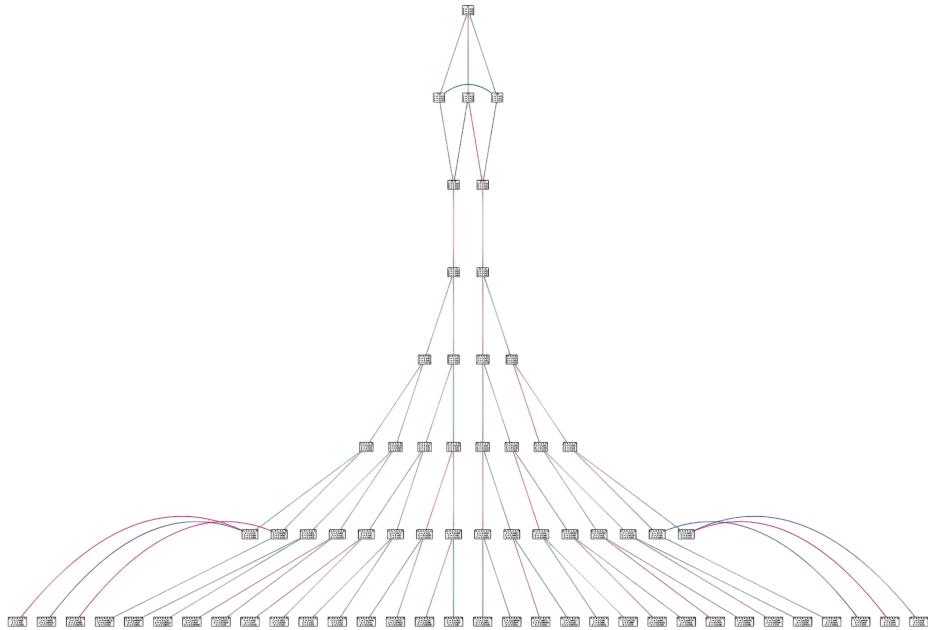


# Math 635: Cluster Varieties

Algebra, Topology, Geometry, Duality

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## Contents

<b>1 Lecture 1</b>	<b>4</b>
1.1 Introduction . . . . .	4
1.2 Total Positivity . . . . .	5
1.3 Plücker Coordinates on Grassmannians . . . . .	6
1.4 Positivity Testing for $\text{Gr}_{2,m}$ . . . . .	6
<b>2 Lecture 2</b>	<b>9</b>
2.1 Flag Positivity . . . . .	9
2.2 Wiring Diagrams . . . . .	10

<b>3 Lecture 3</b>	<b>11</b>
3.1 The Flag Variety and Basic Affine Space . . . . .	11
3.2 Checking Total Positivity for $n \times n$ Matrices . . . . .	12
3.3 Quivers and Their Mutation . . . . .	12
3.4 Triangulations and Quivers . . . . .	13
<b>4 Lecture 4</b>	<b>14</b>
4.1 Review: Triangulations and Quivers . . . . .	14
4.2 Wiring Diagrams and Quivers . . . . .	14
4.3 Plabic Graphs . . . . .	16
4.4 Quivers from Plabic Graphs . . . . .	16
4.5 Moves on Plabic Graphs . . . . .	17
4.6 Mutation Equivalence . . . . .	17
4.7 Finite Mutation Type . . . . .	18
<b>5 Lecture 5</b>	<b>19</b>
5.1 Extended Exchange Matrices . . . . .	19
5.2 Matrix Mutation . . . . .	21
5.3 Skew-Symmetrizable Matrices . . . . .	21
5.4 Diagrams and Uniqueness . . . . .	22
5.5 Mutation Equivalence for Matrices . . . . .	23
5.6 Labeled Seeds . . . . .	24
<b>6 Lecture 6</b>	<b>25</b>
6.1 Labeled Seeds and Seed Mutation . . . . .	25
6.2 Examples . . . . .	25
6.3 Seed Patterns and Cluster Algebras . . . . .	26
6.4 Examples of Cluster Algebras . . . . .	26
<b>7 Lecture 7</b>	<b>27</b>
7.1 Rank 1 Cluster Algebras . . . . .	27
7.2 Rank 2 Cluster Algebras . . . . .	29
<b>8 Lecture 8</b>	<b>32</b>
8.1 Rank 2 examples (continued) . . . . .	32
8.2 The Laurent phenomenon . . . . .	33
8.3 Markov triples . . . . .	34
8.4 The Markov tree . . . . .	35
8.5 The Somos-4 sequence . . . . .	35
<b>9 Lecture 9</b>	<b>37</b>
9.1 The $\hat{y}$ -variables . . . . .	37
9.2 Y-seeds . . . . .	38
9.3 Semifields . . . . .	38
9.4 Coefficient tuples and tropical Y-seed mutation . . . . .	39

<b>10 Lecture 10</b>	<b>41</b>
10.1 Alternative characterization of labeled seeds . . . . .	41
10.2 Example: $A_2$ revisited . . . . .	41
10.3 Finite type classification in rank 2 . . . . .	42
10.4 2-finiteness . . . . .	43

# 1 Lecture 1

*Date: January 12, 2026*

**Main reference:** [FWZ21], §1–2.

## 1.1 Introduction

Roughly speaking:

- A **cluster variety** is a complex algebraic variety obtained by gluing together many copies of  $(\mathbb{C}^*)^n$ , where the gluing maps take a very particular form.
- A **cluster algebra** is the algebra of regular functions  $f: V \rightarrow \mathbb{C}$  on a cluster variety.

**Fomin–Zelevinsky, early 2000s:** Introduced cluster algebras. They arise in many parts of mathematics and physics as a kind of “universal model” for mutation/wall-crossing phenomena:

- Quiver representation theory
- Teichmüller theory
- Poisson geometry
- Grassmannians
- Total positivity
- QFT scattering amplitudes (amplituhedron)
- Integrable systems
- String theory (BPS states)
- etc.

**Gross–Hacking–Keel–Kontsevich (GHKK)** [Gro+18]:

- Constructed canonical bases for cluster algebras.
- Established positivity of the Laurent phenomenon.
- Proof uses mirror symmetry for log Calabi–Yau varieties (which can be thought of as a generalization of toric varieties, related to almost toric fibrations in symplectic geometry).
- Many strong applications in representation theory, e.g., canonical bases for finite-dimensional irreducible representations of  $\mathrm{SL}_n(\mathbb{C})$ .

**Remark 1.1.** The canonical bases were originally found independently by Lusztig and Kashiwara in the early 1990s using quantum groups. Amazingly, the construction of GHKK uses only general geometry—no representation theory!

## 1.2 Total Positivity

**Definition 1.2.** A matrix  $A \in \text{Mat}_{n \times n}(\mathbb{R})$  is **totally positive** (TP) if all of its minors are positive.

**Gantmacher–Krein (1930s):** If  $A$  is TP, then the eigenvalues of  $A$  are real, positive, and distinct.

**Binet–Cauchy theorem:** The TP matrices are closed under multiplication, and hence form a multiplicative semigroup  $G_{>0}$ .

**Lusztig:** Extended the definition of  $G_{>0}$  to other semisimple Lie groups  $G$ .

**More generally:** If a given complex algebraic variety  $Z$  has a distinguished family  $\Delta$  of regular functions  $Z \rightarrow \mathbb{C}$ , we define the **TP variety** by

$$Z_{>0} := \{z \in Z \mid f(z) > 0 \text{ for all } f \in \Delta\}.$$

**Example 1.3.** For  $Z = \text{Mat}_{n \times n}(\mathbb{C})$ ,  $\text{GL}_n(\mathbb{C})$ , or  $\text{SL}_n(\mathbb{C})$ , we recover the above notion of TP, where  $\Delta = \{\text{minors}\}$ .

**Example 1.4.** The **Grassmannian**  $\text{Gr}_{k,m}(\mathbb{C}) = \{k\text{-dimensional linear subspaces of } \mathbb{C}^m\}$ , with  $\Delta = \{\text{Plücker coordinates}\}$ .

**Example 1.5.** Partial flag manifolds, homogeneous spaces for semisimple complex Lie groups, etc. (slight scaling ambiguity).

**Lemma 1.6.** A matrix  $A \in \text{Mat}_{n \times n}$  has  $\binom{2n}{n} - 1$  minors.

*Proof.* The number of minors is

$$\# = \sum_{k=1}^n \binom{n}{k}^2.$$

By Vandermonde's identity:

$$\binom{m+w}{r} = \sum_{k=0}^r \binom{m}{k} \binom{w}{r-k}.$$

Setting  $m = w = r = n$  gives

$$\binom{2n}{n} = \sum_{k=0}^n \binom{n}{k}^2,$$

from which the result follows.  $\square$

**Remark 1.7.** To verify Vandermonde's identity, note that both sides count the number of subcommittees with  $r$  members, given a committee with  $m$  men and  $w$  women.

**Question 1.8.** Can we check that  $A \in \text{Mat}_{n \times n}$  is TP by only testing a subset of the  $\binom{2n}{n} - 1$  minors? How many tests are needed?

**Example 1.9.** Let  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Mat}_{2 \times 2}$ . Define  $\delta := ad - bc$ , so  $d = \frac{\delta+bc}{a}$ . Thus, if  $a, b, c, \delta > 0$ , then  $d$  is automatically positive. This reduces  $\binom{4}{2} - 1 = 5$  checks to 4 checks.

The goal is “efficient TP testing.”

### 1.3 Plücker Coordinates on Grassmannians

Given  $A \in \text{Mat}_{k \times m}$  of rank  $k$ , we have  $\text{rowspan}(A) =: [A] \in \text{Gr}_{k,m}$ .

For  $J \subseteq \{1, \dots, m\}$  with  $|J| = k$ , the **Plücker coordinate** is

$$P_J(A) := k \times k \text{ minor of } A \text{ corresponding to columns } J.$$

**Note 1.10.** For  $A, B \in \text{Mat}_{k \times m}$  with  $[A] = [B]$  (i.e., same row spans), the tuples  $(P_J(A))_{|J|=k}$  and  $(P_J(B))_{|J|=k}$  agree up to common rescaling. We thus get a map

$$\text{Gr}_{k,m} \longrightarrow \mathbb{CP}^{N-1}, \quad N = \binom{m}{k}.$$

In fact, this is an embedding, called the **Plücker embedding**.

Let  $\mathbb{C}[\text{Mat}_{k \times m}]$  denote the coordinate ring of  $\text{Mat}_{k \times m}$ , i.e., the polynomial algebra in variables  $x_{ij}$  for  $1 \leq i \leq k$ ,  $1 \leq j \leq m$ .

**Definition 1.11.** The **Plücker ring**  $R_{k,m}$  is the subring of  $\mathbb{C}[\text{Mat}_{k \times m}]$  generated by  $P_J$  over all  $J \in \{1, \dots, m\}$  with  $|J| = k$ .

**Claim 1.12.** *The ideal of relations in  $R_{k,m}$  is generated by certain quadratic relations called the Grassmann–Plücker relations.*

**Definition 1.13.** The **totally positive Grassmannian**  $\text{Gr}_{k,m}^+$  is the subset of  $\text{Gr}_{k,m}$  consisting of those points whose Plücker coordinates are all positive (up to common scaling).

**Note 1.14.** For  $A \in \text{Mat}_{k \times m}(\mathbb{R})$ , we have  $[A] \in \text{Gr}_{k,m}^+$  if and only if all  $k \times k$  minors of  $A$  have the same sign.

**Question 1.15.** For  $A \in \text{Mat}_{k \times m}(\mathbb{R})$ , can we verify that all  $k \times k$  minors are positive by only checking a subset of the  $\binom{m}{k}$  minors? How many tests are needed?

(We may assume positive WLOG by rescaling.)

### 1.4 Positivity Testing for $\text{Gr}_{2,m}$

**Claim 1.16.** *Given  $A \in \text{Mat}_{2 \times m}$ , put  $P_{ij} := P_{\{i,j\}}$  for  $1 \leq i < j \leq m$ . To check that all  $2 \times 2$  minors  $P_{ij}(A) > 0$ , it suffices to check only the  $2m - 3$  special ones.*

**Note 1.17.**  $2m - 3 = \dim \text{Gr}_{2,m} + 1$ .

**Lemma 1.18.** *For  $1 \leq i < j < k < \ell \leq m$ , we have the three-term Grassmann–Plücker relation:*

$$P_{ik}P_{j\ell} = P_{ij}P_{k\ell} + P_{i\ell}P_{jk}.$$

**Remark 1.19.** For an inscribed quadrilateral (Figure 1), Ptolemy’s theorem (2nd century) gives

$$AC \cdot BD = AB \cdot CD + BC \cdot AD.$$

**Example 1.20.** Let  $A = \begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix}$ . We verify  $P_{13}P_{24} = P_{12}P_{34} + P_{14}P_{23}$ , i.e.,

$$(ag - ce)(bh - df) = (af - be)(ch - dg) + (ah - de)(bg - cf). \quad \checkmark$$

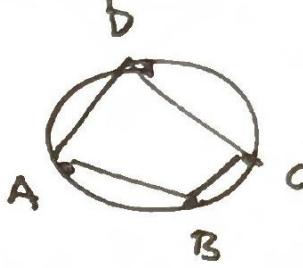


Figure 1: Inscribed quadrilateral for Ptolemy's theorem.

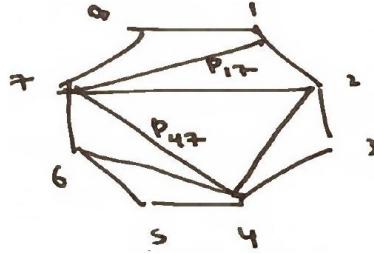


Figure 2: A triangulated polygon  $\mathbb{P}_m$  with vertices labeled  $1, \dots, m$ .

Put  $\mathbb{P}_m$  = regular  $m$ -gon, and let  $T$  be a triangulation.

To each side or diagonal, associate  $P_{ij}$ , where  $i, j$  are the endpoints.

- **Cluster variables:**  $P_{ij}$  ranging over diagonals.
- **Frozen variables:**  $P_{ij}$  ranging over sides.
- **Extended cluster:**  $\{\text{cluster vars}\} \cup \{\text{frozen vars}\} =: \tilde{x}(T)$ .

**Note 1.21.** The extended cluster has  $2m - 3$  variables, and we claim that these are algebraically independent.

**Example 1.22.** In Figure 2, we have cluster variables  $P_{17}, P_{27}, P_{47}, P_{24}$  and frozen variables  $P_{12}, P_{23}, \dots, P_{78}, P_{18}$ .

**Theorem 1.23.** *Each  $P_{ij}$  for  $1 \leq i < j \leq n$  can be written as a subtraction-free rational expression in the elements of a given extended cluster  $\tilde{x}(T)$ .*

**Corollary 1.24.** *If each  $P_{ij} \in \tilde{x}(T)$  evaluates positively on a given  $A \in \text{Mat}_{2 \times m}$ , then all of the  $2m - 3$  of the  $\binom{m}{2}$  minors of  $A$  are positive.*

**Proof of Theorem.** Follows by combining:

- (1) Each  $P_{ij}$  appears as an element of an extended cluster  $\tilde{x}(T)$  for some triangulation  $T$  of  $\mathbb{P}_m$ .
- (2) Any two triangulations of  $\mathbb{P}_m$  are related by a sequence of **flips** (see Figure 3).
- (3) For a flip, replace  $P_{ik}$  with  $P_{j\ell}$ . Using the three-term GP relation, we have

$$P_{ik} = \frac{P_{ij}P_{k\ell} + P_{i\ell}P_{jk}}{P_{j\ell}}.$$

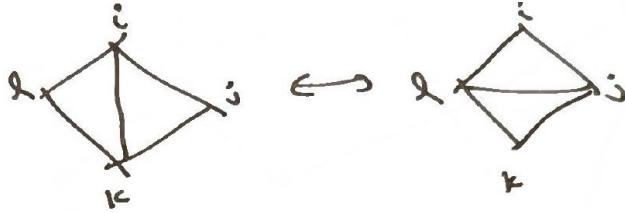


Figure 3: A flip replaces one diagonal with another in a quadrilateral.

**Remark 1.25.** In fact, each Plücker coordinate  $P_{ij}$  can be written as a Laurent polynomial with positive coefficients in the Plücker coordinates from  $\tilde{x}(T)$ . This is an example of the **positive Laurent phenomenon**.

The combinatorics of flips is encoded by a graph:

- Vertices are triangulations.
- Edges are flips.

Each vertex has degree  $m - 3$ . In fact, this is the 1-skeleton of an  $(m - 3)$ -dimensional convex polytope called the **associahedron** (discovered by Stasheff); see Figures 4 and 5.

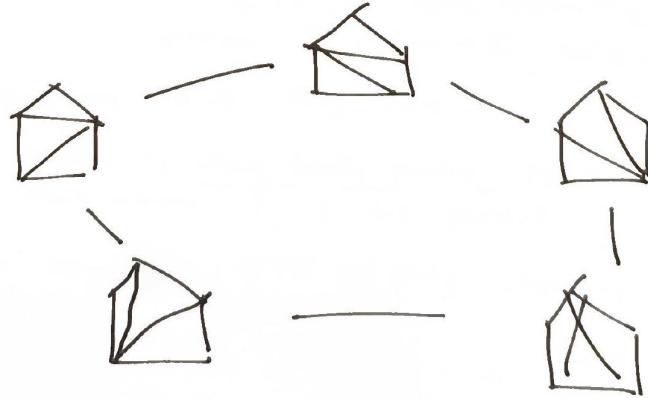


Figure 4: The associahedron for  $m = 5$  (a pentagon).

**Definition 1.26.** A **cluster monomial** is a monomial in the variables of a given extended cluster  $\tilde{x}(T)$ .

**Theorem 1.27** (19th century invariant theory). *The set of all cluster monomials gives a linear basis for the Plücker ring  $R_{2,m}$ .*

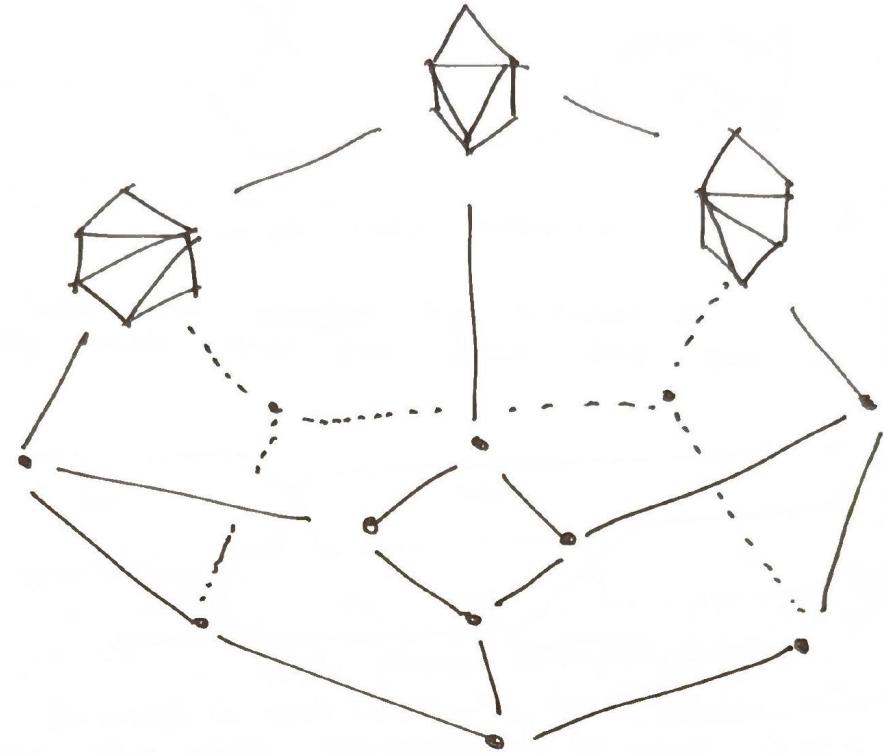


Figure 5: The associahedron for  $m = 6$  (a 3-dimensional polytope).

## 2 Lecture 2

*Date: January 14, 2026*

**Main reference:** [FWZ21], §2–3.

### 2.1 Flag Positivity

Before moving to TP for  $n \times n$  matrices, we discuss an intermediate notion called “flag positivity.” Put  $G = \mathrm{SL}_n$ .

**Definition 2.1.** Given  $J \subsetneq \{1, \dots, n\}$  nonempty, the **flag minor**  $P_J$  is the function  $P_J: G \rightarrow \mathbb{C}$  defined by

$$P_J(z) := z(\vec{e}_J) \mapsto \det(z_{\alpha\beta} \mid \alpha \leq |J|, \beta \in J),$$

i.e., the  $|J| \times |J|$  minor which is “top-justified.”

**Note 2.2.** There are  $2^n - 2$  flag minors.

**Definition 2.3.** An element  $z \in G$  is **flag totally positive** (FTP) if all flag minors  $P_J(z)$  are positive.

**Question 2.4.** Can we check FTP by only checking a subset of the  $2^n - 2$  flag minors?

**Claim 2.5.** It suffices to check only  $\frac{(n-1)(n+2)}{2}$  special flag minors.

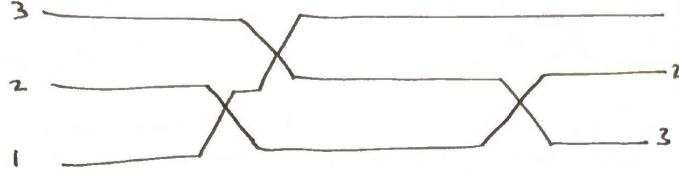


Figure 6: A wiring diagram for  $n = 3$ : each pair of lines intersect exactly once.

## 2.2 Wiring Diagrams

Each pair of lines intersect exactly once (Figure 6).

We label each **chamber** by a subset of  $\{1, \dots, n\}$  indicating which lines pass below that chamber (see Figure 7).

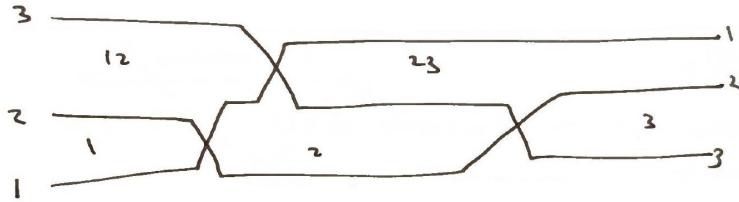


Figure 7: A wiring diagram with chamber labels.

**Note 2.6.** There are always  $\frac{(n-1)(n+2)}{2}$  chambers.

Associated to each chamber is its **chamber minor**  $P_J$ , the flag minor corresponding to its subset  $J \subsetneq \{1, \dots, n\}$ .

**Extended cluster:** All chamber minors of a wiring diagram.

- **Cluster variables:** the chamber minors for bounded chambers. There are  $\frac{(n-1)n}{2}$  of these.
- **Frozen variables:** the chamber minors for unbounded chambers. There are  $2n - 2$  of these.

**Theorem 2.7.** Every flag minor can be written as a subtraction-free rational expression in the chamber minors of a given wiring diagram.

**Corollary 2.8.** If the  $\frac{(n-1)(n+2)}{2}$  chamber minors evaluate positively at a matrix  $z \in \mathrm{SL}_n$ , then  $z$  is **FTP**.

*Proof outline.* Follows by:

- (1) Each flag minor appears as a chamber minor in some wiring diagram.
- (2) Any two wiring diagrams can be transformed into each other by a sequence of local **braid moves** (see Figure 8).
- (3) Under each braid move, the collection of chamber minors changes by exchanging  $Y \leftrightarrow Z$ , and we have

$$YZ = AC + BD.$$

□

**Remark 2.9.** In fact, each flag minor can be written as a Laurent polynomial with positive coefficients in the chamber minors of a given wiring diagram.

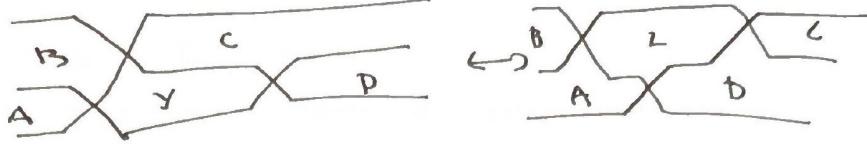


Figure 8: A braid move exchanges two adjacent crossings.

### 3 Lecture 3

*Date: January 23, 2026*

**Main reference:** [FWZ21], §1.3, §1.4, §2.1.

#### 3.1 The Flag Variety and Basic Affine Space

Put  $G = \mathrm{SL}_n(\mathbb{C})$ . Let  $B \subset G$  denote the subgroup of lower triangular matrices (the Borel subgroup), and let  $U \subset G$  denote the subgroup of unipotent lower triangular matrices, i.e., lower triangular matrices with 1's on the diagonal:

$$U = \left\{ \begin{pmatrix} 1 & 0 & \cdots & 0 \\ * & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & 1 \end{pmatrix} \right\}.$$

**Note 3.1.** As a variety,  $U \cong \mathbb{C}^{n(n-1)/2}$ .

Similarly, let  $U^+$  denote the subgroup of unipotent upper triangular matrices.

**Definition 3.2.** The (complete) **flag variety** is

$$\mathcal{F}\ell = B \backslash G = \{V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n \mid \dim V_i = i\}.$$

This is identified with the homogeneous space  $B \backslash G$ , where  $B$  acts on  $G$  by left multiplication.

**Definition 3.3.** The **basic affine space** is  $U \backslash G$ , where  $U$  acts on  $G$  by left multiplication.

**Note 3.4.** There is a natural projection  $U \backslash G \rightarrow B \backslash G$ , which is a  $(\mathbb{C}^*)^{n-1}$ -bundle (a torus bundle) over the flag variety.

Let  $\mathbb{C}[G]$  denote the coordinate ring of  $G = \mathrm{SL}_n(\mathbb{C})$ , and let  $\mathbb{C}[G]^U$  denote the ring of  $U$ -invariant polynomials, where  $U$  acts by left multiplication on matrix entries.

**Claim 3.5** (First and Second Fundamental Theorems of Invariant Theory).

- (1)  $\mathbb{C}[G]^U$  is generated by flag minors.
- (2) The ideal of relations among flag minors in  $\mathbb{C}[G]^U$  is generated by the **generalized Plücker relations**.

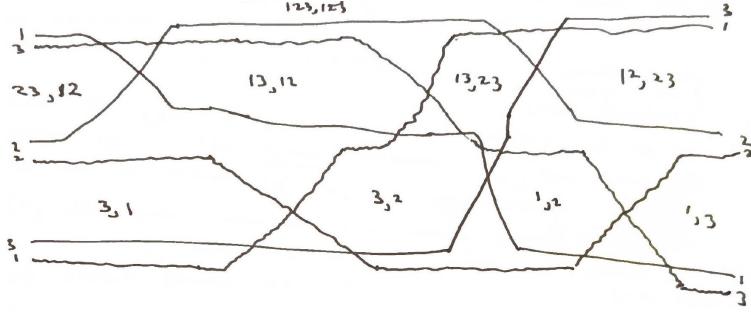


Figure 9: A double wiring diagram for  $n = 3$ .

### 3.2 Checking Total Positivity for $n \times n$ Matrices

Given  $I, J \subseteq \{1, \dots, n\}$  of some cardinality, let  $\Delta_J^I$  denote the minor of an  $n \times n$  matrix determined by rows in  $I$  and columns in  $J$ . This extends to flag minors when  $|I| = |J|$ .

**Double wiring diagrams:** These are a generalization of the wiring diagrams from Lecture 2, used to study total positivity for  $n \times n$  matrices (see Figure 9).

**Claim 3.6.** *Every minor  $\Delta_J^I$  of a chamber can be written as a subtraction-free rational expression in the chamber minors of a given double wiring diagram.*

**Claim 3.7.** *Every minor is a chamber minor for some double wiring diagram.*

The proof follows from:

- (1) Any two double wiring diagrams can be linked by local moves.
- (2) Each local move relates chamber minors of different diagrams.
- (3) Each local double move satisfies a relation of the form  $YZ = AC + BD$ .

**Remark 3.8.** The graph with vertices given by double wiring diagrams and edges given by local moves is related to the theory of cluster algebras.

**Remark 3.9.** In fact, each minor can be written as a Laurent polynomial with positive coefficients in the chamber minors.

### 3.3 Quivers and Their Mutation

**Definition 3.10.** A **quiver**  $Q$  is a finite directed graph (see Figure 10) with:

- No loops (no arrows  $i \rightarrow i$ ).
- No 2-cycles (no pairs of arrows  $i \Rightarrow j$  going both directions).



Figure 10: Examples of quivers (valid examples marked  $\checkmark$ , invalid example marked  $\times$ ).

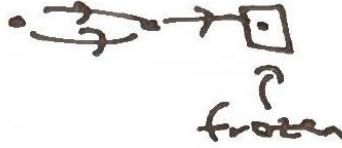


Figure 11: An ice quiver with frozen vertices indicated by boxes.

**Definition 3.11.** An **ice quiver** is a quiver in which some vertices are designated as “frozen” (see Figure 11), and there are no arrows between two frozen vertices. The non-frozen vertices are called **mutable**.

**Definition 3.12.** Let  $Q$  be an ice quiver and let  $k$  be a mutable vertex. The **mutation**  $\mu_k(Q) = Q'$  at vertex  $k$  is defined as follows (see Figure 12):

- (1) For each path  $i \rightarrow k \rightarrow j$ , add an arrow  $i \rightarrow j$  (unless  $i, j$  are both frozen).
- (2) Reverse the direction of all arrows incident to  $k$ .
- (3) Remove any 2-cycles that were created.

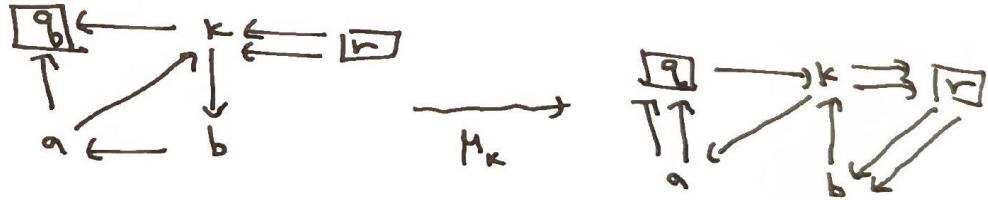


Figure 12: Illustration of quiver mutation at a vertex.

**Exercise 3.13.**

- (1) Mutation is an involution, i.e.,  $\mu_k(\mu_k(Q)) = Q$ .
- (2) Mutation commutes with reversing the orientations of all arrows.
- (3) If  $k, \ell$  are mutable vertices with no arrows between them, then mutations commute:

$$\mu_k(\mu_\ell(Q)) = \mu_\ell(\mu_k(Q)).$$

**Remark 3.14.** If  $k$  is a sink or source, then  $\mu_k$  simply reverses all arrows incident to  $k$ .

**Exercise 3.15.** For any quiver  $Q$  that is a tree with no frozen vertices, show that one can get from any orientation to any other orientation by a sequence of mutations at sources and sinks.

### 3.4 Triangulations and Quivers

We can assign to each triangulation  $T$  of the polygon  $\mathbb{P}_m$  a quiver  $Q(T)$  (see Figure 13).

**Exercise 3.16.** If  $T'$  is obtained from  $T$  by a flip along diagonal  $\gamma$ , then

$$Q(T') = \mu_\gamma(Q(T)).$$

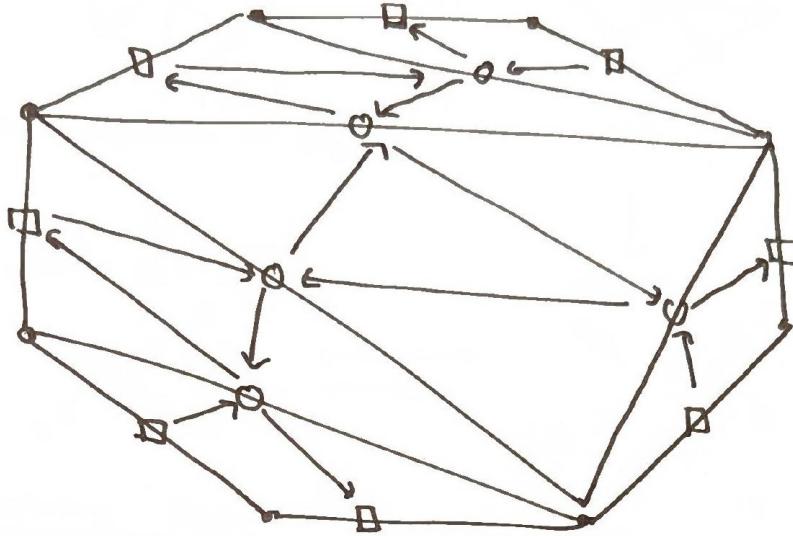


Figure 13: A triangulation  $T$  of  $\mathbb{P}_m$  and its associated quiver  $Q(T)$ .

## 4 Lecture 4

*Date: January 26, 2026*

**Main reference:** [FWZ21], §2.2, §2.3, §2.4, §2.5, §2.6.

### 4.1 Review: Triangulations and Quivers

**Example 4.1.** Let  $T$  be a triangulation of  $\mathbb{P}_4$ . Then a flip along a diagonal gives a new triangulation  $T'$  (see Figure 14):



Figure 14: A flip between triangulations  $T$  and  $T'$  of  $\mathbb{P}_4$ , and the corresponding quivers  $Q(T)$  and  $Q(T')$  related by mutation.

### 4.2 Wiring Diagrams and Quivers

Given a wiring diagram  $D$ , we can associate a quiver  $Q(D)$  (see Figure 15).

**Vertices:** The vertices of  $Q(D)$  are the chambers of  $D$ . A vertex is mutable if the corresponding chamber is bounded, and frozen otherwise.

**Arrows:** For chambers  $c, c'$ , we have an arrow  $c \rightarrow c'$  in  $Q(D)$  if one of the following holds (see Figure 16):

- (i) The right end of  $c$  equals the left end of  $c'$ .

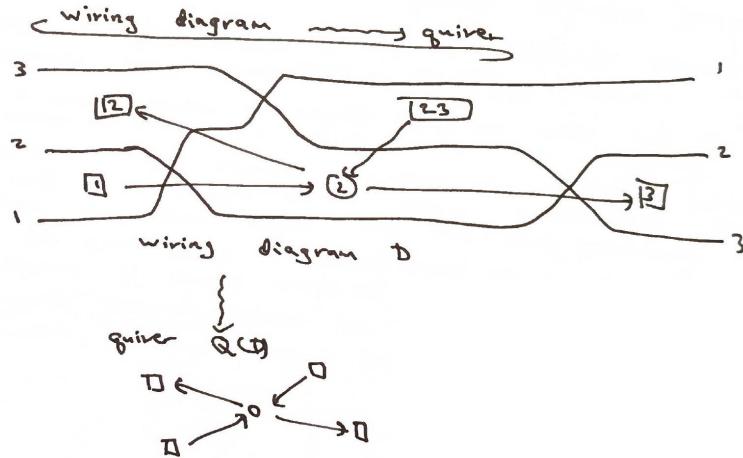


Figure 15: A wiring diagram  $D$  and its associated quiver  $Q(D)$ .

- (ii) The left end of  $c$  is directly above  $c'$ , and the right end of  $c'$  is directly below  $c$ .
- (iii) The left end of  $c$  is directly below  $c'$ , and the right end of  $c'$  is directly above  $c$ .

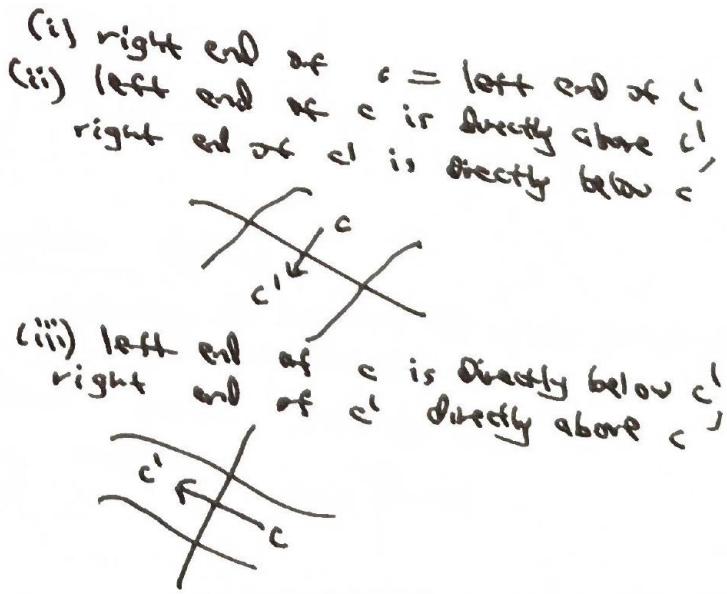


Figure 16: The arrow rules for chambers in a wiring diagram.

**Exercise 4.2.** If  $D, D'$  are wiring diagrams related by a braid move at chamber  $Y$ , then

$$Q(D') = \mu_Y(Q(D)).$$

**Example 4.3.** Figure 17 shows two wiring diagrams related by a braid move, and the corresponding quivers related by mutation at the central chamber.

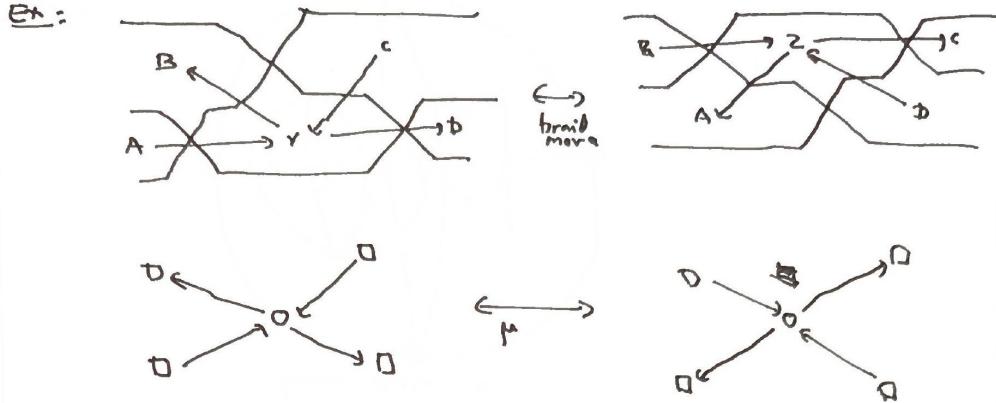


Figure 17: A braid move on wiring diagrams and the corresponding quiver mutation.

### 4.3 Plabic Graphs

**Remark 4.4.** We also have an assignment

$$\text{double wiring diagram } D \rightsquigarrow \text{quiver } Q(D).$$

The description is more complicated, but it is a special case of the quiver associated to a planar bipartite graph.

**Definition 4.5.** A **plabic graph**  $G$  is a connected planar bipartite graph embedded in a disk, where:

- Each vertex is colored black or white and lies either in the interior of the disk or on its boundary.
- Each edge connects vertices of different colors and is a simple curve whose interior is disjoint from the other edges and the disk boundary.
- For each face (connected component of complement), the closure is simply connected.
- Each interior vertex has degree  $\geq 2$ .
- Each boundary vertex has degree 1.

**Note 4.6.** We consider plabic graphs up to isotopy; see Figure 18 for an example.

### 4.4 Quivers from Plabic Graphs

Given a plabic graph  $G$ , we can associate a quiver  $Q(G)$ :

**Vertices:** The vertices of  $Q(G)$  are the faces of  $G$ . A vertex is frozen if the corresponding face is incident to the disk boundary, and mutable otherwise.

**Arrows:** For each edge of  $G$ , we have an arrow joining the two faces it separates, using the orientation rule shown in Figure 19:

Finally, remove oriented 2-cycles.

**Example 4.7.** Figure 20 shows a plabic graph  $G$  and the construction of its quiver  $Q(G)$ .

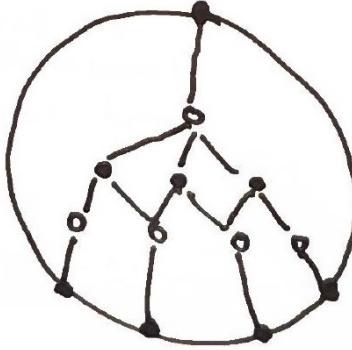


Figure 18: An example of a plabic graph.



Figure 19: The orientation rule for arrows: the arrow points so that the white vertex is on the left.

## 4.5 Moves on Plabic Graphs

**Definition 4.8.** Say a vertex  $v$  is **bivalent** if it is adjacent to two interior vertices.

**Remark 4.9.** Contracting or decontracting a bivalent vertex (Figure 21) does not change the associated quiver.

**Definition 4.10.** Say  $G$  has a **quadrilateral** if it has a face whose vertices have degree  $\geq 3$ .

**Exercise 4.11.** If  $G, G'$  are related by a spider move (Figure 22), then  $Q(G), Q(G')$  are related by mutation.

**Example 4.12.** Figure 23 shows two plabic graphs related by a spider move, and the corresponding quivers.

## 4.6 Mutation Equivalence

**Definition 4.13.** Two quivers  $Q, Q'$  are **mutation equivalent** if  $Q$  becomes isomorphic to  $Q'$  after a sequence of mutations.

**Definition 4.14.** Put

$$[Q] := \{\text{all quivers which are mutation equivalent to } Q\}/\text{isomorphism}.$$

**Example 4.15.** Let  $Q$  be the  $A_3$  quiver (three vertices in a line):

$$\bullet \rightarrow \bullet \rightarrow \bullet$$

Then  $[Q]$  has 4 elements (Figure 24):

**Exercise 4.16.** Show that  $[Q]$  has exactly 4 elements for  $Q$  the  $A_3$  quiver.

**Example 4.17.** Let  $Q$  be the “Markov quiver” (Figure 25):

In fact,  $[Q]$  is just a single element (the Markov quiver is mutation equivalent only to itself).

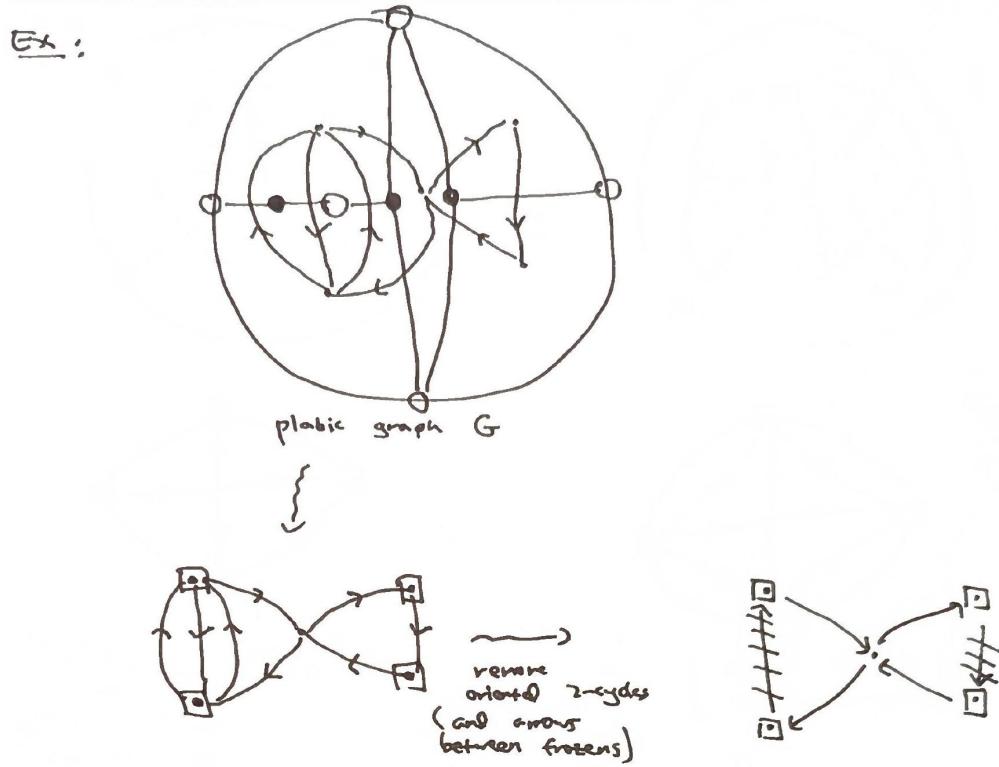


Figure 20: A plabic graph  $G$  and its associated quiver  $Q(G)$ , after removing oriented 2-cycles and arrows between frozen vertices.

#### 4.7 Finite Mutation Type

**Definition 4.18.** A quiver  $Q$  has **finite mutation type** if  $[Q]$  is finite.

**Remark 4.19.** There is a classification theorem for quivers with no frozen vertices and finite mutation type.

**Definition 4.20.** A quiver  $Q$  is **acyclic** if it has no oriented cycles.

**Theorem 4.21** (Caldero–Keller '06). *If  $Q, Q'$  are acyclic and mutation equivalent, then we can transform  $Q$  into  $Q'$  by a sequence of mutations at sources and sinks. In particular,  $Q$  and  $Q'$  have the same underlying undirected graphs.*

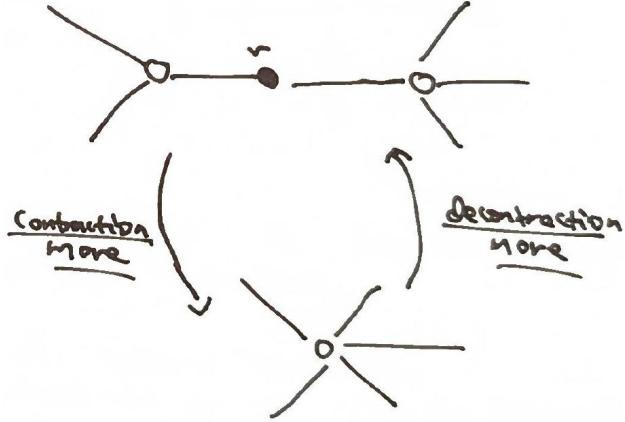


Figure 21: Contraction and decontraction moves on a bivalent vertex.

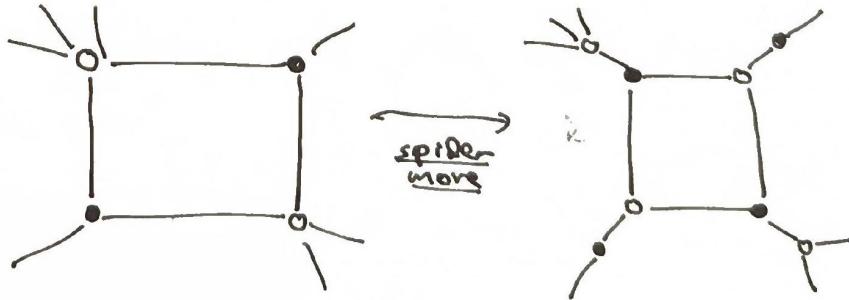


Figure 22: The spider move on a quadrilateral face.

## 5 Lecture 5

*Date: January 28, 2026*

**Main reference:** [FWZ21], §2.7, §2.8.

### 5.1 Extended Exchange Matrices

**Definition 5.1.** Let  $Q$  be a quiver with vertices labeled by  $1, \dots, m$ , such that  $1, \dots, n$  are the **mutable** vertices (with  $n \leq m$ ). The **extended exchange matrix** is

$$\tilde{B}(Q) = (b_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}, \quad \text{where} \quad b_{ij} = \begin{cases} \ell & \text{if } \ell \text{ arrows } i \rightarrow j \\ -\ell & \text{if } \ell \text{ arrows } j \rightarrow i \\ 0 & \text{else} \end{cases}$$

This is an  $m \times n$  matrix. The **exchange matrix** is the submatrix

$$B(Q) := (b_{ij})_{1 \leq i, j \leq n},$$

which is an  $n \times n$  skew-symmetric matrix.

**Example 5.2.** Consider the quiver  $Q$  with mutable vertices 1, 2, 3 and frozen vertices 4, 5 (Figure 26):

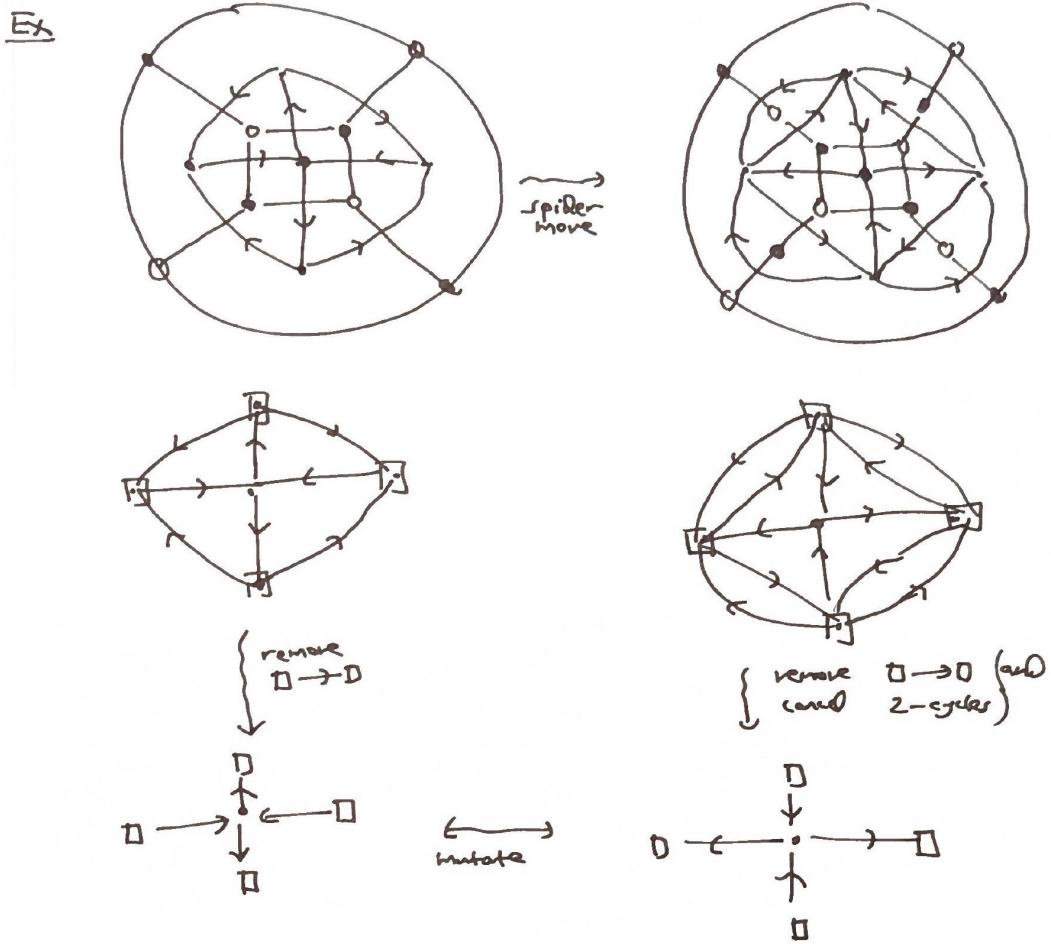


Figure 23: Two plabic graphs related by a spider move, and their quivers related by mutation.

The extended exchange matrix is

$$\tilde{B}(Q) = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \\ -1 & -1 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \quad B(Q) = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}.$$

**Example 5.3.** Let  $Q$  be the Markov quiver. Figure 27 shows the extended exchange matrices for  $Q$  and two of its mutations.

**Remark 5.4.** Reordering the vertices of  $Q$  results in simultaneously reordering the rows  $1, \dots, n$  and reordering the columns  $1, \dots, m$ .

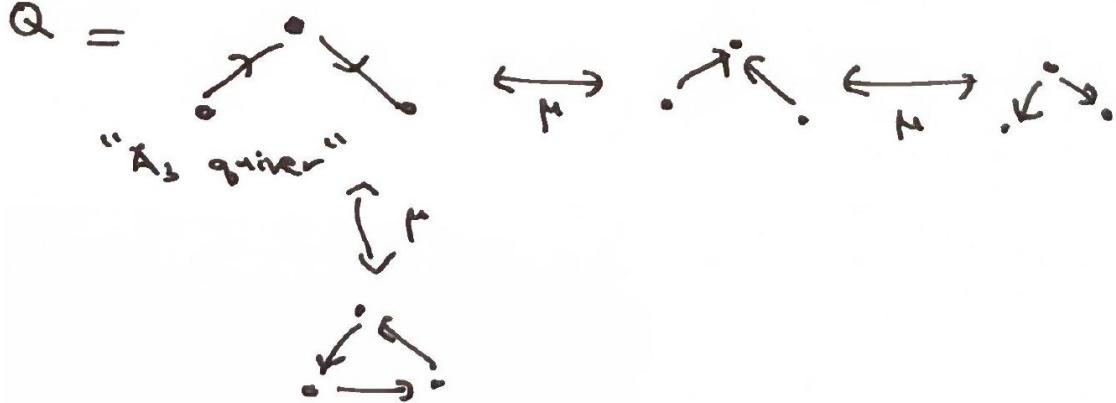


Figure 24: The mutation equivalence class of the  $A_3$  quiver.

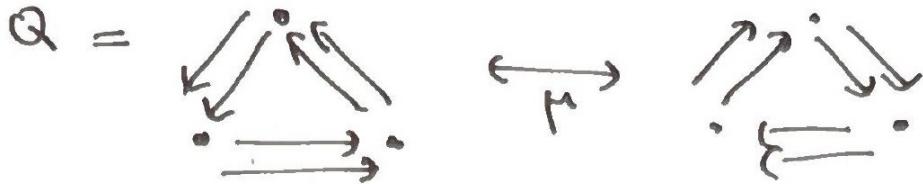


Figure 25: The Markov quiver.

## 5.2 Matrix Mutation

**Lemma 5.5.** For a quiver  $Q$  with  $\tilde{B}(Q) = (b_{ij})$  and  $Q' = \mu_k(Q)$  for a mutable vertex  $k$  of  $Q$ , we have  $\tilde{B}(Q') = (b'_{ij})$  with

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + b_{ik}b_{kj} & \text{if } b_{ik}b_{kj} > 0 \\ b_{ij} - b_{ik}b_{kj} & \text{if } b_{ik}b_{kj} < 0 \\ b_{ij} & \text{else} \end{cases} \quad (*)$$

**Note 5.6.** One can replace the middle two cases with

$$b'_{ij} = b_{ij} + |b_{ik}|b_{kj} \quad \text{if } b_{ik}b_{kj} > 0.$$

**Example 5.7.** Figure 28 shows an example of matrix mutation.

## 5.3 Skew-Symmetrizable Matrices

**Definition 5.8.** An  $n \times n$  matrix  $B = (b_{ij}) \in \text{Mat}_{n \times n}(\mathbb{Z})$  is **skew-symmetrizable** if for some  $d_1, \dots, d_n \in \mathbb{Z}_{>0}$  we have

$$d_i b_{ij} = -d_j b_{ji}.$$

(I.e.,  $B$  becomes skew-symmetric after rescaling the rows by positive integers.)

**Definition 5.9.** An  $m \times n$  matrix is **extended skew-symmetrizable** if the top  $n \times n$  submatrix is skew-symmetrizable.

**Definition 5.10.** For  $\tilde{B} = (b_{ij})$  an extended skew-symmetrizable  $m \times n$  matrix and  $k \in \{1, \dots, n\}$ , we define  $\mu_k(\tilde{B}) = (b'_{ij})$  using the same formula (\*).

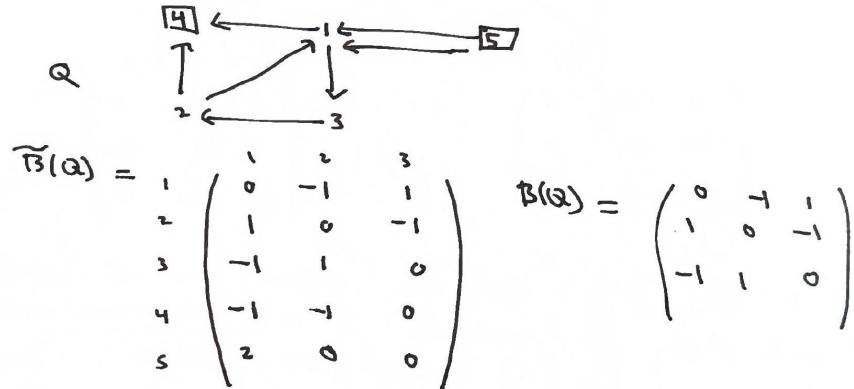


Figure 26: A quiver with frozen vertices 4 and 5 (boxed), and its extended and exchange matrices.

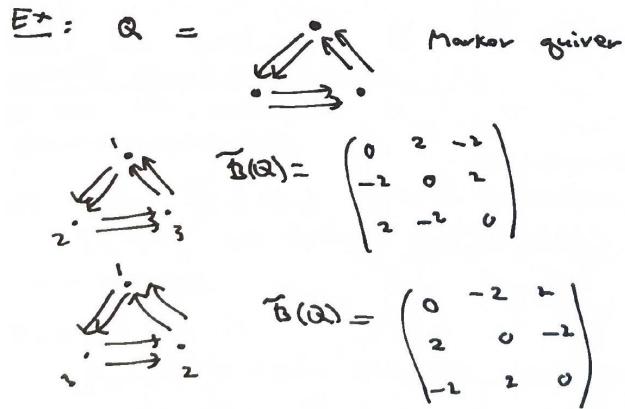


Figure 27: The Markov quiver and extended exchange matrices for mutations.

**Exercise 5.11.** (1)  $\mu_k(\tilde{B})$  is again extended skew-symmetrizable, using the same  $d_1, \dots, d_n$ .

- (2)  $\mu_k(\mu_k(\tilde{B})) = \tilde{B}$ .
- (3)  $\mu_k(-\tilde{B}) = -\mu_k(\tilde{B})$ .
- (4) If  $b_{ij} = b_{ji} = 0$ , then  $\mu_i \mu_j \tilde{B} = \mu_j \mu_i \tilde{B}$ .

#### 5.4 Diagrams and Uniqueness

**Definition 5.12.** For a skew-symmetrizable  $n \times n$  matrix  $B = (b_{ij})$ , its **diagram** is the weighted directed graph  $\Gamma(B)$  with vertices  $1, \dots, n$  and  $i \rightarrow j$  if and only if  $b_{ij} > 0$ , with weight  $|b_{ij}b_{ji}|$ .

**Lemma 5.13.** *If the diagram  $\Gamma(B)$  of an  $n \times n$  skew-symmetrizable matrix  $B$  is connected, then the skew-symmetrizing vector  $(d_1, \dots, d_n)$  is unique up to rescaling.*

*Proof.* By connectedness, there is an ordering  $l_1, \dots, l_n$  of  $\{1, \dots, n\}$  such that for each  $j \geq 2$  we have  $b_{l_i l_j} \neq 0$  for some  $i < j$ .

If  $(d_1, \dots, d_n)$  and  $(d'_1, \dots, d'_n)$  are skew-symmetrizing vectors, we have  $d_i b_{ij} = -d_j b_{ji}$  and  $d'_i b_{ij} = -d'_j b_{ji}$  for all  $i, j$ .

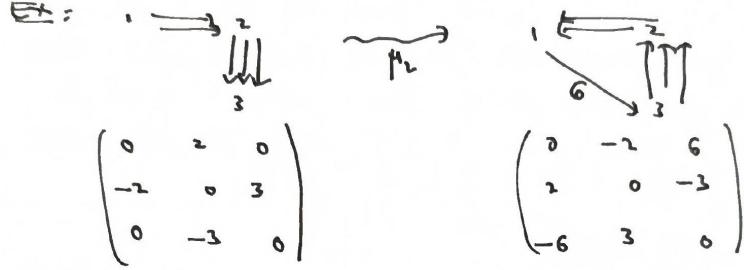


Figure 28: An example of quiver mutation  $\mu_2$  and the corresponding matrix mutation.

If  $b_{ij} \neq 0$ , we have

$$\frac{b_{ij}}{b_{ji}} = \frac{-d_j}{d_i} = \frac{-d'_j}{d'_i}.$$

Thus  $\frac{d_j}{d'_j} = \frac{d_i}{d'_i}$ . □

## 5.5 Mutation Equivalence for Matrices

**Definition 5.14.** Two extended skew-symmetrizable matrices  $\tilde{B}, \tilde{B}'$  are **mutation equivalent** if one can get from  $\tilde{B}$  to  $\tilde{B}'$  by a sequence of mutations followed by a reordering of the rows and columns in the sense from before. Put

$$[B] := \text{mutation equivalence class of } B.$$

**Proposition 5.15.** For an  $n \times n$  skew-symmetrizable matrix, its rank and determinant are preserved by mutations.

*Proof.* One can write

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } k \in \{i, j\} \\ b_{ij} + \max(0, -b_{ik})b_{kj} + b_{ik} \max(0, b_{kj}) & \text{otherwise} \end{cases}$$

We have

$$\begin{aligned} \mu_k(B) &= J_{m,k} \tilde{B} J_{n,k} + J_{m,k} \tilde{B} F_k + E_k \tilde{B} J_{n,k} \\ &= (J_{m,k} + E_k) \tilde{B} (J_{n,k} + F_k) \end{aligned}$$

where:

- $J_{m,k}$  (resp.  $J_{n,k}$ ) is a diagonal  $m \times m$  (resp.  $n \times n$ ) matrix with 1s on the diagonal except for  $-1$  in the  $(k, k)$  entry.
- $E_k = (e_{ij})$  is an  $m \times m$  matrix with  $e_{ik} = \max(0, -b_{ik})$  and all other entries 0.
- $F_k = (f_{ij})$  is an  $n \times n$  matrix with  $f_{kj} = \max(0, b_{kj})$  and all other entries 0.

Note:  $E_k \tilde{B} F_k = 0$  since  $b_{kk} = 0$ .

We have  $\det(J_{m,k} + E_k) = \det(J_{n,k} + F_k) = -1$ . □

## 5.6 Labeled Seeds

**Definition 5.16.** A **labeled seed of geometric type** in  $\mathcal{F} = \mathbb{C}(x_1, \dots, x_m)$  (the field of rational functions) is a pair  $(\mathbf{x}, \tilde{B})$  where:

- $\mathbf{x} = (x_1, \dots, x_m)$  is an  $m$ -tuple of elements of  $\mathcal{F}$  which form a free generating set (i.e.,  $\mathcal{F} = \mathbb{C}(x_1, \dots, x_m)$  and  $x_1, \dots, x_m$  are algebraically independent).
- $\tilde{B} = (b_{ij})$  is an  $m \times n$  extended skew-symmetrizable matrix.

We say:

- $\mathbf{x}$  is the (labeled) **extended cluster** of  $(\mathbf{x}, \tilde{B})$ .
- $(x_1, \dots, x_n)$  is the (labeled) **cluster**.
- $x_1, \dots, x_n$  are the **cluster variables**.
- $x_{n+1}, \dots, x_m$  are the **frozen variables**.
- $\tilde{B}$  is the **extended exchange matrix**.
- Its top  $n \times n$  submatrix  $B$  is the **exchange matrix**.

**Example 5.17.** Figure 29 shows two labeled seeds  $\Sigma$  and  $\Sigma'$  related by mutation, with  $m = 3$  and  $n = 2$ .

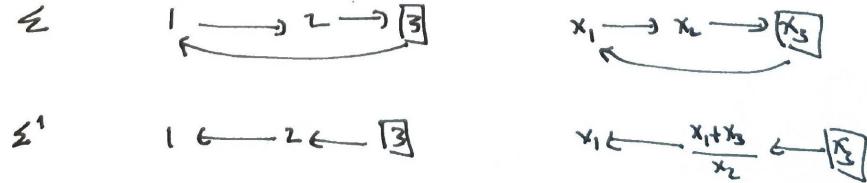


Figure 29: Two labeled seeds  $\Sigma$  and  $\Sigma'$  related by mutation at vertex 1.

For  $\Sigma$ : the extended cluster is  $\mathbf{x} = (x_1, x_2, x_3)$ , the cluster is  $(x_1, x_2)$ , the cluster variables are  $x_1, x_2$ , the frozen variable is  $x_3$ , and

$$\tilde{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For  $\Sigma'$ : the extended cluster is  $\mathbf{x}' = (x'_1, \frac{x_1+x_3}{x_2}, x_3)$ , the cluster variables are  $x'_1, \frac{x_1+x_3}{x_2}$ , and

$$\tilde{B}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

## 6 Lecture 6

Date: February 4, 2026

Main reference: [FWZ21], §3.1.

### 6.1 Labeled Seeds and Seed Mutation

Recall:  $\mathcal{F} = \mathbb{C}(y_1, \dots, y_m)$  is a field of rational functions,  $m \geq n$ . Say  $x_1, \dots, x_m \in \mathcal{F}$  is a **free generating set** if it is algebraically independent and  $\mathcal{F} = \mathbb{C}(x_1, \dots, x_m)$ .

**Definition 6.1.** A **labeled seed** of geometric type in  $\mathcal{F}$  is  $(\tilde{x}, \tilde{B})$  where:

- $\tilde{x} = (x_1, \dots, x_m)$  is a free generating set of  $\mathcal{F}$ .
- $\tilde{B} = (b_{ij})$  is an  $m \times n$  extended skew-symmetrizable integer matrix.

**Terminology:**

- $\tilde{x}$  is the **extended cluster**.
- $x = (x_1, \dots, x_n)$  is the **cluster**;  $x_1, \dots, x_n$  are the **cluster variables**.
- $x_{n+1}, \dots, x_m$  are the **frozen variables**.
- $\tilde{B}$  is the **extended exchange matrix**; the top  $n \times n$  submatrix  $B$  is the **exchange matrix**.

**Definition 6.2.** Given  $(\tilde{x}, \tilde{B})$  a labeled seed,  $k \in \{1, \dots, n\}$ , define a new labeled seed  $\mu_k(\tilde{x}, \tilde{B}) = (\tilde{x}', \tilde{B}')$ , where:

- $\tilde{B}' = \mu_k(\tilde{B})$
- $\tilde{x}' = (x'_1, \dots, x'_m)$ , where  $x'_j = x_j$  for  $j \neq k$  and

$$x_k x'_k = \prod_{b_{ik} > 0} x_i^{b_{ik}} + \prod_{b_{ik} < 0} x_i^{-b_{ik}} \quad (\text{exchange relation})$$

**Remark 6.3.** When  $\tilde{B}$  comes from a quiver, the first product is over arrows ending at  $k$  and the second product is over arrows starting at  $k$ . See Figure 30 for an example.

### 6.2 Examples

Recall the Plücker relation (Figure 31):

$$P_{13}P_{24} = P_{12}P_{34} + P_{14}P_{23}.$$

More generally, a flip gives

$$P_{ik}P_{j\ell} = P_{ij}P_{\ell k} + P_{i\ell}P_{jk},$$

which is a special case of the exchange relation; see also Figure 32 for the wiring diagram case.

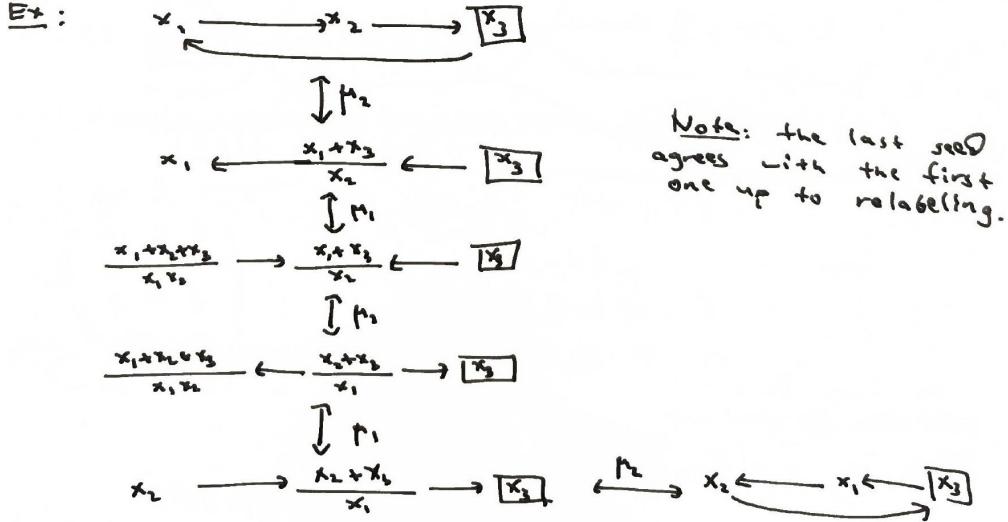


Figure 30: Example of a sequence of seed mutations. Note: the last seed agrees with the first one up to relabeling.

### 6.3 Seed Patterns and Cluster Algebras

**Notation:** Let  $\mathbb{T}_n$  denote the  $n$ -regular tree (Figure 33) with edges labeled by  $1, \dots, n$ , such that the edges incident to each vertex carry distinct labels.

**Definition 6.4.** A **seed pattern** is a choice of labeled seed  $(\tilde{x}(t), \tilde{B}(t))$  for each vertex  $t \in \mathbb{T}_n$ , so that for each labeled edge  $t \xrightarrow{k} t'$ , the corresponding labeled seeds  $(\tilde{x}(t), \tilde{B}(t))$  and  $(\tilde{x}(t'), \tilde{B}(t'))$  differ by  $\mu_k$ .

**Note 6.5.** A seed pattern is determined by any one of its seeds.

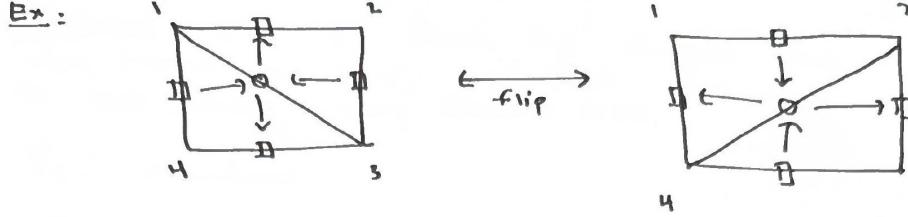
**Definition 6.6.** Let  $(\tilde{x}(t), \tilde{B}(t))_{t \in \mathbb{T}_n}$  be a seed pattern, and put  $R := \mathbb{C}[x_{n+1}, \dots, x_m]$ . Let  $\mathcal{X}$  be the set of all cluster variables appearing in the seeds  $x(t)$  for  $t \in \mathbb{T}_n$ . The **cluster algebra**  $\mathcal{A}$  is the  $R$ -subalgebra of  $\mathcal{F}$  generated by all cluster variables, i.e.,  $\mathcal{A} = R[\mathcal{X}]$ .

**Terminology:** The **rank**  $n$  of any cluster is the rank of a cluster algebra.

**Remark 6.7.** Note that there is an isomorphism of  $\mathcal{F}$  mapping any free generating set to any other. In particular, up to isomorphism  $\mathcal{A}$  depends only on  $\tilde{B}_0$  for any initial seed  $(\tilde{x}_0, \tilde{B}_0)$ , and in fact only on the mutation equivalence class of  $\tilde{B}$ . In particular, each (ice) quiver  $Q$  determines an extended exchange matrix  $\tilde{B}$  and hence a cluster algebra.

### 6.4 Examples of Cluster Algebras

- (1) **Triangulations:** The associated cluster algebra is the Plücker ring.
- (2) **Wiring diagrams:** For a wiring diagram, the associated cluster algebra is the algebra of regular functions on  $\text{Flag}(\text{SL}_n)$  (i.e., on the Borel), generated by flag minors with the Plücker relations.
- (3) **Double wiring diagrams:** For a double wiring diagram, the associated cluster algebra is  $\mathbb{C}[G]^U$  for  $G = \text{SL}_n$ , i.e., the ring of regular functions on the basic affine space.



$$\begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix} \quad P_{13} = ag - ce \quad P_{24} = bh - df$$

Recall:  $P_{13}P_{24} = P_{12}P_{34} + P_{14}P_{23}$

More generally,

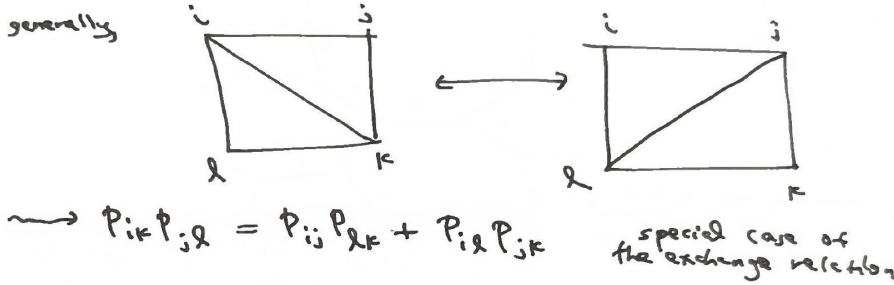


Figure 31: Triangulation flip and Plücker coordinates.

## 7 Lecture 7

Date: February 6, 2026

Main reference: [FWZ21], §3.2.

Recall: a labeled seed  $(\tilde{x}_0, \tilde{B}_0)$  determines a seed pattern  $(\tilde{x}(t), \tilde{B}(t))_{t \in \mathbb{T}_n}$ , and hence a cluster algebra  $\mathcal{A} \subset \mathcal{F}$  generated by all cluster variables and the frozen variables. Here  $\tilde{x}_0 = (x_1, \dots, x_m)$  is a free generating set of  $\mathcal{F} = \mathbb{C}(y_1, \dots, y_m)$ ,  $x_1, \dots, x_n$  are the cluster variables,  $x_{n+1}, \dots, x_m$  are the frozen variables, and the rank of  $\mathcal{A}$  is  $n$ .

### 7.1 Rank 1 Cluster Algebras

**Example 7.1. (Rank  $n = 1$ .)** The 1-regular tree is  $\mathbb{T}_1 = \bullet - \bullet$ . The extended exchange matrix is

$$\tilde{B}_0 = \begin{pmatrix} b_{21} \\ \vdots \\ b_{m1} \end{pmatrix}.$$

The exchange relation is

$$x_1 x'_1 = \prod_{b_{i1} > 0} x_i^{b_{i1}} + \prod_{b_{i1} < 0} x_i^{-b_{i1}} = M_1 + M_2,$$

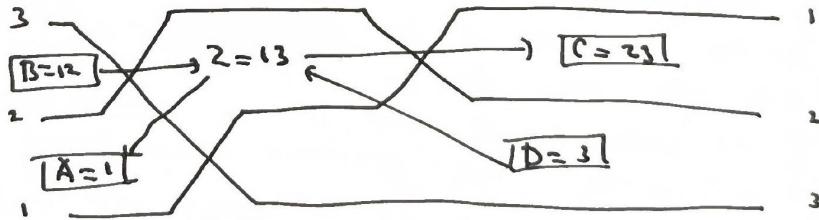
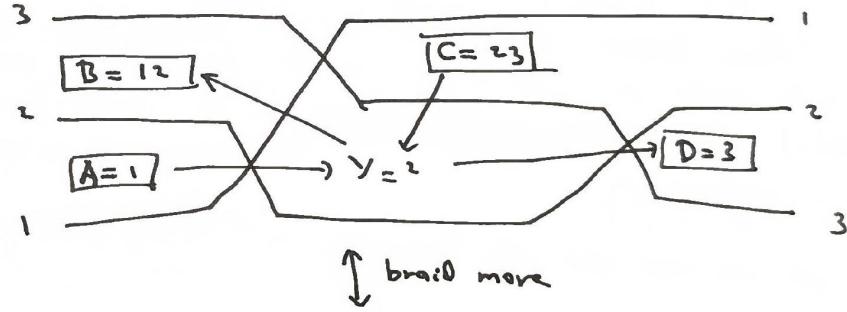
where  $M_1, M_2$  are monomials in the frozen variables  $x_2, \dots, x_m$ . The cluster algebra is

$$\mathcal{A} = \mathbb{C}[x_1, x'_1, x_2, \dots, x_m] \subset \mathcal{F} = \mathbb{C}(x_1, x_2, \dots, x_m),$$

which has the presentation

$$\mathcal{A} \cong \mathbb{C}[z_1, z'_1, z_2, \dots, z_m] / (z_1 z'_1 = M_1 + M_2),$$

where  $M_1, M_2$  are the corresponding monomials in  $z_2, \dots, z_m$ .



$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$\begin{aligned} A &\leftrightarrow a \\ B &\leftrightarrow ae-bd \\ C &\leftrightarrow bf-ce \end{aligned} \quad \text{etc}$$

$$\text{Have } Y_2 = AC + BD$$

special case of  
the exchange relation

Figure 32: Wiring diagram braid move example. The relation  $YZ = AC + BD$  is a special case of the exchange relation.

**Example 7.2.** Let  $G = \mathrm{SL}_3(\mathbb{C})$  and let  $U$  be the subgroup of unipotent lower triangular  $3 \times 3$  matrices. Then  $\mathbb{C}[G]^U$  is a cluster algebra of rank 1.

Recall:  $\mathbb{C}[G]^U$  is generated by flag minors  $P_J$ ,  $J \subsetneq \{1, 2, 3\}$ . Here:

- $\mathcal{F} = \mathbb{C}(P_1, P_2, P_3, P_{12}, P_{23})$
- Frozen variables:  $P_1, P_3, P_{12}, P_{23}$
- Cluster variables:  $P_2, P_{13}$
- Single exchange relation:  $P_2 P_{13} = P_1 P_{23} + P_3 P_{12}$

See Figure 34 for the corresponding wiring diagrams, where a braid move exchanges the cluster variables  $P_2$  and  $P_{13}$ .

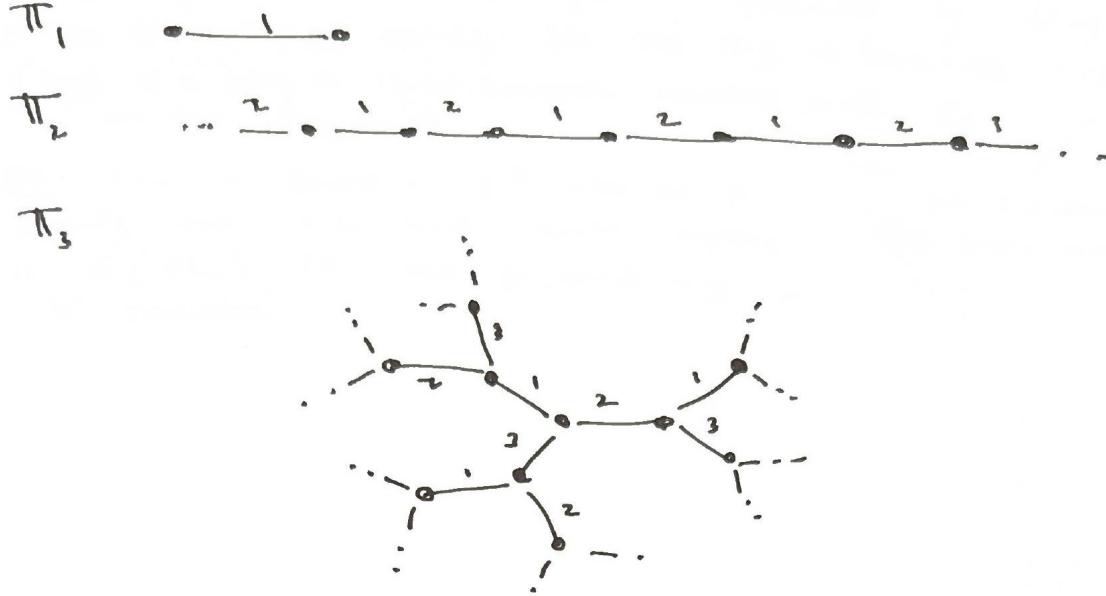


Figure 33: The  $n$ -regular trees  $\mathbb{T}_1$ ,  $\mathbb{T}_2$ , and  $\mathbb{T}_3$ .

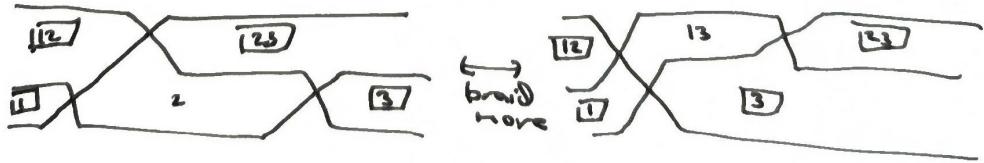


Figure 34: Wiring diagrams for  $SL_3$ : the braid move exchanges the cluster variables  $P_2$  and  $P_{13}$ , corresponding to the exchange relation  $P_2P_{13} = P_1P_{23} + P_3P_{12}$ .

## 7.2 Rank 2 Cluster Algebras

**Example 7.3. (Rank  $n = 2$ .)** The extended exchange matrix has the form

$$\tilde{B}_0 = \begin{pmatrix} 0 & \pm b \\ \mp c & 0 \\ b_{31} & b_{32} \\ \vdots & \vdots \\ b_{m1} & b_{m2} \end{pmatrix},$$

where either  $b, c > 0$  or  $b = c = 0$ .

Suppose there are no frozens, i.e.,  $n = m$ , so  $\tilde{B}_0 = \begin{pmatrix} 0 & \pm b \\ \mp c & 0 \end{pmatrix}$ . Then  $\mu_1(\tilde{B}_0) = \mu_2(\tilde{B}_0) = -\tilde{B}_0$ .

The exchange pattern along  $\mathbb{T}_2$  has seeds

$$\cdots \xrightarrow{2} (z_1, z_0) \xrightarrow{1} (z_1, z_2) \xrightarrow{2} (z_3, z_2) \xrightarrow{1} (z_3, z_4) \xrightarrow{2} \cdots$$

with exchange matrices alternating between  $\begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix}$  and  $\begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}$ , and the exchange relation

gives

$$z_{k-1}z_{k+1} = \begin{cases} z_k^c + 1 & \text{if } k \text{ is even,} \\ z_k^b + 1 & \text{if } k \text{ is odd.} \end{cases}$$

**Example 7.4.** When  $b = c = 0$ , the extended exchange matrix is

$$\tilde{B} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ b_{31} & b_{32} \\ \vdots & \vdots \\ b_{m1} & b_{m2} \end{pmatrix}.$$

Note that  $\mu_k$  flips the sign of the  $k$ th column for  $k = 1, 2$ . The exchange relations are

$$x_1x'_1 = M_1 + M_2, \quad x_2x'_2 = M_3 + M_4,$$

where  $M_1, M_2, M_3, M_4$  are monomials in the frozen variables. The cluster variables are  $x_1, x'_1, x_2, x'_2$ , and this reduces to two rank 1 exchange patterns.

**Notation:** Let  $\mathcal{A}(b, c)$  denote the cluster algebra of rank 2 with exchange matrices  $\begin{pmatrix} 0 & \pm b \\ \mp c & 0 \end{pmatrix}$  and no frozen variables.

**Example 7.5.**  $\mathcal{A}(1, 1)$ : The exchange relation becomes  $z_{k-1}z_{k+1} = z_k + 1$ . We compute:

$$\begin{aligned} z_3 &= \frac{z_2 + 1}{z_1}, \\ z_4 &= \frac{z_3 + 1}{z_2} = \frac{z_1 + z_2 + 1}{z_1 z_2}, \\ z_5 &= \frac{z_4 + 1}{z_3}, \\ z_6 &= z_1, \quad z_7 = z_2, \quad \text{etc.} \end{aligned}$$

So the sequence of cluster variables is **5-periodic**.

**Example 7.6.** Consider  $\tilde{B}_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ p & q \end{pmatrix}$  with rank 2 and 1 frozen variable  $y$ , where  $p, q \geq 0$  are integers. The seed pattern is:

$$\begin{array}{ccccccccc} (z_1, z_2) & \xrightarrow{1} & (z_3, z_2) & \xrightarrow{2} & (z_3, z_4) & \xrightarrow{1} & (z_5, z_4) & \xrightarrow{2} & (z_5, z_6) \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ p & q \end{pmatrix} & & \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ -p & p+q \end{pmatrix} & & \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ q & -(p+q) \end{pmatrix} & & \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ -q & -p \end{pmatrix} & & \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ -q & p \end{pmatrix} \end{array}$$

We have:

$$\begin{aligned} z_3 &= \frac{z_2 + y^p}{z_1}, \\ z_4 &= \frac{y^{p+q}z_1 + z_2 + y^p}{z_1 z_2}, \\ z_5 &= \frac{y^q z_1 + 1}{z_2}, \\ z_6 &= z_1, \quad z_7 = z_2, \quad \text{etc.} \end{aligned}$$

So the cluster variables are still **5-periodic**.

**Remark 7.7.** Although we assumed  $p, q \geq 0$  above, up to mutating and swapping columns, every  $(b, c) \in \mathbb{Z}^2$  can be written in one of the forms

$$(p, q), \quad (p + q, -p), \quad (q, -p - q), \quad (-p, -q), \quad (-q, p).$$

See Figure 35. Later we will view this as a simple example of a **scattering diagram**.

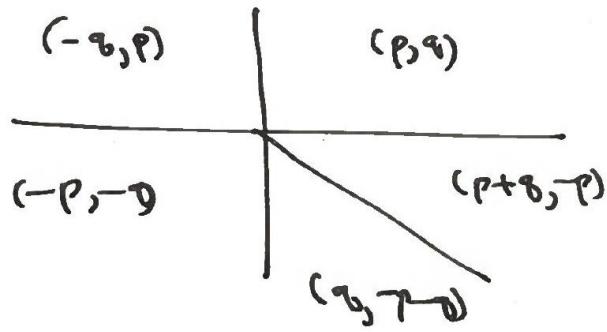


Figure 35: The five mutation forms of the frozen row  $(b, c)$  for a rank 2 cluster algebra with one frozen variable, viewed as a scattering diagram.

## 8 Lecture 8

Date: February 9, 2026

Main reference: [FWZ21], §3.3, §3.4.

### 8.1 Rank 2 examples (continued)

**Example 8.1.**  $\mathcal{A}(1, 2)$ : The exchange relation is

$$z_{k-1}z_{k+1} = \begin{cases} z_k^2 + 1 & \text{if } k \text{ is even,} \\ z_k + 1 & \text{if } k \text{ is odd.} \end{cases}$$

We compute:

$$\begin{aligned} z_3 &= \frac{z_2^2 + 1}{z_1}, \\ z_4 &= \frac{z_3 + 1}{z_2} = \frac{z_1 + z_2^2 + 1}{z_1 z_2}, \\ z_5 &= \frac{z_1^2 + 2z_1 + 1 + z_2^2}{z_1 z_2^2}, \\ z_6 &= \frac{z_1 + 1}{z_2}, \\ z_7 &= z_1, \quad z_8 = z_2, \quad \text{etc.} \end{aligned}$$

So the sequence of cluster variables is **6-periodic**.

**Example 8.2.**  $\mathcal{A}(1, 3)$ : The exchange relation is

$$z_{k-1}z_{k+1} = \begin{cases} z_k^3 + 1 & \text{if } k \text{ is even,} \\ z_k + 1 & \text{if } k \text{ is odd.} \end{cases}$$

Set  $z_1 = z_2 = 1$ . Then:

$$\begin{aligned} z_3 &= \frac{z_2^3 + 1}{z_1} = 2, \\ z_4 &= \frac{z_3 + 1}{z_2} = \frac{2 + 1}{1} = 3, \\ z_5 &= \frac{z_4^3 + 1}{z_3} = \frac{28}{2} = 14, \\ z_6 &= \frac{z_5 + 1}{z_4} = \frac{15}{3} = 5, \\ z_7 &= \frac{z_6^3 + 1}{z_5} = \frac{126}{14} = 9, \\ z_8 &= \frac{z_7 + 1}{z_6} = \frac{10}{5} = 2, \\ z_9 &= \frac{z_8^3 + 1}{z_7} = \frac{9}{9} = 1, \\ z_{10} &= \frac{z_9 + 1}{z_8} = \frac{2}{2} = 1. \end{aligned}$$

So it is **8-periodic** at least after specializing  $z_1 = z_2 = 1$ , and we claim that it is 8-periodic even without this specialization.

**Example 8.3.**  $\mathcal{A}(1, 4)$ : Setting  $z_1 = z_2 = 1$  gives the sequence

$$1, 1, 2, 3, 41, 14, 937, 67, 21506, 321, \dots$$

This is *not* periodic. However, all terms are integers, and in fact each  $z_k$  is a **Laurent polynomial** in  $z_1, z_2$ .

## 8.2 The Laurent phenomenon

**Theorem 8.4** (Laurent phenomenon). *Let  $(\tilde{x}_0, \tilde{B}_0)$  be a labeled seed, with  $\tilde{x}_0 = (x_1, \dots, x_m)$ , and associated cluster algebra  $\mathcal{A}$ . Every cluster variable of  $\mathcal{A}$  is a Laurent polynomial with integer coefficients in the variables  $x_1, \dots, x_m$ . Moreover,  $x_{n+1}, \dots, x_m$  do not appear in the denominators.*

**Remark 8.5.** Note that we can replace  $\tilde{x}_0$  equally with any other extended cluster of  $\mathcal{A}$ .

**Proof idea.** Say  $t_0 \in \mathbb{T}_n$  is the initial vertex with  $(\tilde{x}_0, \tilde{B}_0)$  the initial (labeled) seed. Let  $x = x(t)$  be a cluster variable in the seed at some vertex  $t \in \mathbb{T}_n$ , where  $\tilde{x}_0 = (x_1, \dots, x_m)$ . We want to show that  $x$  is a Laurent polynomial in  $x_1, \dots, x_m$ . We use induction on  $d = \text{dist}(t, t_0)$ .

**Base cases:**

- If  $d = 1$ , then  $x(t_1) = (x_1, \dots, x'_j, \dots, x_m)$  where

$$x'_j = \frac{\prod_{b_{ij} > 0} x_i^{b_{ij}} + \prod_{b_{ij} < 0} x_i^{-b_{ij}}}{x_j},$$

which is already a Laurent polynomial.

- If  $d = 2$ , then  $x(t_2) = (x_1, \dots, x'_j, \dots, x'_k, \dots, x_m)$  where

$$x'_k = \frac{\text{poly in } x_1, \dots, x'_j, \dots, x_m}{x_k} = \frac{\text{Laurent poly in } x_1, \dots, x_m}{x_k}$$

(or swap  $j$  and  $k$ ).

**Inductive step:** Now assume  $d \geq 3$ , and assume for simplicity that  $b_{jk}^0 = b_{kj}^0 = 0$  where  $\tilde{B}_0 = (b_{ij}^0)$ . (The case  $b_{jk}^0 b_{kj}^0 < 0$  is more complicated.)

Put  $t_3 := \mu_k(t_0)$  and  $t_4 := \mu_j \mu_k(t_0)$ . Consider the following portion of  $\mathbb{T}_n$ :

$$\begin{array}{ccccccc} & & k & & j & & \\ & & t_0 & - & t_3 & - & t_4 \\ j & | & & & & & \\ t_1 & -^k & t_2 & - & \dots & - & t \end{array}$$

Note:  $\tilde{x}(t_4) = \tilde{x}(t_2)$ , so both  $t_1, t_3$  lie at distance  $d - 1$  from a seed containing  $x$ . By induction:

$$x = \text{Laurent poly in } \tilde{x}(t_1) = \text{Laurent poly in } \tilde{x}(t_3).$$

Meanwhile,  $x'_j = \frac{M_1 + M_2}{x_j}$  and  $x'_k = \frac{M_3 + M_4}{x_k}$ , for monomials  $M_1, M_2, M_3, M_4$  in  $x_1, \dots, x_m$ .

Substituting:

$$x = \frac{\text{poly in } x_1, \dots, x_m}{(\text{monomial in } x_1, \dots, x_m) \cdot (M_1 + M_2)^a} = \frac{\text{poly in } x_1, \dots, x_m}{(\text{monomial in } x_1, \dots, x_m) \cdot (M_3 + M_4)^b}.$$

It suffices to show that  $a = 0$ .

Let  $\tilde{B}_0^{\text{aug}}$  be  $\tilde{B}_0$  after adding an extra row of the form  $(0, \dots, 1, \dots, 0)$  (with 1 in the  $j$ th entry).

Let  $\mathcal{A}_{\text{aug}}$  be the resulting cluster algebra with coefficient variables  $x_{n+1}, \dots, x_{m+1}$ .

Observe: an expression in  $\mathcal{A}_{\text{aug}}$  for  $x$  in terms of  $x_1, \dots, x_{m+1}$  specializes (setting  $x_{m+1} = 1$ ) to an expression in  $\mathcal{A}$  for  $x$  in terms of  $x_1, \dots, x_m$ .

So  $x$  being a Laurent polynomial in  $x_1, \dots, x_m$  in  $\mathcal{A}_{\text{aug}}$  implies  $x$  is a Laurent polynomial in  $x_1, \dots, x_m$  in  $\mathcal{A}$ , hence WLOG we can assume  $\tilde{B}_0^{\text{aug}}$  instead of  $\tilde{B}_0$ .

But then

$$x'_j = \frac{M_1^{\text{aug}} + M_2^{\text{aug}}}{x_j} = \frac{M_1 x_{m+1} + M_2}{x_j}, \quad x'_k = \frac{M_3^{\text{aug}} + M_4^{\text{aug}}}{x_k} = \frac{M_3 + M_4}{x_k}.$$

Then  $M_1^{\text{aug}} + M_2^{\text{aug}}$  and  $M_3 + M_4$  have no common factors (think about what happens if we specialize  $x_1 = \dots = x_m = 1$ ), which implies  $a = 0$ .  $\square$

### 8.3 Markov triples

**Definition 8.6.** A **Markov triple** is a triple  $(a, b, c) \in \mathbb{Z}_{\geq 1}^3$  which satisfies the **Markov equation**:

$$a^2 + b^2 + c^2 = 3abc.$$

**Example 8.7.**  $(1, 1, 1)$  is a Markov triple, and hence also its permutations. So is  $(1, 2, 5)$  and its permutations:  $(1, 5, 2), (2, 1, 5), (2, 5, 1), (5, 1, 2), (5, 2, 1)$ .

**Lemma 8.8.** If  $(a, b, c)$  is a Markov triple, then so is  $(a, b, c')$  with  $c' = \frac{a^2 + b^2}{c} = 3ab - c$ .

*Proof.* Consider the equation  $a^2 + b^2 + t^2 = 3abt$ , i.e.,  $t^2 - 3abt + (a^2 + b^2) = 0$ . If  $c$  is one root, the other one  $c'$  must satisfy  $c + c' = 3ab$ , i.e.,

$$c' = 3ab - c = \frac{3abc - c^2}{c} = \frac{a^2 + b^2}{c}.$$

This operation is called **Markov mutation**.  $\square$

**Lemma 8.9.** If  $(a, b, c)$  is a Markov triple and  $a \leq b < c$ , then  $c' = 3ab - c < c$ .

*Proof.* Put  $f(t) = t^2 - 3abt + (a^2 + b^2)$ . Then

$$f(b) = b^2 - 3ab^2 + a^2 + b^2 = b^2(2 - 3a) + a^2 \leq -b^2 + a^2 \leq 0.$$

Then  $c'$ , the other root of  $f$ , must satisfy  $c' \leq b < c$ .  $\square$

**Corollary 8.10.** Every Markov triple can be connected to  $(1, 1, 1)$  by a sequence of Markov mutations.

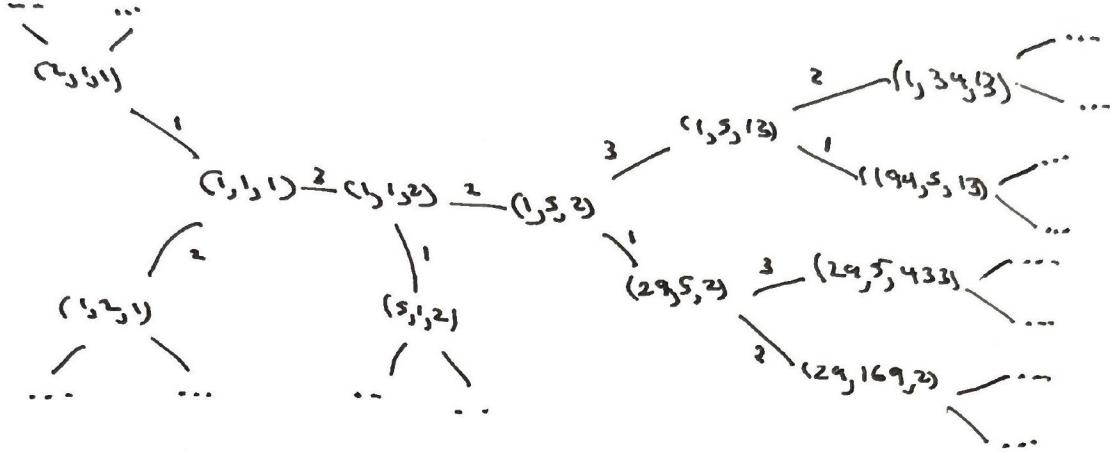


Figure 36: The Markov tree: nodes are Markov triples and edges correspond to Markov mutations.

#### 8.4 The Markov tree

The Markov triples are organized into the **Markov tree**, where the edges correspond to Markov mutations (see Figure 36). Each edge is labeled by the index (1, 2, or 3) of the entry being mutated. For example:

$$(1, 1, 1) \xrightarrow{3} (1, 1, 2) \xrightarrow{2} (1, 5, 2) \xrightarrow{3} (1, 5, 13) \xrightarrow{1} (194, 5, 13), \dots$$

Recall: the **Markov quiver** is the quiver on three vertices with exchange relations (see Figure 37):

$$\begin{aligned} x'_1 x_1 &= x_2^2 + x_3^2, \\ x'_2 x_2 &= x_1^2 + x_3^2, \\ x'_3 x_3 &= x_1^2 + x_2^2. \end{aligned}$$

Thus for any cluster  $\tilde{x} = (x_1, x_2, x_3)$ , specializing the initial cluster variables to  $(1, 1, 1)$  turns the trio into a Markov triple.

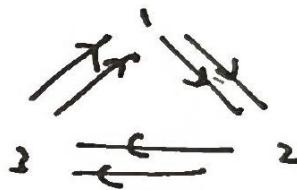


Figure 37: The Markov quiver: three vertices with double arrows forming a cycle.

#### 8.5 The Somos-4 sequence

**Example 8.11.** The **Somos-4 sequence** is defined by  $z_0 = z_1 = z_2 = z_3 = 1$  and the recurrence

$$\tilde{z}_{m+2} \tilde{z}_{m-2} = \tilde{z}_{m+1} \tilde{z}_{m-1} + \tilde{z}_m^2,$$

i.e., the sequence

$$1, 1, 1, 1, 2, 3, 7, 23, 59, 314, 1529, \dots$$

Somos (1980s): these are all integers!

To explain this using cluster algebras, consider the quiver  $Q$  on four vertices with no frozen variables shown in Figure 38. The exchange relation at vertex 1 is

$$z_1 z_5 = z_2 z_4 + z_3^2, \quad Q' = \mu_1(Q).$$

Then  $\mu_1$  rotates  $Q$  by  $\pi/2$ . Applying  $\mu_2$  to  $Q'$  gives

$$z_2 z_6 = z_3 z_5 + z_4^2.$$

Continuing in this way with the mutation sequence  $\mu_1, \mu_2, \mu_3, \mu_4, \mu_1, \mu_2, \mu_3, \mu_4, \dots$  gives

$$\tilde{z}_n = \text{Laurent polynomial in } z_1, z_2, z_3, z_4.$$

Specializing  $z_1 = z_2 = z_3 = z_4 = 1$ , the  $k$ th element of Somos-4 is necessarily an integer by the Laurent phenomenon.

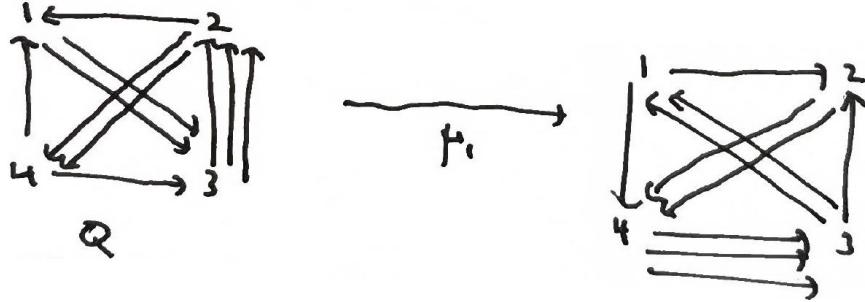


Figure 38: The quiver  $Q$  for the Somos-4 sequence (left) and its mutation  $\mu_1(Q)$  (right), which is  $Q$  rotated by  $\pi/2$ .

## 9 Lecture 9

Date: February 11, 2026

Main reference: [FWZ21], §3.4, §3.5, §3.6.

### 9.1 The $\hat{y}$ -variables

Let  $(\tilde{x}, \tilde{B})$  be a labeled seed, with  $\tilde{x} = (x_1, \dots, x_m)$ ,  $\tilde{B} = (b_{ij})$ . Put  $(\tilde{x}', \tilde{B}') = \mu_k(\tilde{x}, \tilde{B})$ , with  $\tilde{x}' = (x'_1, \dots, x'_m)$ ,  $\tilde{B}' = (b'_{ij})$ .

Put  $\hat{y} := (\hat{y}_1, \dots, \hat{y}_n)$ , where

$$\hat{y}_j = \prod_{i=1}^m x_i^{b_{ij}},$$

and similarly  $\hat{y}' = (\hat{y}'_1, \dots, \hat{y}'_n)$  with  $\hat{y}'_j = \prod_{i=1}^m (x'_i)^{b'_{ij}}$ .

**Proposition 9.1.** We have (for  $j = 1, \dots, n$ ):

$$\hat{y}'_j = \begin{cases} \hat{y}_k^{-1} & \text{if } j = k, \\ \hat{y}_j \left( \hat{y}_k^{-\operatorname{sgn}(b_{kj})} + 1 \right)^{-b_{kj}} & \text{else.} \end{cases}$$

$$\text{Here } \operatorname{sgn}(b) = \begin{cases} 1 & \text{if } b > 0, \\ -1 & \text{if } b < 0. \end{cases}$$

**Remark 9.2.** • Recall that the exchange relation is

$$x_k x'_k = \prod_{b_{ik} > 0} x_i^{b_{ik}} + \prod_{b_{ik} < 0} x_i^{-b_{ik}}.$$

Then  $\hat{y}_k$  is the ratio of these two monomials.

• The above formula for  $\hat{y}'_j$  depends only on the top  $n \times n$  submatrix of  $\tilde{B}$ .

*Proof of Proposition 9.1. Case  $j = k$ :* We have

$$\hat{y}'_k = \prod_{i=1}^m (x'_i)^{b'_{ik}} = \prod_{i \neq k} x_i^{b'_{ik}}.$$

Recall the mutation formula:

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } k \in \{i, j\}, \\ b_{ij} + |b_{ik}|b_{kj} & \text{if } b_{ik}b_{kj} > 0, \\ b_{ij} & \text{else.} \end{cases}$$

Since  $k \in \{i, k\}$ , we have  $b'_{ik} = -b_{ik}$ , so

$$\hat{y}'_k = \prod_{i \neq k} x_i^{-b_{ik}} = \hat{y}_k^{-1}.$$

**Case  $j \neq k$  and  $b_{kj} \leq 0$ :**

$$\begin{aligned}
\hat{y}'_j &= (x'_k)^{b'_{kj}} \prod_{i \neq k} x_i^{b'_{ij}} \\
&= (x'_k)^{-b_{kj}} \left( \prod_{i \neq k} x_i^{b_{ij}} \right) \left( \prod_{b_{ik} < 0} x_i^{-b_{ik}b_{kj}} \right) \\
&= x_k^{b_{kj}} \left( \prod_{b_{ik} > 0} x_i^{b_{ik}} + \prod_{b_{ik} < 0} x_i^{-b_{ik}} \right)^{-b_{kj}} \left( \prod_{i \neq k} x_i^{b_{ij}} \right) \left( \prod_{b_{ik} < 0} x_i^{-b_{ik}b_{kj}} \right) \\
&= \left( \prod_i x_i^{b_{ij}} \right) (\hat{y}_k + 1)^{-b_{kj}} \\
&= \hat{y}_j (\hat{y}_k + 1)^{-b_{kj}}.
\end{aligned}$$

The case  $j \neq k, b_{kj} \geq 0$  is similar.  $\square$

## 9.2 Y-seeds

**Definition 9.3.** A **Y-seed** of rank  $n$  in a field  $\mathcal{F}$  is a pair  $(Y, B)$ , where:

- $Y = (Y_1, \dots, Y_n)$  is an  $n$ -tuple of elements in  $\mathcal{F}$ ,
- $B$  is a skew-symmetrizable  $n \times n$  integer matrix.

We mutate Y-seeds as follows:

$$(Y, B) \xrightarrow{\mu_k} (Y', B'), \quad \text{where } B' = \mu_k(B),$$

$Y' = (Y'_1, \dots, Y'_n)$  with

$$Y'_j = \begin{cases} Y_k^{-1} & \text{if } j = k, \\ Y_j \left( Y_k^{-\text{sgn}(b_{kj})} + 1 \right)^{-b_{kj}} & \text{else.} \end{cases}$$

Thus a labeled seed  $(\tilde{x}, \tilde{B})$  gives rise to a Y-seed  $(\hat{y}, B)$ , where  $B$  is the top  $n \times n$  submatrix of  $\tilde{B}$  and  $\hat{y} = (\hat{y}_1, \dots, \hat{y}_n)$  with  $\hat{y}_i = \prod_{i=1}^m x_i^{b_{ij}}$ .

**Remark 9.4.** The seed mutation at  $k$  leaves  $x_j$  alone for  $j \neq k$  but potentially changes all of  $Y_1, \dots, Y_n$ . However, the formula for  $x'_k$  involves all of  $x_1, \dots, x_m$ , whereas the formula for  $Y'_j$  only involves  $Y_k$  and  $Y_j$ .

## 9.3 Semifields

**Definition 9.5.** A **semifield** is an abelian group  $\mathbb{P}$  (written multiplicatively) endowed with an auxiliary operation  $\oplus$  which is commutative, associative, and distributive with respect to the group operation on  $\mathbb{P}$ . Note that  $(\mathbb{P}, \oplus)$  is only a semigroup (i.e., not necessarily having an identity or inverses).

**Example 9.6.** The multiplicative group  $\mathbb{Q}_{>0}$ , with  $\oplus$  given by ordinary addition.

**Definition 9.7.** The **tropical semifield**  $\text{Trop}(q_1, \dots, q_\ell)$  is defined by:

- the multiplicative group of Laurent monomials in  $q_1, \dots, q_\ell$ ,
- the auxiliary addition (**tropical addition**):

$$\prod_{i=1}^{\ell} q_i^{a_i} \oplus \prod_{i=1}^{\ell} q_i^{b_i} = \prod_{i=1}^{\ell} q_i^{\min(a_i, b_i)}.$$

Check:

- Commutative:  $\min(a_i, b_i) = \min(b_i, a_i)$ .
- Associative:  $\min(\min(a_i, b_i), c_i) = \min(a_i, \min(b_i, c_i))$ .
- Distributive (i.e.,  $p(q \oplus r) = pq \oplus pr$ ):  $\min(a_i, b_i) + c_i = \min(a_i + c_i, b_i + c_i)$ .

#### 9.4 Coefficient tuples and tropical Y-seed mutation

For a labeled seed  $(\tilde{x}, \tilde{B})$  with  $\tilde{x} = (x_1, \dots, x_n, x_{n+1}, \dots, x_m)$  (where  $x_{n+1}, \dots, x_m$  are the frozen variables), we associate the **coefficient tuple**

$$y = (y_1, \dots, y_n), \quad \text{where } y_j = \prod_{i=n+1}^m x_i^{b_{ij}} \in \text{Trop}(x_{n+1}, \dots, x_m)$$

for  $j = 1, \dots, n$ .

**Note 9.8.**  $B = \text{top } n \times n$  submatrix of  $\tilde{B}$ . Together with the coefficient tuple  $y$ , we recover the extended exchange matrix  $\tilde{B}$ .

**Proposition 9.9.** Let  $\tilde{B} = (b_{ij})$  be an extended skew-symmetrizable  $m \times n$  matrix with coefficient tuple  $y = (y_1, \dots, y_n)$ , and  $\tilde{B}' = (b'_{ij}) = \mu_k(\tilde{B})$  with coefficient tuple  $y' = (y'_1, \dots, y'_n)$ . Then

$$y'_j = \begin{cases} y_k^{-1} & \text{if } j = k, \\ y_j \left( y_k^{-\text{sgn}(b_{kj})} \oplus 1 \right)^{-b_{kj}} & \text{else.} \end{cases}$$

This is called **tropical Y-seed mutation**.

**Definition 9.10.** The **universal semifield**  $\mathbb{Q}_{\text{sf}}(x_1, \dots, x_m)$  is

$$\left\{ \frac{P(x_1, \dots, x_m)}{Q(x_1, \dots, x_m)} \in \mathbb{Q}(x_1, \dots, x_m) \mid P, Q \text{ have positive coefficients} \right\}$$

with ordinary multiplication and addition.

**Lemma 9.11.** Given any semifield  $\mathbb{S}$  and elements  $s_1, \dots, s_m \in \mathbb{S}$ , there exists a unique semifield homomorphism  $\mathbb{Q}_{\text{sf}}(x_1, \dots, x_m) \rightarrow \mathbb{S}$  sending  $x_i \mapsto s_i$  for  $i = 1, \dots, m$ .

*Proof of Proposition 9.9.* Let  $f: \mathbb{Q}_{\text{sf}}(x_1, \dots, x_m) \rightarrow \text{Trop}(x_{n+1}, \dots, x_m)$  be the semifield homomorphism sending

$$f(x_i) = \begin{cases} 1 & \text{if } i \leq n, \\ x_i & \text{if } i > n. \end{cases}$$

Note that  $f$  also sends  $x'_k$  to 1, since  $x_k x'_k = M_1 + M_2$  implies

$$1 \cdot f(x'_k) = f(M_1) \oplus f(M_2) = 1$$

(since  $M_1, M_2$  are monomials which share no frozen variables), so  $f(x'_k) = 1$ .

Also,  $\hat{y}_j = \prod_{i=1}^m x_i^{b_{ij}}$  implies

$$f(\hat{y}_j) = \prod_{i=n+1}^m x_i^{b_{ij}} = y_j, \quad \text{for } j = 1, \dots, n,$$

and similarly  $f(\hat{y}'_j) = y'_j$ .

Thus, applying  $f$  to the formula from Proposition 9.1:

$$\hat{y}'_j = \begin{cases} \hat{y}_k^{-1} & \text{if } j = k, \\ \hat{y}_j \left( \hat{y}_k^{-\text{sgn}(b_{kj})} + 1 \right)^{-b_{kj}} & \text{else,} \end{cases}$$

gives

$$y'_j = \begin{cases} y_k^{-1} & \text{if } j = k, \\ y_j \left( y_k^{-\text{sgn}(b_{kj})} \oplus 1 \right)^{-b_{kj}} & \text{else.} \end{cases}$$

□

# 10 Lecture 10

Date: February 13, 2026

Main reference: [FWZ21], §3.5, §3.6, §5.1.

## 10.1 Alternative characterization of labeled seeds

We can now give an alternative characterization of labeled seeds and their mutations. Fix  $\mathcal{F} = \mathbb{C}(b_1, \dots, b_m)$ . A **labeled seed** is a triple  $\Sigma = (x, y, B)$ , where:

- **cluster**  $x = (x_1, \dots, x_n) \in \mathcal{F}^n$  such that  $x \cup \{b_{n+1}, \dots, b_m\}$  freely generates  $\mathcal{F}$ ,
- **exchange matrix**  $B = \text{skew-symmetrizable integer matrix}$ ,
- **coefficient tuple**  $y = (y_1, \dots, y_n)$ , where  $y_j$  is a Laurent monomial in  $\text{Trop}(b_{n+1}, \dots, b_m)$ .

For a mutation

$$(x, y, B) \xrightarrow{\mu_k} (x', y', B'),$$

we have:

- $B' = \mu_k(B)$ ,
- $y'$  is given by the tropical Y-seed mutation rule,
- $x' = (x \setminus \{x_k\}) \cup \{x'_k\}$ , with

$$x_k x'_k = \frac{y_k}{y_k \oplus 1} \prod_{b_{ik} > 0} x_i^{b_{ik}} + \frac{1}{y_k \oplus 1} \prod_{b_{ik} < 0} x_i^{-b_{ik}}.$$

**Key point:** From this perspective, the complexity of the mutation process does not really grow with the number  $m - n$  of frozen variables.

## 10.2 Example: $A_2$ revisited

Consider  $B_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ . This gives a labeled seed pattern on the line:

$$\dots \xrightarrow{1} \Sigma(-1) \xrightarrow{2} \Sigma(0) \xrightarrow{1} \Sigma(1) \xrightarrow{2} \Sigma(2) \xrightarrow{1} \Sigma(3) \xrightarrow{2} \dots$$

Write  $\Sigma(t) = (x(t), y(t), B(t))$ . Then  $B(t) = (-1)^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ .

The cluster variables  $x(t) = (x_1(t), x_2(t))$  and coefficient tuples  $y(t) = (y_1(t), y_2(t))$  are given by:

$t$	$x_1(t)$	$x_2(t)$	$y_1(t)$	$y_2(t)$
0	$x_1$	$x_2$	$y_1$	$y_2$
1	$\frac{y_1 + x_2}{x_1(y_1 \oplus 1)}$	$x_2$	$\frac{1}{y_1}$	$\frac{y_1 y_2}{y_1 \oplus 1}$
2	$\frac{y_1 + x_2}{x_1(y_1 \oplus 1)}$	$\frac{x_1 y_1 y_2 + y_1 + x_2}{(y_1 y_2 \oplus y_1 \oplus 1) x_1 x_2}$	$\frac{y_2}{y_1 y_2 \oplus y_1 \oplus 1}$	$\frac{y_1 \oplus 1}{y_1 y_2}$
3	$\frac{x_1 y_2 + 1}{x_2(y_2 \oplus 1)}$	$\frac{x_1 y_1 y_2 + y_1 + x_2}{(y_1 y_2 \oplus y_1 \oplus 1) x_1 x_2}$	$\frac{y_1 y_2 \oplus y_1 \oplus 1}{y_2}$	$\frac{1}{y_1(y_2 \oplus 1)}$
4	$\frac{x_1 y_2 + 1}{x_2(y_2 \oplus 1)}$	$x_1$	$\frac{1}{y_2}$	$y_1(y_2 \oplus 1)$
5	$x_2$	$x_1$	$y_2$	$y_1$

### 10.3 Finite type classification in rank 2

**Theorem 10.1.** *A seed pattern with initial labeled seed  $\Sigma = (x, y, B)$  with  $B = \pm \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix}$ ,  $b, c \in \mathbb{Z}_{\geq 1}$ , is of finite type if and only if  $bc \leq 3$ .*

Compare:

**Proposition 10.2.** *For  $b, c \in \mathbb{Z}_{\geq 1}$ , the subgroup  $W = \langle R_1, R_2 \rangle \subset \mathrm{GL}_2$  generated by the reflections*

$$R_1 = \begin{pmatrix} -1 & b \\ 0 & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 0 \\ c & -1 \end{pmatrix}$$

*is finite if and only if  $bc \leq 3$ .*

*Proof.* Since  $R_1^2 = R_2^2 = \mathbb{I}$ ,  $W$  is finite if and only if  $R_1 R_2$  has finite order. We compute:

$$R_1 R_2 = \begin{pmatrix} bc - 1 & -b \\ c & -1 \end{pmatrix}.$$

The characteristic equation is

$$\lambda^2 - (bc - 2)\lambda + 1 = 0,$$

giving

$$\lambda = \frac{bc - 2 \pm \sqrt{(bc - 2)^2 - 4}}{2}.$$

For  $bc = 1, 2, 3$ : the roots have order 3, 4, 6 respectively.

If  $bc > 4$ : the roots are real and not  $\pm 1$ , hence  $R_1 R_2$  has infinite order.

If  $bc = 4$ :

$$(R_1 R_2)^k = \begin{pmatrix} 2k + 1 & -kb \\ kc & 1 - 2k \end{pmatrix},$$

which also has infinite order.  $\square$

*Proof of Theorem 10.1.* One can check that in the case  $B = \pm \begin{pmatrix} 0 & 1 \\ -c & 0 \end{pmatrix}$ , the seed pattern has 5 seeds if  $c = 1$ , 6 seeds if  $c = 2$ , and 8 seeds if  $c = 3$ .

Now assume  $bc \geq 4$ . Consider the seed pattern  $(\Sigma(t))_{t \in \mathbb{Z}}$  with  $\Sigma(t) = (x(t), y(t), B(t))$ . Label the cluster variables as a sequence  $z_t$  ( $t \in \mathbb{Z}$ ), where at each mutation step the new cluster variable gets the next index. The exchange relations become:

$$z_{t-1} z_{t+1} = \begin{cases} z_t^c + 1 & t \text{ even,} \\ z_t^b + 1 & t \text{ odd.} \end{cases}$$

Let  $\mathbb{U} = \{u^r \mid r \in \mathbb{R}\}$  be the semifield with

$$u^r \oplus u^s = u^{\max(r,s)}, \quad u^r \cdot u^s = u^{r+s}$$

( $u$  a formal variable).

**Aim:** Construct a semifield homomorphism  $\Psi: \mathcal{F} \rightarrow \mathbb{U}$  such that  $\{\Psi(z_t) \mid t \in \mathbb{Z}\}$  is infinite.

**Case  $bc > 4$ :** Let  $\lambda > 1$  be a real eigenvalue of  $\begin{pmatrix} bc-1 & -b \\ c & -1 \end{pmatrix}$ .

Put  $\Psi(z_1) = u^c$ ,  $\Psi(z_2) = u^{\lambda+1}$ .

The exchange relations become:

$$\Psi(z_{t-1}) \Psi(z_{t+1}) = \begin{cases} \Psi(z_t)^c \oplus 1 & t \text{ even,} \\ \Psi(z_t)^b \oplus 1 & t \text{ odd.} \end{cases}$$

**Claim 10.3.**  $\Psi(z_{2k+1}) = u^{\lambda^k c}$  and  $\Psi(z_{2k+2}) = u^{\lambda^k(\lambda+1)}$ .

Use induction. Writing  $\Psi(z_t) = u^{\alpha_t}$ :

$$\alpha_{2k+3} = c \alpha_{2k+2} - \alpha_{2k+1} = c \cdot \lambda^k(\lambda+1) - \lambda^k c = \lambda^k c(\lambda+1-1) = \lambda^{k+1} c.$$

$$\begin{aligned} \alpha_{2k+4} &= b \alpha_{2k+3} - \alpha_{2k+2} = b \cdot \lambda^{k+1} c - \lambda^k(\lambda+1) \\ &= \lambda^k(bc\lambda - \lambda - 1) \\ &= \lambda^k(\lambda^2 + \lambda) \quad (\text{using } \lambda^2 - (bc-2)\lambda + 1 = 0) \\ &= \lambda^{k+1}(\lambda+1). \end{aligned}$$

Since  $\lambda > 1$ ,  $\alpha_t \rightarrow \infty$ , so  $\{\Psi(z_t)\}$  is infinite.

**Case  $bc = 4$ :** Instead use  $\Psi(z_1) = u$ ,  $\Psi(z_2) = u^b$ .

**Claim 10.4.**  $\Psi(z_{2k+1}) = u^{2k+1}$  and  $\Psi(z_{2k+2}) = u^{(k+1)b}$ .

This also follows by induction. Since  $\alpha_t \rightarrow \infty$ , the seed pattern has infinitely many cluster variables.  $\square$

## 10.4 2-finiteness

**Definition 10.5.** A skew-symmetrizable matrix  $B = (b_{ij})$  is **2-finite** if for any  $B' = (b'_{ij})$  mutation equivalent to  $B$ , we have  $|b'_{ij} b'_{ji}| \leq 3$  for all  $i, j$ .

**Corollary 10.6.** *Finite type seed pattern  $\Rightarrow$  every exchange matrix is 2-finite.*

*Proof.* If  $B \sim B'$  with  $|b'_{ij} b'_{ji}| \geq 4$  for some  $i, j$ , then by freezing all the cluster variables in that seed except for  $x_i, x_j$ , we are reduced to the rank 2 case.  $\square$

**Remark 10.7.** It turns out the converse to the above corollary is also true!

## References

- [FWZ21] Sergey Fomin, Lauren Williams, and Andrei Zelevinsky. *Introduction to Cluster Algebras*. Chapters 1–6, arXiv:1608.05735. 2021.
- [Gro+18] Mark Gross, Paul Hacking, Sean Keel, and Maxim Kontsevich. “Canonical bases for cluster algebras”. In: *J. Amer. Math. Soc.* 31.2 (2018), pp. 497–608.