LECTURE 11

Boolean Algebra

Boolean Functions

The three operations of Boolean Algebra are :

- Complementation: Denoted by a bar and acts as a negation. Example: $\bar{0} = 1$ and $\bar{1} = 0$.
- The Boolean Sum: Denoted by + or 'OR'. Example: 1+1=1; 1+0=1; 0+1=1; 0+0=0
- The Boolean Product: Denoted by '.' or by 'AND'. Example: 1.1=1; 1.0=0; 0.1=0; 0.0=0.

Boolean Functions Examples

1. Find the value of $1.0 + (\overline{0 + 1})$.

   Solution: $1.0 + (\overline{0 + 1}) = 0 + \overline{1} = 0 + 0 = 0$

2. Translate $1.1 + (\overline{0 + 1}) = 0$, into a logical equivalence.

   Solution: $(T \wedge T) \vee \neg F \equiv T$ as T =1 and F = 0.

3. Translate the logical equivalence $(T \wedge T) \vee \neg F \equiv T$ into an identity in Boolean Function

   Solution: $1.0 + (\overline{0 + 1})$

Boolean Expressions and Functions

- Let B= {0, 1}. Then $B^n = \{(x_{1,}\ x_{2,}\ x_{3,} \dots \dots \dots x_n) |\ x_n \in$ B for $1 \leq i\ \leq$ n} is the set of all possible n-tuples of 0s and 1s. The variable x is called Boolean variables if it assumes values only from B, that is, if its only possible values are 0s and 1s. A function from B^n to B is called a Boolean function of degree n.

- Boolean functions can be represented using expression made up from variables and Boolean operations. The Boolean expressions in the variable $x_{1,}\ x_{2,}$ $x_{3,} \dots \dots \dots x_n$ are defined recursively as 0, 1, $x_{1,}\ x_{2,}$ $x_{3,} \dots \dots \dots x_n\ are\ \boldsymbol{boolean\ expressions}$.

Boolean Expressions and Functions

1. Find the values of Boolean functions represented by $F\ (x, y, z) = xy + \ \bar{z}$

x	y	z	xy	$\bar{z}$	$F\ (x, y, z) = xy + \ \bar{z}$
1	1	1	1	0	1
1	1	0	1	1	1
1	0	1	0	0	0
1	0	0	0	1	1
0	1	1	0	0	0
0	1	0	0	1	1
0	0	1	0	0	0
0	0	0	0	1	1

Identities of Boolean Algebra

TABLE 5 Boolean Identities.

Identity	*Name*
$\overline{\overline{x}} = x$	Law of the double complement
$x + x = x$ $x \cdot x = x$	Idempotent laws
$x + 0 = x$ $x \cdot 1 = x$	Identity laws
$x + 1 = 1$ $x \cdot 0 = 0$	Domination laws
$x + y = y + x$ $xy = yx$	Commutative laws
$x + (y + z) = (x + y) + z$ $x(yz) = (xy)z$	Associative laws
$x + yz = (x + y)(x + z)$ $x(y + z) = xy + xz$	Distributive laws
$\overline{(xy)} = \overline{x} + \overline{y}$ $\overline{(x + y)} = \overline{x}\,\overline{y}$	De Morgan's laws
$x + xy = x$ $x(x + y) = x$	Absorption laws
$x + \overline{x} = 1$	Unit property
$x\overline{x} = 0$	Zero property

Show that the distributive law x(y+z) = xy +xz

Solution: Can be proven using a logical equivalence table shown in slide 5.

Representing Boolean Functions

- Sum of Product expressions- Disjunctive normal form
- Product of Sum expressions- Conjunctive normal form

TABLE 2

x	y	z	$x+y$	$\bar{z}$	$(x+y)\bar{z}$
1	1	1	1	0	0
1	1	0	1	1	1
1	0	1	1	0	0
1	0	0	1	1	1
0	1	1	1	0	0
0	1	0	1	1	1
0	0	1	0	0	0
0	0	0	0	1	0

LE 3 Find the sum-of-products expansion for the function $F(x, y, z) = (x + y)\bar{z}$.

es *Solution:* We will find the sum-of-products expansion of $F(x, y, z)$ in two ways. First, we will use Boolean identities to expand the product and simplify. We find that

$$\begin{aligned}
F(x, y, z) &= (x + y)\bar{z} \\
&= x\bar{z} + y\bar{z} && \text{Distributive law} \\
&= x1\bar{z} + 1y\bar{z} && \text{Identity law} \\
&= x(y + \bar{y})\bar{z} + (x + \bar{x})y\bar{z} && \text{Unit property} \\
&= xy\bar{z} + x\bar{y}\,\bar{z} + xy\bar{z} + \bar{x}y\bar{z} && \text{Distributive law} \\
&= xy\bar{z} + x\bar{y}\,\bar{z} + \bar{x}y\,\bar{z}. && \text{Idempotent law}
\end{aligned}$$

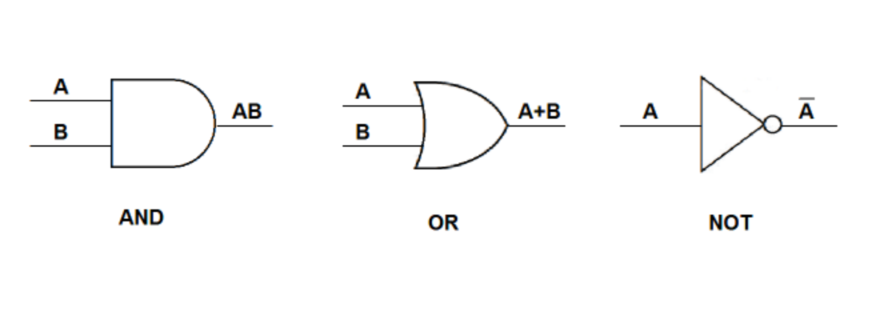
Second, we can construct the sum-of-products expansion by determining the values of F for all possible values of the variables x, y, and z. These values are found in Table 2. The sum-of-products expansion of F is the Boolean sum of three minterms corresponding to the three rows of this table that give the value 1 for the function. This gives

$$F(x, y, z) = xy\bar{z} + x\bar{y}\,\bar{z} + \bar{x}y\bar{z}.$$

It is also possible to find a Boolean expression that represents a Boolean function by taking a Boolean product of Boolean sums. The resulting expansion is called the **conjunctive normal form** or **product-of-sums expansion** of the function. These expansions can be found from sum-of-products expansions by taking duals. How to find such expansions directly is described in Exercise 10.

Logic Gates

- Inverter/ NOT gate: Accept the value of one Boolean variable as input and produces the compliment of this value as output.
- OR gates: The input for this gate are the values of two or more Boolean variables. The output is the Boolean sum of their values.
- AND gates: The input for this gate are the values of two or more Boolean variables. The output is the Boolean product of their values.



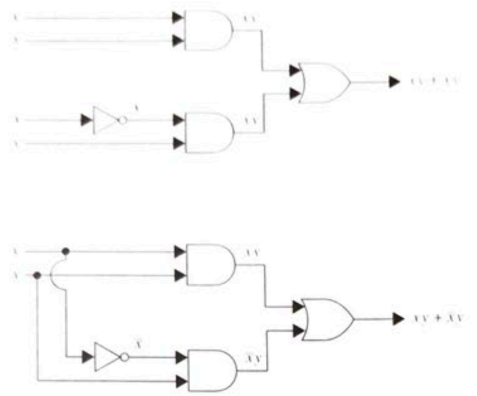


FIGURE 3 Two ways to draw the same circuit.

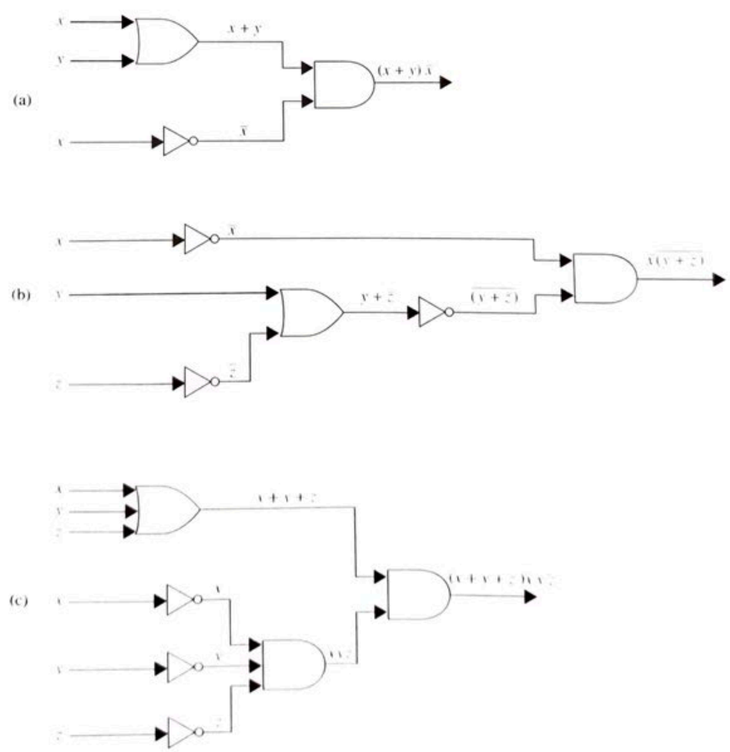


FIGURE 4 Circuits that produce the outputs specified in Example 1.

Combinations of Gates

Construct circuit that produces the following output:

(a) $(x + y)\ \bar{x}$
(b) $\bar{x}\ \overline{(y + \bar{z})}$
(c) $(x + y + z)\ (\bar{x}\ \bar{y}\ \bar{z})$

Minimization of circuits

Karnaugh Maps: https://www.youtube.com/watch?v=PSg5MJLOPRU

Quine-McCluske Methods:
https://www.youtube.com/watch?v=e41p0zoKuXo