Lecture 24

Today:
- more examples $\vec{x}'(t) = A\vec{x}(t)$
- case of repeated evals

Situation w/ repeated evals:

Say A = 2x2 matrix, λ = repeated eval.

Two possibilities:

(a) have two linearly independent evecs with eval λ

(b) have only one evec w/ eval λ up to scaling

prototypical example

$$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

prototypical example

$$\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$$

Ex: $\vec{x}'(t) = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \vec{x}(t)$

find evals:

$$\begin{vmatrix} 3-\lambda & 0 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^2$$

$$\lambda_1 = \lambda_2 = 3$$

find evecs:

$$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 3\begin{pmatrix} a \\ b \end{pmatrix} \iff \begin{cases} 3a = 3a \\ 3b = 3b \end{cases}$$

holds for any choice of a, b

So any $\begin{pmatrix} a \\ b \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is an evect w/ eval 3.

General solution: $\vec{x}(t) = c_1 e^{3t}\begin{pmatrix} 17 \\ 5 \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} \pi \\ 2\pi \end{pmatrix}$

Ex: $\vec{x}'(t) = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} \vec{x}(t)$

find evals:

$$\begin{vmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^2$$

$$\lambda_1 = \lambda_2 = 3.$$

find evecs:

$$\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 3\begin{pmatrix} a \\ b \end{pmatrix} \iff \begin{cases} 3a + b = 3a \Rightarrow b = 0 \\ 3b = 3b \end{cases}$$

So $\begin{pmatrix}1\\0\end{pmatrix}$ is an eigenvector of $\begin{pmatrix}3&1\\0&3\end{pmatrix}$ w/ eval 3

$\longrightarrow$ so $e^{3t}\begin{pmatrix}1\\0\end{pmatrix}$ is <u>a</u> sol$^{\text{n}}$.

How to find a second solution?

$$\vec{x}'(t) = \begin{pmatrix}3&1\\0&3\end{pmatrix}\vec{x}(t) \Longleftrightarrow \begin{cases} x_1'(t) = 3x_1(t) + x_2(t) \\ x_2'(t) = 3x_2(t) \end{cases}$$

We can solve first for x_2, then for x_1.

$x_2(t) = Ce^{3t}$. Let's take $C=1$. So $x_2(t) = e^{3t}$.

Then $\underbrace{x_1'(t) = 3x_1(t) + e^{3t}}$.

ansatz $x_1(t) = Ate^{3t}$

can use integrating factor

or use method of undetermined coeffs

$\longrightarrow Ae^{3t} + \cancel{At\,3e^{3t}} = \cancel{3Ate^{3t}} + e^{3t}$

$\Longrightarrow A = 1.$

So $x_1(t) = te^{3t}$

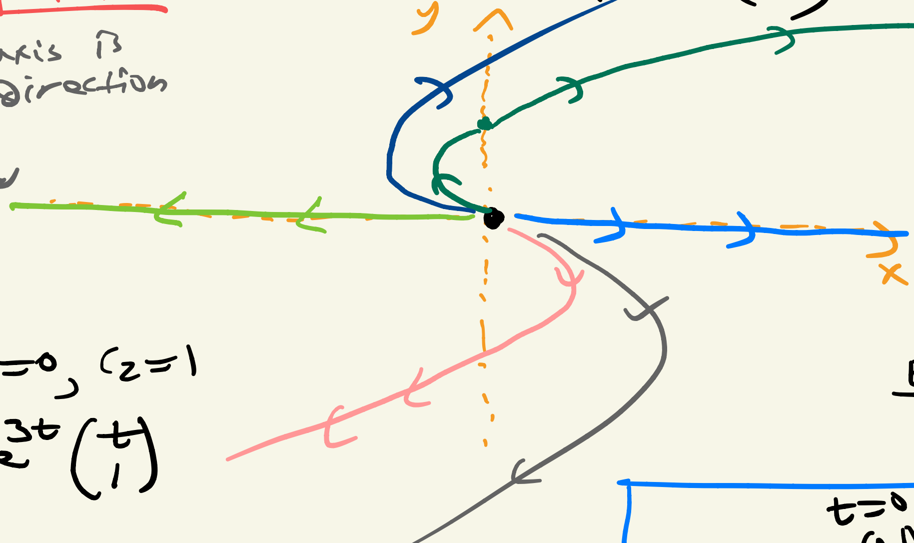
So our second sol$^{\text{n}}$ is $(x_1(t), x_2(t)) = (te^{3t}, e^{3t})$.

General sol'n: $\vec{x}(t) = c_1 e^{3t}\begin{pmatrix}1\\0\end{pmatrix} + c_2\begin{pmatrix}te^{3t}\\e^{3t}\end{pmatrix}$

$$= c_1 e^{3t}\begin{pmatrix}1\\0\end{pmatrix} + c_2 e^{3t}\left(t\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\right).$$

Phase portrait:

x-axis is eigendirection

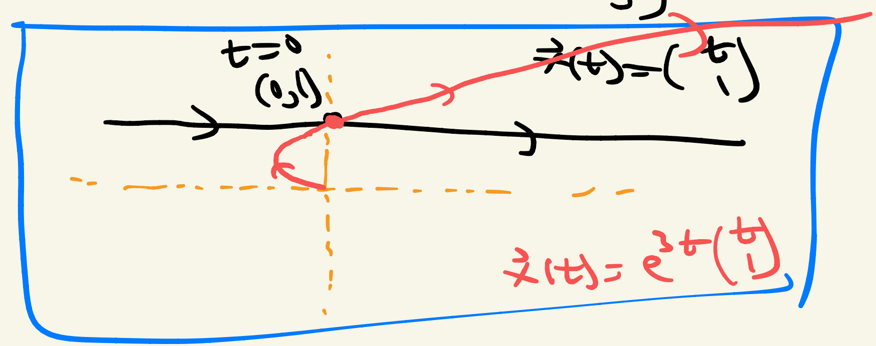


Ex: $c_1 = c_2 = 0$
$\vec{x}(t) = (0,0)$

Ex: $c_1 = 1,\ c_2 = 0$
$\vec{x}(t) = (e^{3t}, 0)$

Ex: $c_1 = -1,\ c_2 = 0$
$\vec{x}(t) = (-e^{3t}, 0)$

Ex: $c_1 = 0,\ c_2 = 1$
$\vec{x}(t) = e^{3t}\begin{pmatrix}t\\1\end{pmatrix}$



$\underline{\text{Ex}}$: $\vec{x}'(t) = \begin{pmatrix} -3 & 5/2 \\ -5/2 & 2 \end{pmatrix} \vec{x}(t)$

$\underline{\text{find evals}}$:

$$\begin{vmatrix} -3-\lambda & 5/2 \\ -5/2 & 2-\lambda \end{vmatrix}$$

$$\overset{\shortparallel}{(\lambda+3)(\lambda-2)} + \frac{25}{4}$$

$$\lambda^2 \overset{\shortparallel}{+} \lambda - 6 + \frac{25}{4} = \lambda^2 + \lambda + 1/4$$

$$= (\lambda + \tfrac{1}{2})^2$$

$$\lambda_1 = \lambda_2 = -1/2$$

$\underline{\text{find evecs}}$:

$$\begin{pmatrix} -3 & 5/2 \\ -5/2 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -1/2 \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} -3a + 5/2\, b = -a/2 \\ -5/2\, a + 2b = -b/2 \end{cases}$$

$$\Leftrightarrow \begin{cases} -5/2\, a = -5/2\, b \\ -5/2\, a = -5/2\, b \end{cases} \implies b = a$$

can take $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ as evec w/ eval $-1/2$

So $e^{-t/2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a soln. How to find another?

$\underline{\text{Ansatz}}$: $\vec{x}(t) = e^{-t/2}\left(t\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix}\right)$.

Need to find c, d to make this a soln.

$$\vec{x}\,'(t) = -\frac{1}{2}e^{-t/2}\left(t\begin{pmatrix}1\\1\end{pmatrix}+\begin{pmatrix}c\\d\end{pmatrix}\right) + e^{-t/2}\begin{pmatrix}1\\1\end{pmatrix}$$

So need

$$\underbrace{-\frac{1}{2}e^{-t/2}\left(t\begin{pmatrix}1\\1\end{pmatrix}+\begin{pmatrix}c\\d\end{pmatrix}\right) + e^{-t/2}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-3 & 5/2\\-5/2 & 2\end{pmatrix}e^{-t/2}\left(t\begin{pmatrix}1\\1\end{pmatrix}+\begin{pmatrix}c\\d\end{pmatrix}\right)}$$

$$\Longleftrightarrow \quad -\frac{1}{2}\left(t\begin{pmatrix}1\\1\end{pmatrix}+\begin{pmatrix}c\\d\end{pmatrix}\right) + \begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-3 & 5/2\\-5/2 & 2\end{pmatrix}\left(t\begin{pmatrix}1\\1\end{pmatrix}+\begin{pmatrix}c\\d\end{pmatrix}\right)$$

$$\Longleftrightarrow \quad \boxed{-\frac{1}{2}\begin{pmatrix}c\\d\end{pmatrix}+\begin{pmatrix}1\\1\end{pmatrix}} - \frac{1}{2}t\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-3 & 5/2\\-5/2 & 2\end{pmatrix}\begin{pmatrix}c\\d\end{pmatrix} + t\begin{pmatrix}-3 & 5/2\\-5/2 & 2\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}$$

must have

$$-\frac{1}{2}\begin{pmatrix}c\\d\end{pmatrix}+\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-3 & 5/2\\-5/2 & 2\end{pmatrix}\begin{pmatrix}c\\d\end{pmatrix}$$

$$\checkmark \quad -\frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-3 & 5/2\\-5/2 & 2\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}$$

says $\begin{pmatrix}1\\1\end{pmatrix}$ is an evec w/ eval $-1/2$ which is true $\checkmark$

So we just need

$$-\frac{1}{2}\begin{pmatrix}c\\d\end{pmatrix}+\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-3 & 5/2\\-5/2 & 2\end{pmatrix}\begin{pmatrix}c\\d\end{pmatrix}$$

$$\Longleftrightarrow \quad \begin{pmatrix}1\\1\end{pmatrix} = \left[\begin{pmatrix}-3 & 5/2\\-5/2 & 2\end{pmatrix} - \left(-\frac{1}{2}\right)\mathbb{1}\right]\begin{pmatrix}c\\d\end{pmatrix}$$

$$\iff \begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-5/2 & 5/2\\ -5/2 & 5/2\end{pmatrix}\begin{pmatrix}c\\d\end{pmatrix}$$

$$\iff \begin{cases} -5/2\,c + 5/2\,d = 1 \\ -5/2\,c + 5/2\,d = 1 \end{cases} \implies \frac{5}{2}d = 1 + 5/2\,c$$

$$d = c + 2/5$$

so $\begin{pmatrix}c\\d\end{pmatrix}$ can be any vector ~~of form~~ $\begin{pmatrix}c\\c+2/5\end{pmatrix}$.

$\rightsquigarrow$ gives soln $e^{-t/2}\left(t\begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}c\\c+2/5\end{pmatrix}\right)$ for any c.

Note: can write as $e^{-t/2}\left(t\begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}0\\2/5\end{pmatrix}\right) + e^{-t/2}c\begin{pmatrix}1\\1\end{pmatrix}$

might as well $c=0$.

$\longrightarrow$ second sol'n: $e^{-t/2}\left(t\begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}0\\2/5\end{pmatrix}\right)$

General soln: $\underbrace{C_1 e^{-t/2}\begin{pmatrix}1\\1\end{pmatrix} + C_2 e^{-t/2}\left(t\begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}0\\2/5\end{pmatrix}\right)}$.

"generalized eigenvector"

Ex: $\vec{x}'(t) = \begin{pmatrix} 0 & -3 \\ 1 & 0 \end{pmatrix} \vec{x}(t)$

find evals:

$$\begin{vmatrix} -\lambda & -3 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 3$$

$$\underbrace{\lambda_1 = i\sqrt{3}, \quad \lambda_2 = -i\sqrt{3}}_{\text{purely imaginary evals}}$$

evecs: $\lambda = i\sqrt{3}$

$$\begin{pmatrix} 0 & -3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = i\sqrt{3} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\Longleftrightarrow \quad -3b = i\sqrt{3}\, a \quad \Longrightarrow \quad b = \frac{-i}{\sqrt{3}} a.$$

can take $\begin{pmatrix} \sqrt{3} \\ -i \end{pmatrix}$

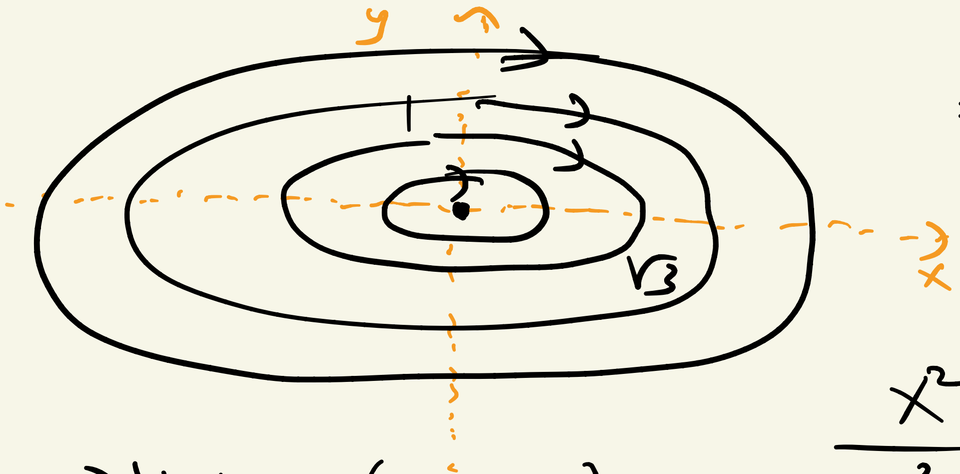
$\lambda = -i\sqrt{3} \longrightarrow$ can take $\begin{pmatrix} \sqrt{3} \\ i \end{pmatrix}$

$$e^{i\sqrt{3}t} \begin{pmatrix} \sqrt{3} \\ -i \end{pmatrix} = \left(\cos(\sqrt{3}t) + i\sin(\sqrt{3}t)\right) \begin{pmatrix} \sqrt{3} \\ i \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} \sqrt{3}\cos(\sqrt{3}t) \\ -\sin(\sqrt{3}t) \end{pmatrix}}_{\text{real part}} + i \underbrace{\begin{pmatrix} \sqrt{3}\sin(\sqrt{3}t) \\ \cos(\sqrt{3}t) \end{pmatrix}}_{\text{imag part}}$$

General solution:

$$\vec{x}(t) = C_1 \begin{pmatrix} \sqrt{3}\cos(\sqrt{3}t) \\ -\sin(\sqrt{3}t) \end{pmatrix} + C_2 \begin{pmatrix} \sqrt{3}\sin(\sqrt{3}t) \\ \cos(\sqrt{3}t) \end{pmatrix}$$



Ex: $c_1 = 0,\ c_2 = 1$

$$\vec{x}(t) = \begin{pmatrix} \sqrt{3}\,\sin(\sqrt{3}t) \\ \cos(\sqrt{3}t) \end{pmatrix}$$

lies on curve

$$\frac{x^2}{3} + y^2 = 1.$$

Ex: $$\vec{x}'(t) = \begin{pmatrix} -1 & -2 \\ 3/2 & 1 \end{pmatrix} \vec{x}(t)$$

evals: $$\begin{vmatrix} -1-\lambda & -2 \\ 3/2 & 1-\lambda \end{vmatrix} = (\lambda-1)(\lambda+1) + 3$$
$$= \lambda^2 + 2$$
$$\lambda_1 = i\sqrt{2},\quad \lambda_2 = -i\sqrt{2}$$

evecs: $$\begin{pmatrix} -1 & -2 \\ 3/2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = i\sqrt{2} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\iff -a-2b = i\sqrt{2}\,a \iff 2b = a(-1-i\sqrt{2})$$

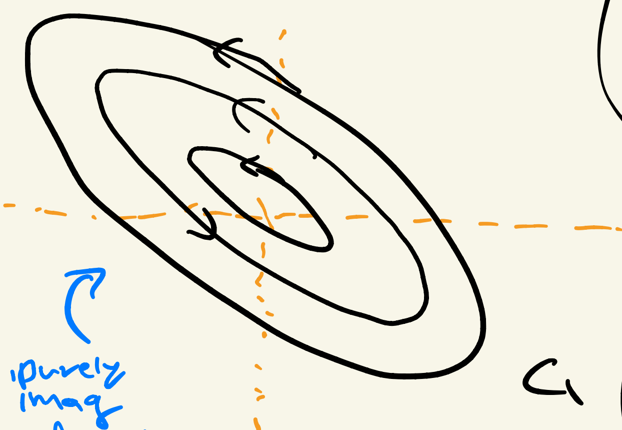
$$b = a\left(\frac{-1-i\sqrt{2}}{2}\right)$$

can take $\begin{pmatrix}2\\-1-i\sqrt{2}\end{pmatrix}$

$$e^{i\sqrt{2}t}\begin{pmatrix}2\\-1-i\sqrt{2}\end{pmatrix} \rightsquigarrow \left[\cos(\sqrt{2}t)+i\sin(\sqrt{2}t)\right]\begin{pmatrix}2\\-1-i\sqrt{2}\end{pmatrix}$$

$$= \underbrace{\begin{pmatrix}\cos(\sqrt{2}t)\\ -\cos(\sqrt{2}t)+\sqrt{2}\sin(\sqrt{2}t)\end{pmatrix}}_{\text{real part}} + i\underbrace{\begin{pmatrix}2\sin(\sqrt{2}t)\\ -\sin(\sqrt{2}t)-\sqrt{2}\cos(\sqrt{2}t)\end{pmatrix}}_{\text{imag part}}$$

phase portrait:



purely imag eigenvalues

General soln:

$$c_1\begin{pmatrix}\cos(\sqrt{2}t)\\ \dots\end{pmatrix} + c_2\begin{pmatrix}2\sin(\sqrt{2}t)\\ \dots\end{pmatrix}$$