Problem 1

(A)

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 3-\lambda & 2 \\ 1 & 4-\lambda \end{bmatrix}$$

$$det(A - \lambda I) = 12 - 4\lambda - 3\lambda + \lambda^2 - 2 = \lambda^2 - 7\lambda + 10$$

$$\lambda = \frac{7 \pm \sqrt{49-40}}{2} = \frac{7 \pm 3}{2} = 2, 5,\ \lambda_1 = 2,\ \lambda_2 = 5$$

$$A - 2I = \begin{bmatrix} 3-2 & 2 \\ 1 & 4-2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$A - 5I = \begin{bmatrix} 3-5 & 2 \\ 1 & 4-5 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$D = P^{-1}AP = \begin{bmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$$

(B)

$$A = \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} -2-\lambda & 2 \\ -2 & 3-\lambda \end{bmatrix}$$

$$det(A - \lambda I) = -6 + 2\lambda - 3\lambda + \lambda^2 + 4 = \lambda^2 - \lambda - 2$$

$$\lambda = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2} = -1, 2,\ \lambda_1 = -1,\ \lambda_2 = 2$$

$$A + I = \begin{bmatrix} -2+1 & 2 \\ -2 & 3+1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -2 & 4 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$A - 2I = \begin{bmatrix} -2-2 & 2 \\ -2 & 3-2 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ -2 & 1 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$D = P^{-1}AP = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

(C)

$$A = \begin{bmatrix} 1 & \frac{1}{16} \\ 1 & \frac{1}{2} \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & \frac{1}{16} \\ 1 & \frac{1}{2}-\lambda \end{bmatrix}$$

$$det(A - \lambda I) = \frac{1}{2} - \frac{1}{2}\lambda - \lambda + \lambda^2 - \frac{1}{16} = \lambda^2 - \frac{3}{2}\lambda + \frac{7}{16}$$

$$\lambda = \frac{\frac{3}{2} \pm \sqrt{\frac{9}{4} - \frac{7}{4}}}{2} = \frac{3 \pm \sqrt{2}}{4},\ \lambda_1 = \frac{3+\sqrt{2}}{4},\ \lambda_2 = \frac{3-\sqrt{2}}{4}$$

$$A - \frac{3+\sqrt{2}}{4}I = \begin{bmatrix} 1 - \frac{3+\sqrt{2}}{4} & \frac{1}{16} \\ 1 & \frac{1}{2} - \frac{3+\sqrt{2}}{4} \end{bmatrix} = \begin{bmatrix} \frac{1-\sqrt{2}}{4} & \frac{1}{16} \\ 1 & \frac{-1-\sqrt{2}}{4} \end{bmatrix}$$

$$v_1 = \begin{bmatrix} 1+\sqrt{2} \\ 4 \end{bmatrix}$$

$$A - \frac{3-\sqrt{2}}{4}I = \begin{bmatrix} 1 - \frac{3-\sqrt{2}}{4} & \frac{1}{16} \\ 1 & \frac{1}{2} - \frac{3-\sqrt{2}}{4} \end{bmatrix} = \begin{bmatrix} \frac{1+\sqrt{2}}{4} & \frac{1}{16} \\ 1 & \frac{-1+\sqrt{2}}{4} \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 1-\sqrt{2} \\ 4 \end{bmatrix}$$

$$P = \begin{bmatrix} 1+\sqrt{2} & 1-\sqrt{2} \\ 4 & 4 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} \frac{1}{2\sqrt{2}} & -\frac{1-\sqrt{2}}{8\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & \frac{1+\sqrt{2}}{8\sqrt{2}} \end{bmatrix}$$

$$D = P^{-1}AP = \begin{bmatrix} \frac{1}{2\sqrt{2}} & -\frac{1-\sqrt{2}}{8\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & \frac{1+\sqrt{2}}{8\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{16} \\ 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1+\sqrt{2} & 1-\sqrt{2} \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} \frac{3+\sqrt{2}}{4} & 0 \\ 0 & \frac{3-\sqrt{2}}{4} \end{bmatrix}$$

(D)

$$A = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} cos(\theta) - \lambda & -sin(\theta) \\ sin(\theta) & cos(\theta) - \lambda \end{bmatrix}$$

$$det(A - \lambda I) = cos^2(\theta) - 2cos(\theta)\lambda + \lambda^2 + sin^2(\theta) = \lambda^2 - 2cos(\theta)\lambda + 1$$

$$\lambda = \frac{2cos(\theta) \pm \sqrt{4cos^2(\theta) - 4}}{2} = cos(\theta) \pm \sqrt{cos^2(\theta) - 1} = cos(\theta) \pm \sqrt{-sin^2(\theta)} = cos(\theta) \pm isin(\theta),\ \lambda_1 = cos(\theta) + isin(\theta),\ \lambda_2 = cos(\theta) - isin(\theta)$$

$$A - (cos(\theta) + isin(\theta))I = \begin{bmatrix} cos(\theta) - (cos(\theta) + isin(\theta)) & -sin(\theta) \\ sin(\theta) & cos(\theta) - (cos(\theta) + isin(\theta)) \end{bmatrix}$$

$$= \begin{bmatrix} -isin(\theta) & -sin(\theta) \\ sin(\theta) & -isin(\theta) \end{bmatrix}$$

$$v_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$A - (cos(\theta) - isin(\theta))I = \begin{bmatrix} cos(\theta) - (cos(\theta) - isin(\theta)) & -sin(\theta) \\ sin(\theta) & cos(\theta) - (cos(\theta) - isin(\theta)) \end{bmatrix}$$

$$= \begin{bmatrix} isin(\theta) & -sin(\theta) \\ sin(\theta) & isin(\theta) \end{bmatrix}$$

$$v_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} -\frac{i}{2} & \frac{1}{2} \\ \frac{i}{2} & \frac{1}{2} \end{bmatrix}$$

$$D = P^{-1}AP = \begin{bmatrix} -\frac{i}{2} & \frac{1}{2} \\ \frac{i}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} cos(\theta) + isin(\theta) & 0 \\ 0 & cos(\theta) - isin(\theta) \end{bmatrix}$$

Problem 2

In [1]:

```
import numpy as np
import numpy.linalg as LA
import networkx as nx
import matplotlib.pyplot as plt
```

In [2]:

```
A = np.array([[1,2,3],[0,4,5],[0,0,6]]);
print(A)
l,v = LA.eig(A)
print(np.round(l,2))
```

```
P = v;
Ad = np.matmul(LA.inv(P),np.matmul(A,P))
print(np.round(Ad,2))
```

```
[[1 2 3]
 [0 4 5]
 [0 0 6]]
[1. 4. 6.]
[[ 1. -0. -0.]
 [ 0.  4. -0.]
 [ 0.  0.  6.]]
```

In [3]:

```
A = np.array([[0,0,1],[0,1,0],[1,0,0]]);
print(A)
l,v = LA.eig(A)
print(np.round(l,2))
P = v;
Ad = np.matmul(LA.inv(P),np.matmul(A,P))
print(np.round(Ad,2))
```

```
[[0 0 1]
 [0 1 0]
 [1 0 0]]
[ 1. -1.  1.]
[[ 1.  0.  0.]
 [-0. -1.  0.]
 [ 0.  0.  1.]]
```

In [4]:

```
A = np.array([[1,1,1,0],[0,1,0,1],[1,0,1,-1],[1,0,0,-1]]);
print(A)
l,v = LA.eig(A)
print(np.round(l,2))
P = v;
Ad = np.matmul(LA.inv(P),np.matmul(A,P))
print(np.round(Ad,2))
```

```
[[ 1  1  1  0]
 [ 0  1  0  1]
 [ 1  0  1 -1]
 [ 1  0  0 -1]]
[ 2. -1. -0.  1.]
[[ 2. -0.  0.  0.]
 [ 0. -1.  0.  0.]
 [ 0. -0.  0.  0.]
 [ 0.  0. -0.  1.]]
```

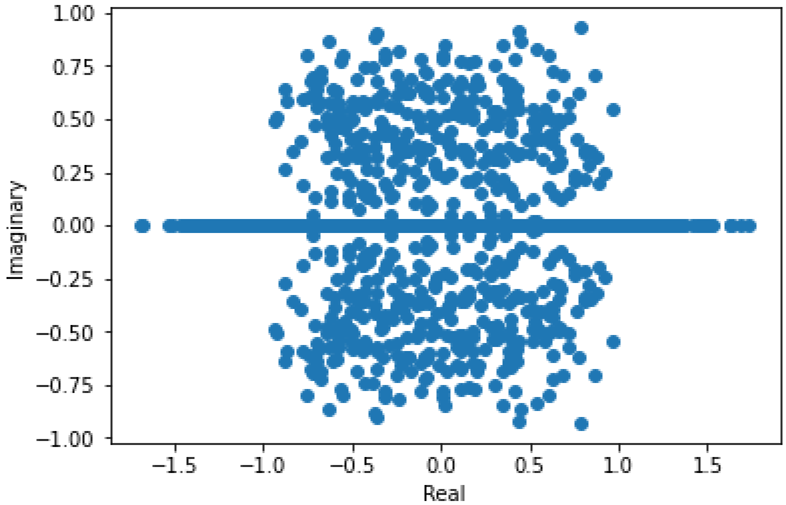
Problem 3

In [23]:

```
x=[]
y=[]
for i in range(1000):
  A = np.random.rand(2,2)*2-1
  # print(A)
  l,v = LA.eig(A)
  # print(np.round(l,2))
  P = v;
  Ad = np.matmul(LA.inv(P),np.matmul(A,P))
  # print(np.round(Ad,2))
  x.append(l.real)
  y.append(l.imag)

# plot the complex numbers
```

```
plt.scatter(x, y)
plt.ylabel('Imaginary')
plt.xlabel('Real')
plt.show()
```

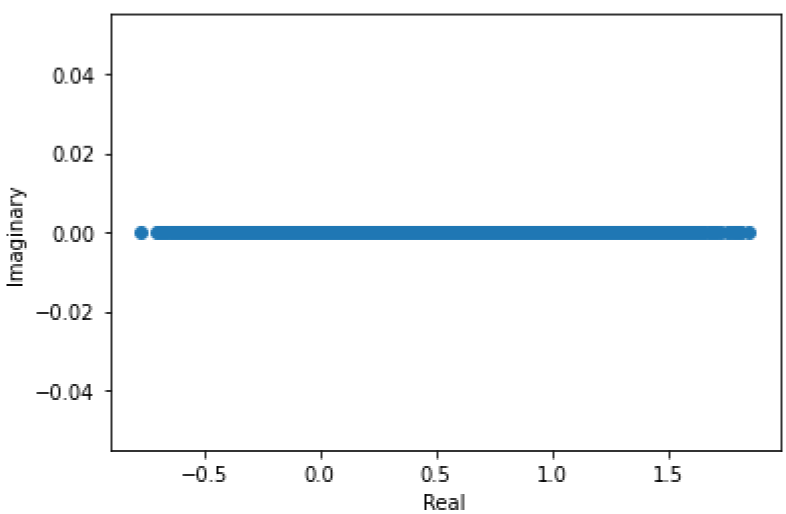


The pure real eigenvalues are on the real axis from -2 to 2, while the eigenvalues with imaginary parts are inbetween -1 to 1 in both the real and imaginary axes.

In [24]:

```
x=[]
y=[]
for i in range(1000):
  A = np.random.rand(2,2)
  # print(A)
  l,v = LA.eig(A)
  # print(np.round(l,2))
  P = v;
  Ad = np.matmul(LA.inv(P),np.matmul(A,P))
  # print(np.round(Ad,2))
  x.append(l.real)
  y.append(l.imag)

# plot the complex numbers
plt.scatter(x, y)
plt.ylabel('Imaginary')
plt.xlabel('Real')
plt.show()
```

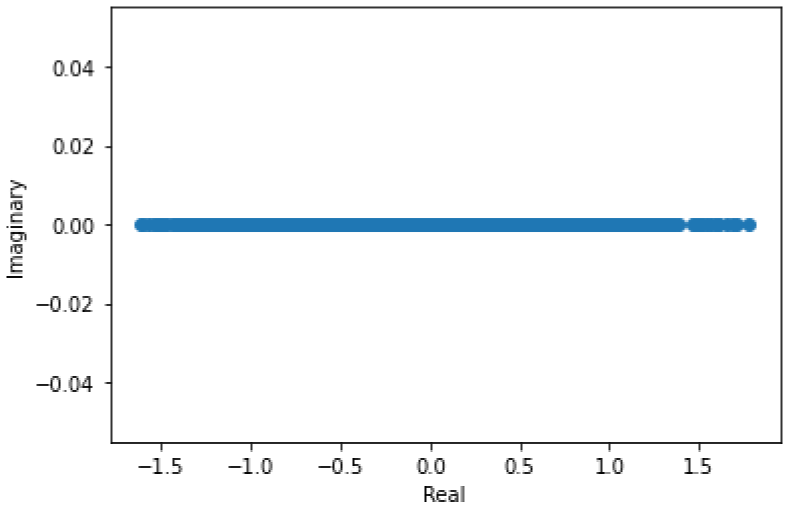


For the 0 to 1 interval, we only get eigenvalues on the real axis from -1 to 2.

In [25]:

```
x=[]
y=[]
for i in range(1000):
  An = np.random.rand(2,2)*2-1
  A = (An+An.T)/2
  # print(A)
  l,v = LA.eig(A)
  # print(np.round(l,2))
  P = v;
  Ad = np.matmul(LA.inv(P),np.matmul(A,P))
  # print(np.round(Ad,2))
  x.append(l.real)
  y.append(l.imag)

# plot the complex numbers
plt.scatter(x, y)
plt.ylabel('Imaginary')
plt.xlabel('Real')
plt.show()
```

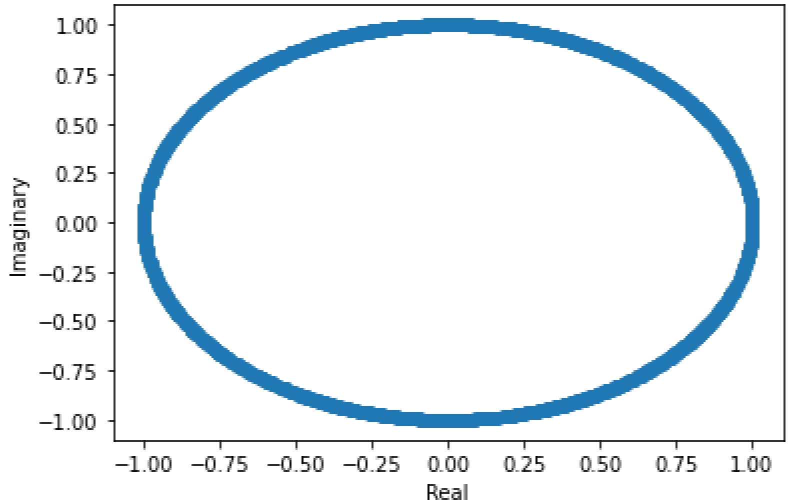


For symmetric matrices, we also only get eigenvalues on the real axis.

In [26]:

```
x=[]
y=[]
for i in range(1000):
  An = np.random.uniform(0,2*np.pi)
  A = np.array([[np.cos(An),-np.sin(An)],[np.sin(An),np.cos(An)]])
  # print(A)
  l,v = LA.eig(A)
  # print(np.round(l,2))
  P = v;
  Ad = np.matmul(LA.inv(P),np.matmul(A,P))
  # print(np.round(Ad,2))
  x.append(l.real)
  y.append(l.imag)

# plot the complex numbers
plt.scatter(x, y)
plt.ylabel('Imaginary')
plt.xlabel('Real')
plt.show()
```



For this matrix, we only get eigenvalues on the unit circle of the real/imaginary graph.

Problem 4

Since v is an eigenvector of T, we know that $T\mathbf{v} = \lambda\mathbf{v}$. This means that repeated application of T to $\mathbf{v}$ is the same as repeated multiplication of $\mathbf{v}$ by λ. Therefore, $T^n\mathbf{v} = \lambda^n\mathbf{v}$.

Problem 5

(A)

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$$

$$det(A - \lambda I) = ad - a\lambda - d\lambda + \lambda^2 - bc = \lambda^2 - (a+d)\lambda + ad - bc$$

$$\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2} = \frac{(a+d) \pm \sqrt{a^2-2ad+d^2+4bc}}{2},\ \lambda_+ = \frac{(a+d)+\sqrt{a^2-2ad+d^2+4bc}}{2},\ \lambda_- = \frac{(a+d)-\sqrt{a^2-2ad+d^2+4bc}}{2}$$

(B)

$$\lambda_+ * \lambda_- = \frac{(a+d)+\sqrt{a^2-2ad+d^2+4bc}}{2} * \frac{(a+d)-\sqrt{a^2-2ad+d^2+4bc}}{2} = \frac{(a+d)^2-a^2+2ad-d^2-4bc}{4} = \frac{a^2+2ad+d^2-a^2+2ad-d^2-4bc}{4} = ad - bc = det(A)$$

$$\lambda_+ + \lambda_- = \frac{(a+d)+\sqrt{a^2-2ad+d^2+4bc}}{2} + \frac{(a+d)-\sqrt{a^2-2ad+d^2+4bc}}{2} = a + d = trace(A)$$

Problem 6

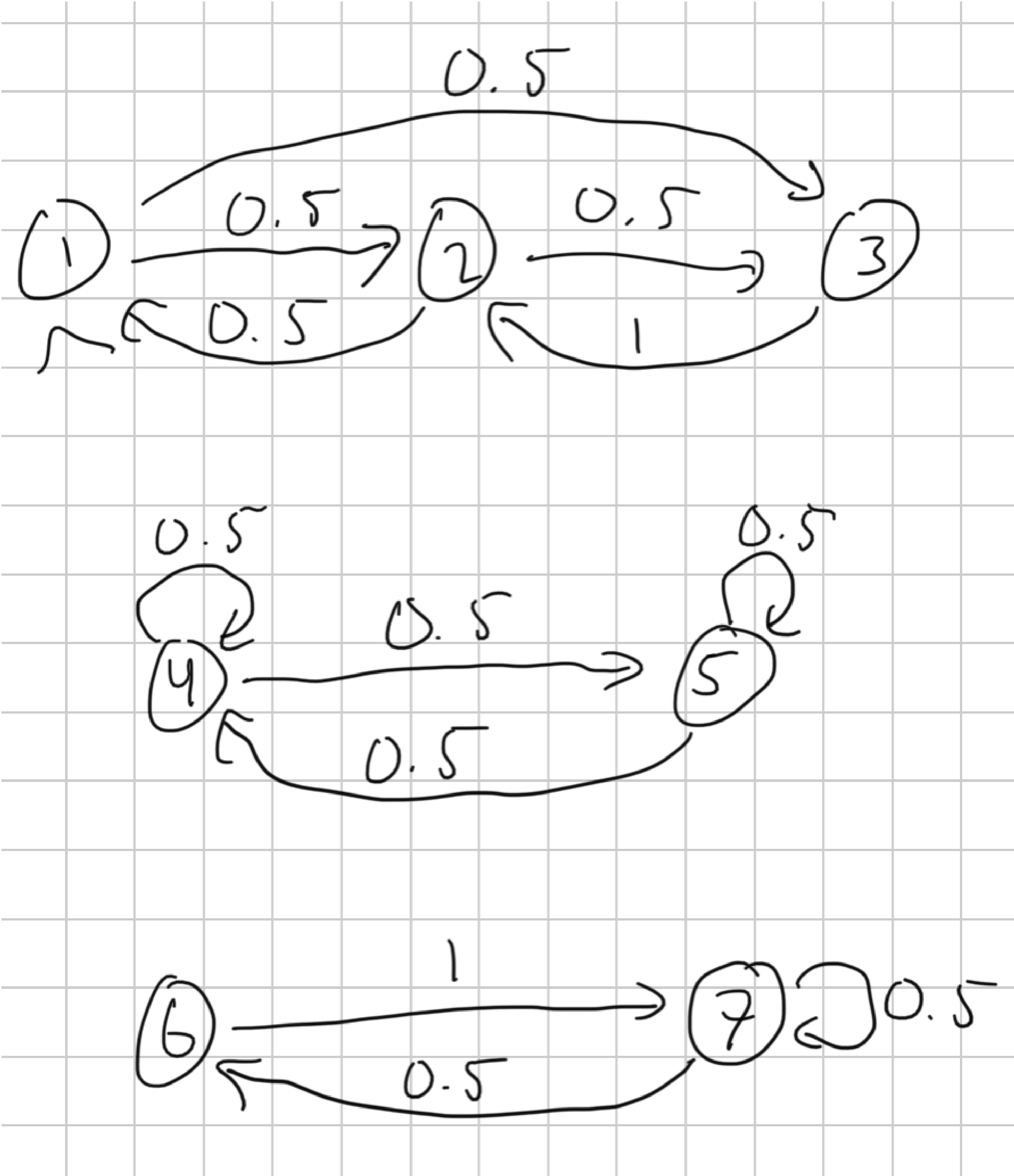
(A)

The sum of each matrix column is 1 because the columns are describing where the material on each node is going; each column needs to sum to 1 so we account for all the material at that node. If one has a certain concentration of a gene active at time t, we can see what genes at what concentrations will be active because of that gene at time t+1.

The row sum is not necessarily 1 because it represents how much material we will be getting for each node. Depending on the network, there may be many genes or very few genes that upregulate a certain gene, so the sum of the row for that gene may be large or small depending on how many genes regulate it and at what level they regulate it.

(B)

This network has three components.



(C)

Since $x(0)$ is the eigenvector of W, we know that $Wx(0) = \lambda x(0)$. This means that repeated application of W to $x(0)$ is the same as repeated multiplication of $x(0)$ by λ. Therefore, $W^n x(0) = \lambda^n x(0)$.

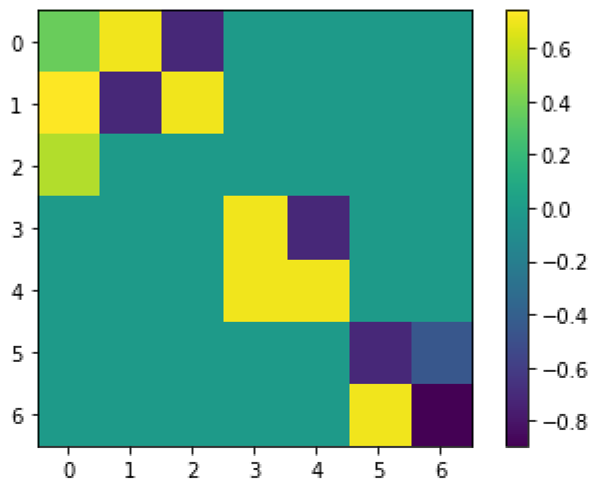
(D)

In [9]:

```
W = np.array([[0,0.5,0,0,0,0,0],[0.5,0,1,0,0,0,0],[0.5,0.5,0,0,0,0,0],[0,0,0,0.5,0.5,0,0],
# print(W)
```

```
l,v = LA.eig(W)
print(np.round(l,2))
P = v;
print(np.round(P,2))
Wd = np.matmul(LA.inv(P),np.matmul(W,P))
# print(np.round(Wd,2))
plt.imshow(P)
plt.colorbar()
plt.show()
```

```
[ 1.   -0.5 -0.5  1.    0.   -0.5  1.  ]
[[ 0.37  0.71 -0.71  0.    0.    0.    0.  ]
 [ 0.74 -0.71  0.71  0.    0.    0.    0.  ]
 [ 0.56  0.    0.    0.    0.    0.    0.  ]
 [ 0.    0.    0.    0.71 -0.71  0.    0.  ]
 [ 0.    0.    0.    0.71  0.71  0.    0.  ]
 [ 0.    0.    0.    0.    0.   -0.71 -0.45]
 [ 0.    0.    0.    0.    0.    0.71 -0.89]]
```



(E)

Since the network has three disconnected components, our eigenvalues and corresponding eigenvectors are also disconnected and correspond to the disconnected components of the network.