HW #4 Solutions

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§ 3.1 : 10, 12, 16

10) $y'' + 4y' + 3y = 0$ $\quad y(0) = 2 \quad y'(0) = -1$

CE: $r^2 + 4r + 3 = 0$

$(r+3)(r+1) = 0$

$r = -3, -1 \quad \Rightarrow y = c_1 e^{-t} + c_2 e^{-3t}$

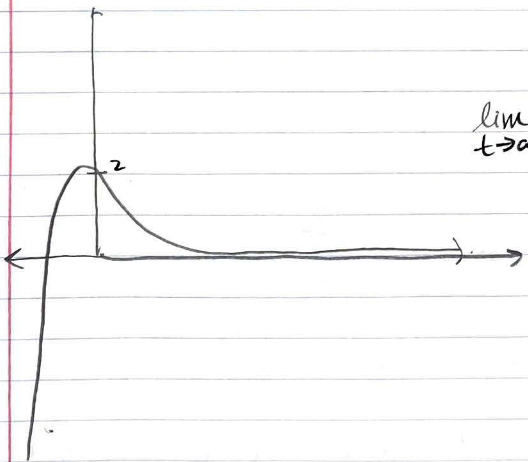
$y(0) = 2 = c_1 + c_2 \quad$ (i)

$y'(0) = -1 = -c_1 - 3c_2 \quad$ (ii)

(i) + (ii) $= -2c_2 = 1 \quad \Rightarrow c_2 = -\frac{1}{2}$

$c_1 = \frac{5}{2}$

$$\boxed{y = 2.5e^{-t} + \left(-\frac{1}{2}\right)ce^{-3t}}$$



$\lim_{t\to\infty} (y) = 0.$

12) $y''+3y'=0$ $\quad y(0)=-2 \quad y'(0)=3$

CE: $r^2+3r=0 \quad r(r+3)=0$

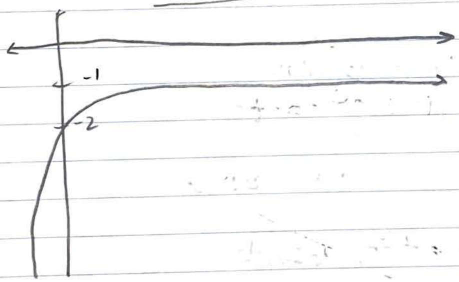
$r=0,-3 \Rightarrow y=c_1+c_2e^{-3t}$

$y(0)=-2=c_1+c_2$

$y'(0)=-3c_2=3$

$c_2=-1, \quad c_1=-1$

$\boxed{y=-1-e^{-3t}}$



$\lim_{t\to\infty}(y)=-1$

16) $4y''-y=0$ $\quad y(-2)=1 \quad y'(-2)=1$

CE: $4r^2-1=0 \quad r^2=\frac{1}{4} \quad r=\frac{1}{2},-\frac{1}{2}$

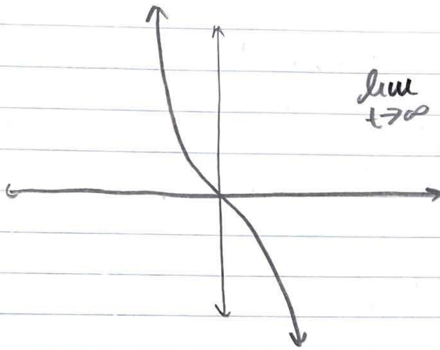
$y=c_1e^{\frac{1}{2}t}+c_2e^{-\frac{1}{2}t}$

$y(-2)=1=c_1e^{-1}+c_2e^{1}$ (i)

$y'(-2)=\frac{1}{2}c_1e^{-1}-\frac{1}{2}c_2e=-1 \Rightarrow c_1e^{-1}-c_2e=-2$ (ii)

(i)+(ii) $=2c_1e^{-1}=-1 \rightarrow c_1=-\frac{e}{2}$

$\boxed{y=-\frac{1}{2}e^{\frac{1}{2}t+1}+\frac{3}{2}e^{-\frac{1}{2}t-1}}$ $\quad c_2=\frac{3}{2}e^{-1}$



$\lim_{t\to\infty} (y) = -\infty$

§ 3.2 : 5, 9, 14, 15, 16, 17

5) $e^t \sin t \quad e^t \cos t$

$y_1 = e^t \sin t \qquad y_1' = e^t \cos t + e^t \sin t$

$y_2 = e^t \cos t \qquad y_2' = -e^t \sin t + e^t \cos t$

$y_1 y_2' - y_1' y_2 = e^t \sin t(-e^t \sin t + e^t \cos t)$

$= -e^{2t} \sin^2 t + e^{2t} \cancel{\sin t \cos t}$

$e^t \cos t(e^t \cos t + e^t \sin t)$

$= e^{2t} \cos^2 t + e^{2t} \cancel{\sin t \cos t}$

$-e^{2t} \sin^2 t - e^{2t} \cos^2 t = -e^{2t} \underbrace{(\sin^2 t + \cos^2 t)}_{=1}$

$= \boxed{-e^{2t}}$

9) $t(t-4)y'' + 3ty' + 4y = 2 \quad y(3) = 0 \quad y'(3) = -1$

$/t(t-4)$: $y'' + \frac{3}{(t-4)} y' + \frac{4}{t(t-4)} y = \frac{2}{t(t-4)}$

$(-\infty, 4)(4, \infty) \quad (-\infty, 0)(0, 4)(4, \infty)$

$t_0 = 3$, so interval $= \boxed{(0, 4)}$

14) verify $y_1(t) = 1$ $y_2(t) = t^{1/2}$

$yy'' + (y')^2 = 0$ for $t > 0$. (*)

$y_1: 1(0) + (0) = 0$ ✓

$y_2 = t^{1/2} \cdot -\frac{1}{4}t^{-3/2} + \left(\frac{1}{2}t^{-1/2}\right)^2$

$= \cancel{\left(-\frac{1}{4}t^{-1}\right)} + \cancel{\left(\frac{1}{4}t^{-1}\right)} = 0$ ✓

check $y = c_1 + c_2 t^{1/2}$

$y' = \frac{1}{2}c_2 t^{-1/2}$

$y'' = -\frac{1}{4}c_2 t^{-3/2}$

$(c_1 + c_2 t^{1/2})\left(-\frac{1}{4}c_2 t^{-3/2}\right) + \left(\frac{1}{2}c_2 t^{-1/2}\right)^2$

$= -\frac{1}{4}c_1 c_2 t^{-3/2} \cancel{- \frac{1}{4}c_2^2 t^{-1} + \frac{1}{4}c_2^2 t^{-1}}$

$= -\frac{1}{4}c_1 c_2 t^{-3/2} \neq 0$

not contradictory b/c (*) is nonlinear, because of the yy'' term

15) Show that if $y = \varphi(t)$ solves $\overset{(*)}{y'' + p(t)y' + q(t)y = \underbrace{g(t)}}$ then $y = c\varphi(t)$ $(c \neq 1)$ not a sol'n.

$g(t)$ not always 0

Suppose $y = c\varphi(t)$ is a solution:

Sub. y: $c\underbrace{(\varphi'' + p\varphi' + q\varphi)}_{= g \text{ b/c } y = \varphi(t) \text{ is a solution.}} = g$

then $cg = g \Rightarrow \boxed{c = 1}$

but $c \neq 1$. CONTRADICTION

not contradicting b/c (*) is not homogenous, as $g(t)$ is not necessarily 0.

16) Can $y = \sin(t^2)$ be a sol'n on an interval containing 0 of (*) $y'' + p(t)y' + q(t)y = 0$

$y' = 2t\cos(t^2)$ $y'' = 2t(2t(-\sin(t^2))) + 2\cos(t^2)$

$= -4t^2\sin(t^2) + 2\cos(t^2) + 2t\cos(t^2)p(t) + \sin(t^2)q(t) = 0$

$= \sin(t^2)(q(t) - 4t^2) + \cos(t^2)(2t \cdot p(t) + 2) = 0$

$t = 0$: $\cancel{\cos(0)}(2(t)p(t) + 2) = 0$

$= p(t) = -\frac{2}{2t} = -\frac{1}{t}$ where $t = 0$.

so $y = \sin(t^2)$ is **not** a solution on this interval.

17) $W = 3e^{4t}$ $f(t) = e^{2t}$ $f' = 2e^{2t}$

$$= e^{2t}g' - 2e^{2t}g = 3e^{4t} \quad / e^{2t}$$

$$= g' - 2g = 3e^{2t} \quad \Rightarrow \mu = e^{(\int -2dt)} = e^{-2t}$$

$$e^{-2t}g' - 2e^{-2t}g = 3$$

$$\Downarrow$$

$$(e^{-2t}g)' = 3 \quad / \int \text{both sides}$$

$$e^{-2t}g = 3t + c$$

$$\boxed{g = 3te^{2t} + ce^{2t}}$$

§3.3: 6, 11, 17, 19, 21

6) write in $a+ib$: π^{-1+2i}

$$\pi^{-1+2i} = e^{\ln(\pi^{-1+2i})} = e^{(-1+2i)\ln\pi}$$

$$= e^{-\ln(\pi) + 2i\ln\pi}$$

$$= e^{\ln(\pi^{-1})} \cdot e^{i(2\ln\pi)}$$

$$= \pi^{-1}\left(\cos(2\ln\pi) + i\sin(2\ln\pi)\right)$$

$$\boxed{= \frac{1}{\pi}\cos(2\ln\pi) + i\frac{1}{\pi}\sin(2\ln\pi)}$$

11) $y'' + 6y' + 13y = 0$ $CE = r^2 + 6r + 13$

$$r = \frac{-6 \pm \sqrt{36-52}}{2} = \frac{-6 \pm 4i}{2} = -3 \pm 2i$$

$$\boxed{y = c_1 e^{-3t}\cos 2t + c_2 e^{-3t}\sin 2t}$$

17) $y'' + 4y = 0 \qquad y(0) = 0 \qquad y'(0) = 1$

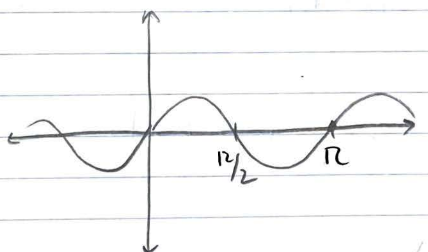
CE: $r^2 + 4 = 0 \qquad r^2 = -4 \; ; \; r = \pm 2i$

$y = c_1 \cos 2t + c_2 \sin 2t$

$y(0) = c_1 = 0$

$y'(0) = 2c_2 \cos(2t) = 2c_2 = 1 \; , \; c_2 = \frac{1}{2}$

sol'n: $\boxed{y = \frac{1}{2} \sin 2t}$



oscillating behavior.

19) $y'' - 2y' + 5y = 0 \qquad y(\pi/2) = 0 \qquad y'(\pi/2) = 2$

CE: $r^2 - 2r + 5 = 0$

$r = \frac{2 \pm \sqrt{4 - 20}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$

$y = c_1 e^t \cos 2t + c_2 e^t \sin 2t$

$y(\frac{\pi}{2}) = c_1 e^{\pi/2} \cos(\pi) + \cancel{c_2 e^{\pi/2} \sin(\pi)}$

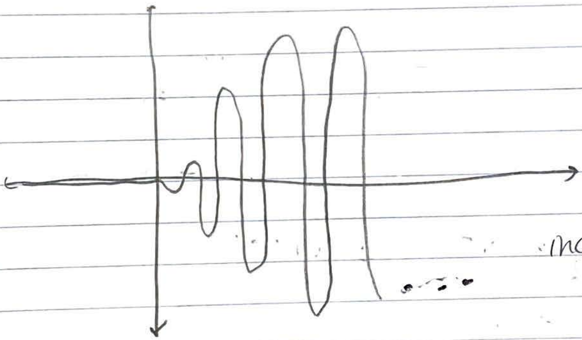
$c_1 e^{\pi/2}(-1) = 0 \Rightarrow c_1 = 0$

$y' = c_2 e^t (2) \cos 2t + c_2 e^t \sin 2t$

$y'(\frac{\pi}{2}) = c_2 e^{\pi/2} \, 2\cos(\pi) + \cancel{c_2 e^t \sin(\pi)} = 2$

$-2c_2 e^{\pi/2} = 2 \quad , c_2 = -e^{-\pi/2}$

$$\boxed{y = -e^{(t-\pi/2)} \sin 2t}$$



increasing oscillation

21) $y'' + y' + 1.25y = 0 \quad y(0) = 3 \quad y'(0) = 1$

CE: $r^2 + r + 1.25 = 0$

$$r = \frac{-1 \pm \sqrt{1-5}}{2} = \frac{-1 \pm 2i}{2} = -\frac{1}{2} \pm i$$

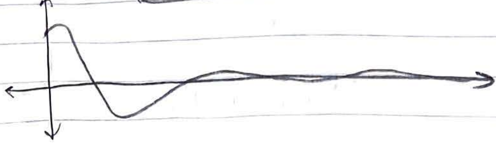
$$y = c_1 e^{-t/2} \cos t + \cancel{c_2 e^{-t/2}} \sin t$$

$$y(0) = c_1 e^0 = 3 \quad , c_1 = 3$$

$$y' = -\frac{3}{2} e^{-t/2} \cos t - \cancel{3e^{t/2} \sin t} + c_2 e^{-t/2} \cos t - \cancel{\frac{c_2}{2} e^{-\frac{t}{2}} \sin t}$$

$$y'(0) = -\frac{3}{2} + c_2 = 1 \quad , c_2 = 5/2$$

sol'n: $\boxed{y = 3e^{-t/2} \cos t + 5/2\, e^{-t/2} \sin t}$



$\lim_{t \to \infty} (y) = 0$

decaying oscillation.

§3.4: 3, 11, 14

3) $4y'' - 4y' - 3y = 0$

CE: $4r^2 - 4r - 3 = 0$ $\qquad \begin{array}{cc} 2 & 1 \\ 2 & -3 \end{array}$

$(2r+1)(2r-3)$

$r = -\frac{1}{2}, \frac{3}{2} \Rightarrow \boxed{y = c_1 e^{3/2 t} + c_2 e^{-1/2 t}}$

11) $9y'' - 12y' + 4y = 0 \qquad y(0) = 2 \quad y'(0) = -1$

CE: $9r^2 - 12r + 4 = 0$ $\qquad \begin{array}{cc} 3 & -2 \\ 3 & -2 \end{array}$

$(3r-2)^2 = 0$

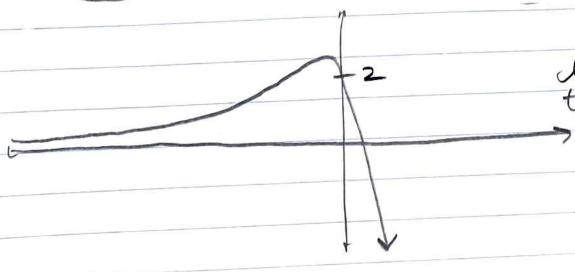
$r_{1,2} = \frac{2}{3} \Rightarrow y = c_1 e^{2/3 t} + c_2 t e^{2/3 t}$

$y(0) = c_1 = 2$

$y'(0) = 2\left(\frac{2}{3} e^{2/3 t}\right) + c_2 e^{2/3 t} + c_2 t\left(\frac{2}{3} e^{2/3 t}\right)$

$= \frac{4}{3} + c_2 = -1 \Rightarrow c_2 = \frac{-7}{3}$

$\boxed{y = 2e^{2t/3} - \frac{7}{3} t e^{2t/3}}$



$\lim_{t \to \infty} (y) = -\infty$

14) $y'' + 4y' + 4y = 0 \qquad y(-1) = 2 \qquad y'(-1) = 1$

$CE: r^2 + 4r + 4 = 0$

$(r+2)^2$

$r_{1,2} = -2$

$y = c_1 e^{-2t} + c_2 t e^{-2t}$

$y(-1) = c_1 e^2 + c_2(-1)e^2 = 2 \Rightarrow c_1 e^2 = 2 + c_2 e^2$

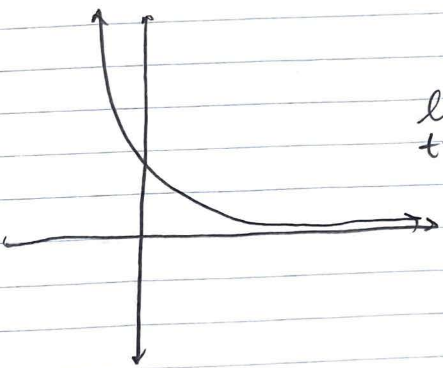
$y' = -2c_1 e^{-2t} + c_2 e^{-2t} - 2c_2 t e^{-2t}$

$y'(-1) = -2\underbrace{c_1 e^2} + c_2 e^2 + 2c_2 e^2 = 1$

$-2(c_2 e^2 + 2) + c_2 e^2 + 2c_2 e^2 = 1$

$c_2 e^2 = 5 \Rightarrow c_2 = 5e^{-2}, \quad c_1 = 7e^{-2}$

soln: $\boxed{y = 7e^{(-2t-2)} + 5te^{(-2t-2)}}$



$\lim_{t \to \infty} (y) = 0$