CHAPTER 10

THEORY OF GRAPHS AND TREES

10.4

Trees: Examples and Basic Properties

Trees: Examples and Basic Properties

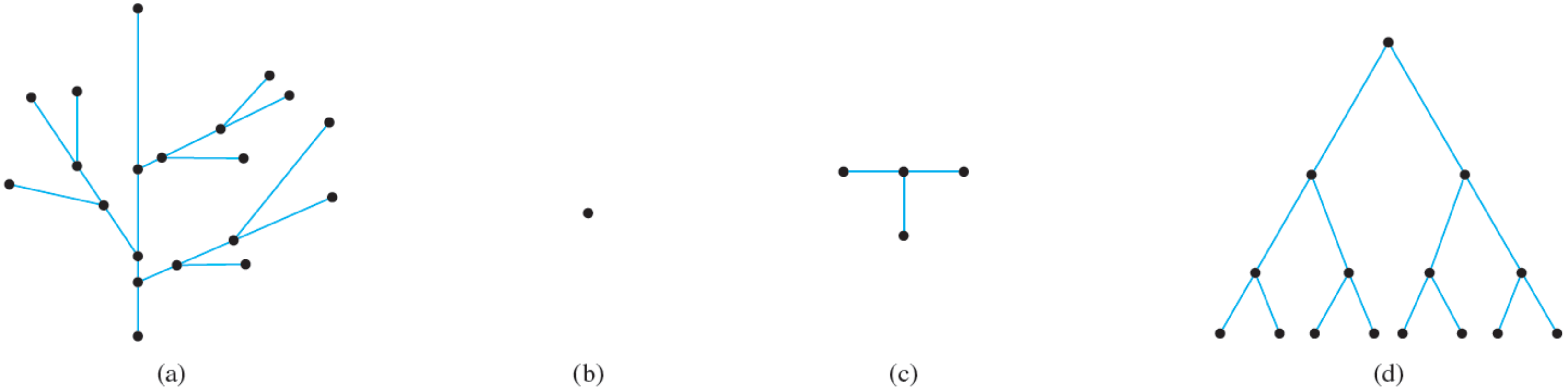
In mathematics, a tree is a connected graph that does not contain any circuits. Mathematical trees are similar in certain ways to their botanical namesakes.

Definition

A graph is said to be **circuit-free** if, and only if, it has no circuits. A graph is called a **tree** if, and only if, it is circuit-free and connected. A **trivial tree** is a graph that consists of a single vertex. A graph is called a **forest** if, and only if, it is circuit-free and not connected.

Example 10.4.1 – *Trees and Non-trees (1/2)*

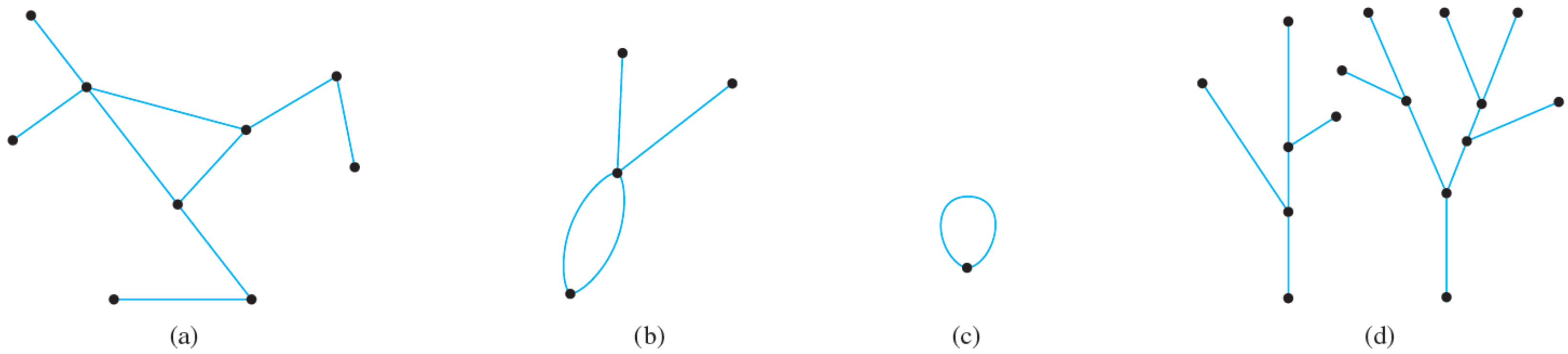
All the graphs shown in Figure 10.4.1 are trees, whereas those in Figure 10.4.2 are not.



Trees. All the graphs in (a)–(d) are connected and circuit-free.

Figure 10.4.1

Example 10.4.1 – *Trees and Non-trees (2/2)*



Non-trees. The graphs in (a), (b), and (c) all have circuits, and the graph in (d) is not connected.

Figure 10.4.2

Examples of Trees

Example 10.4.2 – *A Decision Tree (1/2)*

During orientation week, a college administers a mathematics placement exam to all entering students. The exam consists of two parts, and placement recommendations are made as indicated by the tree shown in Figure 10.4.3.

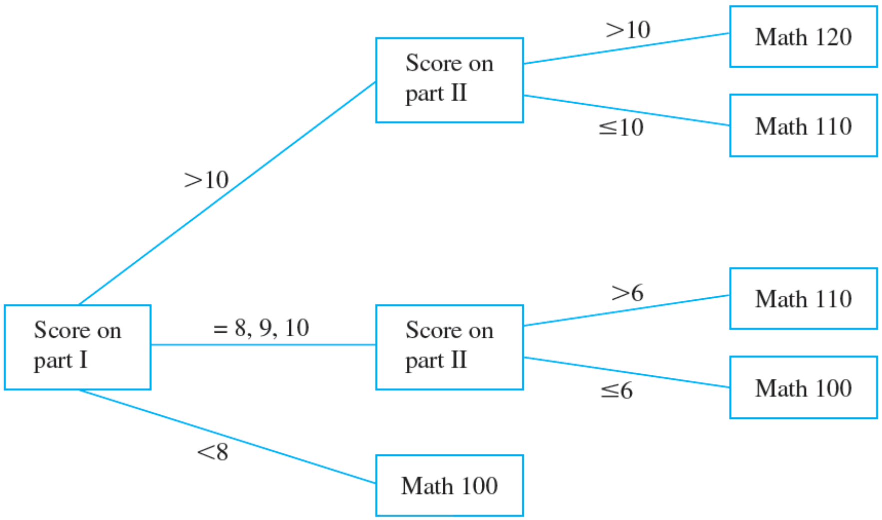


Figure 10.4.3

Example 10.4.2 – *A Decision Tree (2/2)*

Read the tree from left to right to decide what course should be recommended for a student who scored 9 on part I and 7 on part II.

Example 10.4.2 – *Solution*

Since the student scored 9 on part I, the score on part II is checked.

Since it is greater than 6, the student should be advised to take Math 110.

Example 10.4.3 – *A Parse Tree (1/7)*

In the last 30 years, Noam Chomsky and others have developed new ways to describe the syntax (or grammatical structure) of natural languages such as English.

In the study of grammars, trees are often used to show the derivation of grammatically correct sentences from certain basic rules. Such trees are called **syntactic derivation trees** or **parse trees**.

Example 10.4.3 – *A Parse Tree (2/7)*

A very small subset of English grammar, for example, specifies that

1. a sentence can be produced by writing first a noun phrase and then a verb phrase;
2. a noun phrase can be produced by writing an article and then a noun;
3. a noun phrase can also be produced by writing an article, then an adjective, and then a noun;
4. a verb phrase can be produced by writing a verb and then a noun phrase;

Example 10.4.3 – *A Parse Tree (3/7)*

5. one article is “the”;
6. one adjective is “young”;
7. one verb is “caught”;
8. one noun is “man”;
9. one (other) noun is “ball.”

The rules of a grammar are called **productions**. It is customary to express them using the shorthand notation illustrated on the next slide.

Example 10.4.3 – *A Parse Tree (4/7)*

This notation, introduced by John Backus in 1959 and modified by Peter Naur in 1960, was used to describe the computer language Algol and is called the **Backus–Naur notation**.

In the notation, the symbol | represents the word *or*, and angle brackets ⟨ ⟩ are used to enclose terms to be defined (such as a sentence or noun phrase).

1. ⟨sentence⟩ → ⟨noun phrase⟩ ⟨verb phrase⟩

2., 3. ⟨noun phrase⟩ → ⟨article⟩ ⟨noun⟩ | ⟨article⟩ ⟨adjective⟩ ⟨noun⟩

Example 10.4.3 – *A Parse Tree (5/7)*

4. $\langle$verb phrase$\rangle \rightarrow \langle$verb$\rangle$ $\langle$noun phrase$\rangle$

5. $\langle$article$\rangle \rightarrow$ the

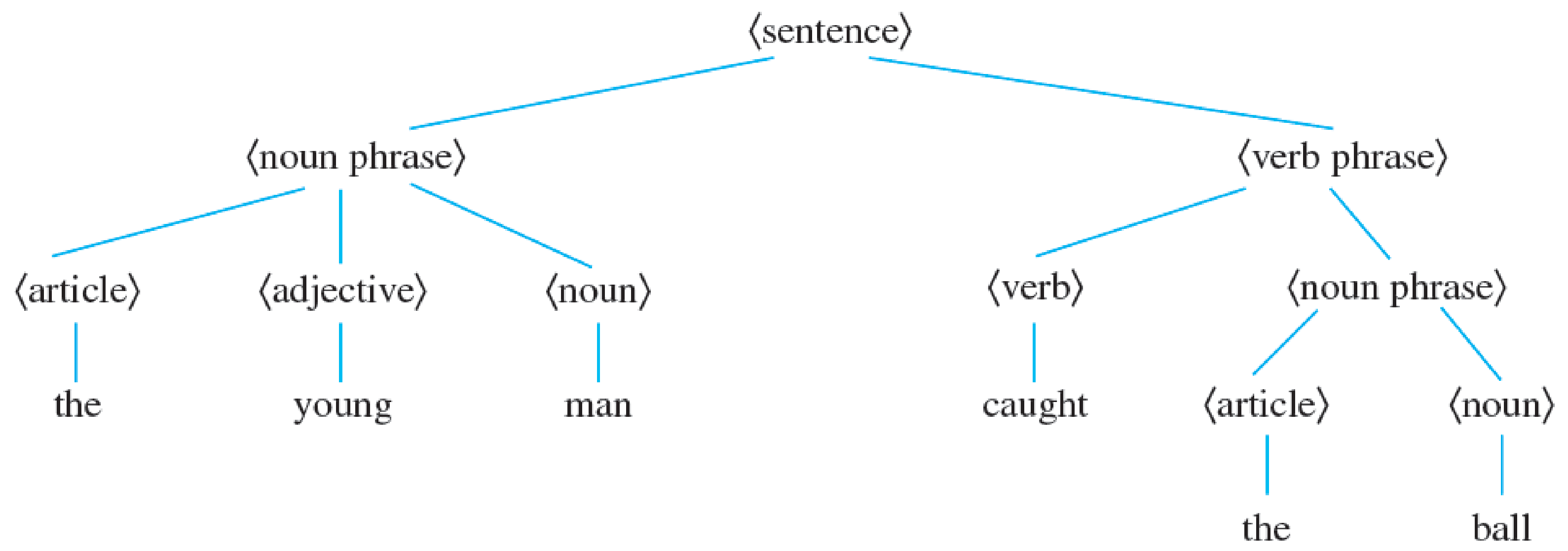
6. $\langle$adjective$\rangle \rightarrow$ young

7. $\langle$verb$\rangle \rightarrow$ caught

8., 9. $\langle$noun$\rangle \rightarrow$ man $|$ ball

Example 10.4.3 – *A Parse Tree (6/7)*

The derivation of the sentence "The young man caught the ball" from the above rules is described by the tree shown below.



Example 10.4.3 – *A Parse Tree (7/7)*

In the study of linguistics, **syntax** refers to the grammatical structure of sentences, and **semantics** refers to the meanings of words and their interrelations.

A sentence can be syntactically correct but semantically incorrect, as in the nonsensical sentence “The young ball caught the man,” which can be derived from the rules given on the previous slides.

Or a sentence can contain syntactic errors but not semantic ones, as, for instance, when a two-year-old child says, “Me hungry!”

Characterizing Trees

Characterizing Trees (1/6)

There is a somewhat surprising relation between the number of vertices and the number of edges of a tree.

It turns out that if n is a positive integer, then any tree with n vertices (no matter what its shape) has $n - 1$ edges.

Perhaps even more surprisingly, a partial converse to this fact is also true—namely, any connected graph with n vertices and $n - 1$ edges is a tree.

Characterizing Trees (2/6)

It follows from these facts that if even one new edge (but no new vertex) is added to a tree, the resulting graph must contain a circuit.

Also, from the fact that removing an edge from a circuit does not disconnect a graph, it can be shown that every connected graph has a subgraph that is a tree.

It follows that if n is a positive integer, any graph with n vertices and *fewer* than $n - 1$ edges is not connected.

Characterizing Trees (3/6)

A small but very important fact necessary to derive the first main theorem about trees is that any nontrivial tree must have at least one vertex of degree 1.

Lemma 10.4.1

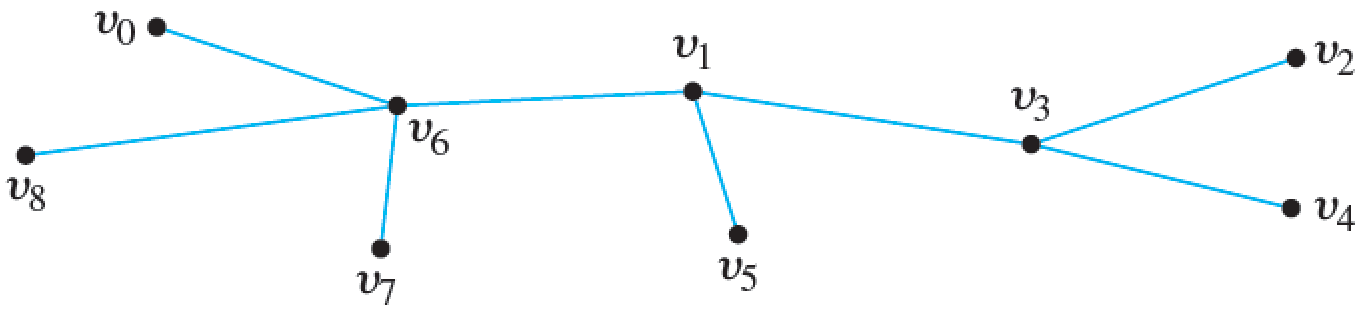
Any tree that has more than one vertex has at least one vertex of degree 1.

Definition

Let T be a tree. If T has at least two vertices, then a vertex of degree 1 in T is called a **leaf** (or a **terminal vertex**), and a vertex of degree greater than 1 in T is called an **internal vertex** (or a **branch vertex**). The unique vertex in a trivial tree is also called a **leaf** or **terminal vertex**.

Example 10.4.5 – *Leaves and Internal Vertices in Trees*

Find all leaves (or terminal vertices) and all internal (or branch) vertices in the following tree:



Example 10.4.5 – *Solution*

The leaves (or terminal vertices) are v_0, v_2, v_4, v_5, v_7, and v_8. The internal (or branch) vertices are v_6, v_1, and v_3.

Characterizing Trees (4/6)

Theorem 10.4.2

For any positive integer n, any tree with n vertices has $n - 1$ edges.

Example 10.4.6 – *Determining Whether a Graph Is a Tree*

A graph G has ten vertices and twelve edges. Is it a tree?

Example 10.4.6 – *Solution*

No. By Theorem 10.4.2, any tree with ten vertices has nine edges, not twelve.

Theorem 10.4.2

For any positive integer n, any tree with n vertices has $n - 1$ edges.

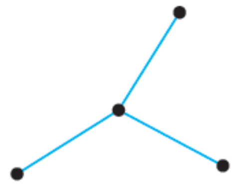
Example 10.4.7 – *Finding Trees Satisfying Given Conditions*

Find all nonisomorphic trees with four vertices.

Example 10.4.7 – *Solution*

By Theorem 10.4.2, any tree with four vertices has three edges. Thus the total degree of a tree with four vertices must be 6. Also, every tree with more than one vertex has at least two vertices of degree 1.

Thus the following combinations of degrees for the vertices are the only ones possible: 1, 1, 1, 3 and 1, 1, 2, 2. There are two nonisomorphic trees corresponding to both of these possibilities, as shown below.



and

Characterizing Trees (5/6)

Lemma 10.4.3

If G is any connected graph, C is any circuit in G, and any one of the edges of C is removed from G, then the graph that remains is connected.

Theorem 10.4.4

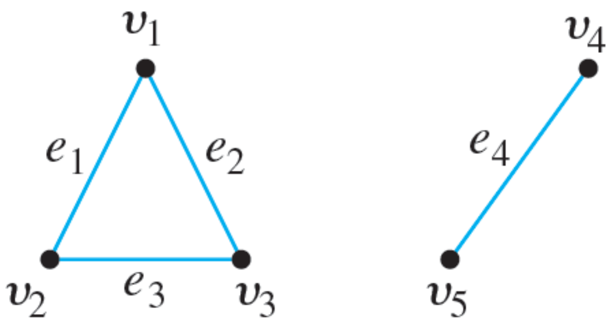
For any positive integer n, if G is a connected graph with n vertices and $n - 1$ edges, then G is a tree.

Example 10.4.8 – *A Graph with n Vertices and n − 1 Edges That Is Not a Tree*

Give an example of a graph with five vertices and four edges that is not a tree.

Example 10.4.8 – *Solution*

By Theorem 10.4.4, such a graph cannot be connected. One example of such an unconnected graph is shown below.



Characterizing Trees (6/6)

Corollary 10.4.5

If G is any graph with n vertices and m edges, where m and n are positive integers and $m \geq n$, then G has a circuit.