Kyle McGraw

ACM 157 Midterm

1. $P = \{x_1, \dots, x_N\}$ $\quad x_i \in \{\alpha, \beta\}$ $\quad \pi = \frac{\#\alpha}{N}$

a)
$$\mu = \frac{1}{N}\sum_{i=1}^{N} x_i = \pi\alpha + (1-\pi)\beta$$
$$\sigma^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - \mu)^2$$
$$= \pi(\alpha-\mu)^2 + (1-\pi)(\beta-\mu)^2$$
$$= \pi(\alpha - \pi\alpha - (1-\pi)\beta)^2 + (1-\pi)(\beta - \pi\alpha - (1-\pi)\beta)^2$$
$$= \pi(1-\pi)^2(\alpha-\beta)^2 + (1-\pi)\pi^2(\beta-\alpha)^2$$

b) $\frac{d}{d\pi}\sigma^2 = 0$

$$\frac{d}{d\pi}\sigma^2 = \frac{d}{d\pi}\left(\pi(1-\pi)^2(\alpha-\beta)^2 + (1-\pi)\pi^2(\beta-\alpha)^2\right)$$
$$0 = (1-\pi)^2(\alpha-\beta)^2 - 2\pi(1-\pi)(\alpha-\beta)^2 - \pi^2(\beta-\alpha)^2 + 2(1-\pi)\pi(\beta-\alpha)^2$$
$$0 = \left((1-\pi)^2 - 2\pi(1-\pi) - \pi^2 + 2\pi(1-\pi)\right)(\alpha-\beta)^2$$
$$0 = (1-\pi)^2 - \pi^2$$
$$0 = 1 - 2\pi + \pi^2 - \pi^2$$
$$\pi = 1/2$$
$$\frac{d^2}{d\pi^2}\sigma^2 = \frac{d}{d\pi}(1-2\pi)(\alpha-\beta)^2$$
$$= -2(\alpha-\beta)^2$$
$(\alpha-\beta)^2 \geq 0$ so $\frac{d^2}{d\pi^2}\sigma^2 = -2(\alpha-\beta)^2 < 0$

Therefore $\pi = 1/2$ maximizes

2. $n_g = 100$ $\quad N_g = 1299$ $\quad N_u = 938$ $\quad \sigma_g^2 = \sigma_u^2$

a) $se[\bar{x}_{n_g}] = se[\bar{x}_{n_u}]$
$$\sqrt{V[\bar{x}_{n_g}]} = \sqrt{V[\bar{x}_{n_u}]}$$
$$V[\bar{x}_{n_g}] = V[\bar{x}_{n_u}]$$
$$\frac{\sigma_g^2}{n_g}\left(1 - \frac{n_g-1}{N_g-1}\right) = \frac{\sigma_u^2}{n_u}\left(1 - \frac{n_u-1}{N_u-1}\right)$$
$$\frac{1}{100}\left(1 - \frac{99}{1298}\right) = \frac{1}{n_u}\left(1 - \frac{n_u-1}{937}\right)$$
$$\frac{1}{100}\left(1 - \frac{99}{1298}\right) = \frac{1}{n_u} - \frac{1}{937} + \frac{1}{937 n_u}$$
$$n_u = \frac{1 + \frac{1}{937}}{\frac{1}{100}\left(1 - \frac{99}{1298}\right) + \frac{1}{937}} = 97.15 \approx 97$$

b) $n_g = N_g$

$$\frac{\sigma_g^2}{n_g}\left(1-\frac{N_g-1}{N_g-1}\right) = \frac{\sigma_u^2}{n_u}\left(1-\frac{n_u-1}{N_u-1}\right)$$

$$0 = \frac{\sigma_u^2}{n_u}\left(1-\frac{n_u-1}{N_u-1}\right)$$

$n_u = N_u = 938$ This does not depend on $\sigma_g^2 = \sigma_u^2$

3. $X_1,\dots,X_n \sim U[0,\theta]$ $Q = X_{(n)}/\theta$ $X_{(n)} = \max\{X_1,\dots,X_n\}$

a)

CDF: $F_Q(x) = P(Q \le x) = P(X_i/\theta \le x \text{ for all } i=1,\dots,n)$

$$= \prod_{i=1}^{n} P(X_i/\theta \le x) = \prod_{i=1}^{n} \frac{x\theta}{\theta} = x^n$$

PDF: $f_Q(x) = \frac{d}{dx} F_Q(x) = nx^{n-1}$

Distribution does not depend on θ so Q is a pivot

b)

$I = (a,b)$ $P(a \le \theta \le b) = 1-\alpha$ $G(x) = x^n$ $G^{-1}(x) = \sqrt[n]{x}$

$$P(a \le \theta \le b) = P(a \le X_{(n)}/Q \le b) = P\left(\frac{1}{b} \le Q/X_{(n)} \le \frac{1}{a}\right)$$

$$= P\left(\frac{X_{(n)}}{b} \le Q \le \frac{X_{(n)}}{a}\right) = G\left(\frac{X_{(n)}}{a}\right) - G\left(\frac{X_{(n)}}{b}\right)$$

$$a = \frac{X_{(n)}}{G^{-1}(1-\frac{\alpha}{2})} = \frac{X_{(n)}}{\sqrt[n]{1-\frac{\alpha}{2}}} \qquad b = \frac{X_{(n)}}{G^{-1}(\frac{\alpha}{2})} = \frac{X_{(n)}}{\sqrt[n]{\alpha/2}}$$

4. $N=900$ $n = n_1 = n_2 = 450$ $\hat{p}_1 = \frac{400}{450}$ $\hat{p}_2 = \frac{375}{450}$ $\theta = p_1 - p_2$

a)

$$\hat{\theta} = \hat{p}_1 - \hat{p}_2 = \frac{400}{450} - \frac{375}{450} = \frac{25}{450} \approx 0.056$$

b)

$$se[\hat{\theta}] = \sqrt{\mathbb{V}[\hat{\theta}]} = \sqrt{\mathbb{V}[\hat{p}_1 - \hat{p}_2]} = \sqrt{\mathbb{V}(\hat{p}_1) + \mathbb{V}(\hat{p}_2)}$$

$$= \sqrt{\hat{p}_1(1-\hat{p}_1)/n + \hat{p}_2(1-\hat{p}_2)/n}$$

$$\approx 0.0230$$

5. $X_1,\dots,X_n \sim F$ μ $X_1^*,\dots,X_n^* \sim \hat{F}_n$ $\hat{F}_n$ is eCDF of $X_1,\dots,X_n$

$$\bar{X}_n^* = \frac{1}{n}\sum_{i=1}^{n} X_i^*$$

a) $\mathbb{E}[\overline{X}_n^* | X_1, \dots, X_n] = \mathbb{E}[\frac{1}{n}\sum_{i=1}^{n} X_i^* | X_1, \dots, X_n]$

$= \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}[X_i^* | X_1, \dots, X_n]$

$= \frac{1}{n}\sum_{i=1}^{n} \hat{\mu}_n$

$= \hat{\mu}_n = \overline{X_n}$

b)

$\mathbb{E}[\overline{X}_n^*] = \mathbb{E}[\mathbb{E}[\overline{X}_n^* | X_1, \dots, X_n]]$

$= \mathbb{E}[\overline{X_n}] = \mu$

6. $X_1, \dots, X_n \sim f(x;\theta)$ $f(1;\theta) = \mathbb{P}(X=1|\theta) = \theta$ $f(2;\theta) = \mathbb{P}(X=2|\theta) = 1-\theta$

$n=5$ $x_1=1$ $x_2=2$ $x_3=1$ $x_4=2$ $x_5=1$

a)

$m_q(\theta) = \hat{m}_q$ $q=1,\dots,k$ 1 unknown so $q=1$

$m_1(\theta) = \mathbb{E}_f[x^1] = \theta + 2(1-\theta) = 2-\theta$

$\hat{m}_1 = \frac{1}{n}\sum_{i=1}^{n} x_i^1 = \frac{1}{5}(1+2+1+2+1) = 1.4$

$2-\theta = 1.4$

$\theta = 0.6$

b) $L_5(\theta | x_1, \dots, x_5) = \prod_{i=1}^{5} f(x_i;\theta)$

$= f(1;\theta)\cdot f(2;\theta)\cdot f(1;\theta)\cdot f(2;\theta)\cdot f(1;\theta)$

$= \theta^3(1-\theta)^2$

$= \theta^3(1-2\theta+\theta^2)$

$2\theta^5 - 2\theta^4 + \theta^3$

$L_5'(\theta) = 0$

$\frac{d}{d\theta}(\theta^5 - 2\theta^4 + \theta^3) = 0$

$5\theta^4 - 8\theta^3 + 3\theta^2 = 0$

$\theta^2(5\theta^2 - 8\theta + 3) = 0$

$\theta^2(5\theta-3)(\theta-1) = 0$

$\theta = 0, \frac{3}{5}, 1$

$L_5''(\theta) < 0$

$\frac{d}{d\theta}(5\theta^4 - 8\theta^3 + 3\theta^2) < 0$

$20\theta^3 - 24\theta^2 + 6\theta < 0$

$L_5''(0) = 0$ $L_5''(1) > 0$

$L_5''(3/5) < 0$

$\theta = 3/5$