Automatica, Vol. 12, pp. 417–425. Pergamon Press, 1976. Printed in Great Britain

Optimal Space Station Detumbling by Internal Mass Motion*

B. G. KUNCIW† and M. H. KAPLAN‡

A movable mass control system may be used to stabilize a tumbling asymmetric space station about a principal axis in minimal time for a given set of constraints.

Summary—This investigation deals with the use of a movable mass control system to stabilize a tumbling asymmetric spacecraft about the maximum inertia axis. A first-order gradient optimization technique is used to minimize angular velocity components along the intermediate and minimum inertia axes, thus, permitting a wide range of initial guesses for mass position history. Motion of the control mass is along a linear track fixed in the vehicle. The control variable is taken as mass acceleration with respect to body coordinates. Motion is limited to defined quantities and a penalty function is used to insure a given range of positions. Numerical solutions of the optimization equations verify that minimum time detumbling is achieved with the largest permissible movable mass, length of linear track, and positions of the mass on the two coordinates perpendicular to the linear motion. The optimal method permits detumbling in about one-fourth the time when compared to a force control law formulation available in the literature.

NOMENCLATURE

- $\mathbf{a}$ inertial acceleration of the origin of coordinates which is fixed at the center of mass of the main body
- $\mathbf{f}_m$ force on the control mass
- f_{m_x} x component of $\mathbf{f}_m$
- $\mathbf{H}$ angular momentum of the system with respect to the origin of coordinates
- $\mathbf{H}_B$ angular momentum of the main body with respect to the origin of coordinates
- $\mathbf{H}_{cm}$ angulr momentum of the system with respect to its own center of mass
- $\mathbf{i}, \mathbf{j}, \mathbf{k}$ orthogonal unit vectors of coordinate frame fixed in the main body
- I_x, I_y, I_z moments of inertia of the main body
- I_{xy}, I_{xz}, I_{yz} products of inertia of the main body
- $\bar{I}$ inertia dyadic of the main body
- I_{max} maximum moment of inertia of the system
- I_{min} minimum moment of inertia of the system
- J performance index to be minimized
- K arbitrary constant associated with the penalty function technique
- K_1, K_2 K for P_1 and P_2, respectively
- m mass of movable object
- M mass of main body
- P auxiliary state variable due to penalty function
- P_1, P_2 P for the two inequality constraints on x_{max} and x_{min}, respectively
- $\mathbf{r}$ position vector from center of mass of main body to the point mass
- $\mathbf{r}_c$ position vector from center of mass of main body to the center of mass of the system
- $\mathbf{R}_o$ position vector from inertial origin to the center of mass of the main body
- $\mathbf{R}_c$ position vector from inertial origin to the center of mass of the system
- $\mathbf{S}$ first moment of mass of the system
- t time
- t_0 initial time
- t_f final time
- T rotational kinetic energy of the system with respect to its own center of mass
- u_1 control variable equal to $\ddot{x}$
- x, y, z coordinates corresponding to $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, respectively
- x_{max}, x_{min} maximum and minimum permitted mass positions on the x axis, respectively
- x_h, x_l actual values used for x_{max} and x_{min}, respectively, during computation
- X, Y, Z coordinate axes fixed at the center of mass of the main body
- β state variable equal to $\dot{x}$
- $\mathbf{\Gamma}$ external moment
- μ equivalent mass equal to $mM/(M+m)$
- $\boldsymbol{\omega}$ angular velocity of spacecraft
- $\omega_x, \omega_y, \omega_z$ components of $\boldsymbol{\omega}$ along $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, respectively
- $\omega_1, \omega_2, \omega_3$ components of $\boldsymbol{\omega}$ along the maximum, intermediate, and minimum inertia axes of the main body, respectively
- $\omega_{2_{max}}, \omega_{3_{max}}$ maximum values of ω_2 and ω_3 desired

*Received 21 October 1974; revised 2 June 1975; revised 15 December 1975; revised 9 February 1976. The original version of this paper was presented at the *4th IFAC Symposium on Control in Space* which was held in Tsakhkadzor, Armenian SSR during August 1974. Information about the published Proceedings of this IFAC Meeting may be obtained from the National Committee of Automatic Control of the U.S.S.R. Profsoyuznaya ul., 81, Moscow GSP-312, U.S.S.R. This paper was recommended for publication in revised form by associate editor D. Tabak.

†Captain, USAF, at the Flight Dynamics Laboratory, Wright Patterson, OH 45433 U.S.A.

‡Associate Professor, Dept. of Aerospace Engineering, The Pennsylvania State University, University Park, PA 16802, U.S.A.

1. INTRODUCTION

IN THE operation of future manned space vehicles there is always a finite probability that an accident will occur which results in uncontrolled tumbling of a spacecraft. On April 3, 1973, Salyut 2 went into a catastrophic tumble believed to be the result of an explosion or a wildly firing thruster[1]. A recent study shows that future manned space bases may be subjected to tumble states with initial angular velocities of up to 2 rpm[2]. Hard docking by a manned rescue craft is not possible because of the hazardous environment to which the rescue crew would be exposed and the excessive accelerations and fuel usage required of the rescue vehicle. Therefore, it is desirable to develop an internal autonomous control system which would become active upon loss of control to either completely detumble the vehicle or lessen the effect of tumbling. Mass explosion devices for this application require onboard storage of propellant and may not be reliable on a long term basis. Some momentum exchange devices may require continuous operation since startup would be difficult once a tumbling situation has occurred. These devices also have a tendency for saturation in large corrective maneuvers. Passive devices have relatively low energy dissipation rates and seem most appropriate for vehicles which have a high nominal spin rate about one axis.

The work presented in this paper deals with the use of a movable mass control system to reduce arbitrary tumbling of an asymmetric spacecraft to simple spin about the axis of maximum inertia in minimal time. Investigations of such control systems for use with symmetric vehicles appear in the literature. Control of an asymmetric spacecraft by a movable mass was investigated by Childs[3]; however, an assumption of small transverse angular velocities relative to the spin velocity was made. The more general case represents a much more complicated problem and requires extensions of previous knowledge. An asymmetric spacecraft with arbitrary initial tumble rates is assumed. A first-order gradient method is used to minimize angular velocity components along the intermediate and minimum inertia axes. This method permits a wide range of initial guesses for mass position history. Motion of the control mass is along a linear track fixed in the vehicle. The control variable is taken as mass acceleration with respect to body coordinates. Motion is limited to defined quantities and a penalty function is used to insure a given range of positions. Numerical solutions of the optimization equations verify that minimum time detumbling is achieved with the largest permissible movable mass, length of linear track, and distance from the spacecraft center of mass. The same technique can be used to determine minimum size control mass to reduce angular velocity components to given values in given time. Results indicate that the optimization permits detumbling in about one-fourth the time obtained when a simple force control law is assumed[4].

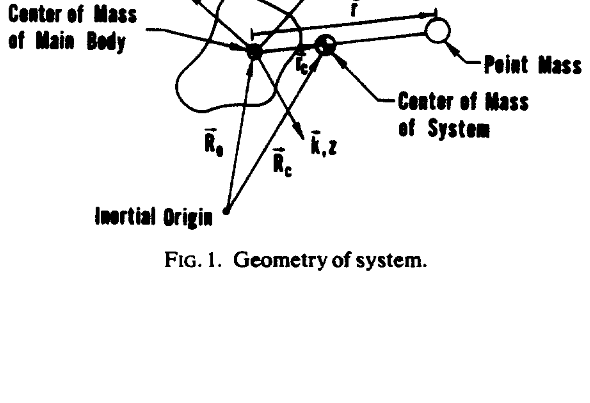

FIG. 1. Geometry of system.

2. EQUATIONS OF MOTION

The angular momentum equation for a rigid body with an arbitrary origin of coordinates[5] is

$$\mathbf{\Gamma} = \dot{\mathbf{H}} + \mathbf{S} \times \mathbf{a}$$

where $\mathbf{\Gamma}$ is the external moment, $\mathbf{H}$ is the angular momentum of the system, $\mathbf{S}$ is the first moment of mass of the system, and $\mathbf{a}$ is the inertial acceleration of the origin of coordinates. The desired equations of motion for a spacecraft with one small movable mass may be obtained by fixing the origin of coordinates at the center of mass of the main body, which is the spacecraft without the movable mass. The geometry of the system is shown in Fig. 1 where x, y, z is a coordinate system, with $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ unit vectors, fixed in the main body, whose origin is at the center of mass of the main body. Assume a moveable mass, m, is a point mass and a circular orbit with no thrusting, or external moments. Noting that the force on the control mass is

$$f_m = \mu \ddot{\mathbf{r}}$$

where μ is the equivalent mass $mM/(M+m)$ with M being the mass of the main body, we may write

$$\dot{\mathbf{H}}_B = -\mathbf{r} \times \mathbf{f}_m \qquad (1)$$

where $\mathbf{H}_B$ is the angular momentum of the main body relative to its own center of mass[6]. Thus, as equation (1) shows, the force on the control mass may be considered as producing a moment on the main body. The mass will be permitted to move along a linear track parallel to the x axis. This equation may be expressed in component form

$$\dot{\omega}_x = \dot{\omega}_x(\omega_x, \omega_y, \omega_z, x, \dot{x}, \ddot{x}),$$
$$\dot{\omega}_y = \dot{\omega}_y(\omega_x, \omega_y, \omega_z, x, \dot{x}, \ddot{x}),$$
$$\dot{\omega}_z = \dot{\omega}_z(\omega_x, \omega_y, \omega_z, x, \dot{x}, \ddot{x}).$$

The full equations for $\dot{\omega}_x$, $\dot{\omega}_y$, and $\dot{\omega}_z$ are given in Appendix A. Since the motion is along an axis parallel to the x axis

$$\dot{y} = \ddot{y} = \dot{z} = \ddot{z} = 0$$

and the y, z positions of the mass may be arbitrarily fixed. Other constants that have to be specified are moments and products of inertia, and masses of the main body and the movable object. The control mass was permitted to have motion parallel to just one axis since then only one control variable will need to be specified; this fact will become important in Appendix B which deals with the optimization technique. However, the direction of the x axis, and, therefore, the direction of mass motion, relative to the spacecraft may be changed arbitrarily by appropriately changing the moments and products of inertia in the equations of motion. The power required for control mass motion is discussed in Appendix C.

3. RESULTS

Several parameters need to be specified before computer simulations may be run for minimum time stabilization. As will become evident, selection of some of these parameters will depend upon the specific satellite to which the movable mass control system is applied; that is, the dimensions of the satellite and the amount of time that can be permitted for detumbling. The choices of minimum time detumbling of the remaining parameters will also become evident; but, these parameters will not be dependent on the type of satellite to be controlled. Specifically, noting that the mass moves along an arbitrary x direction, these parameters are as follows: mass of the movable object, length of the linear track, y and z positions of the mass, point about which the mass oscillates, and direction of the x axis relative to the spacecraft. By interpreting equation (1) in terms of moments applied about each axis, a qualitative preliminary analysis may be made as to the effect of these parameters on the time needed to detumble. Increasing the mass of the object will increase the force; thereby increasing the moment and permitting a decrease in the detumbling time. Increasing the length of the linear track or the y and z magnitudes will increase the moment arm, which should tend to increase the moment and result in a lower minimum time. It should be noted that changes in these parameters will also affect the force; but, the overall effect on the moment will probably be due to the change in the moment arm mentioned previously since the force consists of the relative difference of various terms. By increasing the x value of the point about which the mass oscillates, the moment arm is again increased. However, by permitting the mass to move further on one side of the zero x position than on the other, there may arise difficulties due to a larger moment in one direction than in the opposite. Changing the direction of the x axis relative to the spacecraft will affect the moment arms of the force components producing moments about the intermediate and minimum inertia axes. If the x axis is parallel with the maximum inertia axis, maximum control over the moment arms of the moments about the intermediate and minimum inertia axes will be available; this can be seen by noting that now the x axis is perpendicular to the intermediate and minimum inertia axes. This would seem to indicate that the orientation of the linear track should be parallel to the final spin axis, which is the major principal axis. However, in this case as in the other cases that involved changes in the moment arm, there are also changes in the force itself which are difficult to specify. The above qualitative analysis is not sufficient in itself to arrive at minimum time values for all the parameters being discussed. A quantitative analysis must be made.

The movable mass control system was applied to a manned space station configuration which NASA is considering for the 1980's[7]. This configuration is shown in Fig. 2. The mass of the space station is 99,792 kg; its maximum, intermediate and minimum moments of inertia are $6.74 \times 10^6\ \mathrm{kg \cdot m^2}$, $6.28 \times 10^6\ \mathrm{kg \cdot m^2}$, and $5.15 \times 10^6\ \mathrm{kg \cdot m^2}$. Following is the transformation matrix, body fixed X, Y, Z to principal 1, 2, 3, where 1, 2, 3 are the maximum, intermediate, and minimum inertia axes respectively:

$$\begin{bmatrix} 0.457600 & -0.889132 & 0.00676167 \\ 0.889157 & 0.457597 & -0.00204229 \\ -0.00127826 & 0.00694674 & 0.999975 \end{bmatrix}$$

The optimization procedure permits the fastest detumbling possible with the movable mass. Based on the previous qualitative discussion of

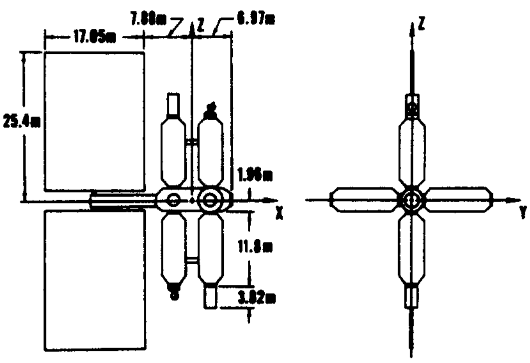

FIG. 2. Manned space station.

various parameters in the differential equations of motion, we will initially fix the parameters to yield the best detumbling sequence; further cases will vary these parameters in order to quantitatively determine the minimum time solution. The mass of the movable object will be set at 499 kg, which is 0.5% of the manned space station mass. It will be permitted to travel approximately ±3.7 m about the zero position on the axis. This axis of motion will be parallel to and have the same sense as the maximum inertia axis. For convenience, the y and z axes will be chosen to be parallel to and have the same sense as the intermediate and minimum inertia axes, respectively. Choosing the y and z positions as large as possible within the limits of the space station, we have 5.55 m and −13.7 m. The initial angular velocity components along the 1-axis of maximum inertia, the 2-axis of intermediate inertia, and the 3-axis of minimum inertia will be chosen as 0.103, −0.199 and 0.000286 rad/sec, respectively. These values are based on a worst case tumbling analysis [2]. They represent the highest tumbling mode of the manned space station and are due to a collision between it and the space shuttle vehicle. If controlled, the manned space station with a fixed 499 kg internal control mass would continue to tumble with ω_1 oscillating between 0.103 and 0.192 rad/sec, ω_2 oscillating between −0.199 and 0.199 rad/sec, and ω_3 oscillating between −0.118 and 0.118 rad/sec. Dissipation effects, of course, will tend to decrease the rotational kinetic energy of the vehicle and alter the envelopes of oscillation. However, in the short time periods involved here, these limits of oscillation can be used as a reference for comparison purposes. The effects of control by an internal movable mass system on oscillations of ω_1, ω_2 and ω_3 are shown in Fig. 3. Curves shown represent envelopes* of oscillations about the principal axes. At 2845 sec the limits of mass motion along the x axis were set at $\pm 10^{-9}$ m in order to zero out the mass position, velocity, and acceleration. By 2893 sec these three variables were essentially zeroed out, having values of 0.0466 m, −0.00322 m/sec, and 0.00141 m/sec^2. After this time, the mass was kept fixed at the zero x position, ω_1 remained at 0.212 rad/sec, ω_2 oscillated between −0.00152 and 0.00142 rad/sec, and ω_3 oscillated between −0.000837 and 0.000909 rad/sec. It should be

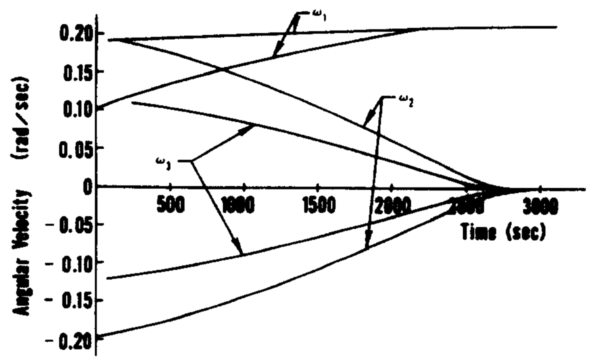

FIG. 3. Envelopes of angular velocity oscillations using 0.5% mass.

noted that the percentage change from peak to peak of ω_2 and ω_3 were reduced by more than 99% of their initial value. Figure 4 shows the corresponding motion of the internal mass. Position, x, at no time exceeds 3.7 m; after 2893 sec, it is essentially equal to zero. The mass oscillates between its maximum permitted limits, utilizing the full linear tract available to it. The velocity of the mass, $\dot{x}$, oscillates between −0.647 and 0.654 m/sec. The greatest mass acceleration, $\ddot{x}$, occurs during the zeroing out of the mass position, velocity, and acceleration; in this period it reaches values of −0.579 and 0.465 m/sec^2. Also during the zeroing out, the force in the x direction, f_{m_x}, acting on the mass reaches its largest magnitude, 283 N. These vaulues for mass velocity and acceleration, and for force, are reasonable. It should further be noted that the mass velocity and force maximum magnitudes occur during energy dissipation, $-\dot{T}$. During energy dissipation, the force in the x direction and the mass velocity are opposite in direction; here the control system is actually restraining the mass. Energy dissipation in excess of that due to damping and friction could be used to provide useful power for the vehicle's systems. Energy must be provided to the mass when $\dot{T}$ is positive. The maximum positive power is 48.4 W, a reasonable value. Figure 5 shows the decrease of rotaitional kinetic energy from its initial value of 1.62×10^5 J to the final value for stable simple spin of 1.54×10^5 J. At various points in this figure, the kinetic energy

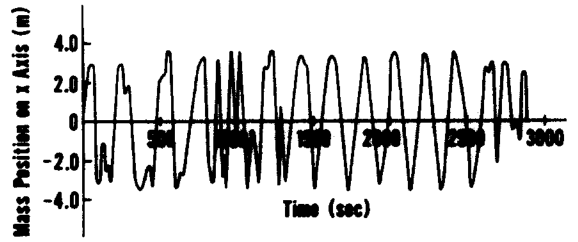

FIG. 4. Motion of 0.5% control mass.

*A natural approach could be to reduce $\omega_i(t_f)$ to a low value instead of the envelope of ω_i. However, that approach does not guarantee a small value of ω_i at another instant of time. The approach here does appear to insure continued low values of ω_i after t_f. Solutions had to be carried out in a piecewise manner.

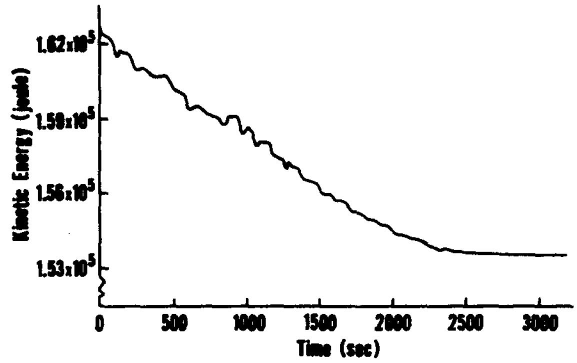

FIG. 5. Decrease of rotational kinetic energy using 0.5% mass.

increases slightly and then resultes its downward curve, corresponding to energy addition to the movable mass control system.

A valid comparison can be made between the optimal control mass motion described in this investigation and the analysis of the force control law formulated by Edwards and Kaplan [4]. All parameters and initial conditions mentioned above were kept the same for both cases, including the permitted extreme positions on both the positive and negative sides of the mass motion axis. The force control law approach yielded a decrease of the ω_2 envelope to a magnitude of 0.00206 rad/sec at 11,050 sec, and the ω_3 envelope to a magnitude of 0.00124 at 11,005 sec. As pointed out above, the optimal analysis of this investigation decreased the magnitudes of the ω_2 and ω_3 envelopes to 0.00152 and 0.000909 rad/sec respectively by about 2900 sec. Thus, optimal techniques permit detumbling in approximately one-fourth of the time required by the simple control law method. Since the one-fourth value means that the crewmen will be subjected to a tumbling state for less than an hour compared to over 3 hr, it is quite significant. The force control analysis required only about one watt of peak positive power. As was shown, the power for the optimal control is greater, but still within reasonable limits.

The effects of changes in the five important parameters mentioned previously were obtained by varying only one at a time. Doubling the mass to one percent of the space station mass caused about a one-half decrease in the time to detumble to simple spin about the maximum inertia axis. A decrease in the extreme limits of mass motion or in the y and z positions caused an increase in the amplitudes of the ω_2 and ω_3 oscillations, resulting in a larger total time to detumble. Control mass oscillation about a non zero position resulted in a faster decrease of the ω_2 and ω_3 envelopes of oscillation; however, the envelopes tended toward non zero final values. Control mass motion along the intermediate inertia axis resulted in slower detumbling and a non zero final value for ω_3.

These results were obtained by running the computer program for simulation times of 100 sec. To bring the extreme mass positions to the permitted magnitudes, in this 100 sec simulation time period, required up to about 35 iterations which used about 100 sec of IBM 370/165 computer time. At the end of each 100 sec simulation run the end values for ω_1, ω_2, ω_3, x and $\dot{x}$ were used as initial conditions for the next 100 sec simulation run. This time increment of 100 sec for simulation was chosen since larger time increments made it difficult to bring the extreme mass positions to within the specific limits. A few times even this time increment was too large; that is, the mass would initially stay within the limits but would then move past, where about 35 iterations were the maximum permitted. Rather than extend the computer time and thereby increase the number of iterations, the acceptable initial part of the run was used. For zeroing out the mass position, velocity, and acceleration in the 0.5% case that was initially investigated, the simulation time increment was arbitrarily chosen at 50 sec. Also, the position limits, as stated previously, were set at $\pm 10^{-9}$ m for the zeroing out simulation time. Normally, limits were set at ± 2.5 m at the beginning of a case and then changed to about ± 3.0 m or higher; the limits of mass position should be set lower than what is desired since the penalty function comes in when there is a violation of the set limits. The iteration that was chosen during each 100 sec simulation time run had the lowest peak magnitudes for the ω_2 and ω_3 oscillations for mass positions within the ± 3.7 m prescribed extreme limits. There was no need to place constraints in the optimization technique on mass velocity and acceleration since, as discussed in the initial 0.5% mass case, these variables did not reach excessive magnitudes. The time steps in the iterations were set at 5.0 sec of simulation time.

4. CONCLUSIONS

An optimal movable mass control system has been applied to a tumbling spacecraft in order to obtain simple spin about the major principal axis. Results indicate that the largest possible magnitudes for the internal mass, length of the linear track, and positions of the mass on the y and z axes will yield the fastest detumbling times. The choice of these values depends upon size and mass of the spacecraft. Results also indicate that the mass should oscillate, about a zero point, on a line parallel to the maximum inertia axis. These results were based on worst case initial conditions for ω_1, ω_2 and ω_3. Tumbling situations that might be encountered in actual space operations will usually be less severe and, therefore, will probably require less time to reach simple spin. Also, these results were based on one vehicle, the modular space station. However, since this vehicle was asymmetric, the optimization technique will apply to any type of spacecraft. Results of various parameter changes, furthermore, were based on motions of a 1% mass and the resultant effects on the peaks of ω_1, ω_2 and ω_3. A one percent mass was used to show the effects of changes of various parameters since the large mass made the effects readily apparent and showed them in a faster time, compared to smaller masses which may be more feasible for the space station due to its considerable mass. The graphs of ω_1, ω_2 and ω_3 were used since the objective is to reduce the peaks of ω_2 and ω_3, with ω_1 tending to one value. Other variables were not discussed since their behavior and magnitudes were comparable to the 0.5% mass case which was examined in detail. This 0.5% mass case showed that the velocity and acceleration of the mass, and the power requirement are low. Therefore, the use of the optimal control system in actual operations is feasible. Compared to the force control law method, detumbling can be achieved in one-fourth the time. This decrease is considerable since stabilization may otherwise require hours. It should be noted that the optimization technique need not only be considered from the standpoint of minimizing time to detumble. Since time increases as mass decreases, a minimum mass solution can be obtained by fixing the time at the largest feasible value. No attempt was made to improve on the first-order gradient solution, because associated decreases in time or mass would probably be slight compared to the values already required.

Optimal control was applied specifically to stabilize a tumbling vehicle about its major principal axis. No simulations were made to achieve simple spin about a specific geometric axis. Such an application may be possible by appropriately varying the performance index and the direction of mass motion. However, further study is required to demonstrate this potential.

It was assumed that the principal moments of inertia of the main body do not change. However, an explosion may result in part of the spacecraft blowing off. Also, tumbling itself may cause loss of part of the vehicle. If this loss results in a large change in inertia properties, it should be included in the optimization. Specifically, the magnitudes of the moments of inertia should be corrected and the direction of the linear track should be altered to correspond with the new major axis if spin is desired about the axis. If direction change is not feasible, a redefinition of the performance index could be considered, which results in a longer time or a larger mass, and residual transverse angular velocity.

Considering actual operations in space, a hybrid computer may be more effective than a digital computer since modeling the vehicle dynamics on an analog will permit faster computation; as was discussed in the previous section, 100 sec of digital computer time are needed to obtain 100 sec of simulation control time. Comparison between computer predicted vehicle motions due to mass movement based on the optimization method and actual motions could then be continuously monitored, thereby permitting updating of the initial conditions for ω_1, ω_2 and ω_3 for the subsequent simulation time increments for optimization. This updating could be done without having to stop active control. Thus, the optimal technique presented here is significant if further study shows that an open loop solution to control a vehicle in real time, regardless of initial conditions, is practical. The highly nonlinear equations of motion preclude the use of an optimal closed loop approach. Normally, a nonoptimal feedback method like the one proposed by Edwards and Kaplan[4] would have to be used.

In regard to the practical application of this optimal controller for detumbling, several additional comments should be noted. First, even though the control mass is a small percentage of the main spacecraft's mass, consideration should be given to the use of an integral part of the main spacecraft itself. The movable mass would then have a function in addition to that of control. Second, the assumption of no external moments about the main spacecraft's center of mass will remain valid since disturbance torques such as caused by gravity gradient and solar radiation pressure are small compared to the torque due to control mass motion. Third, sensing of the angular velocities will cause no difficulties in the optimal control system. The actual control system for moving the control mass according to the optimal controller, including sensing of control mass position and velocity, is not discussed in this paper; however, its design should be straight forward.

Acknowledgement—This research was supported by NASA Grant NGR-39-009-210.

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APPENDIX A
EQUATIONS OF MOTION

The equations of motion are given below. The mass is permitted to move along the x axis which is fixed in the main body.

$$\begin{aligned}\dot{\omega}_x = {} & [B_1(x^2C_1 + x^4C_2 + C_3) \\ & + B_2(xC_4 + x^2C_5 + x^3C_6 + C_7) \\ & + B_3(xC_2 + x^2C_9 + x^3C_{10} + C_{11})] \\ & = [1.0/|\underset{\sim}{A}|]\end{aligned}$$

$$\begin{aligned}\dot{\omega}_y = {} & [B_1(xD_1 + x^2D_2 + x^3D_3 + D_4) \\ & + B_2(xD_5 + x^2D_6 + D_7) \\ & + B_3(xD_8 + x^2D_9 + D_{10})][1.0/|\underset{\sim}{A}|]\end{aligned}$$

$$\begin{aligned}\dot{\omega}_z = {} & [B_1(xE_1 + x^2E_2 + x^3E_3 + E_4) \\ & + B_2(xE_5 + x^2E_6 + E_7) \\ & + B_3(xE_8 + x^2E_9 + E_{10})][1.0/|\underset{\sim}{A}|]\end{aligned}$$

$$\begin{aligned}|\underset{\sim}{A}| = {} & \mu^2x^4I_x + x^3(-2I_{xy}\mu^2y - 2I_{xz}\mu^2z) \\ & + \mu x^2(I_xI_y + I_xI_z - I_{xy}^2 - I_{xy}^2 + I_x\mu y^2 \\ & + I_x\mu z^2 + I_y\mu y^2 + I_z\mu z^2 - 2I_{yz}\mu yz) \\ & + x(-2I_{xy}I_{yz}\mu z - 2I_{xy}\mu^2yz^2 - 2I_{xy}I_z\mu y \\ & - 2I_{xy}\mu^2y^3 - 2I_{yz}I_{xz}\mu y - 2I_{xz}\mu^2y^2z \\ & - 2I_{xz}I_y\mu z - 2I_{xz}\mu^2z^3) \\ & + I_xI_yI_z - I_xI_{yz}^2 - I_{xy}^2I_z - I_{xy}^2I_y \\ & - 2I_{xy}I_{xz}I_{yz} + I_xI_y\mu y^2 + I_xI_z\mu z^2 \\ & - 2I_xI_{yz}\mu yz + I_yI_z\mu y^2 + I_yI_z\mu z^2 \\ & + I_y\mu^2y^4 + I_y\mu^2y^2z^2 + I_z\mu^2y^2z^2 \\ & + I_z\mu^2z^4 - I_{yz}^2\mu y^2 - I_{yz}^2\mu z^2 \\ & - 2I_{yz}\mu^2y^3z - 2I_{yz}\mu^2yz^3 - I_{xy}I_{xz}\mu yz \\ & - I_{xy}^2\mu y^2 - I_{xz}I_{xy}\mu yz - I_{xz}^2\mu z^2\end{aligned}$$

$$\begin{aligned}B_1 = {} & (\omega_y^2 - \omega_z^2)(I_{yz} + \mu_{yz}) \\ & + \omega_x\omega_y(I_{xz} + \mu xz) - \omega_x\omega_z(I_{xy} + \mu xy) \\ & + \omega_y\omega_z(I_y - I_z - \mu y^2 + \mu z^2) + 2\omega_y\mu\dot{x}y \\ & + 2\omega_z\mu\dot{x}z\end{aligned}$$

$$\begin{aligned}B_2 = {} & (\omega_z^2 - \omega_x^2)(I_{xz} + \mu xz) - \omega_x\omega_y(I_{yz} + \mu yz) \\ & + \omega_x\omega_z(I_z - I_x - \mu z^2 + \mu x^2) \\ & + \omega_y\omega_z(I_{xy} + \mu xy) - 2\omega_y\mu x\dot{x} - \mu\ddot{x}z\end{aligned}$$

$$\begin{aligned}B_3 = {} & (\omega_x^2 - \omega_y^2)(I_{xy} + \mu xy) \\ & + \omega_x\omega_y(-I_y + I_x + \mu y^2 - \mu x^2) \\ & + \omega_x\omega_z(I_{yz} + \mu yz) - \omega_y\omega_z(I_{xz} + \mu xz) \\ & - 2\omega_z\mu x\dot{x} + \mu\ddot{x}y\end{aligned}$$

$$C_1 = I_y\mu + I_z\mu + \mu^2y^2 + \mu^2z^2$$

$$C_2 = \mu^2$$

$$C_3 = I_yI_z + I_y\mu y^2 + I_z\mu z^2 - I_{yz}^2 - 2I_{yz}\mu yz$$

$$C_4 = I_{yz}\mu z + \mu^2z^2y + I_z\mu y + \mu^2y^3$$

$$C_5 = I_{xy}\mu$$

$$C_6 = \mu^2y$$

$$C_7 = I_{yz}I_{xz} + I_{xz}\mu yz + I_{xy}I_z + I_{xy}\mu y^2$$

$$C_8 = I_{yz}\mu y + \mu^2y^2z + I_y\mu z + \mu^2z^3$$

$$C_9 = I_{xz}\mu$$

$$C_{10} = \mu^2z$$

$$C_{11} = I_{xy}I_{yz} + I_{xy}\mu yz + I_yI_{xz} + I_{xz}\mu z^2$$

$$D_1 = I_{yz}\mu z + \mu^2yz^2 + I_z\mu y + \mu^2y^3$$

$$D_2 = I_{xy}\mu$$

$$D_3 = \mu^2y$$

$$D_4 = I_{yz}I_{xz} + I_{xz}\mu yz + I_{xy}I_z + I_{xy}\mu y^2$$

$$D_5 = -2I_{xz}\mu z$$

$$D_6 = I_x\mu + \mu^2y^2$$

$$\begin{aligned}D_7 = {} & I_xI_z + I_x\mu y^2 + I_z\mu y^2 + \mu^2y^4 \\ & + I_z\mu z^2 + \mu^2y^2z^2 - I_{xz}^2\end{aligned}$$

$$D_8 = I_{xz}\mu y + I_{xy}\mu z$$

$$D_9 = \mu^2yz$$

$$\begin{aligned}D_{10} = {} & I_{xz}I_{xy} + I_xI_{yz} + I_x\mu yz + I_{yz}\mu y^2 \\ & + \mu^2y^3z + I_{yz}\mu z^2 + \mu^2yz^3\end{aligned}$$

$$E_1 = I_{yz}\mu y + \mu^2y^2z + I_y\mu z + \mu^2z^3$$

$$E_2 = I_{xz}\mu$$

$$E_3 = \mu^2z$$

$$E_4 = I_{xy}I_{yz} + I_{xy}\mu yz + I_yI_{xz} + I_{xz}\mu z^2$$

$$E_5 = I_{xz}\mu y + I_{xy}\mu z$$

$$E_6 = \mu^2yz$$

$$\begin{aligned}E_7 = {} & I_{xz}I_{xy} + I_xI_{yz} + I_x\mu yz + I_{yz}\mu y^2 \\ & + \mu^2y^3z + I_{yz}\mu z^2 + \mu^2yz^3\end{aligned}$$

$$E_8 = -2I_{xy}\mu y$$

$$E_9 = I_x\mu + \mu^2z^2$$

$$E_{10} = I_x I_y + I_x \mu z^2 + I_y \mu y^2 + \mu^2 y^2 z^2 + I_y \mu z^2 + \mu^2 z^4 - I_{xy}^2$$

APPENDIX B

OPTIMAL CONTROL

The problem must be reformulated for application to the first-order gradient method[6, 8]. The differential equations of motion given in the Appendix A can be put into the necessary form by the following substitutions:

$$\dot{x} = \beta$$

and, since $\ddot{x}$ is equal to $\dot{\beta}$,

$$\dot{\beta} = u_1.$$

The state variables, x_{s_i} with $i = 1, \ldots, 5$ are ω_x, ω_y, ω_z, x, and β, respectively. The one control variable is u_1; more control variables would be needed if the mass is not restricted to move parallel to one axis. The equations of motion can now be written as follows:

$$\begin{aligned}
\dot{\omega}_x &= \dot{\omega}_x(\omega_x, \omega_y, \omega_z, x, \beta, u_1),\\
\dot{\omega}_y &= \dot{\omega}_y(\omega_x, \omega_y, \omega_z, x, \beta, u_1),\\
\dot{\omega}_z &= \dot{\omega}_z(\omega_x, \omega_y, \omega_z, x, \beta, u_1),\\
\dot{x} &= \beta,\\
\dot{\beta} &= u_1.
\end{aligned}$$

In order to have simple spin about the maximum inertia axis ω_2 and ω_3, which are angular velocity components along the intermediate and minimum inertia axes, must be minimized. Thus, modeling after regulator problems, the performance index J will be expressed as

$$J = \frac{1}{2}\int_{t_0}^{t_f} \left(\frac{\omega_2^2}{\omega_{2_{max}}^2} + \frac{\omega_3^2}{\omega_{3_{max}}^2}\right) dt$$

where $\omega_{2_{max}}$ and $\omega_{3_{max}}$ are the maximum magnitudes that are desired. Ideally, these values should be zero to have pure, simple spin about the major principal axis. However, in practice these maxima will be set at some very small values. Parameters ω_2 and ω_3, need to be expressed in terms of state variables. If, for example x, y, and z axes are aligned with the maximum, intermediate, and minimum inertia axes, respectively, then ω_1, ω_2, and ω_3 are equal to ω_x, ω_y, and ω_z. The x position of the control mass is limited as follows:

$$x_{min} \leq x \leq x_{max}$$

with the values x_{min} and x_{max} arbitrarily set. In order to apply the penalty function technique, this is expressed as

$$x \leq x_{max}$$

and

$$-x \leq -x_{min}.$$

Thus, there will be two auxiliary state variables associated with the above two equations, P_1 and P_2 respectively. State variables x_{s_6} and x_{s_7} will refer to P_1 and P_2 with additional equations of motion:

$$\dot{P}_1 = \begin{cases} K_1(x - x_h)^2, x \geq x_h \\ 0, x < x_h \end{cases}$$

and

$$\dot{P}_2 = \begin{cases} K_2(-x + x_1)^2, -x \geq -x_1 \\ 0, -x < -x_1 \end{cases}$$

where x_h is less than x_{max} and x_1 is greater than x_{min}. We can now place terminal constraints on the two auxiliary state variables as follows

$$P_1(t_f) = 0$$

and

$$P_2(t_f) = 0.$$

APPENDIX C

ENERGY AND POWER

The total angular momentum vector with respect to the center of mass of the system, $\mathbf{H}_{cm}$ must remain constant during control mass motion. This can be seen by noting that [9]

$$\mathbf{\Gamma} = \dot{\mathbf{H}}_{cm},$$

and, since there is no external moment during mass motion, $\mathbf{H}_{cm}$ is constant. An expression for this total angular momentum vector can be obtained by dividing it into two parts: that due to the rotation of the main body and that due to the motions of the centers of mass of the main body and the movable object about the center of mass of the system. The former is simply $\bar{I} \cdot \boldsymbol{\omega}$ where $\bar{I}$ is the inertia dyadic. The latter can be expressed as $\mu \mathbf{r} \times \dot{\mathbf{r}}$ by considering the two-body problem composed of the main body and the movable mass as an equivalent one-body problem which is the equivalent mass μ moving at a distance $\mathbf{r}$

from the center of mass of the system. Thus, we have

$$\mathbf{H}_{cm} = \boldsymbol{I} \cdot \boldsymbol{\omega} + \mu \mathbf{r} \times \dot{\mathbf{r}}$$

where

$$\dot{\mathbf{r}} = [\dot{\mathbf{r}}] + \boldsymbol{\omega} \times \mathbf{r}.$$

With this total angular momentum vector constant, the rotational kinetic energy, T, can assume the following values

$$\frac{H_{cm}^2}{2I_{max}} \leq T \leq \frac{H_{cm}^2}{2I_{min}}$$

where I_{max} and I_{min} are the spacecraft's maximum and minimum moments of inertia. Specifically, using the same analysis as was used to obtain H_{cm}, we can write the following expression for kinetic energy relative to the center of mass of the system during mass motion

$$T = \frac{1}{2} \boldsymbol{\omega} \cdot \boldsymbol{I} \cdot \boldsymbol{\omega} + \frac{1}{2} \mu \dot{\mathbf{r}} \cdot \dot{\mathbf{r}}$$

To have simple spin about the maximum inertia axis, rotational kinetic energy, which initially is some constant value, must be decreased to the value associated with this simple spin, $H_{cm}^2/2I_{max}$. It should be noted that simple spin about the major principal axis of the spacecraft can be considered spin about the major principal axis of the main body. If the control mass were large and far from the center of mass of the main body, the orientation of the main body maximum inertia axis may be quite different from that of the spacecraft. In that case, the maximum inertia axis of the spacecraft should be used when referring to simple spin about the maximum inertia axis. Since change in kinetic energy of the system requires work to be done by force $\mathbf{f}_m$, acting on the control mass, through a distance $d\mathbf{r}$,

$$d(\text{Work}) = \mathbf{f}_m \cdot d\mathbf{r}.$$

This equation can also be obtained by considering $\mathbf{f}_m$ as acting on the equivalent mass μ and causing a displacement $d\mathbf{r}$. For mass motion along a linear track parallel to the x axis, the right hand side may be written as $f_{m_x}\,dx$. Therefore, we can write the following expression for the rate of change of kinetic energy which is equal to the rate at which work is being done

$$\dot{T} = f_{m_x}\dot{x}.$$

It should be noted that $\dot{T}$ determines the power that will be required.