Computer representation of Sets:

- Elements can be stored in an unordered manner; however, operations such as computing the union, intersection, or difference of two sets would be time-consuming.
- A more efficient method would be to store elements using an arbitrary but consistent ordering derived from the elements of the universal set.
  - Assume the universal set U is finite (of a reasonable size such that the number of elements in U does not exceed the memory capacity).
  - First, specify an arbitrary ordering of the elements of U, for instance $a_1, a_2, \ldots\ldots a_n$
  - Represent a subset A of U with a bit string of length n, where the ith bit in this string is 1 if a_1 belongs to A and is 0 if a_1 does not belong to A.

Examples:

Let U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}, and the ordering of elements of U has the elements in increasing order; that is, $a_i = i$:

a. What bit strings represent the subset of all odd integers in U,

b. the subset of all even integers in U, and

c. the subset of integers not exceeding 5 in U?

Examples:

- Given the universal set U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}, with elements ordered in increasing order such that $a_i = i$, the bit strings for the subsets are as follows:
- The subset of all odd integers in U: Odd integers in U are {1, 3, 5, 7, 9}. For each element a_i that is odd, the ith bit in the string is 1; otherwise, it is 0. Therefore, the bit string representing the subset of all odd integers in U is: 1010101010
- The subset of all even integers in U: Even integers in U are {2, 4, 6, 8, 10}. For each element *ai* that is even, the ith bit in the string is 1; otherwise, it is 0. Therefore, the bit string representing the subset of all even integers in U is: 0101010101
- The subset of integers not exceeding 5 in U: Integers in U not exceeding 5 are {1, 2, 3, 4, 5}. For each element a_i that is less than or equal to 5, the ith bit in the string is 1; otherwise, it is 0. Therefore, the bit string representing the subset of integers not exceeding 5 in U is: 1111100000
- Each bit string is of length 10, corresponding to the 10 elements in the universal set U.

Examples:

We have seen that the bit string for the set {1,3, 5, 7, 9} (with universal set {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} is 10 1010 1010. What is the bit string for the complement of this set?

Examples:

- The complement of a set contains all the elements that are not in the original set with respect to the universal set. Given the universal set U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} and the set {1, 3, 5, 7, 9}, the complement would include all the even numbers that were not in the original set.

- The bit string for the set {1, 3, 5, 7, 9} is 1010101010. To find the complement, you would flip all the bits in this string, turning all 1s to 0s and all 0s to 1s.

- Therefore, the bit string for the complement of the set {1, 3, 5, 7, 9} within the universal set U is 0101010101.

Examples:

The bit strings for the sets {1, 2, 3, 4, 5} and {1, 3, 5, 7, 9} are 111110 0000 and 1010101010, respectively. Use bit strings to find the union and intersection of these sets.

Examples:

- To find the union and intersection of two sets using bit strings, we perform bitwise OR for union and bitwise AND for intersection.
- Given:
  The bit string for {1, 2, 3, 4, 5} is 1111100000.
  The bit string for {1, 3, 5, 7, 9} is 1010101010.
- Union (Bitwise OR):
- We compare each corresponding pair of bits, and if at least one of them is 1, the result is 1.
  1111100000
  OR 1010101010
  = 1111101010
- The bit string for the union of {1, 2, 3, 4, 5} and {1, 3, 5, 7, 9} is 1111101010.
- Intersection (Bitwise AND):
- We compare each corresponding pair of bits, and if both of them are 1, the result is 1.
  1111100000
  AND 1010101010
  = 1010100000
- The bit string for the intersection of {1, 2, 3, 4, 5} and {1, 3, 5, 7, 9} is 1010100000.

LECTURE 7

Cardinality of Sets

- Previously we saw cardinality of sets for finite elements.

- Let’s look at cardinality of sets for infinite elements.

- Let’s define what it means for two infinite sets to have the same cardinality, providing us with a way to measure the relative sizes of infinite sets.

- This concept in this section is important in CS. A function is incomputable if no computer program can be written to find all its values, even with unlimited time and memory.

- If the sets are finite, we can easily claim that $|A| = |B| and\ |A| < |B|$. But can we make such claims with infinite sets?

Let's split infinite sets into two groups, those with same cardinality and those with a different cardinality.

Countable Sets

- A set that is either finite or has the same cardinality as the set of positive integers is called countable. When an infinite set S is countable, we denote the cardinality of S by an *aleph* ($\aleph$).
- Basically, elements can be put/counted in 1-1 correspondence with positive integers.
  - Integers
  - Odd positive integers
  - Hilbert’s Grand Hotel
  - Positive Rational Numbers and more

Uncountable Sets

- An uncountable set is a set that has more elements than the set of natural numbers, such that no one-to-one correspondence exists between the natural numbers and that set.
- A set that is not countable is *uncountable.*
  - The interval [0, 1]
  - Real Numbers: $n(\mathbb{Z}) < n(\mathbb{R})$

Uncountable Sets

- The idea that the real numbers are uncountable was proven by Georg Cantor in the late 19th century using a method known as "Cantor's diagonal argument." Cantor showed that, even if you tried to list all real numbers, you could always find a real number that wasn't on your list, thus proving that the list could never be complete.
- Here's a simplified version of Cantor's argument:
  - Assume you have a list of all real numbers between 0 and 1 (each real number is an infinite decimal).
  - Construct a new number by taking the first digit from the first number, the second digit from the second number, and so on, changing each digit you take (e.g., if the digit is 7, you might change it to 8).
  - This new number differs from every number in the list at least in one decimal place.
  - Therefore, this new number isn't on the list, contradicting the assumption that the list contained all real numbers.

Uncountable Sets

Consequences and Implications

The concept of uncountable sets has profound implications in mathematics. It has led to the understanding that not all infinities are equal and that some infinities are larger than others.

This understanding is a cornerstone of modern set theory and has implications for other areas of mathematics and logic, such as analysis and topology.

Sequences and Summations

Sequences

- Sequences are ordered lists of elements.
  - 2, 4, 6, 8, 10, ……
  - 0, 1, 0, 1, 0, 1 ………
  - 1, 3, 9, 27, 81, 243, …….
- Definition: A sequence is a function from a subset of the integers (usually either the set {0, 1, 2, 3, 4, …..} or {1, 2, 3, 4, ….}) to a set S.

- The notation a_n is used to denote the image of the integer n. We can think of a_n as the equivalent of f(n) where f is a function from {0,1,2,…..} to S. We call a_n a term of the sequence.

Sequences

Example: Consider the sequence $\{a_n\}$ where $a_n = \frac{1}{n}$

$$a_1 = \frac{1}{1}$$
$$a_2 = \frac{1}{2}$$
$$a_3 = \frac{1}{3}$$
$$\dots$$
$$a_n = \frac{1}{n}$$

This is known as the harmonic sequence. Each term in the harmonic sequence is the reciprocal opositive integer. This sequence is important in various fields of mathematics, including number theory and calculus. One of its notable properties is that it diverges, meaning that the sum of its terms grows without bound as n approaches infinity.

Geometric Progression

Definition: A special sequence where each successive number is a fixed multiple of the number before. A geometric progression is a sequence of the form:

$$a, ar, ar^2, \dots\dots, ar^n, \dots.$$

so: $a_n = a_1 \cdot r^{(n-1)}$

where the initial term a and the common ratio r are real numbers.

Examples:

1. Let a = 1 and r = −1. Then:
   $\{b_n\} = \{b_0, b_1, b_2, b_3 \dots\dots\dots\} = \{1, -1, 1, -1, 1, \dots..\}$

2. Let a = 2 and r = 5. Then:
   $\{C_n\} = \{C_0, C_1, C_2, C_3 \dots\dots\dots\} = \{2, 10, 50, 250, 1250, \dots...\}$

3. Let a = 6 and r = 1/3. Then:
   $\{d_n\} = \{d_0, d_1, d_2, d_3 \dots\dots\dots\} = \left\{6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \dots..\right\}$

Theorem: Geometric Series

- Sums of terms of geometric progressions $\sum_{j=0}^{n} ar^j = \begin{cases} \frac{ar^{n+1}-a}{r-1} & r \neq 1 \\ (n+1)a & r = 1 \end{cases}$

Proof:

To compute S_n, first multiply both sides of the equality by r and then manipulate the resulting sum as follows:

Let: $S_n = \sum_{j=0}^{n} ar^j$

$$rS_n = r\sum_{j=0}^{n} ar^j$$ Substituting summation formula for S

$$= \sum_{j=0}^{n} ar^{j+1}$$ By distributive property

Geometric Series

$$= \sum_{k=1}^{n+1} ar^k$$ Shifting the index of summation with $k = j + 1$.

$$= \left(\sum_{k=0}^{n} ar^k\right) + (ar^{n+1} - a)$$ Removing $k = n + 1$ term and adding $k = 0$ term.

$$= S_n + (ar^{n+1} - a)$$ Substituting S for summation formula

$$\therefore \quad rS_n = S_n + (ar^{n+1} - a)$$

$$S_n = \frac{ar^{n+1} - a}{r - 1}$$ if $r \neq 1$

$$S_n = \sum_{j=0}^{n} ar^j = \sum_{j=0}^{n} a = (n+1)a$$ if $r = 1$

Arithmetic Progression

Definition: In this sequence, each element is a fixed number larger than the number before it. An arithmetic progression is a sequence of the form:

$$a, a+d, a+2d, \ldots, a+nd, \ldots$$

where the initial term a and the common difference d are real numbers.

Examples:

1. Let a = −1 and d = 4:

$$\{s_n\} = \{s_0, s_1, s_2, s_3, s_4, \ldots\} = \{-1, 3, 7, 11, 15, \ldots\}$$

2. Let a = 7 and d = −3:

$$\{t_n\} = \{t_0, t_1, t_2, t_3, t_4, \ldots\} = \{7, 4, 1, -2, -5, \ldots\}$$

3. Let a = 1 and d = 2:

$$\{u_n\} = \{u_0, u_1, u_2, u_3, u_4, \ldots\} = \{1, 3, 5, 7, 9, \ldots\}$$

Other Sequences

- Fibonacci Sequence
- Quadratic Sequence
- Cubic Sequence
- Factorial Sequence
- Lucas Sequence
- Pell Sequence
- Triangular Sequence

Strings

- Definition: A string is a finite sequence of characters from a finite set (an alphabet).
- Sequence of the form $a_1, a_2, a_3 \ldots \ldots \ldots a_n$ are often used in CS.
- Length of the string is the number of terms in this string.
- The empty string is denoted by λ and length of the string is zero.
- The string abcde has length 5.

Guessing Sequences

Example: Find a simple formula for a_n if the first 10 terms of the sequence $\{a_n\}$ are 2,6,18,54,162,486,1458,4374,13122,39366

Guessing Sequences

- To find the formula for the given sequence: 2,6,18,54,162,486,1458,4374,13122,39366 we can observe that it appears to be a geometric sequence because each term after the first is obtained by multiplying the previous term by the same number.
- To confirm this, we check if the ratio between consecutive terms is constant. Let's divide the second term by the first, the third term by the second, and so on:
- 6/2=3, 18/6=3, 54/18=3 …
- The formula for the nth term of a geometric sequence is given by: $a_n = a_1 \cdot r^{(n-1)}$
- Given that $a_1=2$ (the first term of the sequence) and $r=3$, the formula becomes:

$$a_n = 2 \cdot 3^{(n-1)}$$

Summations

Sum of the terms for a sequence $\{a_n\}$ $a_m, a_{m+1}, \ldots, a_n$

The notation: $\displaystyle\sum_{j=m}^{n} a_j$ / $\sum_{j=m}^{n} a_j$ / $\sum_{m \le j \le n} a_j$

represents $a_m + a_{m+1} + \cdots + a_n$

The variable j is called the index of summation. It runs through all the integers starting with its lower limit m and ending with its upper limit n.

Examples

FIND

1. $\sum_{i=1}^{5} i$

2. $\sum_{i=3}^{7}(2i + 4)$

3. $\sum_{j=1}^{5} j^2$

4. $\sum_{j=1}^{2} \sum_{i=1}^{3}(i + 2j)$

Examples

1. $\sum_{i=1}^{5} i = 1+2+3+4+5=15$

2. $\sum_{i=3}^{7}(2i + 4) = 70$

3. $\sum_{j=1}^{5} j^2 = 1 + 4 + 9 + 16 + 25 = 55$

4. $\sum_{j=1}^{2}\sum_{i=1}^{3}(i + 2j) = 30$

Product Notation

- Product of the terms $a_m, a_{m+1}, \ldots \ldots \ldots, a_n$ from the sequence $\{a_n\}$

- The notation: $\displaystyle\prod_{j=m}^{n} a_j \quad \prod\nolimits_{j=m}^{n} a_j \quad \prod\nolimits_{m \leq j \leq n} a_j$

represents $a_m \times a_{m+1} \times \cdots \times a_n$

Some Useful Summations

TABLE 2 Some Useful Summation Formulae.	
Sum	***Closed Form***
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x\rvert < 1$	$\frac{1}{1 - x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x\rvert < 1$	$\frac{1}{(1 - x)^2}$

Geometric Series:

Later we will prove some of these by induction.

Proof in text (requires calculus)

Example

FIND $\sum_{k=50}^{100} k^2$

Example

$$\sum_{k=50}^{100} k^2$$

= 100(*100*+1)(2*(100)*+1)/6 – 49(*49*+1)(2*(49)*+1)/6

= 297,925