NESTED QUANTIFIERS

RULES OF INFERENCE

Lecture 3

Nested Quantifiers

- A quantifier is within the scope of another.
- $\forall x \, \exists y (x + y = 0)$

Example

Let x and y be real numbers and p(x, y) denotes "x + y = 0". Find the truth values of:

- a. $\forall x \forall y\ P(x, y)$
- a. $\forall x \exists y\ P(x, y)$
- a. $\exists y \forall x\ P(x, y)$
- a. $\exists x \exists y\ P(x, y)$

Examples

- The statement p(x, y) denotes "x + y = 0", and we are to determine the truth values of various quantified statements about this relation. Here's the analysis of each statement:
- a. $\forall x \forall y\ P(x,y)$: This reads as "For all x and for all y, x + y = 0." This statement is false. It asserts that no matter what x and y are, their sum is always zero, which is clearly not true for all real numbers. For example, if x = 1 and y = 1, then x + y ≠ 0.
- b. $\forall x \exists y\ P(x,y)$: This reads as "For all x, there exists a y such that x + y = 0." This statement is true. For any real number x, we can always find a real number y such that their sum is zero, specifically y = -x.
- c. $\exists y \forall x\ P(x,y)$: This reads as "There exists a y such that for all x, x + y = 0." This statement is false. There does not exist a single y that when added to every x will always result in zero. The value of y that makes the sum zero depends on the choice of x.
- d. $\exists x \exists y\ P(x,y)$: This reads as "There exists an x and there exists a y such that x + y = 0." This statement is true. It is possible to find at least one pair of real numbers (x, y) whose sum is zero. For instance, x = 1 and y = -1 satisfy the condition.

Quantifier Exercise

If R(x, y)="x relies upon y," express the following in unambiguous English:

$\forall x(\exists y\ R(x, y))$= everyone has someone to reply upon

$\exists y(\forall x\ R(x, y))$= There is a y whom everyone relies upon, including himself.

$\exists x(\forall y\ R(x, y))$=There exist an x who relies upon everybody.

$\forall y(\exists x\ R(x, y))$= Everybody has someone who relies upon them

$\forall x(\forall y\ R(x, y))$= Everyone relies upon everybody

English examples with negation

- Example 1: Every integer has a larger integer
- Example 2: Some number in D is the largest

Negation Examples

- **Example 1: "Every integer has a larger integer"**
- This is a true statement in the context of the set of integers. Mathematically, this can be expressed as:
- $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}$ such that $y > x$.
- For any integer x, there exists another integer y that is larger than x. This is an inherent property of the set of integers, which is infinite and unbounded.
- **Example 2: "Some number in D is the largest"**
- The truth of this statement depends on the nature of the set D. If D is a subset of integers, real numbers, or any ordered set, the following interpretations are possible:
  - If D is a finite set, then the statement is true. There exists a number in D that is larger than all the other numbers in that set.
  - If D is an infinite set like the set of all integers, the set of all positive integers, or the set of real numbers, then the statement is false. There is no single "largest" number because for any number you choose, there will always be a larger number.
- For a precise conclusion, the set D would need to be defined clearly. If it is a finite set, there will indeed be a largest element; if it is infinite like the integers or real numbers, then there is no largest element.

Translating from English to Logical Expressions

U = {Frosty, Sweet, tall}

F(x): x is Frosty

S(x): x is Sweet

T(x): x is Tall

Translate “Everything is Frosty”

Solution: $\forall x\, F(x)$

Translation (cont)

U = {Frosty, Sweet, Tall}

F(x): x is Frosty

S(x): x is Sweet

T(x): x is Tall

"Nothing is Sweet."

Solution: $\neg\exists x\ S(x)$

$\forall x \neg S(x)$

Translation (cont)

U = {Frosty, Sweet, Tall}

F(x): x is Frosty

S(x): x is Sweet

T(x): x is Tall

"All Frosty are Sweet"

Solution: $\forall x\ (F(x) \rightarrow S(x))$

RULES OF INFERENCE

Rules of Inference

- Rules of Inference are use for taking a set of premises and getting to a conclusion.
- Deducing Logical arguments
- We have the two premises:

  "If you have a current password, then you can log into the network"

  "You have a current password"

  And the conclusion:

  "You can log onto the network"

Arguments

- The rules of inference are the most important ingredients in producing valid arguments.
- We will show how to construct valid arguments in two stages.

1. Propositional Logic

    Inference rules

2. Predicate Logic

    Inference rules for propositional logic plus additional inference rules to handle variables and quantifiers.

Arguments in Propositional Logic

- An ***argument*** in propositional logic is a sequence of propositions.
- All but the final proposition is called ***premises***.
- The last statement is the ***conclusion***.
- The argument is ***valid*** if the premises imply the conclusion.
  - If the premises are $p_1, p_2, \ldots, p_n$ and the conclusion is q then $(p_1 \land p_2 \land \ldots \land p_n) \rightarrow q$ is a tautology.
- An ***argument form*** is an argument that is valid no matter what propositions are substituted into its propositional variables.

Rules of Inference

- **Inference rules** are all simple argument forms that will be used to construct more complex argument forms.
- Following are the most important rules of inference in propositional logic:

Rules of Inference

- Modus Ponens

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$

- Modus Tollens

$$\begin{array}{l} p \rightarrow q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

- Hypothetical Syllogism

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

- Disjunctive Syllogism

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$$

- Addition

$$\begin{array}{l} p \\ \hline \therefore p \vee q \end{array}$$

- Simplification

$$\begin{array}{l} p \wedge q \\ \hline \therefore q \end{array}$$

- Conjunction

$$\begin{array}{l} p \\ q \\ \hline \therefore p \wedge q \end{array}$$

- Resolution

$$\begin{array}{l} \neg p \vee r \\ p \vee q \\ \hline \therefore q \vee r \end{array}$$

Modus Ponens – Law of Detachment

$$\begin{array}{l} p \to q \\ p \\ \hline \therefore q \end{array}$$

- Modus Ponens is Latin for mode that affirms
- Sometimes referred to as affirming the antecedents
- Corresponding Tautology: $(p \land (p \to q)) \to q$

Example:

Let p be "It is snowing."

Let q be "I will study discrete math."

"If it is snowing, then I will study discrete math." Premises

"It is snowing." Hypotheses

"I will study discrete math." is true. Conclusion.

Modus Tollens

$$\begin{array}{l} p \rightarrow q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

Sometimes known as 'Denying the consequent'

Corresponding Tautology: $(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$

Example:
Let p be "It is snowing."
Let q be "I will study discrete math."

"If it is snowing, then I will study discrete math."
"I will not study discrete math."

"Therefore, it is not snowing."

Hypothetical Syllogism

Corresponding Tautology: $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$

$p \rightarrow q$
$q \rightarrow r$

$\therefore p \rightarrow r$

Example:
Let p be "it snows."
Let q be "I will study discrete math."
Let r be "I will get an A."

"If it snows, then I will study discrete math."
"If I study discrete math, I will get an A."

"Therefore, If it snows, I will get an A."

Disjunctive Syllogism - DS

Corresponding Tautology: $(\neg p \land (p \lor q)) \to q$

$$\begin{array}{c} p \lor q \\ \neg p \\ \hline \therefore q \end{array}$$

Example:
Let p be "I will study discrete math."
Let q be "I will study English literature."

"I will study discrete math, or I will study English literature."
"I will not study discrete math."

"Therefore, I will study English literature."

Addition

Corresponding Tautology: $p \rightarrow (p \vee q)$

$$\begin{array}{c} p \\ \hline \therefore p \vee q \end{array}$$

Example:
Let p be "I will study discrete math."
Let q be "I will visit Las Vegas."

"I will study discrete math."

"Therefore, I will study discrete math, or I will visit Las Vegas."

Basically if P is true, P or anything is true.

Simplification

Corresponding Tautology: $(p \wedge q) \rightarrow p$

$$\begin{array}{c} p \wedge q \\ \hline \therefore p \\ \therefore q \end{array}$$

Example:
Let p be "I will study discrete math."
Let q be "I will study English literature."

"I will study discrete math and English literature"

"Therefore, I will study discrete math."

Conjunction

Corresponding Tautology: $((p) \land (q)) \rightarrow (p \land q)$

$$\begin{array}{c} p \\ q \\ \hline \therefore p \land q \end{array}$$

Example:
Let p be "I will study discrete math."
Let q be "I will study English literature."

"I will study discrete math."
"I will study English literature."

"Therefore, I will study discrete math and I will study English literature."

Resolution

Plays an important role in AI and is used in Prolog

Corresponding Tautology: $((\neg p \vee r) \wedge (p \vee q)) \rightarrow (q \vee r)$

$$\begin{array}{c} \neg p \vee r \\ p \vee q \\ \hline \therefore q \vee r \end{array}$$

Example:
Let p be "I will study discrete math."
Let r be "I will study English literature."
Let q be "I will study databases."

"I will not study discrete math, or I will study English literature."
"I will study discrete math, or I will study databases."

"Therefore, I will study databases, or I will study English literature."

Using the Rules of Inference to Build Valid Arguments

- A ***valid argument*** is a sequence of statements. Each statement is either a **premise** or follows from previous statements by **rules of inference**. The last statement is called **conclusion**.
- A valid argument takes the following form:

Example 1

From the single proposition $p \wedge (p \rightarrow q)$. Show that q is a conclusion.

Rules of Inference

- Modus Ponens
$$\begin{array}{l} p \rightarrow q \\ \underline{p} \\ \therefore q \end{array}$$
- Modus Tollens
$$\begin{array}{l} p \rightarrow q \\ \underline{\neg q} \\ \therefore \neg p \end{array}$$
- Hypothetical Syllogism
$$\begin{array}{l} p \rightarrow q \\ \underline{q \rightarrow r} \\ \therefore p \rightarrow r \end{array}$$
- Disjunctive Syllogism
$$\begin{array}{l} p \vee q \\ \underline{\neg p} \\ \therefore q \end{array}$$
- Addition
$$\begin{array}{l} \underline{p} \\ \therefore p \vee q \end{array}$$
- Simplification
$$\begin{array}{l} \underline{p \wedge q} \\ \therefore q \end{array}$$
- Conjunction
$$\begin{array}{l} p \\ \underline{q} \\ \therefore p \wedge q \end{array}$$
- Resolution
$$\begin{array}{l} \neg p \vee r \\ \underline{p \vee q} \\ \therefore q \vee r \end{array}$$

Example 2

We have three premise R , $R \to\ D\ and\ D \to \neg J$.
Show that 1-3 entails $\neg J$

Rules of Inference

- Modus Ponens
$$\begin{array}{l} p \to q \\ p \\ \hline \therefore q \end{array}$$
- Modus Tollens
$$\begin{array}{l} p \to q \\ \neg q \\ \hline \therefore \neg p \end{array}$$
- Hypothetical Syllogism
$$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$$
- Disjunctive Syllogism
$$\begin{array}{l} p \lor q \\ \neg p \\ \hline \therefore q \end{array}$$
- Addition
$$\begin{array}{l} p \\ \hline \therefore p \lor q \end{array}$$
- Simplification
$$\begin{array}{l} p \land q \\ \hline \therefore q \end{array}$$
- Conjunction
$$\begin{array}{l} p \\ q \\ \hline \therefore p \land q \end{array}$$
- Resolution
$$\begin{array}{l} \neg p \lor r \\ p \lor q \\ \hline \therefore q \lor r \end{array}$$

Example 3

- With these hypotheses:
  - “It is not sunny this afternoon and it is colder than yesterday.”
  - “We will go swimming only if it is sunny.”
  - “If we do not go swimming, then we will take a canoe trip.”
  - “If we take a canoe trip, then we will be home by sunset.”
- Using the inference rules, construct a valid argument for the conclusion:
  - “We will be home by sunset.

Handling Quantified Statements

- Valid arguments for quantified statements are a sequence of statements.
- Each statement is either a premise or follows from previous statements by rules of inference which include:
  - Rules of Inference for Propositional Logic
  - Rules of Inference for Quantified Statements
- The rules of inference for quantified statements are introduced in the next several slides.

Universal Instantiation (UI)

$P(c)$ is true, where c is a particular member of the domain, given the premises $\forall x P(x)$

Example:

Our domain consists of all cats and Simba is a cat.

"All cats are cuddly"

"Therefore, Simba is cuddly."

$$\frac{\forall x P(x)}{\therefore P(c)}$$

Universal Generalization (UG)

$\forall x p(x)$ is true, given the premises that $P(c)$ is true for all the elements of c in the domain.

Used often implicitly in Mathematical Proofs.

Example:

Let p (c) represent “A byte contains 8 bits”

So, for $\forall x p(x)$ ”All bytes contain 8 bits”, it will also be true.

$$\frac{P(c) \text{ for an arbitrary } c}{\therefore \forall x P(x)}$$

Existential Instantiation (EI)

Allows us to conclude that there is an element c in the domain for which p(c) is true if we know that $\exists x p(x)$ is true

Example:

"There is someone who got an A in the course."

"Let's call her a and say that a got an A"

$$\frac{\exists x P(x)}{\therefore P(c) \text{ for some element } c}$$

Existential Generalization (EG)

Example:

"John got an A in the class."

"Therefore, someone got an A in the class."

$$\frac{P(c) \text{ for some element } c}{\therefore \exists x P(x)}$$

Quantifier Inference Rules

- Universal Instantiation (UI)

$$\frac{\forall x P(x)}{\therefore P(c)}$$

- Universal Generalization (UG)

$$\frac{P(c) \text{ for an arbitrary } c}{\therefore \forall x P(x)}$$

- Universal Modus Ponens

$$\frac{\begin{array}{l}\forall x(P(x) \rightarrow Q(x)) \\ P(a), \text{where } a \text{ is a particular} \\ \quad \text{element in the domain}\end{array}}{\therefore Q(a)}$$

- Existential Instantiation (EI)

$$\frac{\exists x P(x)}{\therefore P(c) \text{ for some element } c}$$

- Existential Generalization (EG)

$$\frac{P(c) \text{ for some element } c}{\therefore \exists x P(x)}$$

Universal Modus Ponens

Universal Modus Ponens combines universal instantiation and modus ponens into one rule.

$$\begin{array}{l} \forall x(P(x) \to Q(x)) \\ P(a), \text{where } a \text{ is a particular} \\ \qquad \text{element in the domain} \\ \hline \therefore Q(a) \end{array}$$

Rules of Inference For Quantitative Statement

Example1:

Using the rules of inference, construct a valid argument to show that

"Joey has two eyes"

is a consequence of the premises:

"Every man has two eyes" "John Smith is a man"

Example2:

Use the rules of inference to construct a valid argument showing that the conclusion

“Someone who passed the first exam has not read the book.”

follows from the premises

“A student in this class has not read the book.”

“Everyone in this class passed the first exam.”