NEUTRON STAR PARAMETER ESTIMATION WITH CONTINUOUS GRAVITATIONAL WAVES

Description

The goal of this project is to write a code to estimate the spin-down rate and *ellipticity* of a pulsar (a measure of its deformation from sphericity), given noisy observations of the continuous, quasi-monochromatic gravitational waves (GWs) it emits. According to general relativity, a time-varying mass quadrupole moment Q^{ij} sources gravitational radiation with two polarizations (+,x) whose amplitudes are given by Eq. (3.72) of [Gravitational Waves by Maggiore](). The amplitudes fall off like 1/r, so the amplitude of the strain measured in a gravitational-wave detector depends on the ellipticity of the pulsar and the distance to the source. Because the pulsar spins at a near-constant rate, the GWs have an approximately constant frequency (twice the spin frequency). By comparing a template for the gravitational waveform from a rotating, deformed neutron star to the observed signal, you should be able to determine the best-fit waveform parameters and their uncertainties.

Continuous GWs from rotating pulsars are of interest because several mechanisms—such as strong magnetic fields, accretion, and r-mode oscillations—could cause a non-axisymmetric deformation of the type needed to source gravitational radiation. Their detection would provide observational constraints on the magnitude of these effects. To date, however, we only have upper bounds on ellipticity from non-observation of continuous GWs from known pulsars.

I recommend writing your code in a Python notebook (e.g. with Jupyter or Google Colab). In terms of special Python packages, beyond the usual numpy, scipy and pyplot, I suggest using a pre-packaged Markov Chain Monte Carlo (MCMC) sampler like [emcee]() or [stan]().

Roadmap

1. **Continuous gravitational waves –** A deformed, rotating pulsar can be modeled as a rigidly rotating ellipsoid, with moments of inertia I_x, I_y and I_z about its principal axes. We take the z direction to coincide with the axis of rotation. The ellipticity is defined as

   $$\epsilon = (I_x - I_y)/I_z \,.$$

   For an ellipsoid with semiaxes a, b and c, $\epsilon \approx (b-a)/a$ when $a \approx b$.

   A. Start by calculating the mass quadrupole moment

   $$Q^{ij} = \int \rho \, (x^i x^j - \delta^{ij} r^2) \, d^3x$$

   for the ellipsoid, assuming it is uniform in density ρ and rotates at angular velocity $\omega_{rot} = 2\pi f_{rot}$, and express its components in terms of the ellipticity ϵ. Then, suppose that its rotation is slowing down at a constant rate df_{rot}/dt and correct your expression for ω_{rot} to account for this spin-down.

B. Insert your expression for Q^{ij} into Eq. (3.72) of Maggiore to calculate the amplitudes of the + and x GW polarizations. The strain h(t) measured in a detector is related to these amplitudes by way of *antenna patterns* $F_+(\theta,\phi)$ and $F_x(\theta,\phi)$ that encode the detector's different sensitivity to the two polarizations as a function of the source's sky location (θ,ϕ):

$$h(t) = F_+(\theta,\phi)\, h_+(t) + F_x(\theta,\phi)\, h_x(t) .$$

Look up the antenna patterns for a ground-based interferometer like LIGO in Table 7.1 of Maggiore and write out the expression for h(t): this is your template for the gravitational waveform from a rotating, deformed neutron star. How many parameters are needed to specify the template?

C. The signal-to-noise ratio (SNR) of a continuous GW observed for a duration T in a detector with strain sensitivity $S_n(f)$ is given by

$$SNR^2 = h_0^2\, T/5\, S_n(f) ,$$

where $h_0^2 = h_+^2 + h_x^2$. Look up the rough strain sensitivity $S_n(f)^{½}$ at a reference frequency of, say, 100 Hz for the LIGO detector and its planned next-generation equivalent, Cosmic Explorer, e.g. in Fig. 3.3. of Evans+ arXiv:2109.09882. How large must the ellipticity be to produce a detectable (SNR > 1) signal in LIGO after 10^7 s of observation? What about in Cosmic Explorer? You may specialize the template to the case of a pulsar located in the Galactic centre, at a distance of 10 kpc, and oriented face-on towards us. The neutron star moment of inertia I_z can be approximated as that for a solid 1.4 $M_\odot$ ball of radius 12 km.

2. **Parameter estimation –** A realistic observation of a continuous GW signal will be a superposition s(t) = h(t) + n(t) of the gravitational waveform and detector noise. The best-fit values and uncertainties for waveform parameters $\boldsymbol{\vartheta}_t$ can be estimated by comparing many templates $h_t(t;\, \boldsymbol{\vartheta}_t)$ to s(t) in a process called *Bayesian parameter estimation*. We seek the posterior probability distribution $P(\boldsymbol{\vartheta}_t \mid s) \propto L(s \mid \boldsymbol{\vartheta}_t)\, P(\boldsymbol{\vartheta}_t)$ over the waveform parameters, where $P(\boldsymbol{\vartheta}_t)$ is the prior probability distribution and $L(s \mid \boldsymbol{\vartheta}_t)$ is the likelihood that the waveform parameters produce the signal s(t).

   The likelihood $L(s \mid \boldsymbol{\vartheta}_t)$ is a function of $\boldsymbol{\vartheta}_t$ that measures the noise-weighted match between the signal s(t) and the template $h_t(t;\, \boldsymbol{\vartheta}_t)$. We will model the detector noise $n(t) = n_0$ as stationary and Gaussian, with zero mean. Because the noise has a Gaussian distribution, we will also assume that the likelihood function is Gaussian:

$$L(s \mid \boldsymbol{\vartheta}_t) \propto \exp(-\tfrac{1}{2} < s(t)-h_t(t;\boldsymbol{\vartheta}_t) \mid s(t)-h_t(t;\boldsymbol{\vartheta}_t) >) .$$

   Here, $< s(t)-h(t;\boldsymbol{\vartheta}_t) \mid s(t)-h(t;\boldsymbol{\vartheta}_t) >$ is a noise-weighted inner product of the signal residual that is maximized when $s(t) = h(t;\boldsymbol{\vartheta}_t)$. For the purposes of this project, we will compute it in the time domain as

$$< r(t) \mid r(t) > = \Sigma_i\, r(t_i)^2/b\sigma^2\ ,$$

where $\sigma^2 = S_n(f)/f^2$ and b is the number of data points in the discrete representation of r(t) as a time series of duration T.

One way to calculate $P(\boldsymbol{\vartheta}_t \mid s)$ is to sample from $P(\boldsymbol{\vartheta}_t)$, then construct $h(t;\boldsymbol{\vartheta}_t)$ and compute $L(s \mid \boldsymbol{\vartheta}_t)$ for each sample. This *Monte Carlo algorithm* results in a set of likelihood-weighted parameter samples, from which the posterior distribution can be reconstructed as a weighted histogram or a kernel density estimate.

A. Write a Monte Carlo parameter estimation code to determine the spin-down rate df_{rot}/dt and ellipticity ϵ of the pulsar that produced the simulated continuous GW signal "cw_example.csv" measured by Cosmic Explorer at github.com/landryp/simple-cw. For simplicity, fix the rotational frequency to 182 Hz and the sky location to $(\theta,\phi) = (\pi/4, 0)$—we can assume that f_{rot} is known perfectly if the source is a known pulsar, and the signal is short enough that precession of the source across the sky is a negligible effect. Start by assuming a uniform prior distribution on ϵ and df_{rot}/dt, with some reasonable bounds: ϵ = O(1e-6) and df_{rot}/dt = O(1e-4) in this example. What are the most probable parameter values, and what are their uncertainties (at, say, the 90% credible level)? Make a weighted contour plot of the samples, e.g. with seaborn's kdeplot, to show how well you resolve ϵ and df_{rot}/dt.

B. Suppose now that a pre-existing radio pulsar timing measurement has already established that the pulsar's spin-down rate is 2e-4 Hz/s ± 1e-5 Hz, in addition to determining its spin frequency. How would this information change your assumed prior distribution on the parameters? Repeat step A with the updated prior for f_{rot}. Has the posterior distribution changed?

C. The resolution in waveform parameters that your Monte Carlo algorithm can deliver will depend on how densely you sample the prior distribution. Because the Monte Carlo samples are not equally probable *a posteriori*, the effective number of samples

$$N_{eff} = (\Sigma_i\, w_i)^2/(\Sigma_i\, w_i^2)$$

is a more useful statistic for gauging resolution than just the raw number of Monte Carlo samples N. Compute N_{eff} for the analyses in steps A and B. How efficient is your sampling, i.e. what are the ratios N_{eff}/N?

3. **Parameter estimation with MCMC –** *MCMC* is an algorithm for sampling a posterior distribution directly, and more efficiently than with bare Monte Carlo. Conceptually, rather than sampling from the prior distribution at random, samples are preferentially chosen in the direction in parameter space that maximizes their likelihood (see e.g. Speagle arXiv:1909.12313 for an explainer). For our purposes, you just need to know that the algorithm evaluates the same likelihood as before, but returns equally probable samples from the posterior distribution instead of likelihood-weighted samples.

A. Adapt your Monte Carlo code to interface with a pre-packaged MCMC sampler, like emcee or stan. Repeat the analysis in step 2A. Have your best-fit parameter estimates and their uncertainties changed? How does the number of samples in your MCMC posterior compare to N_{eff} in the Monte Carlo case?

B. The *GW spin-down limit*

$$(df_{rot}/dt)_{sd} = 2\, c^3\, h_0\, d^2\, f_{rot}/5\, G\, I_z$$

for a pulsar is the spin-down rate we would expect if its only source of rotational energy loss was gravitational radiation. Is the source of your continuous GW signal spinning down faster or slower than the GW spin-down limit? What does this imply about its rotational energy budget?

4. **Population inference –** Parameter estimates for an ensemble of continuous GW signals can be used to gain insight into the distribution of parameters across the source population. For example, spin-down measurements of characteristic ages $\tau = f_{rot}/(2\, df_{rot}/dt)$ can constrain the pulsar age distribution and help determine the formation history of the neutron star population. The recipe for inferring properties that are common to all the signals in a statistically consistent way is called *Bayesian hierarchical inference* (see e.g. Thrane+Talbot arXiv:1809.02293). The posterior distribution over population hyperparameters $\mathbf{\Lambda}$ is given by

$$P(\mathbf{\Lambda} \mid s) \propto P(\mathbf{\Lambda})\, \Pi_i \int L(s_i \mid \boldsymbol{\vartheta}_t)\, P(\boldsymbol{\vartheta}_t \mid \mathbf{\Lambda})\, d\boldsymbol{\vartheta}_t$$

for an ensemble of $i = 1,2,\ldots,N_{obs}$ signals, each characterized by a likelihood function $L(s_i \mid \boldsymbol{\vartheta}_t)$ over parameters $\boldsymbol{\vartheta}_t$. $P(\boldsymbol{\vartheta}_t \mid \mathbf{\Lambda})$ is the population model and $P(\mathbf{\Lambda})$ is the prior distribution over population hyperparameters.

A. Run your MCMC parameter estimation code to determine the spin-down rates and ellipticities for the ten simulated continuous GW signals labeled 0 through 9 hosted at github.com/landryp/simple-cw. The simulated data also records the pulsar spin frequency; from the spin frequency and the inferred spin-down rate, calculate the pulsar's characteristic age and its uncertainty. Assuming that the inferred spin-down rates are unbiased draws from a common distribution

$$P(\tau) \propto \tau^{\alpha} \quad \text{for } \tau < \tau_{max}\,,$$

estimate the power-law exponent α for the distribution and the maximum age τ_{max} in the population by performing a hierarchical inference with another MCMC algorithm. What are the uncertainties in your estimates? Is the distribution increasing or decreasing with age? What does this imply about the formation history of the pulsars?