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ACM 157 Set 4

2.

$X_1, \dots, X_n \sim U[\alpha, \beta] \quad \alpha < \beta$

a)

$$m_q(\alpha,\beta) = \hat{m}_q \qquad \hat{m}_1 = \frac{1}{n}\sum_{i=1}^{n} X_i \qquad \hat{m}_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

$$m_1(\alpha,\beta) = \mathbb{E}[x] = \int_\alpha^\beta x \frac{1}{\beta-\alpha}\,dx = \left(\frac{x^2}{2}\frac{1}{\beta-\alpha}\right)\Big|_\alpha^\beta = \frac{\beta^2-\alpha^2}{2(\beta-\alpha)} = \frac{\beta+\alpha}{2}$$

$$m_2(\alpha,\beta) = \mathbb{E}[x^2] = \int_\alpha^\beta x^2 \frac{1}{\beta-\alpha}\,dx = \left(\frac{x^3}{3}\frac{1}{\beta-\alpha}\right)\Big|_\alpha^\beta = \frac{\beta^3-\alpha^3}{3(\beta-\alpha)} = \frac{\alpha^2+\alpha\beta+\beta^2}{3}$$

$$\hat{m}_1 = \frac{\alpha+\beta}{2} \qquad \hat{m}_2 = \frac{\alpha^2+\alpha\beta+\beta^2}{3}$$

$$\alpha = \hat{m}_1 - \sqrt{3(\hat{m}_2 - \hat{m}_1^2)} \qquad \beta = \hat{m}_1 + \sqrt{3(\hat{m}_2 - \hat{m}_1^2)}$$

b)

$$\mathcal{L}(\alpha,\beta \mid X_1,\dots,X_n) = \prod_{i=1}^{n}\left(\frac{1}{\beta-\alpha}\right) = \left(\frac{1}{\beta-\alpha}\right)^n$$

$$\hat{\alpha}_{MLE} = \underset{\alpha \in [X_1,\dots,X_n]}{\arg\max}\left(\frac{1}{\beta-\alpha}\right)^n = \min(X_1,\dots,X_n) = X_{(1)}$$

$$\hat{\beta}_{MLE} = \underset{\beta \in [X_1,\dots,X_n]}{\arg\max}\left(\frac{1}{\beta-\alpha}\right)^n = \max(X_1,\dots,X_n) = X_{(n)}$$

3.

$X_1, \dots, X_n \sim U[\alpha, \beta] \quad \mu = \int x\,dF(x)$

a)

$$\mu = \frac{\alpha+\beta}{2}$$

$$\hat{\mu}_{MLE} = \mu(\hat{\alpha}_{MLE}, \hat{\beta}_{MLE}) = \frac{\hat{\alpha}_{MLE} + \hat{\beta}_{MLE}}{2} = \frac{X_{(1)} + X_{(n)}}{2}$$

b)

$$\hat{\mu}_n = \overline{X}_n \qquad n=10 \quad \alpha=1 \quad \beta=3$$

$$\begin{aligned}
MSE[\hat{\mu}_n] &= \text{bias}[\hat{\mu}_n]^2 + se[\hat{\mu}_n]^2 \\
&= (\mathbb{E}[\hat{\mu}_n] - \mu)^2 + \mathbb{V}(\hat{\mu}_n)/n \\
&= (\mathbb{E}[\overline{X}_n] - \mu)^2 + \sigma^2/10 \\
&= (\mu-\mu)^2 + \tfrac{1}{10}(\mathbb{E}[x^2] - \mathbb{E}[x]^2) \\
&= 0 + \tfrac{1}{10}\left(\int_\alpha^\beta x^2 \tfrac{1}{\beta-\alpha}\,dx - \left(\int_\alpha^\beta x \tfrac{1}{\beta-\alpha}\,dx\right)^2\right) \\
&= \tfrac{1}{10}\left(\tfrac{x^3}{3}\tfrac{1}{3-1}\Big|_1^3 - \left(\tfrac{x^2}{2}\tfrac{1}{3-1}\Big|_1^3\right)^2\right) \\
&= \tfrac{1}{10}\left(\tfrac{27}{3}\tfrac{1}{2} - \tfrac{1}{3}\tfrac{1}{2} - \left(\tfrac{9}{2}\tfrac{1}{2} - \tfrac{1}{2}\tfrac{1}{2}\right)^2\right)
\end{aligned}$$

$$= \frac{1}{10}\left(\frac{13}{3} - (2)^2\right)$$

$$= \frac{1}{10} \cdot \frac{1}{3} = \frac{1}{30}$$

4. $X_1, \dots, X_n \sim N(\theta, 1)$ $\quad Y_i = \begin{cases} 1, & \text{if } X_i > 0 \\ 0, & \text{if } X_i \leq 0 \end{cases}$ $\quad \psi = \mathbb{E}[Y_1]$

a)

$$\psi = \mathbb{E}[Y_1] = \int_{-\infty}^{\infty} Y\, N(\theta,1)\, dY = \int_{-\infty}^{0} 0 \cdot N(\theta,1)\, dY + \int_{0}^{\infty} 1 \cdot N(\theta,1)\, dY$$

$$= \int_0^\infty N(\theta,1)\, dx = \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\theta)^2}{2}}\, dx$$

$\hat{\theta}_{MLE} = \overline{X_n}$ from Maximum Likelihood notes eq 12 with $\mu = \theta$

$$\hat{\psi}_{MLE} = \psi(\hat{\theta}_{MLE}) = \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\hat{\theta}_{MLE})^2}{2}}\, dx = \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\bar{X}_n)^2}{2}}\, dx$$

b)

$$I_\theta = \hat{\theta}_n \pm z_{1-\alpha/2}\, \hat{se} = \bar{X}_n \pm z_{0.975} \frac{\sigma}{\sqrt{n}} = \bar{X}_n \pm \frac{1.96}{\sqrt{n}}$$

$$I_\psi = \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-I_\theta)^2}{2}}\, dx = \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-(\bar{x}_n \pm \frac{1.96}{\sqrt{n}}))^2}{2}}\, dx$$

5.

$X_1, \dots, X_n \sim U[0,\theta]$ $\quad \hat{\theta}_{MLE} = X_{(n)}$

a)

$$CDF = P(X_{(n)} \leq x) = P(X_1 \leq x) \cdot \dots \cdot P(x_n \leq x)$$

$$= \frac{x}{\theta} \cdot \dots \cdot \frac{x}{\theta} = \left(\frac{x}{\theta}\right)^n$$

$$PDF = \frac{n x^{n-1}}{\theta^n}$$