CHAPTER 10

THEORY OF GRAPHS AND TREES

10.2 Matrix Representations of Graphs

Matrices

Matrices (1/4)

Matrices are two-dimensional analogues of sequences.
They are also called two-dimensional arrays.

Definition

An $m \times n$ (read "m by n") **matrix A over a set** S is a rectangular array of elements of S arranged into m rows and n columns:

$$\mathbf{A} = \left[\begin{array}{cccccc} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{array}\right] \begin{array}{l} \\ \\ \\ \leftarrow i\text{th row of } \mathbf{A} \\ \\ \\ \end{array}$$

$$\begin{array}{c} \uparrow \\ j\text{th column of } \mathbf{A} \end{array}$$

We write $\mathbf{A} = (a_{ij})$.

Matrices (2/4)

The ***i* th row of A** is

$$[a_{i1} \quad a_{i2} \;\ldots\; a_{in}]$$

and the ***j* th column of A** is

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}.$$

Matrices (3/4)

The entry a_{ij} in the i th row and j th column of **A** is called the ***i j* th entry of A**. An $m \times n$ matrix is said to have **size** $\boldsymbol{m \times n}$.

If **A** and **B** are matrices, then **A** = **B** if, and only if, **A** and **B** have the same size and the corresponding entries of **A** and **B** are all equal; that is,

$$a_{ij} = b_{ij} \quad \text{for every } i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n.$$

Matrices (4/4)

A matrix for which the numbers of rows and columns are equal is called a **square matrix**.

If **A** is a square matrix of size $n \times n$, then the **main diagonal of A** consists of all the entries a_{11}, a_{22}, . . . , a_{nn}:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1i} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2i} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ii} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{ni} & \cdots & a_{nn} \end{bmatrix}$$

main diagonal of **A**

Example 10.2.1 – *Matrix Terminology*

The following is a 3×3 matrix over the set of integers.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -3 \\ 4 & -1 & 5 \\ -2 & 2 & 0 \end{bmatrix}$$

a. What is a_{23}, the entry in row 2, column 3?

b. What is the second column of **A**?

c. What are the entries in the main diagonal of **A**?

Example 10.2.1 – *Solution*

a. $a_{23} = 5$

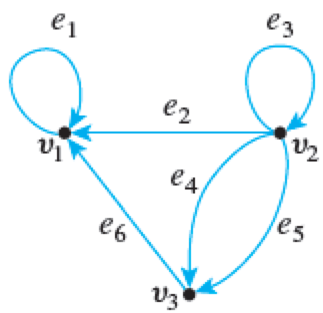
b. $\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$

c. 1, −1, and 0

Matrices and Directed Graphs

Matrices and Directed Graphs (1/2)

Consider the directed graph shown in Figure 10.2.1. This graph can be represented by the matrix $\mathbf{A} = (a_{ij})$ for which a_{ij} = the number of arrows from v_i to v_j, for every i = 1, 2, 3 and j = 1, 2, 3.



Directed Graph G

(a)

$$\mathbf{A} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

Adjacency Matrix of G

(b)

A Directed Graph and Its Adjacency Matrix

Figure 10.2.1

Matrices and Directed Graphs (2/2)

Thus a_{11} = 1 because there is one arrow from v_1 to v_1; a_{12} = 0 because there is no arrow from v_1 to v_2, a_{23} = 2 because there are two arrows from v_2 to v_3, and so forth. **A** is called the *adjacency matrix* of the directed graph. For convenient reference, the rows and columns of **A** are often labeled with the vertices of the graph G.

Definition

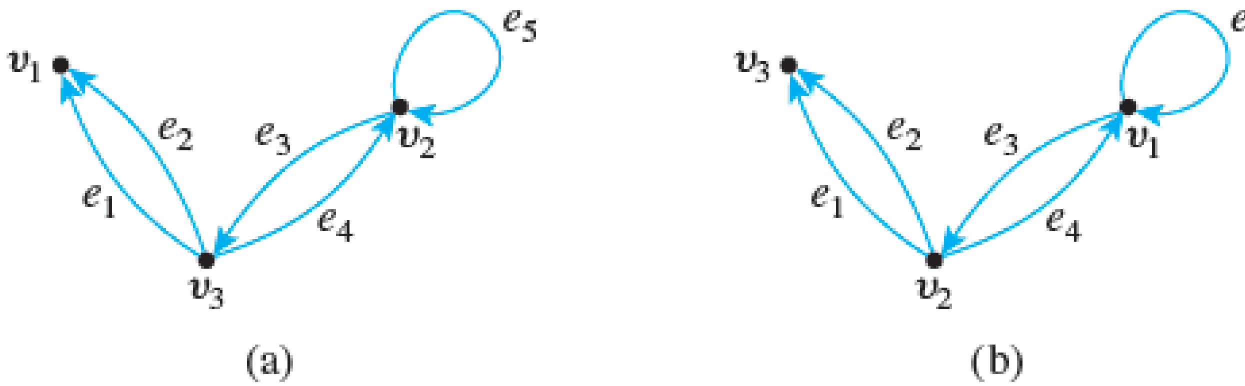
Let G be a directed graph with ordered vertices $v_1, v_2, \ldots, v_n$. The **adjacency matrix of G** is the $n \times n$ matrix $\mathbf{A} = (a_{ij})$ over the set of nonnegative integers such that

$$a_{ij} = \text{the number of arrows from } v_i \text{ to } v_j \quad \text{for all } i, j = 1, 2, \ldots, n.$$

Example 10.2.2 – *The Adjacency Matrix of a Graph*

The two directed graphs shown below differ only in the ordering of their vertices.

Find their adjacency matrices.



Example 10.2.2 – *Solution (1/2)*

Since both graphs have three vertices, both adjacency matrices are 3×3 matrices.

For (a), all entries in the first row are 0 since there are no arrows from v_1 to any other vertex.

For (b), the first two entries in the first row are 1 and the third entry is 0 since from v_1 there are single arrows to v_1 and to v_2 and no arrows to v_3.

Example 10.2.2 – *Solution (2/2)* continued

Continuing the analysis in this way, you obtain the following two adjacency matrices:

$$\mathbf{A} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \end{array} \qquad \mathbf{A} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

(a) (b)

Example 10.2.3 – *Obtaining a Directed Graph from a Matrix*

Let

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 \end{bmatrix}.$$

Draw a directed graph that has **A** as its adjacency matrix.

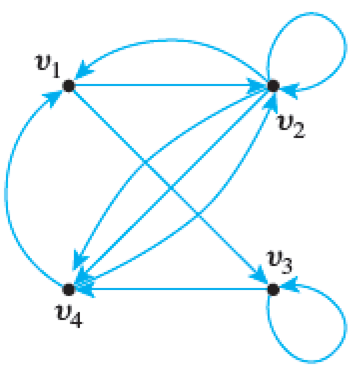
Example 10.2.3 – *Solution (1/2)*

Let G be the graph corresponding to **A**, and let v_1, v_2, v_3, and v_4 be the vertices of G. Label **A** across the top and down the left side with these vertex names, as shown below.

$$\mathbf{A} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \!\!\begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Example 10.2.3 – *Solution (2/2)* continued

Then, for instance, the 2 in the fourth row and the first column means that there are two arrows from v_4 to v_1. The 0 in the first row and the fourth column means that there is no arrow from v_1 to v_4. A corresponding directed graph is shown here (without edge labels because the matrix does not determine those).



Matrices and Undirected Graphs

Matrices and Undirected Graphs (1/2)

Once you know how to associate a matrix with a directed graph, the definition of the matrix corresponding to an undirected graph should seem natural to you. As before, you must order the vertices of the graph, but in this case you simply set the *i j* th entry of the adjacency matrix equal to the number of edges connecting the *i* th and *j* th vertices of the graph.

Definition

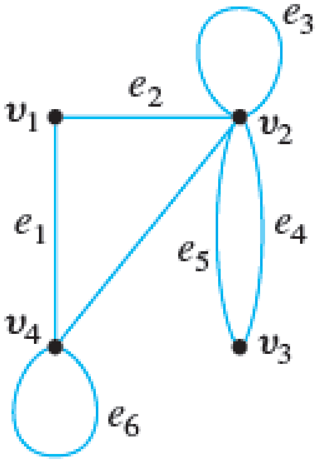
Let G be an undirected graph with ordered vertices $v_1, v_2, \ldots, v_n$. The **adjacency matrix of G** is the $n \times n$ matrix $\mathbf{A} = (a_{ij})$ over the set of nonnegative integers such that

$$a_{ij} = \text{the number of edges connecting } v_i \text{ and } v_j$$

for every i and $j = 1, 2, \ldots, n$.

Example 10.2.4 – *Finding the Adjacency Matrix of a Graph*

Find the adjacency matrix for the graph *G* shown below.



Example 10.2.4 – *Solution*

$$\mathbf{A} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 2 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

Matrices and Undirected Graphs (2/2)

The entries of **A** satisfy the condition, $a_{ij} = a_{ji}$, for every i, $j = 1, 2, \ldots, n$. This implies that the appearance of **A** remains the same if the entries of **A** are flipped across its main diagonal. A matrix, like **A**, with this property is said to be *symmetric*.

Definition

An $n \times n$ square matrix $\mathbf{A} = (a_{ij})$ is called **symmetric** if, and only if, for every i and $j = 1, 2, \ldots, n$,

$$a_{ij} = a_{ji}.$$

Example 10.2.5 – *Symmetric Matrices*

Which of the following matrices are symmetric?

a. $\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$

b. $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 3 \end{bmatrix}$

c. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Example 10.2.5 – *Solution*

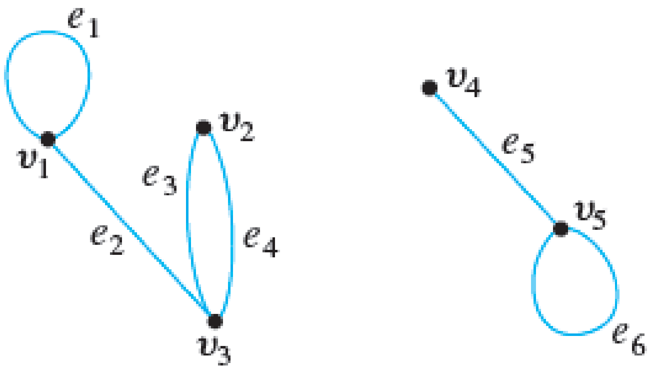
Only (b) is symmetric.

In (a) the entry in the first row and the second column differs from the entry in the second row and the first column; the matrix in (c) is not even square.

Matrices and Connected Components

Matrices and Connected Components (1/3)

Consider a graph G, as shown below, that consists of several connected components.



Matrices and Connected Components (2/3)

The adjacency matrix of *G* is

$$\mathbf{A} = \left[\begin{array}{ccc:cc:cc} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \end{array}\right]$$

Matrices and Connected Components (3/3)

Theorem 10.2.1

Let G be a graph with connected components $G_1, G_2, \ldots, G_k$. If there are n_i vertices in each connected component G_i and these vertices are numbered consecutively, then the adjacency matrix of G has the form

$$\begin{bmatrix} A_1 & O & O & \cdots & O & O \\ O & A_2 & O & \cdots & O & O \\ O & O & A_3 & \cdots & O & O \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ O & O & O & \cdots & O & A_k \end{bmatrix}$$

where each A_i is the $n_i \times n_i$ adjacency matrix of G_i, for every $i = 1, 2, \ldots, k$, and the O's represent matrices whose entries are all 0.

Matrix Multiplication

Matrix Multiplication (1/9)

Matrix multiplication is an enormously useful operation that arises in many contexts, including the investigation of walks in graphs.

Although matrix multiplication can be defined in quite abstract settings, the definition for matrices whose entries are real numbers will be sufficient for our applications.

Matrix Multiplication (2/9)

The product of two matrices is built up of *scalar* or *dot* products of their individual rows and columns.

Definition

Suppose that all entries in matrices **A** and **B** are real numbers. If the number of element, n, in the ith row of **A** equals the number of elements in the jth column of **B**, then the **scalar product** or **dot product** of the ith row of **A** and jth column of **B** is the real number obtained as follows:

$$\begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

Example 10.2.6 – *Multiplying a Row and a Column*

$$\begin{bmatrix} 3 & 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 3 \\ 0 \end{bmatrix} = 3\cdot(-1) + 0\cdot 2 + (-1)\cdot 3 + 2\cdot 0$$

$$= -3 + 0 - 3 + 0 = -6$$

Matrix Multiplication (3/9)

More generally, if **A** and **B** are matrices whose entries are real numbers and if **A** and **B** have *compatible sizes* in the sense that the number of columns of **A** equals the number of rows of **B**, then the product **AB** is defined.

It is the matrix whose *i j* th entry is the scalar product of the *i* th row of **A** times the *j* th column of **B**, for all possible values of *i* and *j*.

Matrix Multiplication (4/9)

Definition

Let $\mathbf{A} = (a_{ij})$ be an $m \times k$ matrix and $\mathbf{B} = (b_{ij})$ a $k \times n$ matrix with real entries. The (matrix) product of $\mathbf{A}$ times $\mathbf{B}$, denoted $\mathbf{AB}$, is that matrix (c_{ij}) defined as follows:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & & \vdots \\ \boxed{a_{i1} \quad a_{i2} \quad \cdots \quad a_{ik}} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mk} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ & \vdots & & \vdots & & \vdots \\ b_{k1} & b_{k2} & \cdots & b_{kj} & \cdots & b_{kn} \end{bmatrix}$$

$$= \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1j} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2j} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ c_{i1} & c_{i2} & \cdots & \boxed{c_{ij}} & \cdots & c_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mj} & \cdots & c_{mn} \end{bmatrix}$$

where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj} = \sum_{r=1}^{k} a_{ir}b_{rj},$$

for each $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Example 10.2.7 – *Computing a Matrix Product*

Let $\mathbf{A} = \begin{bmatrix} 2 & 0 & 3 \\ -1 & 1 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 4 & 3 \\ 2 & 2 \\ -2 & -1 \end{bmatrix}$. Compute $\mathbf{AB}$.

Example 10.2.7 – *Solution (1/3)*

A has size 2×3 and **B** has size 3×2, so the number of columns of **A** equals the number of rows of **B** and the matrix product of **A** and **B** can be computed. Then

$$\begin{bmatrix} 2 & 0 & 3 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 2 & 2 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix},$$

where

$$c_{11} = 2 \cdot 4 + 0 \cdot 2 + 3 \cdot (-2) = 2$$

$$\begin{bmatrix} 2 & 0 & 3 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 2 & 2 \\ -2 & -1 \end{bmatrix}$$

Example 10.2.7 – *Solution (2/3)* continued

$c_{12} = 2 \cdot 3 + 0 \cdot 2 + 3 \cdot (-1) = 3$ $\quad \begin{bmatrix} 2 & 0 & 3 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 2 & 2 \\ -2 & -1 \end{bmatrix}$

$c_{21} = (-1) \cdot 4 + 1 \cdot 2 + 0 \cdot (-2) = -2$ $\quad \begin{bmatrix} 2 & 0 & 3 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 2 & 2 \\ -2 & -1 \end{bmatrix}$

$c_{22} = (-1) \cdot 3 + 1 \cdot 2 + 0 \cdot (-1) = -1$ $\quad \begin{bmatrix} 2 & 0 & 3 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 2 & 2 \\ -2 & -1 \end{bmatrix}.$

Example 10.2.7 – *Solution (3/3)* continued

Hence

$$\mathbf{AB} = \begin{bmatrix} 2 & 3 \\ -2 & -1 \end{bmatrix}.$$

Matrix Multiplication (5/9)

Matrix multiplication is both similar to and different from multiplication of real numbers. One difference is that although the product of any two numbers can be formed, only matrices with compatible sizes can be multiplied. For example, if **A** is a 3×2 matrix and **B** is a 2×4 matrix, then **AB** can be computed because the number of columns of **A** equals the number of rows of **B**.

But **BA** does not exist because **B** has 4 columns, **A** has 3 rows, and $4 \neq 3$.

Matrix Multiplication (6/9)

Another difference is that multiplication of real numbers is commutative (for all real numbers *a* and *b*, *ab* = *ba*), whereas matrix multiplication is not. For instance,

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \quad \text{but} \quad \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

On the other hand, both real number and matrix multiplications are associative: $(ab)c = a(bc)$, for all elements *a*, *b*, and *c* for which the products are defined.

Matrix Multiplication (7/9)

As far as multiplicative identities are concerned, there are both similarities and differences between real numbers and matrices. You know that the number 1 acts as a multiplicative identity for products of real numbers.

It turns out that there are certain matrices, called *identity matrices*, that act as multiplicative identities for certain matrix products.

Matrix Multiplication (8/9)

Definition

For each positive integer n, the $\boldsymbol{n \times n}$ **identity matrix**, denoted $\mathbf{I}_n = (\boldsymbol{\delta}_{ij})$ or just $\mathbf{I}$ (if the size of the matrix is obvious from context), is the $n \times n$ matrix in which all the entries in the main diagonal are 1's and all other entries are 0's. In other words,

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}, \quad \text{for every } i, j = 1, 2, \ldots, n.$$

Matrix Multiplication (9/9)

There are also similarities and differences between real numbers and matrices with respect to the computation of powers. Any number can be raised to a nonnegative integer power, but a matrix can be multiplied by itself only if it has the same number of rows as columns.

Definition

For any $n \times n$ matrix $\mathbf{A}$, the powers of $\mathbf{A}$ are defined as follows:

$$\mathbf{A}^0 = \mathbf{I} \quad \text{where } \mathbf{I} \text{ is the } n \times n \text{ identity matrix}$$

$$\mathbf{A}^n = \mathbf{A}\mathbf{A}^{n-1} \quad \text{for every integer } n \geq 1.$$

Example 10.2.10 – *Powers of a Matrix*

Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$. Compute $\mathbf{A}^0$, $\mathbf{A}^1$, $\mathbf{A}^2$, and $\mathbf{A}^3$.

Example 10.2.10 – *Solution*

$$\mathbf{A}^0 = \text{the } 2 \times 2 \text{ identity matrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{A}^1 = \mathbf{A}\mathbf{A}^0 = \mathbf{A}\mathbf{I} = \mathbf{A}$$

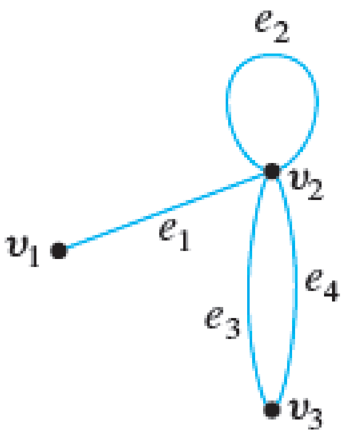
$$\mathbf{A}^2 = \mathbf{A}\mathbf{A}^1 = \mathbf{A}\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 4 \end{bmatrix}$$

$$\mathbf{A}^3 = \mathbf{A}\mathbf{A}^2 = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 5 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 9 & 10 \\ 10 & 4 \end{bmatrix}$$

Counting Walks of Length *N*

Counting Walks of Length *N* (1/2)

A walk in a graph consists of an alternating sequence of vertices and edges. If repeated edges are counted each time they occur, then the number of edges in the sequence is called the **length** of the walk. For instance, the walk $v_2e_3v_3e_4v_2e_2v_2e_3v_3$ has length 4 (counting e_3 twice). Consider the following graph *G*:



Counting Walks of Length *N* (2/2)

If **A** is the adjacency matrix of a graph *G*, the *i j* th entry of $\mathbf{A}^2$ equals the number of walks of length 2 connecting the *i* th vertex to the *j* th vertex of *G*. Even more generally, if *n* is any positive integer, the *i j* th entry of $\mathbf{A}^n$ equals the number of walks of length *n* connecting the *i*th and the *j* th vertices of *G*.

Theorem 10.2.2

If G is a graph with vertices $v_1, v_2, \ldots, v_m$ and $\mathbf{A}$ is the adjacency matrix of G, then for each positive integer n and for all integers $i, j = 1, 2, \ldots, m$,

$$\text{the } ij\text{th entry of } \mathbf{A}^n = \text{the number of walks of length } n \text{ from } v_i \text{ to } v_j.$$