# Runge Kutta

→ Generally a framework for solving ODEs.

## General Form

$$Y_1 = U_n + \Delta t \sum_{j=1}^{s} a_{1j} F(t^n + c_j \Delta t, Y_j)$$

$$Y_2 = U_n + \Delta t \sum_{j=1}^{s} a_{2j} F(t^n + c_j \Delta t, Y_j)$$

$$\vdots$$

$$Y_s = U_n + \Delta t \sum_{j=1}^{s} a_{sj} F(t^n + c_j \Delta t, Y_j)$$

# of stages (s) — weights ($a_{sj}$)

$$\Rightarrow U_{n+1} = U_n + \Delta t \sum_{j=1}^{s} b_j F(t^n + c_j \Delta t, Y_j)$$

step size ($c_j$)

$$U_0 = u_0$$

Coefficients are represented in Butcher Tableau:

$$\begin{array}{c|cccc} c_1 & a_{11} & a_{12} & \cdots & a_{1s} \\ c_2 & a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_s & a_{s1} & a_{s2} & \cdots & a_{ss} \\ \hline & b_1 & b_2 & \cdots & b_s \end{array}$$

Example: Trapezoidal Method as Runge Kutta

$$U_{n+1} = U_n + \Delta t \Big[ \underbrace{\tfrac{1}{2}}_{b_1} F(\underbrace{t^n}_{c_1=0}, \underbrace{U_n}_{Y_1}) + \underbrace{\tfrac{1}{2}}_{b_2} F(t^n + \underbrace{1}_{c_2} \cdot \Delta t, \underbrace{U_{n+1}}_{Y_2}) \Big]$$

$$\rightarrow Y_1 = U_n$$

$$\rightarrow Y_2 = U_{n+1} = U_n + \Delta t \Big[ \underbrace{\tfrac{1}{2}}_{a_1} F(\underbrace{t^n}_{c_1=0}, Y_1) + \underbrace{\tfrac{1}{2}}_{a_2} F(t^n + \underbrace{1}_{c_2} \cdot \Delta t, Y_2) \Big]$$

Tableau:

$$\Rightarrow \begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1/2 & 1/2 \\ \hline & 1/2 & 1/2 \end{array}$$

## Consistency Checks

I) $\boxed{\sum_{j=1}^{s} a_{ij} \overset{!}{=} c_i}$ for every $i$

II) $\boxed{\sum_{j=1}^{s} b_j \overset{!}{=} 1}$

Example Trapezoidal:

I)
$i = 1: \quad a_{11} + a_{12} = c_1 \rightarrow 0 + 0 = 0$ ✓

$i = 2: \quad a_{21} + a_{22} = c_2 \rightarrow \frac{1}{2} + \frac{1}{2} = 1$ ✓

II) $b_1 + b_2 \overset{!}{=} 1 \rightarrow \frac{1}{2} + \frac{1}{2} = 1$ ✓

## Order of accuracy checks

At least 2nd order if:

$$\boxed{\sum_{j=1}^{s} b_j\, c_j = \frac{1}{2}}$$

Example: $b_1 \cdot c_1 + b_2 \cdot c_2 \overset{!}{=} \frac{1}{2}$

$1/2 \cdot 1/2 + 1/2 \cdot 1/2 = 1/2$ ✓ $\rightarrow$ Trapezoidal is at least 2nd order accurate

At least 3rd order if:

$\boxed{\sum_{j=1}^{s} b_j \cdot c_j^2 = \frac{1}{3}}$ $\xrightarrow{\text{ex.}}$ $\frac{1}{2} \cdot \left(\frac{1}{2}\right)^2 + \frac{1}{2} \cdot \left(\frac{1}{2}\right)^2 = \frac{1}{4} \neq \frac{1}{3}$

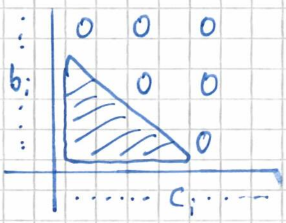
$\boxed{\sum_{j=1}^{s} \sum_{i=1}^{s} b_i \cdot a_{ij} \cdot c_j = \frac{1}{6}}$ $\xrightarrow{\text{ex.}}$ $1 \cdot \frac{1}{2} \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2} \neq \frac{1}{6}$

$\Rightarrow$ Trapezoidal is <u>not</u> 3rd order

## Explicit vs. implicit scheme:

### Explicit:

$\boxed{a_{ij} = 0 \text{ if } j \geq i}$ → A is strictly lower triangular

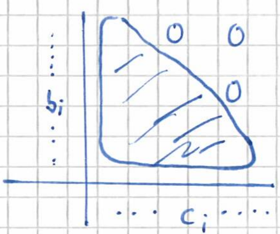



### Diagonally Implicit:

→ a step only requires solving previous and the current step, but not next steps.

$\boxed{a_{ij} = 0 \text{ if } j > i, \quad a_{ii} \neq 0 \text{ for some } i}$

→ A is lower triangular



## Stability

→ generally the smaller dt, the better the numerical scheme approximates our function (→ but also more computation)

→ if the dt is too large, the approx. can go to infinity quickly (→ unstable)

→ Goal: find the threshold for dt, for which a scheme is stable

## Absolute Stability

Let's take the following stiff-ODE:

$$\boxed{\begin{aligned} u'(t) &= \lambda \cdot u(t) \\ u(0) &= 1 \end{aligned}}$$

→ analytic solution: $u(t) = e^{\lambda t}$

→ difficult ODE because it changes quickly

→ good reference, if scheme stable for this ODE, then also for others

For $\lambda \cdot \Delta t < 0$, the function decreases monotonously:

$\Rightarrow u_{n+1} \leq u_n$

$$\boxed{\frac{|u_{n+1}|}{|u_n|} \leq 1}$$

→ check for which $\lambda \cdot \Delta t$, a scheme satifies the inequality above

Example FE:

$$u_{n+1} = u_n + \Delta t \, F(t^n, u_n) = u_n + \Delta t \cdot \lambda \cdot u_n = (1 + \Delta t \lambda) u_n$$

$$\Rightarrow \frac{|u_{n+1}|}{|u_n|} = |1 + \lambda \Delta t| \overset{!}{\leq} 1$$

$$\Rightarrow \lambda \Delta t \in [-2, 0]$$

## A-stable

→ stricter than absolute stability: scheme has to be stable for every $Re(\lambda \Delta t) < 0$

→ FE is i.e. not A-stable!

→ An explicit scheme can never be A-stable, implicits are sometimes.

Tips:

1d) - as discussed in theory
- use $\underbrace{\text{numerical solver}}_{\text{i.e. WolframAlpha}}$ to find the $\lambda \Delta t$ limits

2a) $\underline{u}(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$