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ACM 95A Set 1

1. $z = x + iy$

$$\begin{aligned} a1) \frac{z + \bar{z}}{1 + \bar{z}} &= \frac{x + iy + x - iy}{1 + x + iy} = \frac{2x}{(x+1) + iy} \cdot \frac{(x+1) - iy}{(x+1) - iy} \\ &= \frac{2x(x+1 - iy)}{(x+1)^2 + y^2} = \frac{2x^2 + 2x}{(x+1)^2 + y^2} - i \frac{2xy}{(x+1)^2 + y^2} \end{aligned}$$

q2) $(1+i)^{2021}$

$$\begin{aligned} (1+i)^2 &= 1 + 2i - 1 = 2i \\ (1+i)^3 &= 2i - 2 \\ (1+i)^4 &= -4 \\ (1+i)^5 &= -4 - 4i \\ (1+i)^6 &= -8i \\ (1+i)^7 &= -8i + 8 \\ (1+i)^8 &= 16 \end{aligned}$$

$$1+i$$

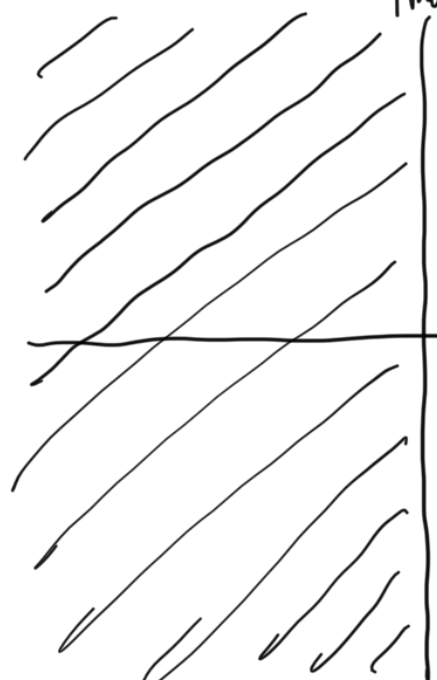
$$\begin{aligned} (1+i)^{2020} &= (1+i)^4^{505} \\ &= (-4)^{505} = -4^{505} \\ &= -2^{1010} \end{aligned}$$

$$\begin{aligned} (1+i)^{2021} &= (1+i)(1+i)^{2020} \\ &= (1+i)(-2^{1010}) \\ &= -2^{1010} - 2^{1010}i \end{aligned}$$

$$b) \left| \frac{z+1}{z-1} \right| < 1 \Rightarrow \frac{|z+1|}{|z-1|} < 1$$

$$\frac{|(x+1) + iy|}{|(x-1) + iy|} < 1 \Rightarrow |(x+1) + iy| < |(x-1) + iy|$$

imaginary



Since y is equal, we only care about the x parts

$$\Rightarrow |x+1| < |x-1|$$

but This is only true

when $x < 0$

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$$a) z_0, z_1 = f(z_0), z_n = f(z_{n-1})$$

$$O_f(z_0)$$

$$a) f(z) = z^2 \quad |z_1 z_2| = |z_1| |z_2|$$

$|z_0| < 1$ Since $|z_0|$ is less than 1, squaring itself will cause it to always decrease in magnitude. This means that $O_f(z_0)$ is bounded by $|z_0|$ and converges to 0.

$|z_0| > 1$ Squaring itself will cause it to always increase in magnitude, so $O_f(z_0)$ will converge to infinity and is unbounded.

$|z_0| = 1$ Squaring itself will keep the magnitude equal, so the magnitude of all points in the orbit will be 1 and the orbit will lie on the unit circle

$$3. a) f(z_0) = g(z_0) = 0 \quad f(z), g(z) \text{ analytic} \\ g'(z_0) \neq 0$$

Taylor expansion around z_0

$$f(z) = f'(z_0)(z - z_0) + \frac{f''(z_0)}{2}(z - z_0)^2 + \dots$$

$$g(z) = g'(z_0)(z - z_0) + \frac{g''(z_0)}{2}(z - z_0)^2 + \dots$$

$$\begin{aligned} \lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} &= \lim_{z \rightarrow z_0} \frac{f'(z_0)(z - z_0) + \frac{f''(z_0)}{2}(z - z_0)^2 + \dots}{g'(z_0)(z - z_0) + \frac{g''(z_0)}{2}(z - z_0)^2 + \dots} \\ &= \lim_{z \rightarrow z_0} \frac{f'(z_0) + \frac{f''(z_0)}{2}(z - z_0) + \dots}{g'(z_0) + \frac{g''(z_0)}{2}(z - z_0) + \dots} \\ &= \frac{f'(z_0)}{g'(z_0)} \end{aligned}$$

$$b) \lim_{z \rightarrow i} \frac{1 + e^{\pi z}}{1 + z^{10}}$$

$$e^{\pi i} = z^{10} = -1$$

$$1 + e^{\pi i} = 1 + z^{10} = 0 \quad \text{Both analytic}$$

$$\lim_{z \rightarrow i} \frac{1 + e^{\pi z}}{1 + z^{10}} = \frac{\pi e^{\pi i}}{10 i^9} = \frac{-\pi}{10 i} = \frac{1}{10} \pi i$$

4.

$$f(z) = u(x, y) + i v(x, y) \quad z = x + iy$$

$$u(x, y) = 3x^2 y^2 \quad v(x, y) = -6x^2 y^2$$

$$u_x = 6xy^2$$

$$v_y = -12x^2 y$$

$$u_y = 6x^2 y$$

$$v_x = -12xy^2$$

continuous everywhere

$$6xy^2 = -12x^2 y$$

$$6x^2 y = 12xy^2$$

$$x = -\frac{1}{2} y$$

$$x = 2y$$

a) differentiable at $x = y = 0$

b) not analytic anywhere

$$5. \quad f(z) = \begin{cases} \frac{\bar{z}^2}{z}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

$$a) \quad \frac{\bar{z}^2}{z} = \frac{(x - yi)^2}{x + yi} = \frac{(x - yi)^3}{x^2 + y^2}$$

$$= \frac{x^3 - 3x^2 yi - 3xy^2 + y^3 i}{x^2 + y^2}$$

$$= \frac{x^3 - 3xy^2}{x^2 + y^2} + \frac{y^3 - 3x^2 y}{x^2 + y^2} i$$

\parallel \parallel
 u v

$$u_x = \frac{3x^2y^2 + 2xy^2 - 3y^4}{(x^2 + y^2)^2}$$

$$u_y = \frac{-6x^3y - 2x^2y}{(x^2 + y^2)^2}$$

$$v_y = \frac{-3x^4 + y^4 + 6x^2y^2}{(x^2 + y^2)^2}$$

$$v_x = \frac{-8xy^3}{(x^2 + y^2)^2}$$

$$u_x = v_y$$

$$\frac{3x^2y^2 + 2xy^2 - 3y^4}{(x^2 + y^2)^2} = \frac{-3x^4 + y^4 + 6x^2y^2}{(x^2 + y^2)^2}$$

$$3x^2y^2 + 2xy^2 - 3y^4 = -3x^4 + y^4 + 6x^2y^2$$

$$0 = 0$$

$$u_y = -v_x \quad \frac{-6x^3y - 2x^2y}{(x^2 + y^2)^2} = \frac{8xy^3}{(x^2 + y^2)^2}$$

$$-6x^3y - 2x^2y = 8xy^3$$

$$0 = 0$$

$$b) \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$\lim_{\Delta z \rightarrow 0} \frac{\frac{(z + \Delta z)^2}{z + \Delta z} - \frac{z^2}{z}}{\Delta z}$$

$$\lim_{\Delta z \rightarrow 0} \frac{\frac{z^2 + 2z\Delta z + \Delta z^2}{z + \Delta z} - \frac{z^2}{z}}{\Delta z}$$

$$\Delta y = 0 \quad \lim_{\Delta x \rightarrow 0} \frac{\frac{z^2 + 2z\Delta x + \Delta x^2}{\Delta x z + \Delta x^2} - \frac{z^2}{z\Delta x}}{\Delta x}$$

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta x^2}{\Delta x^2} = 1$$

$$\Delta y = \Delta x \quad \lim_{\Delta x \rightarrow 0} \frac{\bar{z}^2 + 2\bar{z}(\Delta x - i\Delta x) + (\Delta x - i\Delta x)^2}{z(\Delta x + i\Delta x) + (\Delta x + i\Delta x)^2} - \frac{\bar{z}}{z(\Delta x + i\Delta x)}$$

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta x^2 - 2i\Delta x^2 - \Delta x^2}{\Delta x^2 + 2i\Delta x^2 - \Delta x^2} = -1$$

limits are different from
different directions, so
not differentiable

```
x=linspace(-2,1,10^3);
y=linspace(-1.5,1.5,10^3);
[X,Y]=meshgrid(x,y);
k = zeros(10^3, 10^3);
for a = 1:10^3
    for b = 1:10^3
        z = 0;
        for n = 1:500
            z = z^2 + X(a, b) + Y(a, b) * 1i;
            k(a, b) = n;
            if abs(z) > 2
                break;
            end
        end
    end
end
figure;
imagesc(x, y, k);
axis equal;
axis([-2, 1, -1.5, 1.5]);

x=linspace(-0.748766713922161, -0.748766707771757,10^3);
y=linspace(0.123640844894862, 0.123640851045266,10^3);
[X,Y]=meshgrid(x,y);
k = zeros(10^3, 10^3);
for a = 1:10^3
    for b = 1:10^3
        z = 0;
        for n = 1:500
            z = z^2 + X(a, b) + Y(a, b) * 1i;
            k(a, b) = n;
            if abs(z) > 2
                break;
            end
        end
    end
end
figure;
imagesc(x, y, k);
axis equal;
axis([-0.748766713922161, -0.748766707771757, 0.123640844894862,
    0.123640851045266]);
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