Kyle McLaren

ACM 95a Set 6

1. $(1-t^2)y''-2ty'+2y=0 \qquad 0<t<1$

a) $y''-\frac{2t}{1-t^2}y'+\frac{2}{1-t^2}y=0$

$p(t)$ $q(t)$ continuous

$W(t)=C_1\cdot e^{-\int p(t)dt}$

$=C_1e^{\int\frac{2t}{1-t^2}dt} \qquad u=1-t^2$
$\qquad du=-2t\,dt$

$=C_1e^{-\int\frac{1}{u}du}$

$=C_1e^{-\log(1-t^2)}$

$=C_1\frac{1}{1-t^2}$

b) $y_1(t)=t \quad y_1'(t)=1 \quad y_1''(t)=0$

$(1)\rightarrow 0-\frac{2t}{1-t^2}+\frac{2t}{1-t^2}=0$

$\rightarrow \frac{2t}{1-t^2}=\frac{2t}{1-t^2}$ ✓

$W(t)=\det\begin{pmatrix}y_1(t) & y_2(t)\\ y_1'(t) & y_2'(t)\end{pmatrix}=\det\begin{pmatrix}t & y_2\\ 1 & y_2'\end{pmatrix}$

$=ty_2'-y_2$

$ty_2'-y_2=C_1\frac{1}{1-t^2}$

$y_2'-\frac{1}{t}y_2-C_1\frac{1}{t-t^3}=0 \qquad \alpha(t)=-\frac{1}{t} \quad \beta(t)=C_1\frac{1}{t-t^3}$

$r(t)=e^{\int_{1/2}^{t}-\frac{1}{t}dt}=e^{-\log 2t}=\frac{1}{2t}$

$y_2(t)=t\int_{1/2}^{t}\frac{1}{x}C_1\frac{1}{x-x^3}dx-C_2 2t$

$=C_1t\int_{1/2}^{t}\frac{1}{x^2-x^4}dx-C_2 2t$

$\frac{1}{t^2-t^4}=\frac{1}{t^2}+\frac{1/2}{t+1}-\frac{1/2}{t-1}$

$$Y_2(t) = c_1 t\left(\int_{1/2}^{t} \frac{1}{x^2}dx + \frac{1}{2}\int_{1/2}^{t}\frac{1}{x+1}dx - \frac{1}{2}\int_{1/2}^{t}\frac{1}{x-1}dx\right) - c_2 2t$$

$$= c_1 t\left(-\frac{1}{t} + 2 + \frac{1}{2}\log|t+1| - \log\left(\frac{3}{2}\right) - \frac{1}{2}\log|t-1| - \frac{1}{2}\log\left(\frac{1}{2}\right)\right) - c_2 2t$$

$$= \frac{1}{2}c_1 t\log(t+1) - \frac{1}{2}c_1 t\log(1-t) + c_3 t - c_1$$

2. $t^2y'' - 2y = 3t^2 - 1 \quad t \geq 1 \quad y(1) = 3 \quad y'(1) = 0$

$$y'' - \frac{2}{t^2}y = 3 - \frac{1}{t^2}$$

a) $y_1(t) = t^2 \quad t^2y'' - 2y = 0$

$y_2(t) = v(t)y_1(t) = v(t)t^2 \quad p(t) = 0 \quad y_1'(t) = 2t$

$y_1u' + (2y_1' + py_1)u = 0 \quad u = v' \quad y_2(t) = v(t)t^2$

$t^2u' + 4tu = 0$

$u' + \frac{4}{t}u = 0 \quad \alpha = \frac{4}{t} \quad \beta = 0 \quad r = e^{\int_1^t \frac{4}{x}dx}$

$u = t^{-4}\int_1^t 0\,dx + c_1t^{-4} \quad = e^{4\log t} = t^4$

$= c_1 t^{-4}$

$$v = \int c_1t^{-4}dt = c_1\int t^{-4}dt = c_2t^{-3} + c_3$$

set $c_2 = 1$ $c_3 = 0$ $\quad v(t) = t^{-3}$

$y_2(t) = t^{-3}\cdot t^2 = t^{-1}$

$y_c(t) = c_1t^2 + c_2t^{-1}$

b) $y_1(t) = t^2 \quad y_2(t) = \frac{1}{t} \quad p(t) = 0 \quad y_1' = 2t \quad y_2' = -\frac{1}{t^2}$

$h(t) = 3 - \frac{1}{t^2} \quad w(t) = \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2'\end{pmatrix} = -1 - 2 = -3$

$$y_p(t) = -t^2\int_1^t \frac{1}{x}\left(3 - \frac{1}{x^2}\right)\left(-\frac{1}{3}\right)dx + \frac{1}{t}\int_1^t x^2\left(3 - \frac{1}{x^2}\right)\left(-\frac{1}{3}\right)dx$$

$$= \frac{1}{3}t^2\left(3\int_1^t \frac{1}{x}dx - \int_1^t \frac{1}{x^3}dx\right) - \frac{1}{3t}\left(3\int_1^t x^2dx - \int_1^t 1\,dx\right)$$

$$= \frac{1}{3}t^2\left(3\log t + \frac{1}{2t^2} - \frac{1}{2}\right) - \frac{1}{3t}\left(t^3 - 1 - t + 1\right)$$

$= t^2 \log t - \frac{1}{6}t^2 + \frac{1}{6} - \frac{1}{3}t^2 + \frac{1}{3}$

$= t^2 \log t - \frac{1}{2}t^2 + \frac{1}{2}$

c) $y(t) = t^2 \log t - \frac{1}{2}t^2 + \frac{1}{2} + c_1 t^2 + c_2 \frac{1}{t}$

$y'(t) = 2t \log t + t - t + 2c_1 t - c_2 \frac{1}{t^2}$

$y(1) = 3 \quad y'(1) = 0$

$y(1) = -\frac{1}{2} + \frac{1}{2} + c_1 + c_2 = c_1 + c_2 = 3$

$y'(1) = 2c_1 - c_2 = 0$

$c_1 = 1 \quad c_2 = 2$

$y(t) = t^2 \log t - \frac{1}{2}t^2 + \frac{1}{2} + t^2 + 2\frac{1}{t}$

$= t^2 \log t + \frac{1}{2}t^2 + \frac{2}{t} + \frac{1}{2}$

3.

$y_n(t) \quad t^2 y'' - 2y = 3t^2 - 1 \quad t \geq 1 \quad y(1) = 3 \quad y'(1) = 0$

$[1, 2]$

Matlab

$y'' = \frac{2}{t^2} y + 3 - \frac{1}{t^2}$

$F = [y(2);\ \frac{2}{t^2} y(1) + 3 - \frac{1}{t^2}]$

$y0 = [3; 0]$

4.

$f(t)$ continuous $t \geq 0$ order α_0 as $t \to \infty$

$t_0 > 0 \quad \beta_0 > 0$

$|f(t)| \leq \beta_0 e^{\alpha_0 t}$ for $t \geq t_0$

$F(s) = L(f(t)) \quad \lim_{Re(s) \to \infty} F(s) = 0$

$F(s) = \int_0^\infty e^{-st} f(t)\, dt$

$\lim_{Re(s) \to \infty} F(s) = \lim_{Re(s) \to \infty} \int_0^\infty e^{-st} f(t)\, dt$

$$= \lim_{Re(s)\to\infty} \int_0^{t_0} e^{-st} f(t)\,dt + \int_{t_0}^{\infty} e^{-st} f(t)\,dt$$

$$= \lim_{Re(s)\to\infty} \int_0^{t_0} e^{-st} f(t)\,dt + \lim_{Re(s)\to\infty} \int_{t_0}^{\infty} e^{-st} f(t)\,dt$$

$$\lim_{Re(s)\to\infty} \left| \int_0^{t_0} e^{-st} f(t)\,dt \right|$$

$$\le \lim_{Re(s)\to\infty} \int_0^{t_0} |e^{-st}||f(t)|\,dt$$

$$\le \lim_{Re(s)\to\infty} \int_0^{t_0} e^{-Re(s)t} |f(t)|\,dt$$

$$\le \lim_{Re(s)\to\infty} \int_0^{t_0} e^{-Re(s)t} m\,dt$$ continuous, bounded by m on the range $(0, t_0)$

$$= \lim_{Re(s)\to\infty} m\left(\frac{e^{-Re(s)t_0}}{-Re(s)} - \frac{1}{-Re(s)} \right)$$

$$= 0$$

$$\lim_{Re(s)\to\infty} \left| \int_{t_0}^{\infty} e^{-st} f(t)\,dt \right|$$

$$\le \lim_{Re(s)\to\infty} \int_{t_0}^{\infty} |e^{-st}||f(t)|\,dt$$

$$\le \lim_{Re(s)\to\infty} \int_{t_0}^{\infty} e^{-Re(s)t} \cdot \beta_0 e^{\alpha_0 t}\,dt$$

$$= \lim_{Re(s)\to\infty} \int_{t_0}^{\infty} \beta_0 e^{-(Re(s)-\alpha_0)t}\,dt \qquad Re(s) > \alpha_0$$

$$= \lim_{Re(s)\to\infty} \beta_0 \left. \frac{e^{-(Re(s)-\alpha_0)t}}{(\alpha_0 - Re(s))} \right|_{t_0}^{\infty}$$

$$= \lim_{Re(s)\to\infty} 0 - \beta_0 \frac{e^{-(Re(s)-\alpha_0)t_0}}{(\alpha_0 - Re(s))}$$

$$= 0$$

5. $y^{(4)} - 4y''' + 6y'' - 4y' + y = 0 \quad y(0) = 0 \quad y'(0) = 1 \quad y''(0) = 0 \quad y'''(0) = 1$

$$L[y^{(4)} - 4y''' + 6y'' - 4y' + y]$$

$$= L[y^{(4)}] - 4L[y'''] + 6L[y''] - 4L[y'] + L[y]$$

$$= (s^4 L[y] - s^2 - 1) - 4(s^3 L[y] - s) + 6(s^2 L[y] - 1)$$
$$- 4sL[y] + L[y]$$

$$Y(s) = L(y)$$

$$= Y(s)(s^4 - 4s^3 + 6s^2 - 4s + 1) - s^2 + 4s - 7 = 0$$

$$Y(s) = \frac{s^2-4s+7}{s^4-4s^3+6s^2-4s+1} \qquad \mathcal{L}[t^n e^{\alpha t}] = \frac{n!}{(s-\alpha)^{n+1}}$$

$$= \frac{s^2-4s+7}{(s-1)^4}$$

$$= \frac{(s-1)^2-2s+6}{(s-1)^4}$$

$$= \frac{1}{(s-1)^2} + \frac{-2(s-1)+4}{(s-1)^4}$$

$$= \underset{n=1}{\frac{1}{(s-1)^2}} - \underset{n=2}{\frac{2}{(s-1)^3}} + \frac{4}{6}\underset{n=3}{\frac{6}{(s-1)^4}} \qquad \alpha = 1$$

$$y(t) = te^t - t^2e^t + \frac{2}{3}t^3e^t$$