

Math 635

USC Spring 2026

*Cluster Varieties:
Algebra, Topology, Geometry, Duality*

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Handwritten Lecture Notes

Table of Contents

Lecture 1	2
Lecture 2	8
Lecture 3	11
Lecture 4	15
Lecture 5	20
Lecture 6	24
Lecture 7	28
Lecture 8	31
Lecture 9	35
Lecture 10	38
Lecture 11	41
Lecture 12	44
Lecture 13	52

Cluster varieties: algebra, topology, geometry, duality

Lecture 1

1/12/26

Roughly speaking:

- a cluster variety is a complex algebraic variety obtained by gluing together many copies of $(\mathbb{C}^*)^n$ where the gluing maps take a very particular form
- a cluster algebra is the algebra of regular functions $f: V \rightarrow \mathbb{C}$ on a cluster variety

Fomin-Zelevinsky, early '00s: introduced cluster algebras
Arise in many parts of math and physics as kind of "universal model" for mutation/wall-crossing phenomena:

- quiver representation theory
- ~~Dehn~~ Teichmüller theory
- Poisson geometry
- Grassmannians
- total positivity
- QFT scattering amplitudes (amplitude amplituhedron)
- integrable systems
- string theory (BPS states), etc

Gross-Hacking-Kontsevich 1/9:

- constructed canonical bases for cluster algebras
- established ~~positivity of the Laurent phenomenon~~ positive Laurent phenomenon
- proof uses mirror symmetry for log Calabi-Yau varieties

many strong applications
in representation theory, e.g.
canonical bases for
finite-dimensional irreducible
representations of $SL_n(\mathbb{C})$

can think of as generalization
of toric varieties

(related to almost toric
fibrations in symplectic geometry)

originally found independently
by Lusztig and
Kashiwara in early 90s
using quantum groups

amazingly, the construction
of GTKK uses only
general geometry - no
rep. theory!

Total positivity

Def : $A \in \text{Mat}_{n \times n}(\mathbb{R})$ is totally positive (TP) if all of its minors are positive.

Gantmacher-Krein '30's : $A \text{ TP} \Rightarrow$ eigenvalues are real, positive, and distinct

Binet-Cauchy theorem : The TP matrices in $G = \text{SL}_n(\mathbb{C}) \times \text{GL}_n(\mathbb{C})$ are closed under multiplication, and hence form a multiplicative semigroup $G_{>0}$.

Lusztig : Extended definition of $G_{>0}$ for other semisimple Lie groups G .

More generally : If a given complex algebraic variety Z ~~comes~~ has a distinguished family Δ of regular functions $Z \rightarrow \mathbb{C}$, we define the TP variety by

$$Z_{>0} := \{ z \in Z \mid \begin{matrix} \text{for } z \in Z \\ f(z) > 0 \end{matrix} \forall f \in \Delta \}$$

Ex : For $Z = \text{Mat}_{n \times n}(\mathbb{C})$, $\text{GL}_n(\mathbb{C})$, $\text{SL}_n(\mathbb{C})$, we recover above notion of TP, $\Delta = \text{minors}$,

Ex : Grassmannian $\text{Gr}_{k \times m}(\mathbb{C}) = \{ k\text{-dim linear subspaces of } \mathbb{C}^m \}$
 $\Delta = \text{Plücker coordinates}$

Ex : partial flag manifolds, homogeneous spaces for semisimple complex Lie groups, etc. Slight scaling ambiguity

Lemma : $A \in \text{Mat}_{n \times n}$ has $\binom{2n}{n} - 1$ minors

$$\text{pf} : \# = \sum_{k=1}^n \binom{n}{k} \binom{n}{k}$$

Vandermonde's identity : $\binom{m+w}{r} = \sum_{k=0}^n \binom{m}{k} \binom{w}{r-k}$

$$\text{Setting } m=w=r=n \Rightarrow \binom{2n}{n} = \sum_{k=0}^n \binom{n}{k} \binom{n}{k}$$

(both sides count:
 Given committee with n men
 \sim women,
 how many subcommittees with r members?)

Q : Can we check that $A \in \text{Mat}_{n \times n}$ is TP testing a subset of the $\binom{2n}{n} - 1$ minors? How many tests are needed?

by only

i.e. want
 "efficient
 TP
 testing"

$$\text{Ex} : A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Mat}_{2 \times 2}$$

$$\delta := ad - bc \Rightarrow d = \frac{a + bc}{a}.$$

So if $a, b, c, \delta > 0$, so is δ .

Reduce $\binom{4}{2} - 1 = 5$ checks to 4 checks.

Plücker coordinates on Grassmannians:

Given $A \in \text{Mat}_{k \times m}$ $\rightarrow \text{row span } [A] \in \text{Gr}_{k, m}$
 If rank k

For $J \subset \{1, \dots, m\}$ \rightarrow Plücker coordinates
 $|J|=k$ $P_J(A) := k \times k$ minor of A corresponding
 to J

Note: For $A, B \in \text{Mat}_{k \times m}$ with $[A] = [B]$ (i.e. same row spans)
 $(P_J(A))_{|J|=k}$ and $(P_J(B))_{|J|=k}$ agree up to common rescaling, i.e. get
 $\text{Gr}_{k, m} \rightarrow \mathbb{CP}^N$ for $N = \binom{m}{k} - 1$.

In fact this is an embedding, the Plücker embedding.

Let $\mathbb{C}[\text{Mat}_{k \times m}] = \text{word. ring of } \text{Mat}_{k \times m}$, i.e. the polynomial algebra in variables ~~x_{ij}~~ x_{ij} for $1 \leq i \leq k$
 $1 \leq j \leq m$

Def: The Plücker ring $R_{k, m}$ is the subring of $\mathbb{C}[\text{Mat}_{k \times m}]$ generated by P_J over $J \in \{1, \dots, m\}, |J|=k$.

Claim: the ideal of relations in $R_{k, m}$ is generated by certain quadratic relations called the Grassmann-Plücker relations.

Def: The totally positive Grassmannian $\text{Gr}_{k, m}^+$ is the subset of $\text{Gr}_{k, m}$ of those pts whose Plücker coords are all positive (up to common scaling).

Note: For $A \in \text{Mat}_{k \times m}(\mathbb{R})$, $[A] \in \text{Gr}_{k, m}^+$ iff all $k \times k$ minors of A have the same sign.

Q: For $A \in \text{Mat}_{k \times m}(\mathbb{R})$, can we verify that all $k \times k$ minors are positive by only checking a subset of the $\binom{m}{k}$ minors?
 How many tests are needed? positive wlog

Positivity testing for $\text{Gr}_{2,m}$

Claim: Given $A \in \text{Mat}_{2 \times m}$, put $P_{ij} := P_{\{i,j\}}$ for $1 \leq i, j \leq m$.
 To check that all 2×2 minors $P_{ij}(A) \geq 0$, suffices to check only $2m-3$ special ones.

Note: $2m-3 = \dim \text{Gr}_{2,m} + 1$

Lemma: For $1 \leq i_1 < i_2 < i_3 \leq m$, have three-term Grassmann-Plücker relations:

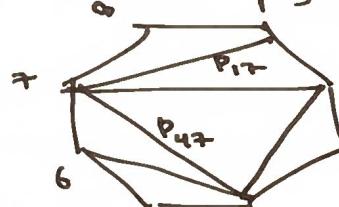
$$P_{i_1 k} P_{i_2 l} = P_{i_1 i_2} P_{k l} + P_{i_1 l} P_{i_2 k}$$

Rmk: For inscribed quadrilateral Ptolemy's thm (2nd century) gives

$$AC \cdot BD = AB \cdot CD + BC \cdot AD$$

Ex: $A = \begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix}$ v/s $P_{13} P_{24} = P_{12} P_{34} + P_{14} P_{23}$, i.e.
 $(ag-ce)(bh-df) = (af-be)(ch-dg) + (ah-de)(bg-cf)$ ✓

Put $P_m = \text{regular } m\text{-gon}$, $T = \text{triangulation}$.



To each side or diagonal associate P_{ij} , where i, j are the end pts

Cluster variables: P_{ij} ranging over diagonals
frozen variables: P_{ij} ranging over sides
extended cluster: $\{\text{cluster vars}\} \cup \{\text{frozen vars}\} =: \tilde{x}(T)$

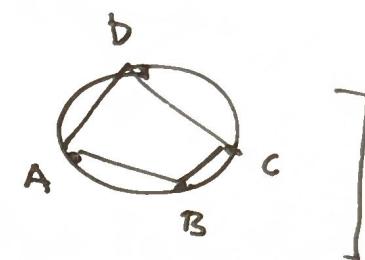
Note: extended cluster has $2m-3$ vars, and we claim that these are algebraically independent.

Ex: In above picture, have cluster variables $P_{13}, P_{15}, P_{17}, P_{19}, P_{24}$
 frozen variables $P_{12}, P_{14}, \dots, P_{28}, P_{16}$

Thm: Each P_{ij} for $1 \leq i < j \leq n$ subtraction-free rational expression can be written as a of a given extended cluster $\tilde{x}(T)$.

Cor: For positivity of each $P_{ij} \in \tilde{x}(T)$ on given $A \in \text{Mat}_{2 \times m}$, then all of the 2×2 minors of A are positive.

(2) of these $\frac{2m-3}{2}$ of these

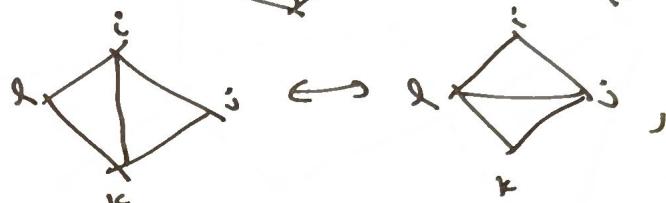


Pf of thm: Follows by combining

- (1) each P_{ij} appears as an elt of an extended cluster $\tilde{x}(T)$ for some triangulation T of \mathbb{P}_m
- (2) any two triangulations of \mathbb{P}_m are related by a sequence of flips



(3) For a flip



replace P_{ik} with P_{li} .

Using three-term GP relation, have $P_{ik} = \frac{P_{ij}P_{lk} + P_{il}P_{jk}}{P_{il}}$

Rank: In fact, each Plücker coordinate P_{ij} can be written as a Laurent polynomial with positive coefficients in the Plücker coordinates from $\tilde{x}(T)$.

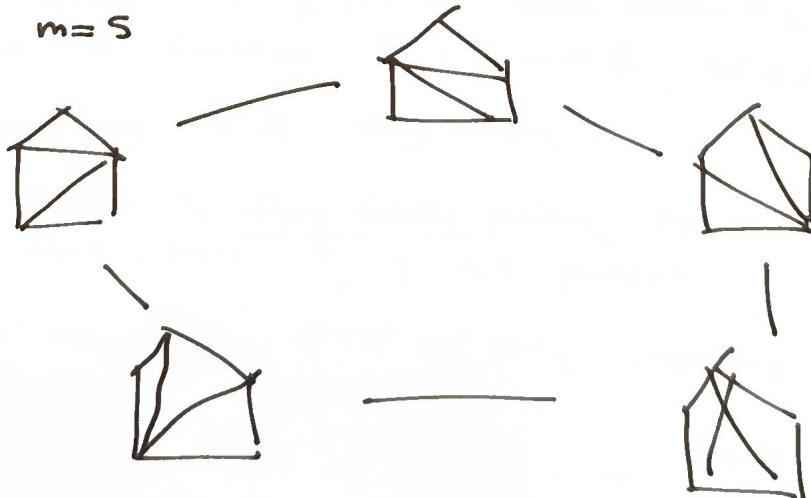
Example of ~~possibly~~ positive Laurent phenomenon.

Combinatorics of flips encoded by graph:

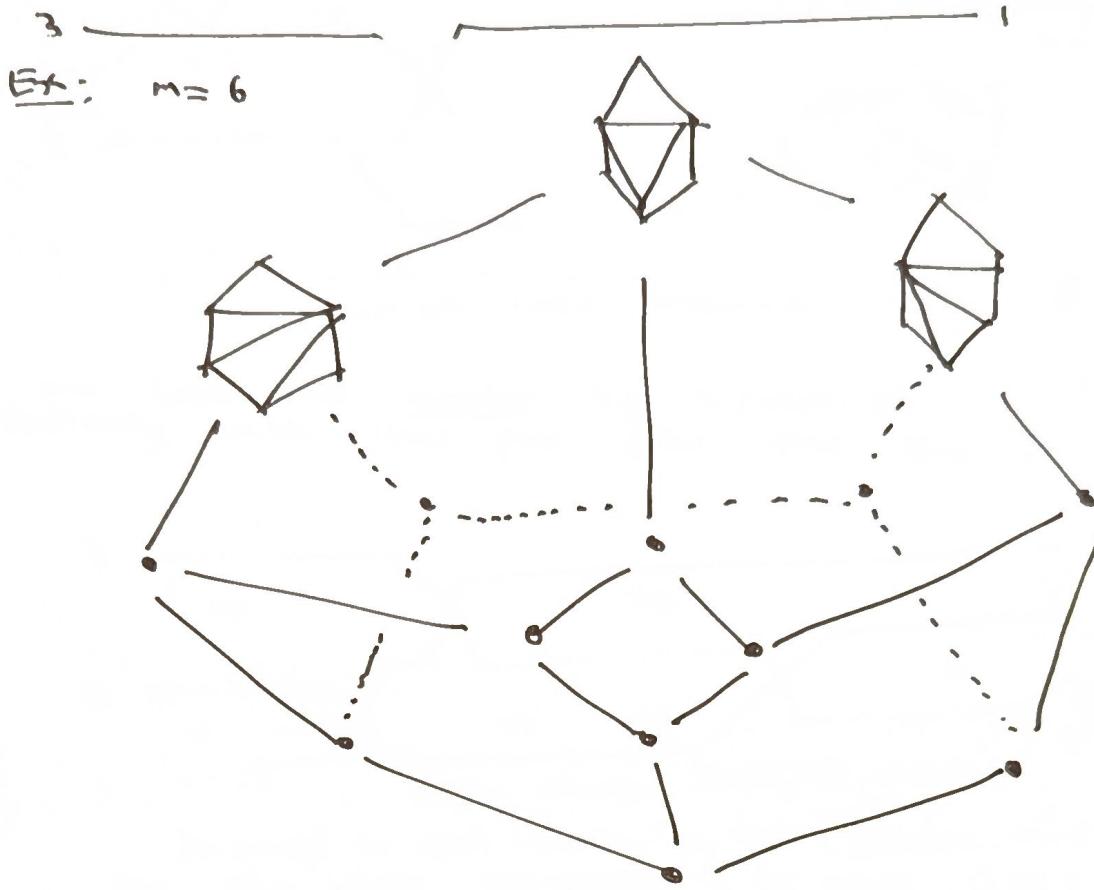
- vertices are ~~triangulations~~ triangulations
- edges are flips

Each vertex has degree $m-3$. In fact, this is the 1-skeleton of an $(m-3)$ -diml convex polytope called the associahedron (discovered by Stasheff).

Ex: $m=5$



Wiring diagrams:



Def: A cluster monomial is a monomial in the variables of a given extended cluster $\tilde{x}(\tau)$.

Thm (19th century invariant theory): The set of all cluster monomials give a linear basis for the Plücker ring $P_{2,n}$.

Lecture 2

11/11/25

Before moving to TP for non matrices, we discuss an intermediate notion called "flag positivity". Put $G = SL_n$.

Def. Given $J \subsetneq \{1, \dots, n\}$ non empty, the flag minor P_J is the function $P_J: G \rightarrow \mathbb{Q}$, $z = (z_{ij}) \mapsto \det(z_{ij}) \mid i \in |J|, j \in J$

Note: there are $2^n - 2$ flag minors.

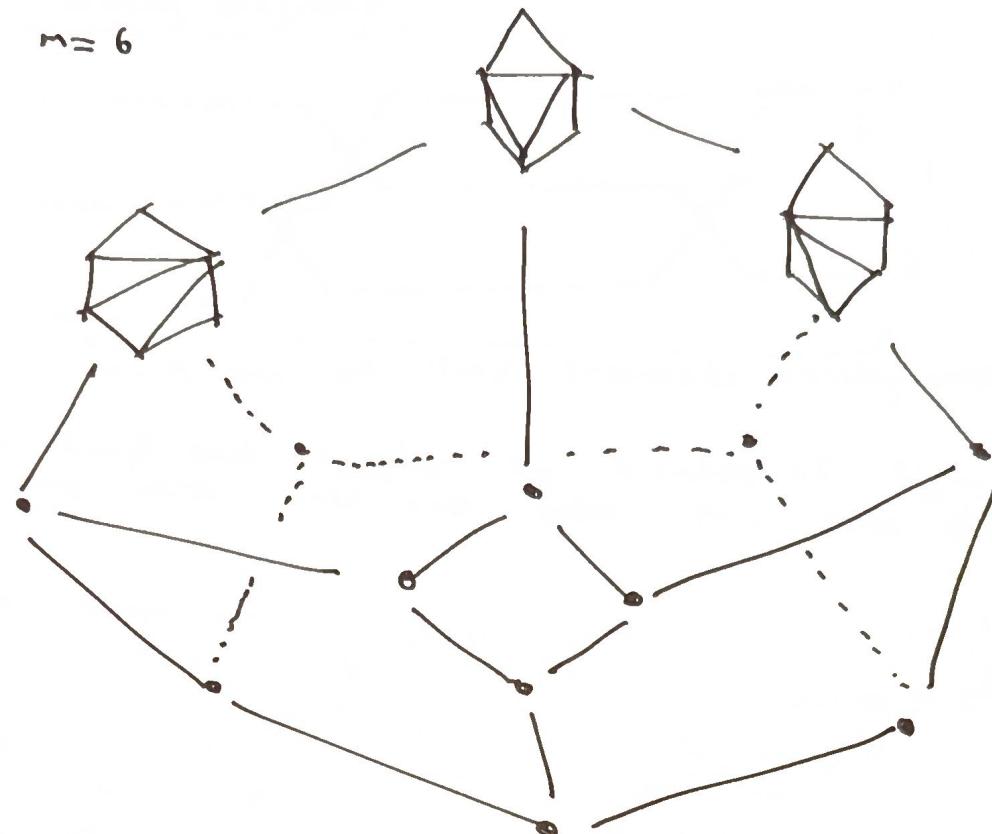
Def: $z \in G$ is flag totally positive (FTP) if all flag minors $P_J(z)$ are positive.

$|J| \times |J|$ minor which is "top-justified"

Q: Can we check FTP by only checking a subset of the $2^n - 2$ flag minors.

Claim: It suffices to check only $\frac{(n-1)(n+2)}{2}$ special flag minors.

Ex: $n=6$



Def: A cluster monomial is a monomial in the variables of a given extended cluster $\tilde{x}(T)$.

Thm (19th century invariant theory): The set of all cluster monomials give a linear basis for the Plücker ring $R_{3, n}$.

Lecture 2

11/11/26

Before moving to TP for $n \times n$ matrices, we discuss an intermediate notion called "flag positivity". Put $G = \text{SL}_n$.

Def: Given $J \subseteq \{1, \dots, n\}$ nonempty, the flag minor P_J is the function $P_J: G \rightarrow \mathbb{Q}$, $z = (z_{ij}) \mapsto \det(z_{ij})$ for $i \in |J|$, $j \in J$.
Note: there are $2^n - 2$ flag minors.

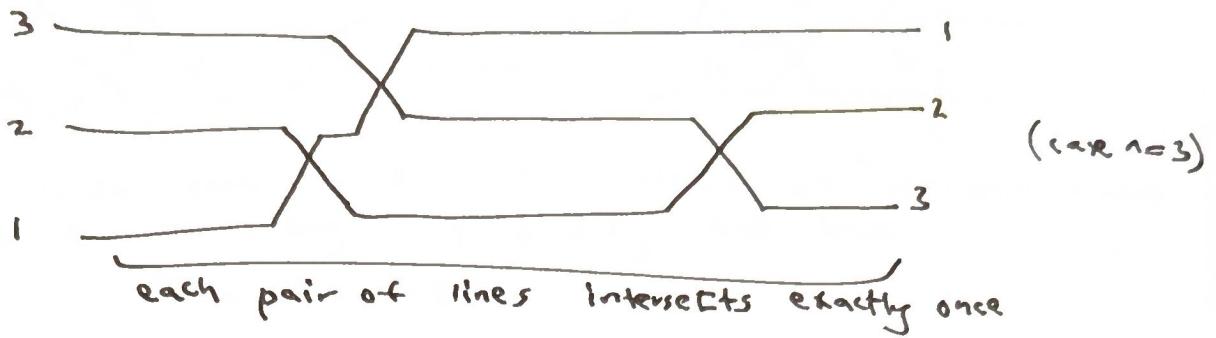
Def: $z \in G$ is flag totally positive (FTP) if all flag minors $P_J(z)$ are positive.

$|J| \times |J|$ minor
which is
"top-justified"

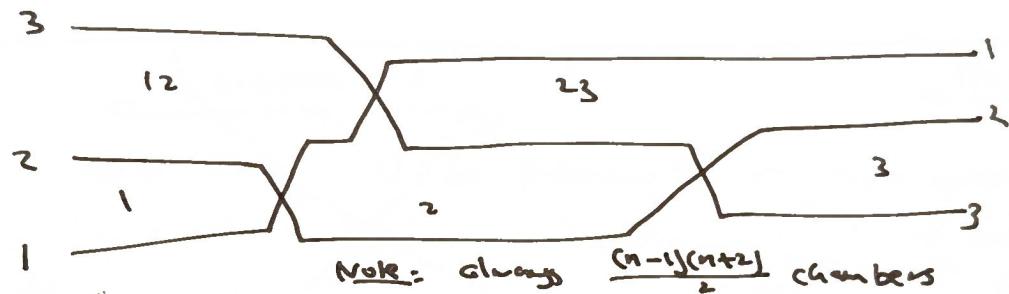
Q: Can we check FTP by only checking a subset of the $2^n - 2$ flag minors.

Claim: It suffices to check only $\frac{(n-1)(n+2)}{2}$ special

Wiring diagrams:



We label each chamber indicating which lines pass by a subset of $\{1, \dots, n\}$ below that ~~one~~ chamber



Associated to each chamber is its chamber minor P_J the flag minor corresponding to its subset $J \subseteq \{1, \dots, n\}$.

extended cluster: all chamber minors of a wiring diagram
cluster variables: the chamber minors for bounded chambers
frozen variables: the chamber minors for unbounded chambers

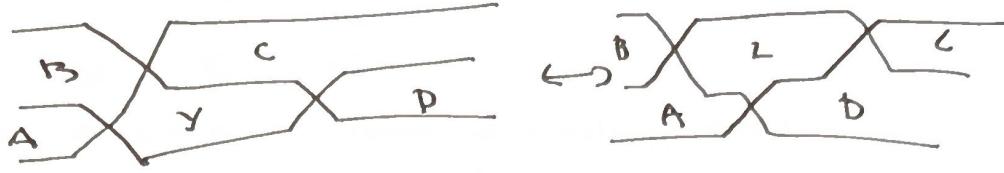
$\frac{n-2}{2}$ of these

$\binom{n-1}{2}$

Thm Every flag minor can be written as a subtraction-free rat'l expr in the chamber minors of a given wiring diag.
Car: If there $\frac{(n-1)(n+2)}{2}$ evaluate positively at a matrix $z \in \text{SL}_n$, then z is FTFP.

Prf: Follows by

- (1) each flag minor appears as a chamber minor in some wiring diagram
- (2) any two wiring diagrams can be transformed into each other by a sequence of local braid moves



(3) Under each braid move, collection of chamber minors changes by exchanging $Y \leftrightarrow Z$, and have
 $YZ = AC + BD$

Point: In fact, each flag minor can be written as a Laurent poly with pos. coeffs in the chamber minors of a given ~~wire~~ wiring diagram.

~~Lecture 3~~ ~~1722126~~

Put $G = SL_n$, $U \times G$ subgroup of unipotent lower-triangulars
i.e. lower triangular matrices with 1s on diagonal

$U \times G$ left multiplication action

$\rightarrow U \times G[G] =$ ring of polynomials in the matrix entries of $A \in G$

$\mathbb{C}[G]^U =$ ring of U -invariant polynomials

Note: $\begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a + \beta c & \alpha b + \beta d \\ \gamma a & \gamma b \end{pmatrix}$

... $\begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} -v_1 \\ -v_2 \end{pmatrix} = \begin{pmatrix} -\alpha v_1 + \beta v_2 \\ \gamma v_1 \end{pmatrix}$

Similarly, $\begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \delta & \varepsilon \\ 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} -v_1 \\ -v_2 \\ -v_3 \end{pmatrix} = \begin{pmatrix} -\alpha v_1 + \beta v_2 + \gamma v_3 \\ -\delta v_1 + \varepsilon v_3 \\ \gamma v_1 \end{pmatrix}$

i.e. $P \in \mathbb{C}[G]$
s.t. $P(yz) = P(z)$
 $y \in U, z \in G$

Def: The full flag variety in \mathbb{C}^n is
 $\{ \sum c_i v_i \in \mathbb{C}^n \mid v_i \text{ is a subspace of dimension } i \text{ for } i=1, \dots, n-1 \}$

This can be identified with the homogeneous space G/B , where $B \subset G$ is the subgroup

Lecture 3

1/23/26

Put $G = \mathrm{SL}_n(\mathbb{C})$

$B \subset G$ subgroup of lower triangular matrices

$V \subset G$ subgroup of unipotent lower triangular matrices

i.e. 1's on
diagonal

Borel
subgroup

Note:

$$\begin{pmatrix} \alpha & 0 \\ \beta & \gamma \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha b \\ \beta a + \gamma c & \beta b + \gamma d \end{pmatrix}, \quad \text{i.e.}$$

$$\begin{pmatrix} \alpha & 0 \\ \beta & \gamma \end{pmatrix} \begin{pmatrix} -r_1 \\ -r_2 \end{pmatrix} = \begin{pmatrix} -\alpha r_1 \\ -\beta r_1 + \gamma r_2 \end{pmatrix}$$

Similarly,

$$\begin{pmatrix} \alpha & 0 & 0 \\ \beta & \gamma & 0 \\ 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} -r_1 \\ -r_2 \\ -r_3 \end{pmatrix} = \begin{pmatrix} -\alpha r_1 \\ -\beta r_1 + \gamma r_2 \\ -\gamma r_1 + \gamma r_2 + \gamma r_3 \end{pmatrix} \quad \text{etc}$$

Def: The (full) flag variety

$\{ \{v_i\} \subset V, v_i \subset \dots \subset V_{n-1} \subset \mathbb{C}^n \mid v_i \text{ is an } i\text{-dimensional subspace for } i=1, \dots, n-1 \}$

Exercise: This is identified with the homogeneous space

Def: The basic affine space is ~~\mathbb{C}^n~~ \mathbb{C}/G

Note that we have

the basic affine space $\mathbb{C}^n \hookrightarrow \mathbb{C}/G \rightarrow B/G$, i.e.

Here $V \times G$ action by left multiplication

$\rightsquigarrow V \times G[G] = \text{ring of polynomials in the entries of } A \in \mathrm{SL}_n$

$\mathbb{C}[G]^V = \text{ring of } V\text{-invariant polynomials}$

Claim:

(1) the flag minors generate $\mathbb{C}[G]^V$ by First and Second Fundamental Theorems of invariant theory

(2) the ideal of quadratic relations among flag minors is generated by quadratic relations called "generalized Plücker relations"

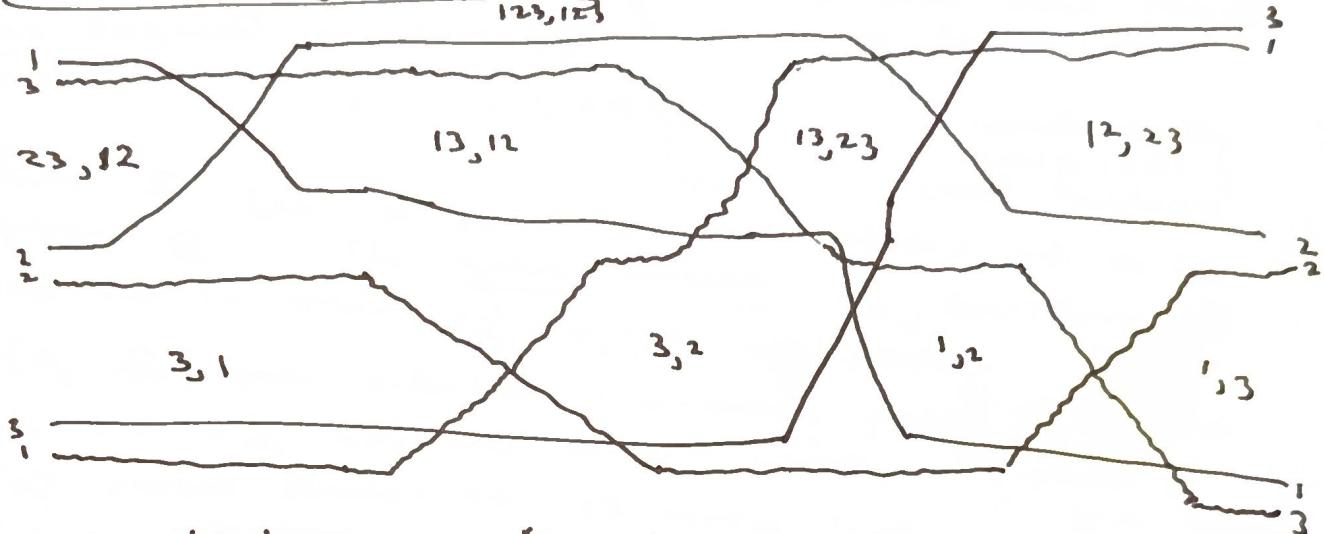
i.e. $P \in \mathbb{C}[G]$
s.t. $P(yz) = P(xy) + P(yz)$
 $y \in V, z \in G$

Checking TP for general nn matrices

Given $I, J \subset \{1, \dots, n\}$ of same cardinality, put
 $\Delta_{I,J} :=$ minor determined by rows in I and columns in J

Thus $\mathbf{z} \in \text{Mat}_{nn}$ is TP $\iff \Delta_{I,J} (\pm) > 0$ for all
 $I, J \subset \{1, \dots, n\}$ with $|I| = |J|$

Double wiring diagrams:



\rightarrow chamber minors $\Delta_{3,1}, \Delta_{3,2}, \Delta_{1,2}, \Delta_{1,3}, \Delta_{23,12}, \Delta_{13,12}, \Delta_{13,23}, \Delta_{12,23}, \Delta_{123,123}$

Claim: number of chamber minors for a double wiring diagram is always n^2 minors

Then: Every minor of an nn matrix can be written as a subtraction-free rational expression in the chamber minors of a given double wiring diagram.

Cor: Only need n^2 tests for positivity.

pt idea:

- (1) every minor is a chamber minor for some double wiring diagram
- (2) any two double wiring diagrams are related by sequence of local moves of three different kinds
- (3) each local move results in an exchange of minors $\gamma \leftrightarrow \gamma'$, where $\gamma' = \gamma - A\gamma + B\gamma$.

Rank: In fact in this we really have Laurent polynomials with positive coefficients.

Rank: The graph with vertices double wiring diagrams and edges local moves is not regular, but this will be rectified by the theory of cluster algebras.

Quivers and their mutations

Def: A quiver is a finite oriented graph with no loops or oriented 2-cycles.

Ex:



Def: An ice quiver is a quiver in which some vertices are designated as "frozen", and no arrows between two frozen vertices.



Def:

Let \mathbb{X} be a mutable quiver Q . The quiver mutation $\mu_{\mathbb{X}}$ transforms Q into new ice quiver

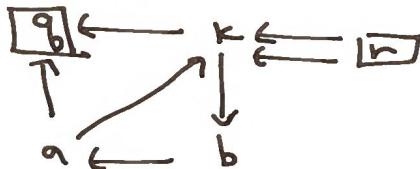
$$Q' = \mu_{\mathbb{X}}(Q)$$

- (1) for each oriented two-arrow path $i \rightarrow k \rightarrow j$, add new arrow $i \rightarrow j$ (unless i, j both frozen)
- (2) reverse direction of all arrows incident to k
- (3) repeatedly reverse any oriented 2-cycles until none left

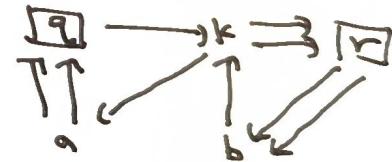
non-frozen vertices will be called "mutable"

vertex of an ice pic transforms Q follows:

Ex:



$$\mu_k$$



Exercise:

- (1) mutation is an involution i.e. $\mu_k(\mu_k(Q)) = Q$
- (2) mutation commutes with reversing orientation of all arrows

(3) if k, l are

mutable vertices with no arrows between them, then

$$\mu_l(\mu_k(Q)) = \mu_k(\mu_l(Q))$$

Rank: If $\mathbb{X} \subseteq \mathbb{K}$ arrows incident

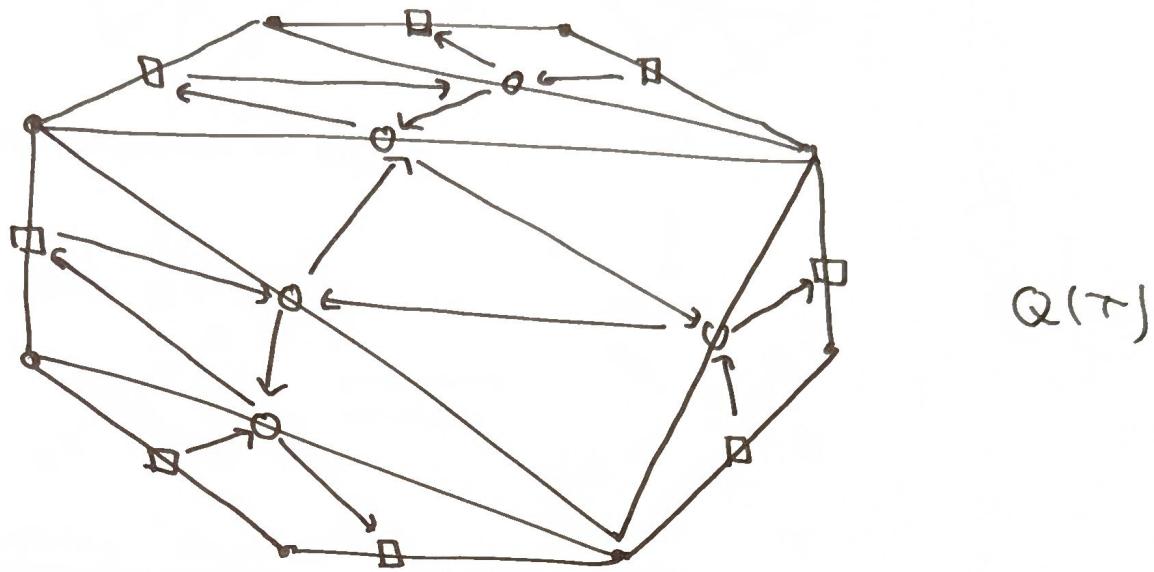
to k are sink or source, μ_k simply reverses all

Exercise: If get from any of mutations

Q is a tree with no frozen, can orientation to any other by a sequence of sinks and sources.

Triangulation and quiver

Can define a quiver from a ~~triangulation~~ triangulation T of P_m .



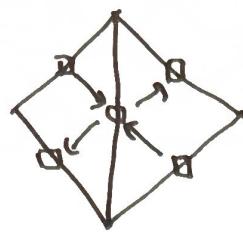
Exercise: If T is a triangulation of P_m and T' obtained by flip along diagonal γ , then

$$Q(T') = \mu_\gamma(Q(T))$$

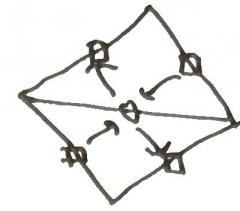
Lecture 34

1/25/26

Ex:

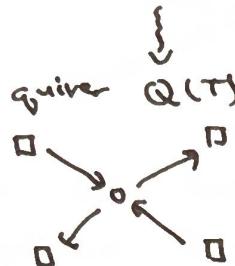


flip

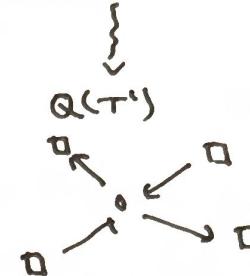


T'

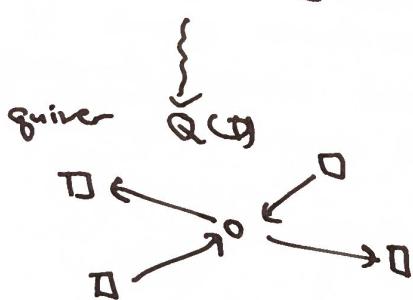
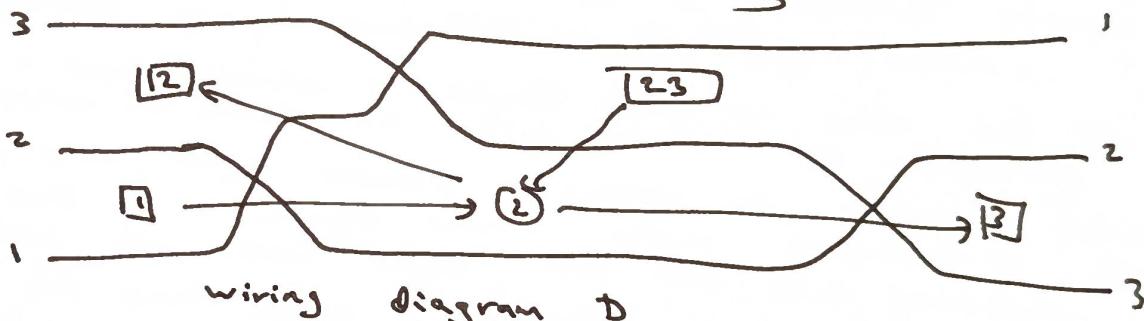
$T = \text{triangulation of } RP_4$



mutation



wiring diagram \rightsquigarrow quiver



vertices: chambers of \mathfrak{p}
(mutable if bounded,
else frozen)

arrows: for chambers c, c'
here $c \rightarrow c'$ in $Q(D)$ if
one of following holds.

- (i) right end of c = left end of c'
- (ii) left end of c is directly above c' ,
right end of c is directly below c'

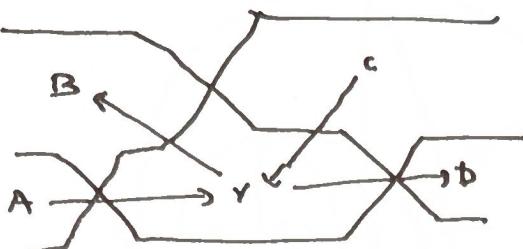


- (iii) left end of right end of c' is directly below c' directly above c

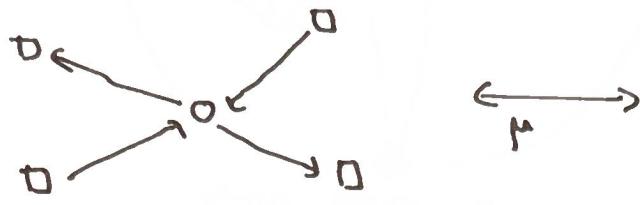
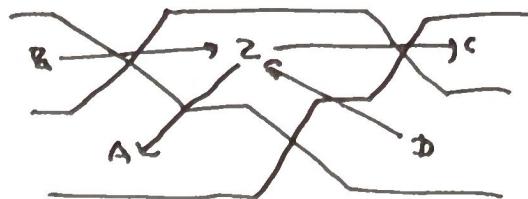


Exercise: If D, D' wiring diagrams related by a braid move at chamber Y , then $(Q(D))' = \mu_Y(Q(D))$.

Ex:



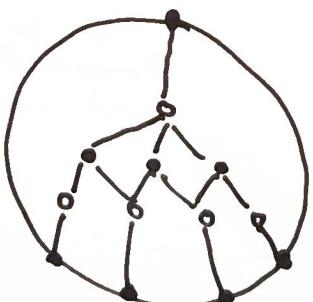
brail
move



Rmk: Also have double wiring diagram \longleftrightarrow quiver $Q(G)$
 Description is more complicated, but quiver associated to a planar bipartite graph is a special case of

Def: A plabic graph G is a connected planar bipartite graph embedded in a disk, where:

- each vertex is colored black or white and lies either in interior of disk or on its boundary
- each edge connects vertices of different colors and is a simple curve whose interior is disjoint from the other edges and the disk boundary
- for each face closure is simply connected
- each internal vertex is simply connected
- each boundary vertex has degree ≥ 2
- each boundary vertex has degree 1



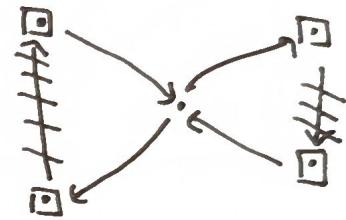
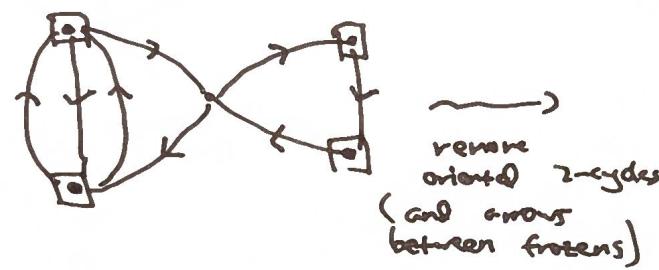
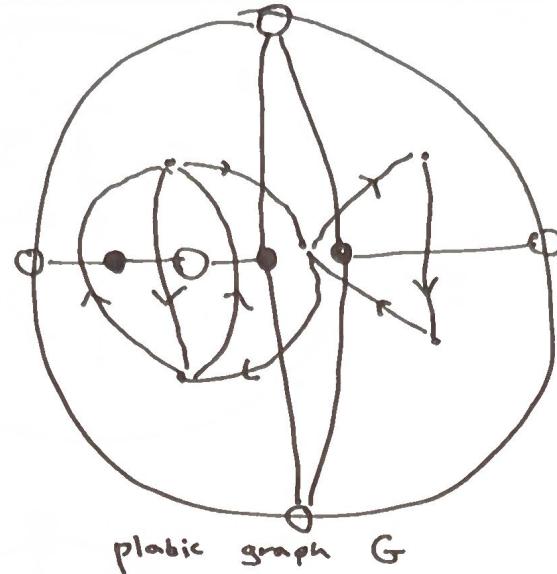
Note: we consider plabic graphs up to isotopy.

plabic graph G \longleftrightarrow quiver $Q(G)$

- vertices are faces of G (frozen if incident to disk boundary, else mutable)
- for each edge of G , have arrow joining the two faces it separates using rule
- remove oriented 2-cycles



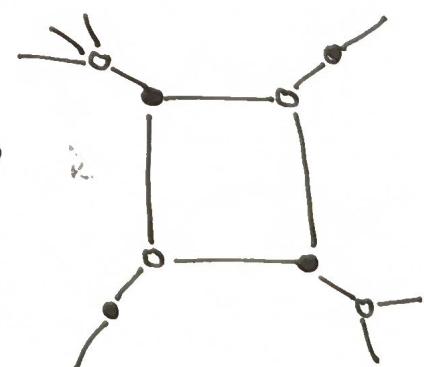
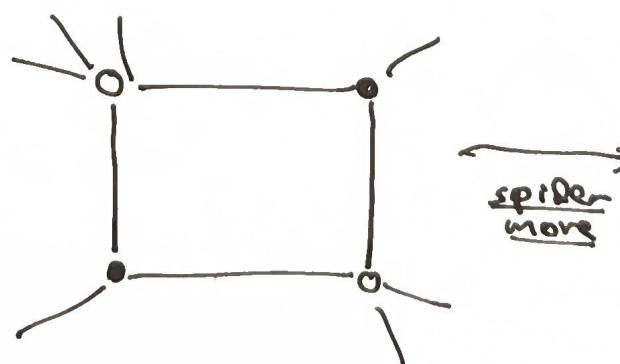
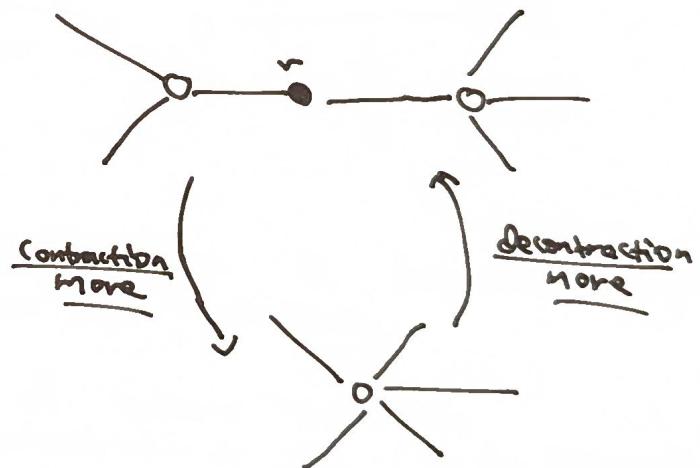
Ex:



Def: Say v bivalent vertex adjacent to two interior vertices

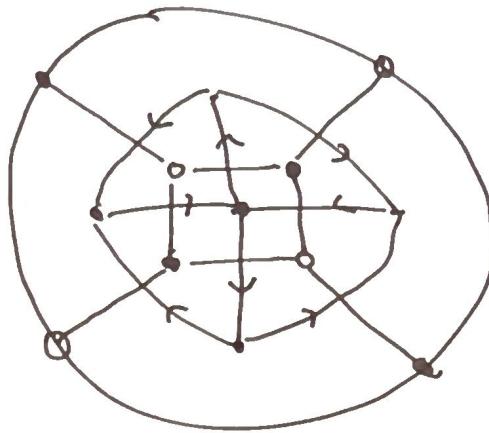
Rule: Does not change associated quiver

Def: Say ~~quadrilater~~ face whose degree ≥ 3 .
G has a quadrilateral vertices have

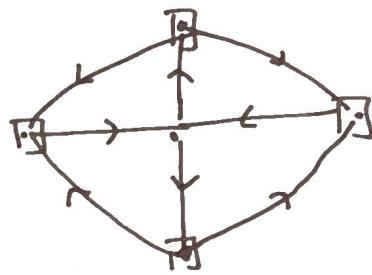
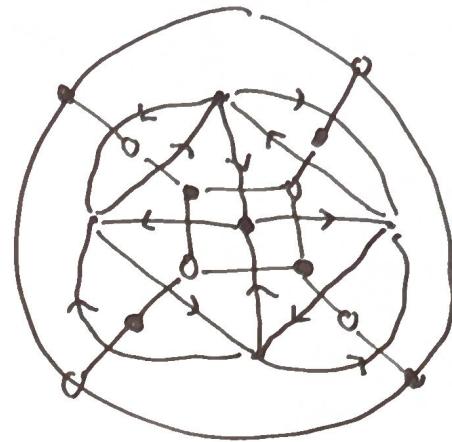


Exercise: If G, G' related by ~~Q(G), Q(G')~~ related by spider move, then mutation

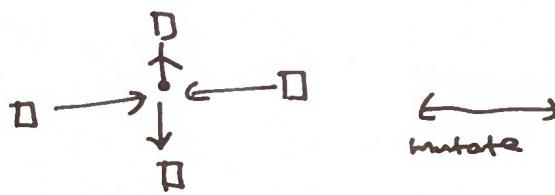
Ex



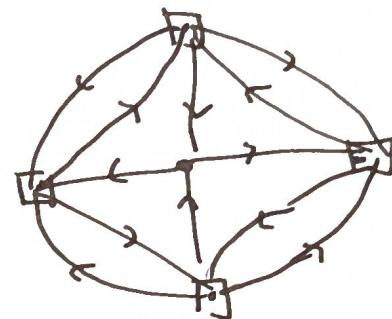
spider move



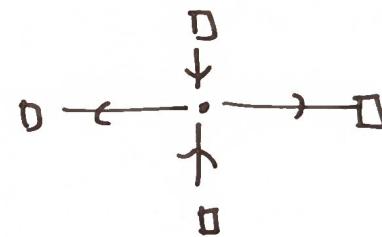
remove $D \rightarrow D$



mutate



remove $D \rightarrow D$ (and
cancel 2-cycles)



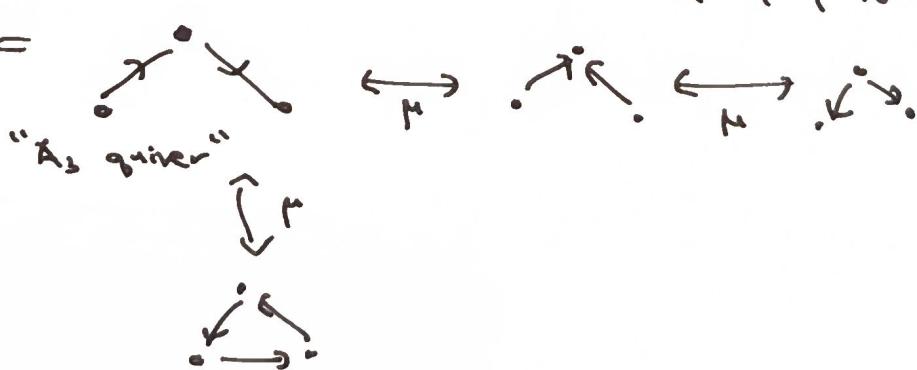
mutation equivalence

Def: Q, Q', Q'' are mutation equivalent if Q becomes isomorphic after a sequence of mutations.

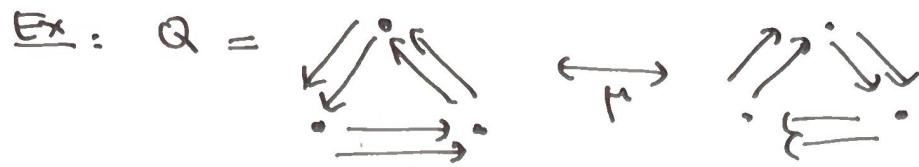
Put $[Q] :=$ set of all quivers which are mutation equivalent to Q (up to isomorphism)

Ex:

$Q =$



Exercise: $[Q]$ has 4 elements



"Markov quiver"

In fact, $[Q]$ is just a single element.

Def. Q has finite mutation type if $[Q]$ is finite.

Rmk: there is a classification theorem for quivers with no frozen vertices and finite mutation type.

Def: Q acyclic if no oriented cycles.

Thm (Caldero-Keller '06): If Q, Q' acyclic and mutation ~~equivalent~~ equivalent, then we can transform Q into Q' by a sequence of mutations at sources and sinks. In particular, Q, Q' have the same underlying undirected graphs.

Lecture 5

11/28/26

Def : Q quiver with vertices labeled by $1, \dots, m$, such that $1, \dots, n$ are the mutable ones ($n \leq m$).

The extended exchange matrix is

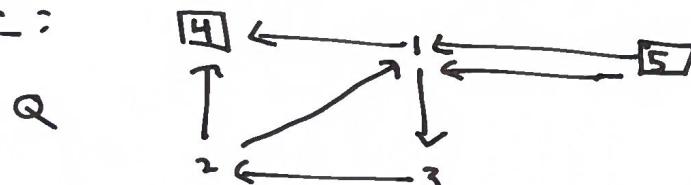
$\tilde{B}(Q) = (b_{ij})_{\substack{1 \leq i, m \\ 1 \leq j \leq n}}$, where $b_{ij} = \begin{cases} 1 & \text{if } \text{arrows } i \rightarrow j \\ -1 & \text{if } \text{arrows } j \rightarrow i \\ 0 & \text{else} \end{cases}$

$m \times n$ matrix

The exchange matrix is the submatrix $B(Q) := (b_{ij})_{1 \leq i, j \leq n}$

$n \times n$ skew-symmetric matrix

Ex :

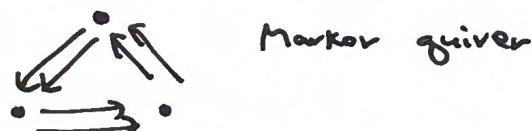


$\tilde{B}(Q) =$

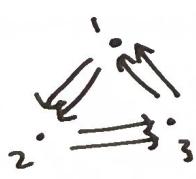
$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \\ -1 & -1 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

$$B(Q) = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

Ex : $Q =$



Marker quiver



$$\tilde{B}(Q) = \begin{pmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{pmatrix}$$



$$\tilde{B}(Q) = \begin{pmatrix} 0 & -2 & 2 \\ 2 & 0 & -2 \\ -2 & 2 & 0 \end{pmatrix}$$

Rank : Rearranging the vertices of Q results in simultaneously rearranging the rows and columns $1, \dots, n$ and reordering the rows $1, \dots, m$.

Lemma : For quiver Q with $\tilde{B}(Q) = (b_{ij})$ and $Q' = \mu_k(Q)$ for a mutable vertex k of Q , have $\tilde{B}(Q') = (b'_{ij})$, with $b'_{ij} = \begin{cases} -b_{ij} & \text{if } i=k \text{ or } j=k \\ b_{ij} + b_{ik}b_{kj} & \text{if } b_{ik}b_{kj} > 0 \\ b_{ij} - b_{ik}b_{kj} & \text{if } b_{ik}b_{kj} < 0 \\ b_{ij} & \text{else} \end{cases}$

Note : Can replace middle two cases with $b'_{ij} = b_{ij} + |b_{ik}|b_{kj}$ if $b_{ik}b_{kj} > 0$

Ex : 

$$\begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 3 \\ 0 & -3 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -2 & 6 \\ 2 & 0 & -3 \\ -6 & 3 & 0 \end{pmatrix}$$

Def : An $n \times n$ matrix $B = (b_{ij}) \in \text{Mat}_{n \times n}(\mathbb{Z})$ is skew-symmetrizable if for some $\alpha_1, \dots, \alpha_n \in \mathbb{Z}_{>0}$ we have $\alpha_i b_{ij} = -\alpha_j b_{ji}$

Def : An $n \times n$ matrix is extended skew-symmetrizable if the top $n-1$ submatrix is skew-symmetrizable.

i.e. becomes skew-symmetric after rescaling the rows by positive integers

Def : For $\tilde{B} = (b_{ij})$ extended skew-sym. $n \times n$ matrix, $k \in \{1, \dots, n\}$, we define $\mu_k(\tilde{B}) = (b'_{ij})$ using same formula $(*)$.

Exercise :

- $\mu_k(\tilde{B})$ is again extended skew-sym., using same $\alpha_1, \dots, \alpha_n$
- $\mu_k(\mu_k(\tilde{B})) = \tilde{B}$
- $\mu_k(-\tilde{B}) = -\mu_k(\tilde{B})$
- ~~$\mu_k(\tilde{B}) = \tilde{B}$~~
- if $b_{ij} = b_{ji} = 0$, then $\mu_i \mu_j \tilde{B} = \mu_j \mu_i \tilde{B}$

Def: For a skew-symmetrizable $n \times n$ matrix $B = (b_{ij})$, its Diagram is the weighted directed graph $\Gamma(B)$ with vertices $1, \dots, n$ and $i \rightarrow j$ iff $b_{ij} > 0$, with weight $|b_{ij}|b_{ji}|$.

Lemma: If the diagram $\Gamma(B)$ of an $n \times n$ skew-symmetrizable matrix B is connected then the skew-symmetrizing vector $(\theta_1, \dots, \theta_n)$ is unique up to rescaling.

pf: By connectedness, there is an ~~strict~~ ordering i_1, \dots, i_n of $\{1, \dots, n\}$ s.t. ~~not~~ for each $j \geq 2$ we have $b_{i_1, i_j} \neq 0$ for some $i < j$.

If $(\theta_1, \dots, \theta_n)$ and $(\theta'_1, \dots, \theta'_n)$ skew-symmetrizing vectors, have $\theta_i \theta_j = -\theta_j \theta_i$ and $\theta'_i \theta_j = -\theta'_j \theta_i \neq \theta_i \theta'_j$.

If $b_{i_1, i_2} \neq 0$, have $\frac{\theta_{i_1}}{\theta_{i_2}} = -\frac{\theta'_1}{\theta'_2} = -\frac{\theta'_j}{\theta'_i}$

$$\Rightarrow \frac{\theta_1}{\theta_2} = \frac{\theta'_1}{\theta'_2}.$$

Def: Two extended

are mutation equivalent if can get from B to B' by a sequence of mutations, followed by a reordering of the rows and columns in the sense from before.

Put $[B] := \text{mutation equivalence class of } B$.

Prop: For an $n \times n$ skew-symmetrizable matrix, its rank and determinant are preserved by mutations.

pf: Can write $b_{ij} = \begin{cases} -b_{ji} & \text{if } k \in \{i, j\} \\ b_{ij} + \max(0, -b_{ik})b_{ki} + b_{ik} \max(0, b_{kj}) & \text{otherwise} \end{cases}$

$$\begin{aligned} \text{Have } f_K(B) &= J_{n, k} \tilde{B} J_{n, k} + J_{n, k} \tilde{B} F_k + E_k \tilde{B} J_{n, k} \\ &= (J_{n, k} + E_k) \tilde{B} (J_{n, k} + F_k) \end{aligned}$$

where $J_{n, k}$ (resp. $J_{n, k}$) is diagonal $n \times n$ (resp. $n \times n$) and has 1s on diagonal except for -1 in (k, k) entry.

$E_k = (e_{ij})$ is $n \times n$ matrix with $e_{ik} = \max(0, -b_{ik})$ and all other entries 0

$F_{1, \kappa} = (f_{ij})$ is the $n \times n$ matrix with $f_{kj} = \max(0, b_{kj})$ and all other entries 0.

Note: $E_{1, \kappa} \tilde{B} F_{\kappa}$ since $b_{ii} = 0$

Have $\det(I_{n, \kappa} + E_{1, \kappa}) = \det(I_{n, \kappa} + F_{\kappa}) = -1$.

Def: A labeled seed of geometric type in $\mathcal{G} = \mathbb{C}(x_1, \dots, x_n)$ is a pair (\tilde{x}, \tilde{B}) where

- $\tilde{x} = (x_1, \dots, x_n)$ is an adapted n -tuple of elts of \mathcal{G} which form a free generating seed
 - $\tilde{B} = (b_{ij})$ is an $n \times n$ extended matrix
- ie $\mathcal{G} = \mathbb{C}(x_1, \dots, x_n)$
and x_1, \dots, x_n alg. indep.
skew-symmetrizable integer

We say:

- \tilde{x} is the labeled extended cluster of (\tilde{x}, \tilde{B})
- $x = (x_1, \dots, x_n)$ is the labeled cluster
- x_1, \dots, x_n are the cluster variables
- x_{n+1}, \dots, x_m are the frozen variables
- \tilde{B} is the extended exchange matrix
- its top $n \times n$ submatrix B is the exchange matrix

	Σ	Σ'
extended cluster	$\tilde{x} = (x_1, x_2, x_3)$	$\tilde{x}' = (x_1, \frac{x_1+x_2}{x_2}, x_3)$
cluster vars	x_1, x_2	$x_1, \frac{x_1+x_3}{x_2}$
frozen vars	x_3	x_3
extended exchange matrix	$\tilde{B} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$	$\tilde{B}' = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
exchange matrix	$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$B' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Here $n=3$, $\kappa=2$.

Σ



Σ'



Lecture 6

Recall: $\mathcal{L} = \mathbb{Q}(q_1, \dots, q_m)$ field of rational functions, $m \geq n$. Say $x_1, \dots, x_m \in \mathcal{L}$ a free generating set if algebraically independent and $\mathcal{L} = \mathbb{Q}(x_1, \dots, x_m)$.

Def: A labeled seed of geometric type in \mathcal{L} is (\tilde{x}, \tilde{B}) , where:

- $\tilde{x} = (x_1, \dots, x_m)$ free generating set of \mathcal{L}
- $\tilde{B} = (b_{ij})$ $m \times n$ extended skew-symmetrizable integer matrix

Terminology:

- \tilde{x} extended cluster
- $x = (x_1, \dots, x_n)$ cluster, x_1, \dots, x_n cluster variables
- x_{n+1}, \dots, x_m frozen variables
- $\tilde{B} \leftrightarrow$ ~~exchange matrix~~ extended exchange matrix
- top $n \times n$ submatrix B is the exchange matrix

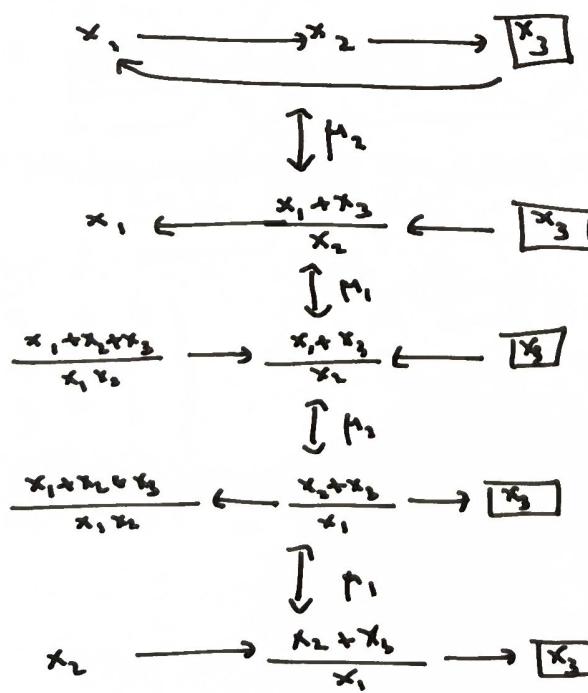
Def: Given (\tilde{x}, \tilde{B}) labeled seed, $k \in \{1, \dots, n\}$, define a new labeled seed $\mu_k(\tilde{x}, \tilde{B}) = (\tilde{x}', \tilde{B}')$, where

- $\tilde{B}' = \mu_k(\tilde{B})$
- $\tilde{x}' = (x'_1, \dots, x'_m)$, where $x'_j = x_j$ for $j \neq k$ and

$$x_k x'_k = \prod_{b_{ik} > 0} x_i^{b_{ik}} + \prod_{b_{ik} < 0} x_i^{-b_{ik}} \quad \text{exchange relation}$$

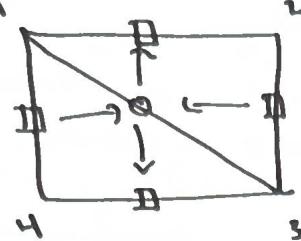
Rmk: When \tilde{B} comes from a quiver, the first product is over arrows ending at k and the second product is over arrows starting at k .

Ex:

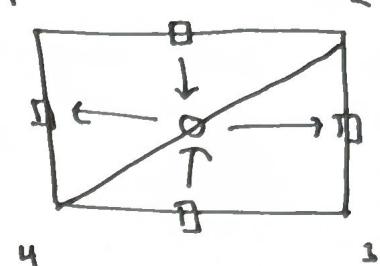


Note: the last seed agrees with the first one up to relabelling.

Ex:



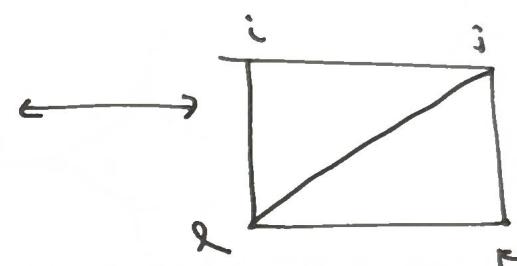
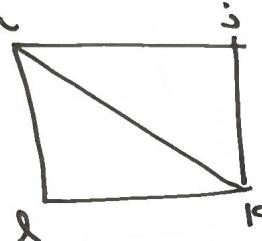
↔ flip



$$\begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix} \quad P_{13} = ag - ce \quad P_{24} = bh - df$$

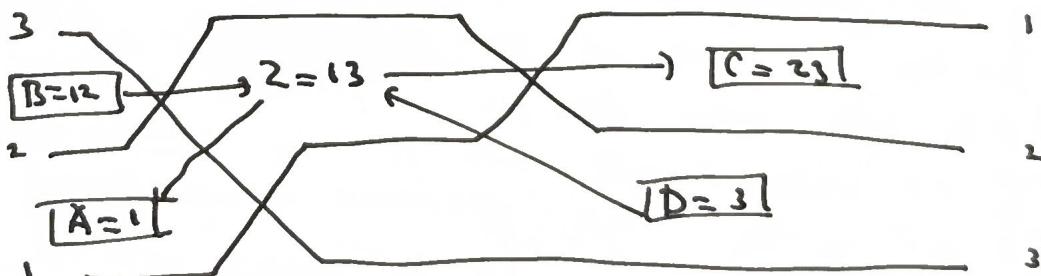
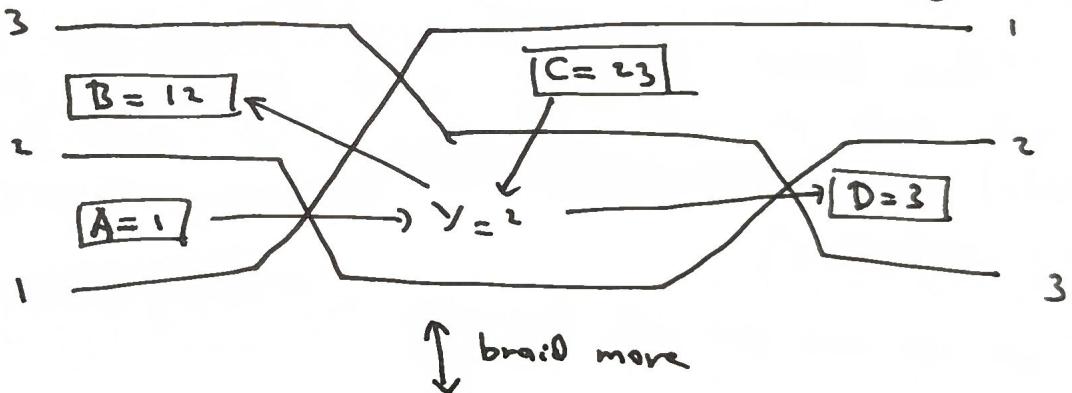
$$\text{Recall: } P_{13} P_{24} = P_{12} P_{34} + P_{14} P_{23}$$

More generally,



$$\rightsquigarrow P_{ik} P_{jl} = P_{ij} P_{lk} + P_{il} P_{jk} \quad \text{special case of the exchange relation}$$

Ex:



$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$\begin{aligned} A &\leftrightarrow a \\ B &\leftrightarrow ae - bf \\ C &\leftrightarrow bf - ce \end{aligned} \quad)$$

etc

$$\text{Have } Y_2 = AC + BD$$

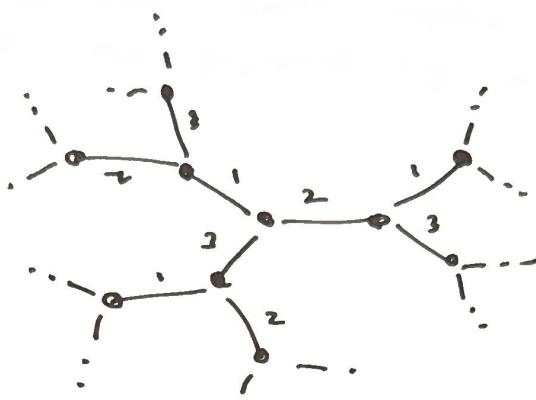
special case of the exchange relation

Notation: Let Π_n denote the n -regular tree with edges labeled by l_1, \dots, l_n such that the edges incident to each vertex carry distinct labels.

$$\Pi_1 \quad \bullet - 1 -$$

$$\Pi_2 \quad \dots - 2 - \bullet - 1 - \bullet - 2 - \bullet - 1 - \bullet - 2 - \bullet - \dots$$

$$\Pi_3$$



Def.: A seed pattern is a choice of labeled seeds $(\tilde{x}(t), \tilde{B}(t))$ for each vertex $t \in \Pi_n$, so that for each labeled edge $t \xrightarrow{\kappa} t'$ the corresponding labeled seeds $(\tilde{x}(t), \tilde{B}(t)), (\tilde{x}(t'), \tilde{B}(t'))$ differ by μ_κ .

Note: a seed pattern is determined by any one of its seeds.

Def: Let $(\tilde{x}(t), \tilde{B}(t))_{t \in \Pi_n}$ be a seed pattern, and put $R := \mathbb{Q}[x_{n_1}, \dots, x_n]$. Let \mathcal{X} be the set of all cluster variables appearing in the seeds $x(t)$ for $t \in \Pi_n$. The cluster algebra A is the R -subalgebra of \mathcal{L} generated by all cluster variables i.e. $A = R[\mathcal{X}]$.

Terminology: The rank n of a cluster algebra is the cardinality of any cluster.

Rank: Note that there is an isomorphism of any free generating set to any other. In particular, up to isomorphism A depends only on \tilde{B}_0 for any initial seed $(\tilde{x}_0, \tilde{B}_0)$, and in fact only on the mutation equivalence class of \tilde{B} . In particular, each (i.e) give Q defines an extended exchange matrix \tilde{B} and hence a cluster algebra.

Ex: For T a triangulation of the regular n -gon P_m , the associated cluster algebra is the Plücker ring $P_{2,m}$.

Ex: For a wiring diagram on k strands, the associated cluster algebra is the algebra generated by flag minors of a $k \times k$ matrix, i.e. the ring of invariants $\mathbb{C}[\mathrm{SL}_k]^U$ (here $U = \text{group of lower-triangular matrices with } 1s \text{ on the diagonal}$)

Ex: For a double wiring diagram on k strands, the associated cluster algebra is $\mathbb{C}[\mathrm{GL}_n]$, i.e. the polynomial ring in k^2 variables. i.e. functions on the basic affine space

Lecture 7

2/6/26

Recall: Labeled seed $(\tilde{x}_0, \tilde{B}_0) \rightsquigarrow$ seed pattern $(\tilde{x}(t), \tilde{B}(t))_{t \in \mathbb{T}_n}$

→ cluster algebra \mathbb{A} of \mathcal{L} ,
generated by all cluster
variables and the frozen
variables

Here $\tilde{x}_0 = (x_1, \dots, x_m)$ free generating set of $\mathcal{L} = \mathbb{C}(x_1, \dots, x_m)$,
cluster variables x_1, \dots, x_n , frozen variables x_{n+1}, \dots, x_m .
The rank of \mathbb{A} is n .

Ex: rank $n=1$ $\mathbb{T}_1 = \{1\}$.

$$\tilde{B}_0 = \begin{pmatrix} 0 \\ b_{11} \\ \vdots \\ b_{m1} \end{pmatrix}$$

$$\text{Exchange relation: } x_i x_i' = \prod_{b_{ii} > 0}^{b_{ii}} x_i + \prod_{b_{ii} < 0}^{-b_{ii}} x_i$$

$$= M_1 + M_2$$

monomials in the
frozen variables x_{n+1}, \dots, x_m

$$\mathbb{A} = \mathbb{C}[x_1, x_1', x_2, \dots, x_m] \subset \mathcal{L}$$

||
 $\mathbb{C}(x_1, x_2, \dots, x_m)$

$$\mathbb{C}[z_1, z_1', z_2, \dots, z_m] / (z_1 z_1' = M_1 + M_2)$$

monomials in z_3, \dots, z_m

Ex: $G = \mathrm{SL}_3(\mathbb{C})$, $U =$ subgroup of unipotent lower-triangular 3×3 matrices

Then $\mathbb{C}[G]^U$ is a cluster algebra of rank 1.

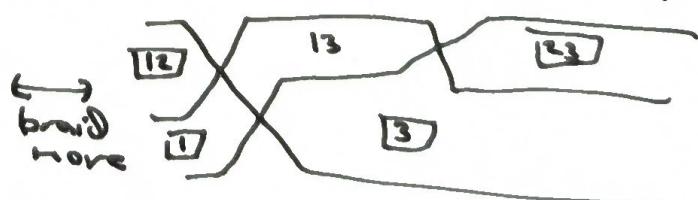
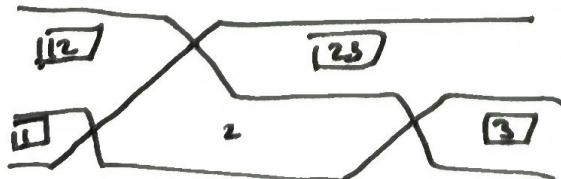
Recall: $\mathbb{C}[G]^U$ generated by flag minors P_J , $J \subset \{1, 2, 3\}$

Here • $\mathcal{L} = \mathbb{C}(P_1, P_2, P_3, P_{12}, P_{23})$

• frozen variables: $P_{11}, P_{33}, P_{12}, P_{23}$

• cluster variables P_{22}, P_{13}

• single exchange relation: $P_2 P_{13} = P_1 P_{23} + P_2 P_{12}$



$$\underline{\text{Ex:}} \text{ rank } n=2, \quad \widetilde{B}_0 = \begin{pmatrix} 0 & \pm b \\ \mp c & 0 \\ b_{31} & b_{32} \\ \vdots & \vdots \\ b_{m1} & b_{m2} \end{pmatrix} \quad \begin{array}{l} \text{either } b, c > 0 \\ \text{or} \\ b = c = 0 \end{array}$$

$$\text{Suppose no frozen, i.e. } n=m, \quad \widetilde{B}_0 = \begin{pmatrix} 0 & \pm b \\ \mp c & 0 \end{pmatrix}$$

$$\text{Then } \mu_1(\widetilde{B}_0) = \mu_2(\widetilde{B}_0) = -\widetilde{B}_0$$

Exchange pattern:

$$\dots - \begin{pmatrix} (z_1, z_0) & \\ (0 & -b) \\ c & 0 \end{pmatrix} \xrightarrow{2} \begin{pmatrix} (z_1, z_2) & \\ (0 & b) \\ -c & 0 \end{pmatrix} \xrightarrow{1} \begin{pmatrix} (z_3, z_2) & \\ (0 & -b) \\ c & 0 \end{pmatrix} \xrightarrow{2} \begin{pmatrix} (z_3, z_4) & \\ (0 & b) \\ -c & 0 \end{pmatrix} \xrightarrow{1} \dots$$

where

$$z_{k-1}, z_{k+1} = \begin{cases} z_k^c + 1 & \text{if } k \text{ even} \\ z_k^b + 1 & \text{if } k \text{ odd} \end{cases}$$

$$\underline{\text{Ex:}} \quad \widetilde{B} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ b_{31} & b_{32} \\ \vdots & \vdots \\ b_{m1} & b_{m2} \end{pmatrix}$$

μ_1 flips sign of k th column
for $k = l_j$

Exchange relations:

$$x_1 x_1' = M_1 + M_2$$

$$x_2 x_2' = M_3 + M_4$$

Cluster variables:

$x_1, x_2, M_1, M_2, M_3, M_4$ monomials in frozen
(reduces to two rank 1)
exchange patterns

Notation: Let $A(b, c)$ denote the
of rank 2 with exchange matrices

$$\underline{\text{Ex:}} \quad A(1, 1)$$

$$z_{k-1}, z_{k+1} = z_k + 1$$

$$z_3 = \frac{z_2 + 1}{z_1}$$

$$z_4 = \frac{z_3 + 1}{z_2} = \frac{z_2 + 1}{z_1} + 1 = \frac{z_1 + z_2 + 1}{z_1 z_2}$$

$$z_5 = \frac{z_4 + 1}{z_3}$$

$$z_6 = z_1, \quad z_7 = z_2 \quad (\text{etc.}) \quad \text{so 5-periodic.}$$

cluster algebra
 $(0 \pm b)$ and no frobns.

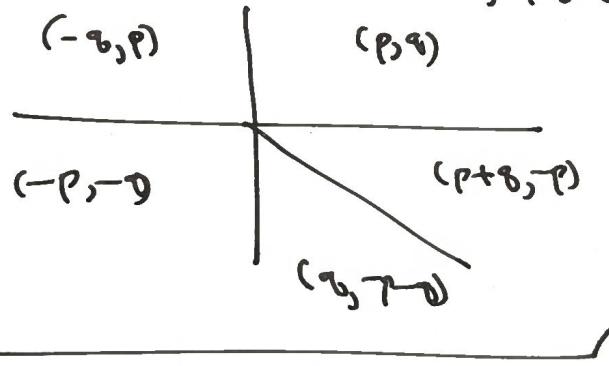
$\text{Ex: } \tilde{B}_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ p & q \end{pmatrix}, \text{ rank} = 2, 1 \text{ frozen variable}$
 $p, q \geq 0 \text{ integers}$

seed pattern:

$$\dots - \begin{pmatrix} z_1, z_2 \\ 0 & 1 \\ -1 & 0 \\ p & q \end{pmatrix} \xrightarrow{1} \begin{pmatrix} z_3, z_4 \\ 0 & -1 \\ 1 & 0 \\ -p & p+q \end{pmatrix} \xrightarrow{2} \begin{pmatrix} z_5, z_6 \\ 0 & 1 \\ -1 & 0 \\ q-p & q \end{pmatrix} \xrightarrow{1} \begin{pmatrix} z_7, z_8 \\ 0 & -1 \\ 1 & 0 \\ -q & -p \end{pmatrix} \xrightarrow{2} \begin{pmatrix} z_9, z_{10} \\ 0 & 1 \\ -1 & 0 \\ -q & p \end{pmatrix} \dots$$

Have $z_3 = \frac{z_2 + y^p}{z_1}, z_4 = \frac{y^{p+q} z_1 + z_2 + y^p}{z_1 z_2},$
 $z_5 = \frac{y^q z_1 + 1}{z_2}, z_6 = z_1, z_7 = z_2, \text{ etc,}$
so still 5-periodic.

Although we assumed $p, q \geq 0$, up to unitating and swapping columns every $(i, j) \in \mathbb{Z}^2$ can be written in one of the forms $(p, q), (p+q, -p), (q, -p-q), (-p, -q), (-q, p)$:



Later we will view this as a ~~scattering~~ simple example of a scattering Diagram.

Lecture 8

2/8/26

Ex : $A_{(1,2)}$

$$z_{k-1} z_{k+1} = \begin{cases} z_k^2 + 1 & k \text{ even} \\ z_k + 1 & k \text{ odd} \end{cases}$$

$$z_3 = \frac{z_2^2 + 1}{z_1} \quad z_4 = \frac{z_3^2 + 1}{z_2} = \frac{\frac{z_2^2 + 1}{z_1}^2 + 1}{z_2} = \frac{z_1^2 + z_2^4 + 2}{z_1 z_2}$$

$$z_5 = \frac{z_1^2 + z_2^2 + 2z_1 + 1}{z_1 z_2} \quad z_6 = \frac{z_1 + 1}{z_2} \quad z_7 = z_1 \quad z_8 = z_2 \quad \text{etc}$$

so it's \mathbb{C} -periodic.

$$\underline{B^+} = A^{(1,3)}$$

$$z_{k-1} z_{k+1} = \begin{cases} z_k^3 + 1 & k \text{ even} \\ z_k + 1 & k \text{ odd} \end{cases}$$

$$\text{Set } z_1 = z_2 = 1.$$

$$z_3 = \frac{z_2 + 1}{z_1} = 2$$

$$z_4 = \frac{z_3 + 1}{z_2} = \frac{2+1}{1} = 3$$

$$z_5 = \frac{z_4^3 + 1}{z_3} = \cancel{z_4} \cancel{z_4} \cancel{z_4} + 1 \cdot \frac{28}{2} = 14$$

$$z_6 = \frac{z_5 + 1}{z_4} = \frac{42}{14} = \frac{15}{3} = 5$$

$$z_2 = \frac{z_6^3 + 1}{z_5} = \frac{126}{14} = 9$$

$$z_8 = \frac{z_7+1}{z_6} = \frac{10}{5} = 2$$

$$z_q = \frac{z_b + 1}{z_a} = \frac{q}{q} = 1$$

$$\frac{z_{16}}{z_8} = \frac{z_9 + 1}{z_9} = \frac{2}{1} = 1$$

So it's θ -periodic at least
 after specializing $\tau_1 = \tau_2 = 1$ and
 we claim that it's θ -periodic
 even without this specialization.

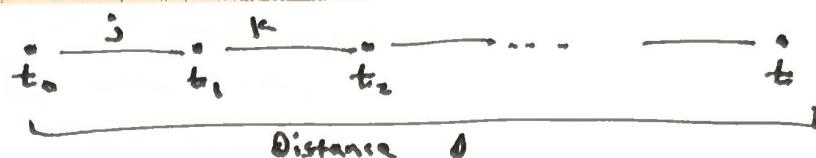
$$E_{\pm} = A^{(1,4)}, \quad \tau_1 = \tau_2 = 1 \implies 1, 1, 2, 3, 41, 14, 937, 67, 21506, 321 \dots$$

each z_k is a Laurent polynomial in z_1, z_2 . However, all integers

Thm Let $(\tilde{x}_0, \tilde{B}_0)$ be a labeled seed, with $\tilde{x}_0 = (x_1, \dots, x_m)$ and associated cluster algebra A . Every cluster variable of A is a Laurent polynomial with integer coefficients in the variables x_1, \dots, x_m . Moreover, $x_{m+1}, \dots, x_n \neq 0$ not appear in the denominators.

Rank: Note that we can replace \mathcal{X}_0 equivalently with any other extended cluster of \mathcal{A} .

proof idea



Distance d

Say $t_0 \in T_n$ initial vertex, $(\tilde{x}_0, \tilde{B}_0)$ initial ((labeled) seed),
cluster variable in the seed

For $\tilde{x}_0 = (x_1, \dots, x_n)$, want to show that x is a
Laurent polynomial in x_1, \dots, x_n . Will use induction on

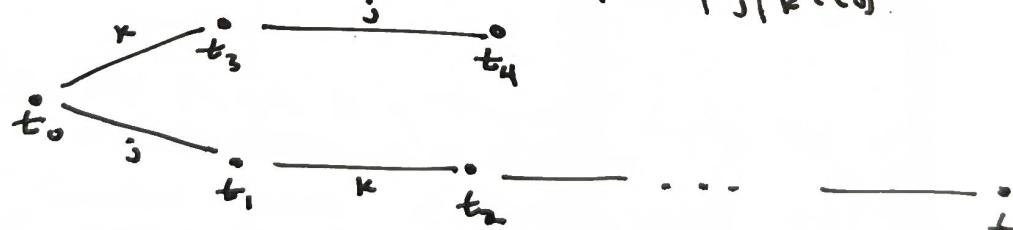
$d = \text{dist}(t_0, t_n)$.

Base cases : if $d=1$, then $x(t) = x(t_1) = (x_1, \dots, x_{j-1}^1, x_j, \dots, x_n)$,
where $x_j^1 = \frac{\prod_{b_{ij} > 0} x_i^{b_{ij}} + \prod_{b_{ij} < 0} x_i^{-b_{ij}}}{x_j}$

if $d=2$, then $x(t) = x(t_1) = x(t_2) = (x_1, \dots, x_{j-1}^1, x_j^1, x_{j+1}^1, \dots, x_n)$,
where $x_k^1 = \frac{\text{poly in } x_1, \dots, x_{j-1}^1, x_j, \dots, x_n}{x_k}$
(or swap)
 $= \frac{\text{Laurent poly in } x_1, \dots, x_n}{x_k}$

Inductive step : Now assume $d \geq 3$, and assume for simplicity
that $b_{j,k}^0 = b_{k,j}^0 = 0$ where $\tilde{B}_0 = (b_{i,j}^0)$
(the case $b_{j,k}^0, b_{k,j}^0 < 0$ is more complicated)

Put $t_3 := \mu_k(t_0)$ and $t_4 := \mu_j \mu_k(t_0)$



Note : $\tilde{x}(t_0) = \tilde{x}(t_3)$, so both t_1, t_3 lie at distance $d-1$
from a seed containing x . By induction:

$$x = \text{Laurent poly in } \tilde{x}(t_0) = \text{Laurent poly in } \tilde{x}(t_3)$$

$$(x_1, \dots, x_{j-1}^1, x_j, \dots, x_n) \quad (x_1, \dots, x_{j-1}^1, x_{j+1}^1, \dots, x_n)$$

Meanwhile, $x_j^1 = \frac{M_1 + M_2}{x_j}$, $x_k^1 = \frac{M_3 + M_4}{x_k}$ for M_1, M_2, M_3, M_4
monomials in x_1, \dots, x_n

$$x = \frac{\text{poly in } x_1, \dots, x_n}{(\text{monomial in } x_1, \dots, x_n) \cdot (M_1 + M_2)^a} = \frac{\text{poly in } x_1, \dots, x_n}{(\text{monomial in } x_3, \dots, x_n) \cdot (M_3 + M_4)^b}$$

(after clearing denominators)

It suffices to show that $a=0$.
 Let \tilde{B}_0^{aug} be \tilde{B}_0 after adding an extra row of the form $(0, \dots, \underbrace{1, \dots, 0}_{i\text{th entry}})$. Let A^{aug} be the resulting cluster algebra with coefficient variables x_{n+1}, \dots, x_m .
Observe: expression in A^{aug} for x in terms of x_{n+1}, \dots, x_m (Specialize $x_{n+1} = 1$) expression in A^{aug} for x in terms of x_1, \dots, x_m

So x Laurent polynomial in x_1, \dots, x_m and in A^{aug} \Rightarrow x Laurent poly in x_1, \dots, x_m in A , hence

WLOG can assume \tilde{B}_0^{aug} instead of \tilde{B}_0 .
 But then $x'_j = \frac{M_1^{\text{aug}} + M_2^{\text{aug}}}{x_j} = \frac{M_1 x_{n+1} + M_2}{x_j}$

$$x'_{j'} = \frac{M_3^{\text{aug}} + M_4^{\text{aug}}}{x_{j'}} = \frac{M_3 + M_4}{x_{j'}}$$

Then $M_1^{\text{aug}} + M_2^{\text{aug}}$ and $M_3 + M_4$ have no common factor
 (think about what happens if we specialize $x_1 = \dots = x_m = 1$)
 $\Rightarrow a=1$ \square

Def: A Markov triple is a triple $(a, b, c) \in \mathbb{Z}_{\geq 1}^3$ which satisfies the Markov equation $a^2 + b^2 + c^2 = 3abc$

Ex: $(1, 1, 1)$ is a Markov triple and hence also its permutations
 $(1, 2, 5)$, $(1, 5, 2)$, $(2, 1, 5)$, $(5, 1, 2)$, $(2, 5, 1)$, $(5, 2, 1)$

Lemma: If (a, b, c) is a ~~Markov triple~~ triple, then so is (a, b, c') with $c' = \frac{a^2 + b^2}{c}$

Pf: Consider equation $a^2 + b^2 + c^2 = 3abc$, i.e. $t^2 - 3abt + (a^2 + b^2) = 0$. If c is one root, the other root c' must satisfy $c + c' = 3ab$, i.e. $c' = 3ab - c = \frac{3abc - c^2}{c} = \frac{a^2 + b^2}{c}$ "Markov mutation"

Lemma: If (a, b, c) is a Markov triple and $a \leq b \leq c$, then $c' = 3abc - c < c$.

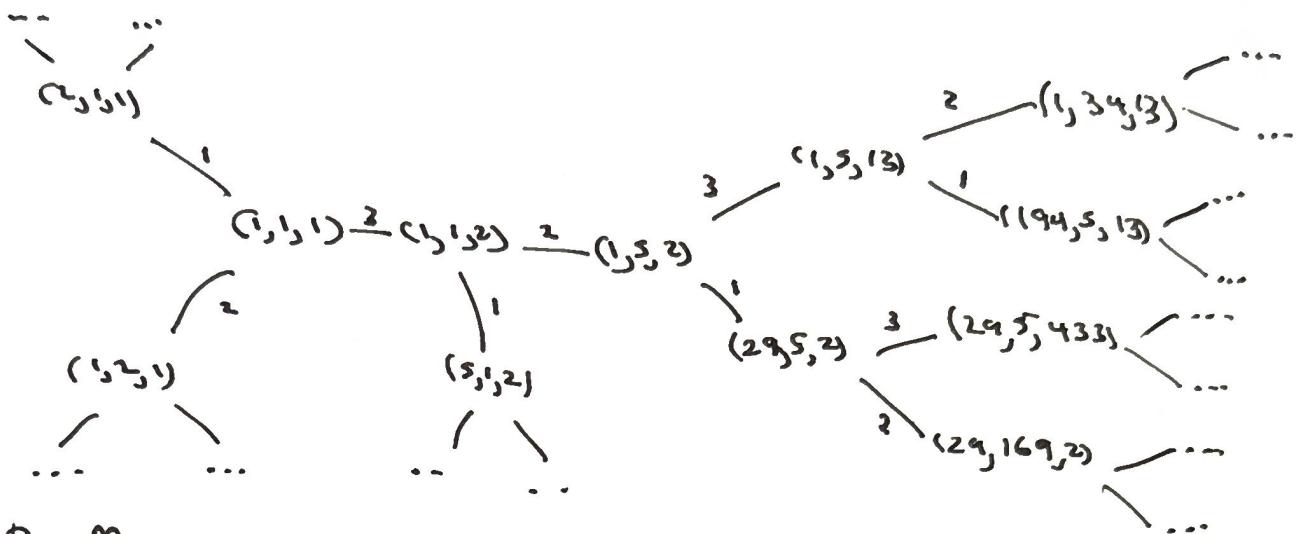
Pf: Put $f(t) = t^2 - 3abt + (a^2 + b^2)$.

$$\begin{aligned} \text{Then } f(b) &= b^2 - 3ab^2 + a^2 + b^2 \\ &= b^2(2 - 3a) + a^2 \\ &\leq -b^2 + a^2 \leq 0 \end{aligned}$$

~~Then c' is the other root of f , must satisfy $c' \leq b < c$.~~

Cor by : Every Markov triple can be connected to $(1, 1, 1)$ by a sequence of Markov mutations.

The Maroon tree :



Recall : The Marian guiver is

Exchange relations:

$$x_1 - x_2 + x_3$$

$$x_1 x_2 = x_1^2 + x_2^2$$

$$x_3^1 x_3 = x_1^2 + x_2^2$$

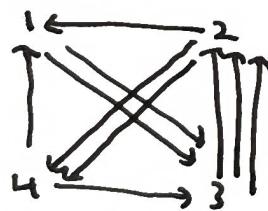
Ex: The Somos-4 sequence $z_0 = z_1 = z_2 = 1$ forms two into a Marker trip.

$\infty = z_1 = z_2 = z_3 = \dots$ sequence is $z_0, z_1, z_2, z_3, \dots$ defined by

Some 1805: these are all i.p. defined by $z_{m+2} z_{m+1} = z_m z_{m-1} + z_{m-2}^2$

To explain using cluster algebras!

1 ← 2 [] algebra consider given



(no ^Q frosen)

$$z_1 z_3 = z_2 z_4 + z_1^2 \quad Q' = \mu_1(Q)$$

The μ_2 rotates Q' by π_{μ_2} ,
 Continue in this way.

Continue in this way with $\frac{2}{5}, \frac{3}{5}, \frac{4}{5}$ etc.

gives a β with mutation

gives $\tilde{z}_n = \text{Lagrange polynomial}$
 $\text{in } t, z_1, z_2, z_3$

- Q rotated by $\pi/3$

$$+ z_1^2 \quad \overrightarrow{Q'} = \mu_1(Q)$$

$$z_1 z_6 = z_1 z_3 + z_4 z_5$$

Sequence 5 3 4

Specialite

1st to 4th term of S_{n+4} necessarily an integer

Lecture 9

2/11/26

Let (\tilde{x}, \tilde{B}) be a labeled seed, with $\tilde{x} = (x_1, \dots, x_m)$, $\tilde{B} = (b_{ij})$.
 Put $(\tilde{x}', \tilde{B}') = \mu_k(\tilde{x}, \tilde{B})$, with $\tilde{x}' = (x'_1, \dots, x'_m)$, $\tilde{B}' = (b'_{ij})$.

Put $\hat{y} := (\hat{y}_1, \dots, \hat{y}_n)$, where $\hat{y}_{j,i} = \prod_{i=1}^m x_i^{b_{ij}}$ and
 similarly $\hat{y}' = (\hat{y}'_1, \dots, \hat{y}'_n)$ with $\hat{y}'_{j,i} = \prod_{i=1}^m (x'_i)^{b'_{ij}}$.

Prop: We have $\hat{y}'_j = \begin{cases} \hat{y}'_k^{-1} & \text{if } j = k \\ \hat{y}'_j (\hat{y}'_k^{-\text{sgn}(b_{kj})} + 1)^{-b_{kj}} & \text{else} \end{cases}$
 (for $j = 1, \dots, n$)

Here $\text{sgn}(b) = \begin{cases} 1 & \text{if } b > 0 \\ -1 & \text{if } b < 0. \end{cases}$

Rank: • recall that the exchange relation is

$$x_k x_{i^*}^{-1} = \underbrace{\prod_{b_{ik} > 0} x_i^{b_{ik}}}_{\text{top non submatrix of } \tilde{B}} + \underbrace{\prod_{b_{ik} < 0} x_i^{-b_{ik}}}_{\text{bottom non submatrix of } \tilde{B}}$$

\hat{y}_k is the ratio of these

• the above formula for \hat{y}'_j depends only on the

Proof: • if $j = k$, $\hat{y}'_k = \prod_{i=1}^m (x'_i)^{b'_{ik}} = \prod_{i \neq k} x_i^{b'_{ik}}$

(recall that we have

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } k \in \{i, j\} \\ b_{ij} + b_{ik} b_{kj} & \text{if } b_{ik} b_{kj} > 0 \\ b_{ij} & \text{else} \end{cases}$$

$$= \prod_{i \neq k} x_i^{-b_{ik}} = \hat{y}_k^{-1}$$

• if $j \neq k$ and $b_{kj} \leq 0$, have

$$\hat{y}'_j = (x'_{i^*})^{b'_{ij}} \prod_{i \neq k} x_i^{b'_{ij}}$$

$$= (x'_{i^*})^{-b_{kj}} \left(\prod_{i \neq k} x_i^{b_{ij}} \right) \left(\prod_{b_{ik} < 0} x_i^{-b_{ik} b_{kj}} \right)$$

$$= x_{i^*}^{b_{ij}} \left(\prod_{b_{ik} > 0} x_i^{b_{ik}} + \prod_{b_{ik} < 0} x_i^{-b_{ik}} \right)^{-b_{kj}} \left(\prod_{i \neq k} x_i^{b_{ij}} \right) \left(\prod_{b_{ik} < 0} x_i^{-b_{ik} b_{kj}} \right)$$

$$= \left(\prod_i x_i^{b_{ij}} \right) \left(\prod_i x_i^{b_{ik}} + 1 \right)^{-b_{kj}}$$

$$= \hat{y}'_j (\hat{y}_k^{-1} + 1)^{-b_{kj}}.$$

• case $j \neq k$, $b_{ik} \geq 0$ similar.

Def : A \mathbb{Y} -seed of rank n in a field \mathbb{L} is (\mathbb{Y}, \mathbb{B}) , where

- \mathbb{Y} = n -tuple of elts in \mathbb{L}
- \mathbb{B} = skew-symmetrizable $n \times n$ integer matrix

We mutate \mathbb{Y} -seeds as follows:

$$(\mathbb{Y}, \mathbb{B}) \xrightarrow{\mu_k} (\mathbb{Y}', \mathbb{B}'), \text{ where } \mathbb{B}' = \mu_k(\mathbb{B}),$$

$$\mathbb{Y}' = (\mathbb{Y}_1, \dots, \mathbb{Y}_n) \text{ with } \mathbb{Y}'_j = \begin{cases} \mathbb{Y}_k & \text{if } j = k \\ \mathbb{Y}_j (\gamma^{-\text{sgn}(b_{jk})} + 1)^{-b_{jk}} & \text{else} \end{cases}$$

Thus labeled seed $(\tilde{\mathbb{Y}}, \tilde{\mathbb{B}})$ \longrightarrow \mathbb{Y} -seed $(\tilde{\mathbb{Y}}, \tilde{\mathbb{B}})$, where

$$\tilde{\mathbb{B}} = \text{top row submatrix of } \tilde{\mathbb{B}}$$

$$\tilde{\mathbb{Y}} = (\tilde{\mathbb{Y}}_1, \dots, \tilde{\mathbb{Y}}_n) \text{ with } \tilde{\mathbb{Y}}_i = \prod_{j=1}^n x_i^{b_{ij}}$$

Part : The seed mutation leaves x_j alone for $j \neq k$ whereas \mathbb{Y} -seed mutation at k only changes \mathbb{Y}_k and \mathbb{Y}_j . \mathbb{Y} -seed mutation at k potentially changes all of $\mathbb{Y}_1, \dots, \mathbb{Y}_n$. However, the formula for x_k involves all of x_1, \dots, x_n , whereas \mathbb{Y}'_j only involves \mathbb{Y}_k and \mathbb{Y}_j .

Def : A semifield is an abelian group P endowed with an auxiliary operation \oplus which is commutative, associative, and distributive with respect to the group operation on P (written multiplicatively). Note that (P, \oplus) is only a semigroup (i.e. not necessarily identity or inverses)

Ex : The multiplicative group $\mathbb{Q}_{>0}$ with \oplus given by ordinary addition.

Def : The tropical semifield $\text{Trop}(\mathbb{Q}_1, \dots, \mathbb{Q}_l)$ is defined by :

- the multiplicative group of Laurent monomials in $\mathbb{Q}_1, \dots, \mathbb{Q}_l$
- $\prod_{i=1}^l \mathbb{Q}_i^{a_i} \oplus \prod_{i=1}^l \mathbb{Q}_i^{b_i} = \prod_{i=1}^l \mathbb{Q}_i^{\min(a_i, b_i)}$ ("tropical addition")

Check :

- commutative : $\min(a_i, b_i) = \min(b_i, a_i)$
- associative : $\min(\min(a_i, b_i), c_i) = \min(a_i, \min(b_i, c_i)) = \min(a_i, b_i + c_i)$ (i.e. $(p \oplus q) \oplus r = p \oplus (q \oplus r)$)
- Distributive : $\min(a_i, b_i) + c_i = \min(a_i + c_i, b_i + c_i)$

For (\tilde{x}, \tilde{B}) labeled seed, \rightarrow coefficient tuple

$$\tilde{x} = (x_1, \dots, \underbrace{x_n, \dots, x_m}_{\text{frozen variables}})$$

$$y = (y_1, \dots, y_m), \text{ where}$$

$$y_j = \prod_{i=n+1}^m x_i^{b_{ij}} \in \text{Trop}(x_{n+1}, \dots, x_m)$$

for $j=1, \dots, n$

Note: $B = \text{top non submatrix of } \tilde{B}$ together with coeff. tuple y recover the extended exchange matrix \tilde{B} .

Prop: $\tilde{B} = (b_{ij})$ extended skew-symmetrizable max matrix with coeff. tuple $y = (y_1, \dots, y_n)$, and $\tilde{B}' = (b'_{ij}) = \mu_k(\tilde{B})$ with coeff. tuple $y' = (y'_1, \dots, y'_n)$. Then

$$y'_j = \begin{cases} y_k^{-1} & \text{if } j=k \\ y_j \left(y_k^{-\text{sgn}(b_{kj})} \oplus 1 \right)^{-b_{kj}} & \text{else.} \end{cases}$$

"tropical Y-seed mutation"

Def: The universal semifield $\mathbb{Q}_{sf}(x_1, \dots, x_m)$ is

$$\left\{ \frac{P(x_1, \dots, x_m)}{Q(x_1, \dots, x_m)} \in \mathbb{Q}(x_1, \dots, x_m) \mid P, Q \text{ have positive coefficients} \right\}$$

with ordinary multiplication and addition.

Lemma: Given any semifield \mathbb{S} , and its $s_1, \dots, s_m \in \mathbb{S}$, $x_i \mapsto s_i$ for $i=1, \dots, m$, $\mathbb{Q}_{sf}(x_1, \dots, x_m) \rightarrow \mathbb{S}$ sending

pf of prop: Let $f: \mathbb{Q}_{sf}(x_1, \dots, x_m) \rightarrow \text{Trop}(x_{n+1}, \dots, x_m)$ be semifield homo. sending $f(x_i) = \begin{cases} 1 & \text{if } i \in n \\ x_i & \text{if } i \in n. \end{cases}$

Note that f also sends x_k^1 to 1, since

$$x_k x_k^1 = M_1 + M_2 \implies 1 \cdot f(x_k^1) = f(M_1) \oplus f(M_2) = 1$$

$$\text{Also, } \hat{y}_j = \prod_{i=1}^n x_i^{b_{ij}} \implies f(\hat{y}_j) = \prod_{i=n+1}^m x_i^{b_{ij}} = y_j \text{ for } j=1, \dots, n,$$

1 since M_1, M_2 monomials which share no frozen variables

\therefore and similarly $f(\hat{y}_j) = y_j$.

$$\text{Thus } \hat{y}'_j = \begin{cases} \hat{y}_k^{-1} & \text{if } j=k \\ \hat{y}_j \left(\hat{y}_k^{-\text{sgn}(b_{kj})} \oplus 1 \right)^{-b_{kj}} & \text{else} \end{cases}$$

$$\hat{y}'_j = \begin{cases} y_k^{-1} & \text{if } j=k \\ y_j \left(y_k^{-\text{sgn}(b_{kj})} \oplus 1 \right)^{-b_{kj}} & \text{else} \end{cases}$$

Lecture 10

21/3/26

We can now give an alternative characterization of labeled seeds and their mutations. Fix $\mathcal{L} = \mathbb{C}(q_1, \dots, q_m)$. A labeled seed is a triple $\mathcal{E} = (x, y, B)$, where

- cluster $x = (x_1, \dots, x_n) \in \mathcal{L}^n$ s.t. $x \cup \{q_{n+1}, \dots, q_m\}$ freely generates \mathcal{L}
- exchange matrix $B = \text{skew-symmetrizable integer matrix}$
- coefficient tuple $y = (y_1, \dots, y_n)$ where y_i is a Laurent monomial in $\text{Trop}(q_{n+1}, \dots, q_m)$

For a mutation $(x, y, B) \xrightarrow{\mu_k} (x', y', B')$, have

$$\bullet B' = \mu_k(B)$$

• y' given by tropical y -seed mutation rule

$$\bullet x' = (x \setminus \{x_k\}) \cup \{x_k'\} \text{ with}$$

$$x_k x_k' = \frac{y_k}{y_k \oplus 1} \prod_{b_{ik} > 0} x_i^{b_{ik}} + \frac{1}{y_k \oplus 1} \prod_{b_{ik} < 0} x_i^{-b_{ik}}$$

Key point: from this mutation process does not really grow with the number $m-n$ of frozen variables

Ex: $(A_2 \text{ revisited})$

$$B_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

→ labeled seed pattern

$$\dots \xrightarrow{1} \mathcal{E}(1) \xrightarrow{2} \mathcal{E}(0) \xrightarrow{1} \mathcal{E}(1) \xrightarrow{2} \mathcal{E}(2) \xrightarrow{1} \mathcal{E}(3) \xrightarrow{2} \dots$$

$$\mathcal{E}(t) = (x(t), y(t), B(t))$$

$$B(t) = -y^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

t	$x(t)$	$y(t)$
0	$x_1 \quad x_2$	$y_1 \quad y_2$
1	$\frac{y_1 + y_2}{x_1(y_1 \oplus 1)} \quad x_2$	$\frac{1}{y_1} \quad y_2$
2	$\frac{y_1 + y_2}{x_1(y_1 \oplus 1)} \quad \frac{x_1 y_2 + y_1 + x_2}{(y_1 y_2 \oplus y_1 \oplus 1) x_1 x_2}$	$y_2 \quad \frac{y_1 y_2 \oplus y_1 \oplus 1}{y_1 y_2}$
3	$\frac{y_1 y_2 + 1}{x_2(y_2 \oplus 1)} \quad \frac{x_1 y_2 + y_1 + y_2}{(y_1 y_2 \oplus y_1 \oplus 1) x_1 x_2}$	$\frac{y_1 y_2 \oplus y_1 \oplus 1}{y_2} \quad \frac{1}{y_1 (y_2 \oplus 1)}$
4	$\frac{x_1 y_2 + 1}{x_2(y_2 \oplus 1)} \quad x_1$	$\frac{1}{y_2} \quad y_1$
5	$x_2 \quad x_1$	$y_2 \quad y_1$

Thm A seed pattern with initial labeled seed $\Sigma = (x_0, y_0, B)$ with $B = \pm \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix}$, $b, c \in \mathbb{Z}_{\geq 1}$, is of finite type if and only if ~~bc ≠ 0~~ only if $bc \leq 3$.

Compare.

Prop: For $b, c \in \mathbb{Z}_{\geq 1}$, the subgroup $W = \langle R_1, R_2 \rangle \subset \text{GL}_2$ generated by reflections $R_1 = \begin{pmatrix} -1 & b \\ 0 & 1 \end{pmatrix}$, $R_2 = \begin{pmatrix} 1 & 0 \\ c & -1 \end{pmatrix}$ is finite if and only if $bc \leq 3$.

Pf: $R_1^2 = R_2^2 = \text{Id}$, so W finite if $R_1 R_2$ has finite order.

$$R_1 R_2 = \begin{pmatrix} bc-1 & -b \\ c & -1 \end{pmatrix}$$

characteristic equation:

$$\lambda^2 - (bc-2)\lambda + 1 = 0 \quad \rightarrow \quad \lambda = \frac{bc-2 \pm \sqrt{(bc-2)^2 - 4}}{2}$$

For $bc = 1, 2, 3$, roots have order 3, 4, 6 respectively.

If $bc > 4$, roots are real and not $\pm 1 \Rightarrow$ infinite order. $bc = 4$, $(\zeta_3, \zeta_2)^k = \begin{pmatrix} 2k+1 & -kb \\ k & 2k+1 \end{pmatrix}$ also infinite order.

Pf of thm:

Can check that in the case $B = \pm \begin{pmatrix} 0 & 1 \\ -c & 0 \end{pmatrix}$ has 5 seeds if $c=1$, 6 seeds if $c=2$, 8 seeds if $c=3$. Now assume $bc \geq 4$, seed pattern $(x(t), y(t), B(t))$.

Put $x(t) = (z_1, z_2)$, $x(t) = (z_3, z_4)$, $x(t) = (z_5, z_6)$, $t \in \mathbb{Z}$.

Let $U = \{u^r \mid r \in \mathbb{R}\}$, $u^r \oplus u^s = u^{\max(r, s)}$ semifield. $u^r \cdot u^s = u^{r+s}$. u formal variable.

Aim: construct semifield homomorphism $\Psi: \mathbb{Z} \rightarrow U$

such that $\{\Psi(tz) \mid t \in \mathbb{Z}\}$ is infinite.

Case $bc > 4$:

Let γ be a real number > 1 which is an eigenvalue of $\begin{pmatrix} bc-1 & -b \\ c & -1 \end{pmatrix}$. Warning: γ is not right domain... should be $\mathbb{Q}(z_1, z_2)$?

Put $\Psi(z_1) = u^c$

Exchange relations become:

$$\begin{pmatrix} bc-1 & -b \\ c & -1 \end{pmatrix}$$

$$\Psi(z_2) = u^{\gamma+1}$$

$$\Psi(z_{t-1}) \Psi(z_{t+1}) = \begin{cases} \Psi(z_t)^{\oplus 1} + \text{even} \\ \Psi(z_t)^b + \text{odd} \end{cases}$$

$$\text{Claim: } \mathcal{N}(z_{2k+1}) = u^{\lambda^k c} \rightarrow \mathcal{N}(z_{2k+2}) = u^{\lambda^k (\lambda+1)} c^{-\lambda^k c}$$

$$\text{Use induction: } \mathcal{N}(z_{2k+3}) = \frac{\mathcal{N}(z_{2k+2})^b \oplus 1}{\mathcal{N}(z_{2k+1})} = u^{\lambda^k (\lambda+1)} c^{-\lambda^k c}$$

$$\mathcal{N}(z_{2k+4}) = \frac{\mathcal{N}(z_{2k+3})^b \oplus 1}{\mathcal{N}(z_{2k+2})} = u^{\lambda^{k+1} b - \lambda^k (\lambda+1)}$$

$$= u^{\lambda^k (\lambda \cdot b - \lambda + 1)}$$

$$= u^{\lambda^{k+1} / (\lambda+1)}$$

$$(\text{using } \lambda^2 - (\lambda \cdot b - \lambda + 1) = 0)$$

(case $bc=2$) : Instead use $\mathcal{N}(z_1) = u$ $\mathcal{N}(z_2) = u^b$.

Claim : $\mathcal{N}(z_{2k-1}) = u^{\lambda^{k-1}}$ $\mathcal{N}(z_{2k+2}) = u^{(\lambda+1)}$ (also by induction)

Def : A skew-symmetrizable matrix $B = (b_{ij})$ is 2-finite if for any $B' = (b'_{ij})$ mutation equivalent to B , we have $(b'_{ij}, b'_{ji} \in \mathbb{Z})$ $\forall i, j$.

Or : Finite type seed pattern \Rightarrow every exchange matrix is 2-finite

pf : If $B \sim B'$ with $|b'_{ij}, b'_{ji}| \geq 4$ for some i, j , then by freezing all the cluster variables to the rank 2 case, we are reduced

Rank : Turns out to converse to above corollary is also true!

Lecture 11

2/18/26

Def: A symmetrizable generalized Cartan matrix is a square integer matrix $A = (a_{ij})$ such that:

- all diagonal entries are 2
- all off-diagonal entries are ≤ 0
- DA is symmetric for some diagonal matrix D with positive entries

Def: A Cartan matrix is a symmetrizable generalized Cartan matrix such that DA is positive definite (i.e. has only > 0 eigenvalues, or equivalently > 0 principal minors).

NR: For a Cartan matrix A , we must have

$$\det \begin{pmatrix} 2 & a_{ij} \\ a_{ji} & 2 \end{pmatrix} = 4 - a_{ij}a_{ji} \geq 0 \quad \text{for all } i \neq j,$$

i.e. $a_{ij}a_{ji} \leq 1$. In particular, $|a_{ij}|, |a_{ji}| \in \{0, 1, 2, 3\}$.

Ex: $A = \begin{pmatrix} 2 & -b \\ -c & 2 \end{pmatrix}$ for $b, c \in \mathbb{Z}_{\geq 0}$ is Cartan if and only if one of:

- $b = c = 0$
- $b = c = 1$
- $b = 1, c = 2$ or $b = 2, c = 1$
- $b = 1, c = 3$ or $b = 3, c = 1$

Note that these "match" our classification of rank 2 cluster algebras of finite type

Given an $n \times n$ Cartan matrix A , its Dynkin diagram $\text{Dynk}(A)$ is the graph with vertices $i \neq j$ where for each $i \neq j$ we put



if $a_{ij} = -1, a_{ji} = -2$



if $a_{ij} = -1, a_{ji} = -3$



if $a_{ij} = a_{ji} = -1$

Ex: $A = \begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix} \rightarrow \text{Dynk}(A) =$



Ex: $A = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \rightarrow \text{Dynk}(A) =$



Note: this is unrelated to the fact that the given

corresponds to the skew-symmetric matrix $\begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}$

Def: A Cartan matrix is indecomposable if its Dynkin diagram is connected. The type of A is its equivalence class up to simultaneous permutations of the rows and columns.

Obs: Any Cartan matrix is equivalent to a block-diagonal matrix with indecomposable blocks, which correspond to the connected components of the corresponding Dynkin diagram. The type of A is determined by the multiplicity of each type of connected Dynkin diagram appearing in such a decomposition.

~~Classification of Cartan matrices~~

Thm (Cartan-Filling): The Dynkin diagrams of indecomposable Cartan matrices are as follows:

A_n ($n \geq 1$)



B_n ($n \geq 2$)



C_n ($n \geq 3$)



D_n ($n \geq 4$)



E_6



E_7



E_8



F_4



G_2



Def: Given an $n \times n$ skew-symmetrizable integer matrix $B = (b_{ij})$, its Cartan counterpart $\text{Cart}(B)$ is the symmetrizable generalized Cartan matrix (a_{ij}) , also $n \times n$, defined by $a_{ij} = \begin{cases} 2 & i=j \\ -b_{ij} & i \neq j \end{cases}$.

Thm: A cluster algebra is of finite type if and only if its seed pattern contains an exchange matrix B such that $\text{Cart}(B)$ is a Cartan matrix.

Thm Suppose that B_1, B_2 are skew-symmetrizable integer matrices s.t. $\text{Cart}(B_1), \text{Cart}(B_2)$ are Cartan. Then $\text{Cart}(B_1), \text{Cart}(B_2)$ have the same type if and only if B_1 and B_2 are mutation equivalent.

Recall: The classification of simple complex Lie algebras (or equivalently compact simply connected Lie groups) is precisely

- A_n ($n \geq 1$) : $sl_{n+1}(\mathbb{C})$ special linear
- B_n ($n \geq 2$) : $so_{2n+1}(\mathbb{C})$ odd orthogonal
- C_n ($n \geq 3$) : $sp_{2n}(\mathbb{C})$ symplectic
- D_n ($n \geq 4$) : $so_{2n}(\mathbb{C})$ even orthogonal
- exceptional algebras : G_2, F_4, E_6, E_7, E_8 sporadic

Note: A Lie algebra is simple if not abelian and no nontrivial ideals.

Lecture 12

2/20/26

Def: A bordered surface with marked points is a pair (S, M) , where

- S = oriented connected surface, possibly with boundary
- $M \subset S$ nonempty subset with at least one point on each boundary cpt

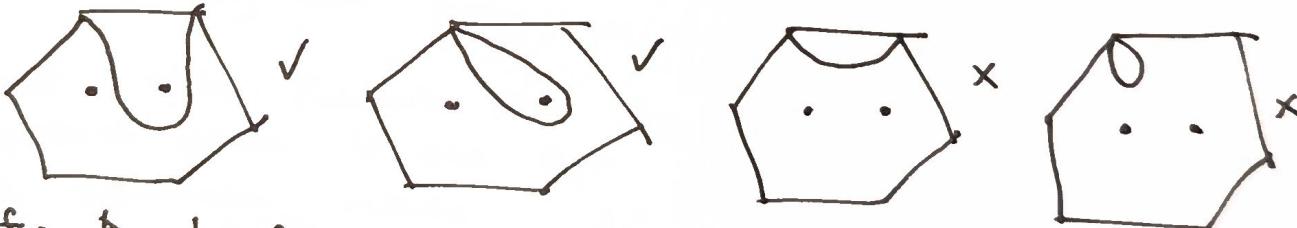
Will refer to M as "marked points" and those in the interior of S as "punctures"

For technical reasons, will assume (S, M) is not a sphere with 1, 2, 3 punctures, a monogon with 0, 1 punctures, or a bigon or triangle without punctures



Def: An arc τ in (S, M) is a curve in S (up to isotopy) such that

- τ does not cross itself (except that endpoints may coincide)
- apart from its endpoints, τ is disjoint from M and ∂S
- τ does not cut out an unpunctured bigon



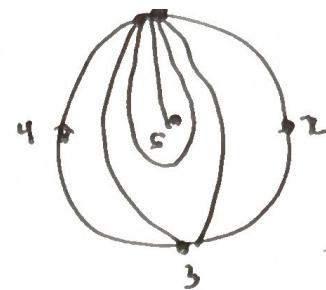
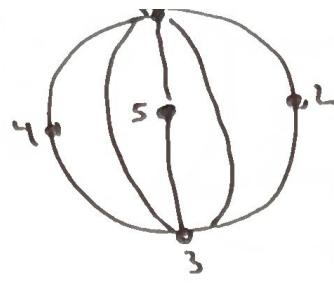
Def: A boundary segment is a curve which connects two marked points and ~~lies~~ lies entirely in ∂S without passing through a third marked point.

Def: Two arcs are compatible if they are isotopic (except possibly at endpoints) and do not cross. A triangulation is a maximal collection of pairwise compatible arcs, along with all boundary segments.

We refer to the components cut out by the arcs of a triangulation as "triangles".

NB: triangles may have either 3 distinct sides or only 2 ("self-folded")

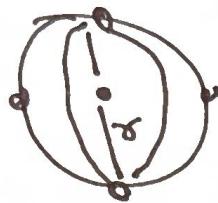
Ex:



self-folded triangle

Def: A flip of a triangulation T is a single arc σ to be another arc σ' for such that $T \setminus \{\sigma\} \cup \{\sigma'\}$ forms a new triangulation. replaces a (unique if it exists)

Ex:



flip along σ



) but we cannot flip along η .

Def: Given a bordered surface S with marked points M , the (cusped) Teichmüller space $T(S, M)$ is the space of all complete finite-area Riemannian metrics with constant curvature -1 on $S \setminus M$ and with geodesic boundary $\partial S \setminus M$. modulo $\text{Diff}_0(S, M)$

Here $\text{Diff}_0(S, M)$ is the group of diffeomorphisms of S which fix M and are isotopic to the identity.

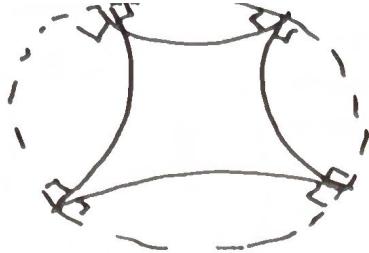
Note: there are cusps at the points M , meaning they are infinitely far away yet the total area is finite.

Recall: the Poincaré disk ~~model~~ for two-dimensional hyperbolic space is the open unit disk \mathbb{D} with the Riemannian metric $ds^2 = \frac{4(dx^2 + dy^2)}{(1 - x^2 - y^2)^2}$ (constant curvature -1).

The geodesics are of the form circles $C \subset \mathbb{R}^2$ meeting $\partial\mathbb{D}$ orthogonally and also $L \cap \mathbb{D}$ where $L \subset \mathbb{R}^2$ is a Euclidean line through the origin.

Ex: Let P be a k -sided hyperbolic polygon cut out by geodesics in \mathbb{D} , equipped with the restriction of the hyperbolic metric. This defines an element of $T(S, M)$ with (S, M) having genus zero, one boundary component, and k boundary marked points.

Ex:

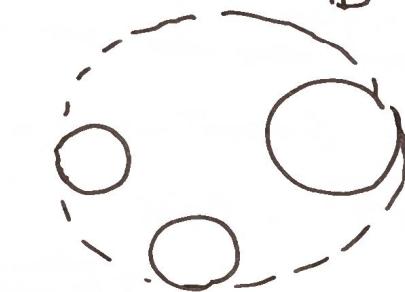


a hyperbolic quadrilateral

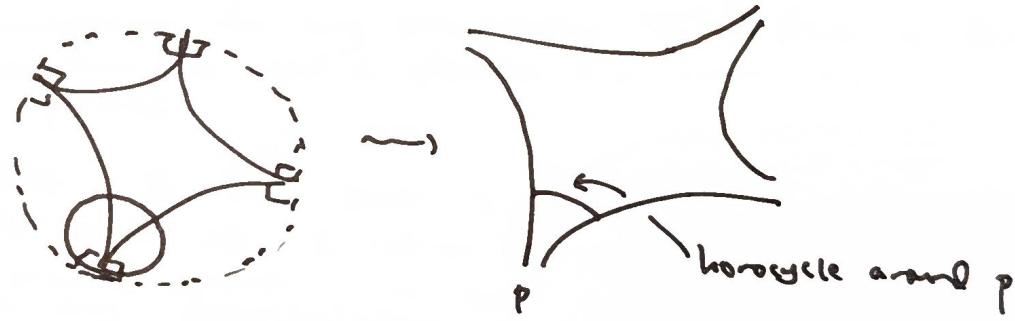
Def: Given $\Sigma \in \mathcal{T}(S, M)$, a horocycle around a puncture p is a closed curve in Σ which is orthogonal to all geodesics asymptotic to p . Similarly, a horocycle around a boundary marked point p is an arc joining two points of $\partial \Sigma$ which is orthogonal to all geodesics asymptotic to p .

Rem: Intuitively the horocycle around p is the set of all points of a fixed distance from p , but this distance is infinite.

Ex: The horocycles in 1D boundary are circles tangent to the



Ex:

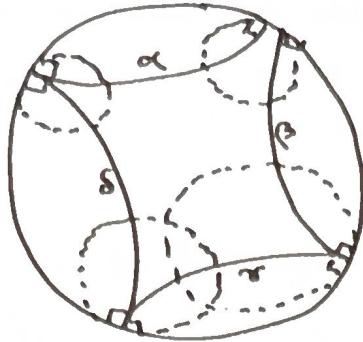


Def: The decorated Teichmüller space $\widetilde{\mathcal{T}}(S, M)$ is defined similarly to $\mathcal{T}(S, M)$ but now we equip $S \setminus M$ with a collection of horocycles, one for each marked point in M .

Def: Fix $\Sigma \in \widetilde{\mathcal{T}}(S, M)$, and let γ be an arc or boundary segment in (S, M) . We define the lambda length $\lambda(\gamma)$ as follows. Let γ_Σ be the unique representative of γ which is geodesic with respect to the hyperbolic metric on Σ . Let $l(\gamma_\Sigma)$ be the signed distance along γ_Σ between the two horocycles at either end of γ_Σ , where the sign is positive if the two horocycles are disjoint and negative otherwise.

Put $\gamma(\gamma) := \exp(l(\gamma_\Sigma)/2)$.

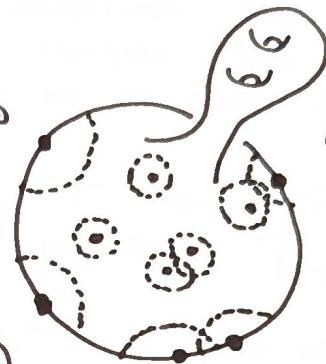
Ex.:



In this example with 4 boundary marked points and no punctures, we have $l(\alpha) l(\beta), l(\gamma) l(\delta) > 0$, but $l(\delta) < 0$.

Note: we can depict a typical element $\Sigma \in \widetilde{\mathcal{T}}(S, M)$ by a cartoon such as

In this example S has genus two and one boundary point, and M consists of 5 boundary marked pts and 5 punctures (i.e. interior marked pts). Some of the horocycles intersect.



this picture is not in any way faithful to the hyperbolic metric on Σ

the boundary segments should be geodesic w.r.t. the hyperbolic metric

Thm (Penner, Fomin-Thurston): the map

$$\pi: \widetilde{\mathcal{T}}(S, M) \xrightarrow{\text{arc or boundary segment of } T} \mathbb{R}_{>0}^{n+k} \text{ is}$$

a homeomorphism for any triangulation T . Here n denotes the number of arcs and c denotes the number of boundary marked points.

Rank: If S has genus g and b boundary points, and M consists of i interior marked points and c boundary marked points in any triangulation, one can compute the number n of arcs in any triangulation to be:

$$n = 6g + 3b + 3i + c - 6$$

$$\text{So } \dim \widetilde{\mathcal{T}}(S, M) = 6g - 6 + 3b + 3i + 2c$$

$$\text{and } \dim \mathcal{T}(S, M) = \dim \widetilde{\mathcal{T}}(S, M) - i - c$$

$$= 6g - 6 + 3b + 2i + c$$

due to choice of horocycles

2 parameters for locations of interior marked pts and 1 parameter for locations of boundary marked pts

Prop ("hyperbolic Ptolemy"): Let $\alpha, \beta, \gamma, \delta$ be arcs or boundary segments which cut out a quadrilateral with diagonals η, θ . Then we have

$$\pi(\eta) \pi(\theta) = \pi(\alpha) \pi(\beta) + \pi(\gamma) \pi(\delta)$$

(here we assume $\alpha, \beta, \gamma, \delta$ are ordered cyclically).

We next associate an extended exchange matrix to any triangulation.

Def : Let T be a triangulation of (S, M) with arcs τ_1, \dots, τ_n and boundary segments $\tau_{n+1}, \dots, \tau_{n+e}$.

Put $b_{ij} = \#\{ \text{triangles with sides } \tau_i, \tau_j \text{ in clockwise order} \}$
— $\#\{ \text{triangles with sides } \tau_i, \tau_j \text{ in counter-clockwise order} \}$.

Put $\tilde{B}_T := (b_{ij})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}} \rightarrow$ the $(n+e) \times n$ extended exchange matrix of T .

Rank : Actually the above definition has to be modified if there are any self-folded triangles. We have $|b_{ij}| \leq 2$ since every arc is a side of at most two triangles.

Prop : Flipping a triangulation T corresponds to mutating the associated extended exchange matrix \tilde{B}_T .

Facts :

- every arc in (S, M) is part of a triangulation
- any two triangulations differ by a sequence of flips

Now let \mathbb{A} denote the cluster algebra associated to \tilde{B}_T .
It follows that:

- each arc τ_j in (S, M) corresponds to a cluster variable $x_j \in \mathbb{A}$
- each triangulation T of (S, M) gives rise to a seed of \mathbb{A}

Rank : There is an injective map

$$\{ \text{arcs in } (S, M) \} \hookrightarrow \{ \text{cluster variables in } \mathbb{A} \}$$

but it is not generally surjective if there are any interior marked points, due to the fact that not all arcs can be flipped. There is a more general notion of "tagged arcs" and "tagged triangulations" which are in bijection with cluster variables (due to Fomin-Shapiro-Thurston and

Rank : At least in the absence of punctures (i.e. interior marked pts), we can view every element in \mathbb{A} as a function $\tilde{T}(S, M) \rightarrow \mathbb{R}$. Are these "all of them" in any sense?

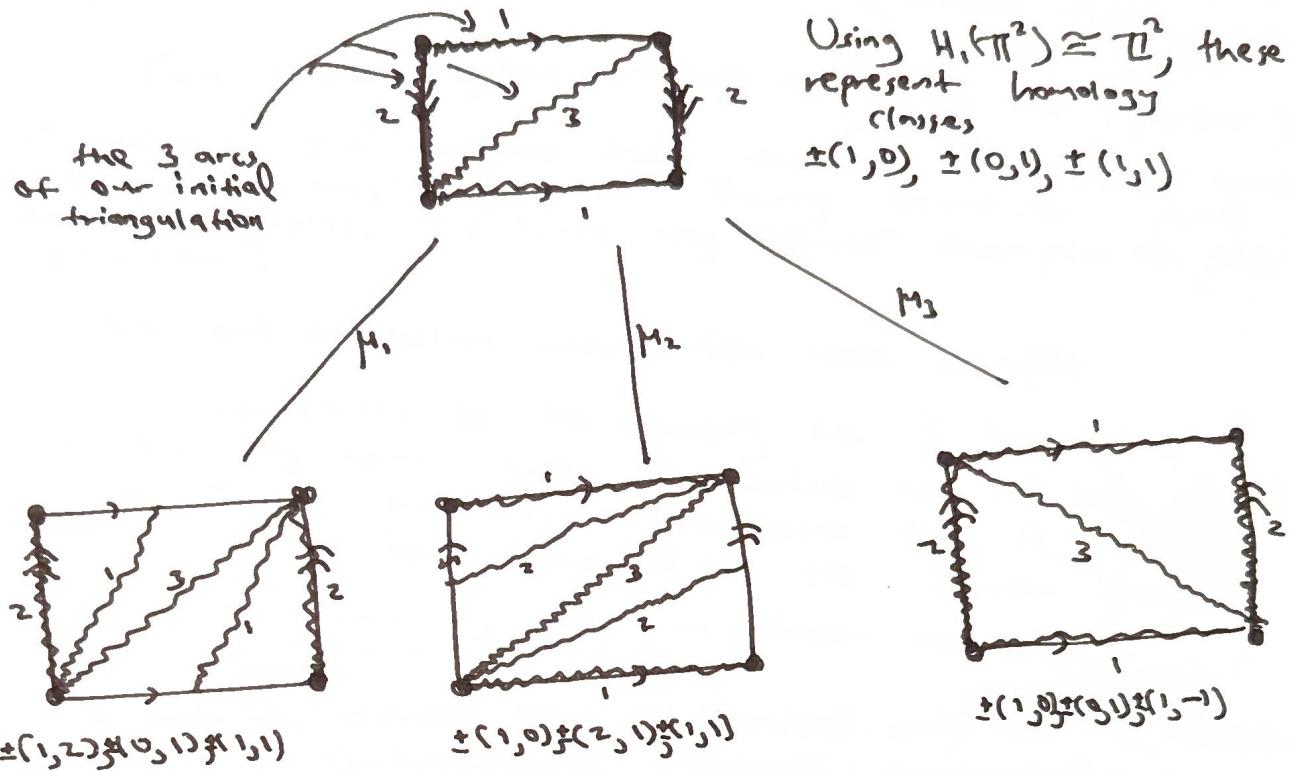
Even though not every seed corresponds to a triangulation (but rather a tagged triangulation), we still have:

Lemma : For each seed of \mathbb{A} , the corresponding (extended) exchange matrix has all entries equal to $0, \pm 1, \pm 2$.

Cor : For any (S, M) , the associated exchange matrix B_T is mutation-finite, i.e. only finitely many mutations appear in its mutation graph.

Ex: Consider the once-punctured torus (S, M) , i.e. $g=1$, $b=0$, $i=1$, $c=0$, and hence every triangulation has $n = 6g + 3b + 3i + c - 6 = 3$ arcs.

Here is the beginning of the exchange graph:

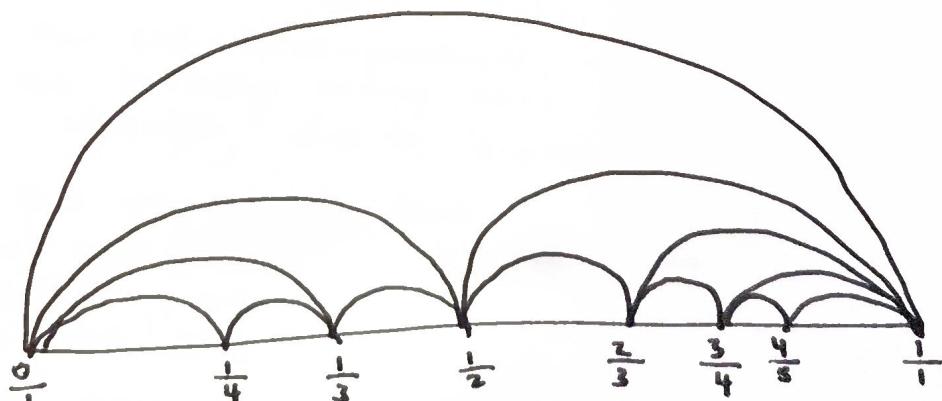


Observe: if a triangulation has arcs with homology classes $\pm(a_1, b_1), \pm(a_2, b_2), \pm(a_3, b_3)$, then must have

$(a_3, b_3) = \pm(a_1 + a_2, b_1 + b_2)$ or $(a_3, b_3) = \pm(a_1 - a_2, b_1 - b_2)$
and μ_3 replaces one option with the other
(and similarly for μ_1, μ_2).

Also, σ_1 and σ_2 intersect in $\det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} = \cancel{a_1 b_2 - a_2 b_1}$
points, so we must have $|a_1 b_2 - a_2 b_1| = 1$, and
similarly $|a_1 b_3 - a_3 b_1| = |a_2 b_3 - a_3 b_2| = 1$.

Upshot: the exchange graph is dual to the Farey tessellation:



Here $p/q, p'/q'$ are connected by an arc iff $|pq' - p'q| = 1$.

Observe : For the triangulation the corresponding exchange matrix is

$$\tilde{B}_T = \begin{pmatrix} 0 & -2 & 2 \\ 2 & 0 & -2 \\ -2 & 2 & 0 \end{pmatrix}$$

(no frosens)

$T =$



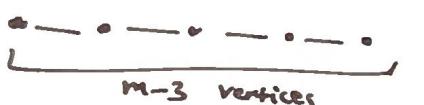
(reversing the orientation on \mathbb{P}^2 would flip these signs)

This is exactly the exchange matrix of the Markov quiver!

Question : It follows that there is a bijective correspondence between triangles in the Farey tessellation and Markov triples. Is there any "direct" description of this bijection?

We end this lecture with a few more examples

Ex : Let (S, M) be the m -gon, i.e. S has genus 0 and one bdy cpt, and M consists of m bdy marked points and no punctures. This gives the A_{m-3} cluster algebra, i.e. the one associated to the Dynkin diagram



plus m frozen variables.

Recall that the exchange graph is identified with the 1-skeleton of the $(m-3)$ -dimensional Stasheff associahedron.

Note that the number of cluster variables is exactly

$$\binom{m}{2} - m = \frac{m(m-3)}{2}$$

and in particular finite.

Ex : Let (S, M) be the once-punctured m -gon. This is the cluster algebra associated with the D_m Dynkin diagram, i.e.



Note that there are only finitely many arcs, but this not a priori imply only finitely many cluster variables.

Ex : For the twice-punctured m -gon, it is easy to see that there are infinitely many arcs, and hence infinitely many cluster variables, due to braiding phenomena:

Prob : One way to prove that

$\{\text{arcs}\} \rightarrow \{\text{cluster variables}\}$ is injective is using hyperbolic Ptolemy and the fact that lambda lengths give a homeomorphism $\tilde{T}(S, M) \cong \mathbb{R}_{\geq 0}^{n_{\text{arcs}}}$.



Another way is as follows. For any arc σ , the cluster variable x_σ is a Laurent polynomial in the initial (extended) cluster variables $x_{u \rightarrow x_m}$.

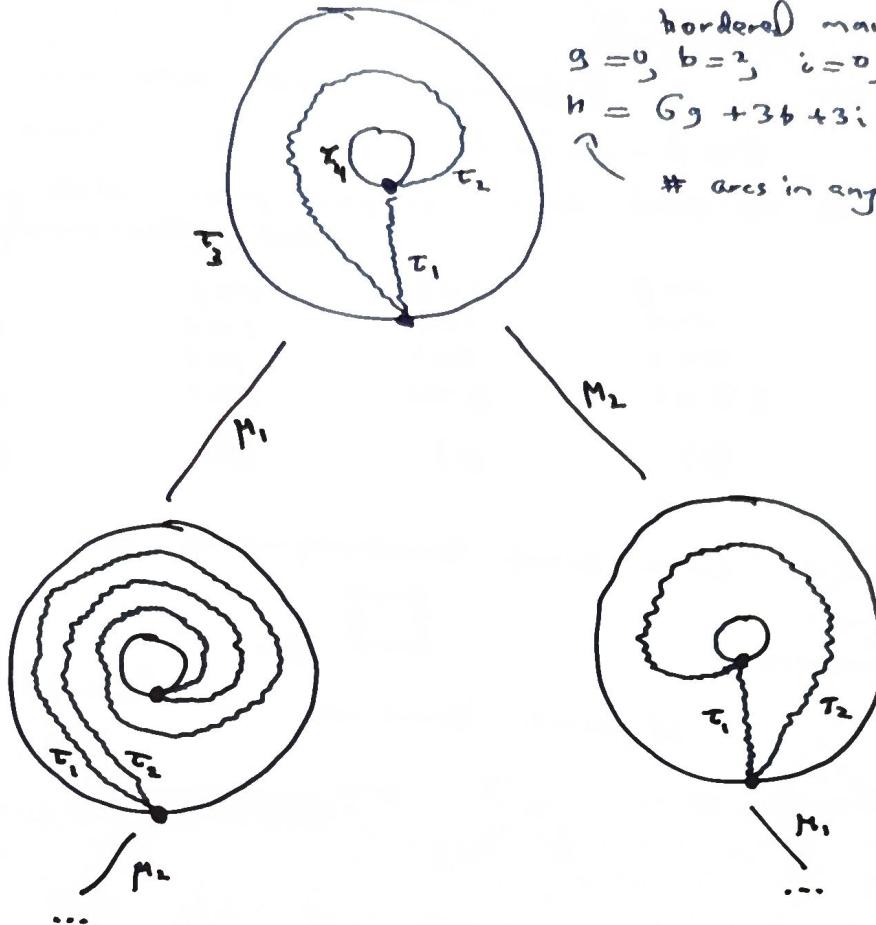
Writing $x_\sigma = \frac{P_\sigma(x_1, \dots, x_m)}{x_1^{d_1} x_2^{d_2} \dots x_m^{d_m}}$, it turns out that the denominator vector (d_1, \dots, d_m) precisely records the intersection numbers of σ with the curves $\sigma_1, \dots, \sigma_m$ of the initial triangulation. Moreover, for ~~distinct~~ distinct arcs σ_1, σ_2 , these intersection numbers cannot all be the same, i.e. $x_{\sigma_1} \neq x_{\sigma_2}$.

Rank: Here intersection number means interior intersections, i.e. σ intersects any boundary segment trivially. This is consistent with the fact that, when writing a cluster variable as a Laurent polynomial in the extended cluster variables of a seed, ~~any~~ the frozen variables do not appear in denominators.

Lecture 13

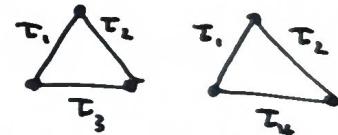
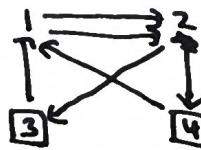
2/23/26

Ex:



bordered marked surface with
 $g=0, b=2, i=0, c=2$
 $n = 6g + 3b + 3i + c - 6 = 2$
arcs in any triangulation

The initial triangulation has two triangles
Corresponding quiver:



This is the rank 2 cluster algebra $A(2,2)$ (plus 2 facets),
with exchange graph and cluster variables

$$\dots, z_1^2, z_1, z_0, z_1, z_2, z_3, \dots$$

Ex: The m -Kronecker quiver
surface-type for $m \geq 3$

$$1 \xrightarrow{m} 2 \text{ is not}$$

Ex:



This is the A_2 cluster algebra, which corresponds to the hexagon.

The number of triangulations

$$C_{m-2} = \frac{1}{m-1} \binom{2m-4}{m-2}$$

of an m -gon is
Catalan number

m	3	4	5	6	7
C_{m-2}	1	2	5	14	42

The number of unlabeled seeds of A_{m-3} is also C_{m-2}
while the number of labeled seeds is $(m-3)!(m-2)$ (is this true??)

Ex:



Q: Does this come from a surface?

Would need

$$n = 6g + 2b + 3i + e - 6 = 3$$

Recalling that every bdy cut must have at least one puncture, the possibilities are:

$$g=1$$

$$b=0$$

$$i=1$$

$$e=0$$

(a)

$$g=0$$

$$b=1$$

$$i=1$$

$$e=3$$

(b)

$$g=0$$

$$b=1$$

$$i=0$$

$$e=6$$

(c)

$$g=0$$

$$b=2$$

$$i=0$$

$$e=3$$

(d)

(not that we explicitly excluded the three-punctured sphere at the outset)

(a) is the once-punctured torus \rightarrow



Markov cluster algebra

(b) is the once-punctured triangle

Triangulation



\rightsquigarrow quiver



This gives the A_3 cluster algebra.

Note: In general the once-punctured n -gon gives D_n , but it happens that

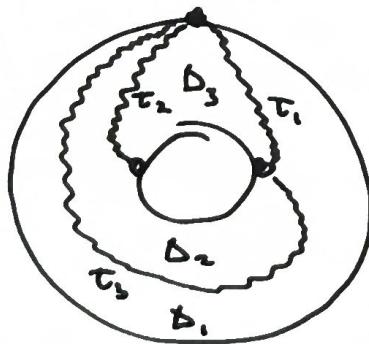
$$D_3 = A_3$$

$\left(\dots \right) = \dots$ coincidence

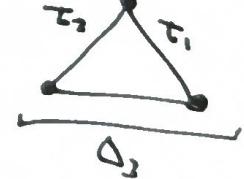
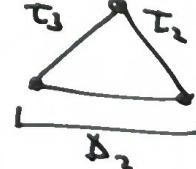
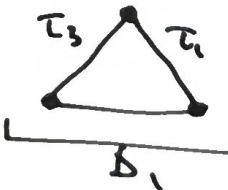
(c) is the hexagon

What about (d)?

\rightsquigarrow also A_3



Ignoring fators (i.e. bdy segments) have 3 triangles



So the corresponding quiver is

It follows that this quiver has finite mutation type, but infinitely many cluster variables (due to arcs like



and so on)



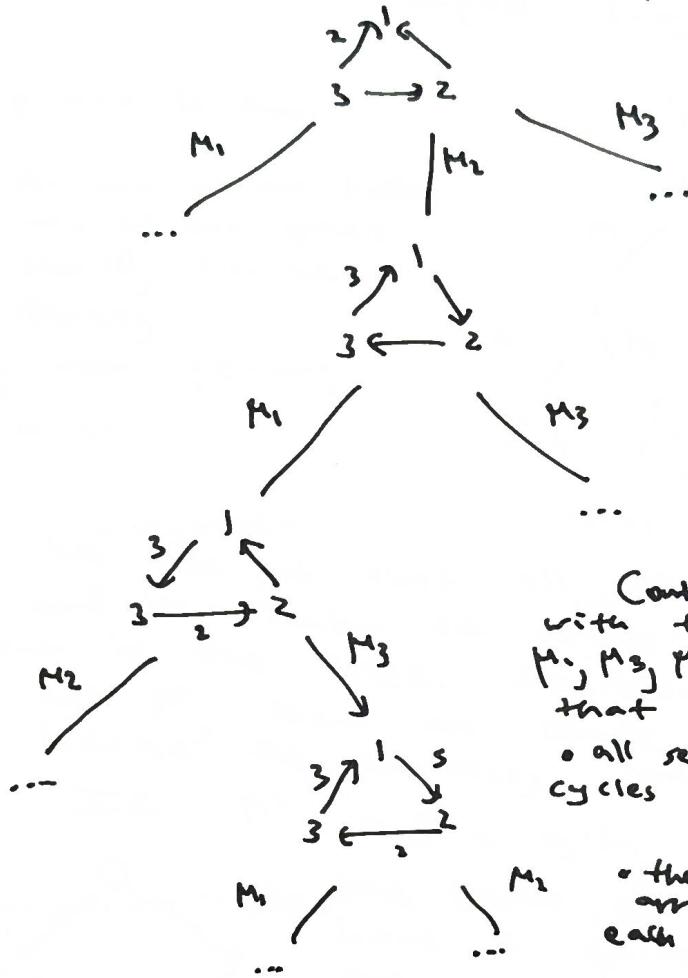
Ex:



Can show that this is not surface type since we already have an exhaustive list of rank 3 surface type cluster algebras.

Alternatively, will show that the quiver itself has infinite mutation type (which is ruled out in the surface case).

Here is part of the mutation graph:



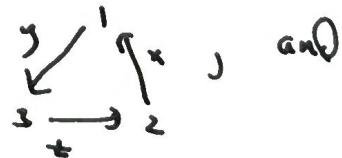
Continuing in this way with the mutation sequence $\mu_1, \mu_3, \mu_1, \mu_3, \mu_1, \mu_3, \dots$, we see that

- all subsequent quivers are cycles (with each mutation flipping the orientation)
- the total number of arrows increases with each subsequent mutation

We will think of the total number of arrows as a measure of the "complexity" of the quiver.

More generally, consider the quiver
assume $z > x, y, 2z$.

Then μ_2 gives $\begin{array}{c} x \\ y \\ z \end{array} \xrightarrow{\quad} \begin{array}{c} x \\ z \\ y \end{array}$
 μ_3 will be very similar

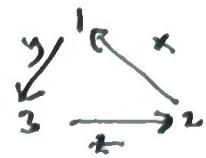


Note that we have $xz - y > z$ (i.e. $z(x-1) > y$)

and the total number of arrows is

$$xz - y + x + z > x + y + z \text{ since } xz \geq 2z > 2y.$$

Lemma: Let Q be the quiver with $x \geq y \geq 2$.

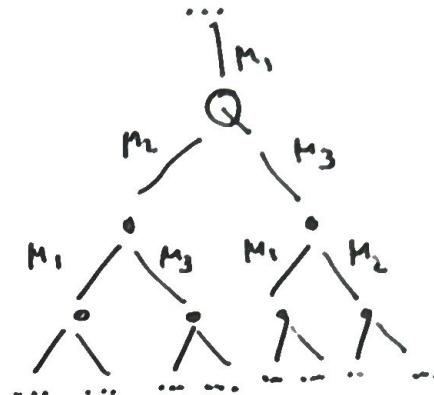


Let Γ be the mutation graph of Q , Γ' be the connected component of Γ containing Q , e is the edge connecting Q with $\Gamma' \setminus Q$. Then Γ' is an infinite complete 2-ary tree rooted in Q .

The picture is thus:

Note: for now we are taking the vertices of our quivers to be labeled, i.e. we do not identify

$i \rightarrow j$ with $i \leftarrow j$
and so on.



pf: We have seen that all quivers in Γ' are cyclic and the number of arrows increases as we go down in the tree. In particular, for each quiver in Γ' there are exactly two mutations which increase the complexity and one which decreases it.

If Γ'

had a cycle, i.e. then at the lower "merge point" we would have a quiver such that only one mutation increases the complexity.

