Kyle McGraw

ACM 95a Set 3

1. $$I(R) = \int_{C_R} \frac{z + \operatorname{Log}(z)}{z^3+1}\, dz \qquad \phi \ \boxed{\text{continuous}}$$

$z = -1$ not in C_R for any R

$C_R: |z| = R \quad -\frac{\pi}{2} \le \operatorname{Arg}(z) \le \frac{\pi}{2}$

a) $|I(R)| \le \alpha(R) \quad \lim_{R\to\infty} = 0$

$L = R\pi$

Triangle inequality

$$\left|\frac{z+\operatorname{Log}(z)}{z^3+1}\right| = \frac{|z+\operatorname{Log}(z)|}{|z^3+1|} \le \frac{|z+\operatorname{Log}(z)|}{|R^3-1|}$$

$$= \frac{|z + \operatorname{Log}|z| + i\operatorname{Arg}(z)|}{|R^3-1|} = \frac{|z + \operatorname{Log} R + i\operatorname{Arg}(z)|}{|R^3-1|}$$

$$\le \frac{|z| + |\operatorname{Log} R| + |i\operatorname{Arg}(z)|}{|R^3-1|} \le \frac{R + |\operatorname{Log} R| + \pi}{|R^3-1|} = M$$

$$|I(R)| \le ML = \frac{R+|\operatorname{Log} R| + \pi}{|R^3-1|}\cdot R\pi = \alpha(R)$$

$$\lim_{R\to\infty} \frac{R+|\operatorname{Log} R|+\pi}{|R^3-1|}\cdot R\pi = \lim_{R\to\infty} \frac{R^2\pi}{|R^3-1|} + \frac{|\operatorname{Log} R|R\pi}{|R^3-1|} + \frac{R\pi^2}{|R^3-1|}$$

$$= \lim_{R\to\infty} 0 + \frac{|\operatorname{Log} R|R\pi}{|R^3-1|} + 0 \qquad |\operatorname{Log} R| \le R$$

$$\le \lim_{R\to\infty} \frac{R^2\pi}{|R^3-1|} = 0$$

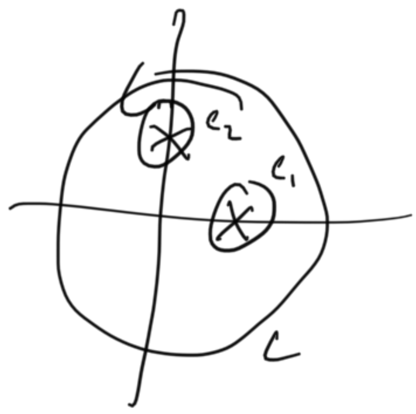
$$\lim_{R\to\infty} |I(R)| \le \lim_{R\to\infty} ML = 0 \qquad \lim_{R\to\infty} |I(R)| = 0$$

b) see Matlab code

$$C_R = \{R\cos(t) + Ri\sin(t),\ -\tfrac{\pi}{2} \le t \le \tfrac{\pi}{2}\}$$

2.

a) $\displaystyle\int_C \frac{\cos z}{(z-i)(z-1)}\,dz \qquad |z| = \pi$



$$\int_C \frac{\cos z}{(z-i)(z-1)}\,dz = \int_{C_1} \frac{\overbrace{\cos z / z-i}^{\text{analytic on and inside } c_1}}{z-1} + \int_{C_2} \frac{\overbrace{\cos z / z-1}^{\text{analytic on and inside } c_2}}{z-i}$$

CG for multiply connected domains

Now, we can use Cauchy Integral Formula

$$= 2\pi i \frac{\cos 1}{1-i} + 2\pi i \frac{\cos i}{i-1}$$

$$= \frac{2\pi i}{1-i} + 0 = \frac{2\pi i}{1-i}$$

b) $\displaystyle\int_C \bar{z}\,dz \qquad z=0 \quad z=2i$

$$\int_C f(z)\,dz = \int_a^b f(z(t))\, z'(t)\,dt$$

$$\int_C \bar{z}\,dz = \int_a^b (u(t) - iv(t))\underbrace{(u'(t) + iv'(t))}_{\shortparallel}\,dt = i\int_a^b u(t)\,dt + \underbrace{\int_a^b v(t)\,dt}_{(-i^2)=1}$$

$0 \to 2i$ paramaterize to take anti-derivative

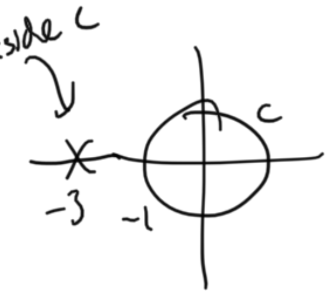
$u(t) = 0$

$v(t) = t \qquad a=0$

$b = 2$

$$= \int_0^2 0\,dt + \int_0^2 t\,dt$$

$$= \int_0^2 t\,dt = \frac{t^2}{2}\Big|_0^2 = 2$$

c) $\displaystyle\int_C \frac{e^{z^3+\pi}}{(z+3)^3}\,dz$ 

$|z|=1$

analytic on and inside C
Simple closed contour

$$\int_C \frac{e^{z^3+\pi}}{(z+3)^3} = 0$$

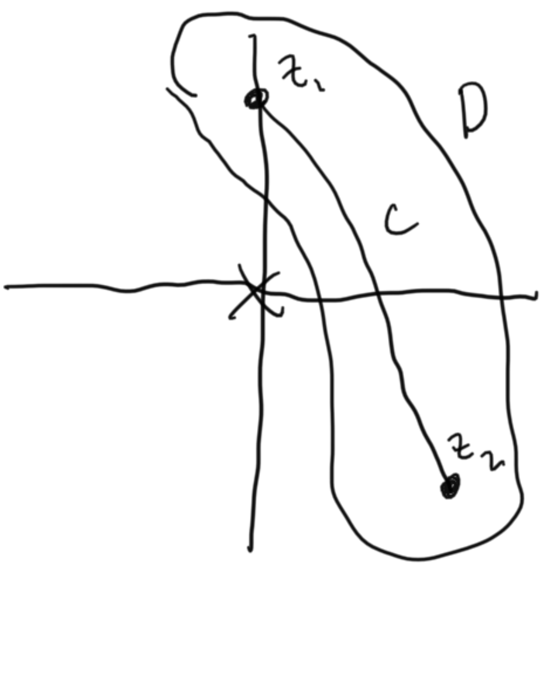
Cauchy-Goursat Theorem

d) $\displaystyle\int_C z^{-1/3}\,dz$ principal branch

$z(t) = t + i\cos(\pi t) \quad 0 \le t \le 1$

$z_1 = i \quad z_2 = 1 - i$

Antiderivative, fun. theorem integrals



$z^{-1/3}$ continuous in D
D does not contain 0

$F(z) = \frac{3}{2} z^{2/3}$

$F'(z) = f(z)$

$\frac{3}{2} z^{2/3} \, dz = z^{-1/3}$

$$\int_C z^{-1/3} = F(z_1) - F(z_2) = \frac{3}{2}(1-i)^{2/3} - \frac{3}{2} i^{2/3}$$

$$= \frac{3}{2}\left(e^{\frac{2}{3}(\mathrm{Log}|1-i| + i\,\mathrm{Arg}(1-i))} - e^{\frac{2}{3}(\mathrm{Log}|i| + i\,\mathrm{Arg}(i))}\right)$$

$$= \frac{3}{2}\left(e^{\frac{2}{3}\mathrm{Log}\sqrt{2}}\, e^{-\frac{2}{3} i \frac{\pi}{4}} - e^{\frac{2}{3}\mathrm{Log}\,1}\, e^{\frac{3}{3} i \frac{\pi}{2}}\right)$$

$$= \frac{3}{2}\Big(e^{\frac{2}{3}\log\sqrt{2}}\left(\cos\left(-\tfrac{1}{6}\pi\right)+i\sin\left(-\tfrac{1}{6}\pi\right)\right)$$
$$- e^{0}\left(\cos\left(\tfrac{1}{3}\pi\right)+i\sin\left(\tfrac{1}{3}\pi\right)\right)\Big)$$

$$= \frac{3}{2}\left(2^{1/3}\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)-\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\right)$$

$$= \frac{3}{2}\cdot\frac{2^{1/3}\sqrt{3}-1}{2} - \frac{3}{2}\,\frac{2^{1/3}+\sqrt{3}}{2}\,i$$

$$\approx 0.887 - 2.244i$$

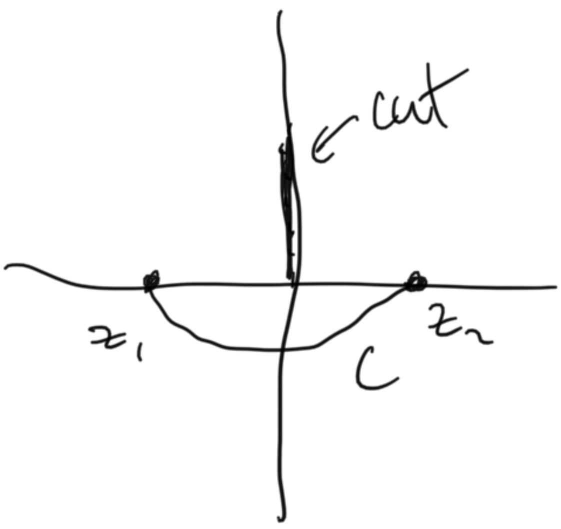
3.

$$\int_C f_1(z)\,dz \neq F_1(z_2)-F_1(z_1)$$

$$\int_C f_1(z)\,dz = \int_C f_2(z)\,dz = F_2(z_2)-F_2(z_1)$$

$f(z) = z^i$ $\quad f_1(z)$ principal branch

$C:\ z_1 = -1 \to z_2 = 1$ below real axis



$$f_2(z) = z^i_{\frac{\pi}{2}}$$

$$\int_C f_1(z) = \int_C f_2(z) = F_2(z_2)-F_2(z_1)$$

$$F_2(z) = \frac{1}{1+i}\,z^{1+i}_{\frac{\pi}{2}}$$
$$F_2'(z) = z^i_{\frac{\pi}{2}}$$

$\operatorname{Log}|-1| = \log|1| = \log 1 = 0$

$\arg_{\frac{\pi}{2}}(-1) = \pi \quad \arg_{\frac{\pi}{2}}(1) = 2\pi$

$$\frac{1}{1+i}(1)^{1+i}_{\frac{\pi}{2}} - \frac{1}{1+i}(-1)^{1+i}_{\frac{\pi}{2}} = \frac{1}{1+i}\left(e^{(1+i)(\operatorname{Log}|1|+i\arg_{\frac{\pi}{2}}(1))} - e^{(1+i)(\operatorname{Log}|-1|+i\arg_{\frac{\pi}{2}}(-1))}\right)$$

$$= \frac{1}{1+i}\left(e^{i2\pi}e^{-2\pi} - e^{i\pi}e^{-\pi}\right)$$

$= \frac{1}{1+i}\left(e^{-2\pi} + e^{-\pi} \right)$ $\qquad \frac{1}{1+i}\cdot\frac{1-i}{1-i} = \frac{1-i}{2}$

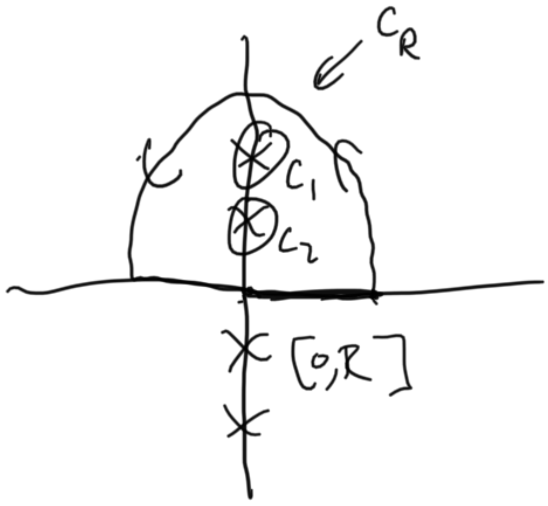
$= \frac{1}{2}\left(e^{-2\pi}+e^{-\pi}\right) - \frac{1}{2}\left(e^{-2\pi}+e^{-\pi}\right)i$

$\approx 0.0225 - 0.0225i$

4. $$I = \int_0^\infty \frac{(x^6-8)}{(x^4+5x^2+4)^2}\,dx$$

$$\int_0^\infty \frac{x^6-8}{\left((x^2+1)(x^2+4)\right)^2}\,dx$$

$x = i, -i, 2i, -2i$



$C = C_R + [0,R]$

$$\int_C \frac{z^6-8}{\left((z^2+1)(z^2+4)\right)^2}dz = \int_0^R \frac{x^6-8}{\left((x^2+1)(x^2+4)\right)^2}dz + \int_{C_R} \frac{z^6-8}{\left((z^2+1)(z^2+4)\right)^2}dz$$

$$\int_C \frac{z^6-8}{\left((z^2+1)(z^2+4)\right)^2}dz = \int_C \frac{z^6-8}{\left((z+i)(z-i)(z+2i)(z-2i)\right)^2}dz$$

$$= \int_{C_1} \frac{z^6-8}{(z+i)^2(z+2i)^2(z-2i)^2}\,\frac{dz}{(z-i)^2} + \int_{C_2} \frac{z^6-8}{(z+i)^2(z-i)^2(z+2i)^2}\,\frac{dz}{(z-2i)^2}$$

$z_0 = i$ $\qquad\qquad$ $z_0 = 2i$

GCIF $\quad$ analytic on and inside C_1, C_2

$$= 2\pi i \frac{d}{dz}\left(\frac{z^6-8}{(z+i)^2\,(z^2+4)^2}\right)\Bigg|_{z=i}$$

$$+ 2\pi i \frac{d}{dz}\left(\frac{z^6-8}{(z^2+1)^2\,(z+2i)^2}\right)\Bigg|_{z=2i}$$

by Wolfram

$$= 2\pi i \cdot \left(-\frac{i}{4}\right) + 2\pi i \cdot \frac{i}{4}$$

$$= \frac{\pi}{2} - \frac{\pi}{2} = 0$$

$$I = 0 - \lim_{R \to \infty} \int_{C_R} \frac{z^6 - 8}{((z^2+1)(z^2+4))^2} \, dz$$

$$|z^6 - 8| \leq ||z|^6 - 8| = |R^6 - 8| = R^6 - 8 \quad R \geq \sqrt{2}$$

$$|((z^2+1)(z^2+4))^2| = |(z^2+1)(z^2+4)|^2 \geq |(|z|^2-1)(|z|^2-4)|^2$$

$$= |(R^2-1)(R^2-4)|^2 = ((R^2-1)(R^2-4))^2$$

$$L = \pi R + R = (\pi+1)R$$

$$\left| \int_{C_R} \frac{z^6 - 8}{((z^2+1)(z^2+4))^2} \, dz \right| \leq \frac{R^6 - 8}{((R^2+1)(R^2+4))^2} \cdot (\pi+1)R \approx \frac{R^7}{R^8} \to 0$$

as $R \to \infty$

$$I = 0 - 0 = 0$$