CHAPTER 9

COUNTING AND PROBABILITY

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9.3 Counting Elements of Disjoint Sets: The Addition Rule

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Counting Elements of Disjoint Sets: The Addition Rule (1/2)

In this section we look at counting problems that can be solved by counting the number of elements in the union of two sets, the difference of two sets, or the intersection of two sets.

The basic rule underlying the calculation of the number of elements in a union or difference or intersection is the addition rule.

Counting Elements of Disjoint Sets: The Addition Rule (2/2)

This rule states that the number of elements in a union of mutually disjoint finite sets equals the sum of the number of elements in each of the component sets.

Theorem 9.3.1 The Addition Rule

Suppose a finite set A equals the union of k distinct mutually disjoint subsets A_1, $A_2, \ldots, A_k$. Then

$$N(A) = N(A_1) + N(A_2) + \cdots + N(A_k).$$

Example 9.3.1 – *Counting the Number of Integers Divisible by 5*

How many three-digit integers (integers from 100 to 999 inclusive) are divisible by 5?

Example 9.3.1 – *Solution (1/3)*

The solution to this problem uses the addition rule. Integers that are divisible by 5 end either in 5 or in 0. Thus the set of all three-digit integers that are divisible by 5 can be split into two mutually disjoint subsets A_1 and A_2 as shown in Figure 9.3.1.

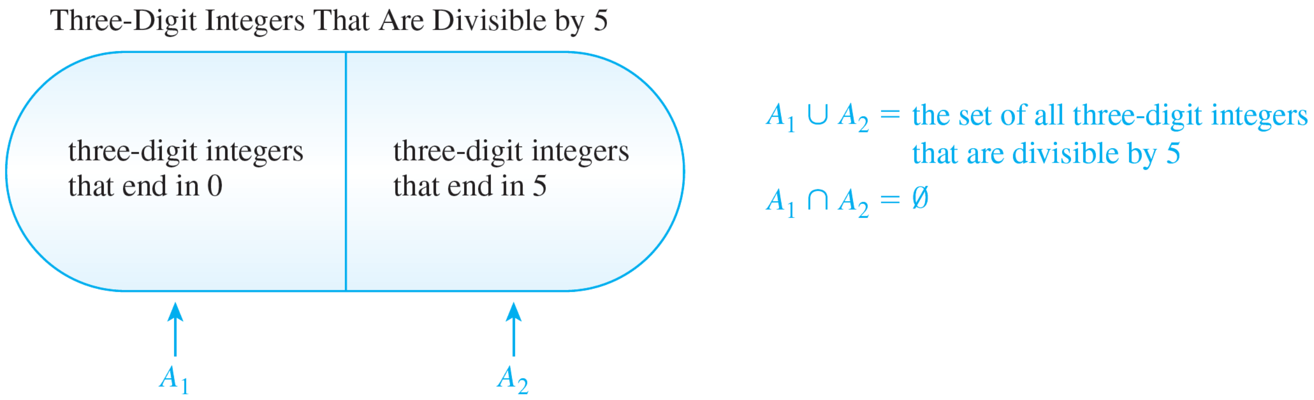


Figure 9.3.1

Example 9.3.1 – *Solution* (2/3)

continued

Now there are as many three-digit integers that end in 0 as there are possible choices for the left-most and middle digits (because the right-most digit must be a 0).

As illustrated below, there are nine choices for the left-most digit (the digits 1 through 9) and ten choices for the middle digit (the digits 0 through 9). Hence $N(A_1) = 9 \cdot 10 = 90$.

$\square$	$\square$	$\square$
$\uparrow$	$\uparrow$	$\uparrow$
9 choices	10 choices	number ends in 0
1, 2, 3, 4, 5, 6, 7, 8, 9	0, 1, 2, 3, 4, 5, 6, 7, 8, 9	

Example 9.3.1 – *Solution (3/3)* continued

Similar reasoning shows that there are as many three-digit integers that end in 5 as there are possible choices for the left-most and middle digits, which are the same as for the integers that end in 0.

Hence, $N(A_2) = 90$. So

$$\begin{bmatrix}\text{the number of} \\ \text{three-digit integers} \\ \text{that are divisible by 5}\end{bmatrix} = N(A_1) + N(A_2) = 90 + 90 = 180.$$

The Difference Rule

The Difference Rule (1/4)

An important consequence of the addition rule is the fact that if the number of elements in a set *A* and the number in a subset *B* of *A* are both known, then the number of elements that are in *A* and not in *B* can be computed.

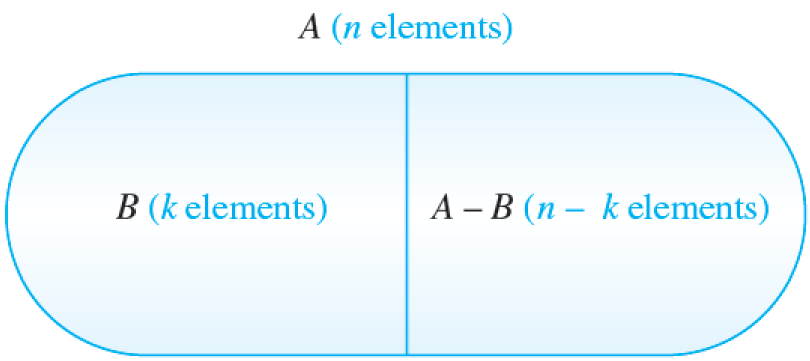
Theorem 9.3.2 The Difference Rule

If A is a finite set and B is a subset of A, then

$$N(A - B) = N(A) - N(B).$$

The Difference Rule (2/4)

The difference rule is illustrated in Figure 9.3.2.



The Difference Rule

Figure 9.3.2

The difference rule holds for the following reason: If B is a subset of A, then the two sets B and $A - B$ have no elements in common and $B \cup (A - B) = A$.

The Difference Rule (3/4)

Hence, by the addition rule,

$$N(B) + N(A - B) = N(A).$$

Subtracting $N(B)$ from both sides gives the equation

$$N(A - B) = N(A) - N(B).$$

Example 9.3.2 – *Counting PINs with Repeated Symbols*

Consider the personal identification numbers (PINs). These are made from exactly four symbols chosen from the 26 uppercase letters of the Roman alphabet and the ten digits.

There are 1,679,616 PINs with repetition allowed and 265,896 PINs with no repeated symbol.

a. How many PINs contain at least one repeated symbol?

b. If all PINs are equally likely, what is the probability that a randomly chosen PIN contains at least one repeated symbol?

Example 9.3.2 – *Solution (1/4)*

a. Let S be the set of all the PINs with repetition allowed, and let A be the set of PINs with no repeated symbol. Then $S - A$ is the set of PINs with at least one repeated symbol, and, by the difference rule,

$$\begin{aligned} N(S - A) &= N(S) - N(A) \\ &= 1{,}679{,}616 - 1{,}413{,}720 \\ &= 265{,}896. \end{aligned}$$

Hence, there are 265,896 PINs that contain at least one repeated symbol.

Example 9.3.2 – *Solution* (2/4)

continued

b. *Solution* 1: Because there are 1,679,616 PINs in all and 265,896 of these contain at least one repeated symbol, by the equally likely probability formula, the probability that a randomly chosen PIN contains a repeated symbol is

$$\frac{265{,}896}{1{,}679{,}616} \cong 0.158 = 15.8\%.$$

Solution 2: $P(A)$ is the probability that a randomly chosen PIN has no repeated symbol, and so $P(S - A)$ is the probability that a randomly chosen PIN has at least one repeated symbol.

Example 9.3.2 – *Solution (3/4)*

continued

Then

$$P(S-A) = \frac{N(S-A)}{N(S)}$$ by definition of probability in the equally likely case

$$= \frac{N(S)-N(A)}{N(S)}$$ by the difference rule

$$= \frac{N(S)}{N(S)} - \frac{N(A)}{N(S)}$$ by the laws of fractions

$$= 1 - P(A)$$ by definition of probability in the equally likely case

Example 9.3.2 – *Solution* (4/4)

continued

$\cong 1 - 0.842$ by Example 9.2.2

$\cong 0.158$

$= 15.8\%.$

The Difference Rule (4/4)

Formula for the Probability of the Complement of an Event

If S is a finite sample space and A is an event in S, then

$$P(A^c) = 1 - P(A),$$

where $A^c = S - A$, the complement of A in S.

Example 9.3.3 – *Passwords with 3-5 Letters*

A certain computer access password consists of 3 through 5 uppercase letters chosen from the 26 letters in the Roman alphabet, with repetitions allowed.

a. How many different passwords are possible?

b. How many different passwords have no repeated letter?

c. How many different passwords have at least one repeated letter?

d. If all passwords are equally likely, what is the probability that a randomly chosen password has at least one repeated letter?

Example 9.3.3 – *Solution (1/6)*

a. The set of all passwords can be partitioned into three subsets consisting of passwords with lengths 3, 4, and 5, as shown in Figure 9.3.3.

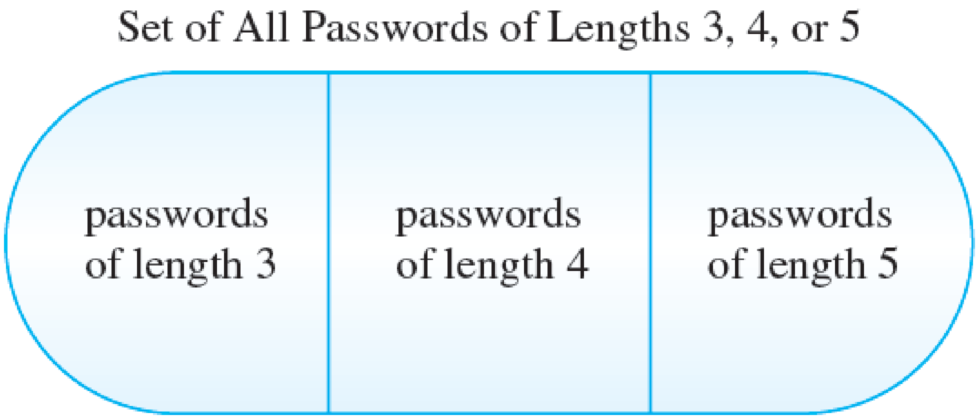


Figure 9.3.3

Example 9.3.3 – *Solution (2/6)* continued

By the addition rule, the total number of passwords equals the number with length 3, plus the number with length 4, plus the number with length 5.

The multiplication rule can be used to compute the number of passwords of each length. Thus the

$$\text{number of passwords with length } 3 = 26^3$$

because forming such a password can be thought of as a three-step process with 26 ways to perform each step

Example 9.3.3 – *Solution* (3/6)

continued

number of passwords with length $4 = 26^4$

because forming such a password can be thought of as a four-step process with 26 ways to perform each step

number of passwords with length $5 = 26^5$

because forming such a password can be thought of as a five-step process with 26 ways to perform each step.

Hence the total number of passwords is

$$26^3 + 26^4 + 26^5 = 12{,}355{,}928.$$

Example 9.3.3 – *Solution (4/6)* continued

b. Constructing a password with length 3 and no repeated letter is a three-step process with 26 choices for step 1, 25 choices for step 2, and 24 choices for step 3.

Thus there are $26 \cdot 25 \cdot 24$ passwords with length three and no repeated letter.

Similarly, there are $26 \cdot 25 \cdot 24 \cdot 23$ passwords with length 4 and no repeated letter and $26 \cdot 25 \cdot 24 \cdot 23 \cdot 22$ passwords with length 5 and no repeated letter.

Example 9.3.3 – *Solution (5/6)*

continued

Hence the total number of passwords with no repeated letter is

$$26 \cdot 25 \cdot 24 + 26 \cdot 25 \cdot 24 \cdot 23 + 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22$$

$$= 8{,}268{,}000.$$

c. By part (a) the total number of passwords is 12,355,928, and by part (b) 8,268,000 of these passwords do not have a repeated letter. Thus, by difference rule, the number of passwords with at least one repeated letter is 4,087,928.

Example 9.3.3 – *Solution (6/6)* continued

d. Given the assumption that all passwords are equally likely, the equally likely probability formula can be used. So the probability that a randomly chosen password has at least one repeated letter is

$$\frac{\begin{bmatrix}\text{\# of passwords with} \\ \text{no repeated letter}\end{bmatrix}}{\text{total \# of passwords}} = \frac{4{,}087{,}928}{12{,}355{,}928} \cong 33.1\%.$$

The Inclusion/Exclusion Rule

The Inclusion/Exclusion Rule (1/2)

The addition rule says how many elements are in a union of sets if the sets are mutually disjoint. Now consider the question of how to determine the number of elements in a union of sets when some of the sets overlap. For simplicity, begin by looking at a union of two sets *A* and *B*, as shown in Figure 9.3.5.

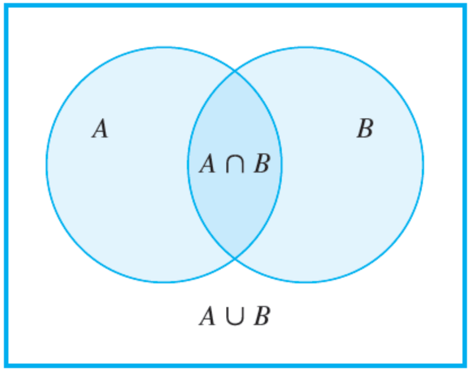


Figure 9.3.5

The Inclusion/Exclusion Rule (2/2)

Theorem 9.3.3 The Inclusion/Exclusion Rule for Two or Three Sets

If A, B, and C are any finite sets, then

$$N(A \cup B) = N(A) + N(B) - N(A \cap B)$$

and

$$\begin{aligned} N(A \cup B \cup C) = N(A) + N(B) + N(C) - N(A \cap B) - N(A \cap C) \\ - N(B \cap C) + N(A \cap B \cap C). \end{aligned}$$

Example 9.3.6 – *Counting Elements of a General Union*

a. How many integers from 1 through 1,000 are multiples of 3 or multiples of 5?

b. How many integers from 1 through 1,000 are neither multiples of 3 nor multiples of 5?

Example 9.3.6 – *Solution (1/6)*

a. Let A = the set of all integers from 1 through 1,000 that are multiples of 3.

Let B = the set of all integers from 1 through 1,000 that are multiples of 5.

Then

$A \cup B$ = the set of all integers from 1 through 1,000 that are multiples of 3 or multiples of 5

and

$A \cap B$ = the set of all integers from 1 through 1,000 that are multiples of both 3 and 5

Example 9.3.6 – *Solution (2/6)* continued

= the set of all integers from 1 through 1,000 that are multiples of 15.

[Now calculate $N(A)$, $N(B)$, *and* $N(A \cap B)$ *and use the inclusion/exclusion rule to solve for* $N(A \cup B)$*.]*

Because every third integer from 3 through 999 is a multiple of 3, each can be represented in the form $3k$, for some integer k from 1 through 333.

Example 9.3.6 – *Solution* (3/6)

continued

Hence there are 333 multiples of 3 from 1 through 1,000, and so $N(A) = 333$.

$$\begin{array}{ccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & \ldots & 996 & 997 & 998 & 999 \\
 & & \updownarrow & & & \updownarrow & & \updownarrow & & & \updownarrow \\
 & & 3\cdot 1 & & & 3\cdot 2 & & 3\cdot 332 & & & 3\cdot 333
\end{array}$$

Similarly, each multiple of 5 from 1 through 1,000 has the form $5k$, for some integer k from 1 through 200.

$$\begin{array}{cccccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \ldots & 995 & 996 & 997 & 998 & 999 & 1{,}000 \\
 & & & & \updownarrow & & & & & \updownarrow & & \updownarrow & & & & & \updownarrow \\
 & & & & 5\cdot 1 & & & & & 5\cdot 2 & & 5\cdot 199 & & & & & 5\cdot 200
\end{array}$$

Example 9.3.6 – *Solution (4/6)*

continued

Thus there are 200 multiples of 5 from 1 through 1,000 and $N(B) = 200$. Finally, each multiple of 15 from 1 through 1,000 has the form $15k$, for some integer k from 1 through 66 (since $990 = 66 \cdot 15$).

$$
\begin{array}{cccccccccccc}
1 & 2 & \ldots 15 & \ldots & 30 & \ldots & 975 & \ldots & 990 & \ldots & 999 & 1{,}000 \\
 & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & & \\
 & & 15 \cdot 1 & & 15 \cdot 2 & & 15 \cdot 65 & & 15 \cdot 66 & & &
\end{array}
$$

Hence there are 66 multiples of 15 from 1 through 1,000, and $N(A \cap B) = 66$.

Example 9.3.6 – *Solution (5/6)*

It follows by the inclusion/exclusion rule that

$$N(A \cup B) = N(A) + N(B) + N(A \cap B)$$

$$= 333 + 200 - 66$$

$$= 467.$$

Thus, 467 integers from 1 through 1,000 are multiples of 3 or multiples of 5.

Example 9.3.6 – *Solution (6/6)*

continued

b. There are 1,000 integers from 1 through 1,000, and by part (a), 467 of these are multiples of 3 or multiples of 5.

Thus, by the set difference rule, there are
$1{,}000 - 467 = 533$ that are neither multiples of 3 nor multiples of 5.