Week 1 Seminar: Calculus

Math Divulged

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Welcome to Math Divulged! We're a team of competitive math veterans, including experienced competitors such as Andrew Cai and Luke Robitaille. Students of any level are encouraged to attend our classes and use our resources! Visit our website at Math Divulged, and email us at mathdivulged@gmail.com.

This handout is a brief outline over the basics of calculus. Concepts covered are further expanded upon in our class. Our teacher and TA for this class are Ben Wright and Kevin Chang. The corresponding YouTube recording is uploaded on our website.

§1 Derivatives

Derivatives appear everywhere, and are a necessary building block for calculus. However, to understand what they are, we first need to define *limits*.

§1.1 Limits

Definition 1.1. The *limit* of a function $f(x)$ as x approaches a (denoted $\lim_{x\to a} f(x)$) is defined as the value that $f(x)$ is infinitely close to in the region infinitely close to x.

If a function f is continuous, then $\lim_{x\to a} f(x) = f(a)$. However, such a limit has no intrinsic requirement on the value of $f(a)$ if the function is not continuous. If $f(a+\delta) \approx L$ for all δ approximately but not exactly equal to 0, but $f(a) = K$, then $\lim_{x\to a} f(x) = L$.

§1.2 Instantaneous Slope

You might be familiar with calculating the slope between two points. *Instantaneous slope* is calculating the slope of two very close points on a function.

Formally, the instantaneous slope of $f(x)$ at $x = a$ is

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

This will yield the slope of a line tangent to $f(x)$ at $x = a$.

Definition 1.2. The *derivative* of a function $f(x)$ is the function that outputs the instantaneous slope of $f(x)$ at the location of the input.

As an example, take the function $f(x) = 1 + 2x$. At every x-value, the slope is 2, and so the derivative of the f is the constant function 2.

§1.3 Notation

The way we notate derivatives varies, though there are a few standard ways. Different notations are helpful in different scenarios.

The derivative of f can be represented as:

- f', useful when taking many derivatives.
- $\frac{df}{dx}$, useful when using chain rule.
- $\frac{d}{dx}f$, useful when treating $\frac{d}{dx}$ as an operator, or a function of functions.
- Df, useful when treating D, the derivative, as a variable.

§1.4 Common Derivatives

Theorem 1.3 (Polynomials)
$\frac{d}{dx}x^n = n * x^{n-1}$, true for *any* value of n.

Theorem 1.4 (Exponentiation)
$\frac{d}{dx}e^x = e^x$, and $\frac{d}{dx}a^x = \ln(a) * a^x$.

Theorem 1.5 (Logarithms)
$\frac{d}{dx}\ln(x) = \frac{1}{x}$.

Theorem 1.6 (Trigonometry)
$\frac{d}{dx}\sin(x) = -\cos(x)$, and $\frac{d}{dx}\cos(x) = \sin(x)$.

§1.5 Chain Rule

Theorem 1.7 (Chain Rule)

$$\frac{df}{dx} = \frac{df}{dg} * \frac{dg}{dx}$$

This theorem makes more intuitive sense if we treat $\frac{df}{dx}$ as "a small change in f over a small change in x". This allows us to divide, manipulate, and cancel terms intuitively.

§1.6 Product Rule

Theorem 1.8 (Product Rule)

$$\frac{d}{dx}(fg) = f * \frac{d}{dx}g + g * \frac{d}{dx}f$$

§1.7 Quotient Rule

Theorem 1.9 (Quotient Rule)

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g * \frac{d}{dx}f - f * \frac{d}{dx}g}{g^2}$$

§2 Integration

§2.1 Anti-derivatives

Definition 2.1. An anti-derivative of a function f, denoted as f_A, is a function such that $\frac{d}{dx}f_A = f$.

Note that adding a constant to a function does not change its derivative; therefore, f_A plus a constant is still an anti-derivative of f.

§2.2 Integrals

Definition 2.2. $\int_a^b f(x)dx$ is the area under the curve of $f(x)$ in the range $a <= x <= b$.

Definition 2.3 (Formal)**.**

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left[f\left(a + i * \frac{b-a}{n}\right) * \frac{b-a}{n} \right]$$

§2.3 Fundamental Theorem of Calculus

Theorem 2.4 (Fundamental Theorem of Calculus)

$$\int_a^b f(x)dx = f_A(b) - f_A(a)$$