Kyle McGraw

ACM 95a Set 2

Partners:
Dallas, Allison, Nate

1. a) $1^{\frac{i}{2\pi}} = e^{\frac{i}{2\pi}\log(1)} = e^{\frac{i}{2\pi}(\mathrm{Log}|1| + i\,\mathrm{Arg}(1) + i2\pi n)}$

$= e^{\frac{i}{2\pi}(0+0+i2\pi n)} = \boxed{e^{-n}} \quad n \in \mathbb{Z}$

b) $\cos(i)^i = \left(\frac{e^{-1}+e^{1}}{2}\right)^i = e^{i\log\left(\frac{e^{-1}+e}{2}\right)}$

$= e^{i\left(\mathrm{Log}\left|\frac{e^{-1}+e}{2}\right| + i\,\mathrm{Arg}\left(\frac{e^{-1}+e}{2}\right) + i2\pi n\right)}$

$= e^{i\left(\mathrm{Log}\left(\frac{e^{-1}+e}{2}\right) + i2\pi n\right)} \quad n \in \mathbb{Z}$

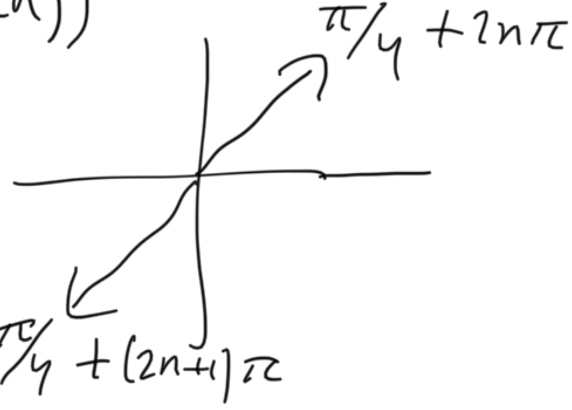
$= \boxed{\cos\left(\mathrm{Log}\left(\frac{e^{-1}+e}{2}\right) + i2\pi n\right) + i\sin\left(\mathrm{Log}\left(\frac{e^{-1}+e}{2}\right) + i2\pi n\right)}$

c) $\log(i^{1/2}) = \log\left(e^{\frac{1}{2}(\mathrm{Log}|i| + i\,\mathrm{Arg}(i) + i2\pi n)}\right)$

$= \log\left(e^{\frac{1}{2}(0 + i\frac{\pi}{2} + i2\pi n)}\right)$

$= \log\left(e^{i\left(\frac{\pi}{4} + \pi n\right)}\right)$

$= \log\left(\cos\left(\frac{\pi}{4}+\pi n\right) + i\sin\left(\frac{\pi}{4}+\pi n\right)\right)$



$\to \log\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$

$= \mathrm{Log}\left|\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right| + i\,\mathrm{Arg}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) + i2\pi n$

$= 0 + i\frac{\pi}{4} + i2\pi n = i\pi\left(2n + \frac{1}{4}\right)$

$\log\left(-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right)$

$= \operatorname{Log}\left|-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right| + i\operatorname{Arg}\left(-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right) + i2\pi n$

$= 0 - i\frac{3\pi}{4} + i2\pi n = i\pi\left(2n - \frac{3}{4}\right)$

$\boxed{i\pi\left(n+\frac{1}{4}\right)}$

2. $\log z^{\alpha} = \alpha \log z$

$\log z^{\alpha} = \log\left(e^{\alpha(\operatorname{Log} z + i2\pi n)}\right) = \log\left(e^{\alpha \operatorname{Log} z} e^{\alpha i 2\pi n}\right)$

$= \log\left(e^{\alpha \operatorname{Log} z}\right) + \log\left(e^{\alpha i 2\pi n}\right)$

$= \alpha \operatorname{Log} z + 2\pi i j + \alpha i 2\pi n + 2\pi i k$

$2\pi i m = 2\pi i(j+k)$ arbitrary constant

$= \alpha \operatorname{Log} z + 2\pi i m + \alpha i 2\pi n$

$\alpha \operatorname{Log} z + 2\pi i m + \alpha i 2\pi n = \alpha \log z$

$\alpha \operatorname{Log} z + 2\pi i m + \alpha i 2\pi n = \alpha \operatorname{Log} z + \alpha 2\pi i k$

$\alpha \operatorname{Log} z + 2\pi i(m+\alpha n) = \alpha \operatorname{Log} z + \alpha 2\pi i k$

$2\pi i(m+\alpha n) = \alpha 2\pi i k$

$m + \alpha n = \alpha k$

$m = \alpha(k-n)$ $\quad j,k,l,m,n \in \mathbb{Z}$

$m = \alpha l$ arbitrary constant

$\alpha = \frac{m}{l}$ $\quad \boxed{\alpha \in \mathbb{Q}}$

3. $z = \tan w = \frac{\sin w}{\cos w} = \frac{e^{iw}-e^{-iw}}{2i} \cdot \frac{2}{e^{iw}+e^{-iw}}$

$= \frac{1}{i} \frac{e^{2iw}-1}{e^{2iw}+1}$

$zi = \frac{e^{2iw}-1}{e^{2iw}+1}$ $\quad zie^{2iw} + zi = e^{2iw} - 1$

$$zi+1 = e^{2iw} - zie^{2iw}$$

$$zi+1 = e^{2iw}(1-zi)$$

$$e^{2iw} = \frac{i-z}{i+z}$$

$$\log(e^{2iw}) = \log\left(\frac{i-z}{i+z}\right)$$

$$\mathrm{Log}(e^{2iw}) + i2\pi n = \mathrm{Log}\left(\frac{i-z}{i+z}\right) + i2\pi m$$

$$2iw = \mathrm{Log}\left(\frac{i-z}{i+z}\right) + i2\pi(m-n)$$

$$w = \frac{1}{2i}\mathrm{Log}\left(\frac{i-z}{i+z}\right) + i2\pi k \qquad k, m, n \in \mathbb{Z}$$

$$\boxed{\tan^{-1}(z) = \frac{1}{2i}\log\left(\frac{i-z}{i+z}\right)}$$

4. z^{1+i} $f(z) = z_{-\pi}^{1+i}$

a) $z=-1$ is not differentiable (Lecture 4, pg. 19) because $\mathrm{Arg}(-1)$ is not continuous and is multivalued $(\pi \neq -\pi)$

$$f(z) = z_{-\pi}^{1+i} = e^{(1+i)\,\mathrm{Log}(z)} = e^{(1+i)(\mathrm{Log}|z| + i\,\mathrm{Arg}(z))}$$

$z=i$ is differentiable because z^{1+i} is analytic in its domain

$D_{-\pi} = \mathbb{C} \setminus (-\infty, 0]$ (Lecture 5, pg. 25)

b) $z = i$

$$\frac{d}{dz} z_{\theta_0}^{\alpha} = \alpha z_{\theta_0}^{\alpha-1} \qquad \text{(Lecture 5, pg. 25)}$$

$$\frac{d}{dz} i_{-\pi}^{1+i} = (1+i)\, i_{-\pi}^{(1+i)-1} = (1+i)\, i_{-\pi}^{i}$$

$$= (1+i)\, e^{i(\text{Log}|i| + i\,\text{Arg}(i))} = \boxed{(1+i)\, e^{-\frac{\pi}{2}}}$$

5. $w = f(z) = \left(\frac{i-z}{i+z}\right)^{1/2}$

a) $\left(\frac{i-z}{i+z}\right)^{1/2} = e^{\frac{1}{2}\log\left(\frac{i-z}{i+z}\right)}$

$\log(z)$ has branch points $z = 0, \infty$

Branch points: $\frac{i-z}{i+z} = 0, \infty$

$\frac{i-z}{i+z} = 0 \quad z = i$

$\frac{i-z}{i+z} = \infty \quad z = -i$

b) $F(0) = -1$

$\frac{i-0}{i+0} = 1$

$e^{\frac{1}{2}(\text{Log}|1| + i\,\text{Arg}(1) + 2\pi i n)} = -1$

$e^{\frac{1}{2}(0 + 0 + 2\pi i n)} = -1$

$e^{i\pi n} = -1$

$\cos(\pi n) + i\sin(\pi n) = -1$

$-\pi + 2\pi n \quad n = (2k+1) \quad k \in \mathbb{Z}$

$n = 1$

$F(z) = f_{\pi}(z)$

$\hookrightarrow$ angle is $[\pi, 3\pi)$

Branch cut

$$c = \frac{i-z}{i+z} \qquad \{c : c \in (-\infty, 0]\}$$

$$ci + cz = i - z$$

$$cz - z = i - ci$$

$$(c-1)z = (1-c)i$$

$$z = \frac{1-c}{c-1} i$$

cut

$$\left\{z : z = \frac{1-c}{c-1} i,\ c \in (-\infty, 0]\right\}$$

c) see Matlab

$$\frac{i - x - iy}{i + x + iy}$$

$$\frac{-x-(y-1)i}{x+(y+1)i} \cdot \frac{x-(y+1)i}{x-(y+1)i}$$

$$\frac{1-x^2-y^2}{x^2+(y+1)^2} + \frac{2x}{x^2+(y+1)^2} i$$

$$e^{\frac{1}{2}(\mathrm{Log}|\text{base}| + i\,\mathrm{Arg}(\text{base}) + 2\pi i n)} \qquad n = 1$$

$$e^{\frac{1}{2}\mathrm{Log}|\text{base}|} \cdot e^{i\left(\frac{1}{2}\mathrm{Arg}(\text{base}) + 2\pi n\right)}$$

$$e^{\frac{1}{2}\mathrm{Log}|\text{base}|} \cos\left(\tfrac{1}{2}\mathrm{Arg}(\text{base}) + 2\pi n\right)$$

$$+\, i\, e^{\frac{1}{2}\mathrm{Log}|\text{base}|} \sin\left(\tfrac{1}{2}\mathrm{Arg}(\text{base}) + 2\pi n\right)$$