Lecture 22

Today:
- evecs and evals
- systems of ODEs, examples

Ex: $A = \begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}$ compute evecs and evals

[motivated by system $\begin{cases} x'(t) = 5x - y \\ y'(t) = -x + 5y \end{cases}$]

Seek $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$, $\lambda \in \mathbb{C}$ s.t. $A\vec{v} = \lambda \vec{v}$

$\vec{v} \neq \vec{0}$

$$A\vec{v} = \lambda\vec{v} \iff \underbrace{(A - \lambda \mathbb{I}_2)\vec{v} = 0}$$

then $(A - \lambda \mathbb{I})$ cannot be invertible,
hence $\det(A - \lambda \mathbb{I}) = 0$.

$$\det(A - \lambda \mathbb{I}) = \left| \begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = \left| \begin{pmatrix} 5-\lambda & -1 \\ -1 & 5-\lambda \end{pmatrix} \right|$$

$$= (5-\lambda)^2 - 1$$

$$= \underbrace{\lambda^2 - 10\lambda + 24}_{\text{"Characteristic polynomial of } A\text{"}} = 0$$

$$\Longrightarrow (\lambda - 4)(\lambda - 6) = 0$$

$$\left. \lambda = 4, 6 \right] \text{ evals of } A$$

Find evecs of A:

$\lambda = 4$, $\rightsquigarrow$ $\begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 4 \begin{pmatrix} a \\ b \end{pmatrix} \Longleftrightarrow \begin{pmatrix} 5a - b \\ -a + 5b \end{pmatrix} = \begin{pmatrix} 4a \\ 4b \end{pmatrix}$

$$\Longleftrightarrow \begin{cases} 5a - b = 4a \\ -a + 5b = 4b \end{cases}$$

$\longrightarrow \begin{cases} a = b \\ a = b \end{cases} \Longrightarrow$ any vector of form $\begin{pmatrix} a \\ a \end{pmatrix}$ w/ $a \neq 0$ is an evec w/ eval 4

$\lambda = 6 \rightsquigarrow \begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = 6\begin{pmatrix} c \\ d \end{pmatrix} \iff \begin{pmatrix} 5c - d \\ -c + 5d \end{pmatrix} = \begin{pmatrix} 6c \\ 6d \end{pmatrix}$

$$\iff \begin{cases} 5c - d = 6c \\ -c + 5d = 6d \end{cases} \iff \begin{cases} -c = d \\ -c = d \end{cases}$$

So any vector of form $\begin{pmatrix} c \\ -c \end{pmatrix}$ w/ $c \neq 0$ is an evec w/ eval 6.

So $\begin{pmatrix} \pi \\ \pi \end{pmatrix}, \begin{pmatrix} 17 \\ -17 \end{pmatrix}$ form a basis of $\mathbb{C}^2$ consisting of eigenvectors of A.

General system of n 1st order ODEs:

$$\begin{cases} x_1'(t) = F_1(t, x_1, \dots, x_n) \\ x_2'(t) = F_2(t, x_1, \dots, x_n) \\ \vdots \qquad\qquad \cdots \\ x_n'(t) = F_n(t, x_1, \dots, x_n) \end{cases}$$

Ex:

$$\begin{cases} x_1'(t) = x_1(t)x_2(t) \\ x_2'(t) = x_1(t)\sin(x_2(t)) \end{cases}$$

Linear system of n 1st order ODEs:

$$(*)\begin{cases} x_1'(t) = P_{11}(t)x_1(t) + \dots + P_{1n}(t)x_n(t) + g_1(t) \\ x_2'(t) = P_{21}(t)x_1(t) + \dots + P_{2n}(t)x_n(t) + g_2(t) \\ \vdots \\ x_n'(t) = P_{n1}(t)x_1(t) + \dots + P_{nn}(t)x_n(t) + g_n(t) \end{cases}$$

Put $\vec{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$ $\quad P(t) = \begin{pmatrix} P_{11}(t) & \dots & P_{1n}(t) \\ P_{n1}(t) & \dots & P_{nn}(t) \end{pmatrix}$ $\quad \vec{g}(t) = \begin{pmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{pmatrix}$

vector-valued function

Can write (*) as $\vec{x}'(t) = \underbrace{P(t)\vec{x}(t)}_{\text{(matrix vector multiplication)}} + \vec{g}(t)$

Constant coeff case: $P_{ij}(t) = $ constant for $1 \le i,j \le n$

$\rightsquigarrow \quad \vec{x}'(t) = P\vec{x}(t) + \vec{g}(t)$

ordinary matrix, e.g. $P = \begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}$

Ex: $x''(t) = t x(t)$ Put $y(t) := x'(t)$

$$\Longleftrightarrow \begin{cases} x'(t) = y(t) \\ y'(t) = t x(t) \end{cases}$$

Ex:

$$\begin{cases} x_1'(t) = 16x_1(t) - 3x_2(t) \\ x_2'(t) = 2x_1(t) + 9x_2(t) \end{cases} \quad (*)$$

Ansatz: $x_1(t) = e^{r_1 t}$ $\quad x_2(t) = e^{r_2 t}$

Need:

$$\begin{cases} r_1 e^{r_1 t} = 16 e^{r_1 t} - 3e^{r_2 t} \\ r_2 e^{r_2 t} = 2e^{r_1 t} + 9e^{r_2 t} \end{cases} \Longleftrightarrow \begin{cases} (r_1 - 16)e^{r_1 t} = -3e^{r_2 t} \\ (r_2 - 9)e^{r_2 t} = 2e^{r_1 t} \end{cases}$$

[can never have $17e^{3t} = 6e^{7t}$ for all t]

$$\Longleftrightarrow \begin{cases} \dfrac{r_1-16}{-3} = e^{(r_2-r_1)t} \\[2ex] \dfrac{r_2-9}{2} = e^{(r_1-r_2)t} \end{cases} \Longrightarrow r_1 = r_2$$

$$\Longrightarrow \begin{array}{l} r_1 - 16 = -3 \\ r_2 - 9 = 2 \end{array}$$

$$\rightsquigarrow \begin{cases} \dfrac{r_1-16}{-3} = 1 \\[2ex] \dfrac{r_2-9}{2} = 1 \end{cases} \Longrightarrow r_1 = 13, \quad r_2 = 11, \quad \text{but } 13 \neq 11.$$

Failure...

[recall: when solving $y'' + 3y = e^t$

Ansatz: $y(t) = Ae^t$]

Modify ansatz: $x_1(t) = C_1 e^{r_1 t}$, $x_2(t) = C_2 e^{r_2 t}$

[Note: if $\vec{x}(t)$ is a soln (*), then $C\vec{x}(t)$ is also a soln. But $C\vec{x}(t) = \begin{pmatrix} Cx_1(t) \\ Cx_2(t) \end{pmatrix}$]

Need:
$$\begin{cases} c_1 r_1 e^{r_1 t} = 16 c_1 e^{r_1 t} - 3 C_2 e^{r_2 t} \\ C_2 r_2 e^{r_2 t} = 2 C_1 e^{r_1 t} + 9 C_2 e^{r_2 t} \end{cases}$$

By similar reasoning $r_1 = r_2$.

$$\begin{cases} c_1 r_1 = 16 c_1 - 3c_2 \\ c_2 r_1 = 2c_1 + 9c_2 \end{cases}$$

Take a step back: Our ansatz is now $\vec{x}(t) = \begin{pmatrix} c_1 e^{rt} \\ c_2 e^{rt} \end{pmatrix} = e^{rt} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

$$(*) \iff \vec{x}'(t) = \begin{pmatrix} 16 & -3 \\ 2 & 9 \end{pmatrix} \vec{x}(t)$$

Plug in our ansatz: $\begin{pmatrix} c_1 r e^{rt} \\ c_2 r e^{rt} \end{pmatrix} = \begin{pmatrix} 16 & -3 \\ 2 & 9 \end{pmatrix} \begin{pmatrix} c_1 e^{rt} \\ c_2 e^{rt} \end{pmatrix}$

$$\iff \underbrace{r \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 16 & -3 \\ 2 & 9 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.}_{\text{says } r \text{ is an eval w/ evec } \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} !}$$

So need $b=2a$, can take $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\left(\text{or } \begin{pmatrix} \pi+3 \\ 2\pi+6 \end{pmatrix} \text{ also works} \right)$

$$\begin{pmatrix} 16 & -3 \\ 2 & 9 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 15\begin{pmatrix} a \\ b \end{pmatrix} \iff \begin{cases} 16a-3b=15a \\ 2a+9b=15b \end{cases}$$

$$\iff \begin{cases} a=3b \\ 2a=6b \end{cases}$$

can take $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

<u>Conclusion</u>: $e^{10t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is a solution to $(*)$, $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} e^{10t} \\ 2e^{10t} \end{pmatrix}$

as is $e^{15t}\begin{pmatrix} 3 \\ 1 \end{pmatrix}$. $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 3e^{15t} \\ e^{15t} \end{pmatrix}$

$x_1 = c_1 e^{10t}$ $x_2 = c_2 e^{10t}$

$c_1 = 1$ $c_2 = 2$

So in fact $\vec{x}(t) = c_1 \begin{pmatrix} e^{6t} \\ 2e^{6t} \end{pmatrix} + c_2 \begin{pmatrix} 3e^{1st} \\ e^{1st} \end{pmatrix}$ is the general soln.

Consider system
$$\begin{cases} x_1'(t) = a_{11}x_1 + \dots + a_{1n}x_n \\ x_2'(t) = a_{21}x_1 + \dots a_{2n}x_n \\ \vdots \qquad \dots \\ x_n'(t) = a_{n1}x_1 + \dots + a_{nn}x_n \end{cases} \quad (*)$$

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

$(*) \iff \vec{x}'(t) = A\vec{x}(t)$

$\boxed{\text{Compare to } y'(t) = 7y(t)}$

Make ansatz $\vec{x}(t) = e^{rt}\vec{c}$,

for $\vec{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ constant vector.

$r\vec{v}\cancel{e^{rt}} = A\vec{v}\cancel{e^{rt}}$

$A\vec{v} = r\vec{v}$

Then after plugging into (*), get $A\vec{c} = r\vec{c}$
and simplifying
(evec/eval eqn).

Recall: evals are given by roots of
char. poly $|A - \lambda I|$.

determinant of an $n \times n$ matrix
always a degree n polynomial in λ

Have roots $r_1, \dots, r_n$, possibly repeated, possibly complex.

Each eval has an associated eigenvector.

But, if have repeated roots, might have fewer than n linearly independent eigenvectors.

To do:

- deal w/ cpx evals
- deal w/ repeated evals (possible deficiency of eigenvecs)
- understand these solutions