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ACM 157 Set 5

1. $X_1, \dots, X_n \sim U[0,\theta]$ $H_0: \theta = \frac{1}{2}$ $H_1: \theta > \frac{1}{2}$ $R = \{X : X_{(n)} > c\}$

a)

$P(\text{Type 1 error}) = P(X \in R \mid \theta \in \Theta_0) = P(X_{(n)} > c \mid \theta = \frac{1}{2})$
$P(\text{Type 2 error}) = P(X \notin R \mid \theta \in \Theta_1) = 1 - P(X_{(n)} > c \mid \theta > \frac{1}{2})$

$\beta(\theta) = P(X \in R \mid \theta) = P(X_{(n)} > c \mid \theta) = 1 - \left(\frac{c}{\theta}\right)^n$ if $\theta \in [0,\theta]$
(1 if $c < 0$, 0 if $c > \theta$)

b)

$\alpha = \sup_{\theta \in \Theta_0} \beta(\theta) = \sup_{\theta = 1/2} 1 - \left(\frac{c}{\theta}\right)^n = 1 - (2c)^n$

$c = \frac{(1-\alpha)^{1/n}}{2}$

c)

$n = 20$ $X_{(n)} = 0.48$ $R_\alpha = \{X : X_{(n)} > c\}$

$p(X) = \inf_{\alpha \in (0,1)} \{\alpha : X \in R\} = \inf_{\alpha \in (0,1)} \{\alpha : X_{(n)} > c\}$

$= \inf_{\alpha \in (0,1)} \{\alpha : X_{(n)} > \frac{(1-\alpha)^{1/n}}{2}\} \Rightarrow \frac{(1-\alpha)^{1/n}}{2} = X_{(n)} \Rightarrow \alpha = 1 - (2X_{(n)})^n$

$\alpha = 1 - (2X_{(n)})^n = 1 - (2 \cdot 0.48)^{20} = 0.558$

2.

$X_1, \dots, X_n \sim N(\mu, \sigma^2)$ $\sigma^2 = 1$ $H_0: \mu = 0$ $H_1: \mu = 1$ $R = \{X : \overline{X}_n > c\}$

a)

$\beta(\mu) = P(\overline{X}_n > c \mid \mu)$ $\overline{X}_n \sim N(\mu, \sigma^2/n)$ $\frac{\sqrt{n}(\overline{X}_n - \mu)}{\sigma} \sim N(0,1)$

$\beta(\mu) = P\left(\frac{\sqrt{n}(\overline{X}_n - \mu)}{\sigma} > \frac{\sqrt{n}(c-\mu)}{\sigma}\right) = 1 - \Phi\left(\frac{\sqrt{n}(c-\mu)}{\sigma}\right) = 1 - \Phi(\sqrt{n}(c-\mu))$

Monotonicity of β:

$\alpha = \beta(0) = 1 - \Phi(\sqrt{n}\, c)$

$c = \frac{\Phi^{-1}(1-\alpha)}{\sqrt{n}}$

b)

$\beta(1) = 1 - \Phi(\sqrt{n}(c-1)) = 1 - \Phi\left[\sqrt{n}\left(\frac{\Phi^{-1}(1-\alpha)}{\sqrt{n}} - 1\right)\right]$

$$= 1 - \phi\left(\phi^{-1}(1-\alpha) - \sqrt{n}\right)$$

c)

$$\lim_{n\to\infty} \beta(0) = \lim_{n\to\infty} \alpha = \alpha$$

$$\lim_{n\to\infty} \beta(1) = \lim_{n\to\infty} 1 - \phi\left(\phi^{-1}(1-\alpha) - \sqrt{n}\right) = 1 - \lim_{n\to\infty} \phi\left(\phi^{-1}(1-\alpha) - \sqrt{n}\right)$$

$$= 1 - \phi(-\infty) = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\infty} e^{-t^2/2} dt = 1 - 0 = 1$$

3.

$X_1, \dots, X_n \sim \text{Poisson}(\lambda) \quad H_0 : \lambda = \lambda_0 \quad H_1 : \lambda \neq \lambda_0 \quad \lambda_0 > 0$

a)

$$W = \left| \frac{\hat{\lambda} - \lambda_0}{\hat{se}} \right| > z_{1-\frac{\alpha}{2}}$$

$$L_n(\lambda | X_1, \dots, X_n) = \prod_{i=1}^{n} f(X_i ; \lambda) = \prod_{i=1}^{n} \frac{\lambda^{X_i} e^{-\lambda}}{X_i!}$$

$$\hat{\lambda}_{MLE} = \underset{\lambda}{\text{argmax}}\, L_n(\lambda | X_1, \dots, X_n) = \underset{\lambda}{\text{argmax}} \log(L_n)$$

$$\frac{d}{d\lambda} \log(L_n) = 0$$

$$\frac{d}{d\lambda} \log\left(\prod_{i=1}^{n} \frac{\lambda^{X_i} e^{-\lambda}}{X_i!}\right) = \frac{d}{d\lambda} \sum_{i=1}^{n} \log\left(\frac{\lambda^{X_i} e^{-\lambda}}{X_i!}\right) = \frac{d}{d\lambda} \sum_{i=1}^{n} \left(X_i \log(\lambda) - \lambda - \log(X_i!)\right)$$

$$= \frac{d}{d\lambda}\left(\log\lambda \sum_{i=1}^{n} X_i - \lambda n - \sum_{i=1}^{n} \log(X_i!)\right)$$

$$= \frac{1}{\lambda} \sum_{i=1}^{n} X_i - n = 0$$

$$\hat{\lambda}_{MLE} = \frac{1}{n} \sum_{i=1}^{n} X_i \qquad \frac{d^2}{d\lambda^2} \log(L_n) = -\frac{1}{\lambda^2} \sum_{i=1}^{n} X_i < 0$$

$$I(\hat{\lambda}) = \mathbb{E}\left[\left(\frac{\partial \log f(X;\lambda)}{\partial \lambda}\bigg|_{\lambda=\hat{\lambda}}\right)^2\right] = \mathbb{E}\left[\left(\frac{\partial}{\partial\lambda} \log\left(\frac{\lambda^X e^{-\lambda}}{X!}\right)\right)^2\right]$$

$$= \mathbb{E}\left[\left(\frac{\partial}{\partial\lambda}(X\log\lambda - \lambda - \log X!)\right)^2\right] = \mathbb{E}\left[\left(\frac{1}{\lambda}X - 1\right)^2\right]$$

$$= \mathbb{E}\left[\left(\frac{X}{\lambda} - \mathbb{E}\left[\frac{X}{\lambda}\right]\right)^2\right] = \text{Var}\left(\frac{X}{\lambda}\right) = \frac{\text{Var}(X)}{\lambda^2} = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$$

$$\hat{se} = \frac{1}{\sqrt{nI(\hat{\lambda})}} = \frac{1}{\sqrt{n/\hat{\lambda}}} = \sqrt{\hat{\lambda}/n}$$

$$W = \left| \frac{\frac{1}{n}\sum_{i=1}^{n} X_i - \lambda_0}{\sqrt{\frac{1}{n^2}\sum_{i=1}^{n} X_i}} \right| > z_{1-\frac{\alpha}{2}}$$