Stellar Structure: A Computer Program for Describing the Internal Structure of a Self-Gravitating, Spherically Symmetrical Polytropic Fluid of Arbitrary Polytropic Index Based on the Lane-Emden Equation

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December 9, 2022

Abstract

The basis of this program is undergirded by the Lane-Emden equation (LEE). This second-order differential equation describes the density of a polytropic fluid as a function of dimensionless radius ξ. Numerical solutions of the LEE can be used to model other properties of a star, such as density, mass enclosed, and internal pressure, all as functions of ξ. With normalization, all of these data sets may be plotted on the same plane.

1 Deriving Equations and Important Parameters

An important quantity that we need for this program is the central density of the star, which is related to the central pressure by the given equation:

$$P = K\rho^{1+1/n} \tag{1}$$

where n is the polytropic index, P is the pressure (a function of r), and ρ is the density (also a function of r) [Wik22a]

The polytropic index is a thermodynamic measure of the work that is done by a system (in this case, the star). [Muk29]

Below is a table describing several polytropic indices and the celestial bodies they best model [Wik22b]:

Polytropic Index n	Celestial Body
0	Rocky/solid or liquid planets; constant density
0.5-1	Neutron Stars
1.5	Fully Convective Star Cores
3	Main Sequence Stars
5	Infinite Radius System

Table 1: Common Polytropic Indices

1.1 Finding the Central Pressure

Borrowing from the work of Vik Dhillon [Dhi10], consider the two equations shown below, commonly known as the Hydrostatic Equilibrium Equation (HEE) and the Mass Continuity Equation (MCE).

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2} \qquad\qquad \frac{dm}{dr} = 4\pi r^2\rho$$

We can divide the HEE by the MCE to obtain the following following equation:

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4} \Rightarrow dP = P_2 - P_1 = -\int_{M_1}^{M_2} \frac{Gm}{4\pi r^4}\, dm$$

If we let P_2 be the pressure of the star at the surface, which we will say is zero, and let P_1 be the pressure at the center of the star (r=0), and invoke the boundaries $M_1 = 0$ and $M_2 = M_{star}$, we can get the central pressure at the center of the star by integrating over these bounds. We obtain the following expression for P_{center}:

$$P_{center} = \frac{GM_{star}^2}{8\pi R_{star}^4} \tag{2}$$

1.2 Finding the Central Density

If we sub this value of P_{center} into equation (1) and rearrange for ρ_{center}, then we get the following expression for the central density of a star:

$$\rho_{center} = \sqrt[1+(1/n)]{\frac{GM_{star}^2}{8\pi K R_{star}^4}} \tag{3}$$

For demo purposes, we will assume K=1, but this value may be experimented with later.

2 Solving the Lane-Emden Equation

2.1 Important Definitions

The Lane-Emden Equation (LEE) is a dimensionless form of Poisson's equation for the gravitational potential of a Newtonian self-gravitating, spherically symmetric, polytropic fluid, or in our case, a star. It reads as follows:

$$\frac{1}{\xi^2}\frac{d}{d\xi}(\xi^2\frac{d\theta}{d\xi}) + \theta^n = 0 \tag{4}$$

Here, θ is a function of dimensionless radius ξ that is related to the density of the star, and r = $\alpha\xi$:

$$\alpha^2 = \frac{(n+1)K\rho_{center}^{1/n-1}}{4\pi G} \qquad\qquad \rho = \rho_{center}\theta^n$$

2.2 Developing an Approach for Solving the LEE

Referring to an approach devised by Vik Dhillon [Dhi13], let us start by expressing $\frac{d^2\theta}{d\xi^2}$ in terms of $\frac{d\theta}{d\xi}$ and θ:

$$\frac{d^2\theta}{d\xi^2} = -\frac{2}{\xi}\frac{d\theta}{d\xi} - \theta^n \tag{5}$$

Using the Euler method, the first derivative could be approximated with the following iterative formula: $\frac{d\theta}{d\xi}_{i+1} = \frac{d\theta}{d\xi}_i + \Delta(\frac{d^2\theta}{d\xi^2}_i)$. If we plug (5) into this equation, we arrive at this palatable formula for the i^{th} term of the first derivative:

$$\frac{d\theta}{d\xi}_{i+1} = \frac{d\theta}{d\xi}_i + \Delta\xi(-\frac{2}{\xi}\frac{d\theta}{d\xi}_i - \theta_i^n) \tag{6}$$

In turn, θ, which of course, is the main interest here, can be approximated with Euler using

$$\theta_{i+1} = \theta_i + \Delta\xi(\frac{d\theta}{d\xi}_{i+1}) \tag{7}$$

We can set up a loop that calculates the value of $\frac{d\theta}{d\xi}_{i+1}$ based on initial conditions $\theta(0) = 1$ $\frac{d\theta}{d\xi}(0) = 0$ (which are definitionally true for the LEE) and small steps of $d\xi$

The algorithm will have to calculate the value of (6), use that value to calculate the value of (7), and then THAT value will be used to calculate the next value of 6. Both of these steps will have to be accounted for in the same loop. We can use the Euler Method to approximate (6) by writing a function for (5) where θ is a parameter, ξ is the independent variable, and $\frac{d\theta}{d\xi}$ is the dependant variable. However, in this function, the parameter θ will be changing with each iteration of the loop, so this must be factored into the algorithm.

2.3 The Algorithm

For the demo version of the program, a two-step Euler approach was used. The pseudo-code algorithm is as follows:

1. $\xi_0 = 0.0001, \theta_0 = 1, \frac{d\theta}{d\xi}_0 = 0$
2. **for** $i \leq nintervals$
3. $\frac{d\theta}{d\xi}_{i+1} = \frac{d\theta}{d\xi}_i + d\xi \frac{d^2\theta}{d\xi^2}(\xi_i, \theta_i, \frac{d\theta}{d\xi}_i)$
4. $\theta_{i+1} = \theta_i + d\xi \frac{d\theta}{d\xi}_{i+1}$
5. $\xi = \xi + d\xi$
6. $i = i + 1$
7. **end for**

By storing all of these values in arrays, we can plot the solution $\theta(\xi)$ with pgplot. The user will specify how far the graph goes out (because the LEE goes to infinity), which will be the variable "XiStar," and the number of steps will also be chosen. In the demo test, we will use 1500 steps.

2.4 Plotting Density and Mass Enclosed as a Function of ξ

If we normalize the function $\rho = \rho_{center}\theta^n$ such that the central pressure is equal to 1 (which would amount to dividing the equation by a constant ρ_{center}, we can plot unitless density as a function of ξ on the same axes as the LEE with the equation $\rho = \theta^n$. We will have all the needed values of θ to carry this function out.

We can also make a crude estimate about the mass enclosed by a shell of radius ξ by multiplying each value in the density array by dξ and summing them for all steps. This function can also be plotted on the same axes.

3 Testing The Demo

3.1 Plots from Acceptable n Values

On the following page, the resulting plots from polytropic indices 1 (top left), 3 (top right), 4 (bottom left), and 5 (bottom right) are shown in figure 1. Figure 2 shows the solutions for the LEE for n values 1 through 5. As can be seen, our results are quite comparable with those found on Wikipedia. Note that the **green line** is the solution of the LEE, the red line is density as a function of ξ, and the blue line is mass as a function of ξ. Note that for n=1, because the polytropic index is equal to one, density is directly proportional to θ. This, the density line (red) is directly imposed on the solution of the Lane Emden equation

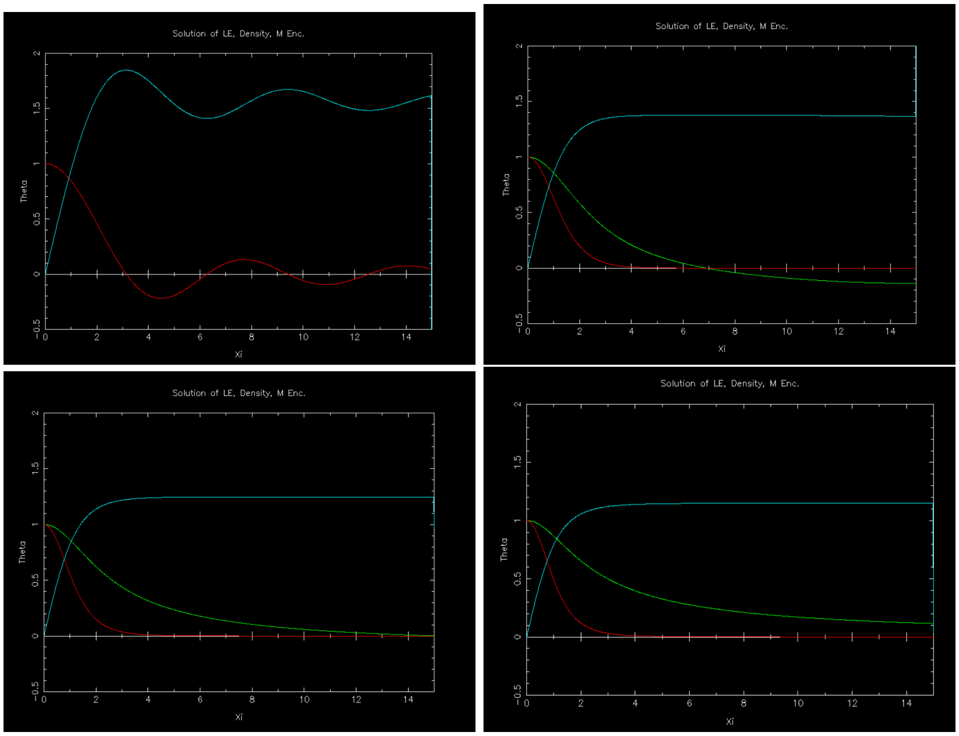


Figure 1: Solutions of the LEE

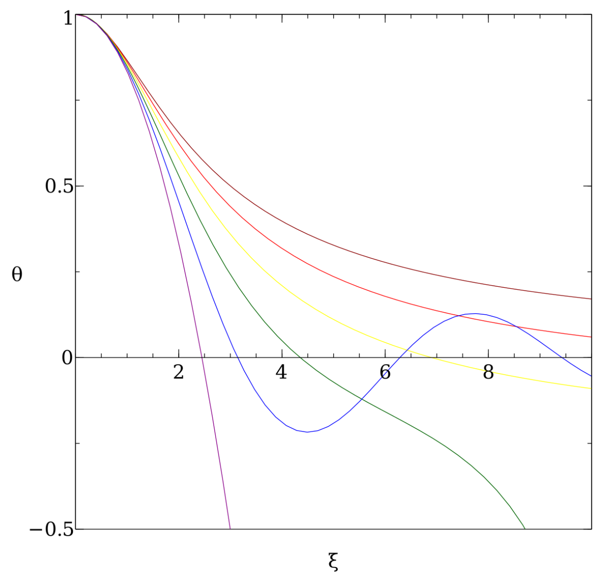
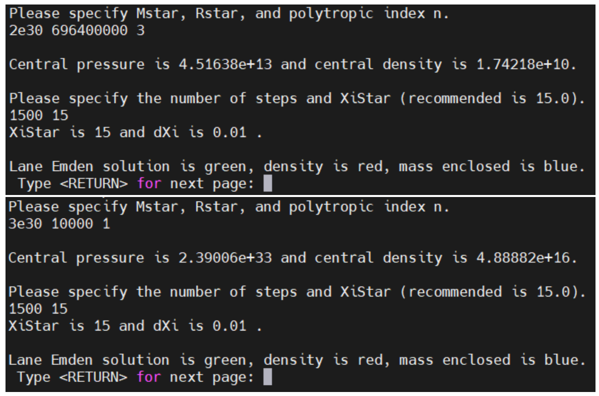


Figure 2: Solution of the LEE from Wikipedia

3.2 Pressure and Density Calculations

Figure 3 shows the terminal after the program was run to model stars; our own Sun (above), and a neutron star with a mass of 1.5 solar masses and a radius of 10km (below). The central pressure at the center of our Sun is known to be about 3.4e11 atm, or 3.44505e16 Pa [Tre97] Our approximation using the mass continuity equation and the pressure gradient is off by about 3 orders of magnitude. According to NASA, the density at the center of our sun is about 150 kilograms per meter cubed [Hat22], meaning our program is off by an order of magnitude of 5.



```
Please specify Mstar, Rstar, and polytropic index n.
2e30 696400000 3

Central pressure is 4.51638e+13 and central density is 1.74218e+10.

Please specify the number of steps and XiStar (recommended is 15.0).
1500 15
XiStar is 15 and dXi is 0.01 .

Lane Emden solution is green, density is red, mass enclosed is blue.
 Type <RETURN> for next page:
Please specify Mstar, Rstar, and polytropic index n.
3e30 10000 1

Central pressure is 2.39006e+33 and central density is 4.88882e+16.

Please specify the number of steps and XiStar (recommended is 15.0).
1500 15
XiStar is 15 and dXi is 0.01 .

Lane Emden solution is green, density is red, mass enclosed is blue.
 Type <RETURN> for next page:
```

Figure 3:

3.3 Shortcomings

The demo fails to graph solutions for positive non-integer values, and for n=2. This is presumably because the graph of n=2 approaches negative infinity rapidly, and results in an error when plotting. Additionally, the estimations for the central pressures and densities of stars seem to be quite off with the current math. Moreover, our rough estimate of the mass enclosed can be improved by using a more accurate approximation technique.

4 Extending The Program

4.1 Improving Accuracy

To make our approximating of the first derivative of $\frac{d\theta}{d\xi}$ more accurate, and in turn, make our approximation of θ more accurate, we can alter the algorithm we used in the demo and make use of the Fourth-Order Runge Kutta technique. Using the same initial conditions as in the demo, we can write the algorithm in pseudo-code as follows:

for $i \leq nintervals$

$k_1 = d\xi \frac{d^2\theta}{d\xi^2}(\xi, \frac{d\theta}{d\xi}, parameters = \theta_i, n)$

$k_2 = d\xi \frac{d^2\theta}{d\xi^2}(\xi + 0.5d\xi, \frac{d\theta}{d\xi} + 0.5k_1, parameters = \theta_i, n)$

$k_3 = d\xi \frac{d^2\theta}{d\xi^2}(\xi + 0.5d\xi, \frac{d\theta}{d\xi} + 0.5k_2, parameters = \theta_i, n)$

$k_4 = d\xi \frac{d^2\theta}{d\xi^2}(\xi + d\xi, \frac{d\theta}{d\xi} + k_3, parameters = \theta_i, n)$

${\frac{d\theta}{d\xi}}_{i+1} = {\frac{d\theta}{d\xi}}_i + (k_1 + 2k_2 + 2k_3 + k_4)/6$

$\theta_{i+1} = \theta_i + d\xi {\frac{d\theta}{d\xi}}_{i+1}$

$\xi = \xi + d\xi, i = i + 1$

end for

4.2 Plotting More Values of n

Code can be added such that a different approach is used when evaluating certain values of n to avoid floating errors. We can make it so that the loop breaks if the value of theta becomes too negative, and alter the number of points we plot.

Using `std::cout` to print the values of theta when using an index of 1.5, for example, it was observed that a floating error would occur after θ reached a value close to 0.004. The solution is to write an if/else statement to make the loop break at different times, depending on the index.

4.3 A More Accurate Representation of Mass Enclosed

Referring to the Wiki on the LEE [Wik22a], one can stumble upon this set of differential equations:

$$\frac{d\varphi}{d\xi} = \theta^n \xi^2 \tag{8}$$

Where φ is dimensionless mass as a function of ξ. We can use this derivative to approximate unitless mass as a function of ξ with the Runge Kutta method. In pseudo-code, the algorithm will look something like this:

1. $\varphi_0 = 0$, i=1.
2. Already have all values of θ and ξ stored in arrays
3. for i $\leq nintervals$
4. $k_1 = d\xi \frac{d\varphi}{d\xi}(\xi_i, \theta_i, paramters = n)$
5. $k_2 = d\xi \frac{d\varphi}{d\xi}(\xi_i + 0.5d\xi, \theta_i, parameters = n)$
6. $k_2 = d\xi \frac{d\varphi}{d\xi}(\xi_i + 0.5d\xi, \theta_i, parameters = n)$
7. $k_2 = d\xi \frac{d\varphi}{d\xi}(\xi_i, \theta_i, parameters = n)$
8. $\varphi_i = \varphi_{i-1} + (k_1 + 2k_2 + 2k_3 + k_4)/6$
9. end for

After plotting this, it is noticed that the graph still goes above the θ=1, so we will normalize the function by dividing every value in the array by the maximum value of the array.

4.4 Plotting Pressure P and $\frac{d\theta}{d\xi}$, and Varying Proportionality Constant K

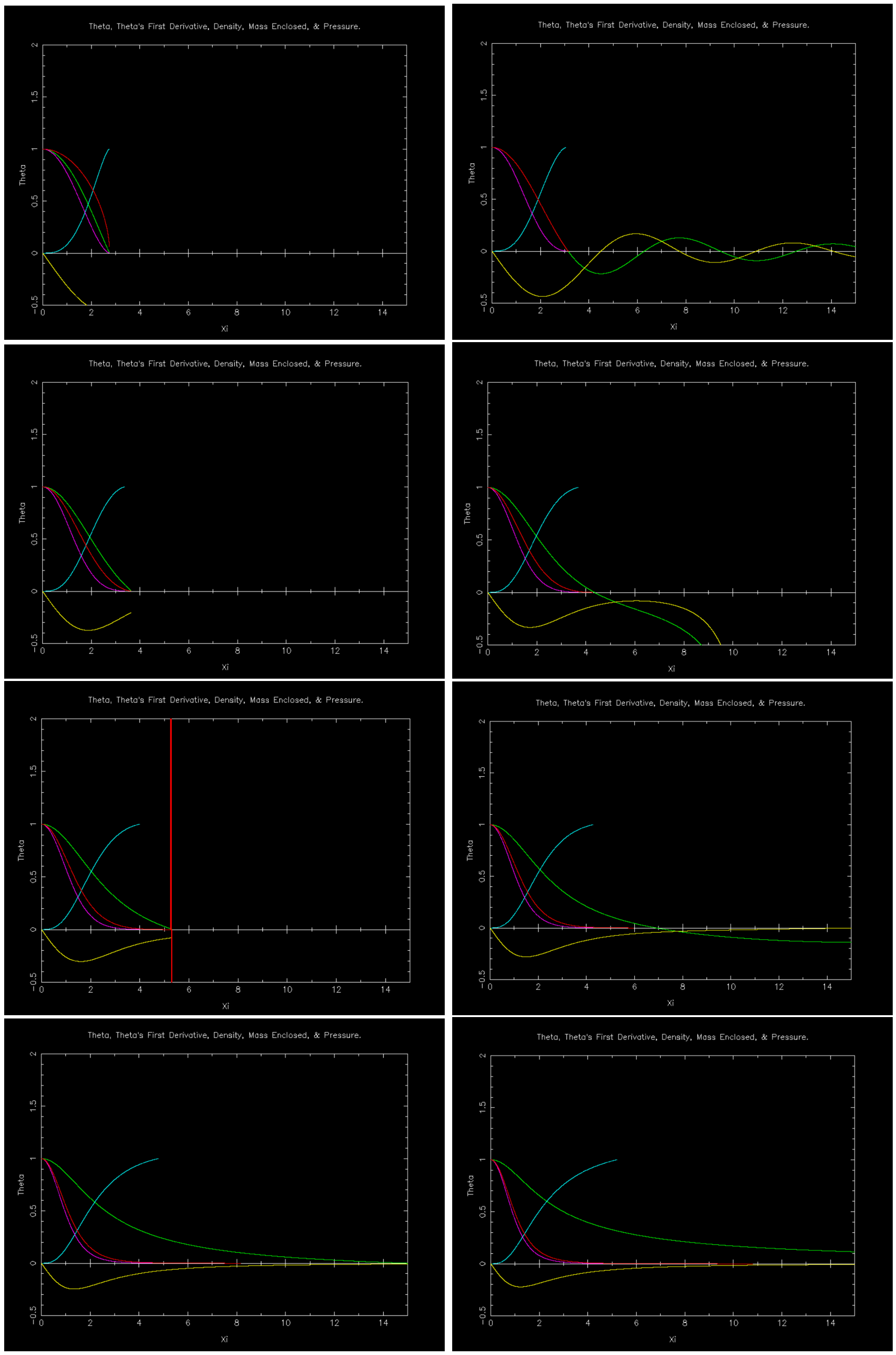
Referring to (1) and realizing that we have a normalized density function, we can plot pressure on the same plot as well. By defining a pressure function with arguments ρ, K, and n, we can plot pressure as a function of ξ. We can also add some basic code to plot the first derivative of theta on the same axes, too.

5 Results

5.1 Plots for n= 0.5, 1, 1.5, 2, 2.5, 3, 4, 5

Note that the solution of the LEE is green, the first derivative is yellow, the density is red, mass enclosed is blue, and the pressure is pink. In this test of the extended program, XiStar was 15.0, and the number of steps was 15000.

See figure 4 on page 7. We now have a valid plot for some of the n values that we were missing in the demo. n=2.5 is slightly strange looking because of the measures we had to take to avoid floating exceptions for non-integer indices, and this could be investigated in the future.



Theta, Theta's First Derivative, Density, Mass Enclosed, & Pressure.
Theta
Xi
−0.5
0
0.5
1
1.5
2
0
2
4
6
8
10
12
14

Figure 4:

5.2 Plots for n=1, 2, 3 and K=2

As can be seen in the below plots, when the proportionality constant is changed from 1 to 2, the pressure (pink line) is doubled in relation to density (red line).

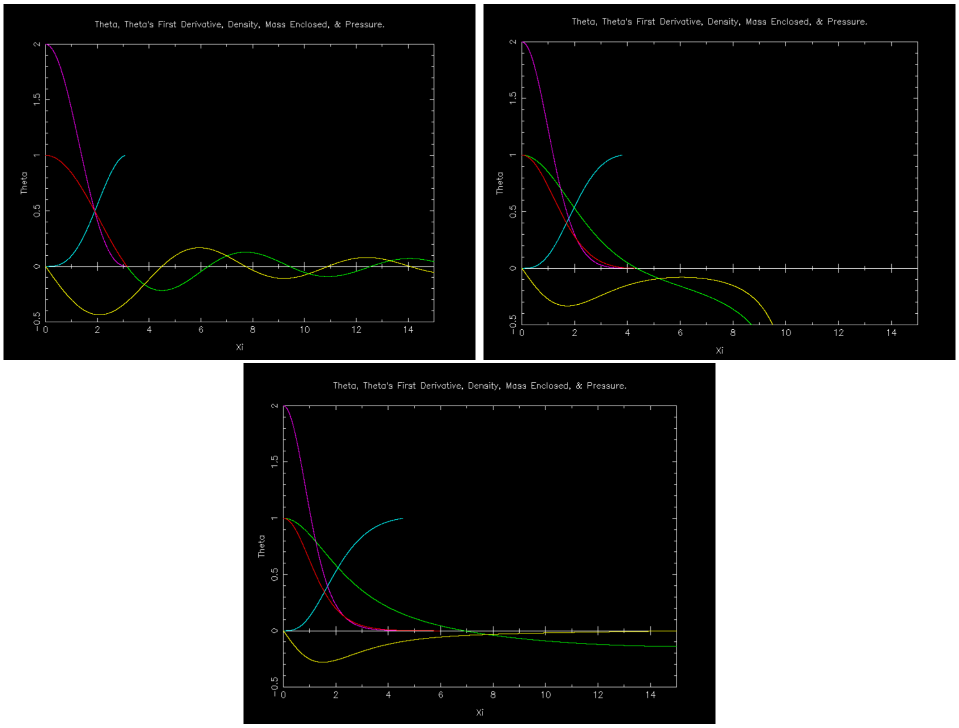


Figure 5:

6 Conclusions

The program does the job of approximating the Lane Emden exceptionally well, especially after employing the Runge Kutta technique. Using the LEE, we were able to determine 3 other properties of a star's internal structure at any point within the star. However, there are still lingering problems. The graph for $n = 2.5$ is unsettling. The graph for $n = 3.5$ yields a similar strange result for the density plot and $n = 4.5$ results in a floating exception. A different integration technique could perhaps be utilized to remedy this; perhaps one besides Euler or Runge Kutta, or further if/else statements could be used to set different boundaries on the loops to avoid floating errors or segmentation artifacts.

Narayan Kumar, Rajesh K. Pandey, and Carlo Cattani devised a method of solving the LEE by employing the "Bernstein Operational Matrix of Integration." [NKC11] This method is incredibly complex, however, and putting it into code would be herculean. The strategy is highly accurate, though.

References

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