A Convex Relaxation Approach to Point Cloud Registration

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Abstract

As an important problem that is significant in many areas, one of the main challenges in the point cloud registration problem is the non-convexity of the objective function, making optimization algorithms susceptible to local optima and/or outliers. In this paper, we adopt a relaxation scheme which turns the problem in to a convex least-square problem, where a global minimum is guaranteed to exist. Our implementation shows that such convex relaxation is able to reconstruct 3D objects into its original form. Our code is open-sourced at `https://github.com/kylewang1999/cse203b_cvx_pcd_reg`.

1. Introduction

1.1. Motivation

Point cloud registration is a well-known task in computer vision and robotics, which involves aligning a source point cloud to a target one using appropriate rotation and translation. Due to the advances in the Internet-of-Things (IoT), point cloud representation has become more and more popular with the increasing use of high precision sensors including LiDAR and Kinect [3]. Having an efficient and robust algorithm for point cloud registration makes possible object pose estimation, robot localization, 3D reconstruction, among other applications.

There are two major challenges in this area, with the first being finding the correspondence between the source point cloud and the target cloud, and the second one being reconstruction of the source from the target by estimating the rotation and translation occurred during the transformation. The first challenge is usually done by techniques including single value decomposition and nearest neighbor. In this paper, we will focus on the reconstruction and assume point-to-point correspondence between source and target.

1.2. Previous works

Various algorithms have been proposed to solve the widely-studied point cloud registration problem. Recent advances in this area focus more on deep learning techniques, whereas there exist many conventional approaches. In this section, we will briefly survey some representative ones from both.

Deep learning has shown outstanding performances in solving complex tasks, and researchers have also applied deep neural networks with point cloud registration [8]. For example, Lu et. al. proposed DeepCVP as an end-to-end neural network that performs point cloud registration [6]. Lee et. al. proposed DeepPRO, an online deep learning framework that performs partial point cloud registration [4]. Liu et. al. proposed PDC-Net as a robust framework with the help of single value decomposition (SVD) [5].

While deep learning approaches are more popular, we will focus on the conventional approaches in this report. Compared to their neural network counterparts, conventional approaches are often faster at the cost of suboptimal performance on complex scenarios. In the class, we derived a naive solution that reconstructs the rotation distortions. However, the major disadvantage of this approach is that it is slow with suboptimal performance. Umeyama [7] has proposed a least-square formulation to the point cloud alignment problem, where they use SVD to speed up the process.

In this report, we are tasked to consider a convex optimization problem. Horowitz et, al [2] proposed a relaxation to the feasible set, i.e. from $\mathbb{SO}(3)$ to its convex hull $\text{conv}(\mathbb{SO}(3))$. While their method requires a one-to-one correspondence between original point cloud and target, it translates a non-convex problem into a convex problem by introducing a positive semi-definite (PSD) matrix in replacement of the non-convex constraints, where a global minima is guaranteed to exist. We apply their method in point cloud registration problem, and present a convex-relaxed form of the problem and its implementation.

1.3. Contribution

In this report, we formulate a convex optimization problem on 3D pose reconstruction. We present the primal, dual, and KKT formulations of the problem. In addition, we present an implementation inspired by a convex relaxation technique introduced by [2]. We show that such relaxed problem formulation is able to accurately reconstruct the rotations and translations posed on the object. We additionally implement two baselines, namely the naive approach and the SVD approach from Umeyama [7].

1.4. Organization of this report

The remaining of the report is organized as follows. Section 2 formally defines the problem with its primal, dual, and KKT formulations. Section 3 presents the convex relaxation method and its counterparts. The results from the implementation is included in Section 4, and Section 5 concludes the paper.

1.5. Task fulfillment

- **Kaiming Kuang**: Implement optimization algorithm.
- **Baichuan Wu**: Implemented constraint 4 in cvxpy, conducted experiments, drafted report, and prepared figures.
- **Kaiyuan Wang**: Derive convex relaxation to the primal formulation. Experiment with 6D pose estimation problem.
- **Dancheng Liu**: Implement optimization algorithm.

2. Problem Statement

Let $\mathbf{X} = [x_i]_{i=1}^N, \mathbf{Y} = [y_i]_{i=1}^N$ be the source and target point cloud respectively, and let $\mathbf{R} \in \mathbb{SO}(3), \mathbf{t} \in \mathbb{R}^\mathbf{3}$ be the rotation matrix and translation vector applied to the source point cloud $\mathbf{X}$. The goal of point cloud registration is to find the optimal $\mathbf{R}$ and $\mathbf{t}$ that minimize the sum of squared L^2 distance $\sum_{i=1}^N \|\mathbf{x_i} + \mathbf{t} - \mathbf{y_i}\|_\mathbf{2}^\mathbf{2}$, subjecting to the constraints $\mathbf{R} \in \mathbb{SO}(3) \equiv \begin{cases} \mathbf{R}\mathbf{R}^\intercal = \mathbf{I}_{3\times 3} & \text{(orthogonal)} \\ \det \mathbf{R} = 1 & \text{(rigid)} \end{cases}, t \in \mathbb{R}^3$.

The *primal formulation* can be summarized by a squared Frobenius norm shown in Equation 2:

$$\min_{\mathbf{R},\mathbf{t}} \|\mathbf{R}\mathbf{X} + \mathbf{t} - \mathbf{Y}\|_F^2$$
$$\text{subject to} \quad \mathbf{R}\mathbf{R}^\intercal = \mathbf{I}_{3\times 3},\ \det \mathbf{R} = 1 \tag{1}$$

The *dual problem* is given by Equation 2:

$$\min_{\mathbf{R},\mathbf{t}} \max_{\Lambda,\lambda} \mathcal{L}(\mathbf{R}, \Lambda, \lambda) = \|\mathbf{R}\mathbf{X} - \mathbf{Y}\|_F^2 + \operatorname{tr}\big(\Lambda(\mathbf{R}\mathbf{R}^\intercal - \mathbf{I}_{3\times 3})\big) + \lambda\big(\det \mathbf{R} - 1\big) \tag{2}$$

where Λ is a symmetric matrix of Lagrange multipliers corresponding to the equality constraint $\mathbf{R}\mathbf{R}^\intercal = \mathbf{I}_{3\times 3}$, and λ is the Lagrange multiplier corresponding to the equality constraint $\det \mathbf{R} = 1$.

The **Karush–Kuhn–Tucker (KKT) conditions** are given by the following:

1. Primal constraints: they are given by the following two equations.

$$\mathbf{R}\mathbf{R}^T = \mathbf{I}_{3\times 3}$$
$$\det \mathbf{R} = 1$$

2. Dual constraints: None for our problem formulation.
3. Complementary slackness: None for our problem formulation.
4. Gradient of the Lagrangian is given by:

$$\nabla_\mathbf{R} \mathcal{L} = 0 = -2\mathbf{Y}\mathbf{X}^\intercal + 2\mathbf{R}\mathbf{X}\mathbf{X}^\intercal + 2\mathbf{R}\Lambda + \lambda\mathbf{R}$$

Note that the problem we just stated is non-convex, but luckily a solution does exist according to [7]. Let $\mathbf{M} = (\mathbf{X}\mathbf{X}^\intercal + \Lambda + \frac{1}{2}\lambda\mathbf{I})$, let $\mathbf{U\Sigma V^T} = \mathbf{XY^T}$ be the singular value decomposition of $\mathbf{X}\mathbf{Y}^\intercal$, and let $\mathbf{S} = \begin{cases} \mathbf{I} & \text{if} \det \mathbf{U} \det \mathbf{V} = 1 \\ \operatorname{diag}(1, 1, \ldots, 1, -1) & \text{if} \det \mathbf{U} \det \mathbf{V} = -1 \end{cases}$, then the solution for the optimal rotation and translation is given by 3, the proof of which is provided by [7]

$$\mathbf{R} = \mathbf{U}\mathbf{S}\mathbf{V}^\intercal,\ \mathbf{t} = \bar{\mathbf{y}} - \mathbf{R}\bar{\mathbf{x}} \tag{3}$$

where $\bar{\mathbf{y}}, \bar{\mathbf{x}} \in \mathbb{R}^3$ are the centroids of the target and the source point clouds $\mathbf{Y}$ and $\mathbf{X}$, respectively.

We refer to the aforementioned closed-form solution in Equation 3 as the *Umeyama's approach*. In the following section, we will detail an alternative convex-relaxation approach to this approach.

3. Methods

3.1. Convex Relaxation of Point Cloud Registration

The primal problem of point cloud registration is not convex. Here we derive a convex relaxation of the problem following [2]. By relaxing the constraint $\mathbf{R}\mathbf{R}^\intercal = \mathbf{I}_{3\times 3}$ and $\det(\mathbf{R}) = 1$ to constraining in the convex hull of the Lie group rather than the Lie group itself, the resulting optimization is a semi-definite program (SDP), and hence convex. Specifically, we replace the $\mathbb{SO}(3)$ constraint in the original problem with one representing the convex hull of $\mathbb{SO}(3)$:

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} & \mathbf{C}_{13} & \mathbf{C}_{14} \\ * & \mathbf{C}_{22} & \mathbf{C}_{23} & \mathbf{C}_{24} \\ * & * & \mathbf{C}_{33} & \mathbf{C}_{34} \\ * & * & * & \mathbf{C}_{44} \end{bmatrix} \succcurlyeq 0 \tag{4}$$

where,

$$\begin{aligned}
\mathbf{C}_{11} &= 1 + \mathbf{R}_{11} + \mathbf{R}_{22} + \mathbf{R}_{33} \\
\mathbf{C}_{12} &= \mathbf{R}_{32} - \mathbf{R}_{23} \\
\mathbf{C}_{13} &= \mathbf{R}_{13} - \mathbf{R}_{31} \\
\mathbf{C}_{14} &= \mathbf{R}_{21} - \mathbf{R}_{12} \\
\mathbf{C}_{22} &= 1 + \mathbf{R}_{11} - \mathbf{R}_{22} - \mathbf{R}_{33} \\
\mathbf{C}_{23} &= \mathbf{R}_{21} + \mathbf{R}_{12} \\
\mathbf{C}_{24} &= \mathbf{R}_{13} + \mathbf{R}_{31} \\
\mathbf{C}_{33} &= 1 - \mathbf{R}_{11} + \mathbf{R}_{22} - \mathbf{R}_{33} \\
\mathbf{C}_{34} &= \mathbf{R}_{32} + \mathbf{R}_{23} \\
\mathbf{C}_{44} &= 1 - \mathbf{R}_{11} - \mathbf{R}_{22} + \mathbf{R}_{33}
\end{aligned}$$

where $*$ denotes symmetric elements.

This constraint relaxation scheme can be derived by reparametrizing $\mathbb{SO}(3)$ using its embeeding into the space of pure quaternions (a subgroup of $\mathbb{SU}(2)$). Namely, we have:

$$\mathbb{SO}(3) = \{\mathbf{U} \in \mathbb{R}^{3\times 3}\}$$

where $\mathbf{U}$ is given by:

$$\begin{bmatrix} 2(u_0^2 + u_1^2) - 1 & 2(u_1u_2 - u_0u_3) & 2(u_1u_3 + u_0u_2) \\ 2(u_1u_2 + u_0u_3) & 2(u_0^2 + u_2^2) - 1 & 2(u_2u_3 - u_0u_1) \\ 2(u_1u_3 - u_0u_2) & 2(u_2u_3 + u_0u_1) & 2(u_0^2 + u_3^2) - 1 \end{bmatrix}$$
$$\mathbf{u} = \begin{pmatrix} u_0 & u_1 & u_2 & u_3 \end{pmatrix}^\intercal, \|\mathbf{u}\|_2^2 = 1$$

from which we define an auxiliary matrix $\mathbf{V}$:

$$\mathbf{V} = \begin{bmatrix} u_0^2 & u_0u_1 & u_0u_2 & u_0u_3 \\ * & u_1^2 & u_1u_2 & u_1u_3 \\ * & * & u_2^2 & u_2u_3 \\ * & * & * & u_3^2 \end{bmatrix} = \mathbf{u}^\intercal \mathbf{u}$$

which is positive semidefinite and satisfies $\mathrm{tr}(\mathbf{V}) = 1$. Since each term in $\mathbf{U}$ is quadratic in elements of $\mathbf{u}$, we can claim that there exists an affine mapping $\mathcal{A}$ such that $\mathbf{U} = \mathcal{A} \cdot \mathbf{V}$. Equivalently, we can express $\mathbb{SO}(3)$ as:

$$\mathbb{SO}(3) = \mathcal{A} \cdot \mathbf{V} \tag{5}$$

for $\mathcal{A} : \mathbb{R}^{4\times 4} \to \mathbb{R}^{3\times 3}$. By taking the convex hull of 5, we can obtain the spectrahedral representation for $\mathrm{conv}(\mathbb{SO}(3))$ in Equation 4.

In the 2-dimensional case, this relaxation can be conceptually perceived as relaxing the admissible trajectory from a unit 1-sphere to a unit 2-ball.

Formulating this problem as a convex problem enables us to take advantage of many popular off-the-shelf solvers such as cvxpy [1]. Using this convex relaxation, the problem can be solved in a few lines of code.

3.2. Counterpart Methods

To evaluate the performance of the convex relaxation method, we introduce two counterpart approaches: the naive least-square relaxation method and the Umeyama's approach [7].

3.2.1 Naive Least-Square Relaxation

The naive relaxation drops the translation term, and only regresses the rotation only. Then the primal problem becomes:

$$\begin{aligned} &\min_{\mathbf{R}} \|\mathbf{R}\mathbf{X} - \mathbf{Y}\|_F^2 \\ &\text{subject to} \quad \mathbf{R}\mathbf{R}^\intercal = \mathbf{I}_{3\times 3},\ \det \mathbf{R} = 1 \end{aligned} \tag{6}$$

Let $\mathbf{R}(\mathbf{R}_0, \omega) = \mathbf{R}_0 \exp([\omega]) \approx \mathbf{R}_0 + \mathbf{R}_0[\omega]$ be a parameterization of rotation matrix $\mathbf{R}$, which is only a small angle $\omega \approx \mathbf{0} \in \mathbb{R}^3$ away from an initial guess $\mathbf{R}_0$, where $[\bullet]$ denotes the skew-symmetric operator. In other words, we limit the change of the rotation matrix $\mathbf{R}$ in each optimization step in a small range so that the approximate parameterization of $\mathbf{R}$ holds. Now the optimization becomes:

$$\min_{\omega} \|\mathbf{A}\omega - \mathbf{B}\|_F^2 \quad \text{s.t. } \|\omega\|_2^2 \le \epsilon \tag{7}$$

where $\epsilon \neq 0$ is a small positive number. Then we derive $\mathbf{A}, \mathbf{B}$ by rewriting the objective function:

$$\begin{aligned} (\mathbf{R}\mathbf{X} - \mathbf{Y})_{:i} &= \mathbf{R}_0 X_{:i} + \mathbf{R}_0[\omega]\mathbf{X}_{:i} - \mathbf{Y}_{:i} \\ &= -\mathbf{R}_0 \left[X_{:i}\right] \omega - (\mathbf{Y}_{:i} - \mathbf{R}_0 \mathbf{X}_{:i}) \\ &= \mathbf{A}_{:i}\omega - \mathbf{B}_{:i} \\ \text{where } \mathbf{A}_{:i} &= -\mathbf{R}_0[X_{:i}],\ \mathbf{B}_{:i} = \mathbf{Y}_{:i} - \mathbf{R}_0\mathbf{X}_{:i} \end{aligned} \tag{8}$$

Due to the small angle constraint on ω, the rotation matrix $\mathbf{R}$ needs to be solved for iteratively using algorithm 1.

Algorithm 1 Naive Linear Least-Square Relaxation

1: $\mathbf{R} \leftarrow \mathbf{I}_{3\times 3}$
2: $\mathbf{t} \leftarrow \bar{\mathbf{y}} - \bar{\mathbf{x}}$
3: **while** not converge **do**
4: $\quad \mathbf{A}_{:i} = -\mathbf{R}[\mathbf{X}_{:i}]\ \forall i = 1, \ldots, N$
5: $\quad \mathbf{B}_{:i} = \mathbf{Y}_{:i} - \mathbf{R}[\mathbf{X}_{:i}]\ \forall i = 1, \ldots, N$
6: $\quad \omega = \arg\min_{\omega} \|\mathbf{A}\omega - \mathbf{B}\|_F^2 \quad \text{s.t.} \|\omega\|_2^2 \le \epsilon$
7: $\quad \mathbf{R} \leftarrow \mathbf{R}(\mathbf{I}_{3\times 3} + [\omega])$
8: **end while**

3.2.2 Umeyama's Approach

The Umeyama approach provides an analytical solution of the original non-convex problem [7]. Assume that $\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^\intercal = \mathbf{X}\mathbf{Y}^\intercal$ is the singular value decomposition of $\mathbf{X}\mathbf{Y}^\intercal$, then the solution of the least-square-error rotation and translation is given by:

$$\mathbf{R} = \mathbf{U}\mathbf{S}\mathbf{V}^\intercal,\ \mathbf{t} = \bar{\mathbf{y}} - \mathbf{R}\bar{\mathbf{x}}$$
$$\mathbf{S} = \begin{cases} \mathbf{I} & \text{if} \det \mathbf{U} \det \mathbf{V} = 1 \\ \operatorname{diag}(1, 1, \dots, 1, -1) & \text{if} \det \mathbf{U} \det \mathbf{V} = -1 \end{cases} \tag{9}$$

4. Results

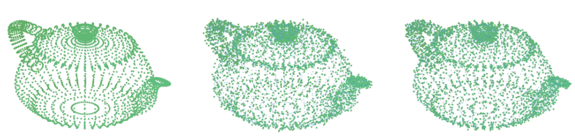
We conducted experiments on point cloud registration tasks using all three aforementioned methods. For each method, we test its performance and robustness under three benchmarks: 1. recovery of a simple rotation, 2. recovery of rotation under Gaussian noise perturbation, and 3. recovery of rotation and translation under Gaussian noise perturbation. We present experiment results both qualitatively and quantitatively in the ensuing sections.

4.1. Naive Least-Square Relaxation

Table 1 qualitatively showcases the convergence process of the naive least-square relaxation approach. It performs robustly in all three cases. Nevertheless, since the Jacobian relies on small incremental rotations, it converges slowly due to the small step size constraint for the sake of numerical stability of the algorithm. We will soon see the other two methods obliterates this naive method in terms of performance.

4.2. Umeyama's Approach

As opposed to the iterative numerical method, Umeyama's formulation gives a closed-form analytical solution to the registration problem. It performs almost perfectly for the rotation-only case, and achieves almost perfect loss to the target point cloud.



(a) R (b) R+G (c) R+G+T

Figure 1. Recovered point clouds using Umeyama's approach [7]

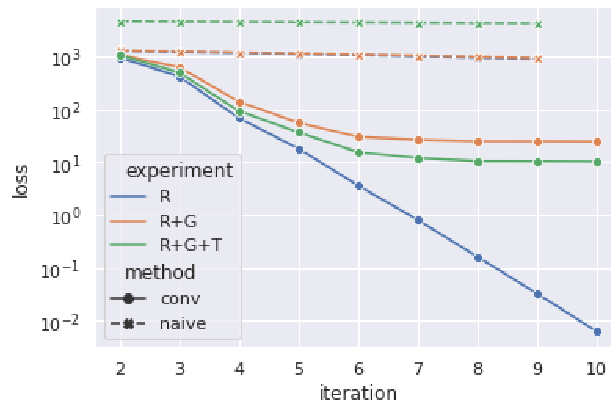


Figure 2. Numerical comparison for convergence performance. Convex-relaxation significantly outperforms naive least-square. Loss is computed using standard L^2 metric.

4.3. Convex Relaxation

Table 2 showcases the convergence process of our method inspired by [2]. It incrementally solves the minimization problem, and finally aligns the input cloud with target cloud with a rotation matrix and a translation vector. Compared to its counterpart convex algorithm (Section 4.1), it has significant visual advantages. We also show the loss after each iteration in Figure 2. The convex relaxation method outperforms the naive baseline on all of the three cases.

5. Conclusion

In this report, we present a convex relaxation approach to the point cloud registration problem. This algorithm effectively transforms the original non-convex problem into a convex approximation, where a global optimum exists. To evaluate our proposed convex relaxation algorithm, we implemented it and compared its performance with a naive convex baseline and an SVD-based algorithm. The experiments demonstrate that the convex relaxation algorithm significantly outperforms the naive baseline, showcasing its potential in handling the point cloud registration.

In conclusion, the convex relaxation approach presented in this report is proved to be promising for solving the point cloud registration problem. It offers improvements over the naive baseline and demonstrates similar performance when compared to analytic method. By leveraging the properties of convex optimization, our algorithm can tackle real-world challenges, such as noisy data and varying constraints. However, given the comparable performance of the SVD-based approach, future work may explore additional regularization techniques or incorporate advanced optimization methods to improve the performance and computational efficiency of the convex relaxation approach.

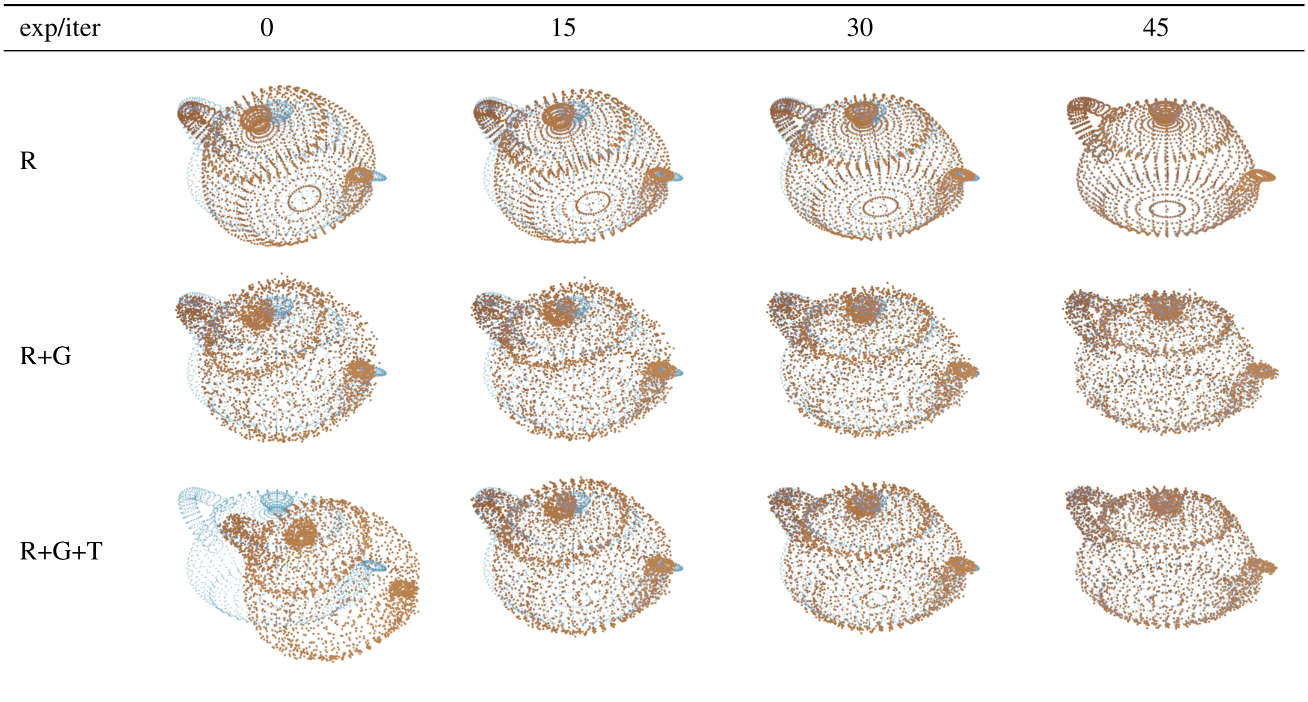


Table 1. Recovery convergence of naive least-square relaxation method. This iterative scheme makes small incremental rotations to match target distribution. R: rotation only, R+G: rotation with Gaussian noise, R+G+T: rotation and translation with Gaussian noise. Orange: source, Blue: target.

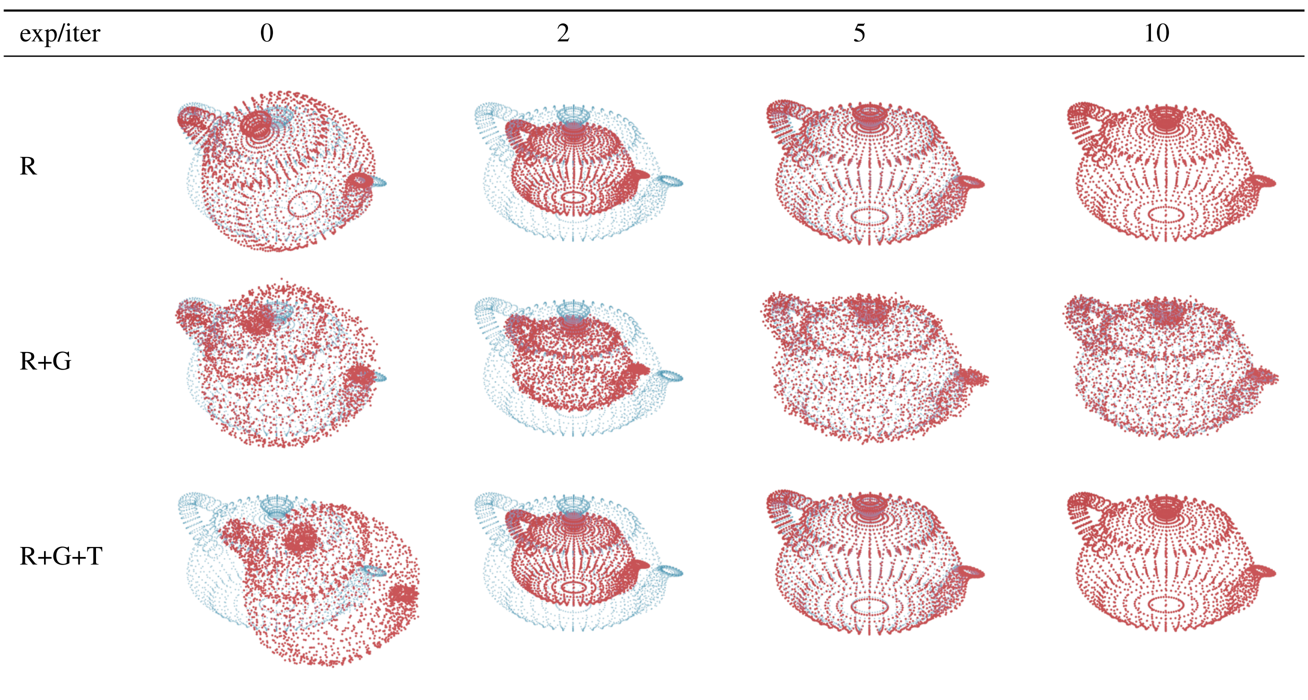


Table 2. Recovery convergence of convex relaxation method. This relaxation relinquishes the constraint on admissible region from $\mathbb{SO}(3)$ to its convex hull. R: rotation only, R+G: rotation with Gaussian noise, R+G+T: rotation and translation with Gaussian noise. Red: source, Blue: target.

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