Final

Modern Algebra 1
Columbia University Fall 2019
Instructor: Kyler Siegel

Instructions:

- Please write your answers **in this printed exam**. You may use the back of pages for additional work. You may also use printer paper if you need additional space, but you must hand in all relevant work. Please turn in all scratch work which is relevant to your submitted answers.
- Suspected cases of copying or otherwise cheating will be taken very seriously.
- Solve as many problems of the following problems as you can in the allotted time, which is *two hours and fifty minutes.* I recommend first solving the problems you are most comfortable with before moving on to the more challenging ones. Note that the problems are not ordered by level of difficulty or topic.
- The exams will be graded on a curve. Therefore the raw score is not important, and you do not necessarily need to solve every problem to achieve a good grade. Just do your best!
- For true or false questions, you will receive $+2$ points for a correct answer, 0 points for no answer, and -3 points for an incorrect answer. This means **you should not make random guesses** unless you are reasonably sure that you know the answer. For the short answer questions, there is no penalty for wrong answers, and you do not need to justify your answers for full credit. For the short proof questions, you should be as precise and rigorous and possible.
- You may use any commonly used notation for standard groups, subgroups, and their elements, as long as it is completely unambiguous. If you are using nonstandard notation you must fully explain it for credit.
- Turn off all electronic devices. You may use the restroom if you must, but you may not take any devices with you.
- Good luck!!

Name: ______________________________

Uni: ______________________________

Question:	1	2	3	4	Total
Points:	46	80	32	0	158
Score:					

1. **True or false questions. Circle one. You do not need to provide any justification. There is a guessing penalty.**

(I) (2 points) Every group of order 73 is cyclic. A. True B. False

(II) (2 points) The groups

$$\mathbb{Z}/(30\mathbb{Z}) \times \mathbb{Z}/(2\mathbb{Z}) \times \mathbb{Z}/(6\mathbb{Z}) \times \mathbb{Z}/(3\mathbb{Z}) \times \mathbb{Z}/(5\mathbb{Z}) \times \mathbb{Z}/(5\mathbb{Z})$$

and

$$\mathbb{Z}/(2\mathbb{Z}) \times \mathbb{Z}/(10\mathbb{Z}) \times \mathbb{Z}/(2\mathbb{Z}) \times \mathbb{Z}/(3\mathbb{Z}) \times \mathbb{Z}/(3\mathbb{Z}) \times \mathbb{Z}/(15\mathbb{Z}) \times \mathbb{Z}/(5\mathbb{Z})$$

are isomorphic. A. True B. False

(III) (2 points) Let H and K be subgroups of a group G. Then $HK = \{hk : h \in H, k \in K\}$ is also a subgroup of G. A. True B. False

(IV) (2 points) Let $\mathbb{Q}$ denote the additive group of rational numbers, and let $\mathbb{Z}$ denote the subgroup of integers. Every element in the quotient group $\mathbb{Q}/\mathbb{Z}$ has finite order. A. True B. False

(V) (2 points) The set of positive integers $\mathbb{Z}_{\geq 1}$ forms of a group under multiplication. A. True B. False

(VI) (2 points) Let G be a group, and suppose that the map $G \to G$ sending $g \in G$ to $g^2 \in G$ is a homomorphism. Then G is an abelian group. A. True B. False

(VII) (2 points) Every simple abelian group is cyclic. A. True B. False

(VIII) (2 points) Any subgroup of order 35 in a group of order 105 is normal. A. True B. False

(IX) (2 points) Let H be a normal subgroup of a group G. Then H is the kernel of a surjective homomorphism whose domain is G. A. True B. False

(X) (2 points) The dihedral group D_{10} is isomorphic to a semidirect product of $\mathbb{Z}/(5\mathbb{Z})$ and $\mathbb{Z}/(2\mathbb{Z})$. A. True B. False

(XI) (2 points) The symmetric group S_4 is isomorphic to a semidirect product of A_4 and $\mathbb{Z}/(2\mathbb{Z})$. A. True B. False

(XII) (2 points) The alternating group A_n is simple for all positive integers n. A. True B. False

(XIII) (2 points) Every group of order 27 is abelian. A. True B. False

(XIV) (2 points) The center of the dihedral group D_{14} is trivial. A. True B. False

(XV) (2 points) Every group of order 49 is abelian. A. True B. False

(XVI) (2 points) Every group of order 77 is cyclic. A. True B. False

(XVII) (2 points) Every group of order 12 is isomorphic to a subgroup of S_{12}. A. True B. False

(XVIII) (2 points) There exists a simple group of order 360. A. True B. False

(XIX) (2 points) The permutation $\sigma \in S_8$ given by $\sigma = (1\ 5)(2\ 6\ 1)(5\ 3)(3\ 7\ 8)(4\ 5)$ lies in the alternating group A_8. A. True B. False

(XX) (2 points) The symmetric group S_{10} is generated by the subset of transpositions. A. True B. False

(XXI) (2 points) The group $(\mathbb{Z}/(16\mathbb{Z}))^{\times}$ is cyclic. A. True B. False

(XXII) (2 points) There exists a simple group of order 56. A. True B. False

(XXIII) (2 points) Let G be a group, let $\{e\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_r = G$ be a composition series, and let $\{e\} = G'_0 \trianglelefteq G'_1 \trianglelefteq \cdots \trianglelefteq G'_{r'} = G$ be another composition series. Then we must have $r = r'$. A. True B. False

2. **Short answer questions. You do not need to provide any justification. There is no penalty for wrong answers but you must box your final answer if it is not clear.**

(I) (4 points) How many subgroups does the group $\mathbb{Z}/(3\mathbb{Z}) \times \mathbb{Z}/(3\mathbb{Z})$ have?

(II) (4 points) Write out a composition series for the group $\mathbb{Z}/(20\mathbb{Z})$.

(III) (4 points) For which $n \in \mathbb{Z}_{\geq 1}$ is there a transitive action of S_4 on a set with n elements?

(IV) (4 points) Let G be a cyclic group of order 100, generated by an element $x \in G$. What is the order of x^{16}?

(V) (4 points) How many generators are there of the group $\mathbb{Z}/(1000\mathbb{Z})$?

(VI) (4 points) How many elements are there in the centralizer of $(1\ 3)(2\ 5)(4)$ in S_5?

(VII) (4 points) How many elements of order 5 are there in a nonabelian group of order 55?

(VIII) (4 points) How many subgroups of index 3 are there in $\mathbb{Z}/(15\mathbb{Z}) \times \mathbb{Z}/(63\mathbb{Z})$?

(IX) (4 points) How many conjugacy classes does S_5 have?

(X) (4 points) How many conjugacy classes does A_5 have?

(XI) (4 points) What is the order of the automorphism group of $\mathbb{Z}/(5\mathbb{Z}) \times \mathbb{Z}/(5\mathbb{Z})$?

(XII) (4 points) What is the order of the automorphism group of S_{12}?

(XIII) (4 points) How many subgroups are there of D_8 which contain the center?

(XIV) (4 points) How many composition series of the quaternion group Q_8 are there?

(XV) (4 points) How many homomorphisms are there from S_3 to D_8?

(XVI) (4 points) How many abelian groups are there of order $432 = 2^4 \cdot 3^3$ up to isomorphism?

(XVII) (4 points) Let H denote the subgroup of S_6 consisting of all permutations which fix 3 under the action of S_6 on $\{1,2,3,4,5,6\}$. What is the order of the normalizer of H in S_6?

(XVIII) (4 points) Let $\sigma \in S_8$ be the permutation given by $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 6 & 1 & 8 & 2 & 5 & 7 \end{pmatrix}$. What is σ^{100}?

(XIX) (4 points) How many actions are there of the group $\mathbb{Z}/(3\mathbb{Z})$ on the set $\{1,2,3,4\}$?

(XX) (4 points) Determine the order of the group with presentation

$$\langle x, y, z \mid x^2y^6z^2 = x^2yz,\ z^5 = z^2,\ xyz^5 = xyz,\ zy^8 = zy,\ yzx^5 = 1\rangle.$$

3. **Short proofs. Make your arguments as precise, rigorous, and clear as possible.**

(I) (8 points) Let G be a group, and let $S = \{xyx^{-1}y^{-1} \ : \ x, y \in G\}$ denote the subset of commutators in G. Let $H = \langle S \rangle$ denote the subgroup of G generated by S. Prove that H is a normal subgroup of G, and that the quotient group G/H is abelian.

(II) (8 points) Let p be a prime number and let G be a group of order p^2. Prove that G is abelian.

(III) (8 points) Prove that there is no simple group of order 72.

(IV) (8 points) Let p be a prime number and let G be a finite p-group, i.e. the order of G is a power of p. Suppose that G acts on a finite set X, and put $X^G := \{x \in X \ : \ g \cdot x = g \text{ for all } g \in G\}$. Prove that we have $|X^G| \equiv |X| \pmod{p}$.

4. **Bonus problems. These are worth very few points, so do not attempt unless you are quite confident with your answers to the previous problems.**

(I) (3 points (bonus)) Let G be a group, and let H_1 and H_2 be subgroups of G such that H_1 is not contained in H_2 and also H_2 is not contained in H_1. Prove that the union $H_1 \cup H_2$ is *not* a subgroup of G.

(II) (3 points (bonus)) Let G be a finite group with $|G| > 2$. Prove that G has a nontrivial automorphism.