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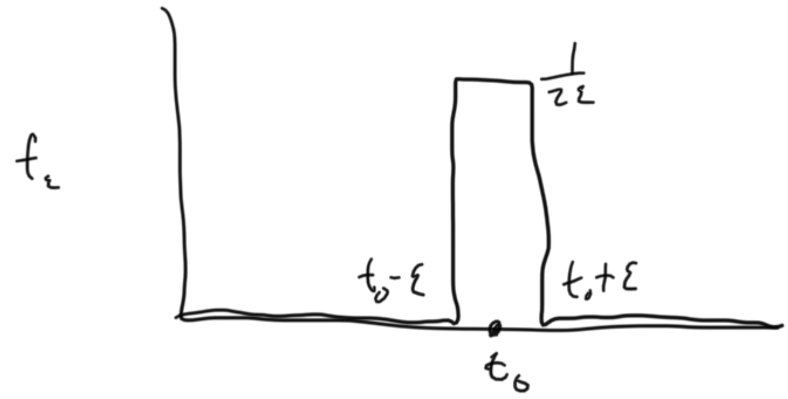
ACM 95a Set 7

1. $y'' + y = f_\varepsilon(t) \quad t \geq 0 \quad y(0) = 0 \quad y'(0) = 0$

$$f_\varepsilon(t) = \frac{H(t-(t_0-\varepsilon)) - H(t-(t_0+\varepsilon))}{2\varepsilon} \quad t_0 > 0 \quad \varepsilon > 0$$

$\lim_{\varepsilon \to 0} f_\varepsilon(t) = \delta(t-t_0)$



a)
$$L(y''+y) = Y(s)(s^2+1)$$

$$L(f_\varepsilon(t)) = L\left(\frac{H(t-(t_0-\varepsilon)) - H(t-(t_0+\varepsilon))}{2\varepsilon}\right)$$

$$= L\left(H(t-(t_0-\varepsilon))\frac{1}{2\varepsilon}\right) - L\left(H(t-(t_0+\varepsilon))\frac{1}{2\varepsilon}\right)$$

$$= \frac{e^{-s(t_0-\varepsilon)}}{2\varepsilon s} - \frac{e^{-s(t_0+\varepsilon)}}{2\varepsilon s}$$

$$Y(s) = \frac{e^{-s(t_0-\varepsilon)}}{s(s^2+1)} - \frac{e^{-s(t_0+\varepsilon)}}{s(s^2+1)}$$

$$Y_\varepsilon(t) = L^{-1}\left(\frac{e^{-s(t_0-\varepsilon)}}{2\varepsilon s(s^2+1)} - \frac{e^{-s(t_0+\varepsilon)}}{2\varepsilon s(s^2+1)}\right)$$

$$= L^{-1}\left(\frac{e^{-s(t_0-\varepsilon)}}{2\varepsilon s(s^2+1)}\right) - L^{-1}\left(\frac{e^{-s(t_0+\varepsilon)}}{2\varepsilon s(s^2+1)}\right)$$

$$= L^{-1}\left(\frac{1}{2\varepsilon s(s^2+1)}\right)(t-(t_0-\varepsilon)) \cdot H(t-(t_0-\varepsilon))$$

$$- L^{-1}\left(\frac{1}{2\varepsilon s(s^2+1)}\right)(t-(t_0+\varepsilon)) \cdot H(t-(t_0+\varepsilon))$$

$$L^{-1}\left(\frac{1}{2\varepsilon s(s^2+1)}\right) = L^{-1}\left(\frac{1}{2\varepsilon s} - \frac{s}{2\varepsilon(s^2+1)}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{1}{2\varepsilon s}\right) - \mathcal{L}^{-1}\left(\frac{s}{2\varepsilon(s^2+1)}\right)$$

$$= \frac{1}{2\varepsilon} H(t) - \frac{1}{2\varepsilon}\cos(t)$$

$$= \frac{H(t-(t_0-\varepsilon))\left(H(t-(t_0-\varepsilon)) - \cos(t-(t_0-\varepsilon))\right)}{2\varepsilon}$$

$$- \frac{H(t-(t_0+\varepsilon))\left(H(t-(t_0+\varepsilon)) - \cos(t-(t_0+\varepsilon))\right)}{2\varepsilon}$$

$$= \begin{cases} 0 & t < t_0 - \varepsilon \\ \dfrac{1-\cos(t-(t_0-\varepsilon))}{2\varepsilon} & t_0 - \varepsilon \leq t < t_0 + \varepsilon \\ \dfrac{\cos(t-(t_0+\varepsilon)) - \cos(t-(t_0-\varepsilon))}{2\varepsilon} & t_0 + \varepsilon \leq t \end{cases}$$

b) $L(y''+y) = Y(s)(s^2+1)$

$$L(\delta(t-t_0)) = \int_0^\infty e^{-st}\,\delta(t-t_0)\,dt = e^{-t_0 s}$$

$$Y(s) = \frac{e^{-t_0 s}}{s^2+1}$$

$$y^*(t) = \mathcal{L}^{-1}\left(\frac{e^{-t_0 s}}{s^2+1}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right)(t-t_0) \cdot H(t-t_0)$$

$$= H(t-t_0)\sin(t-t_0)$$

$$= \begin{cases} 0 & t < t_0 \\ \sin(t-t_0) & t_0 \leq t \end{cases}$$

c)

$$\lim_{\varepsilon\to 0} y_\varepsilon(t) = \lim_{\varepsilon\to 0} \begin{cases} 0 & t < t_0 - \varepsilon \\ \dfrac{1-\cos(t-(t_0-\varepsilon))}{2\varepsilon} & t_0 - \varepsilon \leq t < t_0 + \varepsilon \\ \dfrac{\cos(t-(t_0+\varepsilon)) - \cos(t-(t_0-\varepsilon))}{2\varepsilon} & t_0 + \varepsilon \leq t \end{cases}$$

$$= \begin{cases} 0 & t < t_0 \\ \lim\limits_{\varepsilon \to 0} \dfrac{\cos(t-(t_0+\varepsilon)) - \cos(t-(t_0-\varepsilon))}{2\varepsilon} & t_0 \le t \end{cases}$$

l'Hospital

$$= \begin{cases} 0 & t < t_0 \\ \lim\limits_{\varepsilon \to 0} \dfrac{\sin(t-(t_0+\varepsilon)) + \sin(t-(t_0-\varepsilon))}{2} & t_0 \le t \end{cases}$$

$$= \begin{cases} 0 & t < t_0 \\ \sin(t-t_0) & t_0 \le t \end{cases}$$

$$= y^*(t)$$

2.

$$y' + p(t)y = q(t)y^\alpha$$

a) $\alpha \neq 0,1$ $\quad u = y^{1-\alpha}$ $\quad u' = 1-\alpha y^{-\alpha} y'$

$$y' + p(t)y = q(t)y^\alpha$$

$$y'y^{-\alpha} + p(t)y^{1-\alpha} = q(t)$$

$$\frac{1}{1-\alpha}u' + p(t)u = q(t)$$

$$u' + (1-\alpha)p(t)u = (1-\alpha)q(t)$$

b) $ty' + y = y^2 t^2 \log t \quad t \ge 1$ $\qquad \alpha = 2$

$$y' + \frac{1}{t}y = y^2 t \log t$$

$u = y^{-1} \quad u' = -y^{-2}y'$

$u' - \frac{1}{t}u = -t\ln t$ $\qquad \alpha(t) = -\frac{1}{t} \quad \beta(t) = -t\log t$

$u(t) = t\int_1^t -\log x \, dx + ct$ $\qquad r(t) = e^{\int_1^t -\frac{1}{x}dx}$

$u = \log x \quad dv = 1$ $\qquad = e^{-\log t}$

$du = 1/x \quad v = x$ $\qquad = \frac{1}{t}$

$$= -t\left[x\log x - \int 1 dx\right]_1^t + ct$$

$$= -t\left(t\log(t) - t + 1\right) + ct$$

$$= -t^2\log(t) + t^2 - t + ct$$

$$= -t^2\log(t) + t^2 + ct$$

$$y(t) = \frac{1}{u(t)} = \frac{1}{-t^2\log(t) + t^2 + ct}$$

3.

$$y' = (y^2 + 2ty)/(3+t^2) \quad y(0) = 1/2$$

a) $\Delta t = 0.025 \qquad [t_0, t_f] = [0,1]$

Matlab

b) $\Delta t = 0.025 \quad [t_0, t_f] = [0,1]$

$$y_{n+1} = y_n + \Delta t\left(y_{n+1}^2 + 2t_{n+1}y_{n+1}\right)/(3 + t_{n+1}^2)$$

$$a = \Delta t/(3+t_{n+1}^2) \quad b = \Delta t 2t_{n+1}/(3+t_{n+1}^2) - 1 \quad c = y_n$$

Matlab

c) ode45

Matlab

4.

a)

$$y' = f(t,y) \quad y(t_0) = y_0$$

$$y(t_{n+1}) = y(t_n) + \frac{\Delta t}{2}\left(f(t_n, y(t_n)) + f\left(t_{n+1}, y(t_n) + \Delta t\, f(t_n, y(t_n))\right)\right)$$

$$e_n \alpha (\Delta t)^3$$

$$e_n = y(t_n) - y(t_{n-1}) - \frac{\Delta t}{2}\left(f(t_{n-1}, y(t_{n-1})) + f\left(t_n, y(t_{n-1}) + \Delta t\, f(t_{n-1}, y(t_{n-1}))\right)\right)$$

$$y''(t) = f_t(t,y) + f_y(t,y)f(t,y)$$

$$y'''(t) = f_{tt}(t,y) + f_{ty}(t,y)f(t,y) + f_{ty}(t,y)f(t,y)$$

$$+ f_{yy}(t,y) f(t,y)^2 + f_y(t,y) f_t(t,y) + f_y(t,y)^2 f(t,y)$$

$y(t_n) = y(t_{n-1} + \Delta t)$ Taylor expand + Lagrange Error $\xi_{n-1} \in (t_{n-1}, t_n)$

$$= y(t_{n-1}) + y'(t_{n-1})\Delta t + \tfrac{1}{2} y''(t_{n-1})(\Delta t)^2 + \tfrac{1}{6} y'''(\xi_{n-1})(\Delta t)^3$$

$f(t_{n+1}, y(t_n) + \Delta t\, f(t_n, y(t_n))) =$ Taylor expand + Lagrange Error

$f(t_n + \Delta t, y(t_n) + \Delta y) = f(t_n, y(t_n))$ $\quad \Delta y = \Delta t\, f(t_n, y(t_n))$

$$+ f_t(t_n, y(t_n))\Delta t + f_y(t_n, y(t_n))\Delta t\, f(t_n, y(t_n))$$

$$+ \tfrac{1}{2}\Big(f_{tt}(\xi_n, \eta_n)(\Delta t)^2 + 2 f_{ty}(\xi_n, \eta_n)(\Delta t)^2 f(t_n, y(t_n))$$

$$+ f_{yy}(\xi_n, \eta_n)(\Delta t)^2 f(t_n, y(t_n))^2\Big)$$

$\xi_n \in (t_n, t_{n+1})$ $\quad \eta_n \in (y(t_n), y(t_{n+1}))$

$$= y'(t_n) + y''(t_n)\Delta t + \frac{(\Delta t)^2}{2}\Big(f_{tt}(\xi_n, \eta_n)$$

$$+ 2 f_{ty}(\xi_n, \eta_n) f(t_n, y(t_n)) + f_{yy}(\xi_n, \eta_n) f(t_n, y(t_n))^2\Big)$$

$$e_n = \Big(y(t_{n-1}) + y'(t_{n-1})\Delta t + \tfrac{1}{2} y''(t_{n-1})(\Delta t)^2 + \tfrac{1}{6} y'''(\xi_{n-1})(\Delta t)^3\Big)$$

$$- y(t_{n-1})$$

$$- \frac{\Delta t}{2}\Big(y'(t_{n-1}) + y'(t_{n-1}) + y''(t_{n-1})\Delta t + \frac{(\Delta t)^2}{2}\Big(f_{tt}(\xi_{n-1}, \eta_{n-1})$$

$$+ 2 f_{ty}(\xi_{n-1}, \eta_{n-1}) f(t_{n-1}, y(t_{n-1})) + f_{yy}(\xi_{n-1}, \eta_{n-1}) f(t_{n-1}, y(t_{n-1}))^2\Big)\Big)$$

$$= y'(t_{n-1})\Delta t + \tfrac{1}{2} y''(t_{n-1})(\Delta t)^2 + \tfrac{1}{6} y'''(\xi_{n-1})(\Delta t)^3$$

$$- \Delta t\, y'(t_{n-1}) - \frac{(\Delta t)^2}{2} y''(t_{n-1})$$

$$- \frac{(\Delta t)^3}{4}\Big(f_{tt}(\xi_{n-1}, \eta_{n-1}) + 2 f_{ty}(\xi_{n-1}, \eta_{n-1}) f(t_{n-1}, y(t_{n-1}))$$

$$+ f_{yy}(\xi_{n-1}, \eta_{n-1}) f(t_{n-1}, y(t_{n-1}))^2\Big)$$

$$= (\Delta t)^3\Big(\tfrac{1}{6} y'''(\xi_{n-1}) - \tfrac{1}{4}\Big(f_{tt}(\xi_{n-1}, \eta_{n-1})$$

$$+ 2 f_{ty}(\xi_{n-1}, \eta_{n-1}) f(t_{n-1}, y(t_{n-1})) + f_{yy}(\xi_{n-1}, \eta_{n-1}) f(t_{n-1}, y(t_{n-1}))^2\Big)\Big)$$

$$\propto (\Delta t)^3$$

b) $y' = (t^2 - y^2)\sin y \quad y(0) = 1$

$\Delta t = 0.025 \quad [t_0, t_f] = [0,1] \quad$ Matlab

5.

$y' = (y^2 + 2ty)/(3+t^2) \quad y(0) = 1/2$

a) $y' - \frac{2t}{3+t^2} y = \frac{1}{3+t^2} y^2 \qquad \alpha = 2$

$y' y^{-2} - \frac{2t}{3+t^2} y^{-1} = \frac{1}{3+t^2}$

$u = y^{-1} \qquad u' = -y^{-2} y'$

$u' + \frac{2t}{3+t^2} u = -\frac{1}{3+t^2} \qquad \alpha(t) = \frac{2t}{3+t^2} \quad \beta(t) = -\frac{1}{3+t^2}$

$r(t) = e^{\int_0^t \frac{2x}{3+x^2} dx}$

$= e^{\log(3+t^2) - \log(3)}$

$= \frac{3+t^2}{3}$

$u_e(t) = \frac{3}{3+t^2} \int_0^t -\frac{1}{3} dx + \frac{3c}{3+t^2}$

$= -\frac{t}{3+t^2} + \frac{3c}{3+t^2}$

$= \frac{3c - t}{3+t^2}$

$y_e(0) = \frac{3}{3c} = 1/2 \quad c = 2$

$y_e(t) = \frac{3+t^2}{3c-t} = \frac{3+t^2}{6-t}$

b) $\Delta t = 0.025 \quad [t_0, t_f] = [0,1]$

$E_n = y_e(t_n) - y_n \quad t_n$

Matlab