BIOS760 HOMEWORK II SOLUTION

1. For any open set $O \in \mathcal{O}$, we claim that

$$O = \text{union of all open intervals } (x-r, x+r)$$

$$\text{such that } x \text{ and } r \text{ are rational numbers and } (x-r, x+r) \subset O.$$

Obviously, the right-hand side is contained in O. For any $y \in O$, we can find a positive number ϵ such that $(y-\epsilon, y+\epsilon) \subset O$, we choose x to be a rational number such that $|x-y| < r < \epsilon/2$ where r is another rational number. Then $y \in (x-r, x+r) \subset (y-\epsilon, y+\epsilon) \subset O$. We have proved the claim. Since the union is countable and $(x-r, x+r) = \cup_{n=1}^{\infty}(x-r, x+r-r/n] \in \mathcal{B}$, $O \in \mathcal{B}$. Thus $\mathcal{O} \subset \mathcal{B}$ then $\sigma(\mathcal{O}) \subset \mathcal{B}$. On the other hand, for any left-open and right-close interval $(a, b]$, $(a, b] = \cap_{n=1}^{\infty}(a, b+1/n) \in \mathcal{O} \subset \sigma(\mathcal{O})$. We have $\mathcal{B} \subset \sigma(\mathcal{O})$. The result holds.

2. First, we show $\tilde{\mathcal{A}}$ is a σ-field. Clearly, $\emptyset$ and Ω belong to $\tilde{\mathcal{A}}$. If $\tilde{A} \in \tilde{\mathcal{A}}$, then there exists A and N such that $\tilde{A} = A \cup N, A \in \mathcal{A}$ and $N \subset B \in \mathcal{A}$ with $\mu(B) = 0$. Then $\tilde{A}^c = A^c \cap N^c$. Since $N^c = (B - (B-N))^c = B^c \cup (B-N)$, we have

$$\tilde{A}^c = (A^c \cap B^c) \cup (A^c \cap (B-N)).$$

Clearly, $A^c \cap B^c \in \mathcal{A}$ and $A^c \cap (B-N) \subset B$ with $\mu(B) = 0$. Thus, $\tilde{A}^c \in \tilde{\mathcal{A}}$. If $\tilde{A}_1, \tilde{A}_2, ... \in \tilde{\mathcal{A}}$, then there exists $A_1, A_2, ... \in \mathcal{A}$ and $N_1, N_2, ...$ such that $\tilde{A}_i = A_i \cup N_i$ and $N_i \subset B_i$ for some $B_i \in \mathcal{A}$ with $\mu(B_i) = 0$, $i = 1, 2, ...$ We obtain

$$\cup_{n=1}^{\infty}\tilde{A}_n = \cup_{n=1}^{\infty}\{A_i \cup N_i\} = (\cup_{n=1}^{\infty}A_i) \cup (\cup_{n=1}^{\infty}N_i).$$

Since $\cup_{n=1}^{\infty}A_i \in \mathcal{A}$ and $\cup_{n=1}^{\infty}N_i \subset \cup_{n=1}^{\infty}B_i \in \mathcal{A}$ with $\mu(\cup_{n=1}^{\infty}B_i) \leq \sum_{n=1}^{\infty}\mu(B_i) = 0$, we conclude $\cup_{n=1}^{\infty}A_i \in \tilde{\mathcal{A}}$. So $\tilde{\mathcal{A}}$ is a σ-field. Second, we show $\tilde{\mu}$ is a measure. If $\tilde{A}_1 = A_1 \cup N_1, \tilde{A}_2 = A_2 \cup N_2, ...$ are disjoint in $\tilde{\mathcal{A}}$, so are $A_1, A_2, ...$ From the previous argument, we have

$$\cup_{n=1}^{\infty}\tilde{A}_n = \cup_{n=1}^{\infty}\{A_i \cup N_i\} = (\cup_{n=1}^{\infty}A_i) \cup (\cup_{n=1}^{\infty}N_i),$$

where $\cup_{n=1}^{\infty}N_i$ is contained in a zero-measure set. Thus,

$$\tilde{\mu}(\cup_{n=1}^{\infty}\tilde{A}_n) = \mu(\cup_{n=1}^{\infty}A_n) = \sum_{n=1}^{\infty}\mu(A_n) = \sum_{n=1}^{\infty}\tilde{\mu}(A_n).$$

3. (a) $F(-\infty) = P(\emptyset) = 0$ and $F(\infty) = P(R) = 1$. For any sequence x_n decreasing to x, the sequence of sets $\{(-\infty, x_n]\}$ decreases to $(-\infty, x])$. Moreover, $P((-\infty, x_1]) < 1$.

$$\lim_n F(x_n) = \lim_n P((-\infty, x_n]) = P((-\infty, x]) = F(x).$$

That is, $F(x)$ is right-continuous.

(b) Both P and μ_F are two measures on $\mathcal{B}$. Moreover, $P((a,b]) = P((-\infty, b]) - P((-\infty, a]) = F(b) - F(a) = \mu_F((a,b])$. Therefore, both P and μ_F can be treated as the extension of the same measure in $\mathcal{B}_0$ to $\mathcal{B}$. By the uniqueness in the Caratheodory extension theorem, $P(B) = \mu_F(B)$ for any $B \in \mathcal{B}$.