Lecture 22

Today:
- Evecs / evals
- lots of examples

Recall: **Ex**: $A = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$ find the evecs and evals

Want $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$, $\lambda \in \mathbb{C}$ s.t. $A\vec{v} = \lambda \vec{v}$. and $\vec{v} \neq \vec{0}$

Need $\begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}$.

$$\iff \underbrace{(A - \lambda \mathbb{1})\vec{v}}_{\text{only possible if } A-\lambda\mathbb{1} \text{ is \underline{not} invertible.}} = \vec{0}$$

$$\iff \det(A - \lambda \mathbb{1}) = 0$$

$$\iff \left| \begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = \begin{vmatrix} 5-\lambda & -1 \\ -1 & 5-\lambda \end{vmatrix} = 0$$

$$\iff (5-\lambda)^2 - 1 = 0$$

$$\iff \underbrace{\lambda^2 - 10\lambda + 24}_{\text{characteristic polynomial}} = 0$$

$$\iff (\lambda-4)(\lambda-6) = 0$$

$$\longrightarrow \lambda = 4, 6$$

evals of A

Find eigenvectors of A :

$\lambda = 4$:

$$\begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 4 \begin{pmatrix} a \\ b \end{pmatrix} \iff \begin{pmatrix} 5a - b \\ -a + 5b \end{pmatrix} = \begin{pmatrix} 4a \\ 4b \end{pmatrix}$$

$$\iff \begin{cases} 5a - b = 4a \\ -a + 5b = 4b \end{cases}$$

$\longrightarrow \begin{cases} a = b \\ -a = -b \end{cases} \longrightarrow$ need $b = a$, so can take any vector of form $\begin{pmatrix} a \\ a \end{pmatrix}$ for $a \neq 0$.

$\lambda = 6$: $\begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = 6\begin{pmatrix} c \\ d \end{pmatrix} \iff \begin{cases} 5c - d = 6c \\ -c + 5d = 6d \end{cases}$

$\iff \begin{cases} -c = d \\ -c = d \end{cases}$

so can take any vector of form $\begin{pmatrix} c \\ -c \end{pmatrix}$ for $c \neq 0$.

So $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} \pi \\ -\pi \end{pmatrix}$ form a basis of $\mathbb{R}^2$ consisting of eigenvectors of A.

assoc eval is 4

assoc eval is 6

Note: if $\vec{r}$ is an evec of A w/ eval λ, so is $c\vec{r}$ for $c \neq 0$ scalar.

Systems of ODEs:

General system of 1st order ODEs:

$$\begin{cases} x_1'(t) = F_1(t, x_1, \dots, x_n) \\ x_2'(t) = F_2(t, x_1, \dots, x_n) \\ \dots \\ x_n'(t) = F_n(t, x_1, \dots, x_n) \end{cases}$$

Ex: $\begin{cases} x'(t) = x(t)y(t) \\ y'(t) = x(t)\sin(y(t)) \end{cases}$ (nonlinear)

General system of 1st order linear ODEs:

$$(*) \begin{cases} x_1'(t) = P_{11}(t)x_1(t) + \dots + P_{1n}(t)x_n(t) + g_1(t) \\ \vdots \\ x_n'(t) = P_{n1}(t)x_1(t) + \dots + P_{nn}(t)x_n(t) + g_n(t) \end{cases}$$

inhomog. terms

Introduce $\vec{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$

vector-valued function of t

$$P(t) = \begin{pmatrix} p_{11}(t) & \cdots & p_{1n}(t) \\ \vdots & & \\ p_{n1}(t) & \cdots & p_{nn}(t) \end{pmatrix} \qquad \vec{g}(t) = \begin{pmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{pmatrix}$$

(*) can be written $\underbrace{\vec{x}'(t) = P(t)\vec{x}(t) + \vec{g}(t)}_{\text{matrix-vector multiplication}}$

Ex: Airy eqn: $x''(t) = t\,x(t)$

Put $y(t) := x'(t)$

$$\Longleftrightarrow \begin{cases} x'(t) = y(t) \\ y'(t) = t\,x(t) \end{cases}$$

Constant coefficient case:

$$\vec{x}'(t) = A\vec{x}(t), \quad A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

ordinary matrix (constant)

Ex:

$$\begin{cases} x_1'(t) = 16x_1(t) - 3x_2(t) \\ x_2'(t) = 2x_1(t) + 9x_2(t) \end{cases}$$

Ansatz: $x_1(t) = e^{r_1 t}$ $\qquad x_2(t) = e^{r_2 t}$

plug in:

$$\begin{cases} r_1 e^{r_1 t} = 16e^{r_1 t} - 3e^{r_2 t} \\ r_2 e^{r_2 t} = 2e^{r_1 t} + 9e^{r_2 t} \end{cases}$$

$$\Longleftrightarrow \begin{cases} (r_1 - 16)e^{r_1 t} = -3e^{r_2 t} \\ (r_2 - 9)e^{r_2 t} = 2e^{r_1 t} \end{cases}$$

$$\Longleftrightarrow \begin{cases} \dfrac{r_1 - 16}{-3} = e^{(r_2 - r_1)t} \\ \dfrac{r_2 - 9}{2} = e^{(r_1 - r_2)t} \end{cases}$$

only possible if $r_1 = r_2$.

$$\boxed{\cancel{5e^{3t} = 17e^{6t} \text{ for all } t}}$$

So $r_1 = r_2$. Then $\begin{cases} r_1 - 16 = -3 \\ r_2 - 9 = 2 \end{cases} \Rightarrow \begin{matrix} r_1 = 13 \\ r_2 = 11. \end{matrix}$

But $13 \neq 11$, failure...!

Recall: $y'' + 3y = e^t$

Ansatz $y(t) = Ae^t$

Modified ansatz: $x_1(t) = C_1 e^{r_1 t}$ $x_2(t) = C_2 e^{r_2 t}$

Note: If $\vec{x}(t)$ is a solution to $(*)$, so is

$$c\vec{x}(t) = c\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} cx_1(t) \\ cx_2(t) \end{pmatrix}$$

Need:
$$\begin{cases} C_1 r_1 e^{r_1 t} = 16 C_1 e^{r_1 t} - 3 C_2 e^{r_2 t} \\ C_2 r_2 e^{r_2 t} = 2 C_1 e^{r_1 t} + 9 C_2 e^{r_2 t} \end{cases}$$

Claim: by similar reasoning, must have $r_1 = r_2$.

$$\begin{cases} c_1 r_1 = 16c_1 - 3c_2 \\ c_2 r_1 = 2c_1 + 9c_2 \end{cases}$$

Take step back: Our ansatz is $\vec{x}(t) = \begin{pmatrix} c_1 e^{rt} \\ c_2 e^{rt} \end{pmatrix} = e^{rt}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

Need: $\vec{x}'(t) = \begin{pmatrix} 16 & -3 \\ 2 & 9 \end{pmatrix} \vec{x}(t)$

$$\begin{pmatrix} c_1 r e^{rt} \\ c_2 r e^{rt} \end{pmatrix} = \begin{pmatrix} 16 & -3 \\ 2 & 9 \end{pmatrix} \begin{pmatrix} c_1 e^{rt} \\ c_2 e^{rt} \end{pmatrix}$$

$$\Leftrightarrow \underbrace{r\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 16 & -3 \\ 2 & 9 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}}_{\text{says } r \text{ is an eval of } A \text{ w/ evec } \binom{c_1}{c_2}.}$$

To proceed, compute evals + evecs of A:

Char. poly:
$$\left|\begin{pmatrix}16 & -3\\ 2 & 9\end{pmatrix} - \lambda \mathbb{1}\right| = \begin{vmatrix}16-\lambda & -3\\ 2 & 9-\lambda\end{vmatrix}$$
$$= (16-\lambda)(9-\lambda)+6$$
$$= 144 - 25\lambda + \lambda^2 + 6$$
$$= \lambda^2 - 25\lambda + 150$$
$$= (\lambda-10)(\lambda-15)$$

So the evals are $10, 15$.

Find evecs:

$\lambda = 10$: $\begin{pmatrix}16 & -3\\ 2 & 9\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = 10\begin{pmatrix}a\\ b\end{pmatrix} \iff \begin{cases}16a-3b = 10a\\ 2a+9b = 10b\end{cases}$

$$\iff \begin{cases}6a = 3b\\ 2a = b\end{cases}$$

So can take $\begin{pmatrix}1\\2\end{pmatrix}$ $\left(\text{or } \begin{pmatrix}\pi+1\\2\pi+2\end{pmatrix} \text{ also works}\right)$

$\lambda = 15$: $\begin{pmatrix}16 & -3\\2 & 9\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = 15\begin{pmatrix}a\\b\end{pmatrix} \iff \begin{cases}16a - 3b = 15a\\2a + 9b = 15b\end{cases}$

$\iff \begin{cases}a = 3b\\2a = 6b\end{cases}$

so can take $\begin{pmatrix}3\\1\end{pmatrix}$.

<u>Conclusion</u>: $\vec{x}(t) = e^{10t}\begin{pmatrix}1\\2\end{pmatrix}$ and $\vec{x}(t) = e^{15t}\begin{pmatrix}3\\1\end{pmatrix}$ are both solns to (*)

$x_1(t) = e^{10t}$
$x_2(t) = 2e^{10t}$

$x_1(t) = 3e^{15t}$
$x_2(t) = e^{15t}$

Also, $\tilde{c}_1 e^{10t}\begin{pmatrix}1\\2\end{pmatrix} + \tilde{c}_2 e^{15t}\begin{pmatrix}3\\1\end{pmatrix}$ is the general sol'n

$$\iff x_1(t) = \tilde{c}_1 e^{10t} + 3\tilde{c}_2 e^{15t},$$
$$x_2(t) = 2\tilde{c}_1 e^{10t} + \tilde{c}_2 e^{15t}.$$

Consider $\vec{x}'(t) = A\vec{x}(t)$.

now A = $n\times n$ matrix, $\vec{x}(t)$ = n dim'l vector

Make ansatz $\vec{x}(t) = e^{rt}\begin{pmatrix}c_1\\ \vdots\\ c_n\end{pmatrix} = e^{rt}\vec{c}$

Then need: $\vec{x}'(t) = A\vec{x}(t)$

$$\iff r\cancel{e^{rt}}\begin{pmatrix}c_1\\ \vdots\\ c_n\end{pmatrix} = A\cancel{e^{rt}}\begin{pmatrix}c_1\\ \vdots\\ c_n\end{pmatrix}$$

$$\iff r\begin{pmatrix}c_1\\ \vdots\\ c_n\end{pmatrix} = A\begin{pmatrix}c_1\\ \vdots\\ c_n\end{pmatrix}$$

So $\vec{c}$ must be evec w/ eval r.

Char poly: $|A - \lambda I|$ ← always a poly in λ Degree n

→ has n roots, which are the evals of A.

But:
- could have repeated evals
- could have cpx evals

Simplest scenario: can n distinct real eigenvalues $r_1, \ldots, r_n$. Then have n distinct evecs.

But if we take repeated roots, then we may or may not have a basis of eigenvectors.

Ex: $A =$ 2x2 matrix, char poly $= (\lambda - 3)^2$.

Then $\lambda_1 = \lambda_2 = 3$ repeated eval.

Two possibilities: (a) have two linearly indep. evecs (b) have only one indep eigenvec.

Ex: $A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$. $\iff \begin{cases} x_1'(t) = 3x_1(t) \\ x_2'(t) = 3x_2(t). \end{cases}$

char poly: $\left| \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = \begin{vmatrix} 3-\lambda & 0 \\ 0 & 3-\lambda \end{vmatrix}$

$= (3-\lambda)^2$

So $\lambda_1 = \lambda_2 = 3$ repeated eval.

Evecs:

$\lambda = 3$: $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 3 \begin{pmatrix} a \\ b \end{pmatrix}$

$\iff \begin{cases} 3a = 3a \\ 3b = 3b \end{cases}$ imposes no conditions on a or b.

Upshot: any nonzero vector $\begin{pmatrix} a \\ b \end{pmatrix}$ is an eigenvector.

Basis of evecs: $\begin{pmatrix}1\\0\end{pmatrix}$ $\begin{pmatrix}0\\1\end{pmatrix}$.

General solution: $c_1 e^{3t}\begin{pmatrix}1\\0\end{pmatrix} + c_2 e^{3t}\begin{pmatrix}0\\1\end{pmatrix}$.

<u>Ex</u>: $A = \begin{pmatrix}3 & 1\\0 & 3\end{pmatrix}$. <u>Claim</u>: have repeated eval $\lambda_1 = \lambda_2 = 3$. but <u>no</u> basis of evecs.

<u>To do</u>:

- deal with cpx evals / evecs
- deal rep'd roots
- plot solutions understand their behavior