Lecture 6
Chapter 2

Basic Structures: Sets

Sets

- A set is an unordered collection of objects.
  - the students in this class
  - the chairs in this room
- The objects in a set are called the elements, or members of the set.
- A set is said to contain its elements.
- The notation $a \in A$ denotes that a is an element of the set A.
- If a is not a member of A, we write $a \notin A$
- Repeated elements are listed only once
- A set can contain elements that are sets.
- There is no order to a set. Example $\{\{1,2,3\},a, \{b, c\}\}$ and $\{N,Z,Q,R\}$
- An Empty set doesn't contain any elements '$\emptyset$'
- A Singleton element contains only one element. $\{\emptyset\}$
- The empty set is different from a set containing the empty set. $\emptyset \neq \{ \emptyset \}$

Some Important Sets

N = natural numbers = {0,1,2,3….}

Z = integers = {…,-3,-2,-1,0,1,2,3,…}

Z^+ = positive integers = {1,2,3,…..}

R = set of all real numbers

R^+ = set of all positive real numbers

C = set of all complex numbers.

Q = set of rational numbers

Set-Builder Notation

- Notation: {x | x has a property P} implies "the set of all x such that x has property P"
- Specify the property or properties that all members must satisfy

Example:

Set O of all odd positive integers less than 10:

$$O = \{x \mid x \text{ is an odd positive integer less than } 10\}$$

$$O = \{x \in Z^+ \mid x \text{ is odd and } x < 10\}$$

Set Q^+ of all positive rational numbers:

$$Q^+ = \{x \in R \mid x = p/q, \text{ for some positive integers } p, q\}$$

Interval Notation

$[a, b] = \{x \mid a \leq x \leq b\}$

$[a, b) = \{x \mid a \leq x < b\}$

$(a, b] = \{x \mid a < x \leq b\}$

$(a, b) = \{x \mid a < x < b\}$

closed interval $[a, b]$

open interval (a, b)

Set Equality

Definition: Two sets are equal if and only if they have the same elements. Therefore, If A and B are sets, then A and B are equal if and only if

$$\forall x(x \in A \leftrightarrow x \in B)$$

We write A = B, if A and B are equal sets.
$\{1,3,5\} = \{3, 5, 1\}$
$\{1,5,5,5,3,3,1\} = \{1,3,5\}$

Subsets

Definition: The set A is a subset of B, and B is a superset of A, if and only if every element of A is also an element of B.

- The notation $A \subseteq B$ is used to indicate that A is a subset of the set B.
- $A \subseteq B$ holds if and only if the quantification *below* is true.

$$\forall x (x \in A \to x \in B)$$

Another look at Equality of Sets

- Recall that two sets A and B are equal, denoted by A = B, iff

$$\forall x(x \in A \leftrightarrow x \in B)$$

- Using logical equivalences, we have that A = B iff

$$\forall x[(x \in A \rightarrow x \in B) \wedge (x \in B \rightarrow x \in A)]$$

- This is equivalent to $A \subseteq B$ and $B \subseteq A$

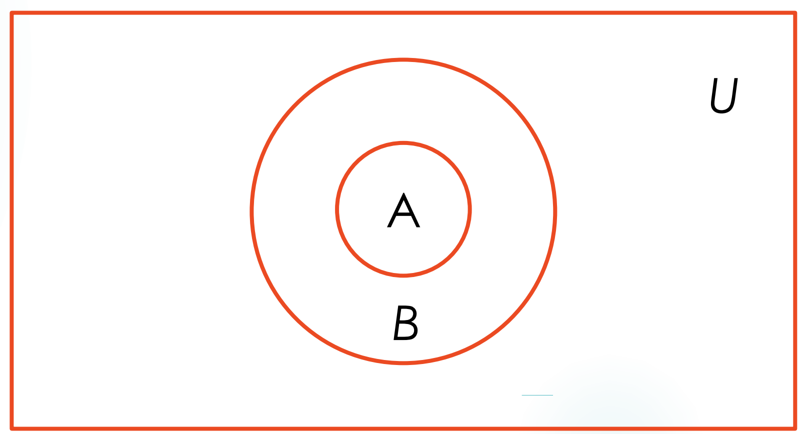
Proper Subsets

Definition: If A ⊆ B, but A ≠ B, then we say A is a proper subset of B, denoted by A ⊂ B. If A ⊂ B, then

$$\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)$$

is **true**.

Venn Diagram:



Set Cardinality

Definition: If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is finite. Otherwise, it is infinite.

The cardinality of a finite set A, denoted by |A|, is the number of (distinct) elements of A.

Examples:

1. |ø| = 0

   Let S be the set of letters of the English alphabet. Then |S| = 26

1. |{1,2,3}| = 3
2. |{ø}| = 1
3. |{ø}, b, {a, b}| = 3
4. The set of integers is infinite.

Power Sets

Definition: The set of all subsets of a set A, denoted P(A), is called the power set of A.

Example:

1. If A = {a, b} then

   P(A) = {ø, {a},{b},{a, b}}

2. If S = {0, 1, 2} then

   P(S) = {ø, {0}, {1}, {2}, {0, 1},{0, 2},{1, 2},{0, 1, 2}}

- If a set has n elements, then the cardinality of the power set is 2^n.
- (In future chapters, we will discuss different ways to show this.)

Cartesian Product

René Descartes
(1596-1650)

Definition: The Cartesian Product of two sets A and B, denoted by A × B is the set of ordered pairs (a, b) where a ∈ A and b ∈ B .

$$A \times B = \{(a,b) | a \in A \wedge b \in B\}$$

Example: Cartesian and its cardinality:

A = {a, b}   B = {1,2,3} = 2 x 3 = 6

A × B = {(a,1),(a,2),(a,3), (b,1),(b,2),(b,3)}

$A^2 = \{(a,a),(a,b),(b,a),(b,b)\}$

$\emptyset \times A = \emptyset$

Cartesian Product

The Cartesian product of more than two sets can be defined as:

The Cartesian products of the sets $A_1,A_2,\ldots\ldots,A_n$, denoted by $A_1 \times A_2 \times \ldots\ldots \times A_n$, is the set of ordered n-tuples $(a_1,a_2,\ldots\ldots,a_n)$ where a_i belongs to A_i for i = 1, 2 … n.

$$A_1 \times A_2 \times \cdots \times A_n = \\ \{(a_1, a_2, \ldots, a_n) | a_i \in A_i \text{ for } i = 1, 2, \ldots n\}$$

Cartesian Product Example

Example:

What is $A \times B \times C$

where $A = \{0, 1\}$, $B = \{1, 2\}$ and $C = \{0, 1, 2\}$?

Example:

What is $A \times B \times C$

where $A = \{0, 1\}$, $B = \{1, 2\}$ and $C = \{0, 1, 2\}$?

Solution:

$A \times B \times C =$

$\{(0, 1, 0), (0, 1, 1), (0, 1, 2), (0, 2, 0), (0, 2,1), (0, 2, 2), (1, 1, 0), (1, 1, 1), (1, 1, 2), (1, 2, 0), (1, 2, 1), (1, 2, 2)\}$

Set Operations

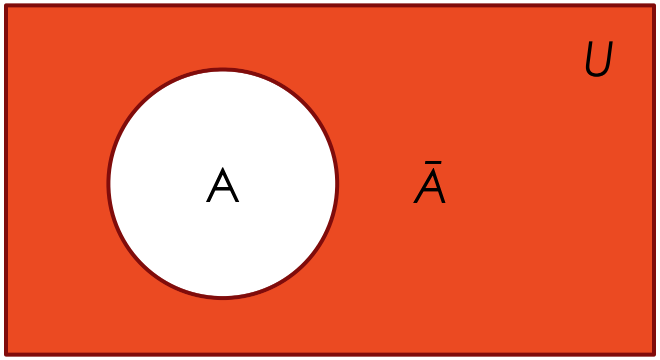
Lecture 5

Complement

Definition: If A is a set, then the complement of the A (with respect to U), denoted by $\bar{A}$ is the set U - A

$\bar{A} = \{x \in U \mid x \notin A\}$. (The compliment of A is sometimes denoted by A^c .)

Venn Diagram



Complement Example

Example:

If U is the positive integers less than 100,
what is the complement of $\{x \mid x > 70\}$?

Complement Example

Example:

If U is the positive integers less than 100, what is the complement of $\{x \mid x > 70\}$?

Solution:

$\{x \mid x \leq 70\}$

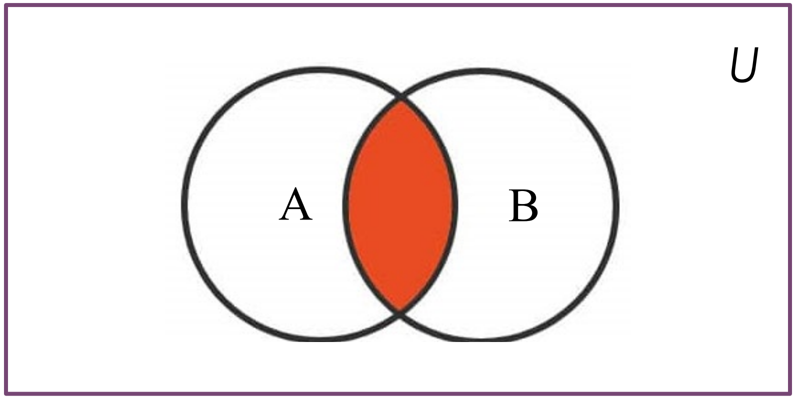
Intersection

Definition:

The intersection of sets A and B, denoted by $A \cap B$, is $\{x | x \in A \wedge x \in B\}$

Note if the intersection is empty, then A and B are said to be disjoint.

Venn Diagram for $A \cap B$



Intersection

Example:

What is $\{1, 2, 3\} \cap \{3, 4, 5\}$?

Solution: $\{3\}$

What is $\{1, 2, 3\} \cap \{4, 5, 6\}$?

Solution: $\emptyset$

Intersection Example

What is $\{1, 2, 3\} \cap \{3, 4, 5\}$?

Solution:

$\{3\}$

What is $\{1, 2, 3\} \cap \{4, 5, 6\}$?

Solution:

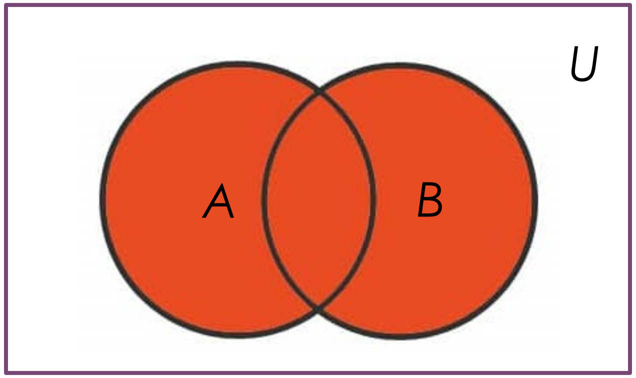
$\emptyset$

Union

Definition: Let A and B be sets. The union of the sets A and B, denoted by A ∪ B, is the set:

$$\{x | x \in A \vee x \in B\}$$

Venn Diagram for A ∪ B



Union Example

What is $\{1,2,3\} \cup \{3, 4, 5\}$?

Union Example

What is $\{1,2,3\} \cup \{3, 4, 5\}$?

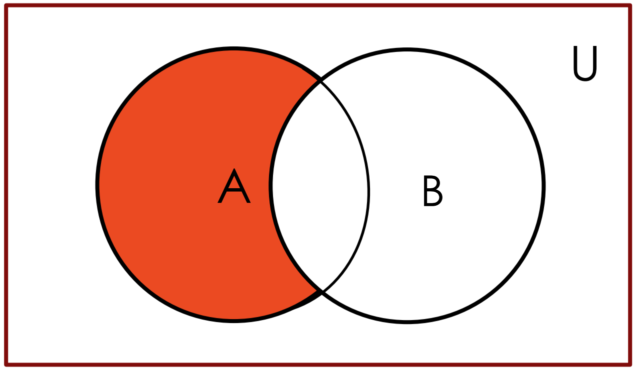
Solution:
$\{1,2,3,4,5\}$

Difference

- Definition: Let A and B be sets. The difference of A and B, denoted by A – B, is the set containing the elements of A that are not in B. The difference of A and B is also called the complement of B with respect to A.

$$A - B = \{x \mid x \in A \wedge x \notin B\} = A \cap \overline{B}$$

Venn Diagram for A − B



Examples

U = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}
A = {1, 2, 3, 4, 5}
B ={4, 5, 6, 7, 8}

1. $A \cup B$
2. $A \cap B$
3. $\bar{A}$
4. $\bar{B}$
5. $A - B$
6. $B - A$

Examples

U = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}
A = {1, 2, 3, 4, 5}
B ={4, 5, 6, 7, 8}

1. A ∪ B: The union of sets *A* and *B* includes all elements that are in *A*, in *B*, or in both.
   Solution: {1,2,3,4,5,6,7,8}
2. A ∩ B: The intersection of sets *A* and *B* includes all elements that are both in *A* and *B*.
   Solution: {4,5}
3. $\bar{A}$: The complement of *A* includes all elements that are in the universal set *U* but not in *A*.
   Solution: {0,6,7,8,9,10}
4. $\bar{B}$: The complement of B includes all elements that are in U but not in B.
   Solution: {0,1,2,3,9,10}
5. A – B: The difference *A*−*B* includes all elements that are in *A* but not in *B*.
   Solution: {1,2,3}
6. B – A: The difference *B*−*A* includes all elements that are in *B* but not in *A*.
   Solution: {6,7,8}

Symmetric Difference

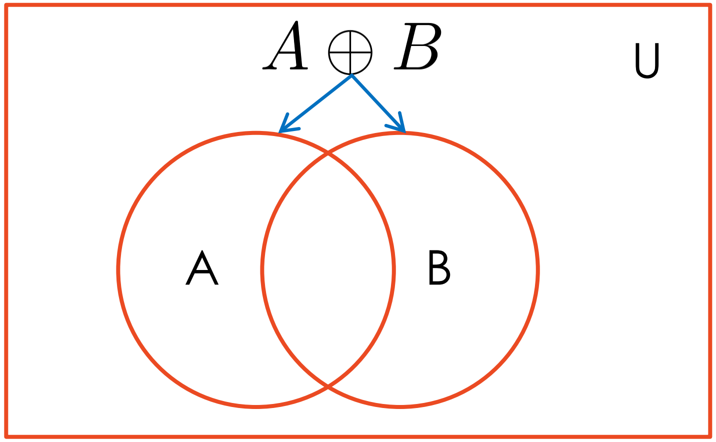
Definition: The symmetric difference of A and B, denoted by $A \oplus B$

which is equivalent to: $(A - B) \cup (B - A)$

The symmetric difference between two sets A and B is a set that contains elements that are in either of the sets A or B but not in their intersection. In other words, it consists of elements that are exclusive to each set.

AKA: Exclusive OR of a set

Venn Diagram



Symmetric Difference Example

Given:
U = {0,1,2,3,4,5,6,7,8,9,10}
A = {1,2,3,4,5}   B ={4,5,6,7,8}

What is: $A \oplus B$?

Symmetric Difference Example

Given:
U = {0,1,2,3,4,5,6,7,8,9,10}
A = {1,2,3,4,5}   B ={4,5,6,7,8}

What is: $A \oplus B$?

Solution: {1,2,3,6,7,8}

Set Identities

Identity	*Name*
$A \cup \emptyset = A$ $A \cap U = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Proving Set Identities

Different ways to prove set identities:

1. Prove that each set (side of the identity) is a subset of the other
2. Use set builder notation and propositional logic
3. Membership Tables:
   - Verify that elements in the same combination of sets always either belong or do not belong to the same side of the identity. Use 1 to indicate it is in the set and a 0 to indicate that it is not.

Computer Representation of Sets

1. Let U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}, and the ordering of elements of U has the elements in increasing order; that is, $a_i = i$. What bit strings represent the subset of all odd integers in U, the subset of all even integers in U, and the subset of integers not exceeding 5 in U?

Computer Representation of Sets

Let U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}, and the ordering of elements of *U* has the elements in increasing order; that is, $a_i = i$. What bit strings represent the subset of all odd integers in *U*, the subset of all even integers in *U*, and the subset of integers not exceeding 5 in *U*?

A bit string representing subsets of a universal set U with ordered elements, each bit in the string corresponds to an element in U. A bit is set to 1 if the corresponding element is included in the subset, and 0 otherwise.

Given U={1,2,3,4,5,6,7,8,9,10} and the ordering $a_i = i$, we can construct bit strings for the following subsets:

1. Subset of all odd integers in U: This subset includes the numbers {1, 3, 5, 7, 9}. The bit string representing this subset would have bits set to 1 at positions corresponding to these odd numbers. Since the ordering is $a_i = i$, the bit string would be "1010101010".
2. Subset of all even integers in U: This subset includes the numbers {2, 4, 6, 8, 10}. The bit string representing this subset would have bits set to 1 at positions corresponding to these even numbers. The bit string would be "0101010101".
3. Subset of integers not exceeding 5 in U: This subset includes the numbers {1, 2, 3, 4, 5}. The bit string representing this subset would have bits set to 1 at positions corresponding to these numbers, and the rest are set to 0. The bit string would be "1111100000".

Computer Representation of Sets

1. We have seen that the bit string for the set (1, 3, 5, 7, 9) (with universal set {1, 2, 3, 4. 5, 6, 7, 8, 9, 10}) is 10 1010 1010. What is the bit string for the compliment of the set?

Computer Representation of Sets

The bit string for the set (1, 3, 5, 7, 9) (with universal set {1, 2, 3, 4. 5, 6, 7, 8, 9, 10}) is 10 1010 1010. What is the bit string for the compliment of the set?

The bit string "1010101010" represents the set of all odd integers in the universal set U={1,2,3,4,5,6,7,8,9,10}. To find the bit string for the complement of the set, you would flip all the bits in the original bit string, changing all 1s to 0s and all 0s to 1s.

Thus, the bit string for the complement of the set {1,3,5,7,9} would be "0101010101", representing the set of all even integers in U, which are {2,4,6,8,10}.

Computer Representation of Sets

1. The bit strings for the sets (1, 2, 3, 4, 5) and (1, 3, 5, 7, 9) are 11 1110 0000 and 10 1010 1010, respectively. Use bit strings to find the union of these sets.

Computer Representation of Sets

The bit strings for the sets (1, 2, 3, 4, 5) and (1, 3, 5, 7, 9) are 11 1110 0000 and 10 1010 1010, respectively. Use bit strings to find the union of these sets.

- Given the bit strings for the sets:
  $A=\{1,2,3,4,5\}$ with the bit string "1111100000“
  $B=\{1,3,5,7,9\}$ with the bit string "1010101010"
- **Union ($A \cup B$) - Bitwise OR:** To find the union of the sets using bit strings, perform a bitwise OR operation on each corresponding pair of bits from the two strings. The rule for OR is that if at least one of the bits is a 1, the result is 1; otherwise, it's 0.
  - A: 1111100000
    B: 1010101010
  - $A \cup B$: 1111101010 (Performing OR operation on each column)
- So, the bit string for the union of sets A and B is "1111101010", which corresponds to the set $\{1,2,3,4,5,7,9\}$.

Computer Representation of Sets

The bit strings for the sets (1, 2, 3, 4, 5) and (1, 3, 5, 7, 9) are 11 1110 0000 and 10 1010 1010, respectively. Use bit strings to find the intersection of these sets..

Computer Representation of Sets

The bit strings for the sets (1, 2, 3, 4, 5) and (1, 3, 5, 7, 9) are 1111100000 and 1010101010, respectively. Use bit strings to find the intersection of these sets.

- **Intersection ($A \cap B$) - Bitwise AND:** To find the intersection of the sets using bit strings, perform a bitwise AND operation. The rule for AND is that if both bits are 1, the result is 1; if either or both are 0, the result is 0.

- A: 1111100000
  B: 1010101010
  $A \cap B$: 1010100000 (Performing AND operation on each column)

- The bit string for the intersection of sets A and B is "1010100000", which corresponds to the set $\{1,3,5\}$.

Generalized Unions and Intersections

Let A_1, A_2 ,…, A_n be an indexed collection of sets.

We define:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$$

These are well defined, since union and intersection are associative.

Example: For i = 1,2,…,n, let A_i = {i, i + 1, i + 2, ….}. Then,

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \{i, i+1, i+2, ...\} = \{1, 2, 3, ...\}$$

$$\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} \{i, i+1, i+2, ...\} = \{n, n+1, n+2, ...\} = A_n$$