Math 635: Cluster Varieties

Algebra, Topology, Geometry, Duality

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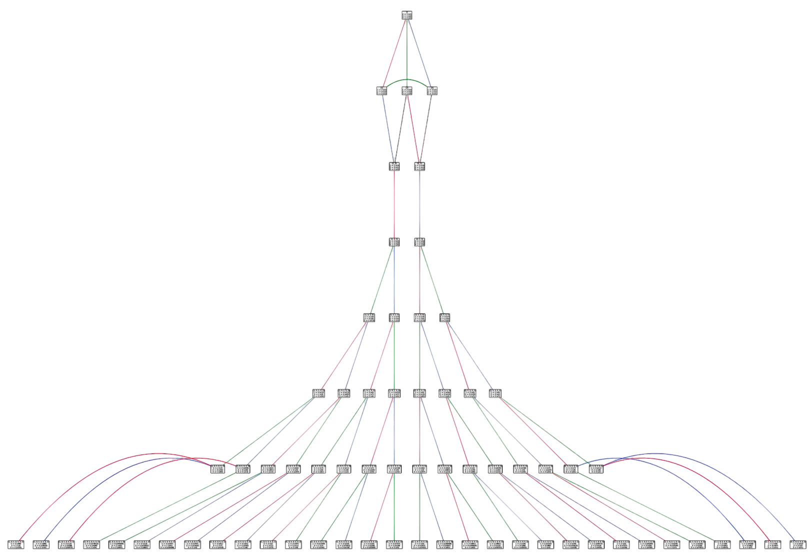
https://kylersiegel.xyz/635_spring_2026.html



Disclaimer: These notes are based on handwritten lecture notes which were typeset and lightly edited with AI assistance. This typesetting process is not perfect and could have introduced some errors.

Contents

1 Lecture 1

Date: January 12, 2026
Main reference: [FWZ21], §1–2.

1.1 Introduction

Roughly speaking:

- A **cluster variety** is a complex algebraic variety obtained by gluing together many copies of $(\mathbb{C}^*)^n$, where the gluing maps take a very particular form.
- A **cluster algebra** is the algebra of regular functions $f\colon V \to \mathbb{C}$ on a cluster variety.

Fomin–Zelevinsky, early 2000s: Introduced cluster algebras. They arise in many parts of mathematics and physics as a kind of "universal model" for mutation/wall-crossing phenomena:

- Quiver representation theory
- Teichmüller theory
- Poisson geometry
- Grassmannians
- Total positivity
- QFT scattering amplitudes (amplituhedron)
- Integrable systems
- String theory (BPS states)
- etc.

Gross–Hacking–Keel–Kontsevich (GHKK) [Gro+18]:

- Constructed canonical bases for cluster algebras.
- Established positivity of the Laurent phenomenon.
- Proof uses mirror symmetry for log Calabi–Yau varieties (which can be thought of as a generalization of toric varieties, related to almost toric fibrations in symplectic geometry).
- Many strong applications in representation theory, e.g., canonical bases for finite-dimensional irreducible representations of $\mathrm{SL}_n(\mathbb{C})$.

Remark 1.1. The canonical bases were originally found independently by Lusztig and Kashiwara in the early 1990s using quantum groups. Amazingly, the construction of GHKK uses only general geometry—no representation theory!

1.2 Total Positivity

Definition 1.2. A matrix $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ is **totally positive** (TP) if all of its minors are positive.

Gantmacher–Krein (1930s): If A is TP, then the eigenvalues of A are real, positive, and distinct.

Binet–Cauchy theorem: The TP matrices are closed under multiplication, and hence form a multiplicative semigroup $G_{>0}$.
Lusztig: Extended the definition of $G_{>0}$ to other semisimple Lie groups G.

More generally: If a given complex algebraic variety Z has a distinguished family Δ of regular functions $Z \to \mathbb{C}$, we define the **TP variety** by

$$Z_{>0} := \{z \in Z \mid f(z) > 0 \text{ for all } f \in \Delta\}.$$

Example 1.3. For $Z = \mathrm{Mat}_{n\times n}(\mathbb{C})$, $\mathrm{GL}_n(\mathbb{C})$, or $\mathrm{SL}_n(\mathbb{C})$, we recover the above notion of TP, where $\Delta = \{\text{minors}\}$.

Example 1.4. The **Grassmannian** $\mathrm{Gr}_{k,m}(\mathbb{C}) = \{k\text{-dimensional linear subspaces of } \mathbb{C}^m\}$, with $\Delta = \{\text{Plücker coordinates}\}$.

Example 1.5. Partial flag manifolds, homogeneous spaces for semisimple complex Lie groups, etc. (slight scaling ambiguity).

Lemma 1.6. *A matrix $A \in \mathrm{Mat}_{n\times n}$ has $\binom{2n}{n} - 1$ minors.*

Proof. The number of minors is

$$\# = \sum_{k=1}^{n} \binom{n}{k}^2.$$

By Vandermonde's identity:

$$\binom{m+w}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{w}{r-k}.$$

Setting $m = w = r = n$ gives

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2,$$

from which the result follows. $\square$

Remark 1.7. To verify Vandermonde's identity, note that both sides count the number of subcommittees with r members, given a committee with m men and w women.

Question 1.8. Can we check that $A \in \mathrm{Mat}_{n\times n}$ is TP by only testing a subset of the $\binom{2n}{n} - 1$ minors? How many tests are needed?

Example 1.9. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_{2\times 2}$. Define $\delta := ad - bc$, so $d = \frac{\delta + bc}{a}$. Thus, if $a, b, c, \delta > 0$, then d is automatically positive. This reduces $\binom{4}{2} - 1 = 5$ checks to 4 checks.

The goal is "efficient TP testing."

1.3 Plücker Coordinates on Grassmannians

Given $A \in \mathrm{Mat}_{k\times m}$ of rank k, we have $\mathrm{rowspan}(A) =: [A] \in \mathrm{Gr}_{k,m}$.

For $J \subseteq \{1,\ldots,m\}$ with $|J| = k$, the **Plücker coordinate** is

$$P_J(A) := k \times k \text{ minor of } A \text{ corresponding to columns } J.$$

Note 1.10. For $A, B \in \mathrm{Mat}_{k\times m}$ with $[A] = [B]$ (i.e., same row spans), the tuples $(P_J(A))_{|J|=k}$ and $(P_J(B))_{|J|=k}$ agree up to common rescaling. We thus get a map

$$\mathrm{Gr}_{k,m} \longrightarrow \mathbb{CP}^{N-1}, \quad N = \binom{m}{k}.$$

In fact, this is an embedding, called the **Plücker embedding**.

Let $\mathbb{C}[\mathrm{Mat}_{k\times m}]$ denote the coordinate ring of $\mathrm{Mat}_{k\times m}$, i.e., the polynomial algebra in variables x_{ij} for $1 \le i \le k$, $1 \le j \le m$.

Definition 1.11. The **Plücker ring** $R_{k,m}$ is the subring of $\mathbb{C}[\mathrm{Mat}_{k\times m}]$ generated by P_J over all $J \in \{1,\ldots,m\}$ with $|J| = k$.

Claim 1.12. *The ideal of relations in $R_{k,m}$ is generated by certain quadratic relations called the **Grassmann–Plücker relations**.*

Definition 1.13. The **totally positive Grassmannian** $\mathrm{Gr}^+_{k,m}$ is the subset of $\mathrm{Gr}_{k,m}$ consisting of those points whose Plücker coordinates are all positive (up to common scaling).

Note 1.14. For $A \in \mathrm{Mat}_{k\times m}(\mathbb{R})$, we have $[A] \in \mathrm{Gr}^+_{k,m}$ if and only if all $k \times k$ minors of A have the same sign.

Question 1.15. For $A \in \mathrm{Mat}_{k\times m}(\mathbb{R})$, can we verify that all $k \times k$ minors are positive by only checking a subset of the $\binom{m}{k}$ minors? How many tests are needed?

(We may assume positive WLOG by rescaling.)

1.4 Positivity Testing for $\mathrm{Gr}_{2,m}$

Claim 1.16. *Given $A \in \mathrm{Mat}_{2\times m}$, put $P_{ij} := P_{\{i,j\}}$ for $1 \le i < j \le m$. To check that all 2×2 minors $P_{ij}(A) > 0$, it suffices to check only the $2m-3$ special ones.*

Note 1.17. $2m - 3 = \dim \mathrm{Gr}_{2,m} + 1$.

Lemma 1.18. *For $1 \le i < j < k < \ell \le m$, we have the three-term Grassmann–Plücker relation:*

$$P_{ik}P_{j\ell} = P_{ij}P_{k\ell} + P_{i\ell}P_{jk}.$$

Remark 1.19. For an inscribed quadrilateral, Ptolemy's theorem (2nd century) gives

$$AC \cdot BD = AB \cdot CD + BC \cdot AD.$$

Example 1.20. Let $A = \begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix}$. We verify $P_{13}P_{24} = P_{12}P_{34} + P_{14}P_{23}$, i.e.,

$$(ag - ce)(bh - df) = (af - be)(ch - dg) + (ah - de)(bg - cf). \quad \checkmark$$

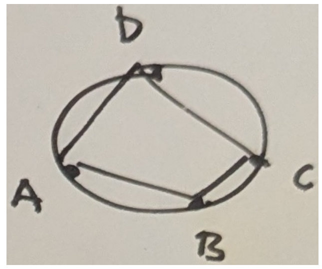


Figure 1: Inscribed quadrilateral for Ptolemy's theorem.

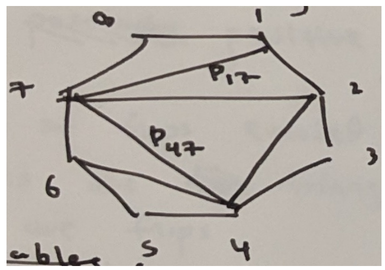


Figure 2: A triangulated polygon $\mathbb{P}_m$ with vertices labeled $1, \dots, m$.

Put $\mathbb{P}_m$ = regular m-gon, and let T be a triangulation.
To each side or diagonal, associate P_{ij}, where i, j are the endpoints.

- **Cluster variables**: P_{ij} ranging over diagonals.
- **Frozen variables**: P_{ij} ranging over sides.
- **Extended cluster**: $\{\text{cluster vars}\} \cup \{\text{frozen vars}\} =: \widetilde{x}(T)$.

Note 1.21. The extended cluster has $2m-3$ variables, and we claim that these are algebraically independent.

Example 1.22. In the above picture, we have cluster variables $P_{17}, P_{27}, P_{47}, P_{24}$ and frozen variables $P_{12}, P_{23}, \dots, P_{78}, P_{18}$.

Theorem 1.23. *Each P_{ij} for $1 \leq i < j \leq n$ can be written as a subtraction-free rational expression in the elements of a given extended cluster $\widetilde{x}(T)$.*

Corollary 1.24. *If each $P_{ij} \in \widetilde{x}(T)$ evaluates positively on a given $A \in \mathrm{Mat}_{2\times m}$, then all of the $2m-3$ of the $\binom{m}{2}$ minors of A are positive.*

Proof of Theorem. Follows by combining:

(1) Each P_{ij} appears as an element of an extended cluster $\widetilde{x}(T)$ for some triangulation T of $\mathbb{P}_m$.

(2) Any two triangulations of $\mathbb{P}_m$ are related by a sequence of **flips**.

(3) For a flip, replace P_{ik} with $P_{j\ell}$. Using the three-term GP relation, we have

$$P_{ik} = \frac{P_{ij}P_{k\ell} + P_{i\ell}P_{jk}}{P_{j\ell}}.$$

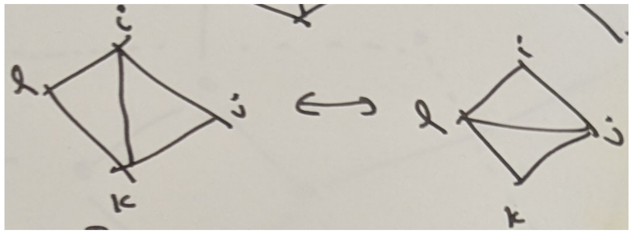


Figure 3: A flip replaces one diagonal with another in a quadrilateral.

Remark 1.25. In fact, each Plücker coordinate P_{ij} can be written as a Laurent polynomial with positive coefficients in the Plücker coordinates from $\widetilde{x}(T)$. This is an example of the **positive Laurent phenomenon**.

The combinatorics of flips is encoded by a graph:

- Vertices are triangulations.
- Edges are flips.

Each vertex has degree $m - 3$. In fact, this is the 1-skeleton of an $(m - 3)$-dimensional convex polytope called the **associahedron** (discovered by Stasheff).

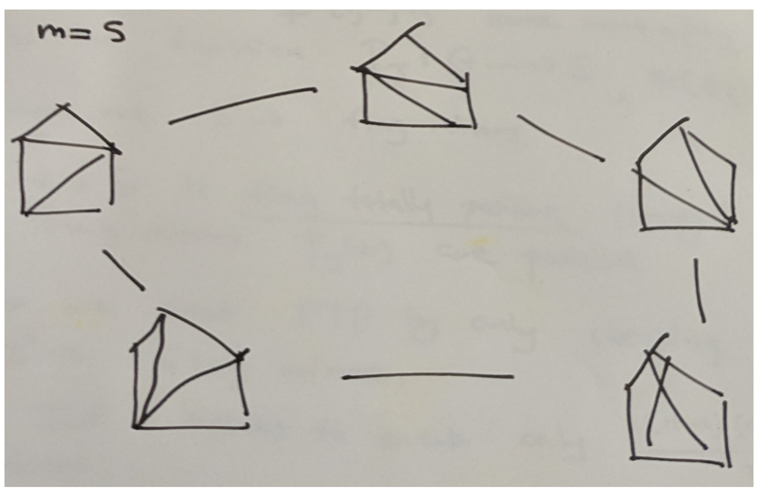


Figure 4: The associahedron for $m = 5$ (a pentagon).

Definition 1.26. A **cluster monomial** is a monomial in the variables of a given extended cluster $\widetilde{x}(T)$.

Theorem 1.27 (19th century invariant theory)**.** *The set of all cluster monomials gives a linear basis for the Plücker ring $R_{2,m}$.*

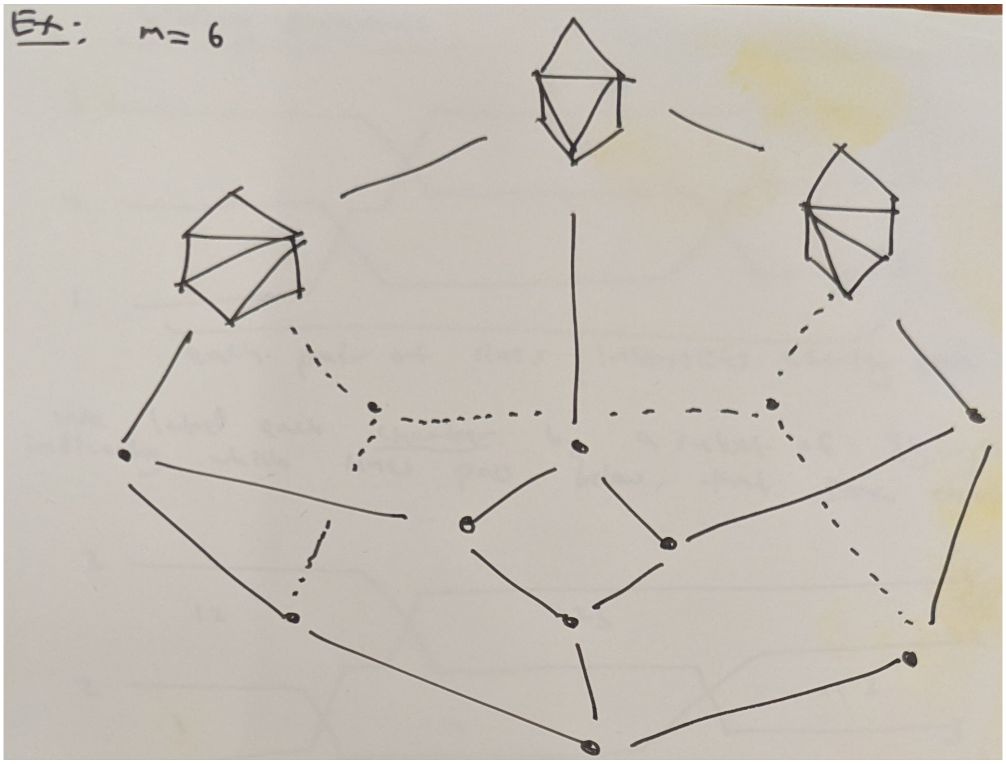


Figure 5: The associahedron for $m = 6$ (a 3-dimensional polytope).

2 Lecture 2

Date: January 14, 2026
Main reference: [FWZ21], §2–3.

2.1 Flag Positivity

Before moving to TP for $n \times n$ matrices, we discuss an intermediate notion called "flag positivity." Put $G = \mathrm{SL}_n$.

Definition 2.1. Given $J \subsetneq \{1, \ldots, n\}$ nonempty, the **flag minor** P_J is the function $P_J : G \to \mathbb{C}$ defined by

$$P_J(z) := z(\vec{e}_J) \mapsto \det\bigl(z_{\alpha\beta} \mid \alpha \leq |J|,\, \beta \in J\bigr),$$

i.e., the $|J| \times |J|$ minor which is "top-justified."

Note 2.2. There are $2^n - 2$ flag minors.

Definition 2.3. An element $z \in G$ is **flag totally positive** (FTP) if all flag minors $P_J(z)$ are positive.

Question 2.4. Can we check FTP by only checking a subset of the $2^n - 2$ flag minors?

Claim 2.5. *It suffices to check only* $\frac{(n-1)(n+2)}{2}$ *special flag minors.*

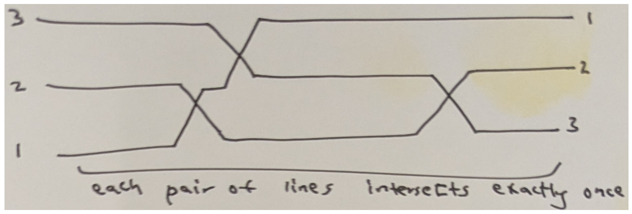


Figure 6: A wiring diagram for $n = 3$: each pair of lines intersects exactly once.

2.2 Wiring Diagrams

Each pair of lines intersects exactly once.

We label each **chamber** by a subset of $\{1, \ldots, n\}$ indicating which lines pass below that chamber.

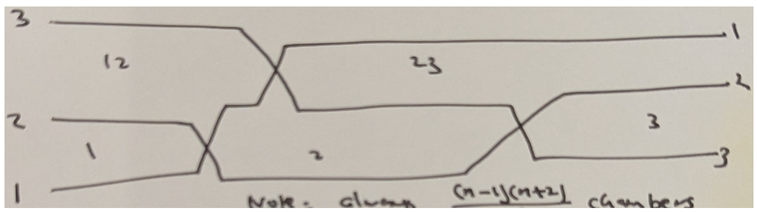


Figure 7: A wiring diagram with chamber labels.

Note 2.6. There are always $\frac{(n-1)(n+2)}{2}$ chambers.

Associated to each chamber is its **chamber minor** P_J, the flag minor corresponding to its subset $J \subsetneq \{1, \ldots, n\}$.

Extended cluster: All chamber minors of a wiring diagram.

- **Cluster variables**: the chamber minors for bounded chambers.
- **Frozen variables**: the chamber minors for unbounded chambers.

There are $\frac{(n-1)n}{2}$ of these (the bounded chambers).

Theorem 2.7. *Every flag minor can be written as a subtraction-free rational expression in the chamber minors of a given wiring diagram.*

Corollary 2.8. *If the $\frac{(n-1)(n+2)}{2}$ chamber minors evaluate positively at a matrix $z \in \mathrm{SL}_n$, then z is FTP.*

Proof outline. Follows by:

(1) Each flag minor appears as a chamber minor in some wiring diagram.

(2) Any two wiring diagrams can be transformed into each other by a sequence of local **braid moves**.

(3) Under each braid move, the collection of chamber minors changes by exchanging $Y \leftrightarrow Z$, and we have
$$YZ = AC + BD.$$

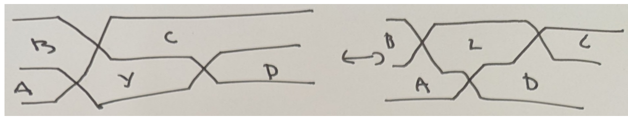


Figure 8: A braid move exchanges two adjacent crossings.

$\square$

Remark 2.9. In fact, each flag minor can be written as a Laurent polynomial with positive coefficients in the chamber minors of a given wiring diagram.

3 Lecture 3

Date: January 23, 2026
Main reference: [FWZ21], §1.3, §1.4, §2.1.

3.1 The Flag Variety and Basic Affine Space

Put $G = \mathrm{SL}_n(\mathbb{C})$. Let $B \subset G$ denote the subgroup of upper triangular matrices, and let $U \subset G$ denote the subgroup of unipotent lower triangular matrices, i.e., lower triangular matrices with 1's on the diagonal:

$$U = \left\{ \begin{pmatrix} 1 & 0 & \cdots & 0 \\ * & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & 1 \end{pmatrix} \right\}.$$

Note 3.1. As a variety, $U \cong \mathbb{C}^{n(n-1)/2}$.

Similarly, let U^+ denote the subgroup of unipotent upper triangular matrices.

Definition 3.2. The (complete) **flag variety** is

$$\mathcal{F}\ell = B\backslash G = \{V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n \mid \dim V_i = i\}.$$

This is identified with the homogeneous space $B\backslash G$, where B acts on G by left multiplication.

Definition 3.3. The **basic affine space** is $U\backslash G$, where U acts on G by left multiplication.

Note 3.4. There is a natural projection $U\backslash G \to B\backslash G$, which is a $(\mathbb{C}^*)^{n-1}$-bundle (a torus bundle) over the flag variety.

Let $\mathbb{C}[G]$ denote the coordinate ring of $G = \mathrm{SL}_n(\mathbb{C})$, and let $\mathbb{C}[G]^U$ denote the ring of U-invariant polynomials, where U acts by left multiplication on matrix entries.

Claim 3.5 (First and Second Fundamental Theorems of Invariant Theory)**.**

(1) $\mathbb{C}[G]^U$ *is generated by flag minors.*

(2) The ideal of relations among flag minors in $\mathbb{C}[G]^U$ *is generated by the* ***generalized Plücker relations****.*

3.2 Checking Total Positivity for $n \times n$ Matrices

Given $I, J \subseteq \{1, \dots, n\}$ of some cardinality, let Δ^I_J denote the minor of an $n \times n$ matrix determined by rows in I and columns in J. This extends to flag minors when $|I| = |J|$.

Double wiring diagrams: These are a generalization of the wiring diagrams from Lecture 2, used to study total positivity for $n \times n$ matrices.

Claim 3.6. *Every minor* Δ^I_J *of a chamber can be written as a subtraction-free rational expression in the chamber minors of a given double wiring diagram.*

Claim 3.7. *Every minor is a chamber minor for some double wiring diagram.*

The proof follows from:

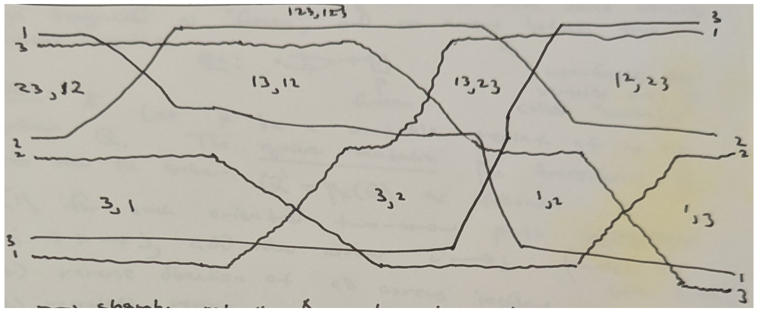


Figure 9: A double wiring diagram for $n = 3$.

(1) Any two double wiring diagrams can be linked by local moves.

(2) Each local move relates chamber minors of different diagrams.

(3) Each local double move satisfies a relation of the form $YZ = AC + BD$.

Remark 3.8. The graph with vertices given by double wiring diagrams and edges given by local moves is related to the theory of cluster algebras.

Remark 3.9. In fact, each minor can be written as a Laurent polynomial with positive coefficients in the chamber minors.

3.3 Quivers and Their Mutation

Definition 3.10. A **quiver** Q is a finite directed graph with:

- No loops (no arrows $i \to i$).
- No 2-cycles (no pairs of arrows $i \rightrightarrows j$ going both directions).

Definition 3.11. Let Q be a quiver with vertices $\{1, \dots, n\}$. The **mutation** $\mu_k(Q) = Q'$ at vertex k is defined by:

(1) Reverse the direction of all arrows incident to k.

(2) For each path $i \to k \to j$, add an arrow $i \to j$.

(3) Remove any 2-cycles that were created.

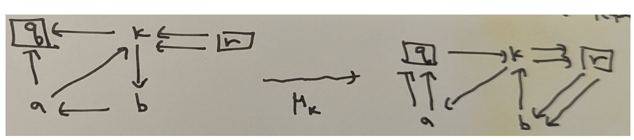


Figure 10: Illustration of quiver mutation at a vertex.

Exercise 3.12. Mutation is an involution, i.e., $\mu_k(\mu_k(Q)) = Q$.

Remark 3.13. If k, ℓ are vertices with no arrows between them, then mutations commute:

$$\mu_k(\mu_\ell(Q)) = \mu_\ell(\mu_k(Q)).$$

Exercise 3.14. For any quiver Q that is a tree with no triangles, show that one can get from any orientation to any other orientation by a sequence of mutations.

3.4 Triangulations and Quivers

We can assign to each triangulation T of the polygon $\mathbb{P}_m$ a quiver $Q(T)$.

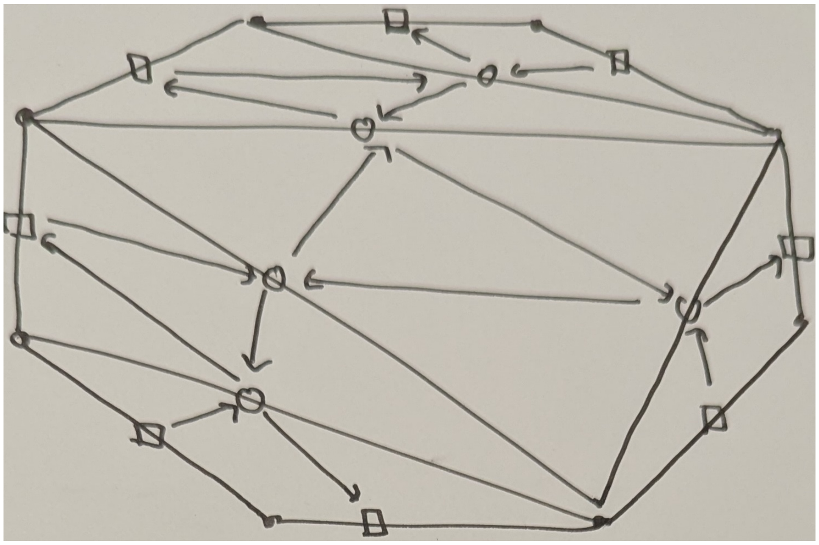


Figure 11: A triangulation T of $\mathbb{P}_m$ and its associated quiver $Q(T)$.

Exercise 3.15. If T' is obtained from T by a flip along diagonal d, then

$$Q(T') = \mu_d(Q(T)).$$

References

[FWZ21] Sergey Fomin, Lauren Williams, and Andrei Zelevinsky. *Introduction to Cluster Algebras*. Chapters 1–6, arXiv:1608.05735. 2021.

[Gro+18] Mark Gross, Paul Hacking, Sean Keel, and Maxim Kontsevich. “Canonical bases for cluster algebras”. In: *J. Amer. Math. Soc.* 31.2 (2018), pp. 497–608.