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ACM 95A Set 1

1. $z = x + iy$

a1) $$\frac{z+\bar{z}}{1+\bar{z}} = \frac{x+iy+x-iy}{1+x+iy} = \frac{2x}{(x+1)+iy} \cdot \frac{(x+1)-iy}{(x+1)-iy}$$

$$= \frac{2x(x+1-iy)}{(x+1)^2+y^2} = \frac{2x^2+2x}{(x+1)^2+y^2} - i\frac{2xy}{(x+1)^2+y^2}$$

a2) $(1+i)^{2021}$

$1+i$

$(1+i)^2 = 1+2i-1 = 2i$

$(1+i)^3 = 2i-2$

$(1+i)^4 = -4$

$(1+i)^5 = -4-4i$

$(1+i)^6 = -8i$

$(1+i)^7 = -8i+8$

$(1+i)^8 = 16$

$(1+i)^{2020} = (1+i)^{4^{505}}$

$= (-4)^{505} = -4^{505}$

$= -2^{1010}$

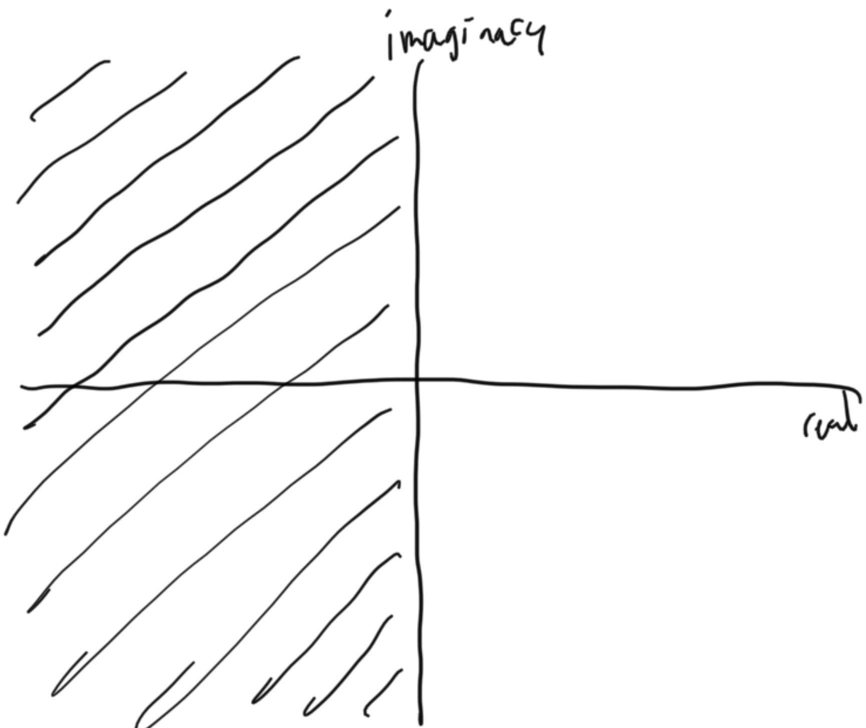
$(1+i)^{2021} = (1+i)(1+i)^{2020}$

$= (1+i)(-2^{1010})$

$= -2^{1010} - 2^{1010}i$

b) $$\left|\frac{z+1}{z-1}\right| < 1 \Rightarrow \frac{|z+1|}{|z-1|} < 1$$

$$\frac{|(x+1)+iy|}{|(x-1)+iy|} < 1 \Rightarrow |(x+1)+iy| < |(x-1)+iy|$$



Since y is equal, we only care about the x parts

$\Rightarrow |x+1| < |x-1|$

This is only true when $x < 0$

2.

a) $z_0, z_1 = f(z_0), z_n = f(z_{n-1})$

$O_f(z_0)$

a) $f(z)=z^2$ $\quad |z_1 z_2| = |z_1||z_2|$

$|z_0|<1$ Since $|z_0|$ is less than 1, squaring itself will cause it to always decrease in magnitude. This means that $O_f(z_0)$ is bounded by $|z_0|$ and converges to 0.

$|z_0|>1$ Squaring itself will cause it to always increase in magnitude, so $O_f(z_0)$ will converge to infinity and is unbounded.

$|z_0|=1$ Squaring itself will keep the magnitude equal, so the magnitude of all points in the orbit will be 1 and the orbit will lie on the unit circle

3. a) $f(z_0)=g(z_0)=0$ $\quad f(z), g(z)$ analytic
$g'(z_0)\neq 0$

Taylor expansion around z_0

$$f(z) = f'(z_0)(z-z_0) + \frac{f''(z_0)}{2}(z-z_0)^2 + \cdots$$
$$g(z) = g'(z_0)(z-z_0) + \frac{g''(z_0)}{2}(z-z_0)^2 + \cdots$$

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \lim_{z\to z_0} \frac{f'(z_0)(z-z_0) + \frac{f''(z_0)}{2}(z-z_0)^2 + \cdots}{g'(z_0)(z-z_0) + \frac{g''(z_0)}{2}(z-z_0)^2 + \cdots}$$

$$= \lim_{z\to z_0} \frac{f'(z_0) + \frac{f''(z_0)}{2}(z-z_0) + \cdots}{g'(z_0) + \frac{g''(z_0)}{2}(z-z_0) + \cdots}$$

$$= \frac{f'(z_0)}{g'(z_0)}$$

b) $\lim_{z\to i} \frac{1+e^{\pi z}}{1+z^{10}}$ $\quad e^{\pi i} = z^{10} = -1$

$1+e^{\pi i} = 1+z^{10} = 0$ Both analytic

$$\lim_{z\to i} \frac{1+e^{\pi z}}{1+z^{10}} = \frac{\pi e^{\pi i}}{10 i^9} = \frac{-\pi}{10i} = \frac{1}{10}\pi i$$

4.

$f(z) = u(x,y) + i\,v(x,y) \quad z = x+iy$

$u(x,y) = 3x^2y^2 \quad v(x,y) = -6x^2y^2$

$u_x = 6xy^2 \qquad v_y = -12x^2y$

$u_y = 6x^2y \qquad v_x = -12xy^2$

continuous everywhere

$6xy^2 = -12x^2y \qquad 6x^2y = 12xy^2$

$x = -\frac{1}{2}y \qquad x = 2y$

a) differentiable at $x=y=0$

b) not analytic anywhere

5. $f(z) = \begin{cases} \frac{\bar{z}^2}{z}, & z \neq 0 \\ 0, & z = 0 \end{cases}$

a) $\frac{\bar{z}^2}{z} = \frac{(x-yi)^2}{x+yi} = \frac{(x-yi)^3}{x^2+y^2}$

$= \frac{x^3 - 3x^2yi - 3xy^2 + y^3 i}{x^2+y^2}$

$= \underbrace{\frac{x^3-3xy^2}{x^2+y^2}}_{u} + \underbrace{\frac{y^3-3x^2y}{x^2+y^2}}_{v} i$

$$u_x = \frac{3x^2y^2 + 2xy^2 - 3y^4}{(x^2+y^2)^2}$$

$$u_y = \frac{-6x^3y - 2x^2y}{(x^2+y^2)^2}$$

$$v_y = \frac{-3x^4 + y^4 + 6x^2y^2}{(x^2+y^2)^2}$$

$$v_x = \frac{-8xy^3}{(x^2+y^2)^2}$$

$$u_x = v_y$$

$$\frac{3x^2y^2 + 2xy^2 - 3y^4}{(x^2+y^2)^2} = \frac{-3x^4 + y^4 + 6x^2y^2}{(x^2+y^2)^2}$$

$$3x^2y^2 + 2xy^2 - 3y^4 = -3x^4 + y^4 + 6x^2y^2$$

$$0 = 0$$

$$u_y = -v_x \qquad \frac{-6x^3y - 2x^2y}{(x^2+y^2)^2} = \frac{8xy^3}{(x^2+y^2)^2}$$

$$-6x^3y - 2x^2y = 8xy^3$$

$$0 = 0$$

b) $$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$\lim_{\Delta z \to 0} \frac{\frac{(\overline{z+\Delta z})^2}{z+\Delta z} - \frac{\bar{z}^2}{z}}{\Delta z}$$

$$\lim_{\Delta z \to 0} \frac{\frac{\bar{z}^2 + 2\bar{z}\overline{\Delta z} + \overline{\Delta z}^2}{z+\Delta z} - \frac{\bar{z}^2}{z}}{\Delta z}$$

$$\Delta y = 0 \quad \lim_{\Delta x \to 0} \frac{\bar{z}^2 + 2\bar{z}\Delta x + \Delta x^2}{\Delta x z + \Delta x^2} - \frac{\bar{z}^2}{z\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\Delta x^2}{\Delta x^2} = 1$$

$$\Delta y = \Delta x \quad \lim_{\Delta x \to 0} \frac{\bar{z}^2 + 2\bar{z}(\Delta x - i\Delta x) + (\Delta x - i\Delta x)^2}{z(\Delta x + i\Delta x) + (\Delta x + i\Delta x)^2} - \frac{\bar{z}^2}{z(\Delta x + i\Delta x)}$$

$$\lim_{\Delta x \to 0} \frac{\Delta x^2 - 2i\Delta x^2 - \Delta x^2}{\Delta x^2 + 2i\Delta x^2 - \Delta x^2} = -1$$

limits are different from different directions, so not differentiable