Problem 1

(A)

$$L * e_x = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$L * e_y = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

(B)

$$L * \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

Qualitatively, L makes y negative and flips the vector.

(C)

$$L^2 * \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -y \\ x \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$$

By applying L twice, we essentially just make the vector the negative of its original value.

$$det(L) = det(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}) = 1$$

$$det(L^2) = det(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}) = det(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}) = 1$$

Problem 2

(A)

$$L = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$y_1 = x_2, \ y_2 = x_1, \ y_3 = x_1 + x_2$$

(B)

$$dim(range(L)) = 3$$

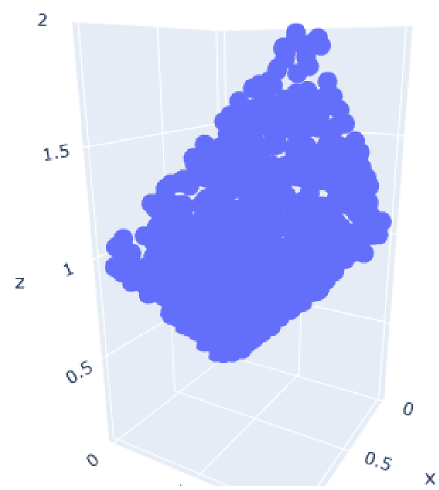
In [3]:

```
import numpy as np
import matplotlib.pyplot as plt
from numpy.linalg import inv
#from operator import itemgetter
#from mpl_toolkits import mplot3d
#from itertools import starmap
import plotly.express as px
#import pandas as pd
```

```python
import plotly.graph_objects as go
import itertools
L = np.array([[0,1],[1,0],[1,1]])
print(L)
data = np.zeros((800,2));
Ldata = np.zeros((800,3));
for i in range(800):

  data[i,:] = (np.random.rand(1,2));
  Ldata[i,:] = np.dot(L,data[i,:]);
fig=go.Figure(data=[go.Scatter3d(x=Ldata[:,0], y=Ldata[:,1], z=Ldata[:,2],
                                   mode='markers')])
fig.show()
```

```
[[0 1]
 [1 0]
 [1 1]]
```



Problem 3

(A)

D = $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

(B)

The dimension of ker(D) is 1 because only contant polynomials have a zero derivative.

Basis of ker(D) is = $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

(C)

The dimension of ker(D^2) is 2 because contant and first order polynomials have a zero derivative.

Basis of ker(D^2) is = $\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\}$

(D)

To show that ker(T) is a closed subspace, we have to show it is closed under addition, show it is closed under scalar multiplication, and show that the vector 0 is in the subset.

Given $u,\ v \in ker(T),\ T(u) = 0,\ T(v) = 0,\ \text{and some scalar}\ a$:

$T(v+u) = T(v) + T(u) = 0$
$T(a * v) = a * T(u) = 0$
$T(0) = T(v - v) = T(v) + T(-v) = T(v) - T(v) = 0$

Problem 4

Using the typer writer that has two letters of uncertainty, A-> A, A-> B, B-> B, B->C, C-> C, C-> D, D-> D, D-> A where each arrow has .5 probability.

(A)

W = $\begin{bmatrix} 0.5 & 0 & 0 & 0.5 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0.5 \end{bmatrix}$

(B)

$$y = Wx = \begin{bmatrix} 0.5 & 0 & 0 & 0.5 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} P(A) \\ P(B) \\ P(C) \\ P(D) \end{bmatrix} = \begin{bmatrix} 0.5 * P(A) + 0.5 * P(D) \\ 0.5 * P(B) + 0.5 * P(A) \\ 0.5 * P(C) + 0.5 * P(B) \\ 0.5 * P(D) + 0.5 * P(C) \end{bmatrix}$$

(C)

The probability of each getting each element of y is just the probability of the two letters that could output that letter multiplied by the probability that each of the input letters give that letter.

(D)

$$W^2x = Wy = \begin{bmatrix} 0.5 & 0 & 0 & 0.5 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0.5 * P(A) + 0.5 * P(D) \\ 0.5 * P(B) + 0.5 * P(A) \\ 0.5 * P(C) + 0.5 * P(B) \\ 0.5 * P(D) + 0.5 * P(C) \end{bmatrix}$$
$$= \begin{bmatrix} 0.25 * P(A) + 0.5 * P(D) + 0.25 * P(C) \\ 0.25 * P(B) + 0.5 * P(A) + 0.25 * P(D) \\ 0.25 * P(C) + 0.5 * P(B) + 0.25 * P(A) \\ 0.25 * P(D) + 0.5 * P(C) + 0.25 * P(B) \end{bmatrix}$$

This map represents two sequential noisy type-writers that make the input more noisy each time. For W^n as n get big enough, the input gets noisy enough that the outputs are all equally likely, so for a single entry $x = (1, 0, 0, 0)$, we would get probabilities around $(0.25, 0.25, 0.25, 0.25)$.

Problem 5

(A)

$$T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

(B)

The dimension of ker(T) is 1 because x_2 must be negative x_1 and x_3 must be half x_1.

(C)

Basis of ker(T) is = $\{ \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \}$

Problem 6

(A)

$$w = \begin{bmatrix} 0.25 & 0.25 \end{bmatrix}$$

(B)

$v_1 w_1$ must be more than 0.5 to for the $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ input to be 1 and $v_2 w_2$ must be more than 0.5 to for the $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ input to be 1. Since we these add for the $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ input, there is no way for this input to be 0 so XOR is impossible.