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ACM 95a Midterm

1. $v(x,y) = xy - x + y \quad z = x + iy$

a) v is harmonic: second partials continuous & $v_{xx} + v_{yy} = 0$

$v_x = y - 1 \quad v_y = x + 1$

$v_{xx} = 0 \quad v_{xy} = 1 \quad v_{yy} = 0$ all continuous in the whole plane

$v_{xx} + v_{yy} = 0$

b) $f(z) = u + iv$ analytic $\quad u(0,0) = 1$

$u(x,y) = \frac{x^2}{2} + x + y - \frac{y^2}{2} + 1$

$u_x = x + 1 \quad u_y = -y + 1$

$u_{xx} = 1 \quad u_{xy} = 0 \quad u_{yy} = -1 \quad u_{xx} + u_{yy} = 0$

$u_x = v_y \quad u_y = -v_x$ all continuous in the whole plane

c) $f(z) = u + iv \quad z = x + iy$

$= \frac{x^2}{2} + x + y - \frac{y^2}{2} + 1 + i(xy - x + y)$

$= \frac{x^2}{2} + x + y - \frac{y^2}{2} + 1 + ixy + ix + iy$

$= \frac{x^2}{2} + y - \frac{y^2}{2} + 1 + ixy + ix + z \qquad y + ix = i(x - iy)$

$= \frac{x^2}{2} - \frac{y^2}{2} + 1 + ixy + z + i\bar{z} \qquad = i\bar{z}$

$= \frac{1}{2}(x^2 + 2ixy - y^2) + 1 + z + i\bar{z}$

$= \frac{1}{2}(x + iy)^2 + 1 + z + i\bar{z}$

$= \frac{1}{2}z^2 + z + i\bar{z} + 1$

2. $$f(z) = \frac{\log(z^2+2z+3)}{2z-\pi}$$

a) $z_0 = 0, \infty$ branch points for $\log(z)$

$\log(z^2+2z+3)$ branch points at

$$z^2+2z+3 = 0, \infty$$
$$(z+1)^2+2=0$$
$$(z+1)^2 = -2$$
$$z_0 = -1+i\sqrt{2}, -1-i\sqrt{2}, \infty, -\infty$$

b) analytic at $z=i-1$, $F(i-1)=0$, $F'(i-1)$

$$f(i-1) = \frac{\log((i-1)^2+2(i-1)+3)}{2(i-1)-\pi}$$
$$= \frac{\log(-1-2i+1+2i-2+3)}{2i-2-\pi}$$
$$= \frac{\log(1)}{2i-2-\pi} = \frac{\text{Log}|1|+i(\text{Arg}(1)+2\pi n)}{2i-2-\pi}$$
$$= \frac{2\pi n i}{2i-2-\pi}$$

choose $n=0$

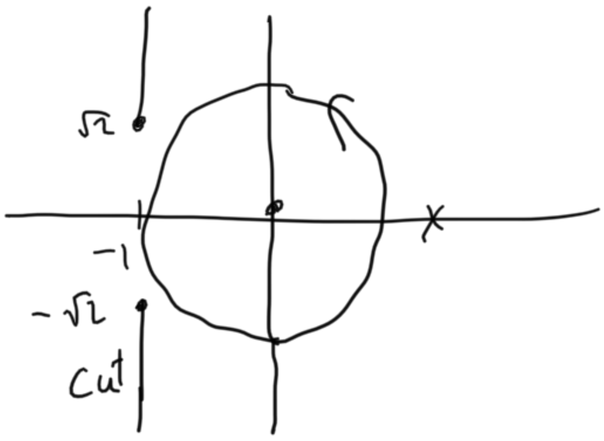
Principal branch $(-\infty, 0]$ cut

$$F(z) = f_{-\frac{\pi}{2}}(z)$$

$$F'(z) = \frac{(\log(z^2+2z+3))'(2z-\pi) - \log(z^2+2z+3)(2z-\pi)'}{(2z-\pi)^2}$$
$$= \frac{\frac{2z+2}{z^2+2z+3}(2z-\pi) - \log(z^2+2z+3)\cdot 2}{(2z-\pi)^2}$$

$$F'(i-1) = \frac{\frac{2i}{1}(2i-2-\pi) - 0}{(2i-2-\pi)^2} = \frac{2i}{2i-2-\pi}$$

c) $\int_C F(z)\,dz \quad |z|=1$



$$\int_C \frac{\log(z^2+2z+3)}{2z-\pi}\,dz$$

$c = z^2+2z+3 \quad c \in [-\infty, 0]$ cut $\quad 2z-\pi=0$
$z = \frac{\pi}{2}$

$(z+1)^2+2 = c \quad z = \pm\sqrt{c-2}-1$

analytic on and inside C

Cauchy - Goursat

$$\int_C F(z)\,dz = 0$$

3.

a) $C_R(z_0) \quad R \quad z_0 \quad C = \{z : |z-z_0| = R\}$

$f(z)$ analytic on and inside $C \quad |f(z)| \le M$ for $z \in C_R(z_0)$

$$\left|f^{(n)}(z_0)\right| \le \frac{Mn!}{R^n} \quad n = 0,1,2,\ldots$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}}$$

$$\left|f^{(n)}(z_0)\right| = \left|\frac{n!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}}\right| \le \left|\frac{n!}{2\pi i}\right| \left|\int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}}\right|$$

ML Bound

$$\le \left|\frac{n!}{2\pi i}\right| \frac{M}{|(z-z_0)^{n+1}|} 2\pi R$$

$$|f(z)| \le M$$

$$= \frac{n!}{|2\pi i|}\frac{M}{|z-z_0|^{n+1}} 2\pi R$$

$$\frac{|f(z)|}{|(z-z_0)^{n+1}|} \le \frac{M}{|(z-z_0)^{n+1}|}$$

$$= \frac{n!}{2\pi}\frac{M}{R^{n+1}} 2\pi R$$

$$L = 2\pi R$$

$$= \frac{Mn!}{R^n}$$

b) $n=1$ $|f'(z_0)| \leq \frac{M}{R}$

let $R \to \infty$

$$\lim_{R\to\infty} |f'(z_0)| \leq \lim_{R\to\infty} \frac{M}{R} = 0$$

$f'(z_0) = 0$ $\quad$ $f(z_0)$ constant

c) $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ $\quad a_n \neq 0$ degree $n \geq 1$

assume there exists a $p(z)$ with degree $n \geq 1, a_n \neq 0$ and no zeros

for $|z| \to \infty$, $|p(z)| \to \infty$, $\left|\frac{1}{p(z)}\right| \to 0$

Since $\left|\frac{1}{p(z)}\right| \to 0$, $\frac{1}{p(z)}$ must be bounded in the complex plane.

Since $p(z)$ has no zeros, $\frac{1}{p(z)}$ is also entire

Therefore by part (b), $\frac{1}{p(z)}$ is constant and $p(z)$ is constant

However, this contradicts our initial assumption that $p(z)$ has degree ≥ 1 with $a_n \neq 0$, so there does not exist a $p(z)$ with degree $n \geq 1$ and $a_n \neq 0$ with no zeros.

This means that any $p(z)$ with degree $n \geq 1$ and $a_n \neq 0$ must have at least 1 zero.

4. $f(z) = \dfrac{e^{1/z}}{(1-z)(i-z)}$

a) $f(z)$ has 3 isolated singularities
$z_0 = 0, 1, i$

$e^{1/z}$ has an essential singularity

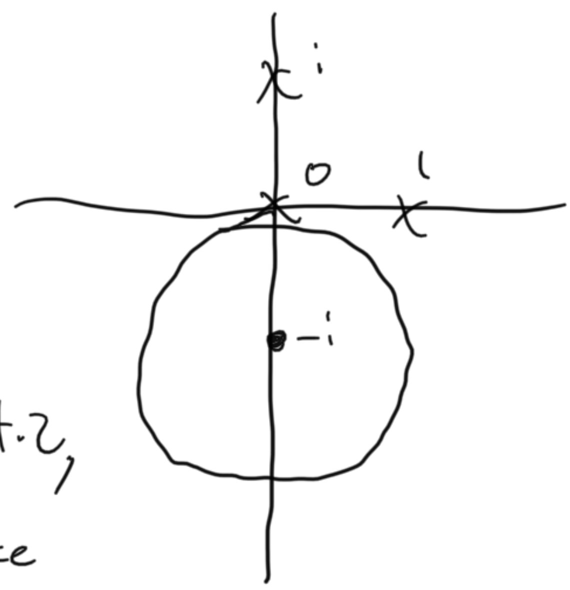
at 0, so $z_0=0$ is an essential singularity

$z_0=1, i$ are both simple poles because $e^{1/z}/(i-z)$ and $e^{1/z}/(1-z)$ are analytic and $\neq 0$ at their respetive points

b) $z_0=-i$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(-i)}{n!}(z+i)^n$$

$f(z)$ analytic in $|z-z_0|<R$



$R=1$

$|z+i|<1$ largest open disk at $z_0=-i$ for which $f(z)$ is analytic, so $R=1$ is the radius of convergence

c) Laurent series $z_0=0$ $|z|>1$

analytic for $|z|>1$

$a_n=0$ $n=1,2,3$

$$e^{1/z}=1+\frac{1}{z}+\frac{1}{2!z^2}+\frac{1}{3!z^3}+\cdots$$

$$\frac{1}{1-z}=-\frac{1}{z}\frac{1}{1-1/z}\overset{|1/z|<1}{=}-\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n$$

$$\frac{1}{i-z}=-\frac{1}{z}\frac{1}{1-i/z}\overset{|i/z|<1}{=}-\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{i}{z}\right)^n$$

$$\frac{e^{1/z}}{(1-z)(i-z)}=\left(\frac{a}{1-z}+\frac{b}{i-z}\right)e^{1/z}$$

$ai-az+b-bz=1$ $\quad ai+b=1$ $\quad -a-b=0$

$a = -\frac{1+i}{2} \quad b = \frac{1+i}{2}$

$$f(z) = \left(1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \cdots\right)\left(\frac{1+i}{2z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n - \frac{1+i}{2z}\sum_{n=0}^{\infty}\left(\frac{i}{z}\right)^n\right)$$

$$f(z) = \left(1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \cdots\right)\underbrace{\left(\frac{1+i}{2z}\right)\left(\sum_{n=0}^{\infty}\frac{1-i^n}{z^n}\right)}_{0 \text{ for } n=0}$$

$$a_0 + a_{-1}\frac{1}{z} + a_{-2}\frac{1}{z^2} + \cdots$$

$n=1 \quad \left(\frac{1+i}{2}\right)\frac{1}{z}\left(\frac{1-i}{z}\right) = \frac{1}{z^2}$

$n=2 \quad \left(\frac{1+i}{2}\right)\frac{1}{z}\frac{2}{z^2} = (1+i)\frac{1}{z^3}$

$n=3 \quad \left(\frac{1+i}{2}\right)\frac{1}{z}\frac{(1+i)}{z^3} = (1+i)^2\frac{1}{z^4}$

$a_0 = 0$

$a_{-1} = 0$

$a_{-2} = 1$

$a_{-3} = (1+i) + 1 = 2+i$

$a_{-4} = (1+i)^2 + (1+i) + \frac{1}{2} = \frac{3}{2} + 3i$

$a_{-2} = 1 \quad a_{-3} = 2+i \quad a_{-4} = \frac{3}{2} + 3i$

5.

a) $I = \int_0^{2\pi} \frac{\sin^2\theta}{5 + 3\cos\theta}\,d\theta \qquad z = e^{i\theta}$

$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - \frac{1}{z}}{2i} \qquad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + \frac{1}{z}}{2}$

$$I = \int_C \frac{\left(\frac{z - \frac{1}{z}}{2i}\right)^2}{5 + 3\,\frac{z + \frac{1}{z}}{2}}\,\frac{1}{iz}\,dz$$

$C:$ 

$|z| = 1$

$$= \int_C \frac{z^2 - 2 + \frac{1}{z^2}}{-20 - 6z - 6\frac{1}{z}}\,\frac{1}{iz}\,dz$$

$$= -\frac{1}{2i}\int_C \frac{z^2 - 2 + \frac{1}{z^2}}{3z^2 + 10z + 3}\,dz$$

$$= -\frac{1}{2i}\int_C \frac{z^4 - 2z^2 + 1}{z^2(z+3)(3z+1)}$$

not in C

$z_0 = -3, -\frac{1}{3}, 0$

$$= -\frac{1}{2i}\, 2\pi i \left(\mathrm{Res}(f;-3) + \mathrm{Res}(f;-\tfrac{1}{3}) + \mathrm{Res}(f;0) \right)$$

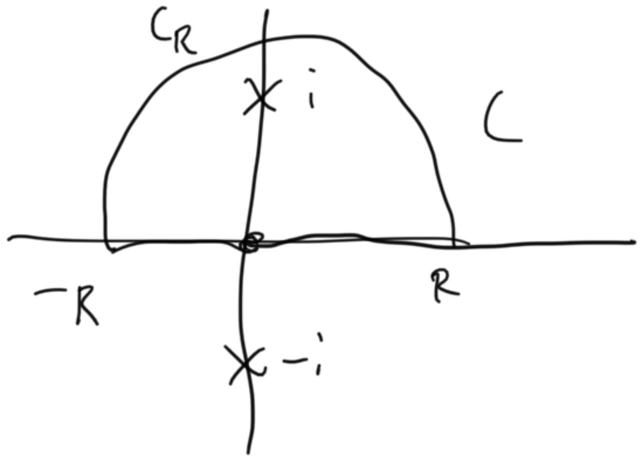
$$\lim_{z \to -\frac{1}{3}} (z+\tfrac{1}{3}) \frac{z^4-2z^2+1}{z^2(z+3)(3z+1)} = \frac{\frac{1}{81} - \frac{2}{9} + 1}{\frac{1}{9} \cdot \left(\frac{8}{3}\right) \cdot 3} = \frac{8}{9}$$

$$\lim_{z \to 0} \frac{d}{dz} \left(z^2 \frac{z^4-2z^2+1}{z^2(z+3)(3z+1)} \right) = \frac{(4z^3-4z)(3z^2+10z+3) - (z^4-2z^2+1)(6z+10)}{(3z^2+10z+3)^2}$$

$$= \frac{0-10}{9} = -\frac{10}{9}$$

$$= -\pi \left(\frac{8}{9} - \frac{10}{9} \right)$$

$$= \frac{2}{9}\pi$$



b) $I = \int_0^\infty \frac{\cos x}{(x^2+1)^2}\, dx$

$$\int_0^\infty \frac{\cos x}{(x^2+1)^2}\, dx$$

$$\int_C \frac{\cos x}{(x^2+1)^2}\, dx = \int_{-R}^0 \frac{\cos x}{(x^2+1)^2}\, dx + \int_0^R \frac{\cos x}{(x^2+1)^2}\, dx + \int_{C_R} \frac{\cos x}{(x^2+1)^2}\, dx$$

f is analytic on and above real axis except $z_0 = i$

$$\left| \frac{\cos x}{(x^2+1)^2} \right| = \frac{|\cos x|}{|x^2+1|^2} \leq \frac{1}{(|x|^2+1)^2} = \frac{1}{(R^2+1)^2} = M$$

$$L = \pi R$$

$$\left| \int_{C_R} \frac{\cos x}{(x^2+1)^2}\, dx \right| \leq \frac{\pi R}{(R^2+1)^2} \to 0 \text{ as } R \to \infty$$

$$\int_{-R}^{0} \frac{\cos x}{(x^2+1)^2}\,dx = \int_{0}^{R} \frac{\cos x}{(x^2+1)^2}\,dx$$

$$\int_C \frac{\cos x}{(x^2+1)^2}\,dx = 2\int_0^R \frac{\cos x}{(x^2+1)^2}\,dx$$

$$\int_C \frac{\cos x}{(x^2+1)^2}\,dx = \int_C \frac{\cos x/(x+i)^2}{(x-i)^2} = 2\pi i\left(\frac{-\sin i}{(i+i)^2} + \frac{-2\cos i}{(i+i)^3}\right)$$

$$= 2\pi i\left(\frac{\sin i}{4} + \frac{\cos i}{4i}\right) = 2\pi i\left(\frac{e^{-1}-e}{8i} + \frac{e^{-1}+e^{1}}{8i}\right)$$

$$= 2\pi i\left(\frac{1}{4ie}\right) = \frac{\pi}{2e}$$