Kyle McGraw

ACM 157 Set 1

1. $x_1, \dots, x_n \quad \bar{x} \quad \tilde{x} \quad s_x \quad IQR_x \qquad y_i = \alpha + \beta x_i$

$$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i = \frac{1}{n}\sum_{i=1}^{n}(\alpha+\beta x_i) = \frac{1}{n}\cdot n\alpha + \frac{1}{n}\cdot\beta\sum_{i=1}^{n} x_i = \alpha + \frac{\beta}{n}\bar{x}$$

$$\tilde{y} = \begin{cases} y_{(\frac{n+1}{2})} & \text{if } n \text{ odd} \\ \frac{1}{2}(y_{(\frac{n}{2})} + y_{(\frac{n}{2}+1)}) & \text{if } n \text{ even} \end{cases}$$

$$= \begin{cases} \alpha + \beta x_{(\frac{n+1}{2})} & \text{if } n \text{ odd} \\ \frac{1}{2}(\alpha + \beta x_{(\frac{n}{2})} + \alpha + \beta x_{(\frac{n}{2}+1)}) & \text{if } n \text{ even} \end{cases}$$

$$= \begin{cases} \alpha + \beta x_{(\frac{n+1}{2})} & \text{if } n \text{ odd} \\ \alpha + \beta \frac{1}{2}(x_{(\frac{n}{2})} + x_{(\frac{n}{2}+1)}) & \text{if } n \text{ even} \end{cases}$$

Since $\tilde{x} = \begin{cases} x_{(\frac{n+1}{2})} & \text{if } n \text{ odd} \\ \frac{1}{2}(x_{(\frac{n}{2})} + x_{(\frac{n}{2}+1)}) & \text{if } n \text{ even} \end{cases}$

$$= \alpha + \beta\tilde{x}$$

$$s_y = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(y_i - \tilde{y})^2} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(\alpha+\beta x_i - \alpha + \beta\tilde{x})^2} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\beta^2(x_i - \tilde{x})^2}$$

$$= \sqrt{\frac{\beta^2}{n}\sum_{i=1}^{n}(x_i - \tilde{x})^2} = |\beta|\, s_x$$

We could also see that $\tilde{y} = \alpha + \beta\tilde{x}$, because we are just linearly scaling x_i's to get y_i's, so the median stays the same data point. Similarly we can see this will be the case for Q_{1y} and Q_{3y}, except if β is negative as the order of the data points switch. Therefore we get:

$$IQR_y = Q_{3y} - Q_{1y}$$

$$= \begin{cases} \alpha + \beta Q_{3x} - (\alpha + \beta Q_{1x}) & \text{if } \beta \geq 0 \\ \alpha + \beta Q_{1x} - (\alpha + \beta Q_{3x}) & \text{if } \beta < 0 \end{cases}$$

$$= \begin{cases} \beta(Q_{3x} - Q_{1x}) & \text{if } \beta \geq 0 \\ \beta(Q_{1x} - Q_{3x}) & \text{if } \beta < 0 \end{cases}$$

$$= \begin{cases} \beta\, IQR_x & \text{if } \beta \geq 0 \\ -\beta\, IQR_x & \text{if } \beta \geq 0 \end{cases}$$

$$= |\beta|\, IQR_x$$

2. $\bar{x} = \underset{\alpha}{\operatorname{argmin}} \sum_{i=1}^{n}(x_i - \alpha)^2 \qquad \tilde{x} = \underset{\alpha}{\operatorname{argmin}} \sum_{i=1}^{n}|x_i - \alpha|$

$\sum_{i=1}^{n}(x_i-\alpha)^2$ minimized when $\frac{d}{d\alpha}\sum_{i=1}^{n}(x_i-\alpha)^2 = 0$ & $\frac{d^2}{d\alpha^2}\sum_{i=1}^{n}(x_i-\alpha)^2 > 0$

$$\frac{d}{d\alpha}\sum_{i=1}^{n}(x_i-\alpha)^2 = 0 \qquad\qquad \frac{d^2}{d\alpha^2}\sum_{i=1}^{n}(x_i-\alpha)^2 > 0$$

$$\sum_{i=1}^{n} 2(x_i-\alpha)\cdot(-1) = 0 \qquad\qquad \frac{d}{d\alpha}\left(2\sum_{i=1}^{n}\alpha - 2\sum_{i=1}^{n} x_i\right) > 0$$

$$2\sum_{i=1}^{n}\alpha - 2\sum_{i=1}^{n}x_i = 0 \qquad 2\sum_{i=1}^{n}1 > 0$$

$$n\alpha = \sum_{i=1}^{n}x_i \qquad 2n > 0$$

$$\alpha = \frac{1}{n}\sum_{i=1}^{n}x_i = \bar{x}$$

From these, we see that α is a minimum and is equal to $\bar{x}$

To show $\tilde{x} = \arg\min_{\alpha} \sum_{i=1}^{n}|x_i - \alpha|$, we will show that the median minimizes $\sum_{i=1}^{n}|x_i - \alpha|$ when n is odd and the median is a minimizer of $\sum_{i=1}^{n}|x_i - \alpha|$ when n is even.

When n is odd, we can see what happens to $\sum_{i=1}^{n}|x_i - \alpha|$ when we move it away from $\tilde{x}$. We know there are an equal amount of points ($\frac{n-1}{2}$) above and below $\tilde{x}$ since $\tilde{x}$ is on the middle point when n is odd. If we move α in eithe direction from $\tilde{x}$ then we can see that $|x_i - \alpha|$ changes by the same amount for each i, increasing or decreasing depending if α moves away from or towards x_i, respectively. However, if we move α away from $\tilde{x}$ in either direction then we can see that it is moving towards fewer points than it is moving away from (initially $\frac{n-1}{2}$ towards and $\frac{n-1}{2}+1$ away and gets worse as we move past more points), so we are increasing $|x_i - \alpha|$ for more points than we are decreasing it. Therefore we can see that $\tilde{x}$ minimizes $\sum_{i=1}^{n}|x_i - \alpha|$ for odd n.

When n is even, we use the same logic as above to show that $\alpha = \tilde{x}$ is in the range of values that minimize $\sum_{i=1}^{n}|x_i - \alpha|$. From the previous case, we know that if we are moving α towards fewer points than we are moving it away from, then $\sum_{i=1}^{n}|x_i - \alpha|$ increases. Unlike the previous case where $\alpha = \tilde{x}$ is the only spot with an equal number of points on either side of α, since n is even, the entire range of values between the two center points of the data set will have an equal number of points on either side. Since $\tilde{x}$ is the average of the two center points, we know it will be in the range of minimum values, $\tilde{x} \in [x_{(\frac{n}{2})}, x_{(\frac{n}{2}+1)}]$, so is a minimizer of $\sum_{i=1}^{n}|x_i - \alpha|$ for even n.

3. $x_1, \dots, x_n$ $\{(z_{\frac{k}{n+1}}, x_{(k)})\}$

If the collection of points falls on the line $y = ax+b$, then we know it still is a normal distribution since departures from normality are indicated by departures from a straight line. We can also see this is the case since $y = ax+b$ is just scaling and shifting $y=x$. This tells us that the distribution is $N(b, a^2)$ since the mean will be shifted by b and the standard deviation scaled by a.

6. $P = \{1, \dots, N\}$ $S = \{s_1, \dots, s_n\}$

a) $P(s_1 = N), \dots, P(s_n = N)$

$P(s_1 = N) = 1/N$ since all values of P are unique

$P(s_2 = N) = 1/(N-1) \cdot (N-1)/N = 1/N$ probability it gets selected given it

$P(s_3 = N) = 1/(N-2) \cdot \cancel{(N-2)}/\cancel{(N-1)} \cdot \cancel{(N-1)}/N = 1/N$ wasn't already picked

For all i, $P(s_i = N) = 1/N$

b) $P(N^{th}$ unit is in sample$)$

We can see it is the sum of probabilities from (a)

$P(s_1 = N) + P(s_2 = N) + \dots + P(s_n = N)$

$= 1/N + 1/N + \dots + 1/N$

$= n/N$

c) $\mathbb{E}[s_1]$

$\mathbb{E}[s_1] = \mu = \frac{1}{N}\sum_{i=1}^{N} x_i = \frac{1}{N}(1+2+\dots+N) = \frac{1}{N} \cdot \frac{1}{2}(N+1)N$

$= \frac{1}{2}(N+1)$

d) $P(s_1 = N, s_2 = 1)$

$= P(s_1 = N) \cdot P(s_2 = 1 \mid s_1 = N)$

$= 1/N \cdot 1/(N-1) = \frac{1}{N(N-1)}$

e) $P(s_i = i$, for all $i = 1, \dots, n)$

$= P(s_1 = 1) \cdot P(s_2 = 2 \mid s_1 = 1) \cdot \dots \cdot P(s_n = n \mid s_1 = 1, s_2 = 2, \dots, s_{n-1} = n-1)$

$= 1/N \cdot 1/(N-1) \cdot \dots \cdot 1/(N-n+1)$

$= \frac{(N-n)!}{N!}$

7. $\bar{X}_n^w = \sum_{i=1}^{n} w_i x_i$ $\mu = \frac{1}{N}\sum_{i=1}^{N} x_i$

a) $\bar{X}_n^w$ unbiased estimate of μ

$\mathbb{E}[\bar{X}_n^w] = \mu$

$\mathbb{E}[\bar{X}_n^w] = \mathbb{E}[\sum_{i=1}^{n} w_i x_i] = \sum_{i=1}^{n} w_i \mathbb{E}[x_i] = \sum_{i=1}^{n} w_i \mu = \mu \sum_{i=1}^{n} w_i$

$\sum_{i=1}^{n} w_i = 1$

b) Minimize $\mathbb{V}[\bar{X}_n^w]$ w/ $\sum_{i=1}^n w_i = 1$

$$\mathbb{V}[\bar{X}_n^w] = \mathbb{V}[\sum_{i=1}^n w_i X_i]$$

$$\boxed{\text{Since } \mathbb{V}[\sum_{i=1}^n X_i] = \sum_{i=1}^n \mathbb{V}[X_i] + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^n Cov(X_i, X_j)}$$

$$\mathbb{V}[\sum_{i=1}^n w_i X_i] = \sum_{i=1}^n \mathbb{V}[w_i X_i] + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^n Cov(w_i X_i, w_j X_j)$$

$$= \sum_{i=1}^n w_i^2 \mathbb{V}[X_i] + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^n w_i w_j Cov(X_i, X_j)$$

$$= \sum_{i=1}^n w_i^2 \sigma^2 + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^n w_i w_j \left(-\frac{\sigma^2}{N-1}\right)$$

$$= \sum_{i=1}^n w_i^2 \sigma^2 - \frac{\sigma^2}{N-1} 2\sum_{i=1}^{n-1}\sum_{j=i+1}^n w_i w_j$$

$$\boxed{\begin{gathered}\left(\sum_{i=1}^n w_i\right)^2 = \sum_{i=1}^n w_i^2 + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^n w_i w_j \quad \& \quad \sum_{i=1}^n w_i = 1 \\ \Rightarrow 2\sum_{i=1}^{n-1}\sum_{j=i+1}^n w_i w_j = 1 - \sum_{i=1}^n w_i^2\end{gathered}}$$

$$= \sum_{i=1}^n w_i^2 \sigma^2 - \frac{\sigma^2}{N-1}\left(1 - \sum_{i=1}^n w_i^2\right)$$

$$= \left(\sigma^2 + \frac{\sigma^2}{N-1}\right)\sum_{i=1}^n w_i^2 - \frac{\sigma^2}{N-1}$$

Since $(\sigma^2 + \frac{\sigma^2}{N-1}) > 0$ and $\left(\sigma^2 + \frac{\sigma^2}{N-1}\right)$ and $\left(-\frac{\sigma^2}{N-1}\right)$ are both constants, we can minimize:

$\sum_{i=1}^n w_i^2$ w/ $\sum_{i=1}^n w_i = 1$

Using Lagrange multipliers:

$$\frac{\partial}{\partial w_j}\left(\sum_{i=1}^n w_i^2 + \lambda\left(\sum_{i=1}^n w_i - 1\right)\right) = 0$$

$$2w_j + \lambda = 0 \quad w_j = -\frac{\lambda}{2} \;\forall j$$

Since all the weights are equal to the same constant and $\sum_{i=1}^n w_i = 1$, then $w_i = 1/n \;\forall i$

We can see $1/n$ is the minimum when $\sum_{i=1}^n w_i = 1$ by looking at the Cauchy-Schwartz inequality:

$$\left(\sum_{i=1}^n u_i v_i\right)^2 \le \left(\sum_{i=1}^n u_i^2\right)\left(\sum_{i=1}^n v_i^2\right)$$

$$\left(\sum_{i=1}^{n} w_i\right)^2 \le \left(\sum_{i=1}^{n} w_i^2\right)\left(\sum_{i=1}^{n} 1^2\right)$$

$$1 \le \left(\sum_{i=1}^{n} w_i^2\right) n$$

$$\sum_{i=1}^{n} w_i^2 \ge \frac{1}{n}$$

we see that $w_i = \frac{1}{n}$ is the minimum

$$\sum_{i=1}^{n}\left(\frac{1}{n}\right)^2 = \frac{1}{n^2}\sum_{i=1}^{n} 1 = \frac{1}{n^2} n = \frac{1}{n} \ge \frac{1}{n}$$

Most efficient estimate is when $w_i = 1/n$
or $\overline{X}_n$