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Optimal Control of Systems Governed by PDEs

Heat Sink Shape Optimization using the Adjoint Method

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Outline

Overview of Shape Optimization

- Shape optimization is a branch of computational mechanics and within the field of optimal control theory.
- Data that describes the behavior of a shape (parameters or control variables) subject to governing laws/equations is obtained and one refines the structure until certain optimal properties are achieved. [5]

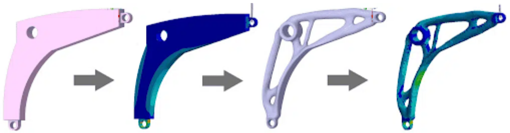


Fig. 1: Optimization of a bell crank to cut down material in low-stress areas[11]

Heat Sink Problem Formulation

- "Constant heat is emitted from a source to the heat sink profile (2D) which is surrounded by room temperature air. We want to determine the profile of the heat sink that's the Pareto optimal (for given weights) with respect to both the total area of the heat sink as well as the average temperature within the shape's domain after sufficient time has passed."
- Objective: Minimize $J_k = \lambda_1 \iint_{\Omega_k} dxdy + \lambda_2 \cdot \frac{1}{|\Omega_k|} \iint_{\Omega_k} u_k(x,y)dxdy$
- Constraints:

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} := \nabla^2 u = 0 & (x,y) \in \Omega_k \\ u = u_{\text{source}} & (x,y) \in \Gamma \\ -k\frac{\partial u}{\partial n} = h(u - u_\infty) & (x,y) \in \partial\Omega_k \backslash \Gamma \end{cases}$$

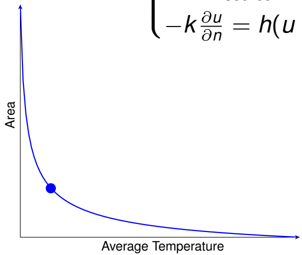


Fig. 2: Schematic of Pareto Front[3]

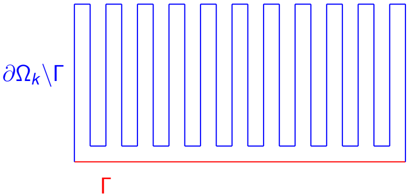


Fig. 3: Boundary of Heat Sink

The Finite Element Method

- The finite element method formulation frames a boundary value problem as a system of algebraic equations, [1]

$$Ku = F$$

where K is called the spring or stiffness matrix, u is the solution to the PDE, and F is called the load vector. Then

$$u = K^{-1}F$$

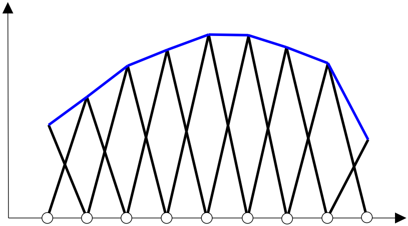


Fig. 4: Linear discretization of the 1D domain in elements and nodes. The approximate solution is described as a piecewise polynomial (blue line)[8]

Primer for the Adjoint Method—A Linear Algebra Perspective

Consider

$$\min_{x}\{C^T x\} \text{ subject to } Ax = b(S)$$

Then, assuming A is invertible,

$$x = A^{-1}b(S) \Rightarrow C^T x = C^T A^{-1} b(S)$$

Rather than solving $Ax = b(S)$ after each change of S and updating $C^T x$ accordingly, one can "pre-compute" $\hat{A} := C^T A^{-1}$ and optimize the original function by only varying S. [12]

$$\min_{S \in \Omega} \left\{\hat{A} b(S)\right\}$$

Overview of the Adjoint Method I

Consider an objective function $J(b_i)$, $i = 1, \ldots, N$ and constraint/state function $R(u_k) \equiv 0$, $k = 1, \ldots, M$. The total derivatives of both functions with respect to the design variables are given by [10, 7]

$$\frac{dJ}{db_i} = \frac{\partial J}{\partial b_i} + \frac{\partial J}{\partial u_k}\frac{du_k}{db_i} \qquad (1) \qquad\qquad \frac{dR}{db_i} = \frac{\partial R}{\partial b_i} + \frac{\partial R}{\partial u_k}\frac{du_k}{db_i} = 0 \qquad (2)$$

which can be equivalently expressed as

$$\frac{dJ}{db_i} = \frac{\partial J}{\partial b_i} + \psi_k \frac{\partial R}{\partial b_i}$$

such that

$$\frac{\partial J}{\partial u_k} + \psi_k \frac{\partial R}{\partial u_k} = 0 \qquad (3)$$

The value ψ_k is referred to as the adjoint, or costate variable. $\frac{dJ}{db_i}$ are then used by a gradient based optimizer to find the next shape.

Overview of the Adjoint Method II

- There are "efficient methods to obtain sensitivities of *many* functions with respect to a *few* design variables" (direct) as well as "efficient methods to obtain sensitivities of a *few* functions with respect to *many* design variables." (adjoint) [2]
- Consider the scenario when $J = \gamma^T u_k$ and u_k satisfies the linear system $Au_k = \phi$. The adjoint equation $A^T \psi = \gamma$ is first solved, from which $J = \psi^T \phi$. [10]
- For the proposed problem,

$$R(u_k) := K_k u_k - F = 0 \Rightarrow \frac{\partial R}{\partial u_k} = K_k \Rightarrow \psi_k = -K_k^{-1} \left(\frac{\partial J}{\partial u_k} \right)$$

$$\frac{\partial J}{\partial b_i} \approx \frac{J_k - J_{k-1}}{x_{i,k} - x_{i,k-1}}$$

$$\frac{\partial R}{\partial b_i} \approx \frac{K_k - K_{k-1}}{x_{i,k} - x_{i,k-1}} + \frac{U_k - U_{k-1}}{x_{i,k} - x_{i,k-1}} - \frac{F_k - F_{k-1}}{x_{i,k} - x_{i,k-1}}$$

Considerations for a Feasible Design I

In order for the model to converge to an optimal profile, the following constraints/choices need to be implemented

- Properly applying the shape gradient
  - Depending on its magnitude, pushing nodes according to their gradient may result in a tangled mesh. (Fig. 4) [4]
  - Defining a threshold for the gradient of the objective function with respect to the design variables, $\left|\frac{dJ}{db_i}\right| > \left|H_{\text{max}}\right|$ which, if met, will remove or add mesh elements.
- Only updating the boundary of the shape, not every point within the shape's domain.

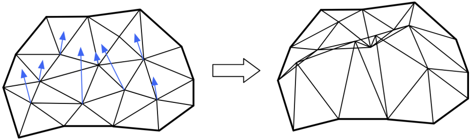


Fig. 5: Mesh tangling

Considerations for a Feasible Design II

- Normalizing the area and the average temperature. Recall the original objective function was:

$$J_k = \lambda_1 \iint_{\Omega_k} dxdy + \lambda_2 \cdot \frac{1}{|\Omega_k|} \iint_{\Omega_k} u_k(x,y)dxdy$$

which now becomes

$$J = \lambda_1 \frac{A_k}{A_{\text{ref}}} + \lambda_2 \frac{\bar{u}_k}{\bar{u}_{\text{ref}}}$$

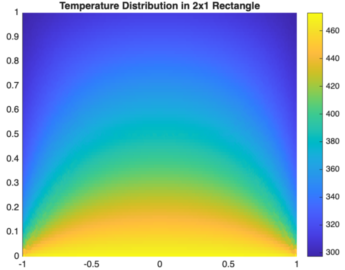


Fig. 6: Area $= 2$, avgTemp $= 372 \Rightarrow J = 374$

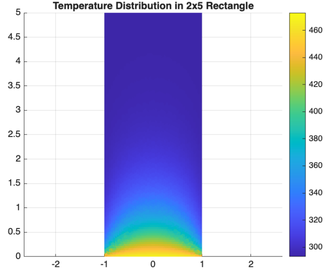


Fig. 7: Area $= 10$, avgTemp $= 317 \Rightarrow J = 327$

Setup for Numerical Scheme

1. Create and import initial 2D domain
2. Mesh domain using the Delaunay Triangulation Algorithm[6] (Fig. 7)
3. Solve primal equation (Laplace) with prescribed boundary conditions using the Finite Element Method[9]
   - $k = 237$ W/m $\cdot$ K (thermal conductivity for aluminum)[13]
   - $h = 7$ W/m^2 $\cdot$ K (natural convective heat transfer coefficient between air and aluminum)
4. Evaluate the objective function, J_0
5. Perturb boundary nodes to enable first gradient calculation

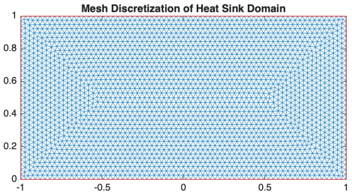


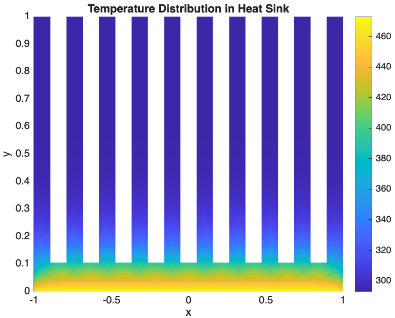
Fig. 8: Mesh with maximum height set to $H_{\max} = 0.03$

Numerical Scheme

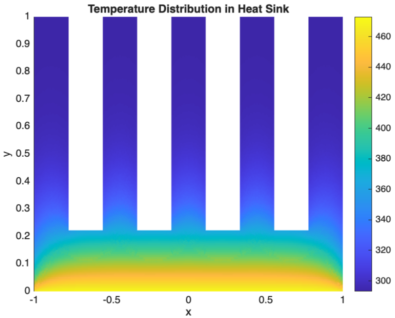
For $k = 1, \dots M$

1. Solve primal equation for u_k (Laplace)
2. Evaluate the objective function, $J[u_k]$
3. Solve the adjoint equation system
4. Compute the sensitivity information for each boundary node
5. Update the geometry based on Step 4 ($b_k = b_{k-1} - \alpha \frac{dJ}{db}$)
   - 5.1 If necessary, remesh domain
6. Repeat if $\left| J[u_k] - J[u_{k-1}] \right| > \epsilon$

Results I



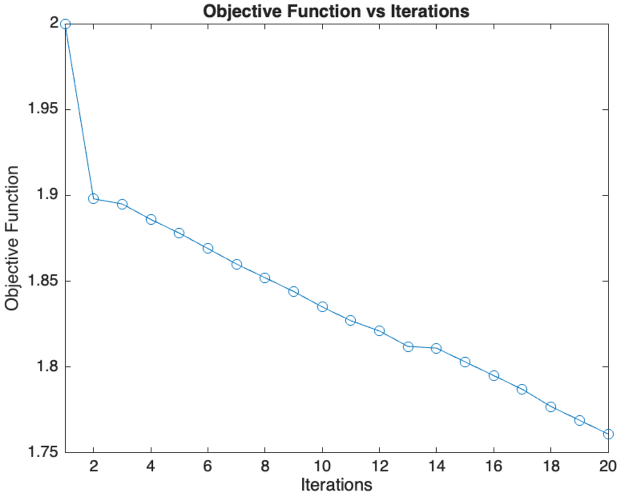
Area: 1.15, Average Temperature: 324.56



Area: 1.31, Average Temperature: 341.40

Results II

Results III


Objective Function vs Iterations
Objective Function
Iterations
2
1.95
1.9
1.85
1.8
1.75
2
4
6
8
10
12
14
16
18
20

Other Applications—Optimizing an Airfoil

- Although the Navier-Stokes Equations best describe the flow, because of the complexity, the flow is often modeled as two-dimensional *incompressible* and *inviscid*. Then the problem is to find a stream function, φ ($v =$ curl(φ) for the velocity, v) where $\varphi := \varphi(b_i)$ and constant $\beta := \beta(b_i)$ such that [5]

$$\begin{cases} -\Delta\varphi = 0 & (x, y) \in \Pi \backslash \Omega \\ \varphi = \beta & (x, y) \in \partial\Omega(b_i) \\ \varphi = \varphi_\infty & (x, y) \in \partial\Pi \\ \nabla\varphi \text{ is regular in a vicinity of the trailing edge, } b_N & \end{cases}$$

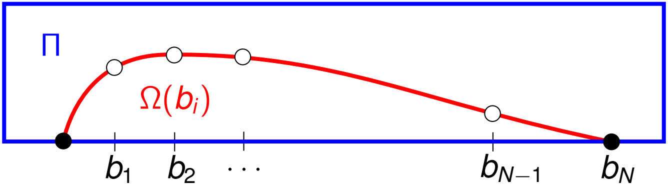


Fig. 9: Shape Parameterization of the upper surface of an airfoil. Solid points are fixed and open points are allowed to move in the y direction

References I

[1] May 2024. URL: https://en.wikipedia.org/wiki/Finite_element_method.

[2] Juan J Alonso. *A Crash-Course on the Adjoint Method for Aerodvnamic Shape Optimization*. Mar. 2005. URL: http://aero-comlab.stanford.edu/aa200b/lect_notes/aa200b-lect18.pdf.

[3] S. Brisset and F. Gillon. *Approaches for Multi-Objective Optimization in the Ecodesign of Electric Systems*. 2015. URL: https://www.sciencedirect.com/topics/engineering/pareto-optimality.

[4] Pascal Frey. *Shape Optimization*. URL: https://www.ljll.fr/~frey/cours/Teheran/P.%20Frey,%20chap7.pdf.

[5] J. Haslinger and R. A. E. Mäkinen. *Introduction to Shape Optimization: Theory, Approximation, and Computation*. SIAM, Society for Industrial and Applied Mathematics, 2003.

[6] *MATLAB PDE Toolbox Initial 2D Mesh*. URL: https://www.mathworks.com/help/pde/ug/initmesh.html.

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[7] Dimitri J. Mavriplis. *Adjoint-Based Sensitivity Analysis for Computational Fluid Dynamics*. URL: https://www.cerfacs.fr/musaf/PDF_MUSAF_DAY3/MAVRIPLIS/Mavriplis.pdf.

[8] M. Rovitto. *Principles of Finite Element Method*. URL: https://www.iue.tuwien.ac.at/phd/rovitto/node63.html.

[9] Gilbert Strang. URL: http://math.mit.edu/~gs/cse/codes/femcode.m.

[10] Dominique Thévenin and Gábor Janiga. *Optimization and Computational Fluid Dynamics*. Springer-Verlag Berlin Heidelberg, 2008.

[11] *Topology Optimization of a Bell Crank*. URL: https://courses.ansys.com/index.php/courses/topology-optimization-using-ansys-mechanical/lessons/topology-optimization-of-a-bell-crank-lesson-1/.

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[12] Qiqi Wang. *Introduction to the Adjoint Method*. May 2018. URL: https://www.youtube.com/watch?v=EybH_Q-QTZ8&list=LL&index=10&t=369s.

[13] Ailing Zhang and Yanxiang Li. *Thermal Conductivity of Aluminum Alloys*. Apr. 2023. URL: https://www.ncbi.nlm.nih.gov/pmc/articles/PMC10144406/.