a_0)(t-1) + (4\cdot 3 a_4 - a_2 - a_1)(t-1)^2 + \cdots \begin{bmatrix}5\cdot 4 a_5 \\ - a_3 - a_2\end{bmatrix}(t-1)^3 + \cdots = 0.$$

Note: $a_4 = \dfrac{a_1 + a_0/2}{12} = \dfrac{a_1}{12} + \dfrac{a_0}{24}$

$a_5 = \dfrac{a_0/2 + \left(\frac{a_1}{12} + \frac{a_0}{24}\right)}{4\cdot 5}$, etc.

Have
$$y(t) = a_0 \underbrace{\left(1 + \tfrac{1}{2}(t-1)^2 + \tfrac{1}{6}(t-1)^3 + \tfrac{1}{24}(t-1)^4 + \cdots\right)}_{y_3} + a_1 \underbrace{\left((t-1) + \tfrac{1}{6}(t-1)^3 + \tfrac{1}{12}(t-1)^4 + \cdots\right)}_{y_4}$$

Rmk: e^t for example can be written as a power series centered at any t_0.

But y_3 and y_4 must be linear combinations of y_1 and y_2.

Note. y_1, y_2 form a fund. set of solns. So do y_3, y_4.

Ansatz $y = \sum_{n=0}^{\infty} a_n t^n \quad y' = \sum_{n=0}^{\infty} a_n n t^{n-1}$

$y'' = \sum_{n=0}^{\infty} a_n n(n-1) t^{n-2}$

Ex: Solve Airy's eqn using ansatz $y(t) = \sum_{n=0}^{\infty} a_n (t-1)^n$.

$y'(t) = \sum_{n=1}^{\infty} a_n n (t-1)^{n-1}$

$y''(t) = \sum_{n=2}^{\infty} a_n n(n-1)(t-1)^{n-2}$

Some terms given in terms of $t-1$, others in terms of t.

Write $t = 1 + (t-1)$

Lazy route: Have

$2\cdot 1 a_2 + 3\cdot 2 a_3 (t-1) + 4\cdot 3 a_4 (t-1)^2 + 5\cdot 4 a_5 (t-1)^3 + \cdots$
$- a_0 - a_1(t-1) - a_2(t-1)^2 - a_3(t-1)^3 - \cdots$
$- a_0(t-1) - a_1(t-1)^2 - a_2(t-1)^3 - \cdots$

This gives: $a_2 = a_0/2$

$a_3 = \dfrac{a_0 + a_1}{2\cdot 3}$, $a_4 = \dfrac{a_1 + a_2}{3\cdot 4}$

$a_5 = \dfrac{a_2 + a_3}{4\cdot 5}$, etc.

$y(t) = a_0 + a_1(t-1) + \frac{a_0}{2}(t-1)^2 + \frac{a_0 + a_1}{6}(t-1)^3 + \left(\frac{a_1}{12} + \frac{a_0}{24}\right)(t-1)^4 + \cdots$

So $y(t) = a_1 y_3(t) + a_2 y_4(t)$.

Ex: $e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots$
$= (e) + (?)(t-1) + (?)(t-1)^2 + (?)(t-1)^3 + \cdots$

Ex: $t = 1 + (t-1)$

Q: What is rel'n b/w y_3, y_4 and y_1, y_2?

Note: $y_3 \neq y_1, y_2$. Why?

$y_3(0) = \underbrace{1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} + \cdots}$

Note: this is neither 0 nor 1

Ex: $y'' - \sin(t) y = 0.$

Plug in:

$$\underbrace{\sum_{n=0}^{\infty} a_n n(n-1) t^{n-2} - \sin(t) \sum_{n=0}^{\infty} a_n t^n = 0}$$

want this as a single power series centered at 0

Recall: $\sin(t) = t - t^3/3! + t^5/5! - t^7/7! + \dots$

$\rightarrow \sum_{n=0}^{\infty} a_n(n)(n-1)t^{n-2} - \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \dots\right)\left(\sum_{n=0}^{\infty} a_n t^n\right)$

$2\cdot 1a_2 + 3\cdot 2a_3 t + 4\cdot 3a_4 t^2 + 5\cdot 4a_5 t^3 + \dots \underbrace{-\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \dots\right)\left(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \dots\right)}_{a_0 t + a_1 t^2 + \left(a_2 - \frac{a_0}{3!}\right)t^3 + \dots} = 0$

So need:

$2a_2 + (3\cdot 2a_3 - a_0)t + (4\cdot 3a_4 - a_1)t^2 + \left(5\cdot 4a_5 - (a_2 - a_0/3!)\right)t^3 + \dots = 0$

This gives $2a_2 = 0$, $3\cdot 2a_3 - a_0 = 0$, $4\cdot 3a_4 - a_1 = 0$, etc.

Ex: $t^2y'' + ty' + y = 0$.

Ansatz: $y = \sum_{n=0}^{\infty} a_n t^n \quad y' = \sum_{n=0}^{\infty} a_n n t^{n-1}$

$y'' = \sum_{n=0}^{\infty} a_n n(n-1)t^{n-2}$

plug in: $\sum_{n=0}^{\infty} a_n n(n-1)t^n + \sum_{n=0}^{\infty} a_n n t^n + \sum_{n=0}^{\infty} a_n t^n = 0$

$\sum_{n=0}^{\infty}\left(a_n n(n-1) + a_n n + a_n\right)t^n = 0.$

gives $(n^2+1)a_n = 0$ for all $n \geq 1$.

$\Rightarrow a_n = 0$ for all n.

Conclusion: $y(t) = 0$.

Clearly not the general soln ...

Recall: considering $P(t)y'' + Q(t)y' + R(t)y = 0$

Could also write as $y'' + p(t)y' + q(t)y = 0$

where $p(t) = Q(t)/P(t)$, $q(t) = R(t)/P(t)$

Note: If $P(t_0) = 0$, then p, q might be ill-defined at $t = t_0$.

Recall: $t = t_0$ is an ordinary point if $P(t_0) \neq 0$, or more generally if $p(t), q(t)$ are analytic at t_0. Otherwise t_0 is called a singular pt.

Ex: $t^2y'' + ty' + y = 0$

sing pts? 0. $p(t) = 1/t \quad q(t) = 1/t^2$

Ex: $y'' - ty = 0$.

sing pts? none ($p(t) = 0$, $q(t) = -t$)

Ex: $(t-3)y'' + (t^2 - 5t + 6)y = 0$

sing pts? none. $p(t) = 0 \quad q(t) = \frac{t^2-5t+6}{t-3} = t-2$

Ex: $t(t-1)y'' + t^3y' + \sin(t)y = 0$.

sing pts? ~~t=0~~, $t = 1$

$p(t) = \frac{t^2}{t-1} \quad q(t) = \frac{\sin(t)}{t(t-1)}$

Thm: Consider the ODE $y'' + p(t)y' + q(t)y = 0$ and suppose t_0 is an ordinary pt. Then the ODE has general soln of form

$\sum_{n=0}^{\infty} a_n(t-t_0)^n = a_0 y_1(t) + a_1 y_2(t)$

Moreover the r.o.c of y is at least

min(r.o.c. of p at t_0, r.o.c. of q at t_0).

can find using ratio test

Or recall that if $p(t)$ is a rational func, then r.o.c. is dist from t_0 to nearest root of denom.

Recall: $\frac{\sin(t)}{t}$ is analytic at 0, since

$\frac{\sin(t)}{t} = \frac{1}{t}\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \dots\right) = 1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \dots$

Ex: Consider $(t^2 + t + 3)y'' + ty' + y = 0$

Suppose $y = \sum_{n=0}^{\infty} a_n(t-\pi)^n$ is a soln. What can we say about the r.o.c of y?

(root means cpx root)

r.o.c. of y is at least

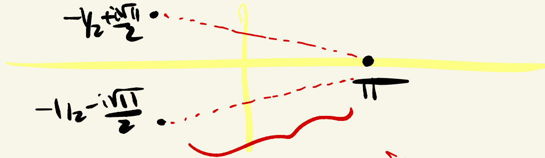
$$\min\left(\text{r.o.c. of } \underbrace{\frac{t}{t^2+t+3}}_{p(t)},\ \text{r.o.c. of } \underbrace{\frac{1}{t^2+t+3}}_{q(t)} \right)$$

roots of t^2+t+3 are $\dfrac{-1 \pm \sqrt{1^2-12}}{2}$

$$= -1/2 \pm i\sqrt{11}/2$$

Both of these are given by

$\mathrm{dist}(\pi,\ \text{nearest root of } t^2+t+3)$



this distance is

$\sqrt{(\pi+1/2)^2 + (\sqrt{11}/2)^2}$.