Finite Element Method

FEM is the discretisation of the variational formulation of a PDE.
→ What is the variational formulation?

Back to Poisson Equation (1D):
The P.Eq. was derived as the solution to the following problem:

$$\min_u J(u)$$

$$\text{where} \quad J(u) = \frac{1}{2}\int_0^1 |u'(x)|^2\,dx - \int_0^1 u(x)f(x)\,dx \qquad (1)$$

$$\text{and} \quad u(0) = u(1) = 0$$

→ reformulate this problem.

define: $H_0^1([0,1]) := \{u : [0,1] \to \mathbb{R} : u(0) = u(1) = 0$ and $\int_0^1 |u'(x)|^2 dx < \infty\}$

⇒ find $u \in H_0^1([0,1])$, such that u minimises the energy $J(v)$ given by (1) for all $v \in H_0^1([0,1])$

i.e. find $u \in H_0^1([0,1])$ such that

$$J(u) = \min_{v \in H_0^1([0,1])} J(v)$$

→ for all $v \in H_0^1([0,1])$ $\quad J'(u,v) = 0 \qquad (2)$

$$\text{where} \quad J'(u,v) = \lim_{\tau \to 0} \frac{J(u+\tau v) - J(u)}{\tau}$$

[non-rigorous analogue: $f'(t) = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$

where $\tau v \mathrel{\hat{=}} \Delta t$]

From (2) we get (see Skript 6.1.1 for details)

$$\int_0^1 u'(x)\, v'(x)\, dx = \int_0^1 v(x) f(x)\, dx$$

→ variational formulation for the Poisson Eq.

How do we recover the P. Eq. from the V.F.?

↳ integration by parts on the left

$$\int_0^1 u'(x)\, v'(x) = \underbrace{u'(x)\, v(x) \Big|_0^1}_{\substack{=0 \\ (v(0)=v(1)=0)}} - \int_0^1 u''(x) v(x)\, dx$$

$$\rightarrow \quad -\int_0^1 u''(x) v(x)\, dx = \int_0^1 v(x) f(x)\, dx$$

$$\int_0^1 \left(f(x) - u''(x)\right) v(x)\, dx = 0$$

$$\rightarrow \quad u''(x) = f(x) \qquad \square$$

Usually when applying FEM we are interested in the opposite question: How can we derive a variational formulation from a PDE?

Example: Given $\Omega = [0,1]$

$$-u'' + cu = f \quad \text{in } \Omega, \quad c \neq 0 \in \mathbb{R}$$

$$u(0) = u(1) = 0$$

Approach: multiply PDE with $v \in H_0^1([0,1])$ and integrate over $\Omega = [0,1]$

$$\hookrightarrow \int_0^1 \left[\underbrace{-u''(x) v(x)} + c\, u(x) v(x)\right] dx = \int_0^1 f(x) v(x)\, dx$$

integration by parts: $\int_0^1 -u''(x)\, v(x)\, dx = \underbrace{-u'(x)\, v(x) \Big|_0^1}_{=0} + \int_0^1 u'(x)\, v'(x)\, dx$

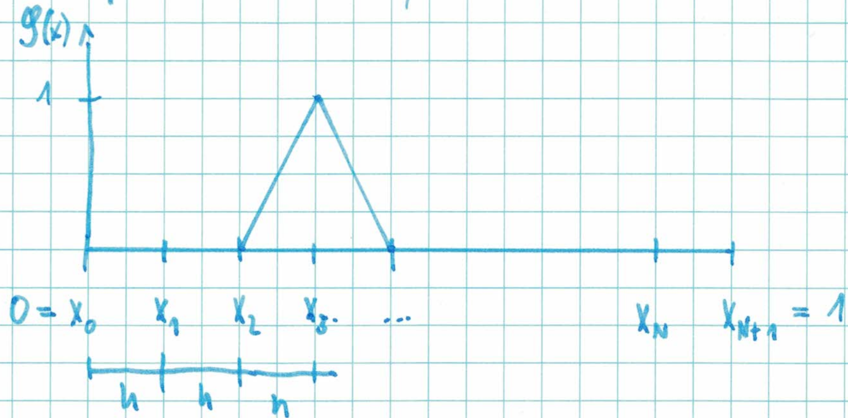
$\Rightarrow \int_0^1 u'(x)\,v'(x)\,dx + \int_0^1 c\,u(x)\,v(x)\,dx = \int_0^1 f(x)\,v(x)\,dx$ □

$\rightarrow$ apply FEM on this variational formulation

An FEM implementation

$V = H_0^1([0,1])$ $\rightarrow$ infinite dimensional!

$\Rightarrow$ first discretize domain $[0,1]$ in x and then approximate V with V^h, where $V^h \subseteq V$ and V^h is the set of piecewise linear functions $\rightarrow$ V^h is N-dimensional



$$\varphi_j(x_i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

$\rightarrow$ example of function which is $\in V^h$ (hat function)

Back to variational formulation

$$\int_0^1 u_h'(x)\,v'(x)\,dx = \int_0^1 v(x)\,f(x)\,dx$$

where $u_h \in V^h$ and $v \in V^h$.

Remember: $\int_0^1 g(x)\,h(x)\,dx = (g, h)$

$\uparrow$ inner product, see LA

the $\varphi_j(x)$ form a basis of V^h

$$\Rightarrow \left.\begin{aligned} v &= \sum_{j=1}^{N} v_j\,\varphi_j(x) \\ u_h &= \sum_{i=1}^{N} u_i\,\varphi_i(x) \end{aligned}\right\}$$

$\rightarrow$ insert into rewritten V.F.:

$(u_h', v') = (v, f)$

we get (skript for more detail)

$$\sum_{i=1}^{N} u_i (\varphi_i', \varphi_j') = (f, \varphi_j) \quad \forall j = 1, \dots, N$$

→ write as Matrix equation:

$$AU = F$$

$$\text{where} \quad A_{ji} = (\varphi_i', \varphi_j') \quad , \quad F_j = (f, \varphi_j)$$

A is the stiffness matrix

F is the load vector

U is the solution vector

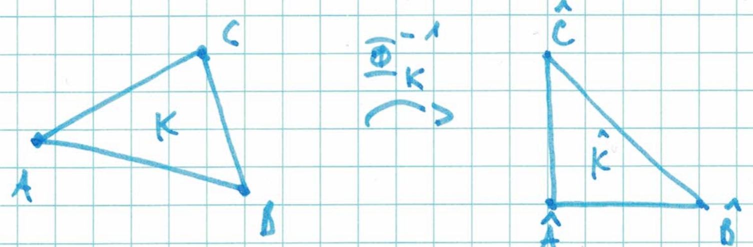
Towards 2D FEM - Integration over reference element

→ see Series 4 Ex. 3 & 4

To assemble the stiffness matrix A and load vector F we need to calculate the inner products, i.e. the integrals.

In 1D this is easy, in 2D more work needed.

Therefore, we transfer the problem from a more challenging domain to an easier domain for the integration.



where $\hat{A} = (0,0)$, $\hat{B} = (0,1)$ and $\hat{C} = (1,0)$

Φ_K : mapping between $\hat{K}$ and K

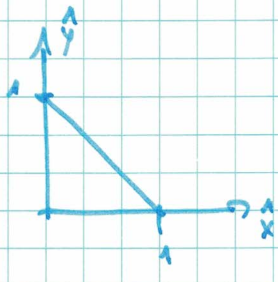
(The triangles are the discretization of the 2D domain)

Then, instead of integrating over the domain K we choose $\hat{K}$

→ $$\int_K g(x)\,dx = \int_{\hat{K}} g(\Phi_K(\hat{x}))\,|\det J_K|\,d\hat{x}$$ for general $g(x)$

⇝ for domain K we need to parametrize the triangle sides to integrate

⇝ for domain $\hat{K}$ we see, that the integral is:



$$\int_{\hat{K}} g(\hat{x})\,d\hat{x} = \int_0^1 \int_0^{1-x} g(\hat{x})\,d\hat{y}\,d\hat{x}$$

→ much easier to compute