Lecture 14

Second order linear ODEs via power series

Review: A power series (aka Taylor series) is a function of the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$$

for some $x_0 \in \mathbb{R}$.

($a_0, a_1, a_2, \ldots \in \mathbb{R}$ "coefficients")

Note: Can also write

$f(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2$

$+a_3(x-x_0)^3+\dots$

Rmk: Often take $x_0 = 0$, so

$f(x) = \underbrace{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots}$

$\rightarrow$ We always have $f(x_0) = a_0$

since $f(x_0) = a_0 + a_1(x_0 - x_0) + a_2(x_0 - x_0)^2 + \dots = a_0$,

but f might not be defined for other values of x.

Ex: $f(x) = \sum_{n=0}^{\infty} n!(x-7)^n$

satisfies $f(7) = 0! = 1$, but

$f(c) = \sum_{n=0}^{\infty} n!(c-7)^n$ diverges for any $c \neq 7$, i.e.

$$\lim_{N\to\infty} \sum_{n=0}^{N} n!(c-7)^n = +\infty$$ for any $c \neq 7$.

, we say that $\sum_{n=0} a_n(x-x_0)^n$ converges pointwise at $x=c$ if $\lim_{N\to\infty} \sum_{n=0}^{N} a_n(c-x_0)^n$ exists and is finite

→ We say it converges absolutely if $\lim_{N\to\infty} \sum_{n=0}^{N} |a_n(c-x_0)^n|$ exists and is finite.

Fact: Have

absolute convergence at $x=c$ $\underset{\not\Leftarrow}{\Longrightarrow}$ ptwise convergence at $x=c$

Ex: $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$ converges ptwise at $x=-1$, but not absolutely.

$f(-1) = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \ldots$

alternating harmonic series converges, but

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$

harmonic harmonic series, does <u>not</u> converge.

→ Every power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ has a <u>radius of convergence</u> R, such that the series converges absolutely at c if $|c-x_0| < R$, the series diverges at c if $|c-x_0| > R$, and at $x_0 - R$ or $x_0 + R$ could converge or diverge.

→ "ratio test": if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ exists and is finite, then the r.o.c. is given by

$$R = \frac{1}{\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|}$$

(and if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0$, then the r.o.c. is ∞)

→ "root test": more generally, the radius is always given by $R = \frac{1}{\limsup_{n\to\infty}\sqrt[n]{|a_n|}}$

(if $\lim_{n\to\infty}\sqrt[n]{|a_n|}$ exists, then limsup = lim, but limsup is <u>always</u> defined)

→ if $f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$ has r.o.c. R and

R_f)

$g(x) = \sum_{n=0}^{\infty} b_n (x-x_0)^n$ has r.o.c. R_g then

$(f+g)(x) = \sum_{n=0}^{\infty} (a_n + b_n)(x-x_0)^n$

has r.o.c. at least $\min(R_f, R_g)$

$\rightarrow$ similarly,

$$f(x)g(x) = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} a_i b_{n-i} \right) (x-x_0)^n$$

$$\boxed{\begin{aligned} &\underline{\text{Check}}: \\ &\left(a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots\right) \\ &\quad \left(b_0 + b_1(x-x_0) + b_2(x-x_0)^2 + \dots\right) \\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0)(x-x_0) \\ &+ (a_0 b_2 + a_1 b_1 + a_2 b_0)(x-x_0)^2 \\ &\qquad + \dots \end{aligned}}$$

and this has r.o.c. <u>at least</u> $\min(R_f, R_g)$.

Def: $f(x)$ is __analytic__ at x_0 if it has a power series at x_0, i.e. $f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$ with r.o.c. > 0.

Rmk: If $f(x)$ is analytic at x_0, then it's __∞ly differentiable__ at x_0.
'smooth'

Rmk: Smooth at x_0 does __not__ imply analytic at x_0.

Fact: If $\sum_{n=0}^{\infty} a_n (x-x_0)^n = 0$ for all x in an open interval, then must have $a_0 = a_1 = a_2 = \dots = 0$.

Ex: Consider $f(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \le 0. \end{cases}$

Then is smooth but not analytic at $x = 0$.

Examples:

Ex: $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\dots$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

r.o.c.?

ratio test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| =$

$$= \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1}$$

$$= 0$$

So $R = +\infty$.

In fact this is e^x.

Ex: $1+x+x^2+x^3+\dots$

$$= \sum_{n=0}^{\infty} x^n$$

ratio test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} 1/1 = 1$

$\Rightarrow R = 1/1 = 1.$

In fact, this $\underbrace{\frac{1}{1-x}}$

Note: the power series diverges for $|x| > 1.$

Ex: $-x - x^2/2 - x^3/3 - x^4/4 - \ldots$

$$= \sum_{n=1}^{\infty} -x^n/n$$

$a_0 = 0$, $a_n = -1/n.$

ratio test: $\lim_{n\to\infty} \frac{(-1/n+1)}{(-1/n)} = \lim_{n\to\infty} \frac{n}{n+1}$

$= 1,$

$R = 1/1 = 1.$

In fact, this is $\ln(1-x)$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$$

integrate term by term:

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots$$

so $\ln(1-x) = -x - x^2/2 - x^3/3 - \dots$

Similarly, $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$

<u>Recall</u>: a <u>rational function</u> is a function of form $f(x) = P(x)/Q(x)$, $P(x), Q(x)$ are polynomials.

<u>Fact</u>: A rational func. $P(x)/Q(x)$ is analytic $x = x_0$ provided that $Q(x_0) \neq 0$, after canceling common factors in $P(x)$ and $Q(x)$.

Moreover, the r.o.c. is then given by the distance from x_0 to the nearest (possibly complex-valued) root of $\alpha(x)$.

Ex: $\dfrac{1+x}{1-x}$ is analytic at all $x \neq 1$

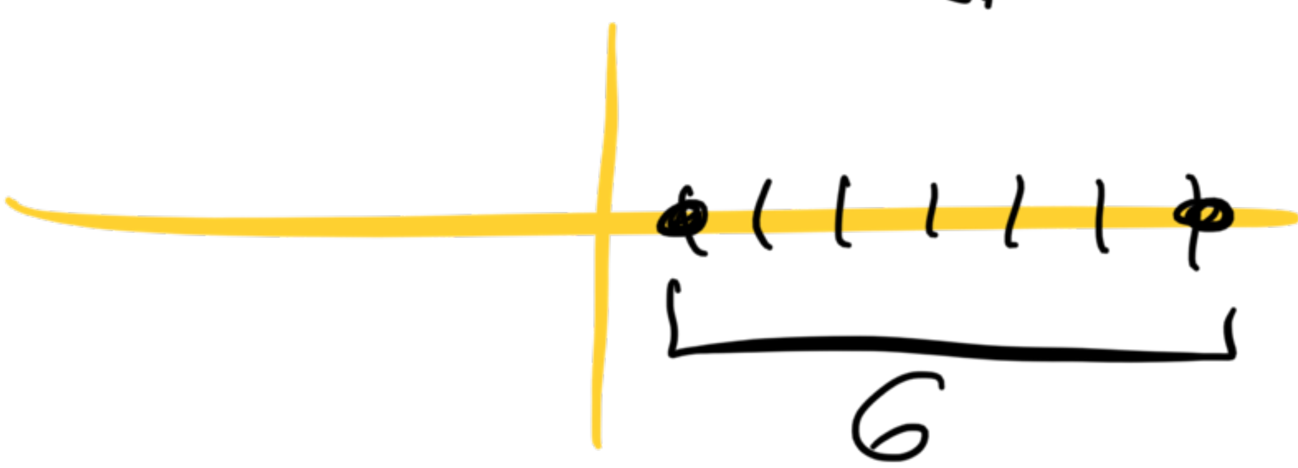
Ex: $\dfrac{x^2+2x+1}{x+1}$ is analytic at all $x \in \mathbb{R}$

since $\dfrac{(x+1)^2}{x+1} = x+1$

Ex: $\dfrac{1}{1-x}$ is analytic at $x=7$

i.e. $\dfrac{1}{1-x} = a_0 + a_1(x-7) + a_2(x-7)^2 + \dots$

and the r.o.c. is 6.



Ex: $\dfrac{x-1}{x^2+1}$ is analytic at

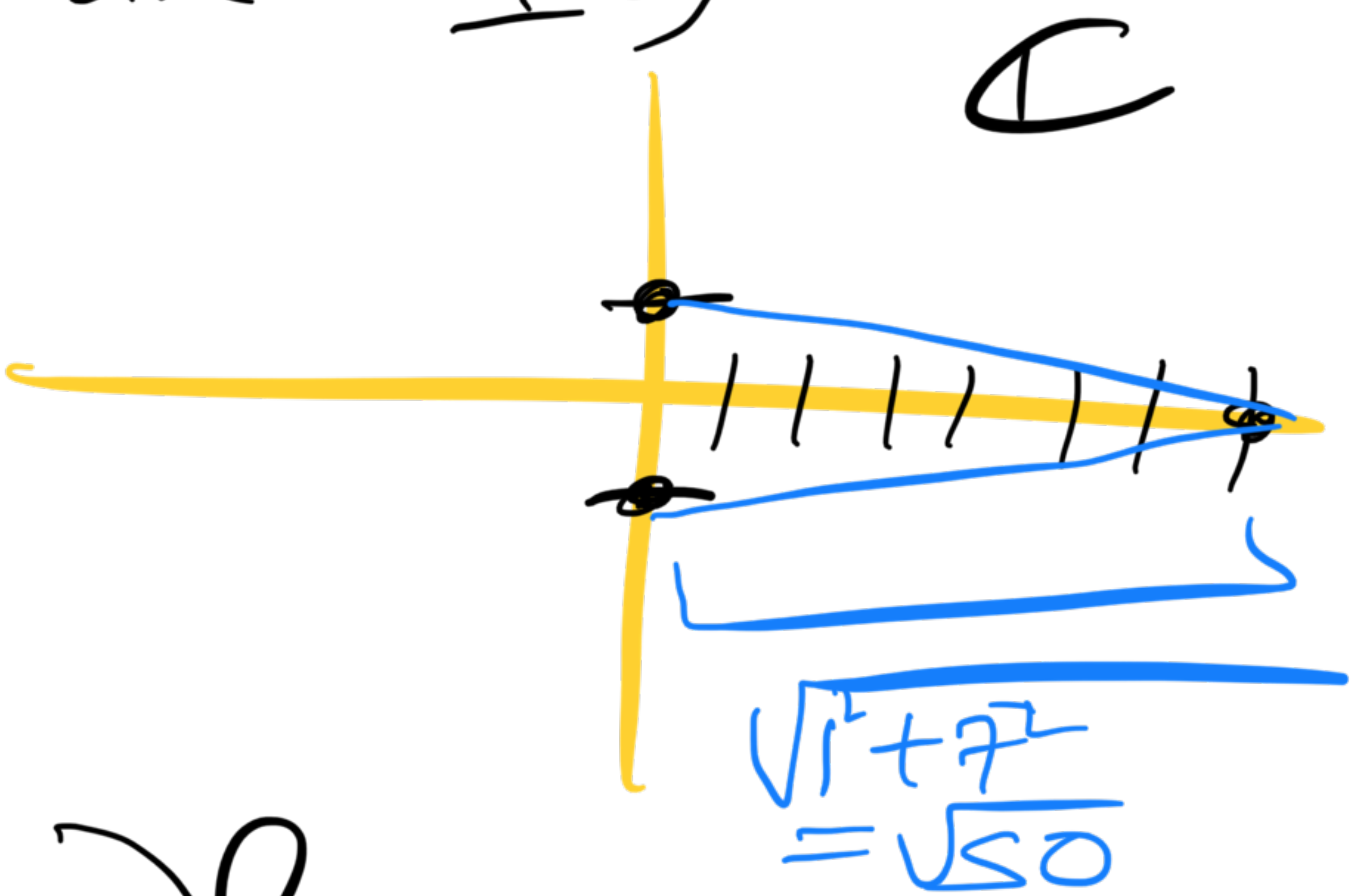
$x = i+7$, so

$$\frac{x-1}{x^2+1} = a_0 + a_1(x-7) + a_2(x-7)^2 + \dots,$$

with r.o.c. $R = \sqrt{50}$.

(So this power series diverges when $x > 7+\sqrt{50}$ or $x < 7-\sqrt{50}$)

Reason: roots of x^2+1 are $\pm i$,



Harmonic oscillator:

$$y'' + y = 0$$

Ansatz:

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

$$y'(t) = \sum_{n=1}^{\infty} a_n n t^{n-1}$$

$$y''(t) = \sum_{n=2}^{\infty} a_n n(n-1) t^{n-2}$$

plug into ODE:

$$\underbrace{\sum_{n=2}^{\infty} a_n n(n-1) t^{n-2} + \sum_{n=0}^{\infty} a_n t^n}_{} = 0$$

$$\left(a_2 2\cdot 1 + a_3 3\cdot 2\cdot t + a_4 4\cdot 3 t^2 + a_5 5\cdot 4\cdot t^3 + \dots\right)$$

$$+ \left(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \dots\right)$$

$$= (a_2 \cdot 2 \cdot 1 + a_0) + (a_3 \cdot 3 \cdot 2 + a_1)t$$

$$+ (a_4 \cdot 4 \cdot 3 + a_2)t^2$$

$$+ (a_5 \cdot 5 \cdot 4 + a_3)t^3$$

$$+ \dots$$

$$= 0$$

So each coefficient must be zero:

$$a_2 \cdot 2 \cdot 1 + a_0 = 0$$

$$a_3 \cdot 3 \cdot 2 + a_1 = 0$$

$$a_4 \cdot 4 \cdot 3 + a_2 = 0$$

$$a_5 \cdot 5 \cdot 4 + a_3 = 0$$

$$\dots$$

System of ∞ly many eqns and ∞ly many variables

$$a_2 = \frac{-a_0}{2 \cdot 1} \qquad a_3 = \frac{-a_1}{3 \cdot 2}$$

$$a_4 = \frac{-a_2}{4 \cdot 3} = \frac{-\left(\frac{-a_0}{2 \cdot 1}\right)}{4 \cdot 3} = \frac{a_0}{4 \cdot 3 \cdot 2 \cdot 1}$$

$$a_5 = \frac{-a_3}{5 \cdot 4} = \frac{a_1}{5 \cdot 4 \cdot 3 \cdot 2}$$

$$a_6 = \frac{-a_4}{6\cdot 5} = \frac{-\left(\frac{a_0}{4\cdot 3\cdot 2\cdot 1}\right)}{6\cdot 5} = \frac{-a_0}{6!}$$

$$a_7 = \frac{-a_5}{7\cdot 6} = \frac{-a_1}{7!}$$

$$a_{2n} = \frac{a_0(-1)^n}{(2n)!}$$

$$a_{2n+1} = \frac{a_1(-1)^n}{(2n+1)!}$$

$$y(t) = a_0\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \dots\right)$$
$$+ a_1\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} - \dots\right)$$

Put $f(t) := 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \dots$

$$g(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \dots$$

<u>Can check</u>: both of these power series have r.o.c. $= \infty$

Note: $f(t), g(t)$ both solve
$y'' + y = 0$.

So the general soln to
$y''+y=0$ is given by
$$C_1 f(t) + C_2 g(t).$$

Note:
$$f(t) = \cos(t)$$
$$g(t) = \sin(t)$$

Some famous 2nd order linear ODEs:

Legendre eqn: $(1-t^2)y'' - 2ty' + k(k+1)y = 0$

Singular pts: $t = -1, 1$ $\quad k \in \mathbb{Z}$

Laguerre eqn: $ty'' + (1-t)y' + ky = 0$ $\quad k \in \mathbb{Z}$

Singular pts: $t = 0$

Bessel eqn: $t^2 y'' + ty' + (t^2 - k^2)y = 0$ $\quad k \in \mathbb{Z}$

Sing pts: $t = 0$

Airy eqn: $y'' + ty = 0$

Sing pts: none!

Hermite eqn: $y'' - 2ty' + 2ky = 0$, $\quad k \in \mathbb{Z}$

Sing pts: none

Terminology:

Consider $\underline{P(t)y''(t) + Q(t)y'(t) + R(t)y(t) = 0}$

Can also write as

$y'' + p(t)y' + q(t)y = 0$, for

$p(t) = Q(t)/P(t)$

$q(t) = R(t)/P(t)$

If $P(t_0) \neq 0$, or more generally, if $p(t)$ and $q(t)$ analytic at t_0,

then t_0 an <u>ordinary pt</u> of the ODE. Otherwise, t_0 is a <u>singular pt</u>.

Ex: $(1+t)y'' + 5y' + (t^2+2t+1)y = 0$

Can write as

$$y'' + \frac{5}{1+t}y' + \frac{(t^2+2t+1)y}{1+t} = 0$$

So $p(t) = \frac{5}{1+t} \leftarrow$ not analytic when $t=-1$

$q(t) = \frac{(t+1)^2}{t+1} = t+1 \leftarrow$ is analytic for all t

So $t=-1$ is a singular pt, all other t are ordinary pts.

Note:

Ex: $\frac{\sin(x)}{x}$ for which x is this analytic?

Recall: $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$

write $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

$$\frac{\sin(x)}{x} = \underbrace{1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots}_{\text{power series with } \infty \text{ radius of convergence}}$$

So $\frac{\sin(x)}{x}$ is analytic for all x.

However,

Ex: $\frac{\sin(x)}{x^2}$ is not analytic at $x = 0$.

$\sin(x)$ $1 - x$ $+ x^3$