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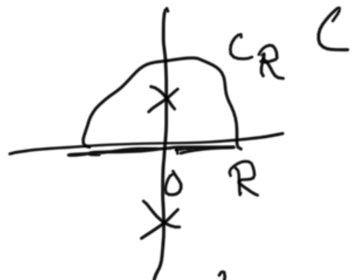
ACM 95a Set 5

1. $$I = \int_0^\infty \frac{x^3 \sin(2x)}{(x^2+1)^2}\,dx$$

$$\sin(2x) = e^{2ix}$$

$$\int_z \frac{z^3 e^{2iz}}{(z^2+1)^2}\,dz$$



$$((z+i)(z-i))^2$$

$$\int_C f(z) e^{i2z}\,dz = \int_{-R}^{R} f(x) e^{i2x}\,dx + \int_{C_R} f(z) e^{i2z}\,dz$$

$$\left|\frac{z^3}{(z^2+1)^2}\right| \le \frac{|z|^3}{|z^2+1|^2} \le \frac{R^3}{(|z|^2+1)^2} \le \frac{R^3}{(R^2+1)^2}$$

$\frac{R^3}{(R^2+1)^2} \to 0$ as $R \to \infty$ so $f(z) \to 0$ as $R \to \infty$

$$\lim_{R\to\infty} \int_{C_R} f(z) e^{i2z}\,dz = 0$$

$$\int_C \frac{z^3 e^{2iz}}{(z^2+1)^2}\,dz = 2\pi i \lim_{z\to i} \frac{d}{dz}\left((z-i)^2 \frac{z^3 e^{2iz}}{(z^2+1)^2}\right)$$

$$= 2\pi i \lim_{z\to i} \frac{d}{dz}\left(\frac{z^3 e^{2iz}}{(z+i)^2}\right)$$

$$= 2\pi i \lim_{z\to i}\left(\frac{3z^2 e^{2iz}}{(z+i)^2} + \frac{z^3 2i e^{2iz}}{(z+i)^2} + \frac{-2z^3 e^{2iz}}{(z+i)^3}\right)$$

$$= 2\pi i\left(\frac{-3e^{-2}}{-4} + \frac{2e^{-2}}{-4} + \frac{2ie^{-2}}{-8i}\right)$$

$$= 2\pi i\left(\frac{3}{4}e^{-2} - \frac{1}{2}e^{-2} - \frac{1}{4}e^{-2}\right)$$

$$= 0$$

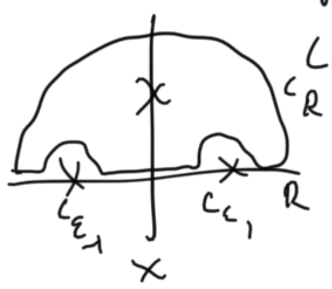
$$0 = \int_{-R}^{R} \frac{x^3 \sin(2x)}{(x^2+1)^2}\,dx + 0$$

$$0 = 2\int_0^R \frac{x^3\sin(2x)}{(x^2+1)^2}\,dx \quad \text{even function}$$

$$\int_0^R \frac{x^3\sin(2x)}{(x^2+1)^2}\,dx = 0$$

$$\int_0^\infty \frac{x^3\sin(2x)}{(x^2+1)^2}\,dx = \lim_{R\to\infty}\int_0^R \frac{x^3\sin(2x)}{(x^2+1)^2}\,dx = \lim_{R\to\infty} 0 = 0$$

2. $$I = P.V. \int_{-\infty}^{\infty} \frac{x^2}{x^4-1}\,dx = P.V.\int_{-\infty}^{0} f(x)\,dx + P.V.\int_0^\infty f(x)\,dx$$



$$\int_C f(z)\,dz = \int_{C_R} f(z)\,dz + P.V.\int_{-\infty}^{0} f(x)\,dx + P.V.\int_{\infty}^{0} f(x)\,dx$$

$$\int_C f(z)\,dz = \int_{C_R} f(z)\,dz + \int_{-R}^{-1-\varepsilon} f(x)\,dx + \int_{-1+\varepsilon}^{0} f(x)\,dx + \int_{C_{\varepsilon_{-1}}} f(z)\,dz$$

$$+ \int_0^{1-\varepsilon} f(x)\,dx + \int_{1+\varepsilon}^{R} f(x)\,dx + \int_{C_{\varepsilon_1}} f(z)\,dz$$

$$\left|\frac{x^2}{x^4-1}\right| \le \frac{|x|^2}{|x|^4-1} \le \frac{R^2}{R^4-1} \to 0 \text{ as } R\to\infty$$

so $f(z) \to 0$ as $R\to\infty$

$$\int_C \frac{z^2}{z^4-1}\,dz = 2\pi i \lim_{z\to i} \frac{z^2}{(z-1)(z+1)(z+i)}$$

$$= 2\pi i \left(\frac{-1}{(i-1)(i+1)2i} \right)$$

$$= 2\pi i \left(\frac{-1}{-4i} \right) = \frac{\pi}{2}$$

$$\int_{C_{\varepsilon_{-1}}} \frac{z^2}{z^4-1}\,dz = -\pi i \lim_{z \to -1} \frac{z^2}{(z-1)(z+i)(z-i)}$$

$$= -\pi i \left(\frac{1}{-2(-1+i)(-1-i)} \right)$$

$$= -\pi i \left(\frac{1}{-4} \right)$$

$$= \frac{1}{4}\pi i$$

$$\int_{C_{\varepsilon_1}} \frac{z^2}{z^4-1}\,dz = -\pi i \lim_{z \to 1} \frac{z^2}{(z+1)(z+i)(z-i)}$$

$$= -\pi i \left(\frac{1}{2(1+i)(1-i)} \right)$$

$$= -\pi i \left(\frac{1}{4} \right)$$

$$= -\frac{1}{4}\pi i$$

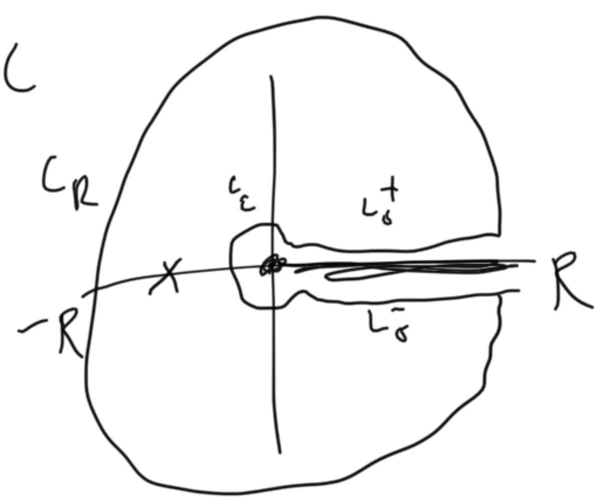
$$\frac{\pi}{2} = 0 + \int_{-R}^{-1-\varepsilon} f(x)\,dx \int_{-1+\varepsilon}^{0} f(x)\,dx + \frac{1}{4}\pi i$$

$$+ \int_{0}^{1-\varepsilon} f(x)\,dx \int_{1+\varepsilon}^{R} f(x)\,dx - \frac{1}{4}\pi i$$

$$I = \text{P.V.} \int_{-\infty}^{\infty} \frac{x^2}{z^4-1}\,dx = \frac{\pi}{2}$$

3.

$$I = \int_0^{\infty} \frac{x^{1/4}}{(x+1)^2}\,dx$$



$$= \int_0^{\infty} \frac{z^{5/4-1}}{(z+1)^2}\,dz$$

$$= \int_0^\infty \frac{r^{5/4-1} e^{i(5/4-1)\theta}}{(re^{i\theta}+1)^2}\, dr \qquad \begin{array}{l} r = |z| \\ \theta = \arg_0(z) \in [0, 2\pi) \end{array}$$

$$\int_C F(z)\,dz = \int_{C_R} F(z)\,dz + \int_{C_\varepsilon} F(z)\,dz + \int_{L_\delta^+} F(z)\,dz + \int_{L_\delta^-} F(z)\,dz$$

$$\int_C \frac{z^{1/4}}{(z+1)^2}\,dz = 2\pi i \lim_{z\to -1} \frac{d}{dz} z^{1/4} = 2\pi i \lim_{z\to -1} 1/4\, z^{-3/4}$$

$$= \frac{1}{2}\pi i \lim_{\substack{r\to 1\\ \theta\to\pi}} r^{-3/4} e^{-3/4 i\theta} = \frac{1}{2}\pi i\, 1^{-3/4} e^{-3/4 i\pi}$$

$$= \frac{1}{2}\pi i\, e^{-3/4 i\pi}$$

$$|F(z)| = \left| \frac{r^{1/4} e^{1/4 i\theta}}{(re^{i\theta}+1)^2} \right| \le \frac{R^{1/4}}{|re^{i\theta}+1|^2} \le \frac{R^{1/4}}{(R-1)^2}$$

$$\left| \int_{C_R} F(z)\,dz \right| \le \frac{R^{1/4}}{(R-1)^2} \cdot 2\pi R \to 0 \text{ as } R\to\infty$$

$$|F(z)| = \left| \frac{r^{1/4} e^{1/4 i\theta}}{(re^{i\theta}+1)^2} \right| \le \frac{\varepsilon^{1/4}}{|re^{i\theta}+1|^2} \le \frac{\varepsilon^{1/4}}{(1-\varepsilon)^2}$$

$$\left| \int_{C_\varepsilon} F(z)\,dz \right| \le \frac{\varepsilon^{1/4}}{(1-\varepsilon)^2} 2\pi\varepsilon \to 0 \text{ as } R\to\infty$$

$$L_\delta^+ \quad \delta\to 0 \quad \theta\to 0 \qquad F(z) \to \frac{r^{1/4}}{(r+1)^2} \qquad L_\delta^+ \to [\varepsilon, R]$$

$$\int_{L_\delta^+} F(z)\,dz \to \int_\varepsilon^R \frac{r^{1/4}}{(r+1)^2}\,dr \to I \quad \varepsilon\to 0 \quad R\to\infty$$

$$L_\delta^- \quad \delta\to 0 \quad \theta\to 2\pi \qquad F(z) \to \frac{r^{1/4} e^{1/4 i 2\pi}}{(re^{i2\pi}+1)^2} = \frac{r^{\alpha-1} e^{5/4 i 2\pi}}{(r+1)^2}$$

$$\int_{L_\delta^-} F(z)\,dz \to \int_R^\varepsilon \frac{r^{\alpha-1} e^{5/4 i 2\pi}}{(r+1)^2}\,dr \underset{\varepsilon\to 0 \;\; R\to\infty}{\to} -e^{5/4 i 2\pi} I \qquad L_\delta^- \to [R, \varepsilon]$$

$$\frac{1}{2}\pi i\, e^{-\frac{3}{4} i\pi} = I\left(1 - e^{\frac{5}{2} i\pi}\right)$$

$$I = \frac{\frac{1}{2}\pi i\, e^{-\frac{3}{4}i\pi}}{1 - e^{\frac{5}{2} i\pi}} = \frac{\frac{1}{2}\pi i}{e^{\frac{3}{4}\pi i} - e^{\frac{13}{4}\pi i}} = \frac{\frac{1}{2}\pi i}{e^{3/4\pi i} - e^{-3/4\pi i}}$$

$$= \frac{1}{4}\pi \frac{1}{\frac{e^{3/4\pi i} - e^{-3/4\pi i}}{2i}} = \frac{\frac{1}{4}\pi}{\sin(3/4\pi)} = \frac{\pi\sqrt{2}}{4}$$

4. $\begin{cases} ty' + 2y = t^2 - t + 1 \\ y(1) = 1/3 \end{cases}$

a) $t_0 = 1$

$$y' + \frac{2}{t}y - t + 1 - \frac{1}{t} = 0 \qquad y' + \frac{2}{t}y = t - 1 + \frac{1}{t}$$

$$r(y' + \frac{2}{t}y) = (ry)' \qquad \alpha = \frac{2}{t} \quad \beta = t - 1 + \frac{1}{t}$$

$\alpha(t), \beta(t)$ continuous $t \neq 0$

$$I = (0, \infty)$$

b)

$$r(t) = e^{\int_1^t \frac{2}{x}dx} = e^{2(\log t - \log 1)} = t^2$$

$$y(t) = t^{-2}\int_1^t x^3 - x^2 + x \, dx + \frac{1}{3}t^{-2}$$

$$= t^{-2}\left(\frac{1}{4}t^4 - \frac{1}{4} - \frac{1}{3}t^3 + \frac{1}{3} + \frac{1}{2}t^2 - \frac{1}{2}\right) + \frac{1}{3}t^{-2}$$

$$= \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} - \frac{1}{12}t^{-2}$$

c) Matlab

5. a) $y'' + 2y' + 5y = 0 \quad y(0) = 0 \quad y'(0) = 4$

$w \neq 0$

$$\alpha_{1,2} = \frac{-2 \pm \sqrt{4 - 20}}{2} = -1 \pm 2i$$

$\mu = -1 \quad \lambda = 2$

$$y(t) = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t)$$

$$y'(t) = -c_1 e^{-t}\cos(2t) - 2c_1 e^{-t}\sin(2t) - c_2 e^{-t}\sin(2t) + 2c_2 e^{-t}\cos(2t)$$

$y(0) = c_1 = 0$

$y'(0) = 2c_2 = 4 \qquad c_2 = 2$

$y(t) = 2e^{-t}\sin(2t)$

b) $y'' - 2y' + y = 0 \quad y(2) = -1 \quad y'(2) = -1 \qquad w \neq 0$

$$\alpha_{1,2} = \frac{2 \pm \sqrt{4-4}}{2} = 1, 1$$

$y(t) = c_1 e^t + c_2 t e^t$

$y'(t) = c_1 e^t + c_2 e^t + c_2 t e^t$

$y(2) = c_1 e^2 + 2c_2 e^2 = -1$

$(c_1 + 2c_2)e^2 = -1$

$y'(2) = c_1 e^2 + c_2 e^2 + 2c_2 e^2$

$(c_1 + 3c_2)e^2 = -1$

$c_2 = 0 \qquad c_1 = -e^{-2}$

$y(t) = -e^{t-2}$