Lecture 20

Today: • Finish up series sol'ns
• Systems of ODEs

In general, consider $y''(t)+p(t)y'(t)+q(t)y(t)=0$, $(*)$
suppose $t=0$ is a regular singular pt.
Means: $tp(t)$ and $t^2q(t)$ are analytic at $t=0$.
Can write $(*)$ as $t^2y''+t(tp)y'+(t^2q)y=0$.

Strategy: Ansatz $y(t)=t^r\sum_{n=0}^{\infty}a_nt^n$, plug in, try to solve for r, $a_0, a_1, a_2, \ldots$ recursively.

Have: $tp(t)=\sum_{n=0}^{\infty}p_nt^n$

$t^2q(t)=\sum_{n=0}^{\infty}q_nt^n$

plug in these series to $(*)$:

$$t^2 \sum_{n=0}^{\infty} a_n (r+n)(r+n-1) t^{n+r-2} + t \underbrace{\left(\sum_{n=0}^{\infty} p_n t^n\right)\left(\sum_{n=0}^{\infty} (r+n) a_n t^{n+r-1}\right)}$$

$$+ \underbrace{\left(\sum_{n=0}^{\infty} q_n t^n\right)\left(\sum_{n=0}^{\infty} a_n t^{n+r}\right)} = 0.$$

can write these each as a single power series by multiplying out

Write out in long form:

$$y = a_0 t^r + a_1 t^{r+1} + a_2 t^{r+2} + \ldots$$

$$y' = a_0 r t^{r-1} + a_1 (r+1) t^r + \ldots$$

$$y'' = a_0 r(r-1) t^{r-2} + a_1 (r+1) r \, t^{r-1} + \ldots$$

$$t^2 \left(a_0 r(r-1) t^{r-2} + a_1 (r+1) r \, t^{r-1} + \ldots \right)$$

$$+ t \left(p_0 + p_1 t + p_2 t^2 + \ldots \right) \left(a_0 r t^{r-1} + a_1 (r+1) t^r + \ldots \right)$$

$$+ \left(q_0 + q_1 t + q_2 t^2 + \ldots \right) \left(a_0 t^r + a_1 t^{r+1} + \ldots \right)$$

$$= t^r \left(a_0 r(r-1) + p_0 a_0 r + q_0 a_0 \right)$$

$$+ t^{r+1} \left(a_1 (r+1) r + p_0 a_1 (r+1) + p_1 a_0 r + q_0 a_1 + q_1 a_0 \right)$$

$$+ t^{r+2}(\cdots) + t^{r+3}(\cdots) = 0$$

Note: t^r term gives $\underbrace{r(r-1) + p_0 r + q_0 = 0}_{\text{indicial equation}}$

If $p_1 = p_2 = p_3 = \cdots = 0$, $q_1 = q_2 = \cdots = 0$, then $(*)$ would be $t^2 y'' + t p_0 y' + q_0 y = 0$. (Euler eqn)

Put $F(r) := r(r-1) + p_0 r + q_0$

(so indicial eqn $\iff F(r) = 0$)

Here:

$$a_0 F(r) t^r + \sum_{n=1}^{\infty} \Big(\underbrace{F(r+n) a_n + \sum_{k=0}^{n-1} a_k \big((r+k) p_{n-k} + q_{n-k} \big)}_{=0} \Big) t^{n+r}$$

good exercise to verify this

(or check book)

So our recursion is:

(1) • $F(r) = 0$

(2) • $a_n = -\dfrac{\left(\sum_{k=0}^{n-1} a_k \left((r+k)p_{n-k} + q_{n-k}\right)\right)}{F(r+n)}$ for $n \geq 1$.

involves $a_0, \dots, a_{n-1}$

By (1), find r_1 & r_2 "exponents of the singularity".

Possible issue: $F(r+n) = 0$. i.e. $r+n = r_1$ or $r+n = r_2$.

If $r = r_1$, then $r+n \neq r_1$ or r_2 for $n \geq 1$.

But, if $r_1 - r_2$ is an integer, then if $r = r_2$, $r+n$ could be r_1.

(read § 5.6 for more details)

Situation:

• If r_1 is real, can find a Frobenius

Series soln $y_1(t) = 1 + \sum_{n=1}^{\infty} t^{r_1} a_n t^n$.

- If $r_1 - r_2$ is <u>not</u> an integer, then can find a second Frob. series soln $y_2(t) = 1 + \sum_{n=1}^{\infty} t^{r_2} b_n t^n$.

[• If $r_1 = r_2$ or $r_1 - r_2$ is an integer or r_1, r_2 are complex then second soln is more complicated.]

Systems of (1st order, ~~linear~~ constant coeff) ODEs

<u>Ex</u> ("SIS system")

$S(t) =$ # of susceptible but not infected individuals

$I(t) =$ # of infected individuals

nonlinear system due to SI term

$$\begin{cases} S'(t) = -\alpha S(t) I(t) + \beta I(t) \\ I'(t) = \alpha S(t) I(t) - \beta I(t) \end{cases}$$

α, β constants

Goal: Solve for $S(t)$ and $I(t)$, given $S(0)$, $I(0)$.

Solve: $S' + I' = 0 \Rightarrow S(t) + I(t) = C$ constant

$\Rightarrow S(t) = C - I(t)$

Then 2nd eqn gives. $\underbrace{I' = \alpha(C - I)I - \beta I}_{\text{separable 1st order ODE!}}$

$\rightarrow$ solve for $I(t)$, and then $S(t) = C - I(t)$

$(C = I(0) + S(0))$

Ex: System $\begin{cases} x'(t) = 4x(t) \\ y'(t) = 6y(t) \end{cases}$ $\leftarrow$ "diagonal" or "uncoupled" system

Solution: $x(t) = C_1 e^{4t}$, $y(t) = C_2 e^{6t}$

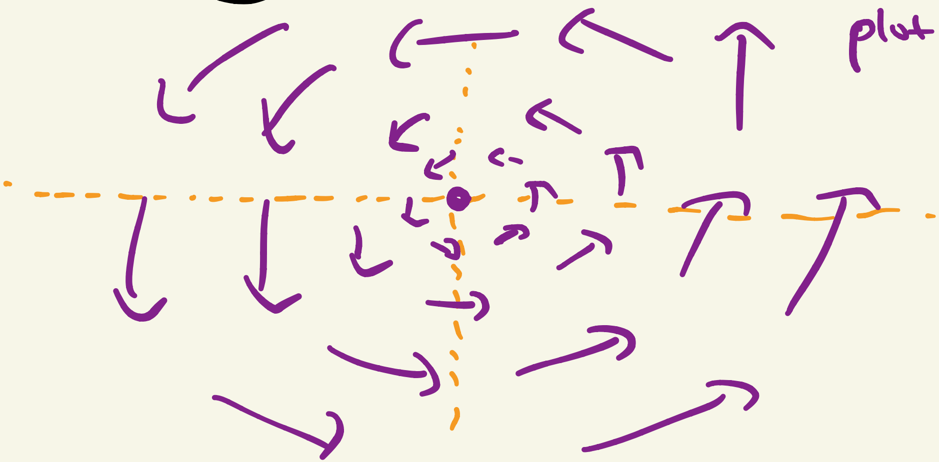
Ex:

$$\begin{cases} x'(t) = -y(t) \\ y'(t) = x(t) \end{cases}$$

Equilibrium? $x(t) \equiv 0,\ y(t) \equiv 0$

plot the vector field $(-y, x)$

(note: $(-y,x)$ is perpendicular to (x,y))



Solve explicitly: $x''(t) = -y'(t) = -x(t)$

So $x''(t) = -x(t) \implies x(t) = C_1 \sin(t) + C_2 \cos(t)$

$\rightarrow y(t) = -x'(t) = -C_1 \cos(t) + C_2 \sin(t)$,

So $(x(t), y(t)) = \left(C_1 \sin(t) + C_2 \cos(t),\ -C_1 \cos(t) + C_2 \sin(t)\right)$

Check: $x(t)^2 + y(t)^2 = c_1^2 \sin^2(t) + c_2^2 \cos^2(t) + c_1^2 \cos^2(t)$
$\cancel{+ 2c_1 c_2 \sin(t)\cos(t)} + c_2^2 \sin^2(t)$
$\cancel{- 2c_1 c_2 \sin(t)\cos(t)}$

$$= c_1^2 + c_2^2$$

Ex: $$x''(t) + \sin(x(t)) = 0 \quad (*)$$ ("pendulum equation")

nonlinear!

Claim: equivalent to a system of 1st order eqns!

Introduce $y(t) = x'(t)$.

Then $(*) \iff \begin{cases} x'(t) = y(t) \\ y'(t) = -\sin(x(t)) \end{cases}$

In fact: any nth order ODE can be converted similarly to a system of n first order ODEs!

Ex:

$$\begin{cases} x'(t) = 5x(t) - y(t) \\ y'(t) = -x(t) + 5y(t) \end{cases}$$

$$\left(\rightsquigarrow x'(t) + 5y'(t) = -y(t) + 25y(t) \right)$$

$$x'(t) = -5y'(t) + 24y(t)$$

$$-5y' + 24y = 5x - y$$

plug in

$$x(t) = 5y - y'$$

$$\rightsquigarrow -5y' + 24y = 5(5y - y') - y$$

$$= 25y - 5y' - y$$

$$= 24y - 5y'$$

$x(t) = e^{r_1 t}$ $\qquad$ $y(t) = e^{r_2 t}$ $\qquad$ ... ??