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ACM 95a Final

1. $t^2y'' - 3ty' + 4y = t^2 \ln t \qquad y(1)=1 \quad y'(1)=1$

a) $t_0 = 1$

$t \neq 0 \quad y'' - \frac{3}{t}y' + \frac{4}{t^2}y = \log t$

$p(t) = -\frac{3}{t} \quad q(t) = \frac{4}{t^2} \quad h(t) = \log t$

$p(t), q(t), h(t)$ continuous on $t > 0$

$I = (0, \infty)$

b) $y'' - \frac{3}{t}y' + \frac{4}{t^2}y = \log t$

$y = t^r \quad y' = rt^{r-1} \quad y'' = (r^2 - r)t^{r-2}$

$t^{r-2}(r^2 - 4r + 4) = 0$

$(r-2)(r-2) = 0$

$r = 2$

$y_c = c_1 t^2 + c_2 \log(t) t^2$

$$W = \begin{vmatrix} t^2 & \log(t)\, t^2 \\ 2t & t + 2\log(t)t \end{vmatrix} = t^3 + 2\log(t)t^3 - 2\log(t)t^3 = t^3$$

$$\int_1^t \frac{\log(x)\, x^2 \log(x)}{x^3}\,dx = \int_1^t \frac{\log(x)^2}{x}\,dx$$

$$= \left[\frac{1}{3}\log(x)^3\right]_1^t = \frac{1}{3}\log(t)^3$$

$$\int_1^t \frac{x^2 \log(x)}{x^3}\,dx = \int_1^t \frac{\log(x)}{x}\,dx$$

$$= \left[\frac{1}{2}\log(x)^2\right]_1^t = \frac{1}{2}\log(t)^2$$

$$Y_p = -t^2 \frac{1}{3}\log(t)^3 + \log(t) t^2 \frac{1}{2}\log(t)^2$$
$$= t^2 \log(t)^3 \left(-\frac{1}{3} + \frac{1}{2}\right)$$
$$= \frac{1}{6} t^2 \log(t)^3$$

$$y(t) = C_1 t^2 + C_2 \log(t) t^2 + \frac{1}{6} t^2 \log(t)^3$$

$$y'(t) = 2C_1 t + C_2 t + 2C_2 \log(t) t + \frac{1}{3} t \log(t)^3 + \frac{1}{2} t \log(t)^2$$

$$Y(1) = C_1 = 1$$
$$y'(1) = 2 + C_2 = 1$$
$$C_1 = 1 \quad C_2 = -1$$
$$y(t) = t^2 - t^2 \log(t) + \frac{1}{6} t^2 \log(t)^3$$

2. $y^{(4)} + 5y'' + 4y = \delta(t - \pi/4) + \delta(t - \pi/2)$

$y(0) = y'(0) = y''(0) = y'''(0) = 0$

$t = 3\pi/4$

$$L[y^{(4)} + 5y'' + 4y] = Y(s)\left(s^4 + 5s^2 + 4s\right)$$
$$L[\delta(t-\pi/4) + \delta(t-\pi/2)] = L[\delta(t-\pi/4)] + L\{\delta(t-\pi/2)\}$$
$$= e^{-\pi/4 s} + e^{-\pi/2 s}$$ shifting property

$$Y(s) = \frac{e^{-\pi/4 s} + e^{-\pi/2 s}}{s^4 + 5s^2 + 4s}$$

$$y(s) = L^{-1}\left[\frac{e^{-\pi/4 s} + e^{-\pi/2 s}}{s^4 + 5s^2 + 4}\right]$$
$$= L^{-1}\left[\frac{e^{-\pi/4 s}}{s^4 + 5s^2 + 4}\right] + L^{-1}\left[\frac{e^{-\pi/2 s}}{s^4 + 5s^2 + 4}\right]$$

t-shifting theorem

$$= \mathcal{L}^{-1}\left[\frac{1}{s^4+5s^2+4}\right](t-\pi/4)\cdot H(t-\pi/4)$$

$$+ \mathcal{L}^{-1}\left[\frac{1}{s^4+5s^2+4}\right](t-\pi/2)\cdot H(t-\pi/2)$$

$$\frac{1}{s^4+5s^2+4} = \frac{1}{(s^2+1)(s^2+4)} = \frac{a}{s^2+1} + \frac{b}{s^2+4}$$

$$as^2+4a+bs^2+b = 1$$

$$a=-b \quad 4a+b=1 \quad 3a=1 \quad a=1/3 \quad b=-1/3$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^4+5s^2+4}\right] = \mathcal{L}^{-1}\left[\frac{1/3}{s^2+1} - \frac{1/3}{s^2+4}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{1/3}{s^2+1}\right] - \mathcal{L}^{-1}\left[\frac{1/3}{s^2+4}\right]$$

$$= \frac{1}{3}\sin(t) - \frac{1}{6}\sin(2t)$$

$$= \left(\frac{1}{3}\sin(t-\pi/4) - \frac{1}{6}\sin(2t-\pi/2)\right)\cdot H(t-\pi/4)$$

$$+ \left(\frac{1}{3}\sin(t-\pi/2) - \frac{1}{6}\sin(2t-\pi)\right)\cdot H(t-\pi/2)$$

$$t = \frac{3\pi}{4}$$

$$y\left(\frac{3\pi}{4}\right) = \left(\frac{1}{3}\sin\left(\frac{\pi}{2}\right) - \frac{1}{6}\sin(\pi)\right)$$

$$+ \left(\frac{1}{3}\sin\left(\frac{\pi}{4}\right) - \frac{1}{6}\sin\left(\frac{\pi}{2}\right)\right)$$

$$= \frac{1}{3} + \frac{1}{3}\frac{\sqrt{2}}{2} - \frac{1}{6}$$

$$= \frac{1}{6} + \frac{\sqrt{2}}{6}$$

$$= \frac{1+\sqrt{2}}{6}$$

3.

$$y' = f(t,y) \quad y(t_0) = y_0$$

$$Y_{n+1} = Y_n + \Delta t\, f\left(t_n + \tfrac{1}{2}\Delta t,\ Y_n + \tfrac{1}{2}\Delta t\, f(t_n, Y_n)\right)$$

$$f(t,y) = \alpha t + \beta y + \gamma$$

$$\Downarrow$$

$$Y_{n+1} = Y_n + \Delta t\left(\alpha\left(t_n + \tfrac{1}{2}\Delta t\right) + \beta\left(Y_n + \tfrac{1}{2}\Delta t\, f(t_n, Y_n)\right) + \gamma\right)$$

$$\Downarrow$$

$$Y_{n+1} = Y_n + \Delta t\left(\alpha t_n + \tfrac{1}{2}\alpha\Delta t + \beta Y_n + \tfrac{1}{2}\beta\Delta t\left(\alpha t_n + \beta Y_n + \gamma\right) + \gamma\right)$$

$$\Downarrow$$

$$Y_{n+1} = Y_n + \Delta t\Big(\tfrac{1}{2} f(t_n, Y_n) + \tfrac{1}{2}\alpha t_n + \tfrac{1}{2}\beta Y_n + \tfrac{1}{2}\gamma + \tfrac{1}{2}\alpha\Delta t$$
$$+ \tfrac{1}{2}\alpha\beta t_n\Delta t + \tfrac{1}{2}\beta^2 Y_n\Delta t + \tfrac{1}{2}\beta\gamma\Delta t\Big)$$

$$\Downarrow$$

$$Y_{n+1} = Y_n + \frac{\Delta t}{2}\Big(f(t_n, Y_n) + \alpha t_n + \beta Y_n + \gamma + \alpha\Delta t$$
$$+ \alpha\beta t_n\Delta t + \beta^2 Y_n\Delta t + \beta\gamma\Delta t\Big)$$

$$\Downarrow$$

$$Y_{n+1} = Y_n + \frac{\Delta t}{2}\Big(f(t_n, Y_n) + \alpha\left(t_n + \Delta t\right) + \gamma$$
$$+ \beta\left(Y_n + \Delta t\left(\alpha t_n + \beta Y_n + \gamma\right)\right)\Big)$$

$$\Downarrow$$

$$Y_{n+1} = Y_n + \frac{\Delta t}{2}\Big(f(t_n, Y_n) + \alpha t_{n+1} + \gamma$$
$$+ \beta\left(Y_n + \Delta t\, f(t_n, Y_n)\right)\Big)$$

$$\Downarrow$$

$$Y_{n+1} = Y_n + \frac{\Delta t}{2}\left(f(t_n, Y_n) + f\left(t_{n+1},\ Y_n + \Delta t\, f(t_n, Y_n)\right)\right)$$

4.

$y'' - 2ty' + 8y = 0 \quad y(0) = 4 \quad y'(0) = 0$

a) $t_0 = 0$

$p(t) = -2t \quad q(t) = 8$

$p(t), q(t)$ analytic at t_0

so $t_0 = 0$ is an ordinary point

b)

$$y(t) = \sum_{n=0}^{a} a_n t^n$$

$$\sum_{n=2}^{a} n(n-1)a_n t^{n-2} - 2t\sum_{n=1}^{a} n a_n t^{n-1} + 8\sum_{n=0}^{a} a_n t^n = 0$$

$$\sum_{n=0}^{a} (n+2)(n+1)a_{n+2}t^n - 2n a_n t^n + 8a_n t^n = 0$$

$$(n+2)(n+1)a_{n+2} - (2n-8)a_n = 0$$

$$a_{n+2} = \frac{2n-8}{(n+2)(n+1)} a_n \qquad n = 0,1,2,\ldots$$

$$y(0) = \sum_{n=0}^{a} a_n 0^n = a_0 = 4$$

$$y'(0) = \sum_{n=0}^{a} n a_n 0^{n-1} = a_1 = 0$$

$a_0 = 4$ $\qquad$ $a_1 = 0$

$a_2 = \frac{-8}{2\cdot 1} a_0 = -16$ $\qquad$ $a_3 = 0$

$a_4 = \frac{-4}{4\cdot 3} a_2 = \frac{16}{3}$ $\qquad$ $a_5 = 0$

$a_6 = \frac{0}{6\cdot 5} a_4 = 0$ $\qquad$ $a_7 = 0$

$a_8 = 0$ $\quad$ all 0 below $\quad$ $a_9 = 0$

$\vdots$ $\qquad\qquad$ $\vdots$

$$y(t) = \sum_{n=0}^{a} a_n t^n = 4 - 16t^2 + \frac{16}{3}t^4$$

c)

$$y'' - 2ty' + 8y = 0$$

$p(t) = -2t \quad q(t) = 8$

both analytic everywhere in $\mathbb{C}$

so the power series converges everywhere

5.

$$4ty'' + 2y' + y = 0 \qquad t > 0$$

a) $t_0 = 0$

$$y'' + \frac{1}{2t}y' + \frac{1}{4t}y = 0$$

$$p(t) = \frac{1}{2t} \quad q(t) = \frac{1}{4t}$$

neither are analytic at t_0
so $t_0 = 0$ is singular

$p(t)\cdot t = \frac{1}{2}$
$q(t)\cdot t^2 = \frac{t}{4}$

both of these are analytic at t_0 so $t_0 = 0$ is regular

b) $$a_n F(r+n) + \sum_{k=0}^{n-1} a_k\left(p_{n-k}(k+r) + q_{n-k}\right) = 0 \quad n = 1, 2, \ldots$$

$$p(t)\cdot t = \frac{1}{2} = \sum_{n=0}^{\infty} p_n t^n \quad p_0 = \frac{1}{2} \quad p_1 = p_2 = \cdots = 0$$

$$q(t)\cdot t^2 = \frac{t}{4} = \sum_{n=0}^{\infty} q_n t^n \quad q_0 = 0 \quad q_1 = \frac{1}{4} \quad q_2 = q_3 = \cdots = 0$$

$$F(r) = r(r-1) + p_0 r + q_0 = r(r-1) + \frac{1}{2}r$$

$$= r\left(r - \frac{1}{2}\right) = 0 \quad r_1 = 1/2 \quad r_2 = 0$$

$$y_1(t) = t^{1/2}\left(1 + \sum_{n=1}^{\infty} a_n(1/2)t^n\right)$$

$$F(n + 1/2) = (n + 1/2)n \quad p_{n-k} = 1/2 \quad k = n \quad q_{n-k} = 1/4 \quad k = n-1$$

$$a_n F(r+n) + \sum_{k=0}^{n-1} a_k\left(p_{n-k}(k+r) + q_{n-k}\right) = 0$$

$$a_n \cdot n \cdot (n + 1/2) + a_{n-1}\frac{1}{4} = 0$$

$$a_n = \frac{-a_{n-1}}{4n(n+1/2)} = \frac{a_{n-2}}{16(n+1/2)n(n-1/2)(n-1)}$$

$$= \ldots = \frac{(-1)^n}{4^n(n+1/2)n(n-1/2)(n-1)\cdots(3/2)(1)}$$

$$= \frac{(-1)^n}{(2n+1)n(2n-1)(2n-2)\cdots(3)(2)} = \frac{(-1)^n}{(2n+1)!}$$

$$y_1(t) = t^{1/2}\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n t^n}{(2n+1)!}\right)$$

$$r_1 - r_2 = 1/2$$

$$Y_2(t) = t^0\left(1 + \sum_{n=1}^{\infty} a_n(0)\, t^n\right)$$

$$a_n F(r+n) + \sum_{k=0}^{n-1} a_k\left(p_{n-k}(k+r) + q_{n-k}\right) = 0$$

$$F(n) = n(n - 1/2) \qquad p_{n-k} = 1/2 \quad k=n \quad q_{n-k} = 1/4 \quad k=n-1$$

$$a_n n(n-1/2) + \sum_{k=0}^{n-1} a_k\left(p_{n-k}(k+r) + q_{n-k}\right) = 0$$

$$a_n \cdot n \cdot (n-1/2) + a_{n-1}\frac{1}{4} = 0$$

$$a_n = \frac{-a_{n-1}}{4n(n-1/2)} = \frac{a_{n-2}}{16(n-1/2)n(n-3/2)(n-1)}$$

$$= \dots = \frac{(-1)^n}{4^n (n-1/2)n(n-3/2)(n-1)\cdots(1/2)(1)}$$

$$= \frac{(-1)^n}{(2n-1)n(2n-3)(2n-2)\cdots(1)(2)} = \frac{(-1)^n}{(2n)!}$$

$$Y_2(t) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n t^n}{(2n)!}$$

$$Y = C_1 t^{1/2}\left(1 + \sum_{n=1}^{\infty}\frac{(-1)^n t^n}{(2n+1)!}\right) + C_2\left(1 + \sum_{n=1}^{\infty}\frac{(-1)^n t^n}{(2n)!}\right)$$

$$\sin t = \sum_{k=0}^{\infty}\frac{(-1)^k t^{2k+1}}{(2k+1)!} \qquad \cos t = \sum_{k=0}^{\infty}\frac{(-1)^k t^{2k}}{(2k)!}$$

$$\sin\sqrt{t} = \sum_{k=0}^{\infty}\frac{(-1)^k t^{k+1/2}}{(2k+1)!} \qquad \cos\sqrt{t} = \sum_{k=0}^{\infty}\frac{(-1)^k t^k}{(2k)!}$$

$$= t^{1/2} + \sum_{k=1}^{\infty}\frac{(-1)^k t^{k+1/2}}{(2k+1)!} \qquad = 1 + \sum_{k=1}^{\infty}\frac{(-1)^k t^k}{(2k)!}$$

$$y(t) = C_1 \sin\sqrt{t} + C_2\cos\sqrt{t}$$

c) $y(0) = 2 \quad y'(0) = -1$

$$y(t) = c_1 \sin\sqrt{t} + c_2 \cos\sqrt{t}$$

$$y'(t) = c_1 \frac{1}{2\sqrt{t}} \cos\sqrt{t} - c_2 \frac{1}{2\sqrt{t}} \sin\sqrt{t}$$

$$y(0) = c_2 = 2$$

$$y'(0) = c_1 \frac{1}{2\sqrt{0}} \cos\sqrt{0} - 2 \frac{1}{2\sqrt{0}} \sin\sqrt{0}$$

$$= c_1 \frac{1}{2\sqrt{0}} - \frac{\sin\sqrt{0}}{\sqrt{0}}$$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$= c_1 \frac{1}{2\sqrt{0}} - 1 = -1$$

$$c_1 = 0$$

$$y(t) = 2\cos\sqrt{t}$$