Chain Rule

If $F(x) = (f \circ g)(x) = f(g(x))$, then $F'(x) = f'(g(x)) \cdot g'(x)$

If $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

1. Use the Chain Rule to compute the following derivatives:

a) $h(t) = \left(\dfrac{2t+3}{6-t^2}\right)^3$

"$f(x) = x^3$"

$g(x) = \dfrac{2x+3}{6-x^2}$

$\Rightarrow f'(x) = 3x^2$

$g'(x) = \dfrac{(6-x^2)(2) - (2x+3)(-2x)}{(6-x^2)^2}$

$$\boxed{h'(t) = 3\left(\frac{2t+3}{6-t^2}\right)^2 \cdot \frac{(6-t^2)(2) - (2t+3)(-2t)}{(6-t^2)^2}}$$

b) $f(t) = \sin(3t^2 + t)$

"$f(x) = \sin(x)$"

$g(x) = 3x^2 + x$

$\Rightarrow f'(x) = \cos(x)$

$g'(x) = 6x + 1$

$$\boxed{f'(t) = \cos(3t^2 + t) \cdot (6t+1)}$$

c) $y = \sec(1-5x)$

"$f(x) = \sec(x)$"

$g(x) = 1 - 5x$

$\Rightarrow f'(x) = \tan(x)\sec(x)$

$g'(x) = -5$

$$\boxed{\frac{dy}{dx} = \tan(1-5x)\sec(1-5x)(-5)}$$

d) $G(x) = \sin(xe^x)$

$f(x) = \sin(x)$

$g(x) = xe^x$

$\Rightarrow f'(x) = \cos(x)$

$g'(x) = xe^x + e^x$

$$\boxed{G'(x) = \cos(xe^x) \cdot (xe^x + e^x)}$$

2. Do each of the following for the equation $x^2 + y^3 = 4$:

a) Find y' by solving the equation for y and differentiating directly.

$y^3 = 4 - x^2$

$\Rightarrow y = \sqrt[3]{4-x^2}$

$$\frac{dy}{dx} = \frac{1}{3}(4-x^2)^{-2/3}(-2x)$$

b) Find y' by implicit differentiation

$$\frac{dy}{dx}(x^2+y^3) = \frac{dy}{dx}(4)$$

$$\Rightarrow 2x + 3y^2 \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2x}{3y^2}$$

c) Check that your answers to (a) and (b) are the same.

$$\frac{-2x}{3y^2} = \frac{-2x}{3(\sqrt[3]{4-x^2})^2} = (-2x)\left(\frac{1}{3}\right)(4-x^2)^{-2/3} \quad ✓$$

d) Find the equation of the line tangent to the curve $x^2 + y^3 = 4$ when $x = \sqrt{3}$.

$$(\sqrt{3})^2 + y^3 = 4 \Rightarrow y^3 = 1 \Rightarrow y = 1$$

$$m = \frac{-2(1)}{3(1)^2} = -\frac{2}{3} \Rightarrow y - 1 = -\frac{2}{3}(x - \sqrt{3})$$

$$\Rightarrow y = -\frac{2}{3}x + \frac{2\sqrt{3}}{3} + 1$$

3. Use implicit differentiation to find an equation of the tangent line to the curve $x^2 + y^2 = (2x^2 + 2y^2 - x)^2$ at the point ~~when $x = 0$~~. $(0, \frac{1}{2})$

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$$\frac{dy}{dx}(x^2+y^2) = \frac{dy}{dx}(2x^2+2y^2-x)^2$$

$$\Rightarrow 2x + 2y\frac{dy}{dx} = 2(2x^2+2y^2-x)\cdot\left(4x + 4y\frac{dy}{dx} - 1\right)$$

Plug in $(0, \frac{1}{2})$ $\Rightarrow$

$$0 + 1\frac{dy}{dx} = 2\left(2\left(\tfrac{1}{2}\right)^2\right)\cdot\left(4\left(\tfrac{1}{2}\right)\frac{dy}{dx} - 1\right)$$

$$\Rightarrow \frac{dy}{dx} = 2\frac{dy}{dx} - 1 \Rightarrow \frac{dy}{dx} = 1$$

$$y - \frac{1}{2} = 1(x - 0) \Rightarrow y = x + \frac{1}{2}$$