CHAPTER 1 – LECTURE 1

THE FOUNDATIONS: LOGICS AND PROOFS

Introduction to the chapter

- The rules of logic specify the meaning of mathematical statements. These rules hel us understand and reason.

- Understand what makes up a correct mathematical argument, that is, proofs; which proven true are called theorems. Knowing the proof of theorem often makes it possible to modify the results to fit new situations.

- Importance of proofs in Computer Science:
  - Verifies that programs produce the correct output for all possible inputs.
  - Shows all algorithms produce the correct results.
  - Establishes computer security.
  - Creates AI

Propositional logic

- This rule of logic is used to distinguish between valid and invalid mathematical arguments.
- A *proposition* is a declarative statement that is either true or false, but not both.
- For example, “Two plus two equals four” and “Two plus two equals five” are both statements first because it is true and the second because it is false
- The truth or falsity of $x^2 + 2 = 11$ depends on the value of x. For some values of x, it is true (x = and x = −3), whereas for other values it is false. This is not a proposition.
- Similarly, the truth or falsity of $x + y > 0$ depends on the values of x and y. For instance, when $x = -1$ and $y = 2$ it is true, whereas when $x = -1$ and $y = 1$ it is false. This is not a proposition.

Answers

a. The Moon is made of green cheese. (Proposition, False)
b. Toronto is the capital of Canada. (Proposition, True)
c. $1 + 1 = 2$ (Proposition, True)
d. $5 > 3$ (Proposition, True)
e. Read this carefully. (Directive)
f. What time is it? (Inquiry)
g. $x + 1 = 2$ (Non-Proposition, unless x is qualified)
h. $x + y = z$ (Non-Proposition, unless x, y, and z are qualified)

Propositional Logic

- Proposition that cannot be expressed in terms of simpler propositions are called atomic propositions.
- Propositional Variablesp, q, r, s,…
- Assume that the variable is set to a proposition that is T or F.
- The area of logic that deals with proposition is called propositional logic.
- It was developed systematically by Greek philosopher Aristotle more than 2300 years ago.

Propositional Logic

- Many mathematical statements are constructed by combining one or more propositions.
- New propositions, called **compound proposition**, are formed using logical operators.
  - Negation $\neg$
  - Conjunction $\wedge$
  - Disjunction $\vee$
  - Implication $\rightarrow$
  - Biconditional $\leftrightarrow$

Compound Propositions

- **Negation (¬)**
  - The *negation* of a proposition *p* is denoted by ¬*p*.
  - ¬*p* is read as 'not *p*'
  - $\bar{p}, \sim p, Np, !\, p, p'$ and $-p$
  - The <u>truth value</u> of the negation (¬*p*) is the opposite of the truth value of *p*. As shown in the following truth table.

p	¬*p*
T	F
F	T

  - *Example*: If *p* denotes "121 is a perfect square.", then ¬*p* denotes "It is not the case that 121 is a perfect square," or more simply "121 is not a perfect square."

Compound Propositions

- **Conjunction (∧)**
  - The conjunction of $p \wedge q$ is true when both p and q are true and is false otherwise.
  - The *conjunction* of p and q is denoted by $p \wedge q$.
  - Note that in conjunction, the word ‘but’ is sometimes used instead of ‘and’.
  - The following is the truth table for conjunction.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

  - *Example*: If p denotes “The sun is shinning” and q denotes “It is raining” then $p \wedge q$ denotes “The sun is shinning, but/and it is raining”

Compound Propositions

- **Disjunction ($\vee$) / Inclusive OR**
  - The disjunction of $p \vee q$ is false when both p and q are false and is true otherwise.
  - The *disjunction* of propositions p and q is denoted by $p \vee q$.
  - The following is the truth table for disjunction.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

  - Example: If p denotes "Adam has a play station" and q denotes "Adam has a PC" then $p \vee q$ denotes "Adam has a play station, or he has a PC"

Compound Propositions

- Exclusive OR ($\oplus$)
  - Exactly one of p and q must is true, but not both.
  - The Exclusive OR of p and q is denoted by $p \oplus q$
  - The truth table for $p \oplus q$ is:

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

  - Example: p and q states, " A student can have a salad with dinner" and " A student can have soup with dinner," then $p \oplus q$ denotes " A student can have a soup or a salad with dinner"

Compound Propositions

- Implication (→) / Conditional statement
  - $p \rightarrow q$ is a conditional statement or implication which reads as “if p, then q ”.
  - $p \rightarrow q$ is false, when p is true, and q is false.
  - Conditional statements play an essential role in mathematic reasoning.
  - In $p \rightarrow q$, p is the hypothesis (antecedent or premise), and q is the conclusion (or consequence)

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Implication

Different way of expressing $p \rightarrow q$

- If p, then q
- If p, q
- p is sufficient for q
- q if p
- q when p
- A necessary condition for p is q
- q unless ¬p
- p implies q
- p only if q
- A sufficient condition for q is p
- q whenever p
- q is necessary for p
- q follows from p
- q provided that p

Example: Think of it as an obligation or contract. "If I am elected, then I will lower taxes"

If p denotes "It is Monday" and q denotes "I have a meeting" then $p \rightarrow q$ denotes "If it is Monday, then I have a meeting"

Compound Propositions

- Biconditional ($\leftrightarrow$)
  - The *Biconditional* of propositions p and q is denoted by $p \leftrightarrow q$, which is true or false when p and q have the same truth value.
  - $p \leftrightarrow q$ is true when each condition is true or false. The two propositions have the same truth value.
  - Also known as Bi-implications and is expressed as "if and only if" (iff).
  - Note this truth table is the exact **opposite** of $\oplus$. $p \leftrightarrow q$ means $\neg(p \oplus q)$

p	q	p$\leftrightarrow$q
F	F	T
F	T	F
T	F	F
T	T	T

Truth Table of Compound Proposition

- Construct the truth table of the compound proposition:

$$(p \vee \neg q) \rightarrow (p \wedge q)$$

Converse, Inverse, Contrapositive

- We can form some new conditional statements starting with implication $p \rightarrow q$. There are three related conditional statements that occurs very often, that they have special names.
- For an implication $p \rightarrow q$ statement:
  - Its *converse* is: $q \rightarrow p$
  - Its *contrapositive*: $\neg q \rightarrow \neg p$
  - Its *inverse* is: $\neg p \rightarrow \neg q$

- The contrapositive always has the same truth value as $p \rightarrow q$
- Can you find the contrapositive of the following conditional statement?

  "The home team wins whenever it is raining"

Rewriting: If it is raining, then the home team wins"

Converse: If the home team wins, then it is raining.

Contrapositive: If the home team does not win, then it is not raining.

Inverse: If it not raining, then home team does not win.

PRECEDENCE OF LOGICAL OPERATORS

- $\neg p \wedge q$;

'¬'operation is applied before all logical operations to reduce parenthesis. $(\neg p) \wedge q$ and not $\neg(p \wedge q)$

- $p \vee q \wedge r$ means $p \vee (q \wedge r)$

- $p \rightarrow q \vee r$ means $p \rightarrow (q \vee r)$

Operator	Priority
$\neg$ (NOT)	1
$\wedge$ (AND)	2
$\vee$ (OR)	3
$\oplus$ (XOR)	3
$\rightarrow$ (Implication)	4
$\leftrightarrow$ (Biconditional)	5

Logic and Bit operations

- Computer represents information in binary digits- bits.
- A bit is a symbol with two possible values 0 and 1.
- A bit can be used to represent truth values like true and false with 1 and 0 respectively.
- A **bit string** is a sequence of zero or more bits. A length of the string is the number of bits in the string.
- Computer bit operations correspond to the logical connectives, OR, AND, XOR.

Example: Find the bitwise OR, bitwise AND, and bitwise XOR of the bit strings 01 1011 0110 and 11 0001 1101.

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Solution:     01 1011 0110
              11 0001 1101
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              11 1011 1111   bitwise OR
              01 0001 0100   bitwise AND
              10 1010 1011   bitwise XOR
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Application Of Propositional Logic

Translating English sentences into logical expressions:

1. You can access the internet from campus only if you are a computer science student major or you are not a freshman. $p \rightarrow (q \lor \neg r)$
   - *p* represents the proposition "You can access the internet from campus,"
   - *q* represents "You are a computer science student major,“
   - *r* represents "You are a freshman."
2. You cannot ride the rollercoaster if you under 4 feet tall unless you are older than 16 years old. $(q \land \neg r) \rightarrow \neg p$
   - *q* represents the proposition "You are under 4 feet tall."
   - *r* represents "You are older than 16 years old."
   - *p* represents "You can ride the rollercoaster."

Application Of Propositional Logic

<u>System Specifications:</u>

Systems and software engineers take requirements in natural languages and produce precise and unambiguous specifications that can be used as the basis of system development.

1. "The automated reply cannot be sent when the file system is full"

System specifications should be consistent. They should not contain any conflicting requirements that could be used to derive contradictions. If specs are not consistent, there will be no way to develop systems that satisfy the requirements.

1. Determine weather these system specifications are consistent:

   "The diagnostic message is stored in the buffer or it is retransmitted"

   "The diagnostic message is not stored in the buffer"

   "If the diagnostic message is stored in the buffer, then it is retransmitted"

Application Of Propositional Logic

Boolean Search:

- Logical operations are used extensively in searches of large collection of information, such as the indexes of the web engine.
- Because these searches employ techniques from propositional logics, they are called Boolean searches.
- AND, OR, AND NOT/XOR is used to search with word filtrations.
- If you search New York Universities, we are looking for pages that are matching NEW AND YORK AND UNIVERSITIES.
- These are also commonly seen in many video games and logic puzzles.

RaymondSmullyan puzzles

Smallyan posed many puzzles about an island that has two kinds of inhabitants. Knights, who always tell the truth and their opposites, knaves, who always lie.

You encounter people A and B. What are A and B if A says, "B is a knight" and B says, "The two of us are opposite types."