CHAPTER 9

COUNTING AND PROBABILITY

9.1 Introduction to Probability

Introduction to Probability (1/7)

Imagine tossing two coins and observing whether 0, 1, or 2 heads are obtained. It would be natural to guess that each of these events occurs about one-third of the time, but in fact this is not the case. Table 9.1.1 shows actual data obtained from tossing two quarters 50 times.

Event	Tally	Frequency (Number of times the event occurred)	Relative Frequency (Fraction of times the event occurred)
2 heads obtained	𝍸 𝍸 |	11	22%
1 head obtained	𝍸 𝍸𝍸 𝍸𝍸 ||	27	54%
0 heads obtained	𝍸𝍸 ||	12	24%

Experimental Data Obtained from Tossing Two Quarters 50 Times

Table 9.1.1

Introduction to Probability (2/7)

To formalize this analysis and extend it to more complex situations, we introduce the notions of random process, sample space, event, and probability.

To say that a process is **random** means that when it takes place, one outcome from some set of outcomes is sure to occur, but it is impossible to predict with certainty which outcome that will be.

Introduction to Probability (3/7)

For instance, if an ordinary person performs the experiment of tossing an ordinary coin into the air and allowing it to fall flat on the ground, it can be predicted with certainty that the coin will land either heads up or tails up (so the set of outcomes can be denoted {heads, tails}), but it is not known for sure whether heads or tails will occur.

Introduction to Probability (4/7)

We restricted this experiment to ordinary people because a skilled magician can toss a coin in a way that appears random but is not, and a physicist equipped with first-rate measuring devices may be able to analyze all the forces on the coin and correctly predict its landing position.

Just a few of many examples of random processes or experiments are choosing winners in state lotteries, selecting respondents in public opinion polls, and choosing subjects to receive treatments or serve as controls in medical experiments.

Introduction to Probability (5/7)

The set of outcomes that can result from a random process or experiment is called a *sample space.*

Definition

A **sample space** is the set of all possible outcomes of a random process or experiment. An **event** is a subset of a sample space.

In the case where an experiment has finitely many outcomes and all outcomes are equally likely to occur, the *probability* of an event (set of outcomes) is just the ratio of the number of outcomes in the event to the total number of outcomes.

Introduction to Probability (6/7)

Equally Likely Probability Formula

If S is a finite sample space in which all outcomes are equally likely and E is an event in S, then the **probability of *E***, denoted ***P*(*E*)**, is

$$P(E) = \frac{\text{the number of outcomes in } E}{\text{the total number of outcomes in } S}.$$

Introduction to Probability (7/7)

Notation

For any finite set A, $N(A)$ denotes the number of elements in A.

With this notation, the equally likely probability formula becomes

$$\boxed{P(E) = \frac{N(E)}{N(S)}.}$$

Example 9.1.1 – *Probabilities for a Deck of Cards (1/3)*

An ordinary deck of cards contains 52 cards divided into four *suits.* The *red suits* are diamonds (♦) and hearts (♥), and the *black suits* are clubs (♣) and spades (♠).

Each suit contains 13 cards of the following *denominations:* 2, 3, 4, 5, 6, 7, 8, 9, 10, J (jack), Q (queen), K (king), and A (ace). The cards J, Q, and K are called *face cards.*

Mathematician Persi Diaconis, working with David Aldous in 1986 and Dave Bayer in 1992, showed that seven shuffles are needed to “thoroughly mix up” the cards in an ordinary deck.

Example 9.1.1 – *Probabilities for a Deck of Cards (2/3)*

continued

In 2000 mathematician Nick Trefethen, working with his father, Lloyd Trefethen, a mechanical engineer, used a somewhat different definition of “thoroughly mix up” to show that six shuffles will nearly always suffice.

Imagine that the cards in a deck have become—by some method—so thoroughly mixed up that if you spread them out face down and pick one at random, you are as likely to get any one card as any other.

Example 9.1.1 – *Probabilities for a Deck of Cards (3/3)*

continued

a. What is the sample space of outcomes?

b. What is the event that the chosen card is a black face card?

c. What is the probability that the chosen card is a black face card?

Example 9.1.1 – *Solution (1/2)*

a. The outcomes in the sample space *S* are the 52 cards in the deck.

b. Let *E* be the event that a black face card is chosen. The outcomes in *E* are the jack, queen, and king of clubs and the jack, queen, and king of spades. Symbolically:

$$E = \{J\clubsuit, Q\clubsuit, K\clubsuit, J\spadesuit, Q\spadesuit, K\spadesuit\}.$$

Example 9.1.1 – *Solution (2/2)*

continued

c. By part (b), $N(E) = 6$, and according to the description of the situation, all 52 outcomes in the sample space are equally likely. Therefore, by the equally likely probability formula, the probability that the chosen card is a black face card is

$$P(E) = \frac{N(E)}{N(S)} = \frac{6}{52} \cong 11.5\%$$

Counting the Elements of a List

Counting the Elements of a List (1/3)

Some counting problems are as simple as counting the elements of a list. For instance, how many integers are there from 5 through 12? To answer this question, imagine going along the list of integers from 5 to 12, counting each in turn.

list:	5	6	7	8	9	10	11	12
	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
count:	1	2	3	4	5	6	7	8

So the answer is 8.

Counting the Elements of a List (2/3)

More generally, if m and n are integers and $m \leq n$, how many integers are there from m through n? To answer this question, note that $n = m + (n - m)$, where $n - m \geq 0$ *[since $n \geq m$]*.

Note also that the element $m + 0$ is the first element of the list, the element $m + 1$ is the second element, the element $m + 2$ is the third, and so forth.

Counting the Elements of a List (3/3)

In general, the element $m + i$ is the $(I + 1)$st element of the list.

$$\begin{array}{llllll} \text{list:} & m\ (= m + 0) & m + 1 & m + 2 & \ldots & n\ (= m + (n - m)) \\ & \updownarrow & \updownarrow & \updownarrow & & \updownarrow \\ \text{count:} & 1 & 2 & 3 & \ldots & (n - m) + 1 \end{array}$$

And so the number of elements in the list is $n - m + 1$.

Theorem 9.1.1 The Number of Elements in a List

If m and n are integers and $m \leq n$, then there are $n - m + 1$ integers from m to n inclusive.

Example 9.1.4 – *Counting the Elements of a Sublist*

a. How many three-digit integers (integers from 100 to 999 inclusive) are divisible by 5?

b. What is the probability that a randomly chosen three-digit integer is divisible by 5?

Example 9.1.4 – *Solution (1/2)*

a. Imagine writing the three-digit integers in a row, noting those that are multiples of 5 and drawing arrows between each such integer and its corresponding multiple of 5.

100	101	102	103	104	105	106	107	108	109	110	. . .	994	995	996	997	998	999
$\updownarrow$					$\updownarrow$					$\updownarrow$			$\updownarrow$				
$5 \cdot 20$					$5 \cdot 21$					$5 \cdot 22$			$5 \cdot 199$				

From the sketch it is clear that there are as many three-digit integers that are multiples of 5 as there are integers from 20 to 199 inclusive.

Example 9.1.4 – *Solution (2/2)* continued

By Theorem 9.1.1, there are 199 - 20 + 1, or 180, such integers. Hence there are 180 three-digit integers that are divisible by 5.

b. By Theorem 9.1.1 the total number of integers from 100 through 999 is 999 − 100 + 1 = 900. By part (a), 180 of these are divisible by 5. Hence the probability that a randomly chosen three-digit integer is divisible by 5 is 180∕900 = 1∕5.

Example 9.1.5 – *Application: Counting Elements of a One-Dimensional Array (1/2)*

Analysis of many computer algorithms requires skill at counting the elements of a one-dimensional array. Let $A[1]$, $A[2], \ldots, A[n]$ be a one-dimensional array, where n is a positive integer.

a. Suppose the array is cut at a middle value $A[m]$ so that two subarrays are formed:

(1) $A[1], A[2], \ldots, A[m]$ and (2) $A[m + 1], A[m + 2], \ldots, A[n]$.

How many elements does each subarray have?

Example 9.1.5 – *Application: Counting Elements of a One-Dimensional Array (2/2)*

continued

b. What is the probability that a randomly chosen element of the array has an even subscript

(i) if n is even? (ii) if n is odd?

Example 9.1.5 – *Solution* (1/5)

a. Array (1) has the same number of elements as the list of integers from 1 through m. So by Theorem 9.1.1, it has m, or $m - 1 + 1$, elements. Array (2) has the same number of elements as the list of integers from $m + 1$ through n. So by Theorem 9.1.1, it has $n - m$, or $n - (m + 1) + 1$, elements.

> **Theorem 9.1.1 The Number of Elements in a List**
>
> If m and n are integers and $m \leq n$, then there are $n - m + 1$ integers from m to n inclusive.

Example 9.1.5 – *Solution (2/5)*

continued

b. (i) If *n* is even, each even subscript starting with 2 and ending with *n* can be matched up with an integer from 1 to *n*/2.

1	2	3	4	5	6	7	8	9	10	$\ldots$	n
	$\updownarrow$		$\updownarrow$		$\updownarrow$		$\updownarrow$		$\updownarrow$		$\updownarrow$
	$2\cdot 1$		$2\cdot 2$		$2\cdot 3$		$2\cdot 4$		$2\cdot 5$		$2\cdot(n/2)$

So there are *n*/2 array elements with even subscripts. Since the entire array has *n* elements, the probability that a randomly chosen element has an even subscript is

$$\frac{n/2}{n} = \frac{1}{2}.$$

Example 9.1.5 – *Solution (3/5)*

continued

(ii) If *n* is odd, then the greatest even subscript of the array is $n - 1$. So there are as many even subscripts between 1 and *n* as there are from 2 through $n - 1$.

Then the reasoning of (i) can be used to conclude that there are $(n - 1)/2$ array elements with even subscripts.

$$
\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & \ldots & n-1 & & n \\
 & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \\
 & 2\cdot 1 & & 2\cdot 2 & & 2\cdot 3 & \ldots & 2\cdot[(n-1)/2)] & &
\end{array}
$$

Example 9.1.5 – *Solution (4/5)* continued

Since the entire array has *n* elements, the probability that a randomly chosen element has an even subscript is

$$\frac{(n-1)/2}{n} = \frac{n-1}{2n}.$$

Observe that as *n* gets larger and larger, this probability gets closer and closer to 1⁄2.

Example 9.1.5 – *Solution* (5/5)

continued

Note that the answers to (i) and (ii) can be combined using the floor notation.

Theorem 4.6.2 The Floor of $n/2$

For any integer n,

$$\left\lfloor \frac{n}{2} \right\rfloor = \begin{cases} \dfrac{n}{2} & \text{if } n \text{ is even} \\[2ex] \dfrac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

By Theorem 4.6.2, the number of array elements with even subscripts is $\lfloor n/2 \rfloor$, so the probability that a randomly chosen element has an even subscript is $\frac{\lfloor n/2 \rfloor}{n}$.