Lecture 15

Airy's equation: $y'' = ty$

Note: this is 2nd order, linear but not constant coeffs

Ansatz: $y(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \dots$

$$= \sum_{k=0}^{\infty} a_k t^k$$

$$y' = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + \dots$$

$$= \sum_{k=1}^{\infty} a_k k t^{k-1}$$

note: first term disappeared

$$y'' = 2a_2 + 3\cdot 2 a_3 t + 4\cdot 3 a_4 t^2 + \dots$$

$$= \sum_{k=2}^{\infty} a_k k(k-1) t^{k-2}$$

$k=2$

plug into our ODE:

need $\displaystyle\underbrace{\sum_{k=2}^{\infty} k(k-1)a_k t^{k-2} - t\sum_{k=0}^{\infty} a_k t^k = 0}_{\text{need to write as single power series}}$ (for all t)

(then we can set all coeffs to be zero)

__Method 1__: long form, try to guess pattern

__Method 2__: keep in everything in summand notation.

__M1__: need

$$2\cdot 1\, a_2 + 3\cdot 2\cdot a_3 t + 4\cdot 3 a_4 t^2 + 5\cdot 4\cdot a_5 t^3 + \dots$$
$$- a_0 t - a_1 t^2 - a_2 t^3 - a_3 t^4 - \dots = 0$$

So need:

$$2a_2 + \left(3\cdot 2 a_3 - a_0\right)t + \left(4\cdot 3 a_4 - a_1\right)t^2 + \left(5\cdot 4 a_5 - a_2\right)t^3$$

$$+ \dots \left((k+2)(k+1)a_{k+2}\right)t^k + \dots$$

$$\left(-a_{k-1}\right. \qquad =$$

"general term"

Set all coeffs to 0:

- $2a_2 = 0$
- $3\cdot 2a_3 - a_0 = 0$
- $4\cdot 3a_4 - a_1 = 0$
- $5\cdot 4a_5 - a_2 = 0$

$\dots$

- $(k+2)(k+1)a_{k+2} - a_{k-1} = 0$

etc.

system of ∞ly many eqns and ∞ly many variables

can solve recursively

M2:

Recall: $\sum_{k=2}^{\infty} k(k-1)a_k t^{k-2} - t\sum_{k=0}^{\infty} a_k t^k = 0$

both power series but terms a bit messed up

In order to combine terms, need to write both in form

$$\sum_{k=0}^{\infty} (\quad) t^k.$$

$$\sum^{\infty} \qquad {}^{k} \qquad \sum^{\infty} \qquad {}^{k+1}$$

Firstly, $t\sum_{k=0} q_k t = \sum_{k=0} q_k t$

$$= \sum_{k=1}^{\infty} \left(q_{k-1}\right) t^{k}$$

Secondly,

$$\sum_{k=2}^{\infty} k(k-1) q_k t^{k-2} = \sum_{k=0}^{\infty} \left(\quad \right) t^{k}$$

power is $k-2$, let's put

$j = k-2$

$\Longrightarrow$ $k = j+2$

when $k=2$, $j=0$
when $k=\infty$, $j=\infty$

$$\sum_{k=2}^{\infty} k(k-1) q_k t^{k-2} = \sum_{j=0}^{\infty} (j+2)\cdot(j+1)\, q_{j+2}\, t^{j}$$

$$\sum_{j=0}^{\infty} (j+2)(j+1) q_{j+2} t^{j} = \sum_{k=0}^{\infty} (k+2)(k+1) q_{k+2} t^{k}$$

So need:

$$\sum_{k=0}^{\infty} (k+2)(k+1) q_{k+2} t^{k} - \sum_{k=1}^{\infty} q_{k-1} t^{k} = 0,$$

i.e.

$$2\cdot 1\cdot q_2 + \sum_{k=1}^{\infty} \begin{pmatrix} (k+2)(k+1) q_{k+2} \\ - q_{k-1} \end{pmatrix} t^{k} = 0$$

Set all coeffs to zero:

$2 \cdot 1 a_2 = 0$

$(k+2)(k+1)a_{k+2} - a_{k-1} = 0$ for $k = 1, 2, 3, \ldots$

System:

$2 \cdot 1 \cdot a_2 = 0$

$3 \cdot 2 \cdot a_3 - a_0 = 0$

$4 \cdot 3 a_4 - a_1 = 0$

$5 \cdot 4 a_5 - a_2 = 0$

etc

$a_2 = 0.$

$$a_3 = \frac{a_0}{3 \cdot 2}$$

$$a_4 = \frac{a_1}{4 \cdot 3}$$

$$a_5 = \frac{a_2}{5 \cdot 4} = 0$$

$$a_6 = \frac{a_3}{6 \cdot 5} = \frac{a_0}{6 \cdot 5 \cdot 3 \cdot 2}$$

Note: behavior "skips 3"

(1) $a_2 = a_5 = a_8 = a_{11} = a_{14} = \ldots = 0$

(2) $a_0 = ??$

$$a_3 = \frac{a_0}{3 \cdot 2}$$

$$a_6 = \frac{a_3}{6 \cdot 5} = \frac{a_0}{6 \cdot 5 \cdot 3 \cdot 2}$$

$$a_9 = \frac{a_6}{9 \cdot 8} = \frac{a_0}{9 \cdot 8 \cdot 6 \cdot 5 \cdot 3 \cdot 2}$$

$$\cdots\cdots$$

$$a_{3k} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3k-1)(3k)}$$

$$\cancel{= \frac{a_0 \, 4 \cdot 7 \cdot 10 \cdots}{(3k)!}}$$

(3) $a_1 = ?$

$$a_4 = \frac{a_1}{4 \cdot 3}$$

$$a_7 = \frac{a_4}{7 \cdot 6} = \frac{a_1}{7 \cdot 6 \cdot 4 \cdot 3}$$

$$a_{10} = \frac{a_7}{10 \cdot 9} = \frac{a_1}{10 \cdot 9 \cdot 7 \cdot 6 \cdot 4 \cdot 3}$$

$$\cdots$$

$$a_{3k+1} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdots (3k)(3k+1)}$$

So overall, have

$$y(t) = \sum_{k=0}^{\infty} a_k t^k$$

$$= a_0 + \frac{a_0}{2\cdot 3}t^3 + \frac{a_0}{2\cdot3\cdot5\cdot6}t^6 + \dots$$

$$+ a_1 + \frac{a_1}{3\cdot 4}t^4 + \frac{a_1}{3\cdot4\cdot6\cdot7}t^7 + \dots$$

$$= a_0\underbrace{\left(1 + \frac{t^3}{2\cdot3} + \frac{t^6}{2\cdot3\cdot5\cdot6} + \frac{t^9}{2\cdot3\cdot5\cdot6\cdot8\cdot9} + \dots\right)}_{y_1(t)}$$

$$+ a_1\underbrace{\left(t + \frac{t^4}{3\cdot4} + \frac{t^7}{3\cdot4\cdot6\cdot7} + \frac{t^{10}}{3\cdot4\cdot6\cdot7\cdot9\cdot10} + \dots\right)}_{y_2(t)}$$

So $y(t) = a_0 y_1(t) + a_1 y_2(t)$

this solves Airy's eqn for any a_0, a_1 !

In particular $y_1(t), y_2(t)$ are solns by themselves.

Questions:

(1) what are the radii of convergence ~~of~~ y_1 and y_2?

(2) do they form a fundamental set?

(3) what are these functions?? if not familiar, what do they

look like'?

Radius of convergence of y_1?

→ **Recall**: any power series $\sum_{k=0}^{\infty} a_k (t-t_c)^k$ has a radius of convergence, R, s.t.

the series converges (absolutely) for $|t-t_c|<R$,

i.e. $t_c - R < t < t_c + R$

i.e. $\sum_{k=0}^{\infty} |a_k (t-t_c)^k|$ converges

the series diverges for $|t-t_c|>R$.

Ratio test: If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists and is finite, then

$$R = \frac{1}{\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|}$$

(and $R=\infty$ if

$\left(\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 0 \right)$

Let's compute r.o.c. for

$$y_1(t) = 1 + \frac{t^3}{2\cdot 3} + \frac{t^6}{2\cdot 3\cdot 5\cdot 6} + \dots + \frac{t^{3k}}{2\cdot 3\cdot 5\cdot 6 \cdots (3k-1)(3k)} + \dots$$

$1 + 0t + 0t^2 + \frac{t^3}{2\cdot 3} + \dots \longrightarrow$

Slight issue:

what is $\frac{a_{n+1}}{a_n}$? $\frac{a_3}{a_2} = \frac{(1/2\cdot 3)}{0}$

$\frac{a_4}{a_3} = \frac{0}{(1/6)} = 0$

So $\frac{a_{n+1}}{a_n}$ oscillates b/w 0 and $\frac{\#}{0}$ ...

Fix: Put $s = t^3$.

Then $y_1(s) = 1 + \frac{s}{2\cdot 3} + \frac{s^2}{2\cdot 3\cdot 5\cdot 6} + \frac{s^3}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9} + \dots$

$$= \sum_{s=0}^{\infty} b_n s^n$$

So $\frac{b_{n+1}}{b_n} = \frac{2\cdot 3\cdot 5\cdot 6 \cdots (3n-1)(3n)}{2\cdot 3\cdot 5\cdot 6 \cdots (3n-1)(3n)(3n+2)(3n+3)}$

$= \frac{1}{\quad}$

$(3n+2)(3n+3)$

So $\lim_{n \to \infty} \left| \frac{b_{n+1}}{b_n} \right| = 0.$

So $y_1(s)$ converges for $|s| < \infty$, i.e. $y_1(t)$ converges for $|t^3| < \infty$, i.e. $|t| < \infty$, i.e. for all t.

So Roc for $y_1(t)$ is ∞!

Similarly can show R.o.c. of $y_2(t)$ is ∞!

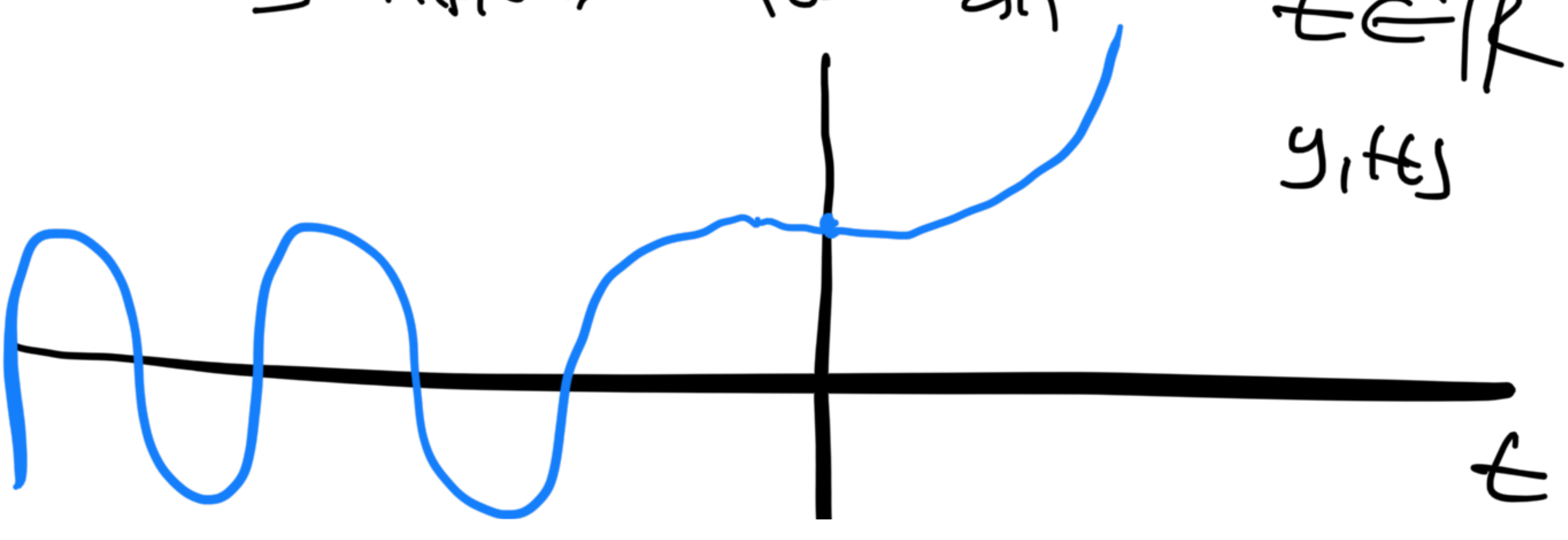
Lets compute $W(y_1, y_2)(0)$.

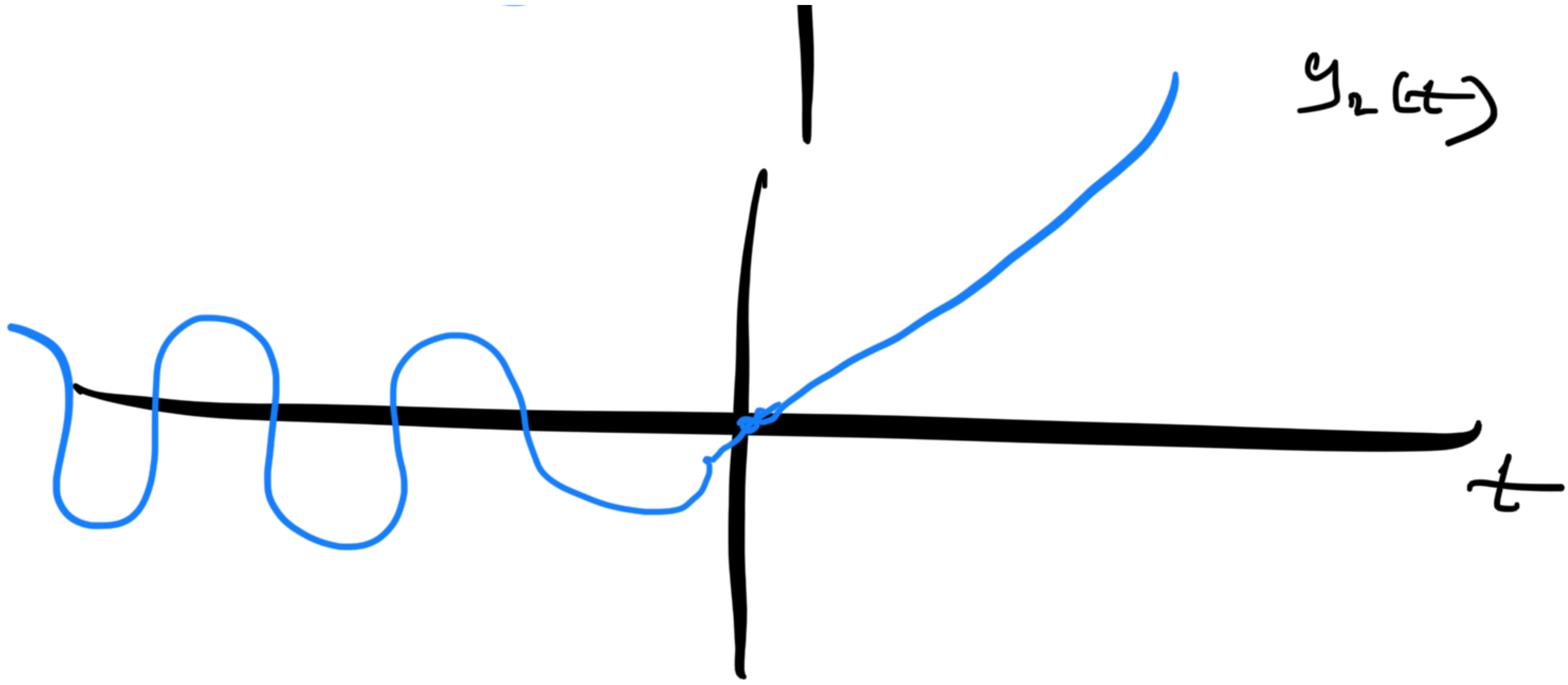
$y_1(0) = 1 \qquad y_1'(0) = 0$

$y_2(0) = 0 \qquad y_2'(0) = 1$

$$W(y_1, y_2)(0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$\Longrightarrow y_1, y_2$ form a fundamental set of solutions for all $t \in \mathbb{R}$.





In practice, to understand these functions, can approximate by taking first 1000 terms.