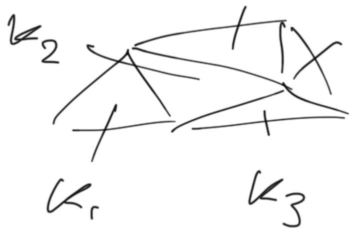
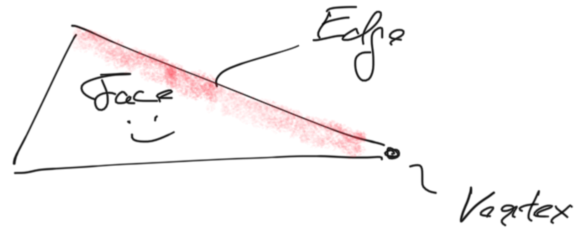
CHEA II E. Strauch 26th Nov

Mesh

- triangulation τ
- domain Ω



$\Omega = \bigcup_{K \in \tau} K$

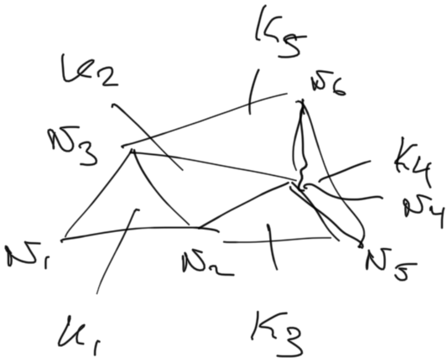


let $h = \min_{\tau} (\text{diam}(K))$

$V^h \in H_0^1(\Omega)$

$V^h = \{ v \in H_0^1(\Omega) \mid v \text{ is cont}, v|_K \text{ linear}$

i.e.



$\{ (v_1, v_2, v_3)$

(v_2, v_4, v_3)

$(\qquad)$

$\}$

z_j i.e. $((0,2,0,3),(\cdot\ 7,$

Array of all vertices

$$Z = (v_1, v_2, \dots, v_n)$$

$$T = ((1,2,3),(2,4,3),(\ \ ))$$

$\to 2 \times v$ $\quad Z(q, i)$ $\quad q^{th}$ comp. of vert. v_i

$T(\alpha, m)$ $\quad \alpha^{th}$ vertex of K_m

$$\to Z(q, T(\alpha, m))$$

Quadratic Integration

$x \to (x_1, x_2)$

$$\begin{cases} -\Delta u = f(x) & \text{in } \Omega \subset \mathbb{R}^2 \\ u(x) = 0 & \text{on } \partial\Omega \end{cases}$$

where $f \in L^2(\Omega)$

Var. Form.

Find $u \in H_0^1$ s.t. $\forall v \in H_0^1$ it holds that

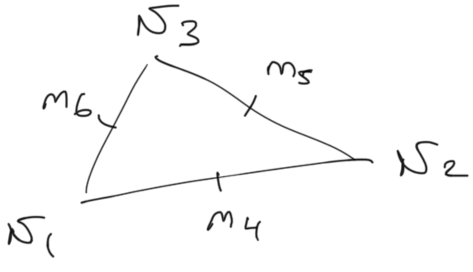
$$(u, v)_{H_0^1(\Omega)} = (f, v)_{L^2(\Omega)}$$

$$V^h = \{ w : \Omega \to \mathbb{R} \mid w \text{ is cont},$$
$$w|_{\partial\Omega} = 0,\ w|_K \text{ is a } 2^{nd}$$
$$\text{order polynomial } \forall K \in \mathcal{T} \}.$$

we assume

$$w(x,y) = a_{0,0} + a_{1,0}\,x + a_{0,1}\,y$$
$$+ a_{1,1}\,xy + a_{2,0}x^2 + a_{0,2}y^2$$

all $a_{i,j} \in \mathbb{R}$



· contr. at vertices

· contr. at midpoints

$x_j \to N_j$

N_V # of vertices

N_E # of edges

$$\delta_i(N_j) := \begin{cases} 1 & i=j \\ 0 & \text{otherwise} \end{cases} \quad \forall \{i\}_0^{N_V-1}$$

$$\psi_i(m_j) := \begin{cases} 1 & i=j \\ 0 & \text{otherwise} \end{cases} \quad \forall \{i\}_0^{N_E-1}$$

6 eqns for 6 coeff.

3 from the vertices

3 from midpoints

$N = N_V + N_E$

$$\varphi_i^N := \begin{cases} \delta_i & i \in [0, \dots, N_V-1 \\ \psi_{i-N_V} & i \in [N_V, \dots, N-1 \end{cases}$$

$\{\varphi_i\}_{i=1}^N$ form a basis for V^h

$$\to \quad u_N(x) = \sum_{i=1}^{N} \mu_i \varphi_i^N(x)$$

$$\mu = \{\mu_i\}_{i=0}^{N-1}$$

Var. form $\qquad \forall i \in E$

$$(\nabla u_N, \nabla \varphi_i) = (f, \varphi_i)$$

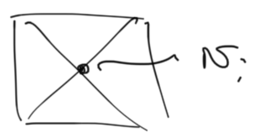
$$\Rightarrow (\nabla(\sum_j \mu_j \varphi_j), \nabla \varphi_i) = (f, \varphi_i)$$

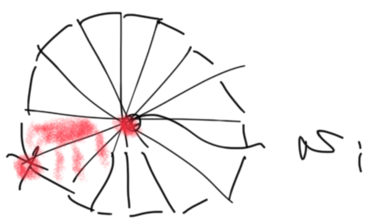
$$= (\sum_j \mu_j \nabla \varphi_j, \nabla \varphi_i)$$

$$= \sum_j \mu_j (\nabla \varphi_j, \nabla \varphi_i)$$

$$\to \sum_{j=1}^{N} \mu_j \underbrace{(\nabla \varphi_j, \nabla \varphi_i)}_{A_{ij}} = \underbrace{(f, \varphi_i)}_{F_i}$$

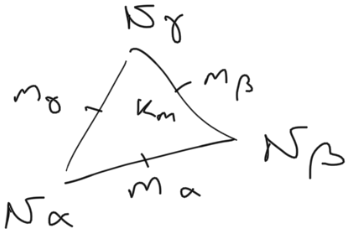
$$A\mu = F$$





$$A_{ij} = \int_\Omega (\nabla \varphi_i, \nabla \varphi_j)\, dx \quad ; \Omega = \bigcup_{m=}^{n}$$

$$= \sum_{m=1}^{n} \int_{K_m} (\nabla \varphi_i, \nabla \varphi_j)\, dx$$



$$\int_{K_m} (\nabla \varphi_i, \nabla \varphi_j) \neq 0$$

only iff $i, j \in K_m$

local base fns φ_α^K for $\alpha = \begin{cases} 1, \ldots 3 & \text{lin} \\ 1, \ldots 6 & \text{quad} \end{cases}$

$$\varphi_\alpha^K(N_i) = \begin{cases} 1 & T(\alpha, K) \\ 0 & \text{otherwise} \end{cases}$$

$$\int_K (\nabla \varphi_\alpha^K, \nabla \varphi_\beta^K)\, dx \quad , \alpha, \beta \in \{1, \ldots, 6\}$$

local st. matrix A^K

$$A^K_{\alpha,\beta} = \int_K (\quad)\, dx \quad ;$$

$\longrightarrow A^K$ is symmetric

Pseudo-Alg.

- Computing A

For $m = 1, \ldots, M$ // looping T

Compute A^{K_m}

For each $\alpha, \beta = 1, 2, 3, (4,$

$$A^K_{\alpha,\beta} = \int_{\hat{K}} \ldots\, dx$$

// global assembly

$$A_{T(\alpha,m), T(\beta,m)} \mathrel{+}= A^{K_m}_{\alpha,}$$

$T(\alpha, m)$ α^{th} vert. of K_m
node

Ex.

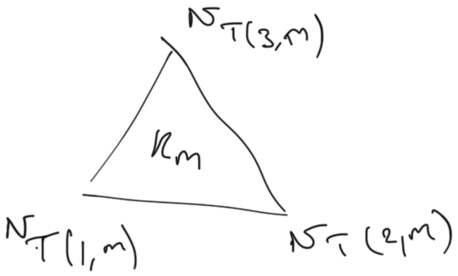
$$A_{ij} = \sum \int (\nabla \varphi_i, \nabla \varphi_j)\, dx$$

$v \quad m \quad K_m$

$$A^{K_m}_{\alpha,\beta} = \int_{K_m} (\nabla \varphi^K_\alpha , \nabla \varphi^K_\beta)\, dx$$

αth vert. of K_m $\Downarrow$ $T(\alpha, m) \to$ globally i

βth vert. K_m $\Downarrow$ $T(\beta, m) \to$ globally



RHS $\to$ load vector

$$(f, \varphi_i) = \sum_m \int_K (f, \varphi_i)\, dx$$

$(A\mu = F)$

· Computing F

Compute local load vector

$$F_m^\alpha = \int_{\hat{K}} f(\Phi_K(y)) \cdot \hat{\varphi}^K_\alpha(y) \cdot |d \quad dy$$

$$F_{T(\alpha,m)} \mathrel{+}= F_m^\alpha$$

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triplet$(\underbrace{i, j}_{\text{index}}, 7.67)$

$A_{ij} = 7.67$

triplet$(i, j, e^{\wedge 2})$

$\wedge \qquad \wedge 2$

$$\Longrightarrow A_{ij} = 7.67 + c \ldots$$

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10 FEM with Adaptive Mesh

$$\begin{cases} -u''(x) + \alpha u'(x) = f(x) \ ; \ \Omega = (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

$f : \mathbb{R} \to \mathbb{R}$ is smooth, $\alpha \in \mathbb{R}$ is const

a) Derive var. form.

Let $v \in H_0^1([0,1]) =$

$$\left\{ u : [0,1] \to \mathbb{R} \ \middle| \ u(0) = u(1) = 0 \text{ and } \int_0^1 |u'(x)|^2 \, dx < \infty \right\}.$$

mult. by test fn v and multipl. / int.

$$\underbrace{\int_0^1 -u''(x) \cdot v(x) \, dx}_{\text{int. by parts}} + \int_0^1 \alpha \cdot u' v \, dx = \int_0^1 f \cdot v$$

u' v

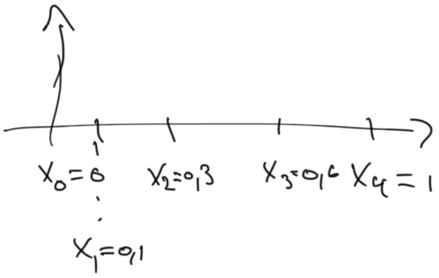
$$\int_0^1 -u''(x) \, v(x) \, dx = -\underbrace{[u'(x) \cdot v(x)]_0^1}_{=0} + \int_0^1 u'(x) \, v'(x)$$

Find $u \in H_0^1([0,1])$ s.t. for all $v \in H_0^1([0,1])$

it holds that

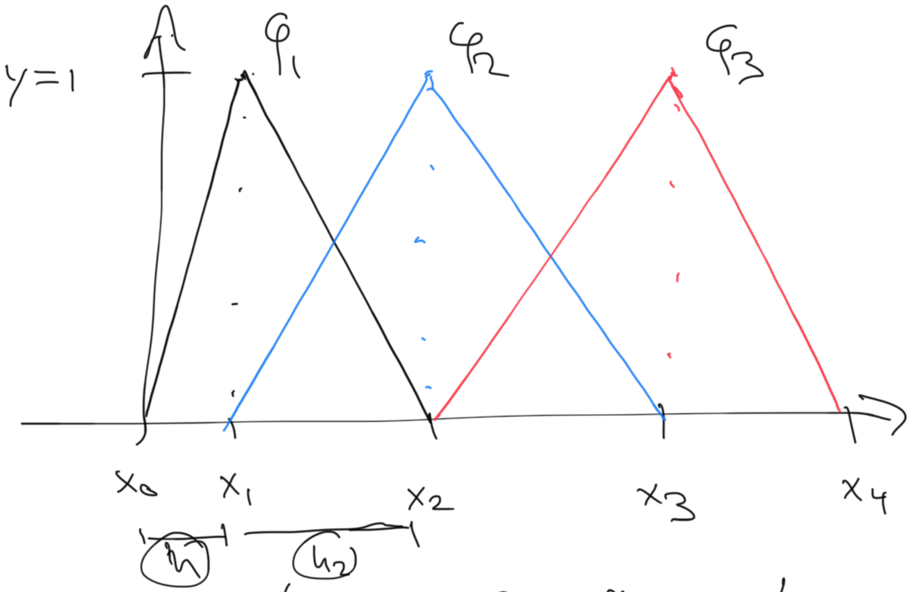
$$\int_0^1 u'(x) \cdot v'(x) \, dx + \alpha \cdot \int_0^1 u'(x) \, v(x) \, dx = \int_0^1 f(x) \, v(x)$$

b) Adaptive Mesh (1D)



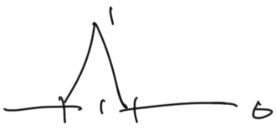
→ not uniform

· lin. base f'ns



c) φ_i , φ_i' ; base f'ns $\in V^h$

$$\varphi_i = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & x \in [x_{i-1}, x_i) \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i} & x \in [x_i, x_{i+1}) \\ 0 & \text{otherwise} \end{cases}$$



$$\varphi_i'(x) = \begin{cases} \dfrac{1}{x_i - x_{i-1}} & x \in [x_{i-1}, x_i) \\ \dfrac{-1}{x_{i+1} - x_i} & x \in [x_i, x_{i+1}) \\ 0 & \text{otherwise} \end{cases}$$

$x_0 = 0$, $x_1 = 0,1$, $x_2 = 0,3$, $x_4 = 0,6$, $x_4 = 1$

$$G(x) = \begin{cases} 10x & x \in [0, 0,1) \\ (0,3 - x)\cdot 5 & x \in [0,1 , 0,3) \end{cases}$$

$$\varphi_1'(x) = \begin{cases} 10 & x \in \ldots \\ -5 & \ldots \\ 0 & \ldots \end{cases}$$

$\varphi_2 \quad \varphi_2'$

$\varphi_3 \quad \varphi_3'$

d) Stiffness Matrix $\quad AU = F \qquad -u'' + \alpha u' = f$

$$\Rightarrow A = \underbrace{A_1}_{-u''} + \underbrace{\alpha A_2}_{\alpha u'}$$

$$u_n(x) := a \cdot \varphi_1(x) + b\, \varphi_2(x) + c\, \varphi_3(x)$$

$$U = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

$v = \varphi_i$
$$\int_0^1 u'v'\, dx + \alpha \int_0^1 u'v\, dx = \int_0^1 f \cdot v\, dx$$

$$\sum_{j=1}^{N=3} \int_0^1 \underline{z_j}\, \varphi_j' \cdot \varphi_i'\, dx + \alpha \cdot \sum_{i=1}^{N=3} \int \ldots$$

$a \cdot \varphi_1(x)$ is for $j=1$ (fixed) for φ_i $i=1$
$\Rightarrow$ first

$$\Rightarrow a \left[\int_0^1 \varphi_1' \varphi_{i=1}'(x)\, dx + \alpha \int \varphi_1'(x)\, \varphi_{i=1}(x)\, dx \right]$$

$$+ b \cdot (\ldots \varphi_2) + c\,(\ldots \varphi_3) = \int f \cdot \varphi_{i=1}$$

$$\Rightarrow \sum \int z\, \varphi' \varphi' + \alpha \cdot \sum \int z\, \varphi' \varphi = \int f \varphi$$

2)
$$A = \begin{array}{c c} & \begin{array}{ccc} z_1 & z_2 & z_3 \\ u_1 & u_2 & u_3 \end{array} \\ \begin{array}{c} u_1^h \\ \vdots \\ u_{15}^h \end{array} & \begin{pmatrix} \; & \; & \; \\ & & \\ & & \end{pmatrix} \end{array}$$

$$A = A(-u'') + \alpha A(u)$$

A_{ij}

$$= \begin{pmatrix} \int \varphi_1' \varphi_1' & \int \varphi_2' \varphi_1' & \int \varphi_3' \varphi_1' \\ & \ddots & \\ \int \varphi_1' \varphi_3' & & \ddots \end{pmatrix}$$

$$+ \alpha \cdot \begin{pmatrix} \int \varphi_1' \varphi_1 & \int \varphi_2' \varphi_1 & \int \varphi_3' \varphi_1 \\ \int \varphi_1' \varphi_2 & \ddots & \\ \int & & \ddots \end{pmatrix}$$

$$F = \begin{pmatrix} \int f \varphi_1 \\ \int f \varphi_2 \\ \int f \varphi_3 \end{pmatrix}, \quad u = \begin{pmatrix} 9 \\ 5 \\ c \end{pmatrix} \to Au =$$

e) A_1