CO2020 : Computer-Aided Numerical Methods II

Course Instructors: Niranjan S. Ghaisas and Raja Banerjee

Homework 4

Due Date: 5 April 2023

Topic: Elliptic PDEs

Question 1: The 2D steady heat conduction equation,

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\dot{q}=0,$$

is to be solved using a second-order accurate finite difference method on the domain $[0, Lx] \times [0, Ly]$. The domain is discretized into $nx \times ny$ grid points. The grid points are $x_i = i \times h_x$ with $i = 0, 1, 2, ...nx-1$ and $h_x = L_x/(nx-1)$ in x and similarly in y.

We assume that homogeneous Dirichlet conditions are specified at the left, right and top boundaries, i.e. $T(x = 0) = T(x = Lx) = T(y = Ly) = 0$, and homogeneous Neumann condition is specified at the bottom, i.e. $(\partial T/\partial y)\,(y = 0) = 0$.

(a) Determine analytically the expression for $\dot{q}$ for the exact solution to be $T_{ex} = x(1-x)cos(\pi y)$ on the domain $Lx = 1$ and $Ly = 0.5$ and a constant diffusivity, $k = 1$. Note that the exact solution satisfies the boundary conditions prescribed above.

(b) Starting from the skeleton code given to you, write a code to solve the problem described in part (a) numerically.

(c) Do the calculations for computing the coefficients by hand for a small system, e.g. $nx = 5$ and $ny = 3$. Ensure that your hand calculations agree with the answers computed by the code. You need not report anything for this part. Once you have ensured correctness of the code, you can comment out any unnecessary 'printf' statements so that the output of the code remains clean.

(d) Run the code for about 5 or 6 grids such as 9×5, 17×9, 33×17, etc. For a few selected grids, plot the contours of the numerical and exact solutions.

(e) Calculate the error norm for different grid sizes and hence verify the order of accuracy. This should be reported as a plot of the error norm vs the number of grid points along one direction (or vs the grid spacing).

Question 2: Consider a problem with the same domain size and the same exact solution but with a varying diffusivity, $k = A + BT$ with $A = 1$ and $B = 0.8$. Note that setting $A = 1$ and $B = 0$ reduces this to the previous problem.

(a) Determine analytically the expression for $\dot{q}$ for the same exact solution as before.

(b) Change the 'calc_source' and 'calc_diffusivity' functions to calculate the new diffusivity and source term. Then make appropriate changes to the 'main' and 'get_coeffs' functions to solve this nonlinear problem. Note that to solve a nonlinear problem, you need to perform several iterations of the 'get_coeffs' and 'solve_gssor' functions until convergence.

(c) Plot the contours of the numerical and exact solutions, the source term and the diffusivity for the finest grid that you select.

(d) Verify the order of accuracy as for Question 1.

Question 3: (Optional) Solve the constant-diffusivity (linear) problem on a stretched grid. Use any of the grids discussed in class. Derive the expressions for the metric terms and implment them in the code. Determine the order of accuracy for different stretching parameter values (e.g. β).

General instructions:

1. Use Matlab or any other postprocessing software to generate line plots and contour plots. Ensure that the font size of the legend and labels is large enough to be easily visible. Ensure that the line thickness is appropriately large. Export your image as a png or eps file and include it in your report.

2. Prepare a short report documenting the results (mainly figures with captions) and brief comments as directed. To reduce your work, do not repeat the problem statement.

3. Please put some thought in preparing the report so that it is easily readable. For example, do not simply dump every png or eps file that you generate into your pdf. Group similar figures together to create a meaningful figure, say Fig. 1 with several sub-panels such as Fig. 1(a), Fig. 1(b), etc. One png file should not span the entire width of the page. Two (or sometimes three) figures can easily fit side-by-side on one page. Please prepare a report you would like to read yourself a few years from now!

4. Please submit your report as one pdf file as per the directions above. In addition, submit a zip or tar file with all codes, any input files or Makefile you may have used.