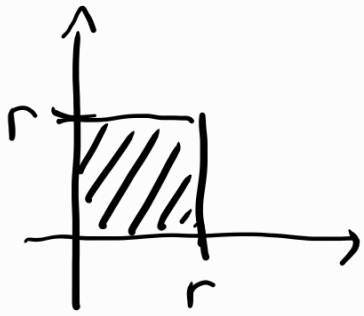
Lagrangian Classification & Applications

I.) Background: The Lagr. Markov tori

II.) Superpotential & Mutation

III.) Applications: Non-isotopic packings of <u>cubes</u> $X \overset{\text{sympl}}{\hookrightarrow} \mathbb{CP}^1 \times \mathbb{CP}^1$

$[0,r]^2$



I. Background

$$\mu_{\mathbb{CP}^2} : (\mathbb{CP}^2, \omega_{FS}) \to \mathbb{R}^2 \qquad \int_{\text{line}} \omega_{FS} = \pi$$

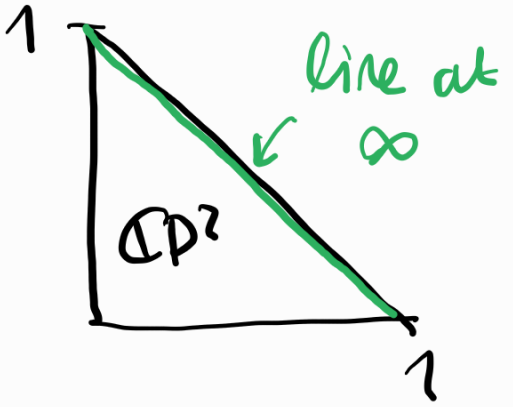
$$(z_1, z_2) \mapsto \left(\frac{\|z_1\|^2}{1+\|\bar{z}\|^2}, \frac{\|z_2\|^2}{1+\|\bar{z}\|^2} \right)$$

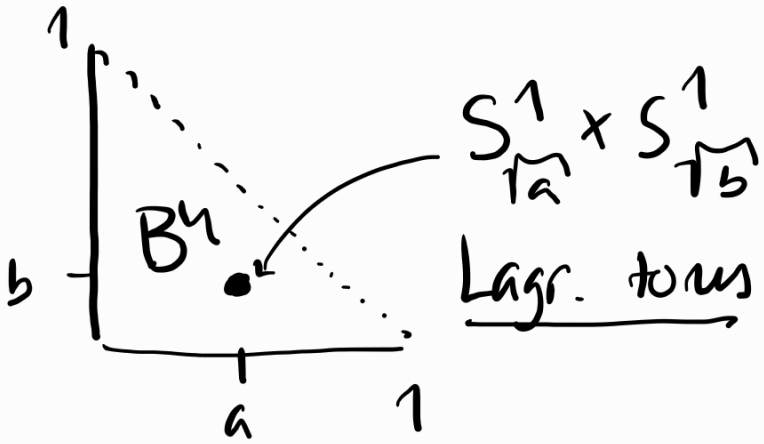
The standard <u>momentum map</u>

$$\mu : (B^4, \omega_0) \overset{\text{symplectically}}{\cong} (\mathbb{CP}^2 \setminus \infty, \omega_{FS}) \to \mathbb{R}^2$$

$$(z_1, z_2) \longmapsto (\|z_1\|^2, \|z_2\|^2)$$

OBS: <u>not</u> the affine coord^s





To have well-def. enumerative invariants (e.g. superpotential) we want: $\frac{\pi}{6}$ for $\mathbb{CP}^2$

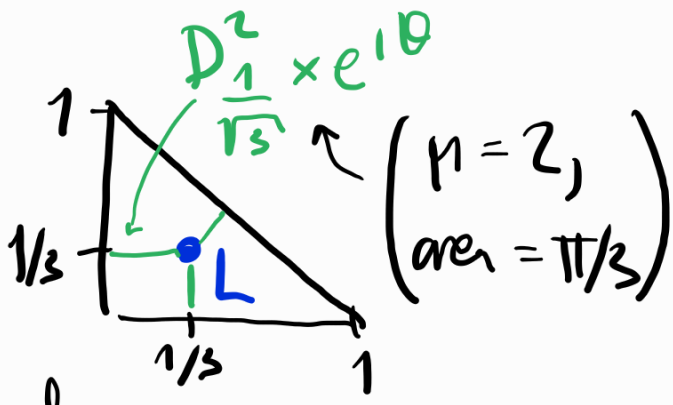
$$\boxed{L \subseteq (X, \omega) \ \underline{\text{monotone}} \iff \int_A \omega = c \cdot \mu(A) \quad A \in H_2(X, L)}$$

μ: Maslov class

$\mu(\Sigma) = 2c_1(\Sigma)$, Σ closed

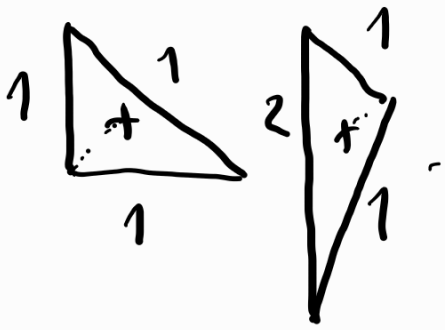
$\mu(D^2) = 2$

The <u>Clifford torus</u> is the <u>unique monotone fibre</u> of $\mu_{\mathbb{CP}^2}$



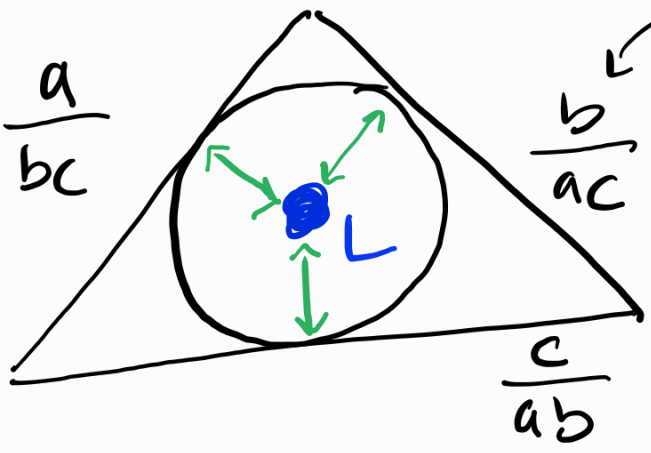
Ex [Symington, Galkin-Ushnich, Vianna]

$(\mathbb{CP}^2, \omega_{FS})$ has many <u>almost toric fibrations</u>, namely obtained after repeated mutation of the standard toric fibration $\mu_{\mathbb{CP}^2}$



They are classified by the <u>Markov triples</u>

$(a,b,c) \in (\mathbb{Z}_{>0})^3$, $a \leq b \leq c$, $a^2+b^2+c^2 = 3abc$



<u>affine</u> length of side

$L = L_{abc}$ fibre w. same affine distance to all sides (necc: $\frac{1}{3}$) is <u>monotone</u>

The Lagrangian classification problem

- Lots of Lagr. tori (no volume obstruction), e.g. in Darboux balls
- <u>Monotone tori</u> are harder to find
- All Lagr. tori in $\mathbb{C}^2$, $\mathbb{CP}^2$, $\mathbb{CP}^1 \times \mathbb{CP}^1$ are isotopic through Lagrangians

  [— — Goodman — Ivrii]
- Homologically essential Lagr. tori in $\mathbb{CP}^2 \setminus \left(\begin{smallmatrix} 2 \\ \text{or} \\ 3 \end{smallmatrix} - \text{nodal cubic}\right)$ are classified

  [— GI, —] : (one resp. two monotone)
- We want a gen. classification up to Ham. diffeo (= sympl. diffeo. if $\pi_1(X) = 1$)

<u>Thm</u> [Vianna] L_{abc} all live in distinct classes up to sympl. diffeo.

The invariant: The superpotential, which was later completely computed in

<u>Thm</u> [Pascaleff-Tonkonog] The superpotential of $L_{a,b,c}$ is obtained from that of $L_{1,1,1}$ (Clifford) by <u>mutation</u> of Laurent poly. $\leftrightarrow$ mutation of ATFs.

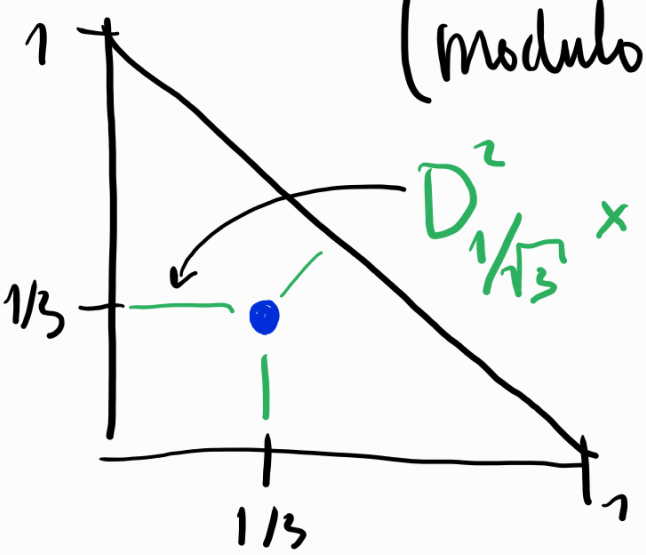
This confirms what [Galkin-Ushnich] concluded using the homological mirror symmetry conjecture.

II. Superpotential & mutation

For monotone Lagr. tori, the superpotential is well-behaved

- Can be defined $/\mathbb{C}$
- Is invariant under sympl. diffeo

$\mathcal{M}^J_{pt}(A)$ = moduli space of J-hol discs in class $A \in H_2(X,L)$ w. bdy passing through a generic pt $\in L$, (modulo reparam.)



$D^2_{1/\sqrt{3}} \times pt = u_1$

std a.c.s.

$\mathcal{M}^{i}_{pt}([u_1]) = \{u_1\}$

The superpotential of a monotone torus L:

$$P_L = \sum_{\mu(A)=2} \# \mathcal{M}^J_{pt}(A) \cdot e^{S(A)} \in \mathbb{C}[H_1(L;\mathbb{Z})]$$

$$\mathbb{C}[H_1(L;\mathbb{Z})] = \mathcal{O}(\mathrm{Hom}_{\mathbb{Z}}(H_1(L), \mathbb{C}^*)) = \mathcal{O}(\text{Flat } \mathbb{C}^*\text{-conn. on } L)$$

$\cap$

$\mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}]$ for any framing $H_1(L) = \mathbb{Z}x_1 \oplus \mathbb{Z}x_2$

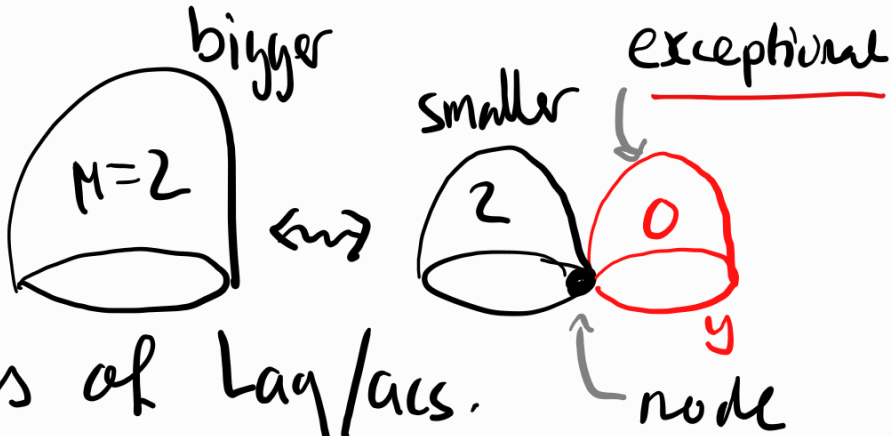
<u>Thm</u> [Arnol'd-Liouville] Any (local) momentum map $\mu : \mu^{-1}(U) \to U \overset{\text{open}}{\subseteq} \mathbb{R}^n$ w. <u>U simply conn.</u> and all fibres smooth (e.g. <u>part</u> of ATF):

$$\exists \quad (\mathbb{T}^n_\theta \times U_p, d(p \wedge \theta)) \xrightarrow{\cong} \mu^{-1}(U) \subseteq (X, \omega)$$

$$\begin{tikzcd} (\mathbb{T}^n_\theta \times U_p, d(p \wedge \theta)) \arrow[r, "\cong"] \arrow[dr, "\mathrm{proj}"'] & \mu^{-1}(U) \subseteq (X, \omega) \arrow[d, "\mu"] \\ & \mathbb{R}^n \end{tikzcd}$$

$\Rightarrow$ $\boxed{\text{fibres are framed tori}}$

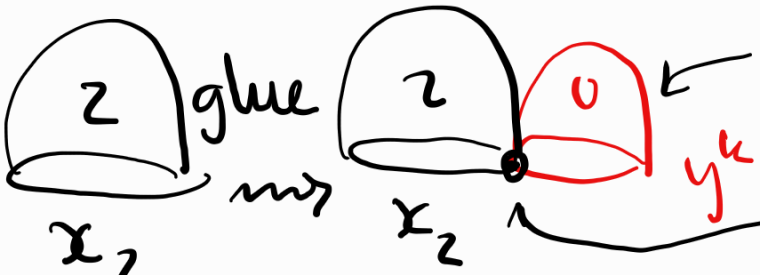
<u>Mutation</u>: For a non-monotone torus appearance of pseudohol. $\mu=0$ discs break invariance of $\mathcal{P}$ under deformations of Lag/acs.



<u>Assume</u>: single emb. psh. $\mu=0$ disc $y=\frac{x_1}{x_2}$

[Pascaleff-Tonkonog]: assume D has normal form: <u>sympl. push-off of a Lagr. disc</u>

Then



k: foled cover

int. nr $=+k>0$

$$\rightsquigarrow x_2\left(1-y+y^2-y^3+\dots\right)$$

$$=\frac{x_2}{1+y} \quad \text{(still Laurent!)}$$

We get a birational transformation:

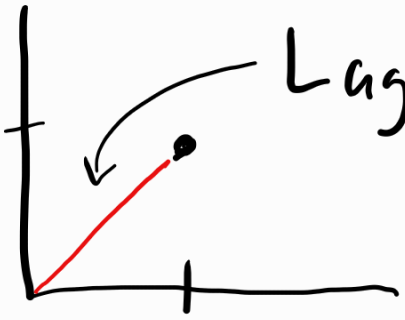
$$(\tilde{x}_2, y) = T_y(\tilde{x}_2, \tilde{y}) = \left(\frac{\tilde{x}_2}{1+\tilde{y}}, \tilde{y}\right)$$

<u>Rmk</u> The formula only dep^s on the boundary of the $\mu=0$ disc, i.e. y. <u>Indeed</u>:

$x \in H_1(L)$ s.t. $x \bullet y = 1 \Rightarrow x = x_2 \cdot y^m$

<u>Ex</u> Mutating Clifford to Chekanov

$\mu=0$ disc comes from emb. Lagr. disc



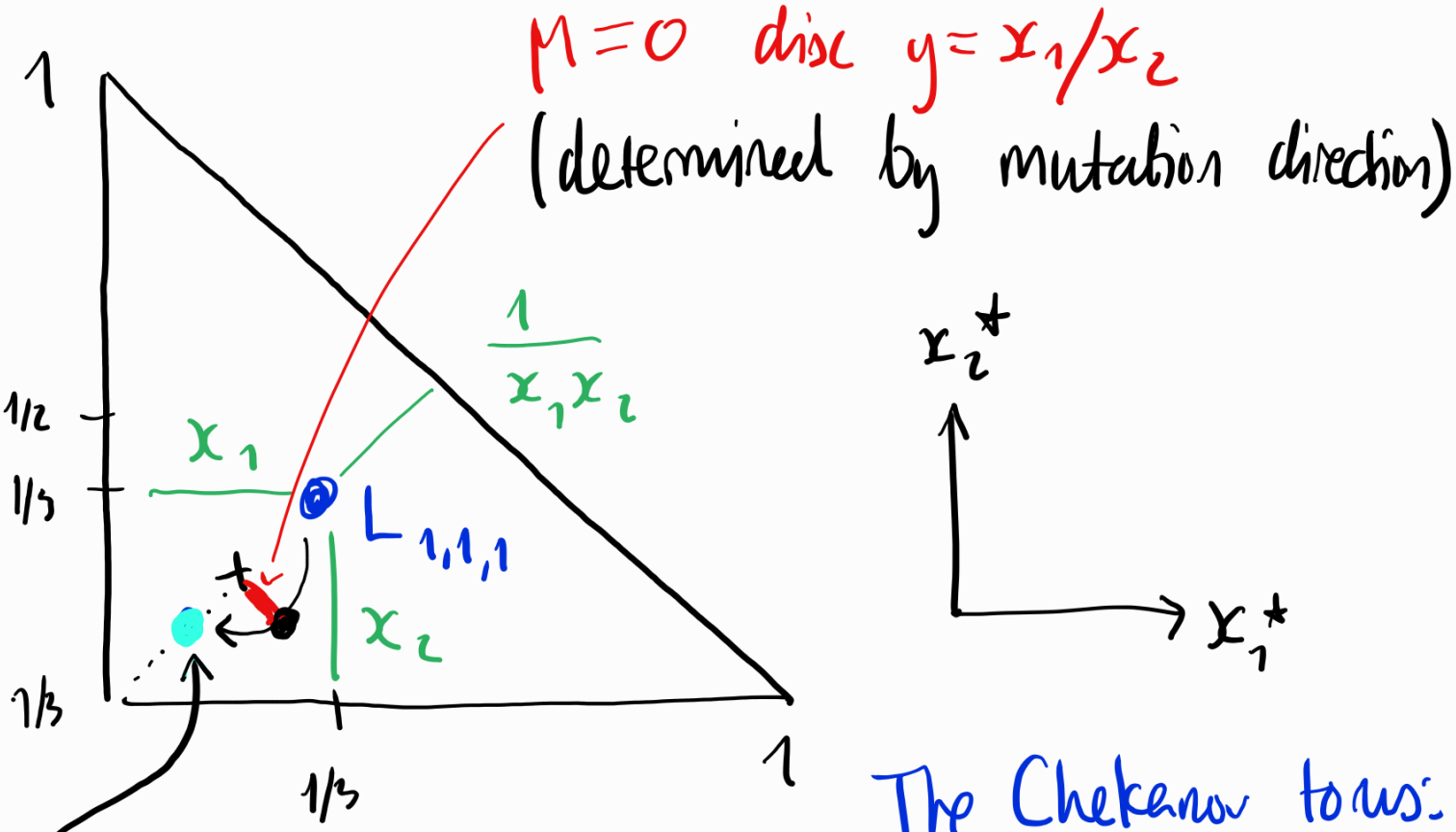
Lagr disc with bdy $y = \frac{x_1}{x_2}$

$$\mathcal{P}_{L_{1,1,1}} = x_2(1+y) + \frac{1}{x_2^2 y}$$

mutate

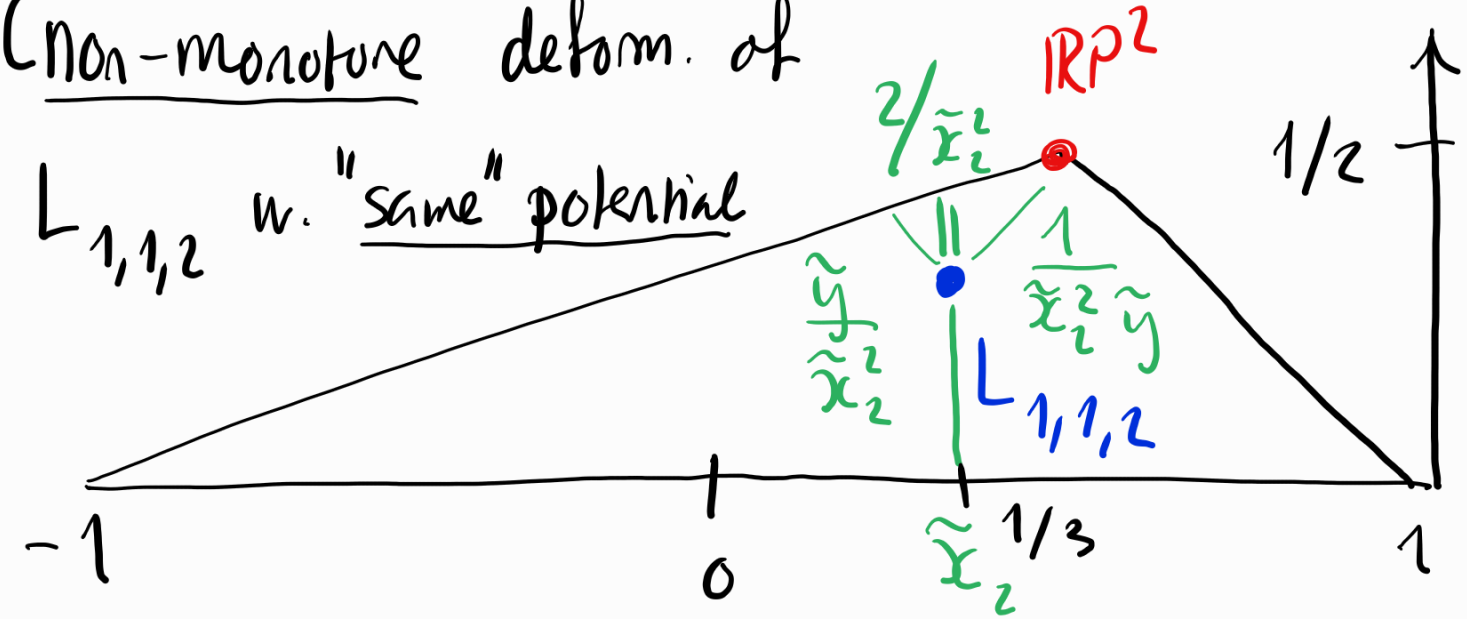
$$\mathcal{P}_{L_{1,1,2}} = \mathcal{P}_{L_{1,1,1}} \circ T = \tilde{x}_2 + \frac{(1+\tilde{y})^2}{\tilde{x}_2^2 \tilde{y}}$$

As depicted in the ATFs:



Non-monotone deform. of $L_{1,1,2}$ w. "same" potential

The Chekanov torus:



Conjectures

- Periods of P_L for $L \subseteq X$ monotone

$$\sum_{n \geq 0} \frac{1}{n!} \underbrace{\left[P_L^n\right]_0}_{\text{const. part}} \cdot t^n = \sum_{n \geq 0} \underbrace{GW_{g=0}(\psi^n)}_{\text{rat}^l \text{ GW-invts w. descendents}} \cdot t^n$$

Consequence: $\left[P_L^n\right]_0$ dep^s only on X

[See work by Tonkonog, but incomplete]

- The periods recover P_L / mutation

<u>In part</u>: same periods as $L_{1,1,1}$

$\Rightarrow P_L = P_{L_{abc}}$ for some Markov torus

(Opposite implication is known to be true)

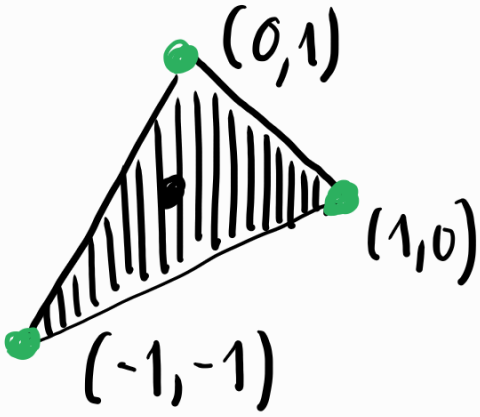
(N.B: This is a purely algebraic conj.)

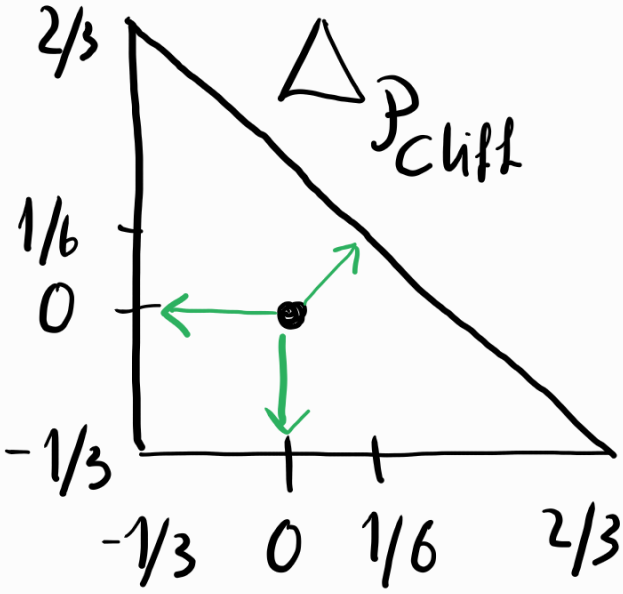
- The Newton polytope N_{P_L} of P_L determines a full vol. embedding of $\mathbb{T}^2 \times \Delta \subseteq \mathbb{T}^2 \times \mathbb{R}^2 = T^*\mathbb{T}^2$ (even ATF fibration on X?) where

$$\Delta_{P_L} = \{ \bar{p} \mid \bar{p} \bullet \bar{v} \geq -\tfrac{1}{3} \text{ for } \bar{v} \in N_{P_L} \}$$

$\sim$ polar of N

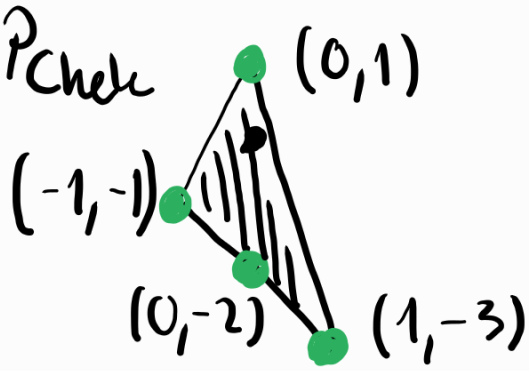
$N_{P_{Cliff}}$





$\Delta_{P_{Cliff}}$

$N_{P_{Chek}}$

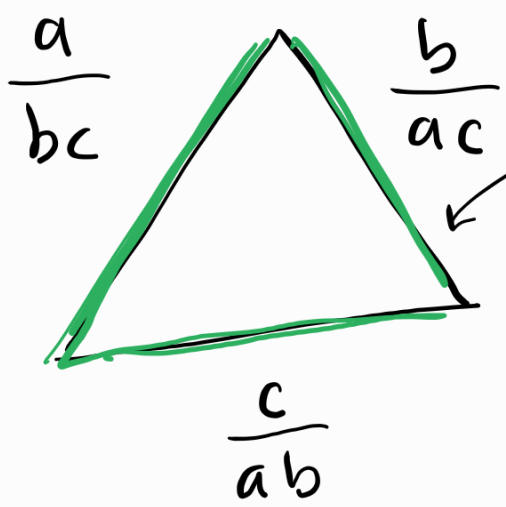


$\Delta_{P_{Chek}}$

- If $P_L = P_{L_{abc}}$, then $\exists$ Ham. diffeo ϕ

s.t. $\phi(L) \subseteq \mathbb{CP}^2 \setminus$ (smooth elliptic curve)

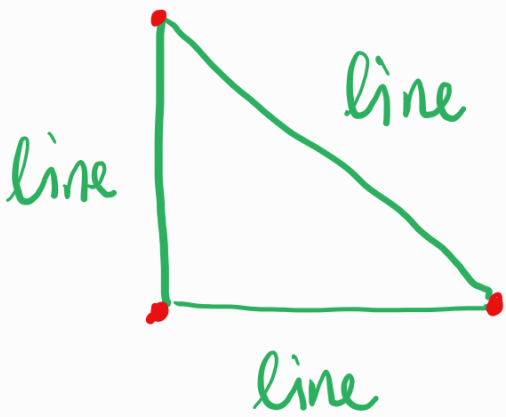
homologically essential



Elliptic curve

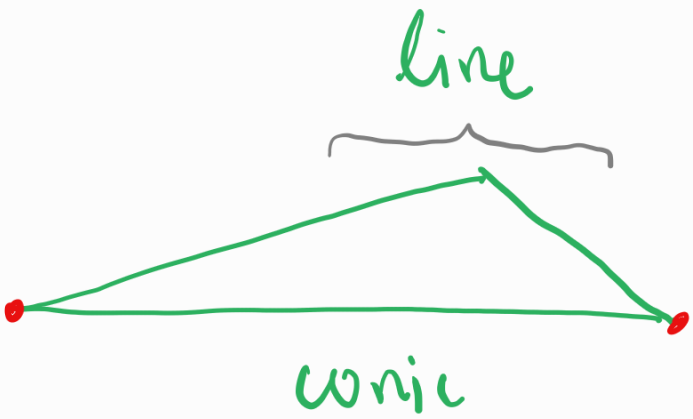
smooth if $a, b, c > 1$

(otherwise nodal)



3-nodal

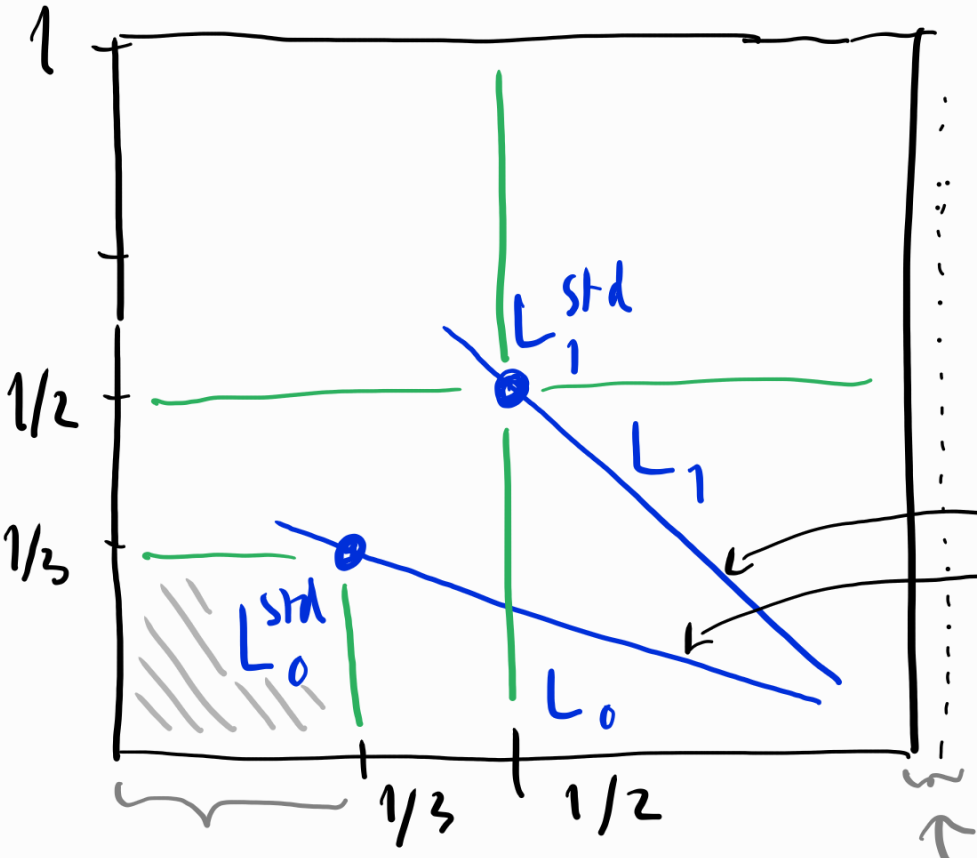
$(1, 1, 1)$



2-nodal

$(1, 1, 2)$

III. Applications to cube packings



$\mathbb{CP}^1 \times \mathbb{CP}^1$

these are not product tori

inflation makes L_i^{std} & L_i Hamiltonian iso.

$$\underbrace{X_{\Omega_0} = \left(D^2_{1/\sqrt{3}}\right)^2}$$

$$X_{\Omega_1} = \left(D^2_{1/\sqrt{2}}\right)^2$$

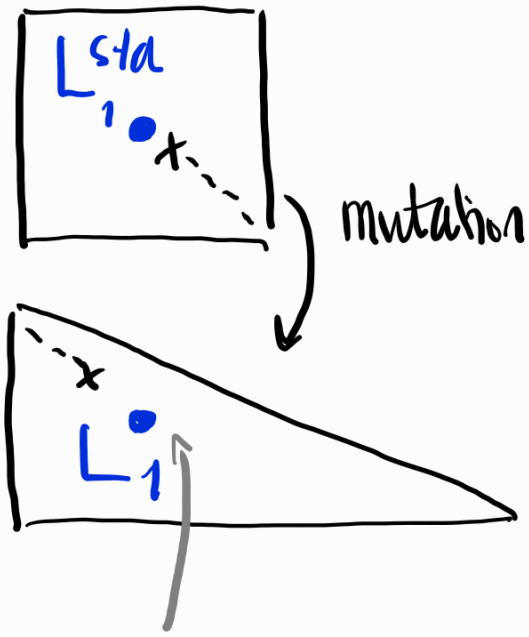
We can produce an <u>exotic</u> sympl. emb. of X_{Ω_i} whose "corner" is given by L_i.

$$X_{\Omega_i} \hookrightarrow (\mathbb{CP}^1)^2$$

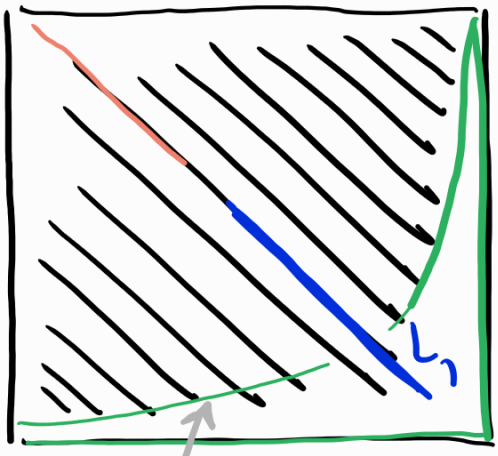
L_1 can be seen to be exotic by a <u>standard mutation</u>

$$\mathcal{P}_{L_1^{std}} = x(1+y) + \frac{1+y}{xy}$$

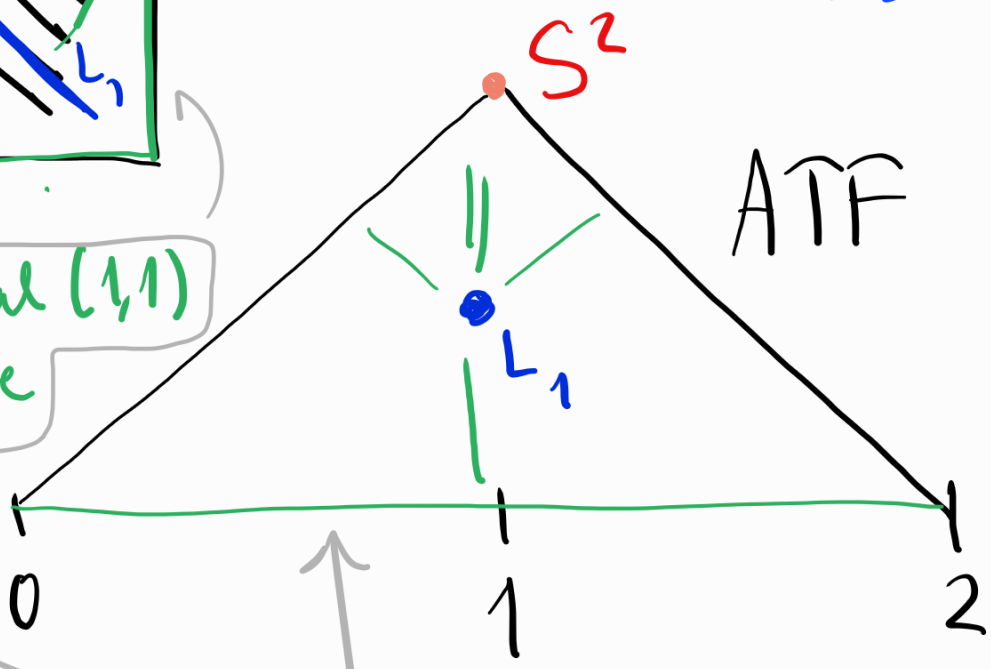
$$\mathcal{P}_{L_1} = x + \frac{(1+y)^2}{xy}$$



Chekanov-Schlenk
Eliashberg-Polterovich
torus



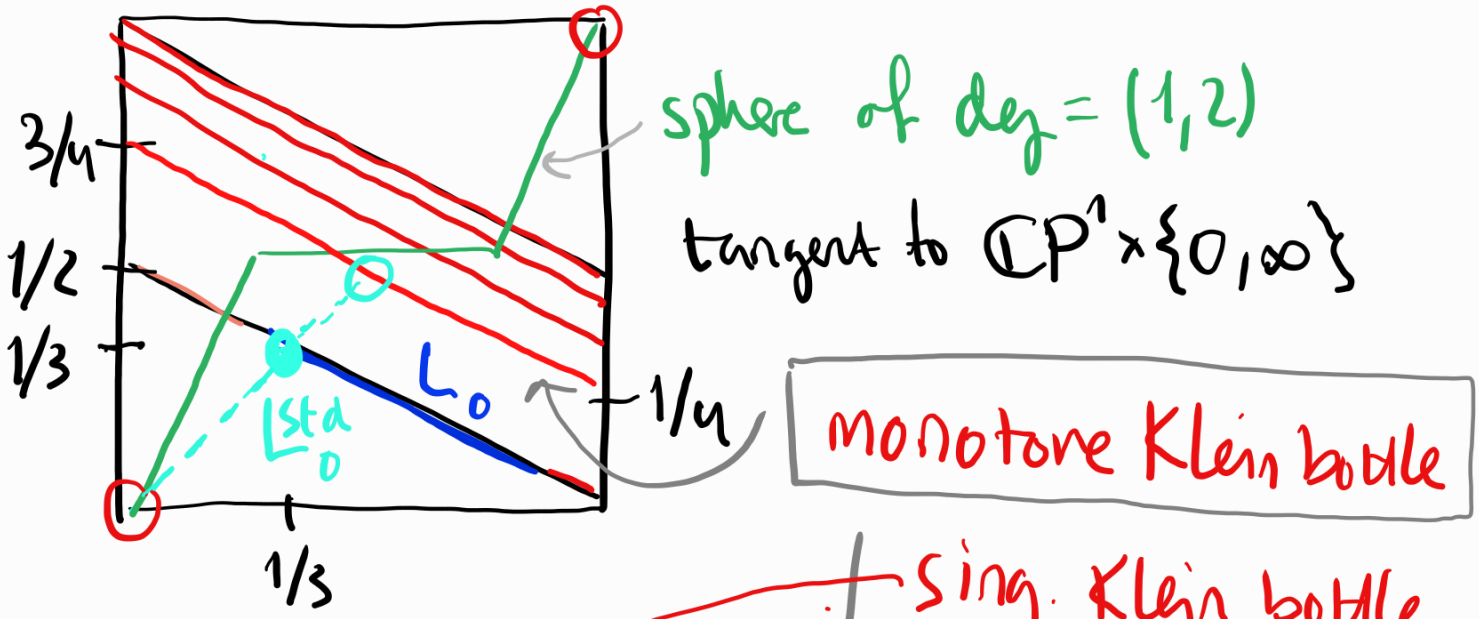
toric

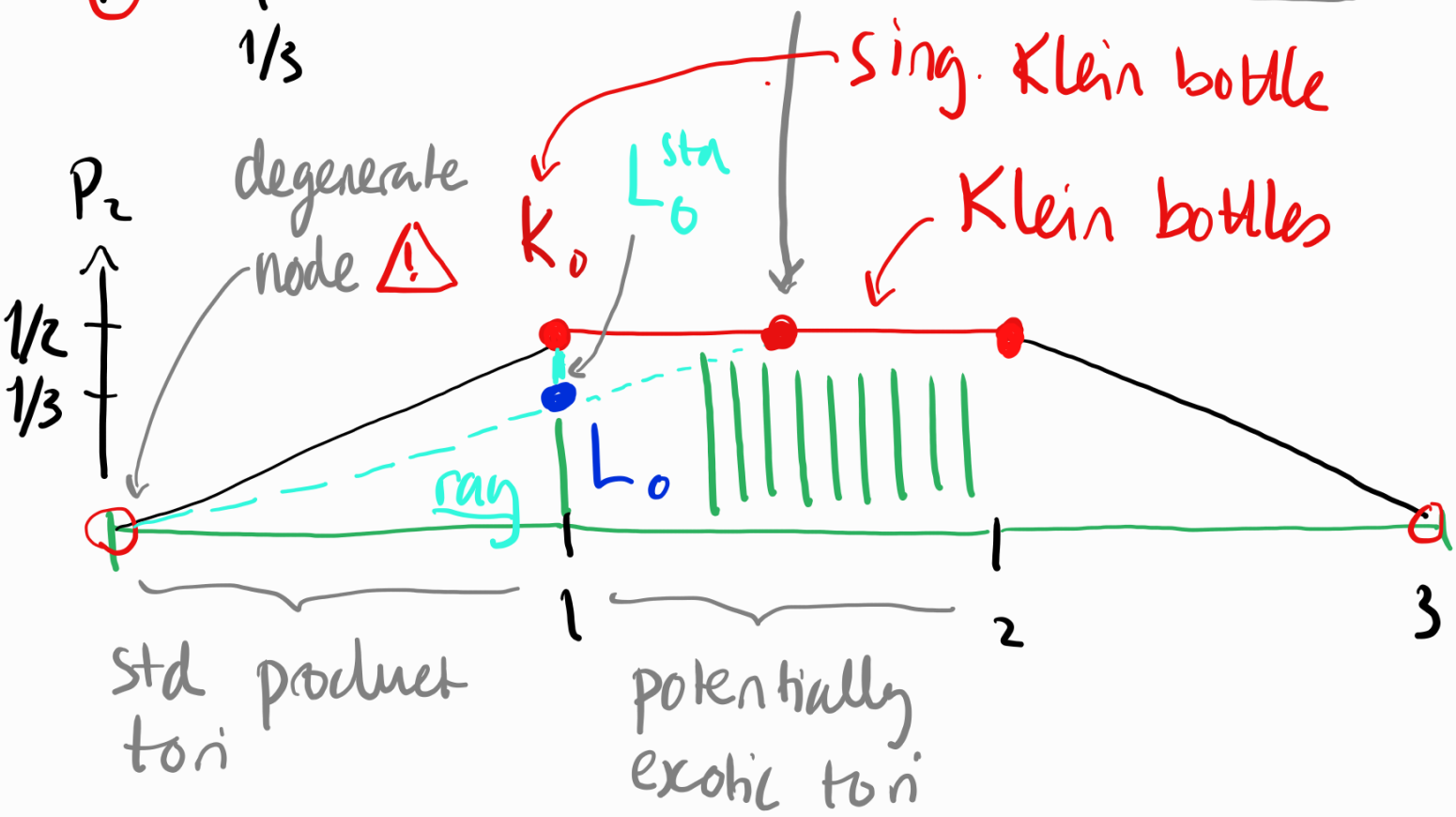


nodal (1,1) sphere

smooth (1,1)-sphere

L_0 is non-monotone, and not obtained by a standard mutation.

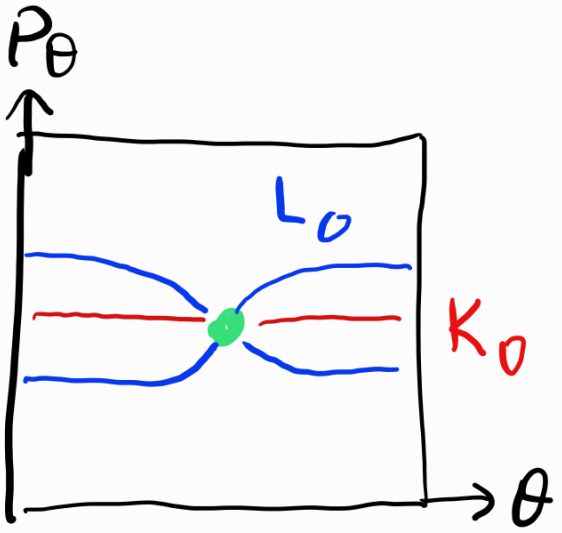


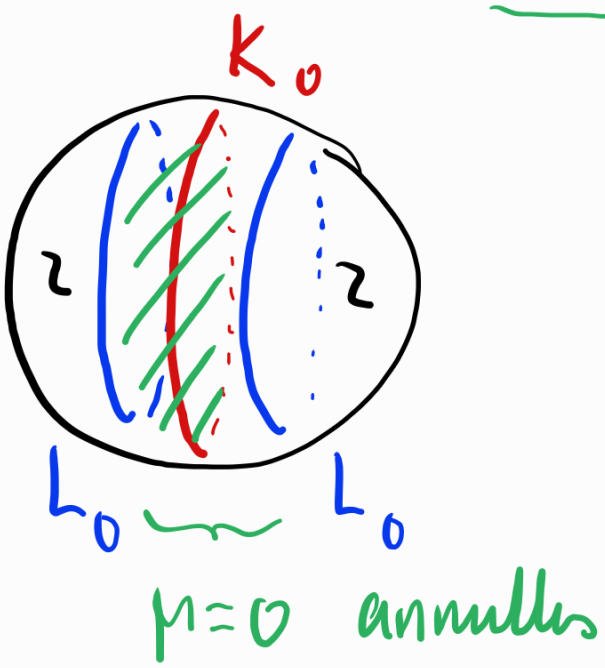


<u>Idea</u>: L_0^{std} can be deformed to std monotone torus $S^1_{1/\sqrt{2}} \times S^1_{1/\sqrt{2}}$ along ray

<u>Claim</u>: No such deformation exists for L_0

<u>Reason:</u> • L_0 admits a single psh
(sketch) $\mu=0$ annulus of <u>area $= 1/3$</u>





• This annulus can appear/reappear.
by breaking into $\mu=0$ disc w. self-node

• The area of these annulli $\to 0$ along <u>ray</u>

$\Rightarrow$ <u>if</u> tori stay embedded annulus breaks

• However, the $\mu=0$ disc deforms the potential

$\Rightarrow$ monotone limit is <u>not</u> $S^1_{1/\sqrt{2}} \times S^1_{1/\sqrt{2}}$