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ACM 95a Set 4

1. $f(z) = \frac{1}{(z-1)^2}$ $\quad z_0 = 0$

Analytic at z_0

$$f'(z) = \frac{-2}{(z-1)^3}$$

$$f^{(n)}(z) = \frac{(-1)^n (n+1)!}{(z-1)^{(n+2)}} \qquad f^{(n)}(0) = \frac{(-1)^n (n+1)!}{(-1)^{n+2}}$$

$$f^{(n)}(0) = (-1)^{-2}(n+1)! = (n+1)!$$

$$\frac{1}{(z-1)^2} = \sum_{n=0}^{\infty} \frac{(n+1)!}{n!} z^n = \sum_{n=0}^{\infty} (n+1) z^n$$

$z_0 = 1$

$\frac{1}{0}$ is bad

$|z| < 1$ largest open disk at $z_0 = 0$ for which $\frac{1}{(z-1)^2}$ is analytic

2. $f(z)$ principal branch of $(1+z)^i$

a) $g(z) = a_0 + a_1 z + a_2 z^2$ $\quad z_0 = 0$

$$(1+z)^i = e^{i\,\text{Log}(1+z)} \qquad f^{(n)}(z) = \left(\frac{i}{1+z}\right)^n e^{i\,\text{Log}(1+z)}$$

$$f^{(n)}(0) = i^n e^{i\,\text{Log}(1)} = i^n$$

$$(1+z)^i = \sum_{n=0}^{\infty} \frac{i^n}{n!} z^n = 1 + iz - \frac{z^2}{2} + \dots$$

$a_0 = 1 \quad a_1 = i \quad a_2 = -\frac{1}{2}$

$z_0 = -1$

$e^{i\,\text{Log}(0)}$ is bad

$|z| < 1$ largest open disk at $z_0 = 0$ for which $(1+z)^i$ is analytic

b) See Matlab code

3.

a) $\sum_{n=1}^{\infty} \frac{n!}{n^n} z^n \qquad z_0 = 0 \qquad a_n = \frac{n!}{n^n}$

$$R = \lim_{n\to\infty} \left| \frac{n!}{n^n} \frac{(n+1)^{(n+1)}}{(n+1)!} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^n}{n^n} \right|$$

$$= \lim_{n\to\infty} \left| \left(\frac{n+1}{n}\right)^n \right| = \lim_{n\to\infty} \left| \left(1+\frac{1}{n}\right)^n \right| = 1$$

$|z| < 1$ is the circle of convergence

b) $\sum_{n=0}^{\infty} (n+1)^2 (z+5i)^{2n}$

$z_0 = -5i$

$(n+1)^2 (z+5i)^{2n}$

$\sum_{n=0}^{\infty} a_n (z+5i)^n \qquad a_n = \begin{cases} \left(\frac{n}{2}+1\right)^2, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$

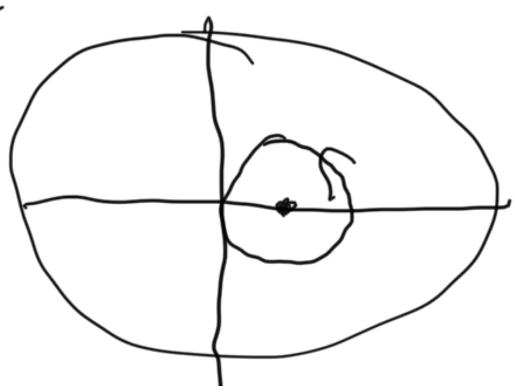
$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}} = \frac{1}{\lim_{n\to\infty} \left(\frac{n}{2}+1\right)^{\frac{2}{n}}} \qquad n \text{ even}$$

$$\lim_{n\to\infty} \left(\frac{n}{2}+1\right)^{\frac{2}{n}} = 1$$

$R = 1 \qquad |z| < 1$ is the circle of convergence

4. $f(z) = \sum_{n=0}^{\infty} n^4 \left(\frac{z}{4}\right)^n = \sum_{n=0}^{\infty} \frac{n^4}{4^n} z^n$

a) $\int_C \sin(iz) f(z)\, dz \qquad C$



$$= \sum_{n=0}^{\infty} \frac{n^4}{4^n} \int_C \sin(iz) z^n \, dz$$

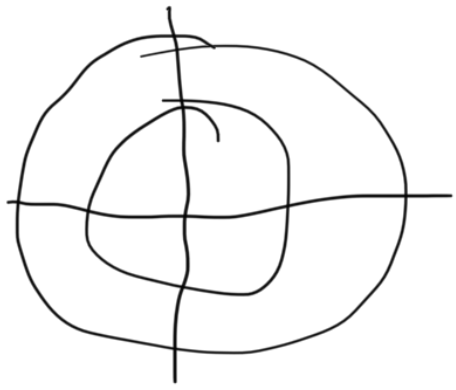
$$R = \lim_{n\to\infty} \left| \frac{n^4}{4^n} \frac{4^{n+1}}{(n+1)^4} \right| = \lim_{n\to\infty} \left| 4 \left(\frac{n}{n+1}\right)^4 \right| = 4$$

$$\int_C \sin(iz) z^n \, dz = 0$$

analytic on and inside C

$$\sum_{n=0}^{\infty} \frac{n^4}{4^n} \cdot 0 = 0$$

b) $\int_C \frac{f(z)}{z^3} \, dz$ $\quad C \; |z| = \pi$



$$= \sum_{n=0}^{\infty} \frac{n^4}{4^n} \int_C \frac{z^n}{z^3} \, dz$$

z^n analytic

GCIF

$$\int_C \frac{z^n}{z^3} \, dz = \frac{2\pi i}{2} f^{(2)}(0) = 0$$

$$f^{(2)}(n) = \frac{d^2}{dz^2} z^n = n(n-1) z^{n-2}$$

$$f^{(2)}(0) = 0$$

$$\sum_{n=0}^{\infty} \frac{n^4}{4^n} 0 = 0$$

5. $f(z) = \frac{1}{z(1-z)}$ $\quad$ not analytic at $z = 0, 1$

a) $0 < |z| < 1$ $\quad z_0 = 0$ $\quad |z| < 1$

$$f(z) = \frac{1}{z} + \frac{1}{1-z} \qquad \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

$$f(z) = \frac{1}{z} + (1 + z + z^2 + \dots) = \sum_{n=-\infty}^{\infty} a_n z^n$$

$$a_n = \begin{cases} 1, & n \geq -1 \\ 0, & n < -1 \end{cases}$$

b) $|z| > 1$ $\quad z_0 = 0$

$|1/z|<1$

$$f(z)=\frac{1}{z}-\frac{1}{z-1} \qquad \frac{1}{z}\frac{1}{1-1/z}=\frac{1}{z}\sum_{n=0}^{\infty}(1/z)^n$$

$$f(z)=\frac{1}{z}-\left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right)$$

$$=-\frac{1}{z^2}-\frac{1}{z^3}-\cdots=\sum_{n=-\infty}^{\infty}a_n z^n$$

$$a_n=\begin{cases}0, & n\geq -1\\ -1, & n<-1\end{cases}$$

c) $|z-1|>1 \quad z_0=1$

$$f(z)=\frac{1}{z}-\frac{1}{z-1} \qquad \frac{1}{z}=\frac{1}{z-1}\frac{1}{1-\frac{-1}{z-1}} \qquad \left|\frac{-1}{z-1}\right|<1$$

$$=\frac{1}{z-1}\sum_{n=0}^{\infty}\left(\frac{-1}{z-1}\right)^n$$

$$f(z)=-\frac{1}{z-1}+\frac{1}{z-1}-\frac{1}{(z-1)^2}+\frac{1}{(z-1)^3}-\frac{1}{(z-1)^4}+\cdots$$

$$f(z)=\sum_{n=0}^{\infty}a_n(z-1)^n \qquad a_n=\begin{cases}0, & n\geq -1\\ (-1)^{n+1}, & n<-1\end{cases}$$

6. $\hat{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$

a) $f_1(z)=\frac{z}{z^2+1}$

$z_0=\pm i$ are isolated singularities

$$f_1(z)=\frac{z}{(z-i)(z+i)}=\frac{1}{2}\left(\frac{1}{z-i}+\frac{1}{z+i}\right)$$

$$\frac{1}{z+i}=\frac{1}{z-i}\frac{1}{1-\frac{-2i}{z-i}}=\frac{1}{z-i}\sum_{n=0}^{\infty}\left(\frac{-2i}{z-i}\right)^n$$

$$f_1(z)=\frac{1}{2}\frac{1}{z-i}+\frac{1}{2}\frac{1}{z-i}-\frac{i}{(z-i)^2}-\frac{2}{(z-i)^3}+\frac{4i}{(z-i)^4}$$

$z_0=\pm i$ are essential singularities

b) $f_2(z)=z\sin\left(\frac{1}{z}\right)$

$$f_2(z)=z\left(\frac{1}{z}-\frac{1}{z^3 3!}+\frac{1}{z^5 5!}-\cdots\right)$$

$$= 1 - \frac{1}{3!\, z^2} + \frac{1}{5!\, z^4} - \cdots$$

$z_0 = 0$ is an essential isolated singularity