Motion Planning and Control for Robotics

Trajectory Optimization

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October 5, 2020

RoboJackets

Getting Started

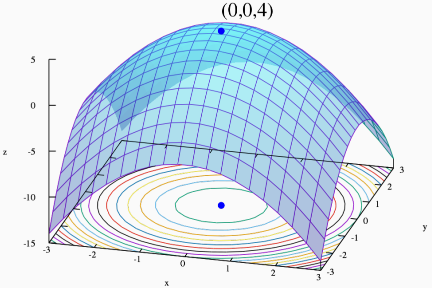
Exercises: *https://bit.ly/2S2v58j*

Review

Review: Optimization

Mathematical optimization is the process of finding the minimum or maximum value of some function, as well as the argument that gives that minimum or maximum.

$$4 = \max_{x,y} 4 - x^2 - y^2$$

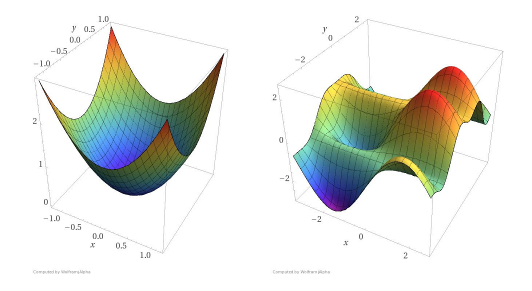


Review: Convexity

A function is **convex** if for any two points p and q on the function, the line between p and q lies entirely above the graph of the function.

A convex function has a single global minimum.

$$q \leq x \leq p \implies f(x) \leq (1-t)f(q) + tf(p)$$

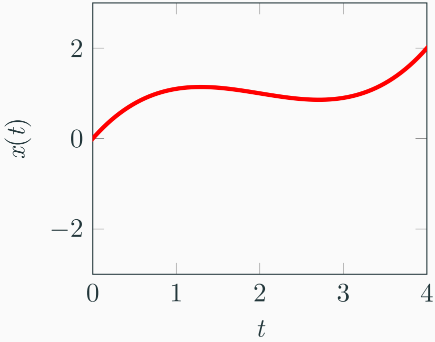


Review: Terminology

Name	Symbol	Description
Dynamics	$f(x_t, u_t)$	Evolution of the state.
Running cost	$\ell(x, u)$	The instantaneous cost (rate).
Horizon	T	How many timesteps to compute.
Terminal cost	$\Phi(x)$	One-time cost for the final state.
Cost-to-go	$J_t(x)$	The minimum remaining total cost if the system is in state x at time t.

Review: Bellman Equation and Cost-to-go

If we have an optimal trajectory $x^*(t)$...



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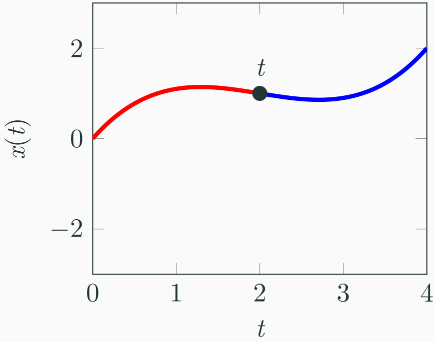
If we have an optimal trajectory $x^*(t)$...

And we break it up into parts...

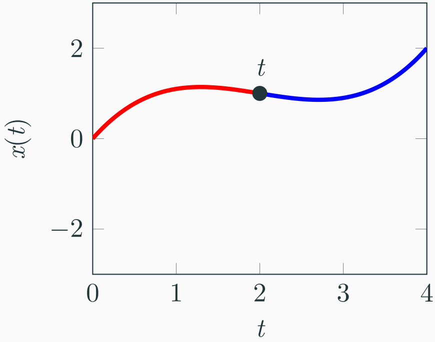


Review: Bellman Equation and Cost-to-go

If we have an optimal trajectory $x^*(t)$...

And we break it up into parts...

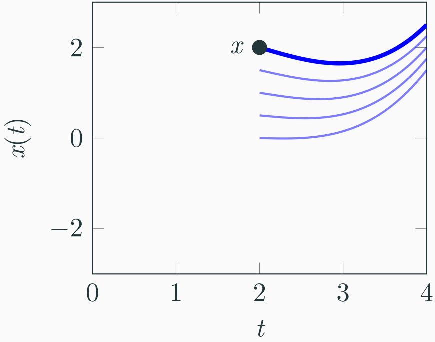
The piece starting at t and ending at T is optimal, for any trajectory starting in x at t.



Review: Bellman Equation and Cost-to-go

The **cost-to-go** at time t is the function $J_t(x)$ that gives you the optimal cost, for starting in an arbitrary state x, at time t.

$$J_t(x) = \min_u \left[\int_t^T \ell(x, u)\, dt + \Phi(x_T) \right]$$



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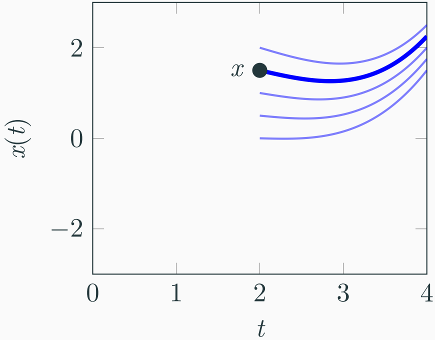

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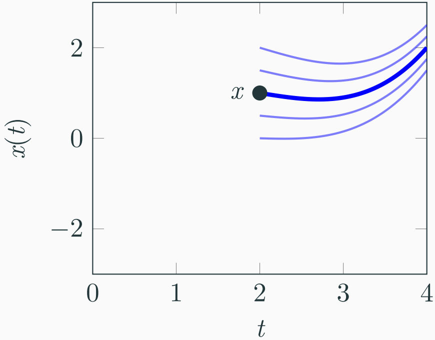

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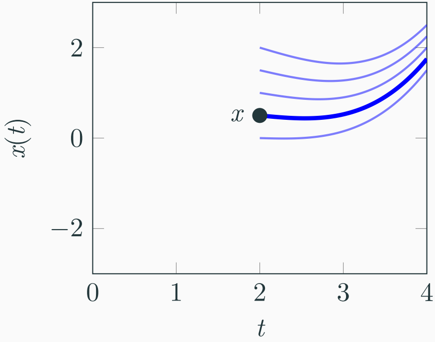

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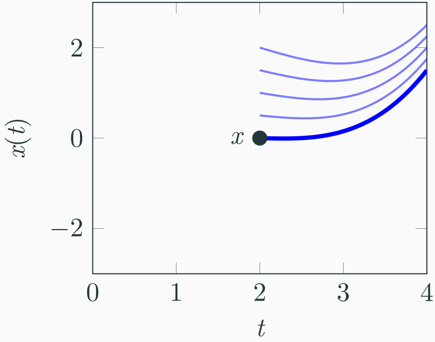
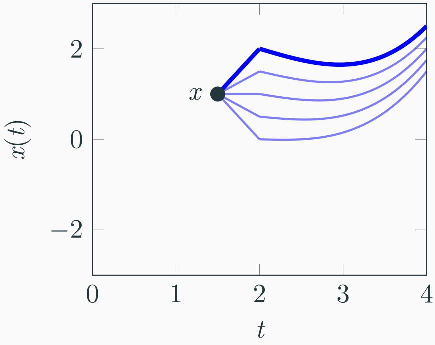


Review: Bellman Equation and Cost-to-go

If we know all possible best-case paths at t, we can find the best cases for a particular state at x_{t-1} (i.e., calculate $J_{t-1}(x)$ for a particular value of x).

$$J_{t-1}(x) = \min_u \left[\ell(x, u) + J_t(f(x, u))\right]$$



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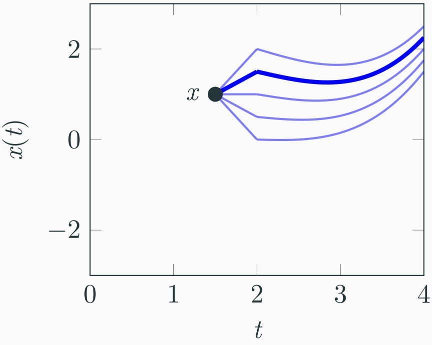

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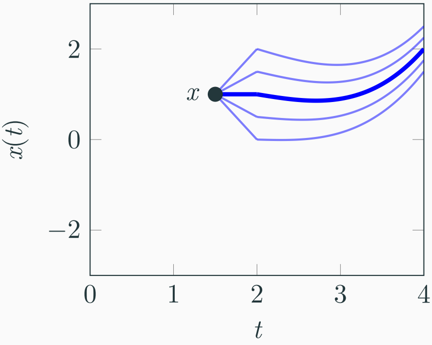

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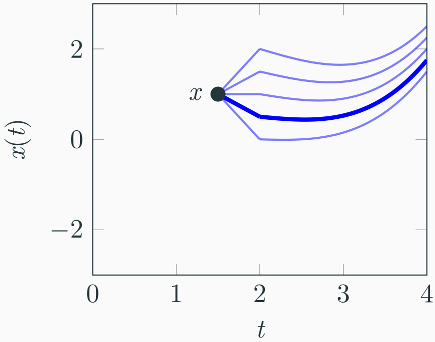

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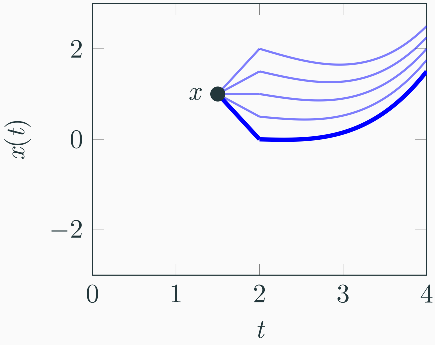


Review: LQR

$$x_{t+1} = f(x_t, u_t) = Ax_t + Bu_t$$

$$\ell(x, u) = \frac{1}{2}x^T Qx + \frac{1}{2}u^T Ru$$

$$\Phi(x_T) = \frac{1}{2}x_t^T Q_f x_t$$

$$J_t(x) = \frac{1}{2}x_t^T S_t x_t$$

Update equation:

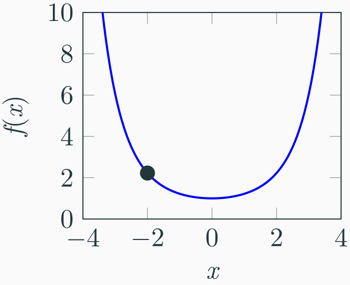
$$S_T = Q_f$$

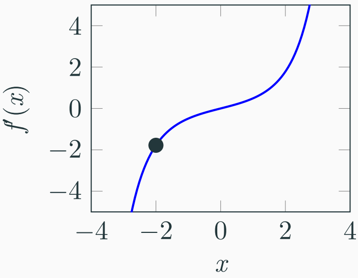
$$S_{t-1} = A^T S_t A - A^T S_t B(R + B^T S_t B)^{-1} B^T S_t A + Q$$

Iterative Methods and iLQR

Newton's Method: Iterative Approximation

To minimize a function, we find its derivative and find where it is zero. Newton's method is a way of *iteratively* solving that (or any) equation.





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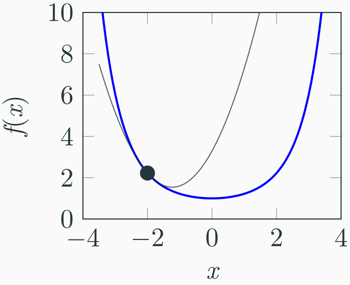

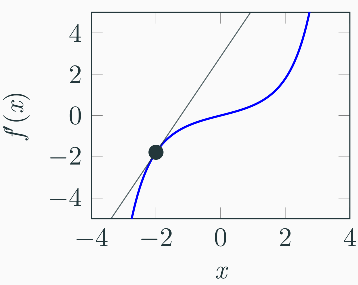

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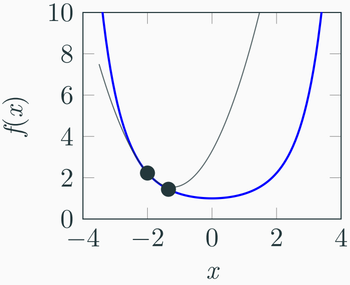

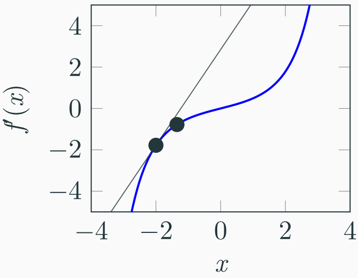

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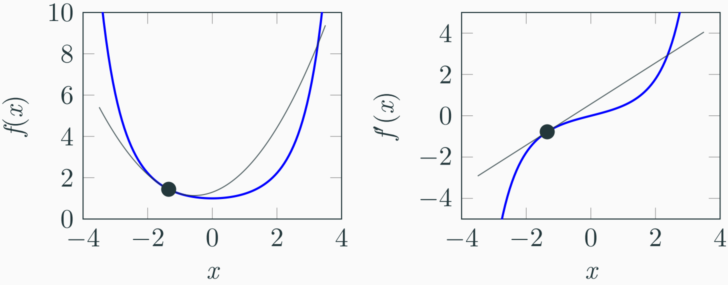

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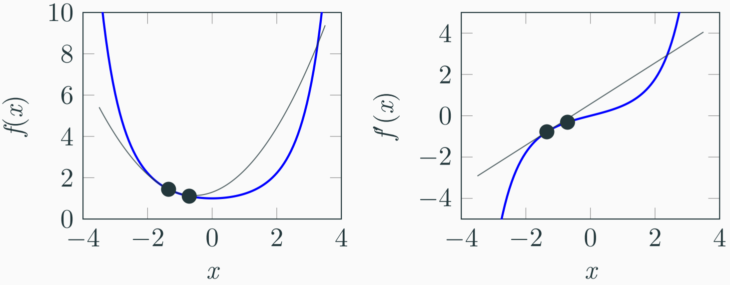


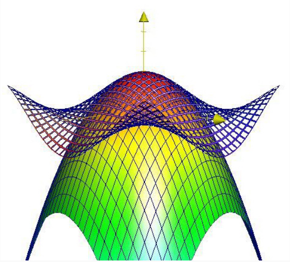
Taylor Series in Multiple Dimensions

For a function $f : \mathbb{R}^n \to \mathbb{R}$, we can approximate it around a point x_0 as...

$$f(x) \approx \frac{1}{2}(x - x_0)^T \text{hess}(f)(x - x_0) + \text{grad}(f)(x - x_0) + f(x_0)$$

In this case, hess f is like the "second derivative" (matrix of partial derivatives $\frac{\partial^2 f}{\partial x_i \partial x_j}$):

$$\text{hess} f = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{pmatrix}$$



Jacobian Matrix

We can also approximate a function $f\colon \mathbb{R}^n \to \mathbb{R}^n$ with the *Jacobian matrix* $\frac{\partial f}{\partial x}$, which is the matrix $\frac{\partial f_i}{\partial x_j}$.

$$\frac{\partial}{\partial x} f = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

This gives us $f(x) \approx \frac{\partial f}{\partial x}(x - x_0) + f(x_0)$.

Iterative LQR

Idea: approximate dynamics as linear and cost as quadratic, then solve LQR!

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Problem: Where do we linearize/quadratize around?

Answer: The trajectory from the last iteration!

Iterative LQR

Intuition: set up some "nominal trajectory" that follows the full nonlinear dynamics. Then create a "delta trajectory" $\delta x, \delta u$ that evolves based on the *linearized* dynamics (around the nominal trajectory).

1. $\delta x = 0, \delta u = 0$ is a fixed point
2. $\delta \ell$ is the *change* in running cost from the nominal trajectory
3. δJ_t is the *change* in cost-to-go from the nominal trajectory

We still want to minimize δJ_t (make as large of change as possible, in the negative direction).

We will propagate *cost-to-go* backwards (as usual), and then propagate *dynamics* forwards to find a new nominal trajectory.

Iterative LQR

Notation: drop time index subscripts for t ($x = x_t$), let prime denote $t+1$ (i.e. $J_{t+1} = J'$). Denote partial derivatives with subscript x and u.

$$\delta u = u - u^*, \delta\ell(\delta x, \delta u) = \ell(x, u) - \ell(x^*, u^*), \delta J = J(x) - J(x^*)$$

$$\delta x_{t+1} = x_{t+1} - x^*_{t+1} \approx f_x \delta x + f_u \delta u$$

$$\delta\ell(x, u) \approx \frac{1}{2}\delta x^T \ell_{xx}\delta x + \ell_x \delta x + \frac{1}{2}\delta u^T \ell_{uu} + \ell_u \delta u + \delta x^T \ell_{xu}\delta u$$

Iterative LQR

Bellman Equation:

$$\delta J_t(x) \approx \min_{\delta u}[\frac{1}{2}\delta x^T \ell_{xx}\delta x + \ell_x \delta x + \frac{1}{2}\delta u^T \ell_{uu}\delta u + \ell_u \delta u + \delta x^T \ell_{xu}\delta u + \delta J_{t+1}(f_x \delta x + f_u \delta u)]$$

Iterative LQR

We can solve in the same way we solved LQR, but it's just a bunch of messy algebra...

Define Q as the the inside of the $\min$ above:

$$Q(\delta x, \delta u) = \frac{1}{2}\delta x^T \ell_{xx} \delta x + \ell_x \delta x + \frac{1}{2}\delta u^T \ell_{uu} \delta u + \ell_u \delta u + \dots$$

And group terms:

$$\begin{aligned} Q_{uu} &= \ell_{uu} + f_u^T J'_{xx} f_u \quad & Q_u &= \ell_u + J'_x f_u \\ Q_{xx} &= \ell_{xx} + f_x^T J'_{xx} f_x \quad & Q_x &= \ell_x + J'_x f_x \end{aligned}$$

$$Q_{xu} = \ell_{xu} + f_x^T J'_{xx} f_u$$

Exercise (4 minutes): Calculate Q in the notebook.

Find the minimizing value of δu by taking the gradient and setting it to zero:

$$0 = \partial_u Q(x, u) = \delta u^T Q_{uu} + \delta x^T Q_{xu} + Q_u$$

$$\delta u = -Q_{uu}^{-1}(Q_{xu}^T \delta x + Q_u^T)$$

We call $K\delta x = -Q_{uu}^{-1} Q_{xu}^T \delta x$ the *feedback* term and $k = -Q_{uu}^{-1} Q_u^T$ the *feedforward* term.

Exercise (2 minutes): Calculate K and k in the notebook.

Then substitute the our new value for δu into the Bellman equation for δJ:

$$\begin{aligned}\delta J_t(x) &= \frac{1}{2}\delta u^T Q_{uu}\delta u + Q_u\delta u + \frac{1}{2}\delta x^T Q_{xx}\delta x + Q_x\delta x + \delta x^T Q_{xu}\delta u \\ &= \frac{1}{2}\delta x^T(Q_{xx} - K^T Q_{uu}K)\delta x + (Q_x + k^T Q_{xu}^T)\delta x - k^T Q_{uu}k\end{aligned}$$

Iterative LQR

The forward pass is easy: just simulate the system forwards to calculate new values for x^* and u^* using dynamics equations and the feedback rule $u = Kx + k$. We keep a running copy of x called x^*, this is the simulated state (x_t^* is updated in-place after we calculate u^*).

Algorithm 1: iLQR Forwards Pass

x^* = initial value of x;
$K_t = 0$; $k_t = 0$;
for *each timestep* $t = 0..T-1$ **do**
    $\delta u_t^* = K_t(x^* - x_t^*) + k_t$;
    $u_t^* \mathrel{+}= \delta u_t^*$;
    $x_t^* = x^*$;
    $x^* = f(x_t^*, u_t^*)$;
end

The backwards pass is a little longer, but it's mostly just combining things we already have. We also have to keep track of $J_{xx,t}$ and $J_{x,t}$.

Algorithm 2: iLQR Backwards Pass

$\delta J_{xx,T} = \frac{\partial^2 \Phi}{\partial x^2}|_{x_T^*}$;
$\delta J_{x,T} = \frac{\partial \Phi}{\partial x}|_{x_T^*}$;
for *each timestep* $t = T - 1..0$ *(backwards)* **do**
  $Q_{xx}, Q_x, Q_{uu}, Q_u, Q_{xu}$ = CalculateQ($J_{xx,t+1}, J_{x,t+1}, x_t^*, u_t^*$);
  K_t, k_t = OptimizeQ($Q_{xx}, Q_x, Q_{uu}, Q_u, Q_{xu}$);
  $J_{xx,t}, J_{x,t}$ = CalcPrevC2G($Q_{xx}, Q_x, Q_{uu}, Q_u, Q_{xu}, K_t, k_t$);
end

Addendum: Off-the-Shelf Solvers and Model-Predictive Control

Constrained Optimization

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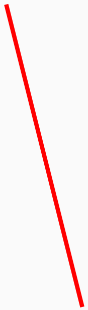

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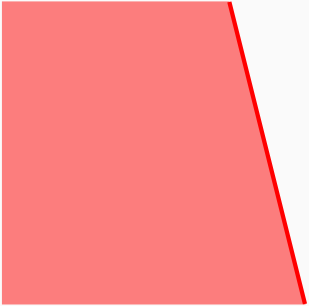

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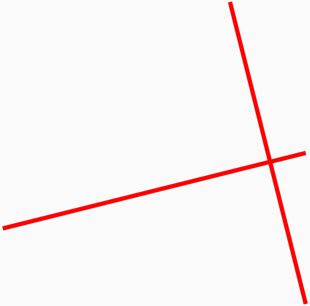

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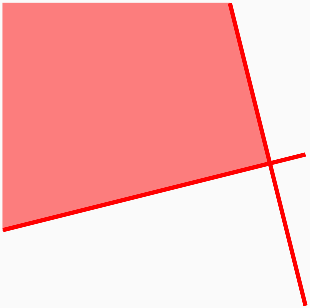

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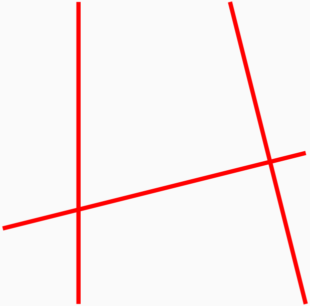

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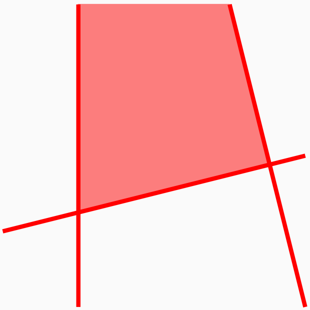

Constrained Optimization for Linear Problems

We can write our original optimal control problem as a “Quadratic Program” (quadratic cost, linear inequality and inequality constraints):

$$\min_{x,u} \sum_{i=1}^{T} \left[\frac{1}{2} x^T Q x + q^T x + \frac{1}{2} u^T R u + r^T u \right]$$

Subject to:

$$x_{t+1} = A x_t + B u_t + c$$

$$Hu \leq k$$

Off-the-Shelf Solvers

There are solvers that will just *solve* non-quadratic optimization problems with non-linear constraints!

(Remember: if we can express nonlinear constraints we can have nonlinear dynamics. If we can express non-quadratic cost, we can have better cost functions)

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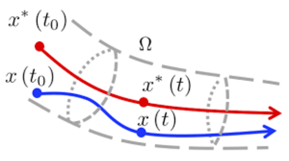
- Can fail (return "infeasible"), even if there is a solution that satisfies all of your constraints.
- **Doesn't give us a feedback policy.** (Open-loop only)



Replanning as Control

Strategy:

- Plan a trajectory from $t = 1$ to $t = T$

(Model-Predictive Control)

Replanning as Control

Strategy:

- Plan a trajectory from $t = 1$ to $t = T$
- Execute u_1

(Model-Predictive Control)

Replanning as Control

Strategy:

- Plan a trajectory from $t = 1$ to $t = T$
- Execute u_1
- Find the new state x_2

(Model-Predictive Control)

Replanning as Control

Strategy:

- Plan a trajectory from $t = 1$ to $t = T$
- Execute u_1
- Find the new state x_2
- Plan a trajectory from $t = 2$ to $t = T + 1$, starting at x_2

(Model-Predictive Control)

Replanning as Control

Strategy:

- Plan a trajectory from $t = 1$ to $t = T$
- Execute u_1
- Find the new state x_2
- Plan a trajectory from $t = 2$ to $t = T + 1$, starting at x_2
- Execute u_2 (the first control input from the new sequence)

(Model-Predictive Control)

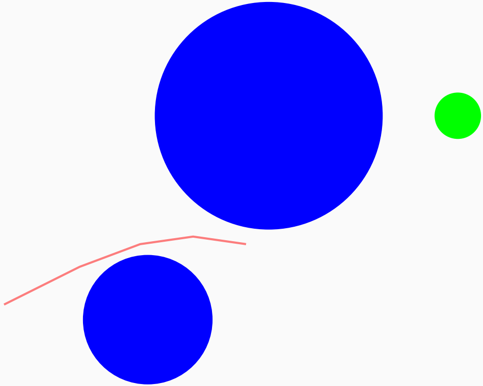
Replanning as Control

Strategy:

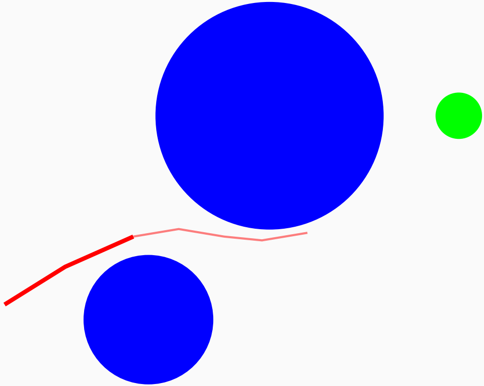
- Plan a trajectory from $t = 1$ to $t = T$
- Execute u_1
- Find the new state x_2
- Plan a trajectory from $t = 2$ to $t = T + 1$, starting at x_2
- Execute u_2 (the first control input from the new sequence)
- Repeat...

(Model-Predictive Control)

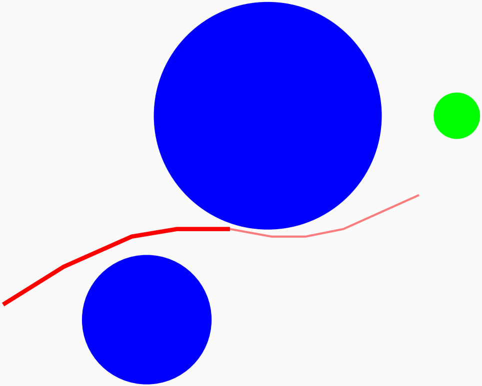
Model-Predictive Control



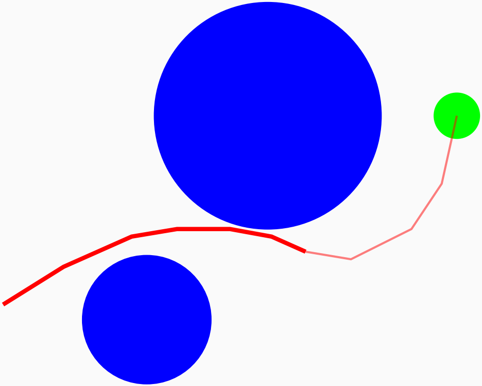
Model-Predictive Control



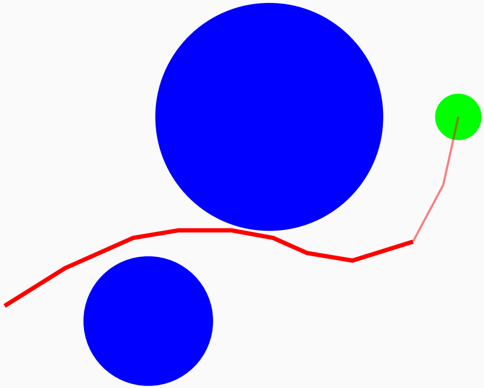
Model-Predictive Control



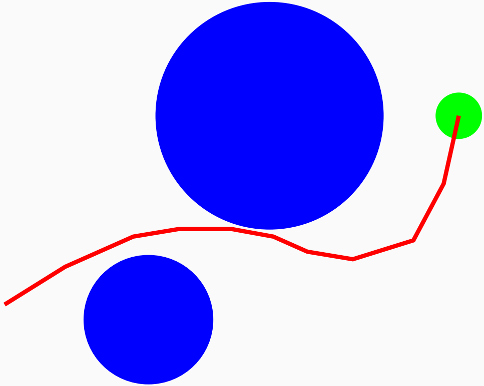
Model-Predictive Control



Model-Predictive Control



Model-Predictive Control



Conclusion

- Optimization-based motion planning is a powerful technique
- We can use LQR (repeatedly) to find good *local* solutions for nonlinear systems
- Off-the-shelf optimizers can be excellent…but sometimes slow
- Constant replanning (MPC) can turn open-loop sequences into a feedback technique

Resources

- Russ Tedrake's lectures: *http://underactuated.mit.edu*
- UIUC motion planning notes: *http://planning.cs.uiuc.edu*