CHAPTER 10

THEORY OF GRAPHS AND TREES

10.1 Trails, Paths, and Circuits

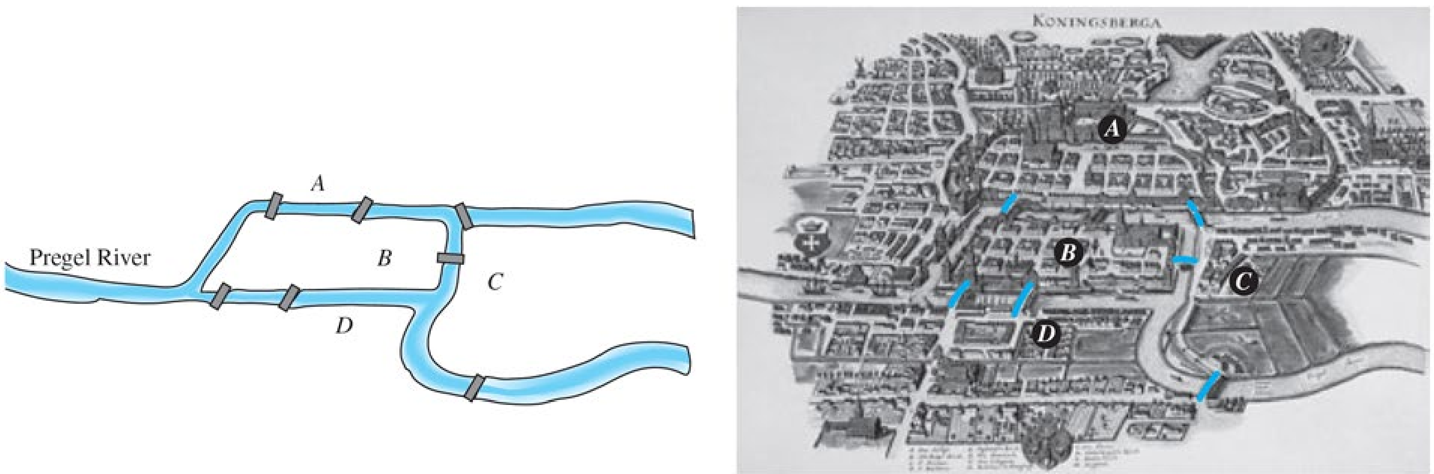
Trails, Paths, and Circuits (1/6)

The subject of graph theory began in the year 1736 when the great mathematician Leonhard Euler published a paper giving the solution to the following puzzle:

The town of Königsberg in Prussia (now Kaliningrad in Russia) was built at a point where two branches of the Pregel River came together. It consisted of an island and some land along the river banks.

Trails, Paths, and Circuits (2/6)

These were connected by seven bridges as shown in Figure 10.1.1.



The Seven Bridges of Königsberg

Figure 10.1.1

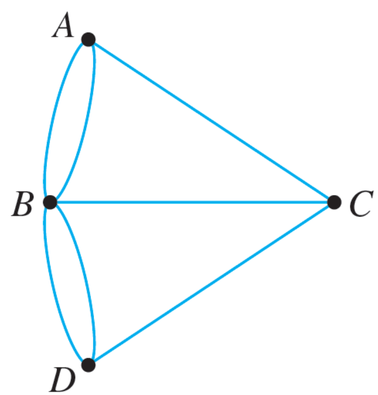
Trails, Paths, and Circuits (3/6)

The question is this: Is it possible for a person to take a walk around town, starting and ending at the same location and crossing each of the seven bridges exactly once?

To solve this puzzle, Euler translated it into a graph theory problem. He noticed that all points of a given land mass can be identified with each other since a person can travel from any one point to any other point of the same land mass without crossing a bridge.

Trails, Paths, and Circuits (4/6)

Thus for the purpose of solving the puzzle, the map of Königsberg can be identified with the graph shown in Figure 10.1.2, in which the vertices *A*, *B*, *C*, and *D* represent land masses and the seven edges represent the seven bridges.



Graph Version of Königsberg Map

Figure 10.1.2

Trails, Paths, and Circuits (5/6)

Definition

Let G be a graph, and let v and w be vertices in G.

A **walk from v to w** is a finite alternating sequence of adjacent vertices and edges of G. Thus a walk has the form

$$v_0 e_1 v_1 e_2 \cdots v_{n-1} e_n v_n,$$

where the v's represent vertices, the e's represent edges, $v_0 = v$, $v_n = w$, and for each $i = 1, 2, \ldots n$, v_{i-1} and v_i are the endpoints of e_i. The **trivial walk from v to v** consists of the single vertex v.

A **trail from v to w** is a walk from v to w that does not contain a repeated edge.

A **path from v to w** is a trail that does not contain a repeated vertex.

A **closed walk** is a walk that starts and ends at the same vertex.

A **circuit** is a closed walk that contains at least one edge and does not contain a repeated edge.

A **simple circuit** is a circuit that does not have any other repeated vertex except the first and last.

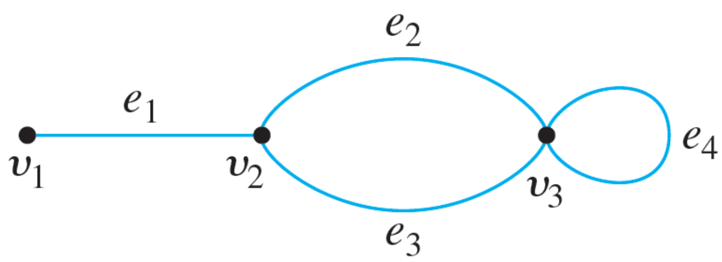
Trails, Paths, and Circuits (6/6)

For ease of reference, these definitions are summarized in the following table:

	Repeated Edge?	Repeated Vertex?	Starts and Ends at the Same Point?	Must Contain at Least One Edge?
Walk	allowed	allowed	allowed	no
Trail	no	allowed	allowed	no
Path	no	no	no	no
Closed walk	allowed	allowed	yes	no
Circuit	no	allowed	yes	yes
Simple circuit	no	first and last only	yes	yes

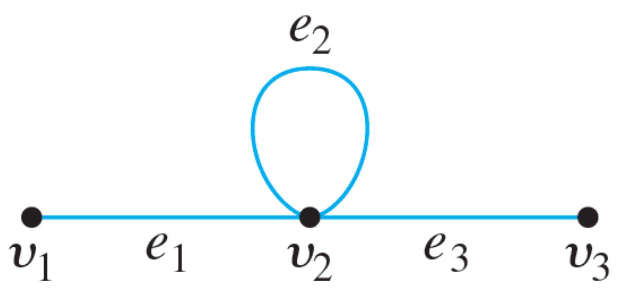
Example 10.1.1 – *Notation for Walks (1/2)*

a. In the graph below, the notation $e_1e_2e_4e_3$ refers unambiguously to the following walk: $v_1e_1v_2e_2v_3e_4v_3e_3v_2$. On the other hand, the notation e_1 is ambiguous if used by itself to refer to a walk. It could mean either $v_1e_1v_2$ or $v_2e_1v_1$.



Example 10.1.1 – *Notation for Walks (2/2)*

b. In the graph of part (a), the notation v_2v_3 is ambiguous if used to refer to a walk. It could mean $v_2e_2v_3$ or $v_2e_3v_3$. On the other hand, in the graph below, the notation $v_1v_2v_2v_3$ refers unambiguously to the walk $v_1e_1v_2e_2v_2e_3v_3$.



Example 10.1.2 – *Walks, Trails, Paths, and Circuits*

In the graph below, determine which of the following walks are trails, paths, circuits, or simple circuits.

a. $v_1e_1v_2e_3v_3e_4v_3e_5v_4$

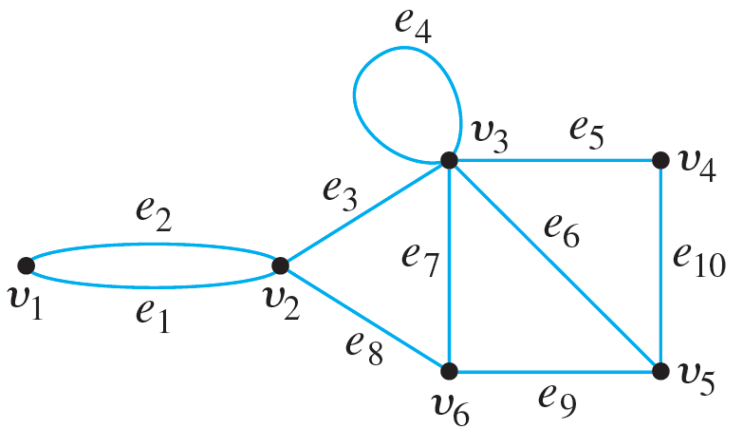
b. $e_1e_3e_5e_5e_6$

c. $v_2v_3v_4v_5v_3v_6v_2$

d. $v_2v_3v_4v_5v_6v_2$

e. $v_1e_1v_2e_1v_1$

f . v_1



Example 10.1.2 – *Solution (1/2)*

a. This walk has a repeated vertex but does not have a repeated edge, so it is a trail from v_1 to v_4 but not a path.

b. This is just a walk from v_1 to v_5. It is not a trail because it has a repeated edge.

c. This walk starts and ends at v_2, contains at least one edge, and does not have a repeated edge, so it is a circuit. Since the vertex v_3 is repeated in the middle, it is not a simple circuit.

Example 10.1.2 – *Solution (2/2)* continued

d. This walk starts and ends at v_2, contains at least one edge, does not have a repeated edge, and does not have a repeated vertex. Thus it is a simple circuit.

e. This is just a closed walk starting and ending at v_1. It is not a circuit because edge e_1 is repeated.

f. The first vertex of this walk is the same as its last vertex, but it does not contain an edge, and so it is not a circuit. It is a closed walk from v_1 to v_1. (It is also a trail from v_1 to v_1.)

Subgraphs

Subgraphs (1/1)

Definition

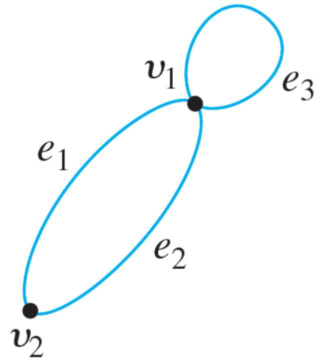
A graph H is said to be a **subgraph** of a graph G if, and only if, every vertex in H is also a vertex in G, every edge in H is also an edge in G, and every edge in H has the same endpoints as it has in G.

Example 10.1.3 – *Subgraphs*

List all subgraphs of the graph G with vertex set $\{v_1, v_2\}$ and edge set $\{e_1, e_2, e_3\}$, where the endpoints of e_1 are v_1 and v_2, the endpoints of e_2 are v_1 and v_2, and e_3 is a loop at v_1.

Example 10.1.3 – *Solution (1/2)*

G can be drawn as shown below.



There are 11 subgraphs of *G*, which can be grouped according to those that do not have any edges, those that have one edge, those that have two edges, and those that have three edges.

Example 10.1.3 – *Solution (2/2)* continued

The 11 subgraphs are shown in Figure 10.1.3.

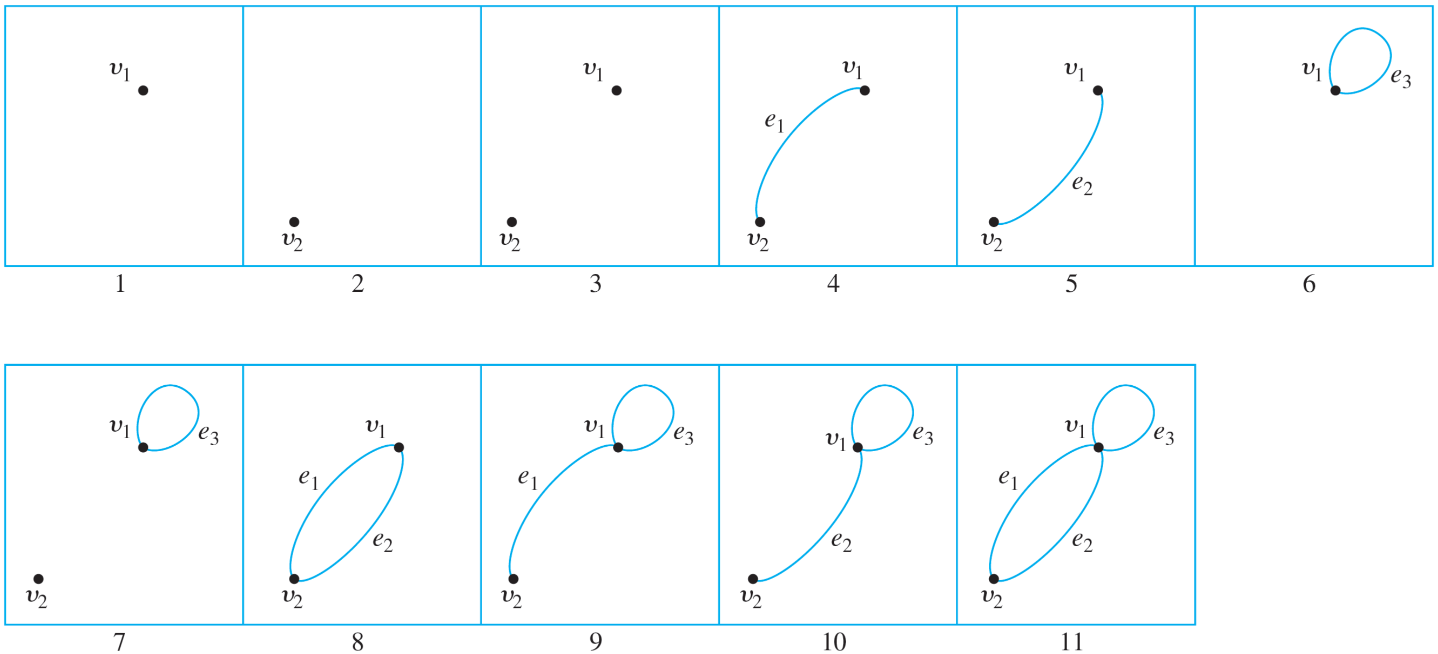


Figure 10.1.3

Connectedness

Connectedness (1/4)

It is easy to understand the concept of connectedness on an intuitive level. Roughly speaking, a graph is connected if it is possible to travel from any vertex to any other vertex along a sequence of adjacent edges of the graph.

The formal definition of connectedness is stated in terms of walks.

Connectedness (2/4)

Definition

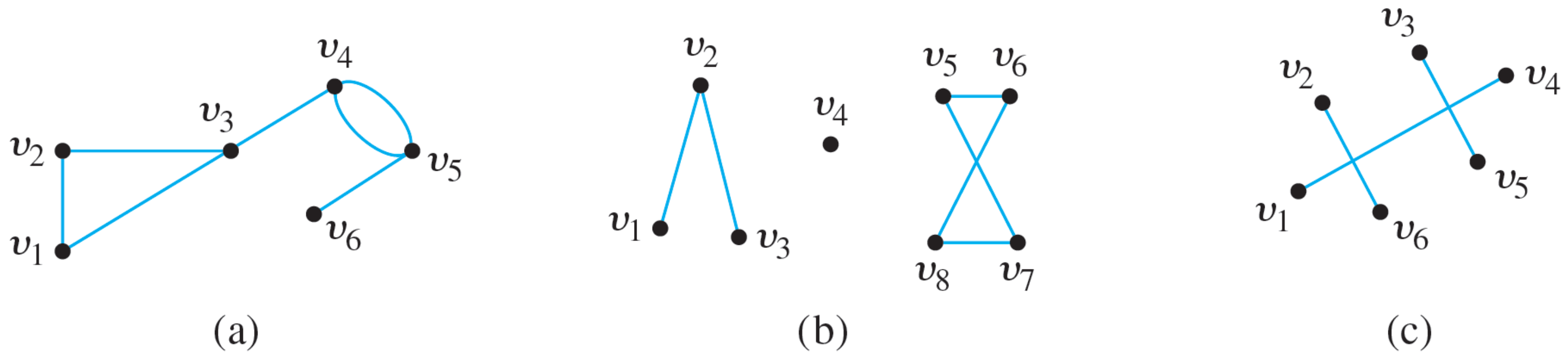
Let G be a graph. Two **vertices v and w of G are connected** if, and only if, there is a walk from v to w. The **graph G is connected** if, and only if, given *any* two vertices v and w in G, there is a walk from v to w. Symbolically:

$$G \text{ is connected} \quad \Leftrightarrow \quad \forall \text{ vertices } v \text{ and } w \text{ in } G, \exists \text{ a walk from } v \text{ to } w.$$

If you take the negation of this definition, you will see that a graph G is *not connected* if, and only if, there exist two vertices of G that are not connected by any walk.

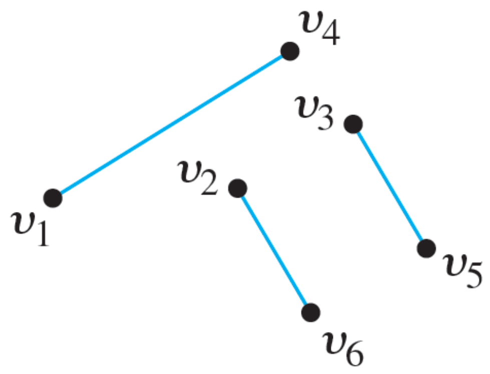
Example 10.1.4 – *Connected and Disconnected Graphs*

Which of the following graphs are connected?



Example 10.1.4 – *Solution*

The graph represented in (a) is connected, whereas those of (b) and (c) are not. To understand why (c) is not connected, recall that in a drawing of a graph, two edges may cross at a point that is not a vertex. Thus the graph in (c) can be redrawn as follows:



Connectedness (3/4)

Some useful facts relating circuits and connectedness are collected in the following lemma.

Lemma 10.1.1

Let G be a graph.

a. If G is connected, then any two distinct vertices of G can be connected by a path.

b. If vertices v and w are part of a circuit in G and one edge is removed from the circuit, then there still exists a trail from v to w in G.

c. If G is connected and G contains a circuit, then an edge of the circuit can be removed without disconnecting G.

Connectedness (4/4)

A *connected component* of a graph is a connected subgraph of largest possible size.

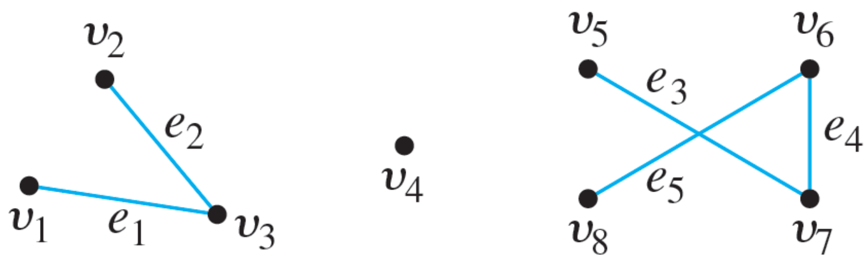
Definition

A graph H is a **connected component** of a graph G if, and only if,

1. H is subgraph of G;
2. H is connected; and
3. no connected subgraph of G has H as a subgraph and contains vertices or edges that are not in H.

Example 10.1.5 – *Connected Components*

Find all connected components of the following graph *G*.



Example 10.1.5 – *Solution*

G has three connected components: H_1, H_2, and H_3 with vertex sets V_1, V_2, and V_3 and edge sets E_1, E_2, and E_3, where

$$
\begin{aligned}
V_1 &= \{v_1, v_2, v_3\}, & E_1 &= \{e_1, e_2\}, \\
V_2 &= \{v_4\}, & E_2 &= \varnothing, \\
V_3 &= \{v_5, v_6, v_7, v_8\}, & E_3 &= \{e_3, e_4, e_5\}.
\end{aligned}
$$

Euler Circuits

Euler Circuits (1/5)

Now we return to consider general problems similar to the puzzle of the Königsberg bridges. The following definition is made in honor of Euler.

Definition

Let G be a graph. An **Euler circuit** for G is a circuit that contains every vertex and every edge of G. That is, an Euler circuit for G is a sequence of adjacent vertices and edges in G that has at least one edge, starts and ends at the same vertex, uses every vertex of G at least once, and uses every edge of G exactly once.

Euler Circuits (2/5)

The analysis used earlier to solve the puzzle of the Königsberg bridges generalizes to prove the following theorem:

Theorem 10.1.2

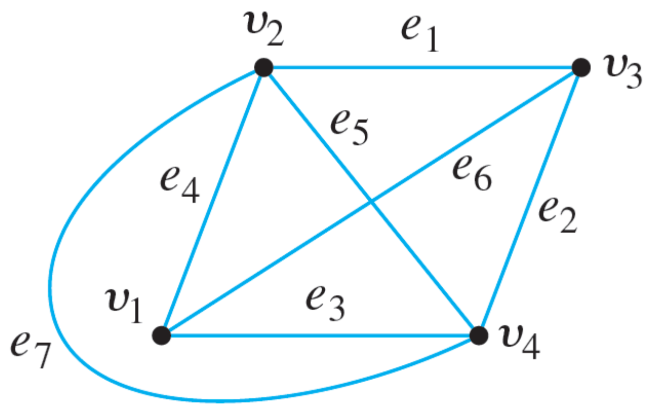
If a graph has an Euler circuit, then every vertex of the graph has positive even degree.

Contrapositive Version of Theorem 10.1.2

If some vertex of a graph has odd degree, then the graph does not have an Euler circuit.

Example 10.1.6 – *Showing That a Graph Does Not Have an Euler Circuit*

Show that the graph below does not have an Euler circuit.



Example 10.1.6 – *Solution*

Vertices v_1 and v_3 both have degree 3, which is odd. Hence by (the contrapositive form of) Theorem 10.1.2, this graph does not have an Euler circuit.

Theorem 10.1.2

If a graph has an Euler circuit, then every vertex of the graph has positive even degree.

Contrapositive Version of Theorem 10.1.2

If some vertex of a graph has odd degree, then the graph does not have an Euler circuit.

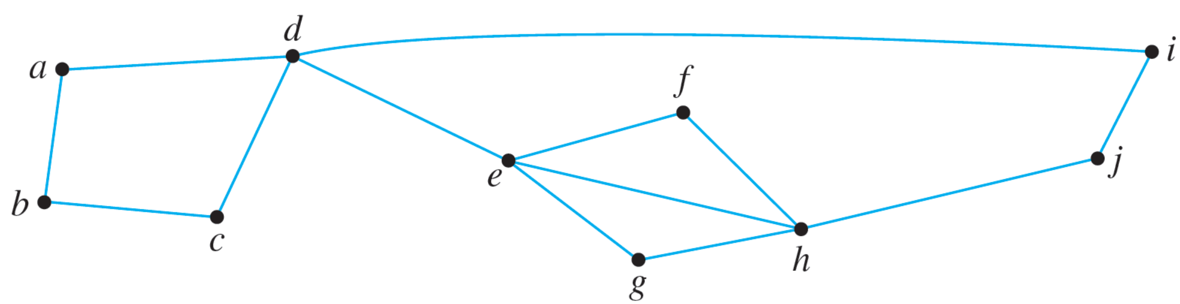
Euler Circuits (3/5)

Theorem 10.1.3

If a graph G is connected and the degree of every vertex of G is a positive even integer, then G has an Euler circuit.

Example 10.1.7 – *Finding an Euler Circuit*

Use Theorem 10.1.3 to check that the graph below has an Euler circuit. Then use the algorithm from the proof of the theorem to find an Euler circuit for the graph.



Example 10.1.7 – *Solution (1/5)*

Observe that

$\deg(a) = \deg(b) = \deg(c) = \deg(f) = \deg(g) = \deg(i) = \deg(j) = 2$

and that $\deg(d) = \deg(e) = \deg(h) = 4.$

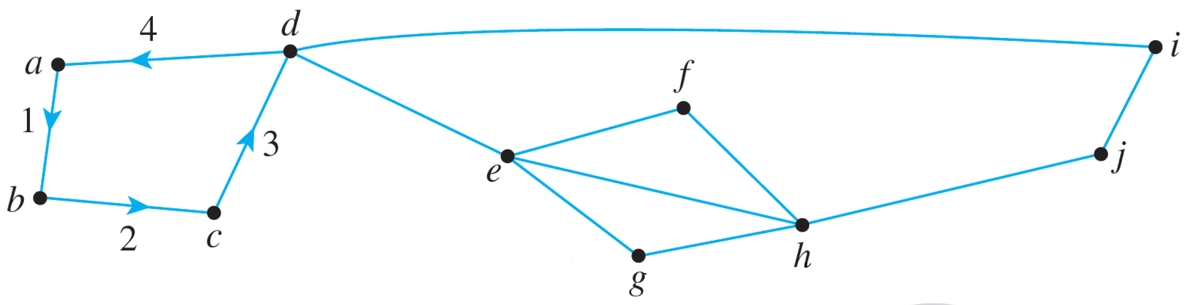
Hence all vertices have even degree. Also, the graph is connected. Thus, by Theorem 10.1.3, the graph has an Euler circuit.

Example 10.1.7 – *Solution (2/5)* continued

To construct an Euler circuit using the algorithm of Theorem 10.1.3, let $v = a$ and let C be

$$C: abcda.$$

C is represented by the labeled edges shown below.

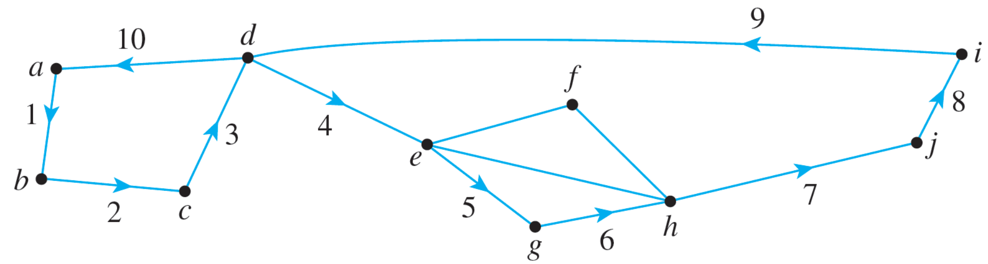


Example 10.1.7 – *Solution (3/5)* continued

Observe that C is not an Euler circuit for the graph but that C intersects the rest of the graph at d. Let C' be C': *deghjid*.

Patch C' into C to obtain C'': *abcdeghjida*.

Set $C = C''$. Then C is represented by the labeled edges shown below.



Example 10.1.7 – *Solution (4/5)* continued

Observe that C is not an Euler circuit for the graph but that it intersects the rest of the graph at e and h. Let C' be

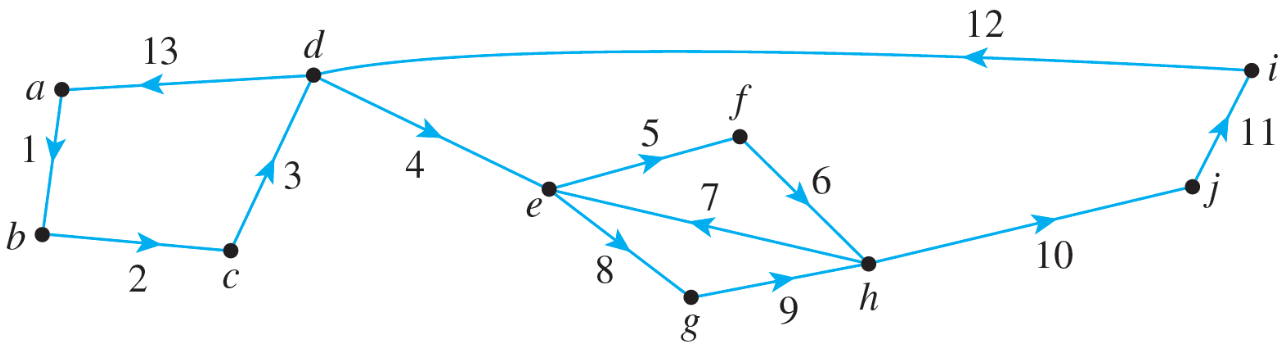
$$C': efhe.$$

Patch C' into C to obtain

$$C'': abcdefheghjida.$$

Example 10.1.7 – *Solution* (5/5) continued

Set $C = C''$. Then C is represented by the labeled edges shown below.



Since C includes every edge of the graph exactly once, C is an Euler circuit for the graph.

Euler Circuits (4/5)

Theorem 10.1.4

A graph G has an Euler circuit if, and only if, G is connected and every vertex of G has positive even degree.

A corollary to Theorem 10.1.4 gives a criterion for determining when it is possible to find a walk from one vertex of a graph to another, passing through every vertex of the graph at least once and every edge of the graph exactly once.

Euler Circuits (5/5)

Definition

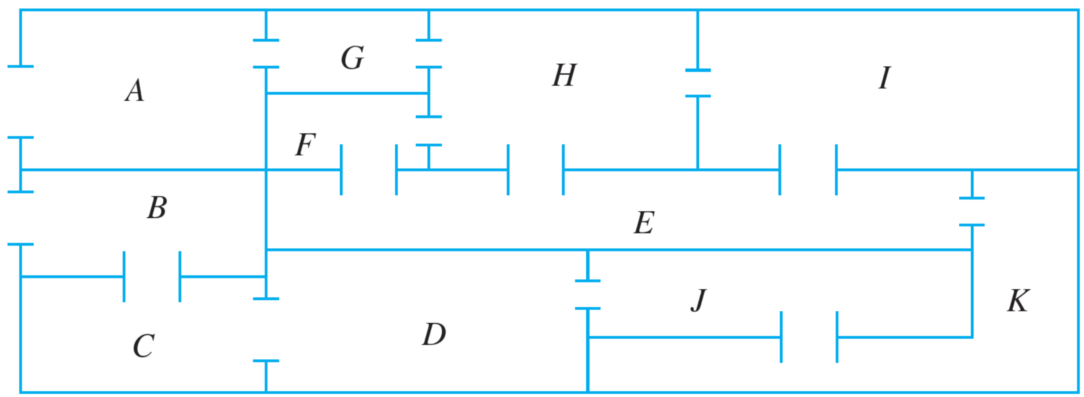
Let G be a graph, and let v and w be two distinct vertices of G. An **Euler trail from *v* to *w*** is a sequence of adjacent edges and vertices that starts at v, ends at w, passes through every vertex of G at least once, and traverses every edge of G exactly once.

Corollary 10.1.5

Let G be a graph, and let v and w be two distinct vertices of G. There is an Euler trail from v to w if, and only if, G is connected, v and w have odd degree, and all other vertices of G have positive even degree.

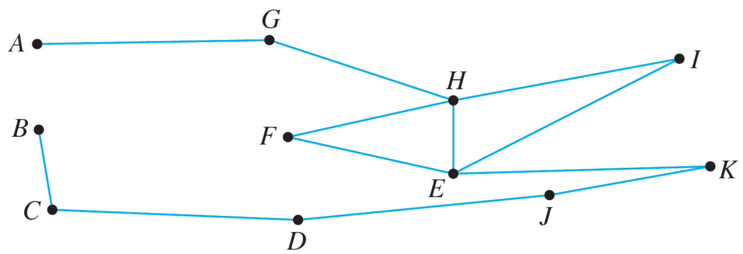
Example 10.1.8 – *Finding an Euler Trail*

The floor plan shown below is for a house that is open for public viewing. Is it possible to find a trail that starts in room *A*, ends in room *B*, and passes through every interior doorway of the house exactly once? If so, find such a trail.



Example 10.1.8 – *Solution*

Let the floor plan of the house be represented by the graph below, where the edges indicate the openings between the rooms.



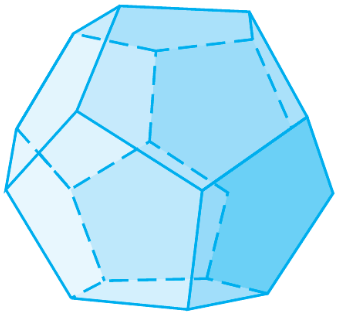
Each vertex of this graph has even degree except for *A* and *B*, each of which has degree 1. Hence by Corollary 10.1.5, there is an Euler trail from *A* to *B*. One such trail is

A G H F E I H E K J D C B.

Hamiltonian Circuits

Hamiltonian Circuits (1/6)

In 1859 the Irish mathematician Sir William Rowan Hamilton introduced a puzzle in the shape of a dodecahedron (DOH-dek-a-HEE-dron). (Figure 10.1.7 contains a drawing of a dodecahedron, which is a solid figure with 12 identical pentagonal faces.)



Dodecahedron

Figure 10.1.7

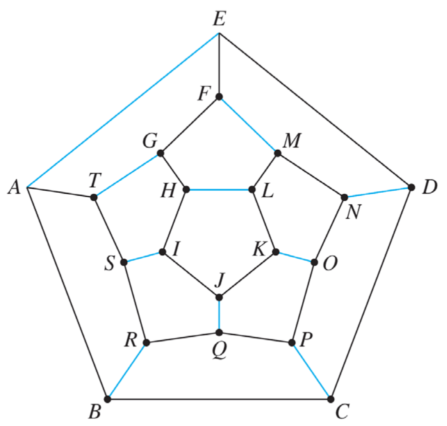
Hamiltonian Circuits (2/6)

Each vertex was labeled with the name of a city—London, Paris, Hong Kong, New York, and so on.

The problem Hamilton posed was to start at one city and tour the world by visiting each other city exactly once and returning to the starting city.

Hamiltonian Circuits (3/6)

One way to solve the puzzle is to imagine the surface of the dodecahedron stretched out and laid flat in the plane, as follows:



One solution is the circuit

$$A\ B\ C\ D\ E\ F\ G\ H\ I\ J\ K\ L\ M\ N\ O\ P\ Q\ R\ S\ T\ A,$$

whose edges are indicated with black lines.

Hamiltonian Circuits (4/6)

Note that although every city is visited, many edges are omitted from the circuit. (More difficult versions of the puzzle required that certain cities be visited in a certain order.)

The following definition is made in honor of Hamilton.

Definition

Given a graph G, a **Hamiltonian circuit** for G is a simple circuit that includes every vertex of G. That is, a Hamiltonian circuit for G is a sequence of adjacent vertices and distinct edges in which every vertex of G appears exactly once, except for the first and the last, which are the same.

Hamiltonian Circuits (5/6)

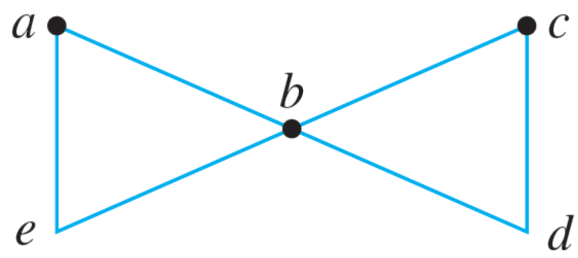
Proposition 10.1.6

If a graph G has a Hamiltonian circuit, then G has a subgraph H with the following properties:

1. H contains every vertex of G.
2. H is connected.
3. H has the same number of edges as vertices.
4. Every vertex of H has degree 2.

Example 10.1.9 – *Showing That a Graph Does Not Have a Hamiltonian Circuit*

Prove that the graph *G* shown below does not have a Hamiltonian circuit.



Example 10.1.9 – *Solution (1/2)*

If *G* has a Hamiltonian circuit, then by Proposition 10.1.6, *G* has a subgraph *H* that (1) contains every vertex of *G*, (2) is connected, (3) has the same number of edges as vertices, and (4) is such that every vertex has degree 2. Suppose such a subgraph *H* exists.

In other words, suppose there is a connected subgraph *H* of *G* such that *H* has five vertices (*a*, *b*, *c*, *d*, *e*) and five edges and such that every vertex of *H* has degree 2.

Example 10.1.9 – *Solution (2/2)* continued

Since the degree of *b* in *G* is 4 and every vertex of *H* has degree 2, two edges incident on *b* must be removed from *G* to create *H*. Edge {*a*, *b*} cannot be removed because if it were, vertex *a* would have degree less than 2 in *H.* Similar reasoning shows that edges {*e*, *b*}, {*b*, *a*}, and {*b*, *d*} cannot be removed either.

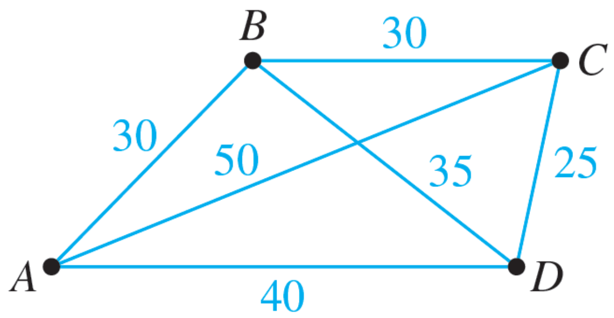
It follows that the degree of *b* in *H* must be 4, which contradicts the condition that every vertex in *H* has degree 2 in *H.* Hence no such subgraph *H* exists, and so *G* does not have a Hamiltonian circuit.

Hamiltonian Circuits (6/6)

The next example illustrates a type of problem known as a **traveling salesman problem**. It is a variation of the problem of finding a Hamiltonian circuit for a graph.

Example 10.1.10 – *A Traveling Salesman Problem*

Imagine that the drawing below is a map showing four cities and the distances in kilometers between them. Suppose that a salesman must travel to each city exactly once, starting and ending in city *A*. Which route from city to city will minimize the total distance that must be traveled?



Example 10.1.10 – *Solution*

This problem can be solved by writing all possible Hamiltonian circuits starting and ending at A and calculating the total distance traveled for each.

Route	Total Distance (In Kilometers)	
$A\ B\ C\ D\ A$	$30 + 30 + 25 + 40 = 125$	
$A\ B\ D\ C\ A$	$30 + 35 + 25 + 50 = 140$	
$A\ C\ B\ D\ A$	$50 + 30 + 35 + 40 = 155$	
$A\ C\ D\ B\ A$	140	[$A\ B\ D\ C\ A$ backward]
$A\ D\ B\ C\ A$	155	[$A\ C\ B\ D\ A$ backward]
$A\ D\ C\ B\ A$	125	[$A\ B\ C\ D\ A$ backward]

Thus either route $A\ B\ C\ D\ A$ or $A\ D\ C\ B\ A$ gives a minimum total distance of 125 kilometers.