CO2020 : Computer-Aided Numerical Methods II

Homework 4

Elliptic PDEs

Question 1:

a) Solving for q.

$$\frac{\partial T}{\partial x}\left(\frac{k\partial T}{\partial x}\right) + \frac{\partial T}{\partial y}\left(\frac{k\partial T}{\partial y}\right) + \dot{q} = 0.$$

$T = x(1 - x)cos(\pi y)$ and for $K = 1$,

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \dot{q} = 0$$

$$\frac{\partial T}{\partial x} = 1 - 2x\ cos(\pi y)$$

$$\frac{\partial T}{\partial y} = x(x - 1)sin(\pi y)$$

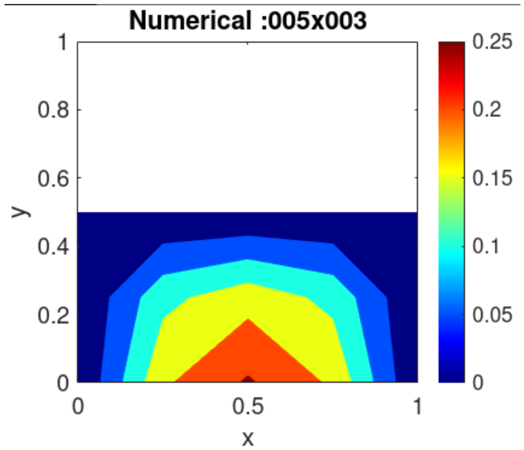
$$\frac{\partial^2 T}{\partial x^2} =- 2cos(\pi y)$$

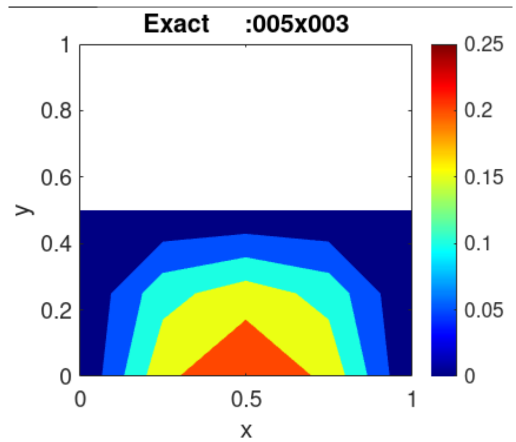
$$\frac{\partial^2 T}{\partial y^2} = x(1 - x)\pi^2 cos(\pi y)$$

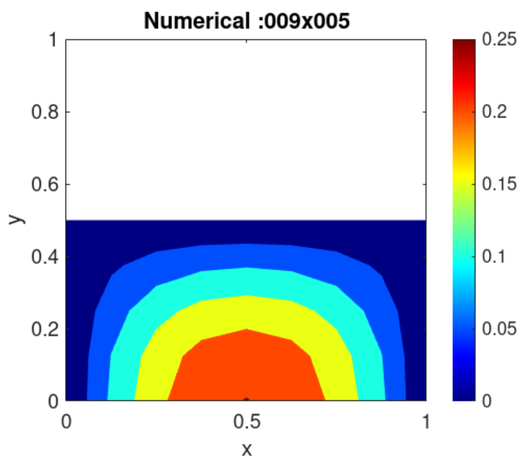
$$This\ gives\ us\ q = 2cos(\pi y) - \pi^2 x(1 - x)cos(\pi y)$$

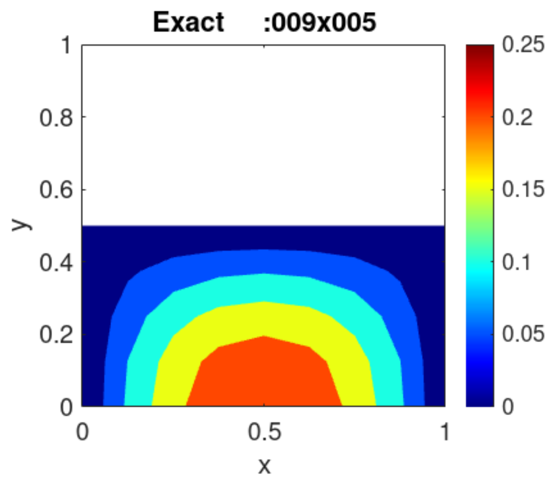
b) The code is in the file “ellip2d_skel.c”.

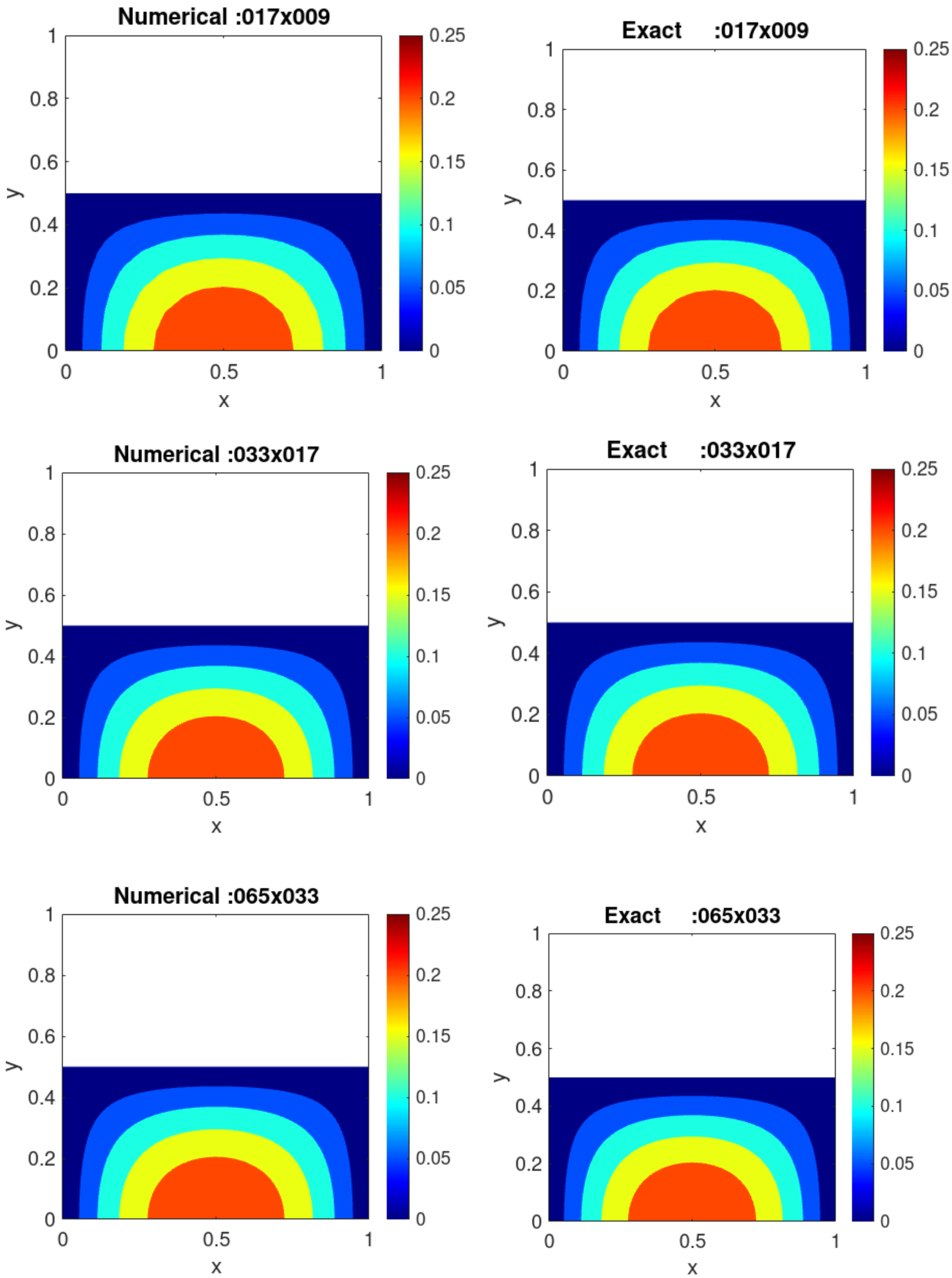
d) Figures for exact and calculated T with various grid sizes:

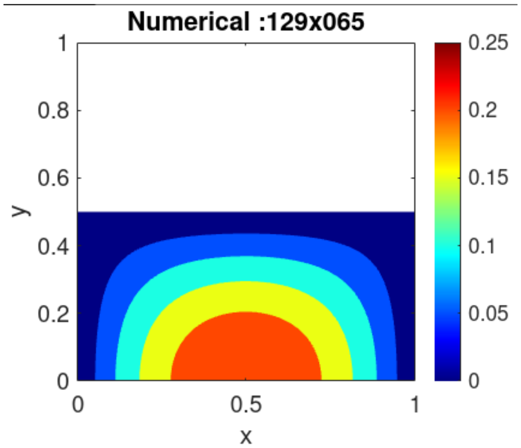

Numerical :005x003
y
x
0
0.5
1
0.2
0.4
0.6
0.8
0.05
0.1
0.15
0.25

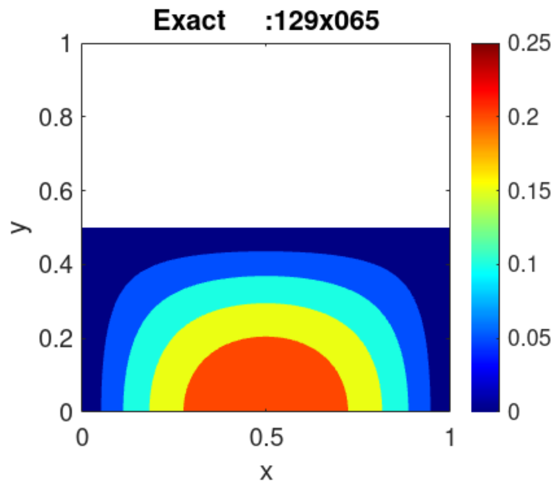

Exact :005x003
y
x
0
0.5
1
0.2
0.4
0.6
0.8
0.05
0.1
0.15
0.25


Numerical :009x005
y
x
0
0.5
1
0.2
0.4
0.6
0.8
0.05
0.1
0.15
0.25

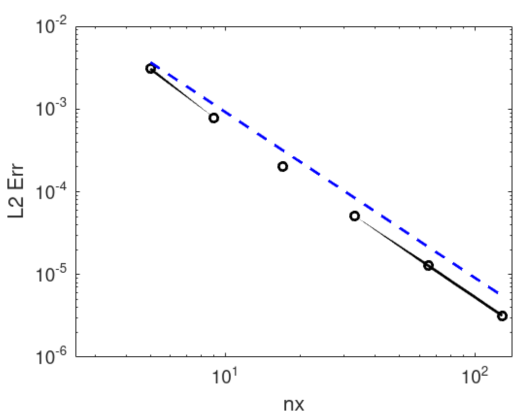

Exact :009x005
y
x
0
0.5
1
0.2
0.4
0.6
0.8
0.05
0.1
0.15
0.25

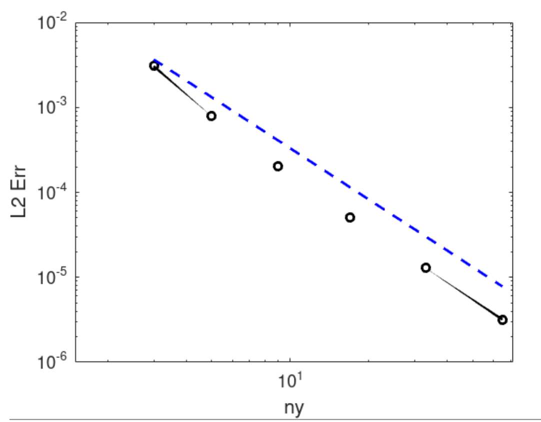

Numerical :017x009
Exact :017x009
Numerical :033x017
Exact :033x017
Numerical :065x033
Exact :065x033
x
y
0
0.5
1
0.2
0.4
0.6
0.8
0.05
0.1
0.15
0.25





e)





Slope of the curve = -0.0002566 (approximately). The slope of the curve is order of accuracy.

Question 2:

a) Solving for q.

$$\frac{\partial T}{\partial x}\left(\frac{k\partial T}{\partial x}\right)+\frac{\partial T}{\partial y}\left(\frac{k\partial T}{\partial y}\right)+\dot{q}\ =\ 0.$$

$T\ =\ x(1\ -\ x)cos(\pi y)$ and for $K\ =\ 1\ +\ 0.8T$

$$=> \frac{\partial T}{\partial x}\left(\frac{(1+0.8T)\partial T}{\partial x}\right) + \frac{\partial T}{\partial y}\left(\frac{(1+0.8T)\partial T}{\partial y}\right) + \dot{q} = 0$$

$$=> 0.8(\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial y}\right)^2) + (1 + 0.8T)\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \dot{q} = 0$$

$$\frac{\partial T}{\partial x} = 1 - 2x\, cos(\pi y)$$

$$\frac{\partial T}{\partial y} = x(x - 1)sin(\pi y)$$

$$\frac{\partial^2 T}{\partial x^2} =- 2cos(\pi y)$$

$$\frac{\partial^2 T}{\partial y^2} = x(1 - x)\pi^2 cos(\pi y)$$

$$=> q = (1 + 0.8x(1 - x)cos(\pi y))(2cos(\pi y) - \pi^2 x(1 - x)cos(\pi y))$$

$$- 0.8((1 - 2\, x\, cos(\pi y))^2 + (x(x - 1)\pi sin(\pi y))^2)$$

b)

$$K\frac{\partial^2 T}{\partial x^2} + K\frac{\partial^2 T}{\partial y^2} + \frac{\partial K}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial K}{\partial y}\frac{\partial T}{\partial y} =- q$$

Discretizing the above equation gives us the following.

$$=> 2K(1 + h^2/k^2)T_{i,j} - (K + \frac{\partial K}{\partial x}h/2)T_{i+1,j} - (K - \frac{\partial K}{\partial x}h/2)T_{i-1,j} - h^2/k^2(K + \frac{\partial K}{\partial y}k/2)T_{i,j+1}$$

$$- h^2/k^2(K - \frac{\partial K}{\partial y}k/2)T_{i,j-1} = h^2 q$$

Left boundary:

$$T_{i-1,j} = T_{i+1,j} + (2hq_j/p_j)T_{i,j} - 2hr_j/p_j$$

$$=> (2K(1 + h^2/k^2)p_j - (K - \frac{\partial K}{\partial x}h/2)(2hq_j))T_{i,j} - 2p_j(K + \frac{\partial K}{\partial x}h/2)T_{i+1,j}$$

$$- h^2/k^2(K + \frac{\partial K}{\partial y}k/2)p_j T_{i,j+1} - h^2/k^2(K - \frac{\partial K}{\partial y}k/2)p_j T_{i,j-1} = h^2 p_j q - (K - \frac{\partial K}{\partial x}h/2)\, 2hr_j$$

Right boundary:

$$T_{i+1,j} = T_{i-1,j} - (2hq_j/p_j)T_{i,j} + 2hr_j/p_j$$

$$=> (2Kp_j(1 + h^2/k^2) + (K + \frac{\partial K}{\partial x}h/2)(2hq_j))T_{i,j} - (2Kp_j)T_{i-1,j}$$

$$- h^2/k^2(K + \frac{\partial K}{\partial y}k/2)p_j T_{i,j+1} - h^2/k^2(K - \frac{\partial K}{\partial y}k/2)p_j T_{i,j-1} = h^2 qp_j + (K + \frac{\partial K}{\partial x}h/2)(2hr_j)$$

Top boundary:

$$T_{i,j+1} = T_{i,j-1} - (2kq_j/p_j)T_{i,j} + 2kr_j/p_j$$

$$=> (2K(1 + h^2/k^2)p_j + h^2/k^2(K + \frac{\partial K}{\partial y}k/2)(2kq_j))T_{i,j} - (K + \frac{\partial K}{\partial x}h/2)p_j T_{i+1,j} - (K - \frac{\partial K}{\partial x}h/2)p_j T_{i-1,j}$$

$$- 2h^2/k^2 Kp_j T_{i,j-1} = h^2 qp_j + h^2/k^2(K + \frac{\partial K}{\partial y}k/2)2kr_j$$

Bottom boundary:

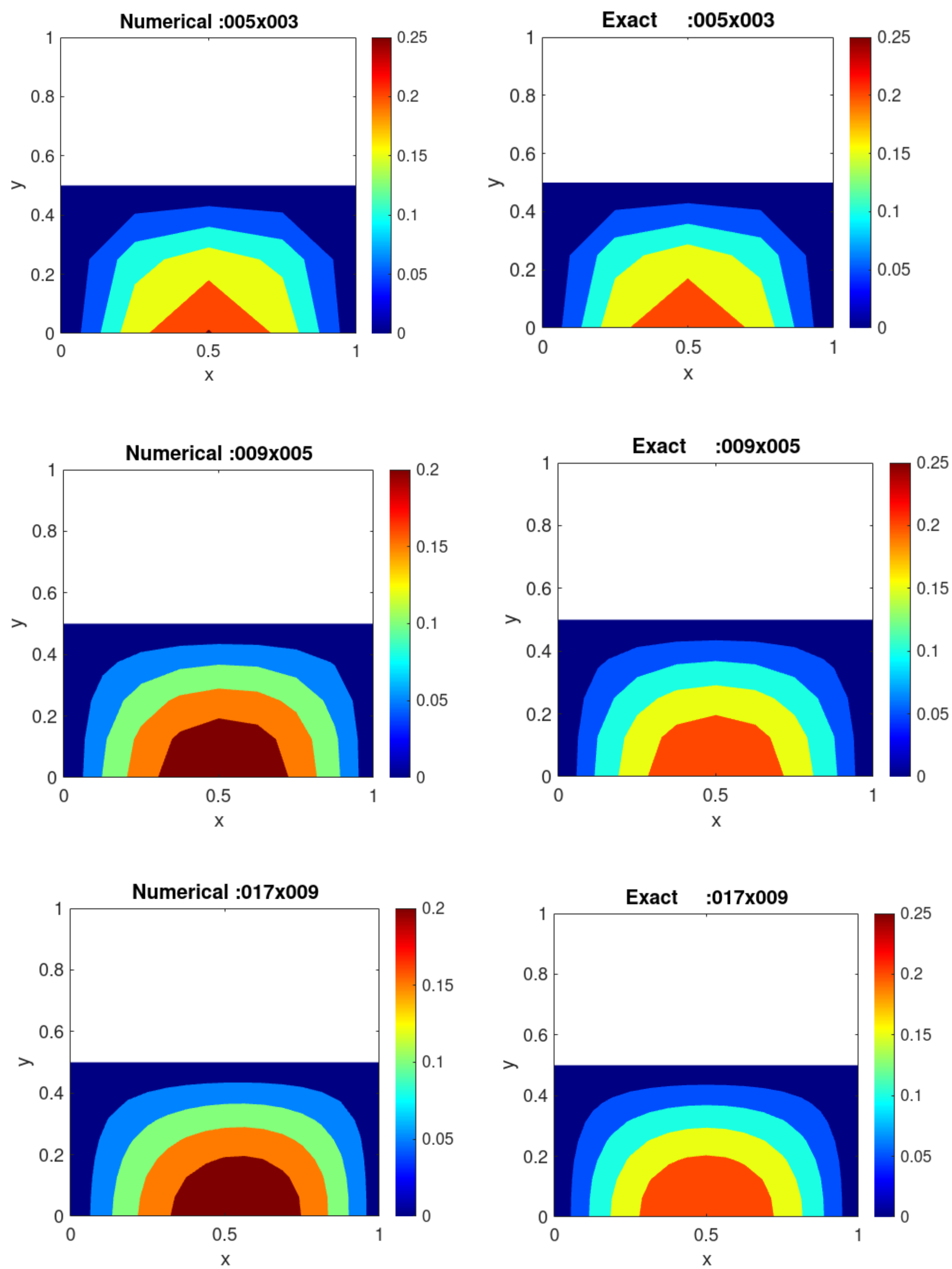
$$T_{i,j-1} = T_{i,j+1} + (2kq_j/p_j)T_{i,j} - 2kr_j/p_j$$

$$=> (2K(1 + h^2/k^2)p_j - h^2/k^2(K - \frac{\partial K}{\partial y}k/2)(2kq_j))T_{i,j} - (K + \frac{\partial K}{\partial x}h/2)p_j T_{i+1,j} - (K - \frac{\partial K}{\partial x}h/2)p_j T_{i-1,j}$$

$$- 2h^2/k^2 Kp_j T_{i,j+1} = h^2 qp_j + h^2/k^2(K - \frac{\partial K}{\partial y}k/2)2kr_j$$

The code is in the file “ellip2d_vark.c”.

c)Figures for exact and calculated T with various grid sizes:


Numerical :005x003
Exact :005x003
Numerical :009x005
Exact :009x005
Numerical :017x009
Exact :017x009
x
y
0
0.5
1
0.2
0.4
0.6
0.8
0.05
0.1
0.15
0.25