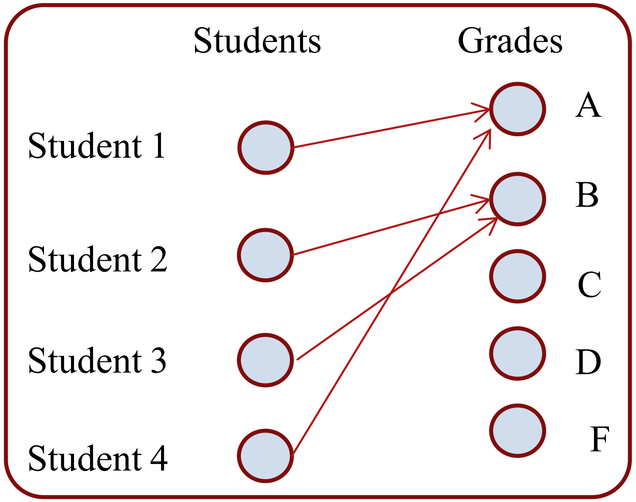
LECTURE 8

Functions

Functions

Definition: Let A and B be nonempty sets. A function f from A to B, denoted as f: A → B is an assignment of each element of A to an element of B. We write f(a) = b if b is an element of B assigned by the function f to the element a of A.

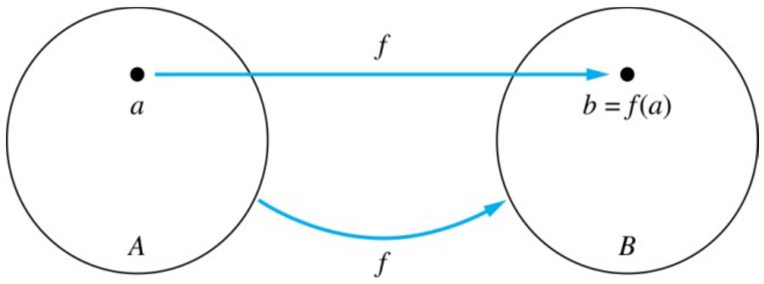
Functions are also called mappings or transformations.



Note: A function f from A to B contains one, and only one ordered pair (a, b) for every element a∈ A.

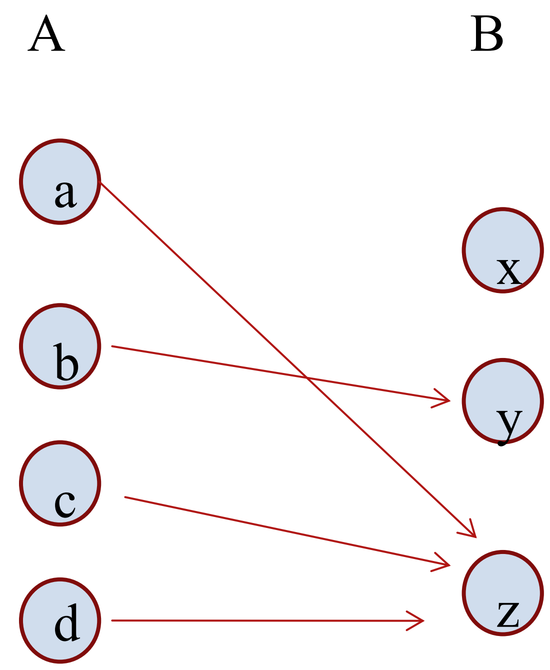
Functions

- Given a function f: A → B
- We say f maps A to B or f is a mapping from A to B.
- A is called the domain of f.
- B is called the codomain of f.
- If f(a) = b,
  - then b is called the image of a under f.
  - a is called the preimage of b.
- The range of f is the set of all images of points in A under f.
  - We denote it by f(A).
- Two functions are equal when they have the same domain and codomain
  - Maps each element of the domain to the same element of the codomain



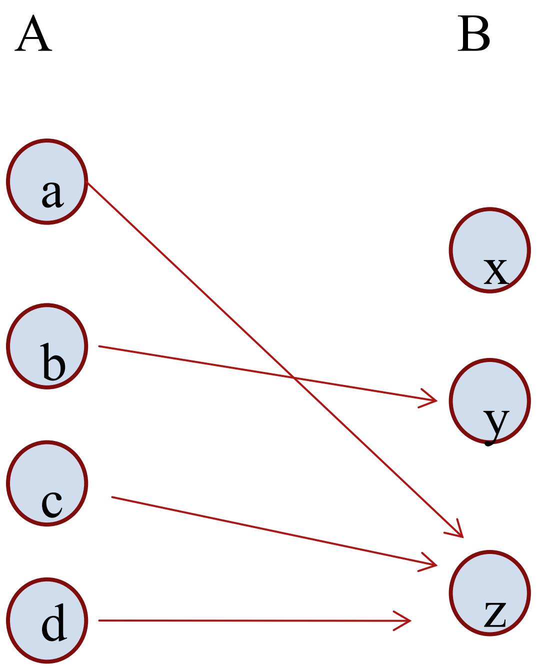
Examples

1. f(a) =
2. The image of d is
3. The domain of f is
4. The codomain of f is
5. The preimage of y is
6. f(A) =
7. The preimage(s) of z is
8. The range of f is



Answers

1. $f(a) = z$
2. The image of d is z
3. The domain of f is $A = \{a,b,c,d\}$
4. The codomain of f is $B = \{x,y,z\}$
5. The preimage of y is b
6. $f(A) = \{x,y,z\}$
7. The preimage(s) of z is $\{a,c,d\}$
8. The range of f is $\{y,z\}$



Examples

- **Function and Mapping**:
- Function: Consider a function f that represents the process of squaring a real number. It maps from the set of real numbers A to the set of non-negative real numbers B.
- Domain: A is the set of real numbers.
- Codomain: B is the set of non-negative real numbers.
- Mapping: $f(x) = x^2$ for all x in A.
- Example: If we take A = {-2, -1, 0, 1, 2}, then f maps these elements to B = {0, 1, 4}.

Image and Preimage:

- Image: If we take an element a = 3 from the domain A, the image of 3 under f is b = 9. So, f(3) = 9, and 9 is the image of 3.
- Preimage: If we take an element b = 16 from the codomain B, the preimage of 16 under f is a = -4 and a = 4 because f(-4) = 16 and f(4) = 16.

Examples

- **Range**:
- Consider a function g that maps from the set of integers A = {1, 2, 3, 4, 5} to the set of even numbers B = {2, 4, 6, 8, 10}.
- Range: The range of g, denoted as g(A), is the set of all images of points in A under g. So, g(A) = {2, 4}.
- Explanation: In this case, the range consists of the even numbers 2 and 4, which are the images of 2 and 4 in the domain.
- The range is the set of outputs a function produces.

Equality of Functions:

•Consider two functions h: P → Q and i: P → Q, where P represents the set of prime numbers and Q represents the set of positive integers.

•Both functions map each prime number to itself.

•Function h: h(p) = p for all prime numbers p in P.

•Function i: i(p) = p for all prime numbers p in P.

•These two functions are equal because they have the same domain (P) and codomain (Q) and map each element of the domain to the same element in the codomain.

Injective

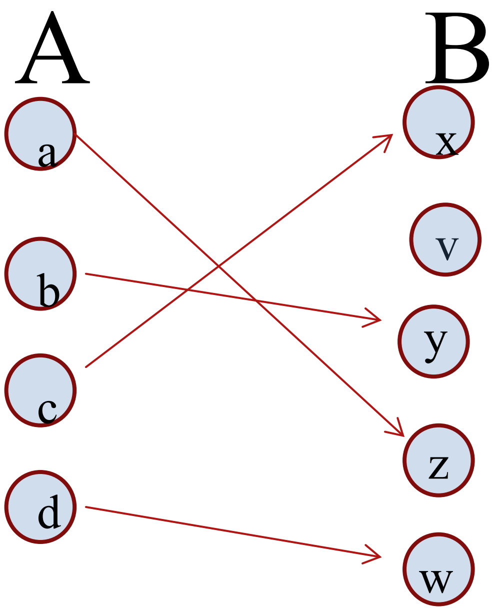
Definition: A function f is said to be one-to-one, or injective, if and only if **f(a) = f(b) implies that a = b** for all a and b in the domain of f.

- An injective function $f: A \rightarrow B$ is one in which each distinct element in the domain A maps to a distinct element in the codomain B. In other words:
- For every pair of different elements a1 and a2 in A ($a1 \neq a2$), if $f(a1) = f(a2)$, then it must be the case that $a1 = a2$.

A function is said to be an injection if it is one-to-one.

Informally: Unique input to unique output

An injective function does not map two different elements in its domain to the same element in its codomain. It ensures that there are no collisions or multiple elements in the domain that produce the same image in the codomain.



Example:

- Consider a function g: $\{1, 2, 3\} \rightarrow \{4, 5, 6\}$ defined as follows:
- $g(1) = 4$
- $g(2) = 5$
- $g(3) = 6$
- This function g is injective because each element in the domain $\{1, 2, 3\}$ maps to a distinct element in the codomain $\{4, 5, 6\}$. There are no two different elements in the domain that produce the same output in the codomain.

One-to-one examples

- Show f(x) = 3x – 2 is injective.
- Show f(x) = x^2 is NOT one-to-one.
- Determine whether function f(x) = x + 1 from the set of real numbers to itself is one-to-one.
- Determine whether the function *f* from {a, b, c, d} to {1, 2, 3, 4, 5} with *f*(a) = 4, *f*(b) = 5, *f*(c) = 1, and *f*(d) = 3 is one-to-one.

One-to-One

- To show that the function $f(x) = 3x - 2$ is injective (one-to-one), we need to demonstrate that for any distinct pair of elements $x1$ and $x2$ in the domain, $f(x1) \neq f(x2)$. In other words, we need to prove that if $f(x1) = f(x2)$, then it must be the case that $x1 = x2$.
- Assume that $f(x1) = f(x2)$ for some $x1$ and $x2$ in the domain. This means: $f(x1) = f(x2)$
- Substitute the expression for $f(x)$: $3x1 - 2 = 3x2 - 2$
- Now, let's simplify this equation: $3x1 = 3x2$
- Divide both sides by 3: $x1 = x2$
- We have shown that if $f(x1) = f(x2)$, then $x1 = x2$. This proves that the function $f(x) = 3x - 2$ is injective because it does not map two different elements in its domain to the same element in its codomain, ensuring that each distinct input corresponds to a distinct output.

Not One-to-One

- The function $f(x) = x^2$ is not one-to-one (injective). To demonstrate this, we need to show that there exist two distinct elements $x1$ and $x2$ in the domain such that $f(x1) = f(x2)$, violating the one-to-one property.
- Let $x1 = 2$ and $x2 = -2$, both in the domain of real numbers. Then:
  $f(x1) = (2)^2 = 4$
  $f(x2) = (-2)^2 = 4$
- As you can see, $f(x1) = f(x2)$, but $x1 \neq x2$. This means that we have found two distinct elements in the domain ($x1$ and $x2$) that map to the same element in the codomain (4). Since there exist such elements, the function $f(x) = x^2$ is not one-to-one (injective).

One-to-One

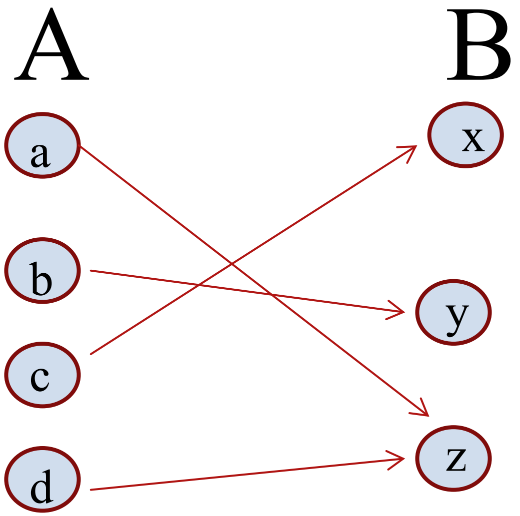
- To determine whether the function $f(x) = x + 1$ from the set of real numbers to itself is one-to-one (injective), we need to check if it satisfies the one-to-one property. In other words, we need to determine if for any distinct pair of real numbers x1 and x2, $f(x1) \neq f(x2)$ implies $x1 \neq x2$.
- Assume that $f(x1) = f(x2)$, where x1 and x2 are distinct real numbers.
  $f(x1) = x1 + 1$
  $f(x2) = x2 + 1$
- Now, let's set them equal: $x1 + 1 = x2 + 1$
- Subtract 1 from both sides of the equation: $x1 = x2$
- So, we have shown that if $f(x1) = f(x2)$, then $x1 = x2$. This means that for any distinct pair of real numbers x1 and x2, if they have the same image under the function f, then they must be the same real number. Therefore, the function $f(x) = x + 1$ is one-to-one (injective).

One-to-One

- To determine if the function f from the set {a, b, c, d} to the set {1, 2, 3, 4, 5} is one-to-one (injective), we need to check if it satisfies the one-to-one property. In other words, we need to determine if for any distinct pair of elements in the domain, their images under the function are distinct.
- Given the function:
  $f(a) = 4$
  $f(b) = 5$
  $f(c) = 1$
  $f(d) = 3$
- Now, let's check for one-to-one (injective) property:
  - $f(a) \neq f(b)$: $4 \neq 5$ (Distinct)
  - $f(a) \neq f(c)$: $4 \neq 1$ (Distinct)
  - $f(a) \neq f(d)$: $4 \neq 3$ (Distinct)
  - $f(b) \neq f(c)$: $5 \neq 1$ (Distinct)
  - $f(b) \neq f(d)$: $5 \neq 3$ (Distinct)
  - $f(c) \neq f(d)$: $1 \neq 3$ (Distinct)
- In all cases, the images of distinct elements in the domain are distinct in the codomain. Therefore, no two different elements in the domain {a, b, c, d} map to the same element in the codomain {1, 2, 3, 4, 5}.
- Since all distinct elements in the domain have distinct images, the function f is indeed one-to-one (injective).

Surjections

Definition: A function f from A to B is called onto or surjective, if and only if for every element $b \in B$ there is an element $a \in A$ with $\boldsymbol{f(a) = b}$.



A function f is called a surjection if it is onto.

Informally: codomain = range

Surjections

Codomain and Range: The codomain of a function is the set B, which represents all possible values that the function can output.

The range of a function, also known as the image, is the subset of the codomain B consisting of all actual values that the function takes when applied to elements from the domain A.

Surjective Function (Onto): A function is called surjective or onto if it covers its entire codomain B. In other words, it maps elements from the domain to every possible element in the codomain.

For every element b in the codomain B, there is at least one element a in the domain A such that f(a) = b.

This means that the function doesn't miss any values in its codomain; it reaches all of them through its mappings.

Surjections

- **Informal Statement: Codomain = Range**:
- Informally, a surjective function is one where its codomain is essentially equal to its range because it covers all the possible values in the codomain without leaving any out.
- While the range is a subset of the codomain, in the case of a surjective function, it is as if the range fills up the entire codomain, making them practically equal.

Onto example

1. Let f be the function from {a, b, c, d} to {1, 2, 3} defined by f(a) = 3, f(b) = 2, f(c) = 1 and f(d) = 3. Is *f* an onto function?

2. Show that f(x) = x + 1.

3. Show that f(x) = 5x + 2 for all $x \in \mathbb{R}$. What about Integers?

Onto example

To determine if the function f from {a, b, c, d} to {1, 2, 3} is onto (surjective), we need to check whether it covers the entire codomain {1, 2, 3} without missing any elements.

Given the function:
f(a) = 3
f(b) = 2
f(c) = 1
f(d) = 3

The codomain is {1, 2, 3}, and the function's range (the set of actual values it takes) is {1, 2, 3}, which includes all the elements in the codomain. In other words, every element in the codomain is covered by the function's mappings.

$x \in \mathbb{R}$.

Therefore, since the function f maps elements from its domain to every possible element in its codomain {1, 2, 3}, it is onto (surjective). It doesn't miss any values in the codomain, fulfilling the definition of an onto function.

Onto example

- To show that the function $f(x) = x + 1$ is onto (surjective), we need to demonstrate that for every element b in the codomain, there exists at least one element a in the domain such that $f(a) = b$.
- The function $f(x) = x + 1$ maps from the set of real numbers to itself. Assume we have an arbitrary element b in the codomain, which is a real number. We need to find an element a in the domain (real numbers) such that $f(a) = b$, or in other words:
- $a + 1 = b$
- Now, we'll isolate 'a':
- $a = b - 1$
- So, for any real number b, we can find an element a (in the domain) that satisfies the equation above. This shows that for every element b in the codomain (real numbers), there is at least one element a in the domain (real numbers) such that $f(a) = b$.
- Since we've demonstrated that the function covers the entire codomain without missing any elements, the function $f(x) = x + 1$ is onto (surjective).

Onto example

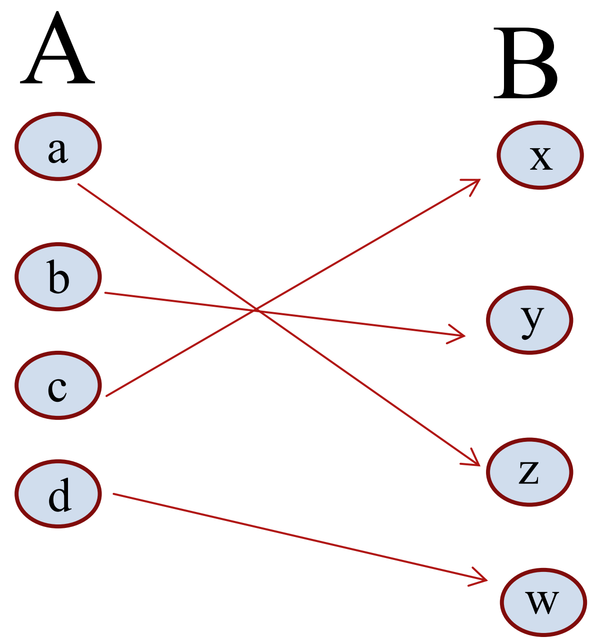
- To show that the function $f(x) = 5x + 2$ is onto (surjective) for all real numbers, we need to demonstrate that for every real number b, there exists at least one real number a such that $f(a) = b$.
- Let's prove this for all real numbers, so assume we have an arbitrary real number b.
- We need to find a real number a such that $f(a) = b$, which means: $5a + 2 = b$
- Now, we'll isolate 'a': $5a = b - 2$
- $a = (b - 2)/5$
- For any real number b, we can find a real number a that satisfies the equation above. This shows that the function $f(x) = 5x + 2$ is onto (surjective) for all real numbers.
- However, if we consider the set of integers, the situation changes. In this case, not all integers can be expressed as $f(a) = 5a + 2$, where 'a' is an integer. This is because dividing an integer by 5 might not always result in an integer. For example, if you choose $b = 3$, there is no integer 'a' such that $5a + 2 = 3$. Therefore, the function $f(x) = 5x + 2$ is not onto (surjective) when considering the set of integers as the codomain.

Bijections

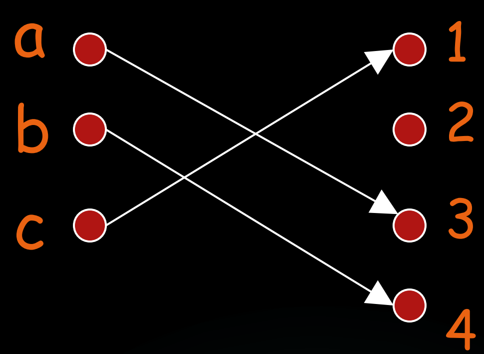
Definition: A function f is a one-to-one correspondence, or a bijection, if it is both one-to-one and onto (surjective and injective).

Both set cardinalities are equal.

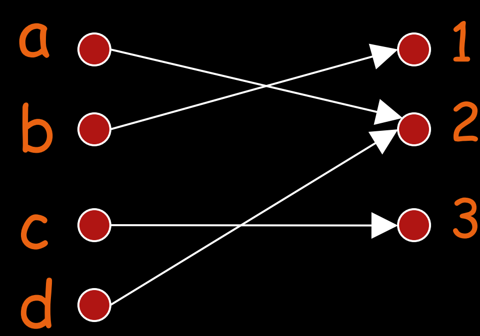
Bijections are highly important in mathematics because they establish a perfect correspondence between elements of two sets. In a bijection, each element in the domain is paired with exactly one element in the codomain, and each element in the codomain has exactly one preimage in the domain. This property allows for a one-to-one and onto relationship, making bijections useful for various mathematical applications, including set theory, combinatorics, and the study of functions.

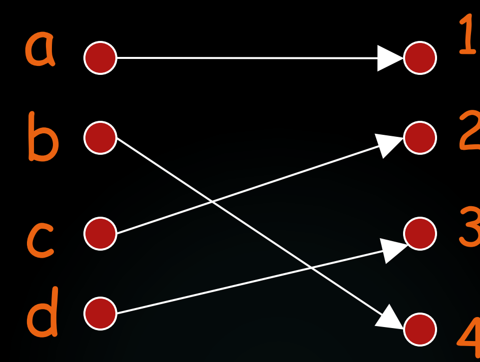


Injective
Not Surjective



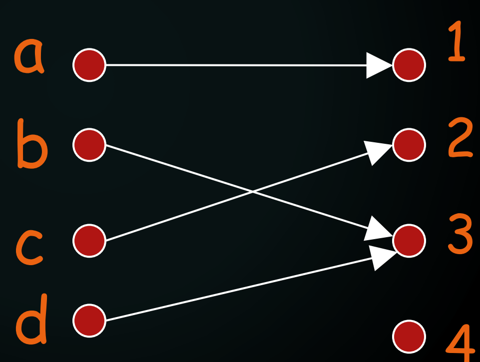
Not Injective
Surjective



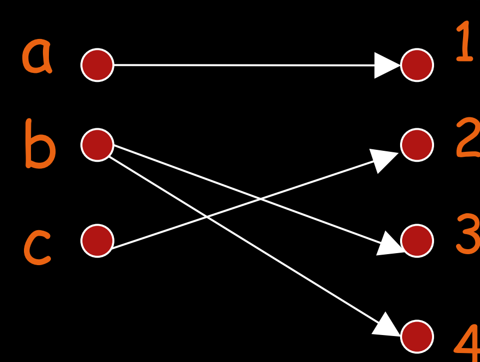


Injective and Surjective
Bijective

Not Injective and not surjective



Not a function

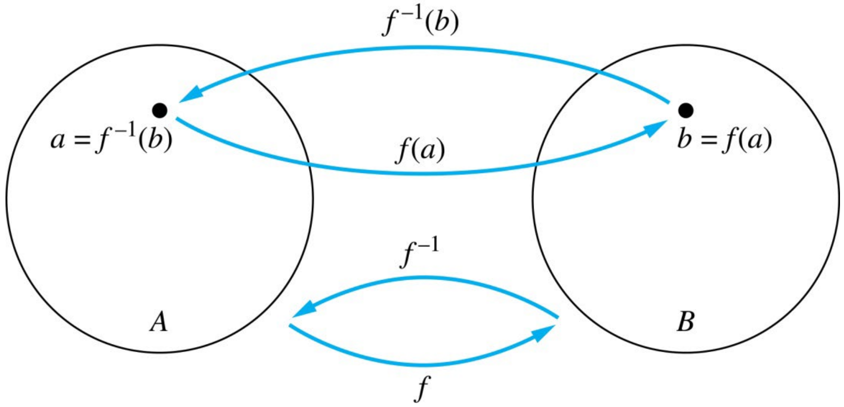


Inverse Functions

Definition: Let f be a bijection from A to B. Then the inverse of f, denoted f^{-1} , is the function from B to A defined as

$$f^{-1}(y) = x \text{ iff } f(x) = y$$

No inverse exists unless f is a bijection.

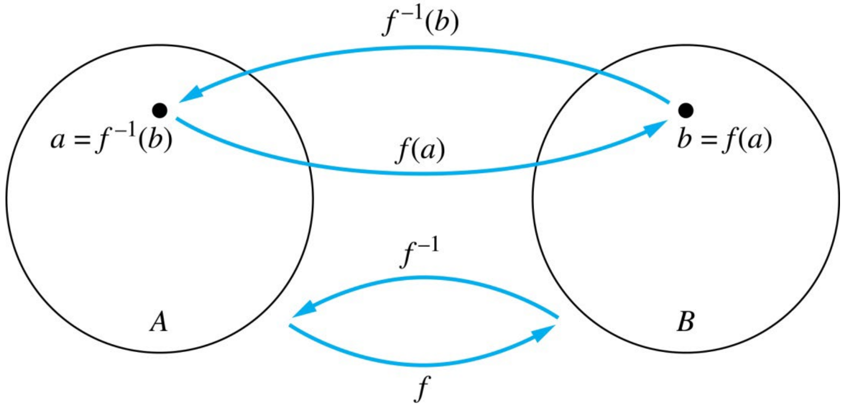


Inverse Functions

An inverse function is a function that "undoes" the work of another function. If you have a function f(x), its inverse function is often denoted as $f^{-1}(x)$.

For any input x, applying the function f, and then applying its inverse function f^{-1}, should bring you back to the original input: $f^{-1}(f(x)) = x$.

Not all functions have inverse functions. For an inverse function to exist, the original function must be one-to-one (injective), meaning it maps distinct inputs to distinct outputs.



Inverse examples

1. Let *f* be the function from {a, b, c} to {1,2,3} such that f(a) = 2, f(b) = 3, and f(c) = 1. Is *f* invertible and if so, what is its inverse?

2. Let *f*: Z → Z be such that f(x) = x + 1. Is *f* invertible, and if so, what is its inverse?

3. Let *f*: R → R be such that $f(x) = x^2$. Is *f* invertible, and if so, what is its inverse?

Inverse examples

To determine if the function f from {a, b, c} to {1, 2, 3} is invertible, we need to check if it is a bijection, which means it must be both one-to-one (injective) and onto (surjective).

Given the function:

$f(a) = 2$
$f(b) = 3$
$f(c) = 1$

Injective (One-to-One): To be injective, no two distinct elements in the domain should map to the same element in the codomain. In this case, all three elements in the domain map to distinct elements in the codomain. So, the function is injective.

Surjective (Onto): To be surjective, the function should cover the entire codomain. In this case, it does because all three elements in the codomain {1, 2, 3} are mapped to by elements in the domain {a, b, c}. So, the function is surjective.

Since the function f is both injective and surjective (a bijection), it is invertible. The inverse function f^{-1} can be determined as follows:

We know that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. To find the inverse function f^{-1}, we need to reverse the roles of elements in the domain and codomain:

$f^{-1}(1) = c$
$f^{-1}(2) = a$
$f^{-1}(3) = b$

So, the inverse function f^{-1} maps 1 to c, 2 to a, and 3 to b.

Inverse examples

The function f: $Z \rightarrow Z$ defined as $f(x) = x + 1$ maps an integer x to the integer $x + 1$. To determine if this function is invertible, we need to check if it is a bijection, which means it must be both one-to-one (injective) and onto (surjective).

Injective (One-to-One): To be injective, no two distinct elements in the domain should map to the same element in the codomain. In this case, $f(x) = x + 1$ is indeed injective because adding 1 to distinct integers produces distinct results.

Surjective (Onto): To be surjective, the function should cover the entire codomain. In this case, it does because for any integer y in the codomain Z, you can always find an integer x (specifically, $x = y - 1$) such that $f(x) = y$.

Since the function f is both injective and surjective (a bijection), it is invertible. The inverse function f^{-1} can be determined as follows:

Given $f(x) = x + 1$, to find $f^{-1}(y)$, where y is an integer, we subtract 1 from y: $f^{-1}(y) = y - 1$

So, the inverse function f^{-1} maps any integer y to $y - 1$.

Inverse examples

The function f: R → R defined as $f(x) = x^2$ maps a real number x to its square, x^2. To determine if this function is invertible, we need to check if it is a bijection, which means it must be both one-to-one (injective) and onto (surjective).

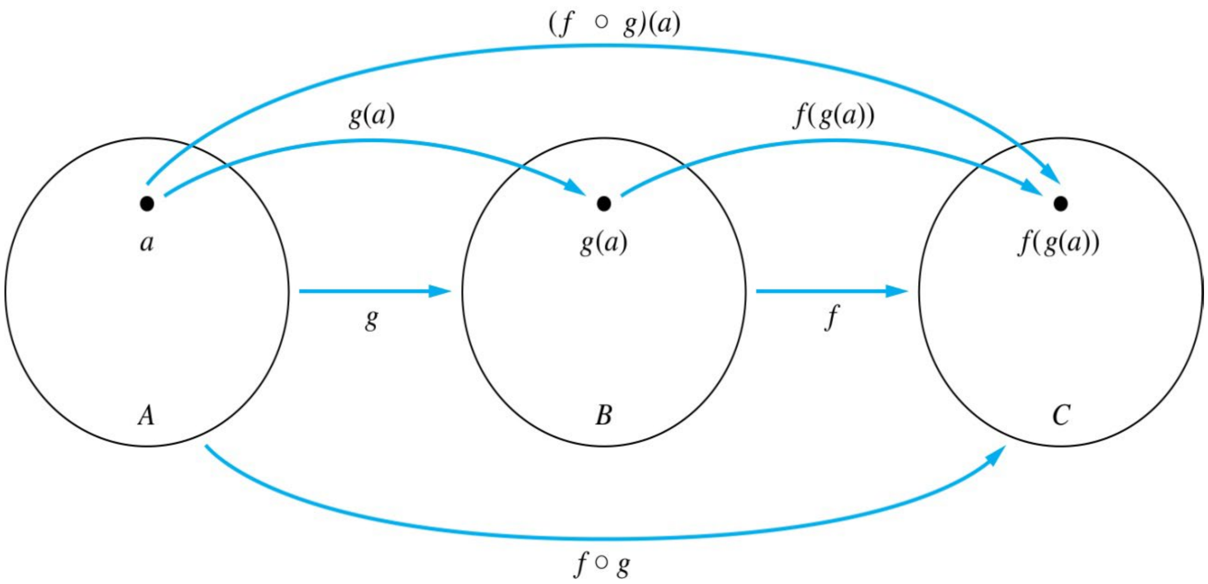
Injective (One-to-One): To be injective, no two distinct elements in the domain should map to the same element in the codomain. In the case of $f(x) = x^2$, it is not injective because, for example, both -2 and 2 in the domain map to the same value, which is 4, in the codomain.

Surjective (Onto): To be surjective, the function should cover the entire codomain. In this case, it does because every non-negative real number in the codomain can be obtained as the square of some real number from the domain.

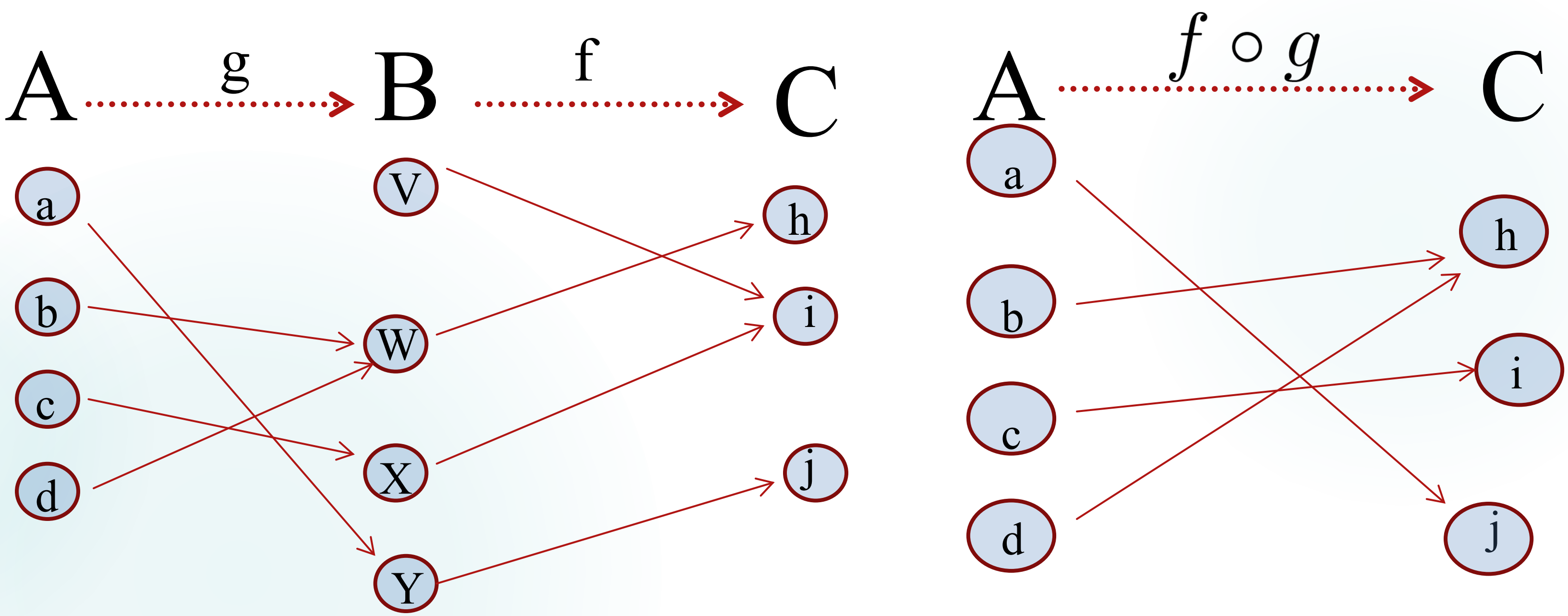
Since the function f is not injective (it fails the one-to-one condition), it is not invertible. An inverse function for $f(x) = x^2$ does not exist because there is no unique way to "undo" the squaring operation, given that multiple values from the domain can result in the same value in the codomain.

Composition

Definition: Let f: B → C, g: A → B. The composition of f with g, denoted $f \circ g$ is the function from A to C defined by $f \circ g\ (x) = f(g(x))$



Composition



Composition

Composition of functions is a fundamental concept in mathematics, particularly in the study of functions and relations. It involves combining two or more functions to create a new function. The composition of functions is denoted by the symbol "(o)" or by simply placing one function inside another.

Given two functions:
Function f: Maps elements from a set A to a set B. It is represented as $f: A \to B$.
Function g: Maps elements from a set C to a set D. It is represented as $g: C \to D$.

The composition of these two functions, denoted as $(g \circ f)$, results in a new function that maps elements from set A to set D. The composition is defined as follows:

$$(g \circ f)(x) = g(f(x))$$

Composition

Here's a breakdown of what this means:

- Start with an element x from set A.
- Apply function f to x, which yields a new element in set B: f(x).
- Take f(x), and apply function g to it, which yields an element in set D: g(f(x)).
- g(f(x)), is the output of the composition of functions g and f for the input x.

In essence, (g o f) combines the two functions by applying them sequentially, first applying f and then applying g to the result of f.

It's important to note that the order of composition matters. In general, (g o f) is not the same as (f o g), unless both functions are inverses of each other.

The composition of functions is a powerful tool used in various mathematical and scientific fields for modeling and solving problems involving transformations and relationships between sets.

Composition

1. If $f(x) = x^2$ and $g(x) = 2x + 1$, then what is fog and gof?

2. Let g be the function from the set {a,b,c} to itself such that g(a) = b, g(b) = c, and g(c) = a. Let f be the function from the set {a,b,c} to the set {1,2,3} such that f(a) = 3, f(b) = 2, and f(c) = 1.What is the composition of f and g, and what is the composition of g and f.

3. Let f and g be functions from the set of integers to the set of integers defined by f(x) = 2x + 3 and g(x) = 3x + 2. What is the composition of f and g, and the composition of g and f ?

Composition

Let's find both the compositions, f o g and g o f, for the given functions $f(x) = x^2$ and $g(x) = 2x + 1$.

f o g (f composed with g):

Start with (f o g)(x), which means apply g to x first and apply f to the result.
First, apply g to x: $g(x) = 2x + 1$
Now, apply f to the result: $f(2x + 1) = (2x + 1)^2$
So, $(f \circ g)(x) = (2x + 1)^2$.

Now: g o f (g composed with f):

Start with the composition (g o f)(x), which means applying f to x first and then applying g to the result.
First, apply f to x: $f(x) = x^2$
Now, apply g to the result: $g(x^2) = 2(x^2) + 1$
So, $(g \circ f)(x) = 2x^2 + 1$.

In summary:
$(f \circ g)(x) = (2x + 1)^2$
$(g \circ f)(x) = 2x^2 + 1$

These are the compositions of the functions $f(x) = x^2$ and $g(x) = 2x + 1$.

Composition

Let g be the function from the set {a,b,c} to itself such that g(a) = b, g(b) = c, and g(c) = a. Let f be the function from the set {a,b,c} to the set {1,2,3} such that f(a) = 3, f(b) = 2, and f(c) = 1.What is the composition of f and g, and what is the composition of g and f.

f o g (f composed with g):

- Start with the composition (f o g)(x), which means applying g to x first and then applying f to the result.
- First, apply g to each element in the set {a, b, c}:
  g(a) = b
  g(b) = c
  g(c) = a
- Now, apply f to the results:
  f(g(a)) = f(b) = 2
  f(g(b)) = f(c) = 1
  f(g(c)) = f(a) = 3
- So, (f o g)(x) = {2, 1, 3}.

Composition

Let g be the function from the set {a,b,c} to itself such that g(a) = b, g(b) = c, and g(c) = a. Let f be the function from the set {a,b,c} to the set {1,2,3} such that f(a) = 3, f(b) = 2, and f(c) = 1.What is the composition of f and g, and what is the composition of g and f.

g o f (g composed with f):

Start with the composition (g o f)(x), which means applying f to x first and then applying g to the result.

First, apply f to each element in the set {a, b, c}:
f(a) = 3
f(b) = 2
f(c) = 1

Now, apply g to the results:
g(f(a)) = g(3) = a
g(f(b)) = g(2) = b
g(f(c)) = g(1) = c
So, (g o f)(x) = {a, b, c}.

In summary:

(f o g)(x) = {2, 1, 3}
(g o f)(x) = {a, b, c}

These are the compositions of the functions f and g, as well as g and f.

Composition

1. Let f and g be functions from the set of integers to the set of integers defined by f(x) = 2x + 3 and g(x) = 3x + 2. What is the composition of f and g, and the composition of g and f ?

f o g (f composed with g):

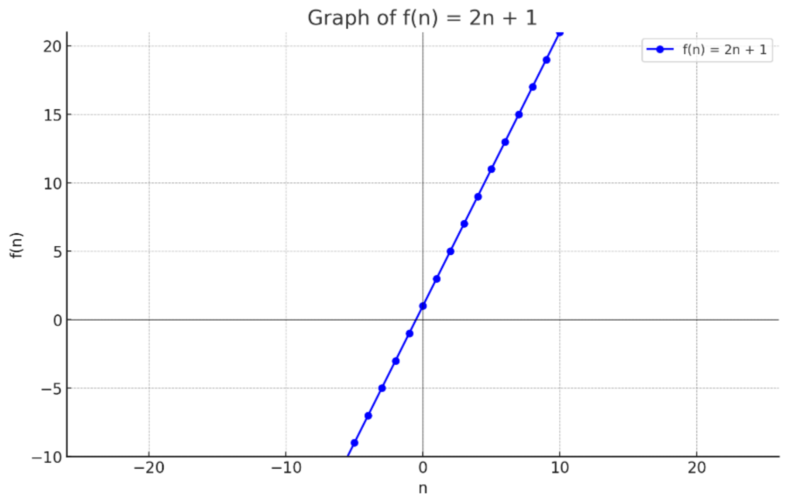
1. Start with the composition (f o g)(x), which means applying g to x first and then applying f to the result.
2. First, apply g to x: g(x) = 3x + 2
3. Now, apply f to the result: f(3x + 2) = 2(3x + 2) + 3 = 6x + 4 + 3 = 6x + 7
4. So, (f o g)(x) = 6x + 7.

g o f (g composed with f):

1. Start with the composition (g o f)(x), which means applying f to x first and then applying g to the result.
2. First, apply f to x: f(x) = 2x + 3
3. Now, apply g to the result: g(2x + 3) = 3(2x + 3) + 2 = 6x + 9 + 2 = 6x + 11
4. So, (g o f)(x) = 6x + 11.

Graphs of Functions

- Let f be a function from the set A to the set B. The graph of the function f is the set of ordered pairs $\{(a,b) \mid a \in A \text{ and } f(a) = b\}$.



Graph of $f(n) = 2n + 1$

from Z to Z

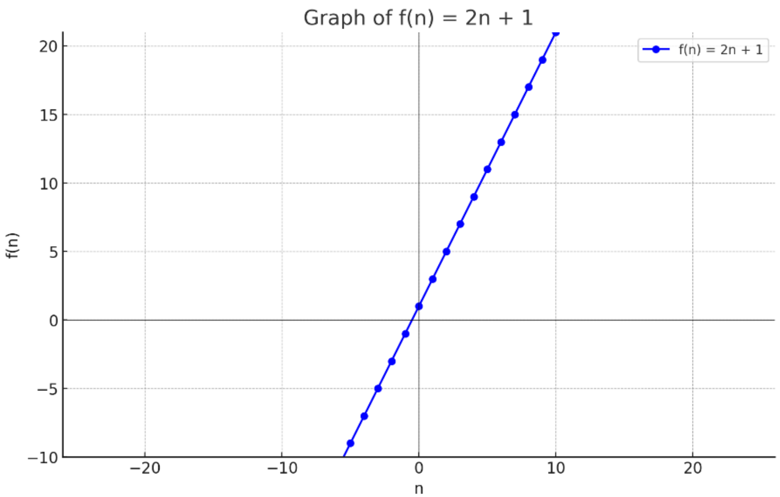
Graph of $f(x) = x^2$

from Z to Z

Graph the function $f(x)= 5x-4$

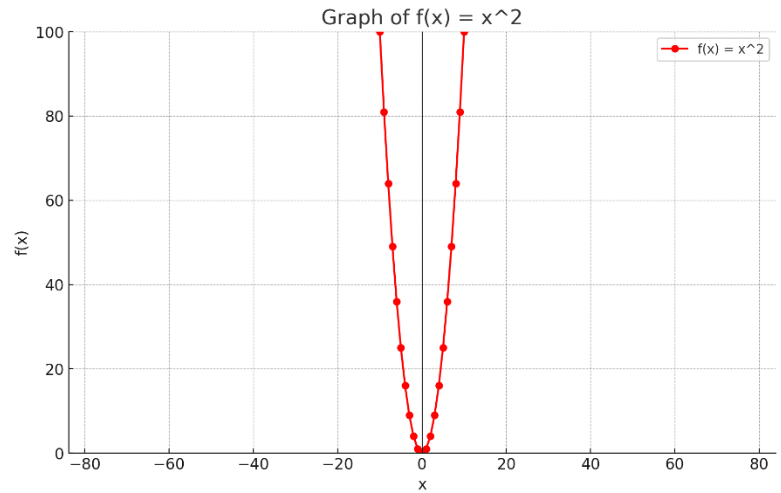
Graphs of Functions

- Let f be a function from the set A to the set B. The graph of the function f is the set of ordered pairs $\{(a,b) \mid a \in A \text{ and } f(a) = b\}$.



Graph of $f(n) = 2n + 1$

from Z to Z



Graph of $f(x) = x^2$

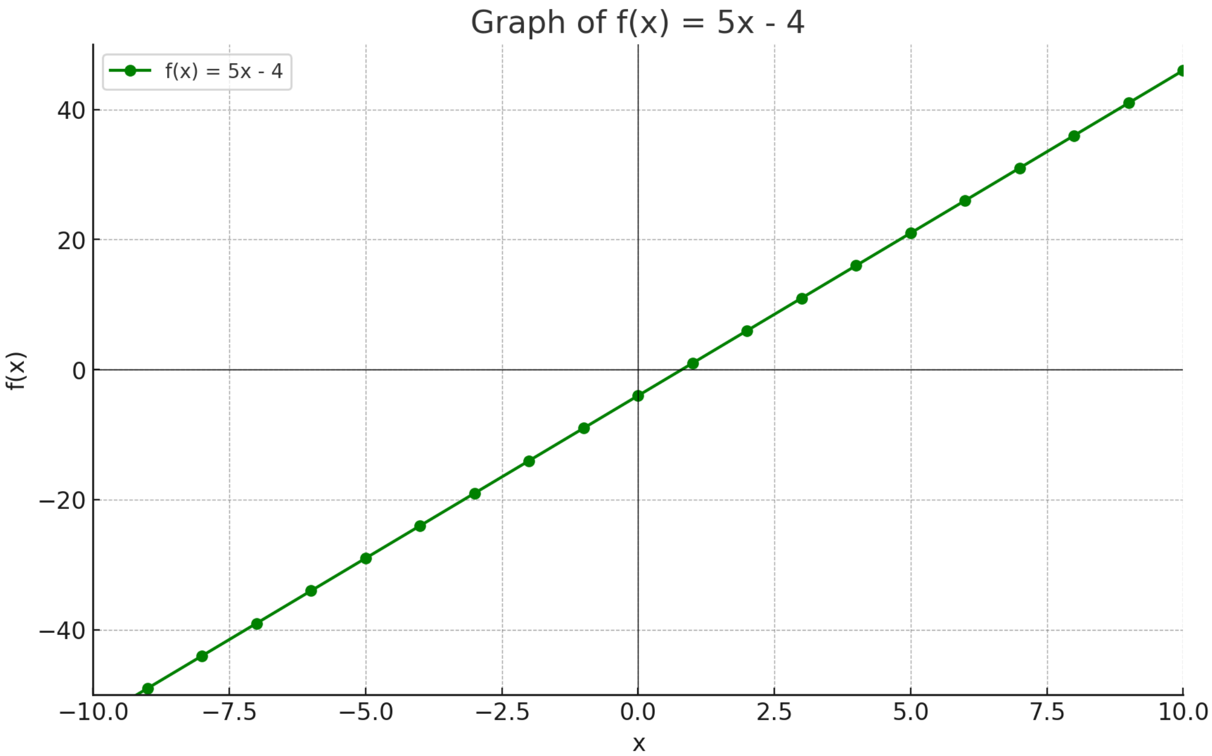
from Z to Z

Graphs of Functions

Graph the function $f(x)= 5x-4$

Graphs of Functions

Graph the function f(x)= 5x-4



Some Important Functions

- The floor function, denoted as: $f(x) = \lfloor x \rfloor$

is the largest integer less than or equal to x.

Is math.floor(x) in Python

- The ceiling function, denoted $f(x) = \lceil x \rceil$

is the smallest integer greater than or equal to x.

Is math.ceil(x) in Python

Examples

Solve the following:

$$\lceil 3.5 \rceil = 4 \qquad \lceil -1.5 \rceil = -1$$

$$\lfloor 3.5 \rfloor = 3 \qquad \lfloor -1.5 \rfloor = -2$$

Examples

Solve the following:

Data stored on a computer disk or transmitted over a data network are usually represented as a string of bytes. Each byte is made up of 8 bits. How many bytes are required to encode 100 bits of data?

Examples

Solve the following:

Data stored on a computer disk or transmitted over a data network are usually represented as a string of bytes. Each byte is made up of 8 bits. How many bytes are required to encode 100 bits of data?

To encode 100 bits of data, we need to calculate how many full bytes are required. Since a byte consists of 8 bits, we can divide 100 by 8 to determine the number of bytes needed. However, since we can only have whole bytes, if there is any remainder after this division, we will need to round up to the next whole byte.

$$100/8 = 12.5 = 13$$

TABLE 1 Useful Properties of the Floor and Ceiling Functions.
(n is an integer, x is a real number)

(1a)	$\lfloor x \rfloor = n$ if and only if $n \leq x < n + 1$
(1b)	$\lceil x \rceil = n$ if and only if $n - 1 < x \leq n$
(1c)	$\lfloor x \rfloor = n$ if and only if $x - 1 < n \leq x$
(1d)	$\lceil x \rceil = n$ if and only if $x \leq n < x + 1$
(2)	$x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$
(3a)	$\lfloor -x \rfloor = -\lceil x \rceil$
(3b)	$\lceil -x \rceil = -\lfloor x \rfloor$
(4a)	$\lfloor x + n \rfloor = \lfloor x \rfloor + n$
(4b)	$\lceil x + n \rceil = \lceil x \rceil + n$

Proving Properties of Functions

Prove or disprove that for all real numbers x and y. $\lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil$

Solution: Although this statement may appear reasonable, it is false. A counterexample is supplied by x = ½ and y = ½ . With these values we find that,

$\lceil x + y \rceil = \left\lceil \frac{1}{2} + \frac{1}{2} \right\rceil = \lceil 1 \rceil = 1$ But $\lceil x + y \rceil = \lceil 1/2 \rceil + \lceil 1/2 \rceil = 1 + 1 = 2$

Proving Properties of Functions

Example: Prove that x is a real number, then

$$\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor$$

Solution: Let $x = n + \varepsilon$, where n is an integer and $0 \leq \varepsilon < 1$.

Case 1: $\varepsilon < ½$

- $2x = 2n + 2\varepsilon$ and $\lfloor 2x \rfloor = 2n$, since $0 \leq 2\varepsilon < 1$.
- $\lfloor x + 1/2 \rfloor = n$, since $x + ½ = n + (1/2 + \varepsilon)$ and $0 \leq ½ + \varepsilon < 1$.
- Hence, $\lfloor 2x \rfloor = 2n$ and $\lfloor x \rfloor + \lfloor x + 1/2 \rfloor = n + n = 2n$.

Case 2: $\varepsilon \geq ½$

- $2x = 2n + 2\varepsilon = (2n + 1) + (2\varepsilon - 1)$ and $\lfloor 2x \rfloor = 2n + 1$, since $0 \leq 2\varepsilon - 1 < 1$.
- $\lfloor x + 1/2 \rfloor = \lfloor n + (1/2 + \varepsilon) \rfloor = \lfloor n + 1 + (\varepsilon - 1/2) \rfloor = n + 1$, since $0 \leq \varepsilon - 1/2 < 1$.
- Hence, $\lfloor 2x \rfloor = 2n + 1$ and $\lfloor x \rfloor + \lfloor x + 1/2 \rfloor = n + (n + 1) = 2n + 1$.

Proving Properties of Functions

Example: Prove $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor$ when x is a real number, using Python.

Use a range of -5 to 5 step 0.2