Q_4 Solution:

(a) $\mu_F(a,b] = F(b) - F(a)$

$$= \sum_{n=1}^{\infty} 2^{-n} I(b \geq n^{-1}) + (e^{-1} - e^{-b}) I(b \geq 1) + \sum_{n=1}^{\infty} 2^{-n} I(a \geq n^{-1}) - (e^{-1} - e^{-a}) I(a \geq 1)$$

$$= \begin{cases} \sum_{n=1}^{\infty} 2^{-n} I(b \geq n^{-1}) - \sum_{i=1}^{n} 2^{-n} I(a \geq n^{-1}), & a < b < 1 \\ 1 + e^{-1} - e^{-b} - \sum_{i=1}^{n} 2^{-n} I(a \geq n^{-1}), & a < 1 \leq b \\ e^{-a} - e^{-b}\,), & 1 \leq a < b \end{cases}$$

(b) For any $B \in \mathcal{B}$, we define

$$\hat{\mu}(B) = \sum_{n=1}^{\infty} I(n^{-1} \in B) 2^{-n} + \int_B e^{-x} I(x \geq 1) \, d\lambda(x).$$

If we can show $\hat{\mu}$ is a measure in $\mathcal{B}$, then since both μ_F and $\hat{\mu}$ have the same values on $\mathcal{B}_0$, by the unique extension to $\mathcal{B}$ in the Caratheodory extension theorem, we conclude that $\mu_F(B) = \hat{\mu}(B) = \sum_{n=1}^{\infty} I(n^{-1} \in B) 2^{-n} + \int_B e^{-x} I(x \geq 1) d\lambda(x)$ for any $B \in \mathcal{B}$. To show $\hat{\mu}$ is a measure, we note that for any disjoint sets $B_1, B_2, \cdots \in \mathcal{B}$.

$$\hat{\mu}\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{n=1}^{\infty} I\left(n^{-1} \in \bigcup_{i=1}^{\infty} B_i\right) 2^{-n} + \int_{\bigcup_{i=1}^{\infty} B_i} e^{-x} I(x \geq 1) \, d\lambda(x)$$

$$= \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} I(n^{-1} \in B_i) 2^{-n} \right) + \int \left(\sum_{i=1}^{\infty} I_{B_i} \right) e^{-x} I(x \geq 1) \, d\lambda(x)$$

$$= \sum_{n=1}^{\infty} \left(\lim_{m \to \infty} \sum_{i=1}^{m} I(n^{-1} \in B_i) \right) 2^{-n} + \int \lim_{m \to \infty} \left(\sum_{i=1}^{m} I_{B_i} \right) e^{-x} I(x \geq 1) \, d\lambda(x).$$

By the monotone convergence theorem, the last is equal to

$$\lim_{m \to \infty} \left[\sum_{n=1}^{\infty} \sum_{i=1}^{m} I(n^{-1} \in B_i) 2^{-n} \right] + \lim_{m \to \infty} \int \left(\sum_{i=1}^{m} I_{B_i} \right) e^{-x} I(x \geq 1) \, d\lambda(x).$$

$$= \sum_{i=1}^{\infty} \left(\sum_{n=1}^{\infty} I(n^{-1} \in B_i) 2^{-n} \right) + \sum_{i=1}^{\infty} \int_{B_i} e^{-x} I(x \geq 1) \, d\lambda(x) = \sum_{i=1}^{\infty} \hat{\mu}(B_i).$$

Therefore, $\hat{\mu}$ is a measure.

(c) Choose a sequence of simple functions Y_n increasing to X. By definition.

$$\int X(x)\, d\mu_F(x) = \lim_n \int Y_n(x)\, d\mu_F(x).$$

Since each $Y_n(x)$ has the form $\sum_{k=1}^{m} x_k I_{A_k}$, where $A_1, \cdots, A_m$ are disjoint sets in $\mathcal{B}$, it holds.

$$\int Y_n(x)\, d\mu_F(x) = \sum_{k=1}^{m} x_k \mu_F(A_k) = \sum_{k=1}^{m} x_k \left[\sum_{n=1}^{\infty} I(n^{-1} \in A_k) 2^{-n} + \int_{A_k} e^{-x} I(x \geq 1)\, d\lambda(x) \right]$$

$$= \sum_{n=1}^{\infty} \sum_{k=1}^{m} x_k I(n^{-1} \in A_k) 2^{-n} + \int \left(\sum_{k=1}^{m} x_k I_{A_k} \right) e^{-x} I(x \geq 1)\, d\lambda(x)$$

$$= \sum_{n=1}^{\infty} Y_n(n^{-1}) 2^{-n} + \int Y_n(x) e^{-x} I(x \geq 1)\, d\lambda(x).$$

By the monotone convergence theorem

$$\lim_n \int Y_n(x)\, d\mu_F(x) = \sum_{n=1}^{\infty} X(n^{-1}) 2^{-n} + \int X(x) e^{-x} I(x \geq 1)\, d\lambda(x).$$

(d) From previous result,

$$\int x^2 \mu_F(x) = \sum_{n=1}^{\infty} n^{-2} 2^{-n} + \int x^2 e^{-x} I(x \geq 1)\, d\lambda(x).$$

$$= \sum_{n=1}^{\infty} n^{-2} 2^{-n} + \int_1^{\infty} x^2 e^{-x}\, dx$$

$$= \sum_{n=1}^{\infty} n^{-2} 2^{-n} + \left(-x^2 e^{-x} \Big|_1^{\infty} - 2x e^{-x} \Big|_1^{\infty} - 2 e^{-x} \Big|_1^{\infty} \right)$$

$$= \sum_{n=1}^{\infty} n^{-2} 2^{-n} + 5e^{-1}$$

Q5. Solution:

(a) From construction of measure in the Carotheodory extension theorem,

$$\lambda(B) = \inf\{\sum_{i=1}^{\infty}\lambda(a_i,b_i];\ B\subset \bigcup_{i=1}^{\infty}(a_i,b_i]\}.$$

Therefore, $\forall\ \varepsilon>0$, $\exists$ intervals of $(a_i,b_i]$'s, $-\infty<a_1<b_1<\cdots<a_i<b_i<\cdots<\infty$, s.t

$$\sum_{i=1}^{\infty}\lambda(a_i,b_i]-\frac{\varepsilon}{2}\le\lambda(B)\le\sum_{i=1}^{\infty}\lambda(a_i,b_i],$$

Notice that we can assume $(a_i,b_i]$'s to be disjoint, i.e. by taking

$$(a_1',b_1']=(a_1,b_1],\ (a_2',b_2']=(a_2,b_2]/(a_1,b_1],\ \cdots,$$

we have $\bigcup_{i=1}^{\infty}(a_i',b_i']=\bigcup_{i=1}^{\infty}(a_i,b_i]$ and $(a_i',b_i']$s are disjoint.

Since $\lambda(B)<\infty$, then $\sum_{i=1}^{\infty}\lambda(a_i,b_i]<\infty$. For the above ε, $\exists n$, s.t

$$\sum_{i=n}^{\infty}\lambda(a_i,b_i]\le\frac{\varepsilon}{2},$$

Therefore, we have

$$\begin{aligned}
&\lambda\big((B-\bigcup_{i=1}^{n}(a_i,b_i])\cup(\bigcup_{i=1}^{n}(a_i,b_i]-B)\big)\\
&\le\lambda(B-\bigcup_{i=1}^{n}(a_i,b_i])+\lambda(\bigcup_{i=1}^{n}(a_i,b_i]-B)\\
&\le\lambda(\bigcup_{i=1}^{\infty}(a_i,b_i]-\bigcup_{i=1}^{n}(a_i,b_i])+\lambda(\bigcup_{i=1}^{\infty}(a_i,b_i]-B)\\
&\le\lambda(\bigcup_{i=n}^{\infty}(a_i,b_i])+\big(\sum_{i=1}^{\infty}\lambda(a_i,b_i]-\lambda(B)\big)\\
&\le\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon
\end{aligned}$$

(b) First, we assume $f(x)$ to be non-negtive.

By definition, $\forall\varepsilon>0$, $\exists$ a simple function $h(x)$, s.t

$$\int|f(x)-h(x)|\,d\lambda(x)<\frac{\varepsilon}{2}, \qquad ①$$

and $h(x)$ has the form $\sum_{k=1}^{\ell}x_k I_{A_k}$, where $A_1\cdots A_\ell$ are disjoint sets.

Let $h_k(x)=x_k I_{A_k}$.

From (a), we have for each A_k, $\exists$ disjoint and bounded intervals $(a_{ki},b_{ki}]$s, $i=1,\cdots,\infty$, s.t

i> $A_k\subset\bigcup_{i=1}^{\infty}(a_{ki},b_{ki}]$s, and $\sum_{i=1}^{\infty}\lambda(a_{ki},b_{ki}]-\lambda(A_k)\le\frac{\varepsilon}{2\ell|x_k|}$,

ii> $\exists m$, s.t. $\sum_{m}^{\infty}\lambda(a_{ki},b_{ki}]\le\frac{\varepsilon}{4\ell|x_k)}$

$$\int \left| h_k(x) - \sum_{i=1}^{m} x_k I_{(a_{ki}, b_{ki}]} \right| d\lambda(x)$$

$$= \int \left| x_k I_{A_k} - \sum_{i=1}^{\infty} x_k I_{(a_{ki}, b_{ki}]} + \sum_{i=1}^{\infty} x_k I_{(a_{ki}, b_{ki}]} - \sum_{i=1}^{m} x_k I_{(a_{ki}, b_{ki}]} \right| d\lambda(x)$$

$$\leq |x_k| \left(\sum_{i=1}^{\infty} \lambda(a_{ki}, b_{ki}] - \lambda(A_k) \right) + |x_k| \sum_{m}^{\infty} \lambda(a_{ki}, b_{ki}]$$

$$\leq \frac{\varepsilon}{2\ell}$$

Therefore, $\int \left| h(x) - \sum_{k=1}^{\ell} \sum_{i=1}^{m} x_k I_{(a_{ki}, b_{ki}]} \right| d\lambda(x) \leq \sum_{k=1}^{\ell} \int \left| h_k(x) - \sum_{i=1}^{m} x_k I_{(a_{ki}, b_{ki}]} \right| d\lambda(x) < \frac{\varepsilon}{2}$ ②

The desired result follows from ① and ② for non-negtive function

For general $f(x)$. $f(x) = f^+(x) - f^-(x)$, $\exists\ g_1(x), g_2(x)$, s.t

$$\begin{cases} \int |f^+(x) - g_1(x)| \, d\lambda(x) < \frac{\varepsilon}{2} \\ \int |f^-(x) - g_2(x)| \, d\lambda(x) < \frac{\varepsilon}{2} \end{cases}$$

Let $g(x) = g_1(x) - g_2(x)$, $g(x)$ is a step function, we have

$$\int |f(x) - g(x)| \, d\lambda(x) \leq \int |f^+(x) - g_1(x)| \, d\lambda(x) + \int |f^-(x) - g_2(x)| \, d\lambda(x) < \varepsilon.$$

(c) From (b), $\exists\ g(x) = \sum_{i=1}^{n} x_i I_{(a_i, b_i]}$, s.t $\int |f(x) - g(x)| \, d\lambda(x) < \frac{\varepsilon}{2}$ ③

where $(a_i, b_i]$s are disjoint

Define $\ell(x)$

$$\ell(x) = \begin{cases} 0 & \text{if } \bar{x}\ \cup (a_k, b_k] \\ x_k & \text{if } x \in \left(a_k + \frac{\varepsilon}{2\sum_{k=1}^{n}|x_k|},\ b_k - \frac{\varepsilon}{2\sum_{k=1}^{n}|x_k|}\right] \\ x_k \cdot \frac{x - a_k}{\varepsilon/2\sum|x_k|} & \text{if } x \in \left(a_k,\ a_k + \frac{\varepsilon}{2\sum|x_k|}\right] \\ x_k - \frac{x - b_k + \frac{\varepsilon}{2\sum|x_k|}}{\varepsilon/2\sum|x_k|} & \text{if } x \in \left(b_k - \frac{\varepsilon}{2\sum|x_k|},\ b_k\right] \end{cases}$$

$\ell(x)$ is continuous

$$\int |g(x) - \ell(x)| \, d\lambda(x) \leq \sum_{k=1}^{n} |x_k| \frac{\varepsilon}{2\sum_{k=1}^{n}|x_k|} = \frac{\varepsilon}{2} \qquad ④$$

The desired result follows from ③ and ④.