1 Introduction

The purpose of this article is to analytically solve the Black-Scholes diffusion equation, and demonstrate using the Euler-Maruyama method to model the stochastic system. The Black-Scholes equation is a classic model for representing a single stock price using Brownian motion. My contribution is to model drift and volatility by drawing from the sample distribution created by historical data, thus modeling the uncertainty in our estimates for those values considered constant by the model.

Brownian motion is the integral of white noise, and is a set of normally distributed random variables W_t for each $t \in \mathbb{R}^+$ where $W_t - W_{t-1}$ is independent of W_{t-1} (as well as all previous instances $W_1, \ldots, W_{t-2}$).

2 Analytic solution

The Black-Scholes equation takes the following form (Sauer 2012):

$$\begin{cases} dX = \mu X\, dt + \sigma X\, dW_t \\ X(0) = X_0 \end{cases} \tag{1}$$

where μ, σ, and X_0 are constants. Here, μ represents the drift (or expected change in value of the stock), σ represents the diffusion (the variance of that change), and X_0 is the initial value (stock price).

If $Y = f(t, X)$, Itô's lemma is defined by Sauer (2012) as

$$dY = \frac{\partial f}{\partial t}(t, X)\, dt + \frac{\partial f}{\partial x}(t, X)\, dx + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X)\, dx\, dx, \tag{2}$$

where dx is the differential with respect to the random variable X. $dx\, dx$ can be interpreted using the following identities.

$$\begin{gathered} dt\, dt = 0 \\ dt\, dW_t = dW_t\, dt = 0 \\ dW_t\, dW_t = dt \end{gathered} \tag{3}$$

Using Itô's lemma with the inner random variable as W_t, the Black-Scholes equation can be shown to be satisfied by $X = f(t, Y) = X_0 e^Y$, where $Y = \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t$.

$$\begin{aligned} dX &= X_0 e^Y \left(\mu - \frac{1}{2}\sigma^2\right) dt + X_0 e^Y \sigma \, dW_t + \frac{1}{2} X_0 e^Y \sigma^2 \, dW_t \, dW_t \\ &= X \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma X \, dW_t + \frac{1}{2}\sigma^2 X \, dt \\ &= \mu X \, dt + \sigma X \, dW_t \end{aligned}$$

3 Parameter estimation

To estimate the parameter μ, I use finite time steps Δt to approximate dt. The stock data I'm using is given by day, so I can use $\Delta t = 1$. The *average daily drift* $\hat{\mu}$ is estimated from the data by calculating

$$\hat{\mu} = \frac{1}{N-1} \sum_{i=2}^{N} (X_i - X_{i-1}); \tag{4}$$

daily drift models the average daily change of the value of X. *Average daily volatility* is estimated similarly:

$$\hat{\sigma}^2 = \frac{1}{N-2} \sum_{i=2}^{N} (X_i - X_{i-1} - \hat{\mu})^2. \tag{5}$$

Note that this is the unbiased variance estimator.

4 Simulation

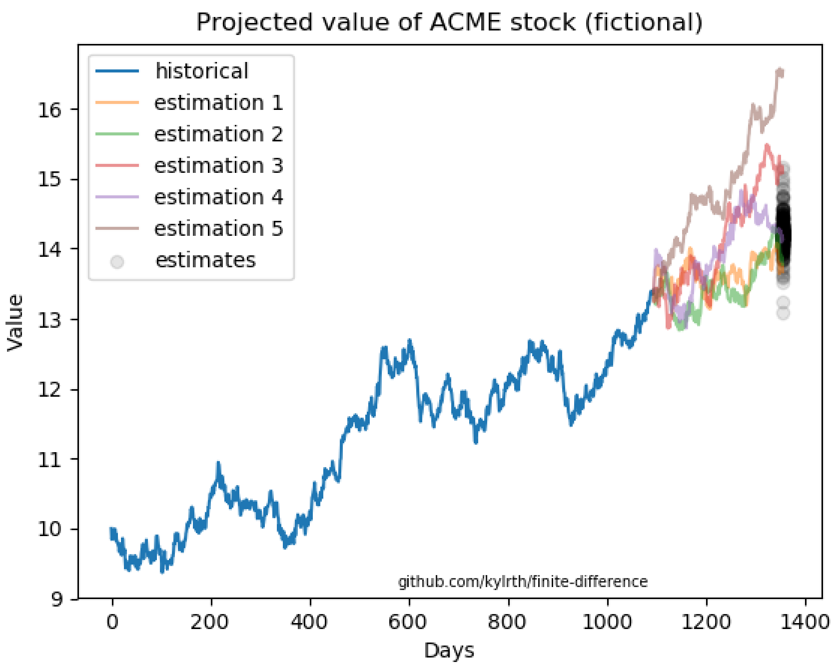
The Euler-Maruyama method estimates a solution to the system by iteratively calculating the following values, adapted from Sauer (2012):

$$\begin{aligned} x_0 &= X_0 \\ x_{i+1} &= x_i + a(t_i, x_i)\Delta t_{i+1} + b(t_i, x_i)\Delta W_{i+1} \end{aligned} \tag{6}$$

where each ΔW_i is modeled as $\Delta W_i = z_i\sqrt{\Delta t_i}$, with $z_i \sim \mathcal{N}(0, 1)$. a and b are functions that model the distribution of sample estimates of μ and σ. Given data sampled from the true distribution of X, we can approximate the true values of μ and σ by sampling from $\{\overline{\mu}_i\}_{i=2}^N$ and $\{\overline{\sigma}_i\}_{i=2}^N$ respectively, where $\overline{\mu}_i = X_i - X_{i-1}$ and $\overline{\sigma}_i = X_i - X_{i-1} - \hat{\mu}$. (Optionally, samples from the latter can be multiplied by $(n-1)/(n-2)$ to remove bias.) In my implementation, functions representing a and b return these estimated values in place of returning the constant estimators $\hat{\mu}$ and $\hat{\sigma}$.

5 Results

I estimate the value of a stock on December 31, 2019, given data that ends on December 15, 2018. There are 10 trading days left in 2018, and 252 trading days in 2019, so we want to project results 262 trading days from December 15, 2018. Five runs of my implementation of the Euler-Maruyama method are displayed below, alongside the historical data.



The line estimations are projections using the Euler-Maruyama method, sampling estimates of μ and σ as described. The gray points at the end are Euler-Maruyama estimates without calculating intermediate points. For these projections, we use $\hat{\mu}$ from the data but sample from $\{\overline{\sigma}_i\}_{i=2}^N$ as before. I found this to be optimal in describing the possible range that the stock price could reach, because we can observe the distribution of results assuming the same experimental mean drift while under no strong assumption about volatility. The mean and variance of the projections were 14.206 and 0.019904 for this run.

Code to produce this plot is hosted at `github.com/kylrth/finite-difference/` in `black_scholes.py`.

6 Future work

This model assumed constant true values for μ and σ over the time domain, a poor assumption for most real-world systems. An improvement left for future work is to avoid this assumption by projecting the change in the sample estimates $\overline{\mu}_i$ and $\overline{\sigma}_i$ out into the domain being estimated.

7 Acknowledgements

Jason Gardiner pointed me toward Sauer's explanation of SDEs, and gave general pointers about the goal of the project.

I read an explanation of parameter estimation for a similar problem in a StackExchange question entitled "Estimating the historical drift and volatility".

8 References

Timothy Sauer. 2012. Numerical solution of stochastic differential equations in finance. In *Handbook of Computational Finance*, Springer (4). 529–550.