CHAPTER 10

THEORY OF GRAPHS AND TREES

10.3 Isomorphisms of Graphs

Isomorphisms of Graphs (1/3)

Two graphs that are the same except for the labeling of their vertices and edges are called *isomorphic.* The word *isomorphism* comes from the Greek, meaning "same form." Isomorphic graphs are those that have essentially the same form.

Definition

Let G and G' be graphs with vertex sets $V(G)$ and $V(G')$ and edge sets $E(G)$ and $E(G')$, respectively. **G is isomorphic to G'** if, and only if, there exist one-to-one correspondences $g: V(G) \to V(G')$ and $h: E(G) \to E(G')$ that preserve the edge-endpoint functions of G and G' in the sense that for each $v \in V(G)$ and $e \in E(G)$,

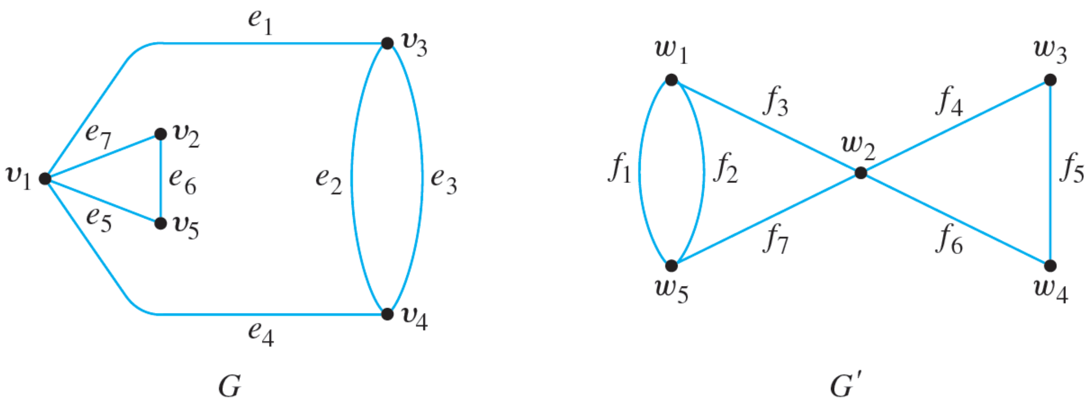
$$v \text{ is an endpoint of } e \quad \Leftrightarrow \quad g(v) \text{ is an endpoint of } h(e). \qquad 10.3.1$$

Isomorphisms of Graphs (2/3)

In words, G is isomorphic to G' if, and only if, the vertices and edges of G and G' can be matched up by one-to-one, onto functions in such a way that the edges between corresponding vertices correspond to each other.

Example 10.3.1 – *Showing That Two Graphs Are Isomorphic*

Show that the following two graphs are isomorphic.



Example 10.3.1 – *Solution (1/4)*

To solve this problem, you must find functions $g : V(G) \to V(G')$ and h: $E(G) \to E(G')$ such that for each $v \in V(G)$ and $e \in E(G)$, v is an endpoint of e if, and only if, $g(v)$ is an endpoint of $h(e)$.

Setting up such functions is partly a matter of trial and error and partly a matter of deduction.

For instance, since e_2 and e_3 are parallel *[have the same endpoints]*, $h(e_2)$ and $h(e_3)$ must be parallel also.

Example 10.3.1 – *Solution (2/4)* continued

So $h(e_2) = f_1$ and $h(e_3) = f_2$ or $h(e_2) = f_2$ and $h(e_3) = f_1$. Also, the endpoints of e_2 and e_3 must correspond to the endpoints of f_1 and f_2, and so $g(v_3) = w_1$ and $g(v_4) = w_5$ or $g(v_3) = w_5$ and $g(v_4) = w_1$.

Similarly, since v_1 is the endpoint of four distinct edges (e_1, e_7, e_5, and e_4), $g(v_1)$ must also be the endpoint of four distinct edges *[because every edge incident on* $g(v_1)$ *is the image under h of an edge incident on* v_1 *and h is one-to-one and onto]*.

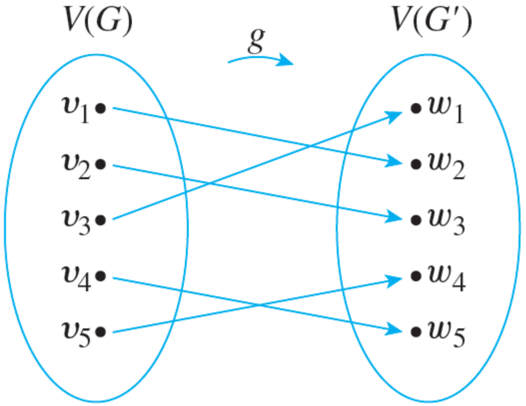
Example 10.3.1 – *Solution (3/4)* continued

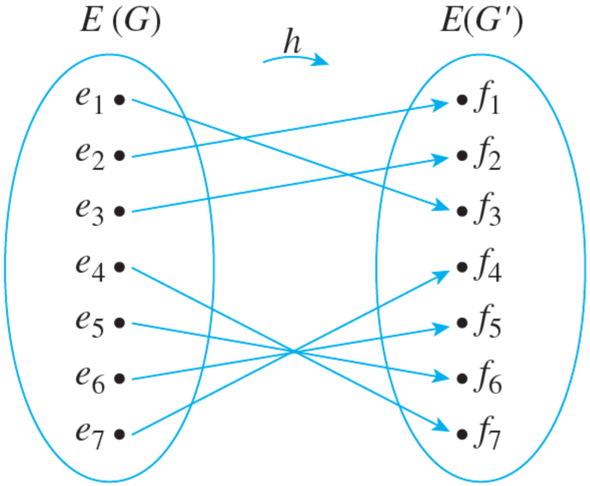
But the only vertex in G' that has four edges coming out of it is w_2, and so $g(v_1) = w_2$. Now if $g(v_3) = w_1$, then since v_1 and v_3 are endpoints of e_1 in G, $g(v_1) = w_2$ and $g(v_3) = w_1$ must be endpoints of $h(e_1)$ in G'. This implies that $h(e_1) = f_3$.

By continuing in this way, possibly making some arbitrary choices as you go, you eventually can find functions g and h to define the isomorphism between G and G'.

Example 10.3.1 – *Solution (4/4)* continued

One pair of functions (there are several) is the following:





Isomorphisms of Graphs (3/3)

Theorem 10.3.1 Graph Isomorphism Is an Equivalence Relation

Let S be a set of graphs and let R be the relation of graph isomorphism on S. Then R is an equivalence relation on S.

Definition

A property P is called an **invariant for graph isomorphism** if, and only if, given any graphs G and G', if G has property P and G' is isomorphic to G, then G' has property P.

Theorem 10.3.2

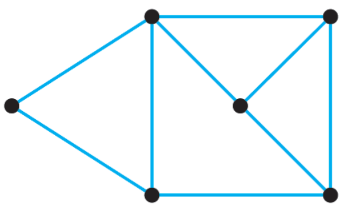
Each of the following properties is an invariant for graph isomorphism, where n, m, and k are all nonnegative integers:

1. has n vertices
2. has m edges
3. has a vertex of degree k
4. has m vertices of degree k
5. has a circuit of length k
6. has a simple circuit of length k
7. has m simple circuits of length k
8. is connected
9. has an Euler circuit
10. has a Hamiltonian circuit.

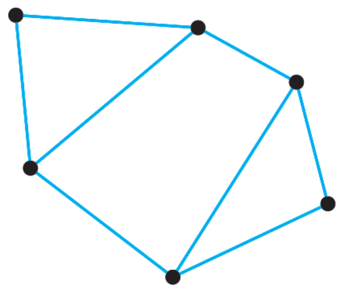
Example 10.3.3 – *Showing That Two Graphs Are Not Isomorphic*

Show that the following pairs of graphs are not isomorphic by finding an isomorphic invariant that they do not share.

a.

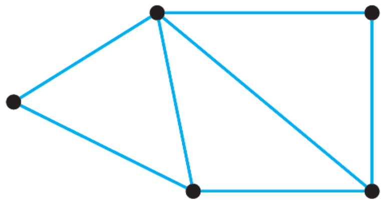


G

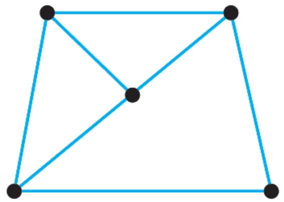


G'

b.



H



H'

Example 10.3.3 – *Solution*

a. G has nine edges; G' has only eight.

b. H has a vertex of degree 4; H' does not.

Graph Isomorphism for Simple Graphs

Graph Isomorphism for Simple Graphs (1/1)

When graphs G and G' are both simple, the definition of G being isomorphic to G' can be written without referring to the correspondence between the edges of G and the edges of G'.

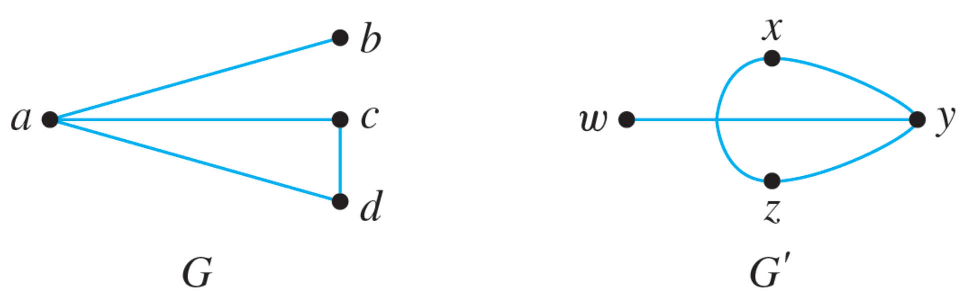
Definition

If G and G' are simple graphs, then **G is isomorphic to G'** if, and only if, there exists a one-to-one correspondence g from the vertex set $V(G)$ of G to the vertex set $V(G')$ of G' that preserves the edge-endpoint functions of G and G' in the sense that for all vertices u and v of G,

$$\{u, v\} \text{ is an edge in } G \quad \Leftrightarrow \quad \{g(u), g(v)\} \text{ is an edge in } G'. \qquad 10.3.2$$

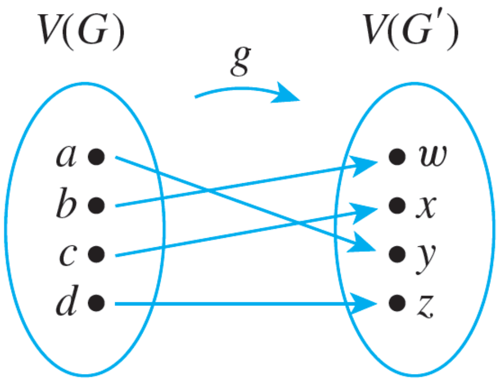
Example 10.3.5 – *Isomorphism of Simple Graphs*

Are the two graphs shown below isomorphic? If so, define an isomorphism.



Example 10.3.5 – *Solution (1/2)*

Yes. Define $g : V(G) \to V(G')$ by the arrow diagram shown below.



Then g is one-to-one and onto by inspection.

Example 10.3.5 – *Solution (2/2)* continued

The fact that g preserves the edge-endpoint functions of G and G' is shown by the following table:

Edges of G	Edges of G'
$\{a, b\}$	$\{y, w\} = \{g(a), g(b)\}$
$\{a, c\}$	$\{y, x\} = \{g(a), g(\mathrm{c})\}$
$\{a, d\}$	$\{y, z\} = \{g(a), g(d)\}$
$\{c, d\}$	$\{x, z\} = \{g(c), g(d)\}$