BIOS760 HOMEWORK I SOLUTION

1. (a) $Y_1, ..., Y_{n+1}$ are independent with $\exp(\theta)$. Let $U = Y_1 + ... + Y_i, V = Y_{i+1} + ... + Y_{n+1}$. Then $U \sim Gamma(i, \theta), V \sim Gamma(n+1-i, \theta)$. Let

$$Z_i = U/(U+V), W = U + V.$$

Consider the transformation $(U, V)' \mapsto (Z_i, W)'$. Note that the transformation is one-to-one with the Jacobian

$$|det\left(\frac{\partial(U,V)}{\partial(Z_i,W)}\right)| = |det\left(\begin{pmatrix} W & Z_i \\ -W & 1-Z_i \end{pmatrix}\right)| = |W|.$$

From the joint density of $(U, V)'$,

$$\frac{1}{\Gamma(i)}\theta\exp\{-\theta u\}(\theta u)^{i-1}I(u>0) \times \frac{1}{\Gamma(n+1-i)}\theta\exp\{-\theta v\}(\theta v)^{n-i}I(v>0),$$

we obtain the joint density of (Z_i, W) as

$$\frac{1}{\Gamma(i)}\theta\exp\{-\theta z_i w\}(\theta z_i w)^{i-1} \times \frac{1}{\Gamma(n+1-i)}\theta\exp\{-\theta(1-z_i)w\}(\theta(1-z_i)w)^{n-i}w$$
$$\times I(0<z_i<1)I(w>0),$$

Thus, the marginal density of $Z_i = X/(X+Y)$ is equal to

$$(1-z_i)^{n-i}z_i^{i-1}I(0<z_i<1)\frac{\Gamma(n+1)}{\Gamma(i)\Gamma(n+1-i)}\int_w \theta\exp\{-\theta w\}\{\theta w\}^n I(w>0)dw$$
$$= \frac{\Gamma(n+1)}{\Gamma(i)\Gamma(n+1-i)}z_i^{i-1}(1-z_i)^{n-i}I(0<z_i<1).$$

That is, $Z_i \sim Beta(i, n+1-i)$.

(b) Let

$$W_1 = Y_1, W_2 = Y_1 + Y_2, ..., W_n = Y_1 + ... + Y_n, S = Y_1 + ... + Y_{n+1}.$$

Consider the transformation $(Y_1, ..., Y_{n+1})' \mapsto (W_1, ..., W_n, S)'$. Note that the transformation is one-to-one with the Jacobian

$$|det\left(\frac{\partial(Y_1,...,Y_{n+1})}{\partial(W_1,...,W_n,S)}\right)| = |det\begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ -1 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & \ldots & -1 & 1 \end{pmatrix}| = 1$$

From the joint density of $(Y_1, ..., Y_{n+1})'$,

$$\prod_{i=1}^{n+1} \frac{1}{\theta} \exp\{-\frac{1}{\theta} y_i\} I(0 < y_i < \infty) = (\frac{1}{\theta})^{n+1} \exp\{-\frac{\sum_{i=1}^{n+1} y_i}{\theta}\} I(0 < y_i < \infty),$$

we obtain the joint density of $(W_1, ..., W_n, S)$ as

$$(\frac{1}{\theta})^{n+1} \exp\{-\frac{s}{\theta}\} I(0 < w_1 < w_2 < ... < w_n < s < \infty),$$

Let

$$Z_1 = W_1/S, Z_2 = W_2/S, ..., Z_n = W_n/S, S.$$

Consider the transformation $(W_1, ..., W_n, S)' \mapsto (Z_1, ..., Z_n, S)'$. Note that the transformation is one-to-one with the Jacobian

$$|det \left(\frac{\partial(W_1, ..., W_n, S)}{\partial(Z_1, ..., Z_n, S)} \right)| = |det \begin{pmatrix} s & 0 & \dots & 0 & z_1 \\ 0 & s & \dots & 0 & z_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & s & z_n \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} | = s^n$$

From the joint density of $(Z_1, ..., Z_n, S)'$,

$$= (\frac{1}{\theta})^{n+1} \exp\{-\frac{s}{\theta}\} s^n I(0 < z_1 < z_2 < ... < z_n < 1) I(s > 0)$$

We obtain the joint density of $(Z_1, ..., Z_n)'$,

$$f(z_1, ..., z_n) = \int_s f(z_1, ..., z_n, S) ds = n! I(0 < z_1 < z_2 < ... < z_n < 1)$$

which is the joint density of order statistics of n uniform(0,1) random variables.

2. $Cov(X, Y|Z) = E(XY|Z) - E(X|Z)E(Y|Z)$, thus

$$E[Cov(X, Y|Z)] = E(XY) - E[E(X|Z)E(Y|Z)].$$

Since

$$\begin{aligned} Cov(E(X|Z), E(Y|Z)) &= E[E(X|Z)E(Y|Z)] - E[E(X|Z)]E[E(Y|Z)] \\ &= E[E(X|Z)E(Y|Z)] - E(X)E(Y), \end{aligned}$$

we have

$$E[Cov(X, Y|Z)] + Cov(E(X|Z), E(Y|Z)) = E(XY) - E(X)E(Y) = Cov(X, Y)$$

3. (a) For $z < 0$, $P(Z \le z, \Delta = 1) = P(Z \le z, \Delta = 0) = 0$.

For $z \ge 0$,

$$P(Z \le z, \Delta = 1) = P(X \le Y, X \le z)$$

$$= E[I(X \le Y, X \le z)] = E[E[I(X \le Y, X \le z)|X]]$$

$$= E[I(X \le z)(1 - G(X-))] = \int_0^z (1 - G(x-))dF(x).$$

By the symmetry,

$$P(Z \le z, \Delta = 0) = \int_0^z (1 - F(x-))dG(x).$$

(b) Note

$$\int_0^z (1 - G(x-))dF(x) = \int_0^z \lambda \exp\{-\mu x\} \exp\{-\lambda x\} dx = \frac{\lambda}{\lambda + \mu}(1 - \exp\{-(\lambda + \mu)z\})$$

and

$$\int_0^z (1 - F(x-))dG(x) = \int_0^z \mu \exp\{-\mu x\} \exp\{-\lambda x\} dx = \frac{\mu}{\lambda + \mu}(1 - \exp\{-(\lambda + \mu)z\}).$$

In other words,

$$P(Z \le z, \Delta = \delta) = (1 - \exp\{-(\lambda + \mu)z\}) \left\{\frac{\lambda}{\lambda + \mu}\right\}^{\delta} \left\{\frac{\mu}{\lambda + \mu}\right\}^{1-\delta}.$$

Thus Z and Δ are independent.

4. (a) Let Σ be an orthogonal matrix with the first row bing $(\sqrt{\omega_1}, ..., \sqrt{\omega_n})$. Define $(Z_1, ..., Z_n)^T = \Sigma(X_1, ..., X_n)^T$. Then $(Z_1, ..., Z_n)^T$ follows a multivariate normal distribution with mean zeros and covariance $\sigma^2 I_{n \times n}$. Note $Y_n = Z_1/\sigma$. Thus, $Y_n \sim N(0, 1)$.

(b) Note $Z_1^2 + ... + Z_n^2 = X_1^2 + ... + X_n^2$. Thus,

$$(n-1)S_n^2/\sigma^2 = (\sum_{i=1}^n Z_i^2 - Z_1^2)/\sigma^2 = \sum_{i=2}^n Z_i^2/\sigma^2 \sim \chi^2_{n-1}.$$

(c) Since Z_1 is independent of $Z_2, ..., Z_n$, so are $\bar{X}_{nw}$ and S_n^2. $T_n \sim N(0,1)/\sqrt{S_n^2} = t_{n-1}/\sigma$.

(d) When $\omega_1 = ... = \omega_n = 1/n$, $Y_n = \sqrt{n}\bar{X}_n/\sigma$ and $S_n^2 = (n-1)^{-1}\sum_{i=1}^n (X_i - \bar{X}_n)^2$. We obtain the result.

5. (a) $P(W_m = k) = \binom{k-1}{m-1}p^m(1-p)^{k-m} = \binom{k-1}{m-1}\exp\{k\log(1-p) + m\log(p/(1-p))\}$. This is a 1-parameter exponential family with $T(x) = x$, $\eta = \log(1-p)$ and $A(\eta) = -m\log(p/(1-p)) = m\log\{e^\eta/(1-e^\eta)\}$.

(b) The moment generating function for W_m, equivalently T, is

$$M_T(t) = \exp\{A(\eta+t) - A(\eta)\} = e^{mt}\frac{(1-e^\eta)^m}{(1-e^{\eta+t})^m} = e^{mt}\frac{p^m}{(1-(1-p)e^t)^m}.$$

(c) The cumulant generating function for W_m is equal to

$$mt + m\log p - m\log(1-(1-p)e^t).$$

After differentiation, we obtain the first two cumulants as

$$\mu_1 = p,\ \mu_2 = m/p^2 - m/p.$$