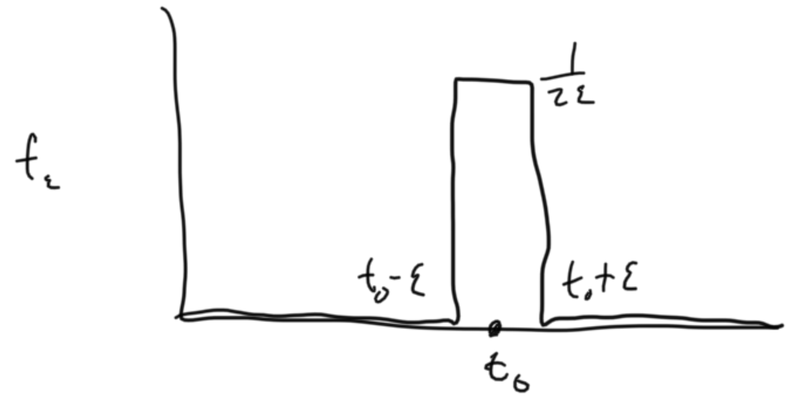
Kyle McGraw

ACM 95a Set 7

1. $y'' + y = f_\varepsilon(t) \quad t \geq 0 \quad y(0) = 0 \quad y'(0) = 0$

$$f_\varepsilon(t) = \frac{H(t-(t_0-\varepsilon)) - H(t-(t_0+\varepsilon))}{2\varepsilon} \qquad t_0 > 0 \qquad \varepsilon > 0$$

$$\lim_{\varepsilon \to 0} f_\varepsilon(t) = \delta(t - t_0)$$



a)

$$L(y''+y) = Y(s)(s^2+1)$$

$$L(f_\varepsilon(t)) = L\left(\frac{H(t-(t_0-\varepsilon)) - H(t-(t_0+\varepsilon))}{2\varepsilon}\right)$$

$$= L\left(H(t-(t_0-\varepsilon))\frac{1}{2\varepsilon}\right) - L\left(H(t-(t_0+\varepsilon))\frac{1}{2\varepsilon}\right)$$

$$= \frac{e^{-s(t_0-\varepsilon)}}{2\varepsilon s} - \frac{e^{-s(t_0+\varepsilon)}}{2\varepsilon s}$$

$$Y(s) = \frac{e^{-s(t_0-\varepsilon)}}{s(s^2+1)} - \frac{e^{-s(t_0+\varepsilon)}}{s(s^2+1)}$$

$$Y_\varepsilon(t) = L^{-1}\left(\frac{e^{-s(t_0-\varepsilon)}}{2\varepsilon s(s^2+1)} - \frac{e^{-s(t_0+\varepsilon)}}{2\varepsilon s(s^2+1)}\right)$$

$$= L^{-1}\left(\frac{e^{-s(t_0-\varepsilon)}}{2\varepsilon s(s^2+1)}\right) - L^{-1}\left(\frac{e^{-s(t_0+\varepsilon)}}{2\varepsilon s(s^2+1)}\right)$$

$$= L^{-1}\left(\frac{1}{2\varepsilon s(s^2+1)}\right)(t-(t_0-\varepsilon)) \cdot H(t-(t_0-\varepsilon))$$

$$- L^{-1}\left(\frac{1}{2\varepsilon s(s^2+1)}\right)(t-(t_0+\varepsilon)) \cdot H(t-(t_0+\varepsilon))$$

$$L^{-1}\left(\frac{1}{2\varepsilon s(s^2+1)}\right) = L^{-1}\left(\frac{1}{2\varepsilon s} - \frac{s}{2\varepsilon(s^2+1)}\right)$$

$$= L^{-1}\left(\frac{1}{2\varepsilon s}\right) - L^{-1}\left(\frac{s}{2\varepsilon(s^2+1)}\right)$$

$$= \frac{1}{2\varepsilon} H(t) - \frac{1}{2\varepsilon}\cos(t)$$

$$= \frac{H(t-(t_0-\varepsilon))\left(H(t-(t_0-\varepsilon)) - \cos(t-(t_0-\varepsilon))\right)}{2\varepsilon}$$

$$- \frac{H(t-(t_0+\varepsilon))\left(H(t-(t_0+\varepsilon)) - \cos(t-(t_0+\varepsilon))\right)}{2\varepsilon}$$

$$= \begin{cases} 0 & t < t_0 - \varepsilon \\ \frac{1-\cos(t-(t_0-\varepsilon))}{2\varepsilon} & t_0 - \varepsilon \le t < t_0 + \varepsilon \\ \frac{\cos(t-(t_0+\varepsilon)) - \cos(t-(t_0-\varepsilon))}{2\varepsilon} & t_0 + \varepsilon \le t \end{cases}$$

b) $L(y''+y) = Y(s)(s^2+1)$

$$L(\delta(t-t_0)) = \int_0^\infty e^{-st}\,\delta(t-t_0)\,dt = e^{-t_0 s}$$

$$Y(s) = \frac{e^{-t_0 s}}{s^2+1}$$

$$y^*(t) = L^{-1}\left(\frac{e^{-t_0 s}}{s^2+1}\right)$$

$$= L^{-1}\left(\frac{1}{s^2+1}\right)(t-t_0) \cdot H(t-t_0)$$

$$= H(t-t_0)\sin(t-t_0)$$

$$= \begin{cases} 0 & t < t_0 \\ \sin(t-t_0) & t_0 \le t \end{cases}$$

c)

$$\lim_{\varepsilon\to 0} y_\varepsilon(t) = \lim_{\varepsilon\to 0} \begin{cases} 0 & t < t_0 - \varepsilon \\ \frac{1-\cos(t-(t_0-\varepsilon))}{2\varepsilon} & t_0 - \varepsilon \le t < t_0 + \varepsilon \\ \frac{\cos(t-(t_0+\varepsilon)) - \cos(t-(t_0-\varepsilon))}{2\varepsilon} & t_0 + \varepsilon \le t \end{cases}$$

$$= \begin{cases} 0 & t < t_0 \\ \lim\limits_{\varepsilon\to 0} \dfrac{\cos(t-(t_0+\varepsilon)) - \cos(t-(t_0-\varepsilon))}{2\varepsilon} & t_0 \leq t \end{cases}$$

l'Hospital

$$= \begin{cases} 0 & t < t_0 \\ \lim\limits_{\varepsilon\to 0} \dfrac{\sin(t-(t_0+\varepsilon)) + \sin(t-(t_0-\varepsilon))}{2} & t_0 \leq t \end{cases}$$

$$= \begin{cases} 0 & t < t_0 \\ \sin(t-t_0) & t_0 \leq t \end{cases}$$

$$= y^*(t)$$

2.

$$y' + p(t)y = q(t)y^\alpha$$

a) $\alpha \neq 0,1$ $\quad u = y^{1-\alpha}$ $\quad u' = 1-\alpha y^{-\alpha} y'$

$$y' + p(t)y = q(t)y^\alpha$$

$$y'y^{-\alpha} + p(t)y^{1-\alpha} = q(t)$$

$$\frac{1}{1-\alpha}u' + p(t)u = q(t)$$

$$u' + (1-\alpha)p(t)u = (1-\alpha)q(t)$$

b) $ty' + y = y^2 t^2 \log t \quad t \geq 1$ $\qquad \alpha = 2$

$$y' + \frac{1}{t}y = y^2 t \log t$$

$$u = y^{-1} \qquad u' = -y^{-2}y'$$

$$u' - \frac{1}{t}u = -t\ln t \qquad \alpha(t) = -\frac{1}{t} \quad \beta(t) = -t\log t$$

$$u(t) = t\int_1^t -\log x \, dx + ct \qquad r(t) = e^{\int_1^t -\frac{1}{x}dx}$$

$u = \log x \quad dv = 1$

$du = 1/x \quad v = x$

$$= e^{-\log t}$$

$$= \frac{1}{t}$$

$$= -t\left[x\log x - \int 1 dx\right]_1^t + ct$$

$= -t\left(t\log(t) - t + 1\right) + ct$

$= -t^2\log(t) + t^2 - t + ct$

$= -t^2\log(t) + t^2 + ct$

$$y(t) = \frac{1}{u(t)} = \frac{1}{-t^2\log(t) + t^2 + ct}$$

3.

$y' = (y^2 + 2ty)/(3 + t^2) \quad y(0) = 1/2$

a) $\Delta t = 0.025 \quad [t_0, t_f] = [0, 1]$

Matlab

b) $\Delta t = 0.025 \quad [t_0, t_f] = [0, 1]$

$$y_{n+1} = y_n + \Delta t\left(y_{n+1}^2 + 2t_{n+1}y_{n+1}\right)/(3 + t_{n+1}^2)$$

$a = \Delta t/(3 + t_{n+1}^2) \quad b = dt\,2t_{n+1}/(3 + t_{n+1}^2) - 1 \quad c = y_n$

Matlab

c) ode45

Matlab

4.

a)

$y' = f(t, y) \quad y(t_0) = y_0$

$$y(t_{n+1}) = y(t_n) + \frac{\Delta t}{2}\left(f(t_n, y(t_n)) + f\left(t_{n+1}, y(t_n) + \Delta t\, f(t_n, y(t_n))\right)\right)$$

$e_n \propto (\Delta t)^3$

$$e_n = y(t_n) - y(t_{n-1}) - \frac{\Delta t}{2}\left(f(t_{n-1}, y(t_{n-1})) + f\left(t_n, y(t_{n-1}) + \Delta t\, f(t_{n-1}, y(t_{n-1}))\right)\right)$$

$y''(t) = f_t(t, y) + f_y(t, y) f(t, y)$

$y'''(t) = f_{tt}(t, y) + f_{ty}(t, y) f(t, y) + f_{ty}(t, y) f(t, y)$

$$+ f_{yy}(t,y) f(t,y)^2 + f_y(t,y) f_t(t,y) + f_y(t,y)^2 f(t,y)$$

$y(t_n) = y(t_{n-1}+\Delta t)$ Taylor expand + Lagrange Error $\xi_{n-1} \in (t_{n-1}, t_n)$

$$= y(t_{n-1}) + y'(t_{n-1})\Delta t + \frac{1}{2} y''(t_{n-1})(\Delta t)^2 + \frac{1}{6} y'''(\xi_{n-1})(\Delta t)^3$$

$f(t_{n+1}, y(t_n) + \Delta t\, f(t_n, y(t_n))) =$ Taylor expand + Lagrange Error

$f(t_n+\Delta t, y(t_n)+\Delta y) = f(t_n, y(t_n))$ $\quad \Delta y = \Delta t f(t_n, y(t_n))$

$$+ f_t(t_n, y(t_n))\Delta t + f_y(t_n, y(t_n))\Delta t f(t_n, y(t_n))$$

$$+ \frac{1}{2}\left(f_{tt}(\xi_n, \eta_n)(\Delta t)^2 + 2 f_{ty}(\xi_n,\eta_n)(\Delta t)^2 f(t_n, y(t_n))\right.$$

$$\left. + f_{yy}(\xi_n, \eta_n)(\Delta t)^2 f(t_n, y(t_n))^2\right)$$

$\xi_n \in (t_n, t_{n+1})$ $\quad \eta_n \in (y(t_n), y(t_{n+1}))$

$$= y'(t_n) + y''(t_n)\Delta t + \frac{(\Delta t)^2}{2}\left(f_{tt}(\xi_n, \eta_n)\right.$$

$$\left. + 2 f_{ty}(\xi_n,\eta_n) f(t_n, y(t_n)) + f_{yy}(\xi_n, \eta_n) f(t_n, y(t_n))^2\right)$$

$$e_n = \left(y(t_{n-1}) + y'(t_{n-1})\Delta t + \frac{1}{2} y''(t_{n-1})(\Delta t)^2 + \frac{1}{6} y'''(\xi_{n-1})(\Delta t)^3\right)$$

$$- y(t_{n-1})$$

$$- \frac{\Delta t}{2}\left(y'(t_{n-1}) + y'(t_{n-1}) + y''(t_{n-1})\Delta t + \frac{(\Delta t)^2}{2}\left(f_{tt}(\xi_{n-1}, \eta_{n-1})\right.\right.$$

$$\left.\left. + 2 f_{ty}(\xi_{n-1},\eta_{n-1}) f(t_{n-1}, y(t_{n-1})) + f_{yy}(\xi_{n-1}, \eta_{n-1}) f(t_{n-1}, y(t_{n-1}))^2\right)\right)$$

$$= y'(t_{n-1})\Delta t + \frac{1}{2} y''(t_{n-1})(\Delta t)^2 + \frac{1}{6} y'''(\xi_{n-1})(\Delta t)^3$$

$$- \Delta t\, y'(t_{n-1}) - \frac{(\Delta t)^2}{2} y''(t_{n-1})$$

$$- \frac{(\Delta t)^3}{4}\left(f_{tt}(\xi_{n-1}, \eta_{n-1}) + 2 f_{ty}(\xi_{n-1},\eta_{n-1}) f(t_{n-1}, y(t_{n-1}))\right.$$

$$\left. + f_{yy}(\xi_{n-1}, \eta_{n-1}) f(t_{n-1}, y(t_{n-1}))^2\right)$$

$$= (\Delta t)^3\left(\frac{1}{6} y'''(\xi_{n-1}) - \frac{1}{4}\left(f_{tt}(\xi_{n-1}, \eta_{n-1})\right.\right.$$

$$\left.\left. + 2 f_{ty}(\xi_{n-1},\eta_{n-1}) f(t_{n-1}, y(t_{n-1})) + f_{yy}(\xi_{n-1}, \eta_{n-1}) f(t_{n-1}, y(t_{n-1}))^2\right)\right)$$

$$\propto (\Delta t)^3$$

b) $y' = (t^2 - y^2)\sin y \quad y(0) = 1$

$\Delta t = 0.025 \quad [t_0, t_f] = [0, 1] \quad$ Matlab

5.

$y' = (y^2 + 2ty)/(3+t^2) \quad y(0) = 1/2$

a) $y' - \frac{2t}{3+t^2} y = \frac{1}{3+t^2} y^2 \quad \alpha = 2$

$y' y^{-2} - \frac{2t}{3+t^2} y^{-1} = \frac{1}{3+t^2}$

$u = y^{-1} \quad u' = -y^{-2} y'$

$u' + \frac{2t}{3+t^2} u = -\frac{1}{3+t^2} \qquad \alpha(t) = \frac{2t}{3+t^2} \quad \beta(t) = -\frac{1}{3+t^2}$

$r(t) = e^{\int_0^t \frac{2x}{3+x^2} dx}$

$u_e(t) = \frac{3}{3+t^2} \int_0^t -\frac{1}{3} dx + \frac{3c}{3+t^2} \qquad = e^{\log(3+t^2) - \log(3)}$

$= -\frac{t}{3+t^2} + \frac{3c}{3+t^2} \qquad = \frac{3+t^2}{3}$

$= \frac{3c - t}{3+t^2} \qquad y_e(0) = \frac{3}{3c} = 1/2 \quad c = 2$

$y_e(t) = \frac{3+t^2}{3c-t} = \frac{3+t^2}{6-t}$

b) $\Delta t = 0.025 \quad [t_0, t_f] = [0, 1]$

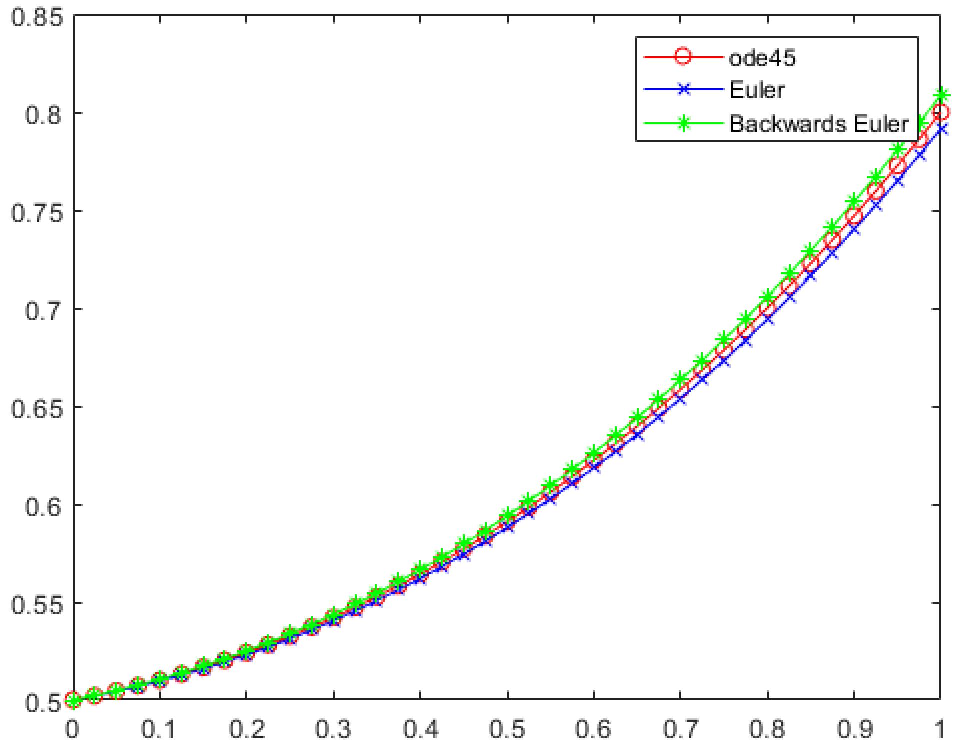
$E_n = y_e(t_n) - y_n \quad t_n$

Matlab

```
t0=0;
tf=1;
dt=0.025;
t=0:dt:1;
N=(tf-t0)/dt;
f=@(t,y) (y^2+2*t*y)/(3+t^2);
y=zeros(2,N+1);
y(1,1)=1/2;
y(2,1)=1/2;

for n = 1:N
    y(1,n+1)=y(1,n)+dt*f(t(n),y(1,n));
    a=dt/(3+t(n+1)^2);
    b=dt*2*t(n+1)/(3+t(n+1)^2)-1;
    c=y(2,n);
    y(2,n+1)=(-b-sqrt(b^2-4*a*c))/(2*a);
end
[tOde,yOde] = ode45(f,[0,1],1/2);

figure
p1 = plot(t,yOde,'r-o');
hold on;
p2 = plot(t,y(1,:),'b-x');
p3 = plot(t,y(2,:),'g-*');
hold off;
legend([p1(1);p2;p3],'ode45','Euler','Backwards Euler');
```



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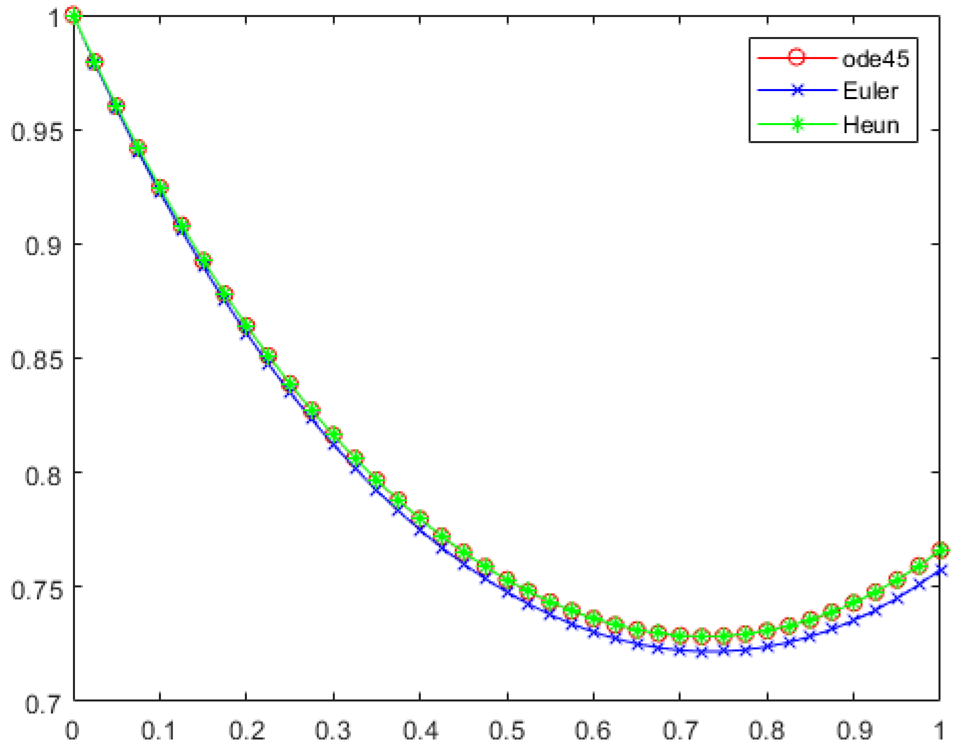
```
t0=0;
tf=1;
dt=0.025;
t=0:dt:1;
N=(tf-t0)/dt;
f=@(t,y) (t^2-y^2)*sin(y);
y=zeros(2,N+1);
y(1,1)=1;
y(2,1)=1;

for n = 1:N
    y(1,n+1)=y(1,n)+dt*f(t(n),y(1,n));
    y(2,n+1)=y(2,n)+dt/2*(f(t(n),y(2,n))+f(t(n
+1),y(2,n)+dt*f(t(n),y(2,n))));
end
[tOde,yOde] = ode45(f,[0,1],1);

figure
p1 = plot(t,yOde,'r-o');
hold on;
p2 = plot(t,y(1,:),'b-x');
p3 = plot(t,y(2,:),'g-*');
hold off;
legend([p1(1);p2;p3],'ode45','Euler','Heun');
```



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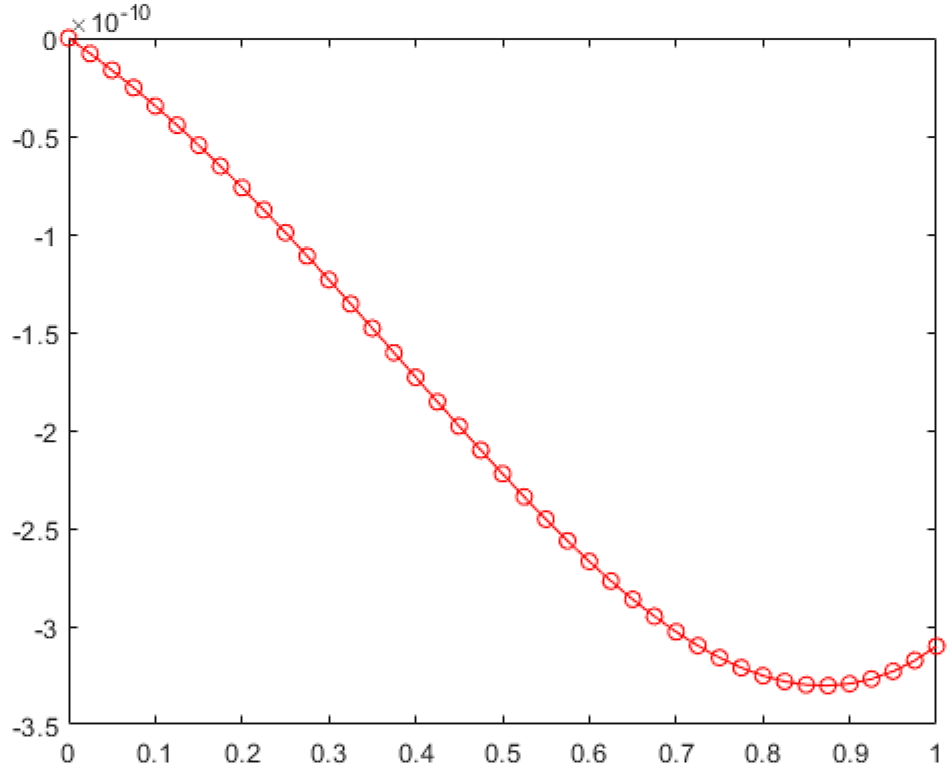
```
t0=0;
tf=1;
dt=0.025;
t=0:dt:1;
N=(tf-t0)/dt;
f=@(t,y) (y^2+2*t*y)/(3+t^2);
yExact=@(t) (3+t.^2)./(6-t);
y=zeros(1,N+1);
y(1)=1/2;

for n = 1:N
    k1=f(t(n),y(n));
    k2=f(t(n)+dt/2,y(n)+dt/2*k1);
    k3=f(t(n)+dt/2,y(n)+dt/2*k2);
    k4=f(t(n)+dt,y(n)+dt*k3);
    y(n+1)=y(n)+dt*(k1+2*k2+2*k3+k4)/6;
end
[tOde,yOde] = ode45(f,[0,1],1);
disp(yExact(t(1:10)))
disp(y(1:10))
p1 = plot(t,yExact(t)-y,'r-o');
```

```
  Columns 1 through 7

    0.5000    0.5022    0.5046    0.5073    0.5102    0.5133    0.5167

  Columns 8 through 10

    0.5203    0.5241    0.5282

  Columns 1 through 7

    0.5000    0.5022    0.5046    0.5073    0.5102    0.5133    0.5167

  Columns 8 through 10

    0.5203    0.5241    0.5282
```



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