CHAPTER 9

COUNTING AND PROBABILITY

9.4 The Pigeonhole Principle

The Pigeonhole Principle (1/2)

The pigeonhole principle states that if *n* pigeons fly into *m* pigeonholes and $n > m$, then at least one hole must contain two or more pigeons. This principle is illustrated in Figure 9.4.1 for $n = 5$ and $m = 4$.

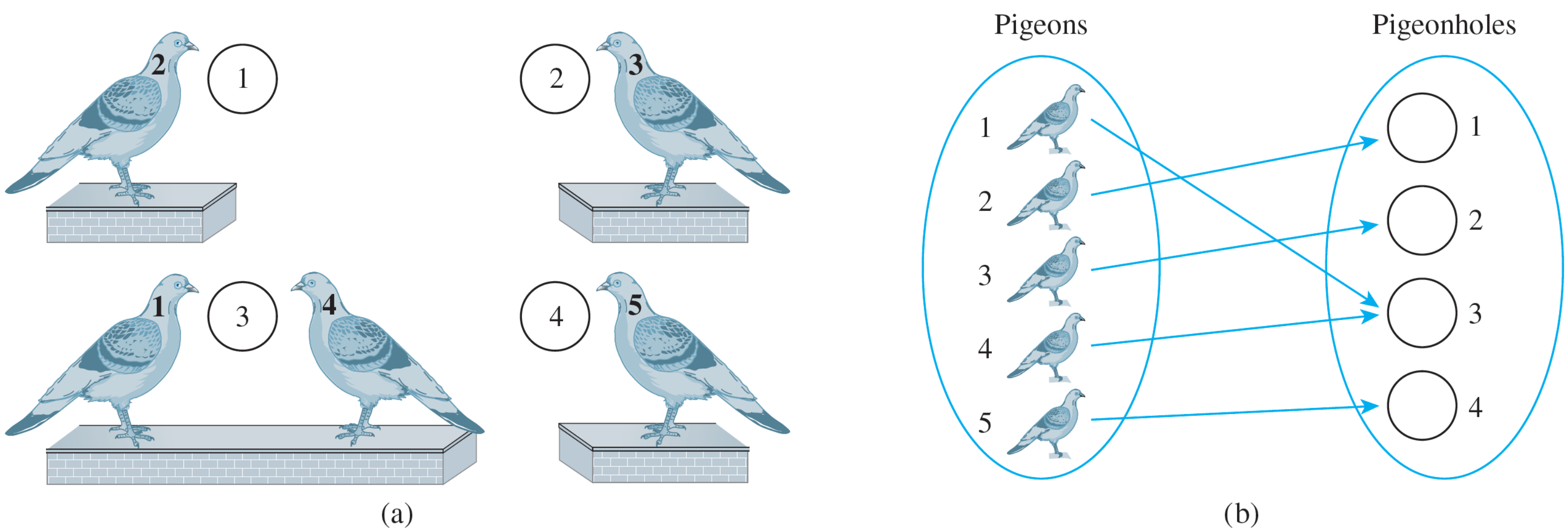


Figure 9.4.1

The Pigeonhole Principle (2/2)

Illustration (a) shows the pigeons perched next to their holes, and (b) shows the correspondence from pigeons to pigeonholes. The pigeonhole principle is sometimes called the *Dirichlet box principle* because it was first stated formally by J. P. G. L. Dirichlet (1805–1859).

Pigeonhole Principle

A function from one finite set to a smaller finite set cannot be one-to-one: There must be at least two elements in the domain that have the same image in the co-domain.

Example 9.4.1 – *Applying the Pigeonhole Principle*

a. In a group of six people, must there be at least two who were born in the same month? In a group of thirteen people, must there be at least two who were born in the same month? Why?

b. Among the residents of New York City, must there be at least two people with the same number of hairs on their heads? Why?

Example 9.4.1 – *Solution (1/5)*

a. A group of six people need not contain two who were born in the same month. For instance, the six people could have birthdays in each of the six months January through June.

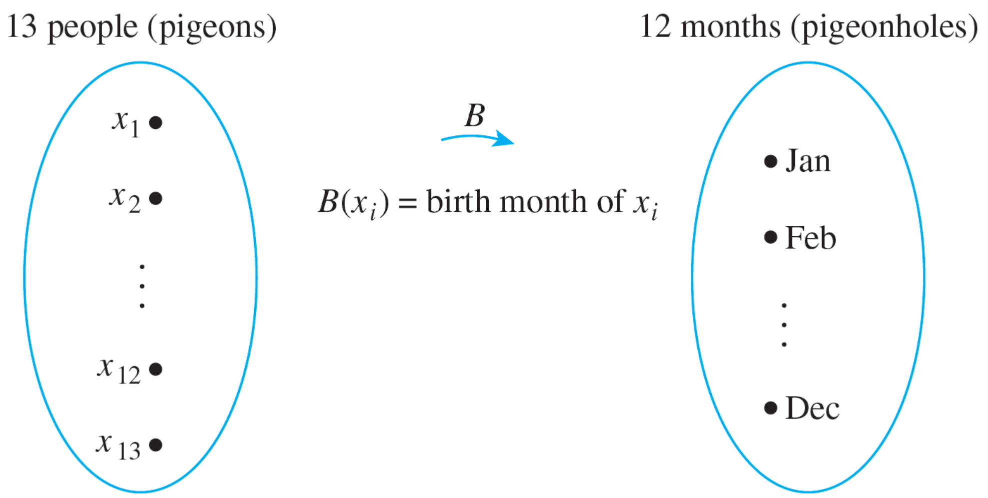
A group of 13 people, however, must contain at least two who were born in the same month, for there are only 12 months in a year and $13 > 12$.

To get at the essence of this reasoning, think of the thirteen people as the pigeons and the twelve months of the year as the pigeonholes.

Example 9.4.1 – *Solution* (2/5)

continued

Denote the thirteen people by the symbols $x_1, x_2, \ldots, x_{13}$ and define a function B from the set of people to the set of twelve months as shown in the following arrow diagram.



Example 9.4.1 – *Solution* (3/5) continued

The pigeonhole principle says that no matter what the particular assignment of months to people, there must be at least two arrows pointing to the same month. Thus at least two people must have been born in the same month.

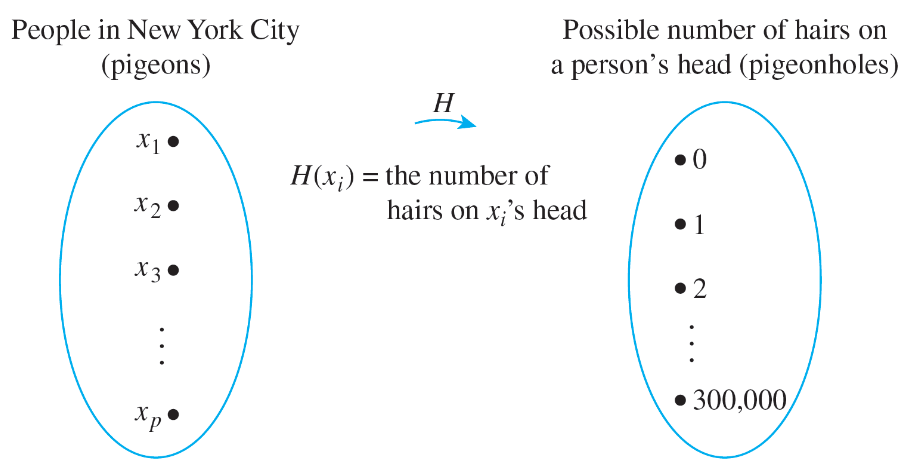
b. The answer is yes. In this example the pigeons are the people of New York City and the pigeonholes are all possible numbers of hairs on any individual's head.

Call the population of New York City P. It is known that P is at least 8,000,000.

Example 9.4.1 – *Solution* (4/5)

continued

Also, the maximum number of hairs on any person's head is known to be less than 300,000. Define a function H from the set of people in New York City $\{x_1, x_2, \ldots, x_p\}$ to the set $\{0, 1, 2, 3, \ldots, 300{,}000\}$, as shown in the arrow diagram.



Example 9.4.1 – *Solution (5/5)*

continued

Since the number of people in New York City is larger than the number of possible hairs on their heads, the function H is not one-to-one; at least two arrows point to the same number. And this means that at least two people have the same number of hairs on their heads.

Application to Decimal Expansions of Fractions

Application to Decimal Expansions of Fractions (1/1)

One important consequence of the pigeonhole principle is the fact that

> *the decimal expansion of any rational number either terminates or repeats.*

A terminating decimal is one like 3.625, and a repeating decimal is one like

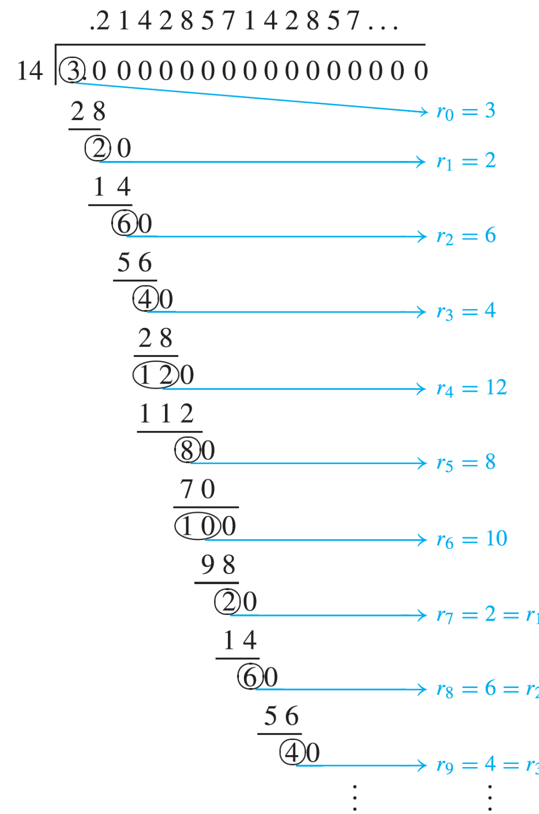
$$2.38\overline{246},$$

where the bar over the digits 246 means that these digits are repeated forever.

Example 9.4.4 – *The Decimal Expansion of a Fraction (1/4)*

Let *a* and *b* be integers and consider a fraction *a/b*, where for simplicity *a* and *b* are both assumed to be positive.

The decimal expansion of *a/b* is obtained by dividing *a* by *b* as illustrated here for *a* = 3 and *b* = 14.



Example 9.4.4 – *The Decimal Expansion of a Fraction (2/4)*

continued

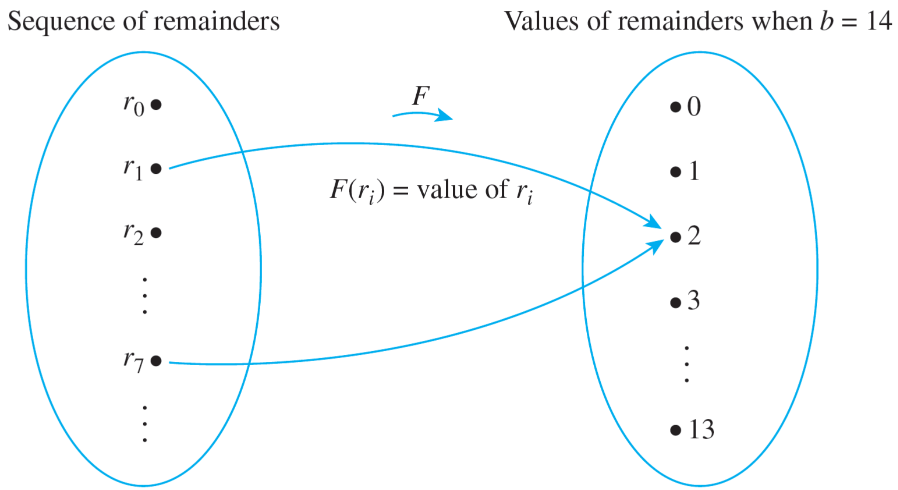
Let $r_0 = a$ and let $r_1, r_2, r_3, \ldots$ be the successive remainders obtained in the long division of a by b. By the quotient-remainder theorem, each remainder must be between 0 and $b - 1$.

(In this example, a is 3 and b is 14, and so the remainders are from 0 to 13.) If some remainder $r_i = 0$, then the division terminates and a/b has a terminating decimal expansion. If no $r_i = 0$, then the division process and hence the sequence of remainders continues forever.

Example 9.4.4 – *The Decimal Expansion of a Fraction (3/4)*

continued

By the pigeonhole principle, since there are more remainders than values that the remainders can take, some remainder value must repeat: $r_j = r_k$, for some indices j and k with $j < k$. This is illustrated below for $a = 3$ and $b = 14$.



Example 9.4.4 – *The Decimal Expansion of a Fraction (4/4)*

continued

If follows that the decimal digits obtained from the divisions between r_j and r_{k-1} repeat forever.

In the case of 3⁄14, the repetition begins with $r_7 = 2 = r_i$ and the decimal expansion repeats the quotients obtained from the divisions from r_1 through r_6 forever:

$$3/14 = 0.2\overline{142857}.$$

Generalized Pigeonhole Principle

Generalized Pigeonhole Principle (1/4)

A generalization of the pigeonhole principle states that if n pigeons fly into m pigeonholes and, for some positive integer k, $km < n$, then at least one pigeonhole contains $k + 1$ or more pigeons.

Generalized Pigeonhole Principle (2/4)

This is illustrated in Figure 9.4.2 for $m = 4$, $n = 9$, and $k = 2$.

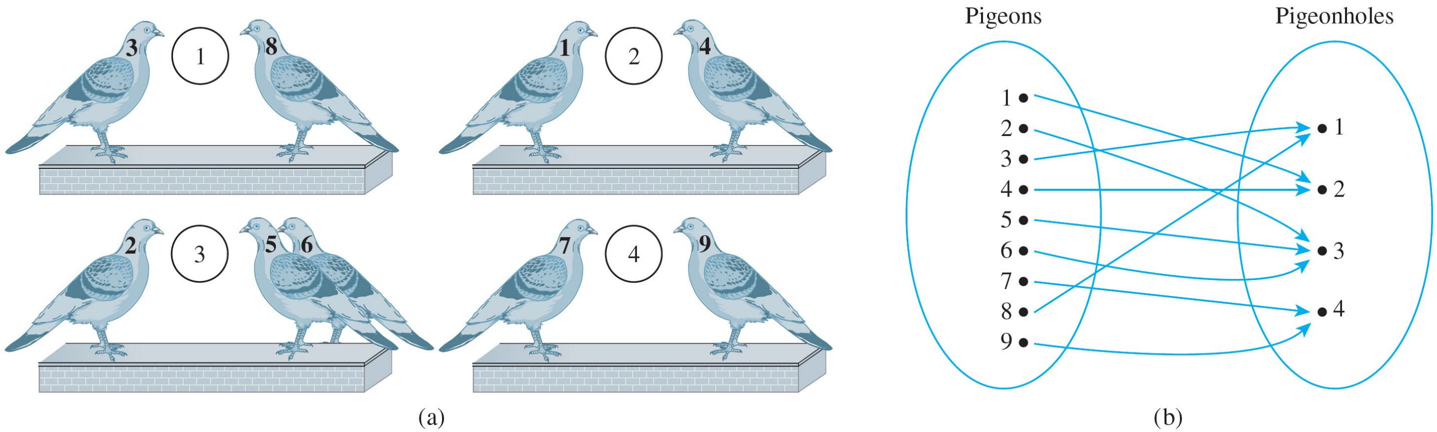


Figure 9.4.2

Since $2 \cdot 4 < 9$, at least one pigeonhole contains three (2 + 1) or more pigeons. (In this example, pigeonhole 3 contains three pigeons.)

Generalized Pigeonhole Principle (3/4)

Generalized Pigeonhole Principle

For any function f from a finite set X with n elements to a finite set Y with m elements and for any positive integer k, if $km < n$, then there is some $y \in Y$ such that y is the image of at least $k + 1$ distinct elements of X.

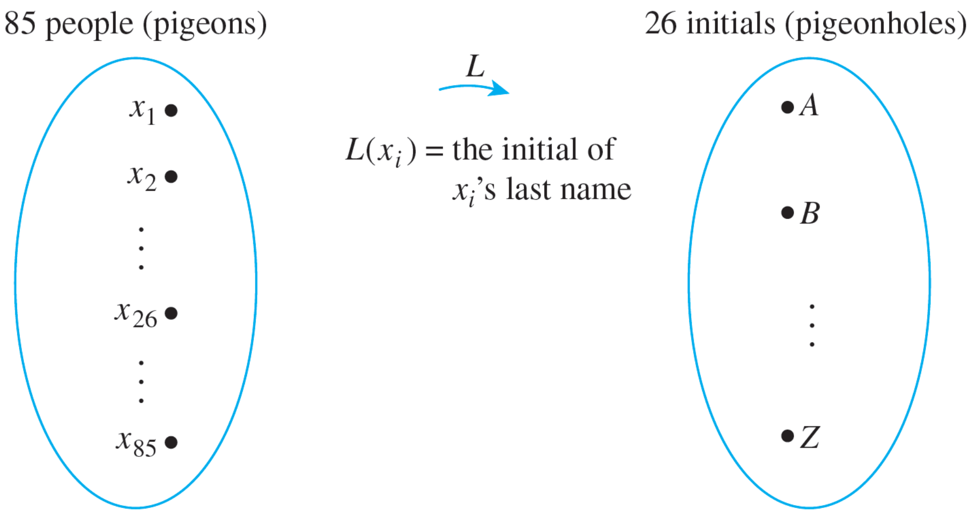
Example 9.4.5 – *Applying the Generalized Pigeonhole Principle*

Show how the generalized pigeonhole principle implies that in a group of 85 people, at least 4 must have the same last initial.

Example 9.4.5 – *Solution (1/2)*

In this example the pigeons are the 85 people and the pigeonholes are the 26 possible last initials of their names.

Consider the function *L* from people to initials defined by the following arrow diagram.



Example 9.4.5 – *Solution (2/2)* continued

Since $3 \cdot 26 = 78 < 85$, the generalized pigeonhole principle states that some initial must be the image of at least four $(3 + 1)$ people. Thus at least four people have the same last initial.

Generalized Pigeonhole Principle (4/4)

Consider the following contrapositive form of the generalized pigeonhole principle.

Generalized Pigeonhole Principle (Contrapositive Form)

For any function f from a finite set X with n elements to a finite set Y with m elements and for any positive integer k, if for each $y \in Y$, $f^{-1}(y)$ has at most k elements, then X has at most km elements; in other words, $n \leq km$.

Proof of the Pigeonhole Principle

Proof of the Pigeonhole Principle (1/3)

The truth of the pigeonhole principle depends essentially on the sets involved being finite. We know that a set is called **finite** if, and only if, it is the empty set or there is a one-to-one correspondence from $\{1, 2, \ldots, n\}$ to it, where n is a positive integer.

In the first case the **number of elements** in the set is said to be 0, and in the second case it is said to be n. A set that is not finite is called **infinite**.

Proof of the Pigeonhole Principle (2/3)

Thus any finite set is either empty or can be written in the form $\{x_1, x_2, \ldots, x_n\}$ where n is a positive integer.

Theorem 9.4.1 The Pigeonhole Principle

For any function f from a finite set X with n elements to a finite set Y with m elements, if $n > m$, then f is not one-to-one.

Proof of the Pigeonhole Principle (3/3)

An important theorem that follows from the pigeonhole principle states that a function from one finite set to another finite set of the same size is one-to-one if, and only if, it is onto.

This result does not hold for infinite sets.

Theorem 9.4.2 One-to-One and Onto for Finite Sets

Let X and Y be finite sets with the same number of elements and suppose f is a function from X to Y. Then f is one-to-one if, and only if, f is onto.