Recap

Differential Equation is of the form

$$\mathcal{F}(x, t, u, u_{x_i}, u_t, u_{x_i x_j}, u_{tt}, \dots) = 0$$

Remember ODEs

the unknown $u(t)$ dependent only on one variable i.e. time

$$\mathcal{F}(t, u, u_t, u_{tt}, \dots) = 0$$

Initial Value Problem (IVP) for ODEs

$$\begin{cases} \mathcal{F}(t, u, u_t, u_{tt}, \dots) = 0 \\ u(0) = u_0 \\ u'(0) = u_1 \\ \vdots \\ u^{(k-1)}(0) = u_{k-1} \end{cases}$$

→ The generic form

Given the IVP

$$\begin{cases} \theta''(t) = -\sin(\theta(t)) \\ \theta(0) = \theta_0 \\ \theta'(0) = v_0 \end{cases}$$

can be rewritten as

$$\begin{cases} \theta'(t) = v(t) \\ v'(t) = -\sin(\theta(t)) \\ \theta(0) = \theta_0 \\ v(0) = v_0 \end{cases}$$

and rewriting again

$$\begin{cases} U = \begin{pmatrix} \theta \\ v \end{pmatrix} \\ F = \begin{pmatrix} v \\ -\sin(\theta) \end{pmatrix} \end{cases}$$

→ the generic form

in general:

$$U'(t) = F(u(t))$$

Types of ODEs

- Autonomous ODE

$$\text{if } \forall t \in [0,T] \text{ it holds that } \mathcal{F}(t, u(t)) \equiv \mathcal{F}(u(t))$$

$\rightarrow$ Non-autonomous ODEs can be converted

Given such ODE

$$\begin{cases} u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix} \\ F = \begin{pmatrix} F_1 \\ \vdots \\ F_m \end{pmatrix} \end{cases}$$

we simply introduce an additional $u_{m+1}(t) = t$

hence $u'_{m+1}(t) = 1$ (i.e. const. in time)

Rewriting the ODE

$$\begin{cases} u^{\text{enhanced}} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \\ u_{m+1} \end{pmatrix} \\ G = \begin{pmatrix} F_1 \\ \vdots \\ F_m \\ F_{m+1} \end{pmatrix} \end{cases}$$

where by $u'(t) = F(u(t))$; $F_{m+1} = 1$

and all $F_1, \dots, F_{m+1}$ are $\mathcal{F}(u_1, u_2, \dots, u_{m+1})$.

Ex. $u'(t) = u(t)^2 + t$ a scalar, non-autonomous ODE

introducing $u_2(t) = t$

$$\rightarrow u'(t) = \begin{pmatrix} u_1'(t) \\ u_2'(t) \end{pmatrix} = \begin{pmatrix} u_1(t)^2 + u_2(t) \\ 1 \end{pmatrix} \; ; \; u(0) = \begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix} = \begin{pmatrix} u_{1,0} \\ 0 \end{pmatrix}$$

where we already rewrote the Scalar ODE as a System of ODEs.

Ex. $u'' + u' + u = 0$

let $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u \\ u' \end{pmatrix}$, then $u' = \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} u' \\ u'' \end{pmatrix}$; where $u'' = -u' - u$

and $u(0) = \begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix} = \begin{pmatrix} u_0 \\ u_0' \end{pmatrix}$

$\rightarrow$ Generic Form: $u' = F(u) \overset{\text{linearity}}{=} A \cdot u$ and hence $A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$.

- Linear and Non-linear ODEs

ODEs are linear if r.h. function $\mathcal{F}$ is linear in u which allows for rewriting (in general)

$$\mathcal{F}(t, u(t)) = A(t) \cdot u(t) + c(t)$$

Ex.

$u'(t) = u(t) + t$ linear

$u'(t) = u(t)^2 + t$ non-linear

Explicit Solutions

- Linear Scalar ODE

$$\begin{cases} u'(t) = \lambda u(t) \\ u(0) = u_0 \end{cases} \Rightarrow u(t) = u_0 e^{\lambda t}$$

- Linear System of ODEs

$$\begin{cases} u'(t) = A u(t) \\ u(0) = u_0 \end{cases} ; \; A = R D R^{-1},$$

note the dimensions $\begin{cases} u(t) \in \mathbb{R}^m \\ u_0 \in \mathbb{R}^m \\ A \in \mathbb{R}^{m \times m} \end{cases}$

and $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_m)$ containing the Eigenvalues of A

obtained via $\det(A - \lambda \mathbb{I}) = 0$.

and $R = (r_1, \ldots, r_m)$ containing the corresponding Eigenvectors $(EV_1, EV_2, \ldots)$

(Note: if λ is complex write EV in the form $\binom{1}{\lambda_1}, \binom{1}{\lambda_2}, \ldots$

the solution writes

$$u(t) = R \, e^{Dt} R^{-1} u_0 \quad ; \; e^{Dt} = \operatorname{diag}(e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots)$$

OR

$$u(t) = c_1 e^{\lambda_1 t} E_{\lambda_1} + \ldots + c_m \cdot e^{\lambda_m t} E_{\lambda_m}$$

Well-posedness

Lipschitz cont. on the set $[0, T] \times U$, then there exists a $T^* \in [0, T]$ s.t. there is a unique solution in $[0, T^*]$ to an IVP.

→ see lecture

Numerical Methods

Considering the std. 1^{st} order IVP for ODEs

$$\begin{cases} u'(t) = F(t, u(t)) \\ u(0) = u_0 \end{cases}$$

where $u_0 \in \mathbb{R}^m$ is const,

$u : [0,T] \to \mathbb{R}^m$

$F : [0,T] \times \mathbb{R}^m \to \mathbb{R}^m$

(Note: dep. on dimension both u and F can be scalar or vector-valued f'ns)

- Time discretisation

Consider the IVP in the time interval $[0,T]$ where $T \in (0, \infty)$ is some fixed, finite time.

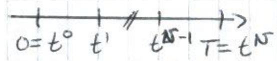
We discretise the time domain:

let $\Delta t > 0$ the time step 'size' $\left(\Delta t = \frac{T}{N}\right)$

then $N = \frac{T}{\Delta t}$ the # of time steps

So we divide $[0,T)$ into many intervals $[t^n, t^{n+1})$

We define $t^n = n \Delta t$

$\to$ equally spaced intervals 

for notation: $\{t^n\}_{n=0}^{N}$ (time steps)

t^n refers to the n^{th} time level

We want to approximate the exact solution u of the IVP at these time levels.

$\{U_n\}_{n=0}^{N}$ the approx. solution ; $U^n \in \mathbb{R}^m$

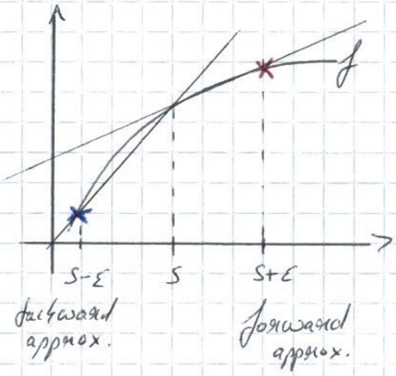
i.e. $U_n \approx u(t^n)$

$(U_0 \approx u(t^0) = u(0) = u_0)$

Syntax: let $U_n = U^n$ the approx. solution at t^n

Forward Euler

Let's take a look at the derivative of a fn f and a point s



$$f'(s) = \lim_{\varepsilon \to 0} \frac{f(s+\varepsilon) - f(s)}{\varepsilon}$$

for a reasonably small ε

$$f'(s) \approx \frac{f(s+\varepsilon) - f(s)}{\varepsilon}$$

$\Rightarrow$ FE for $u'(t^n) = F(t^n, u(t^n))$

$$u'(t^n) \approx \frac{u(t^{n+1}) - u(t^n)}{\Delta t} \overset{\text{approx.}}{\approx} \frac{u^{n+1} - u^n}{\Delta t}$$

and

$$F(t^n, u(t^n)) \overset{\text{approx.}}{\approx} F(t^n, u^n)$$

$$\Rightarrow \quad u^{n+1} = u^n + \Delta t\, F(t^n, u^n) \qquad \underline{\text{Forward Euler}}$$

(from $s = t^n$, $\varepsilon = \Delta t$)

how can we improve this?

$\Rightarrow$ BE

now let $s = t^{n+1}$, $\varepsilon = -\Delta t$

then for $u'(t^{n+1}) = F(t^{n+1}, u(t^{n+1}))$

$$u'(t^{n+1}) \approx \frac{u(t^n) - u(t^{n+1})}{(-\Delta t)} = \frac{u(t^{n+1}) - u(t^n)}{\Delta t} \overset{\text{approx.}}{\approx} \frac{u^{n+1} - u^n}{\Delta t}$$

and

$$F(t^{n+1}, u(t^{n+1})) \overset{\text{approx.}}{\approx} F(t^{n+1}, u^{n+1})$$

$$\Rightarrow \quad u^{n+1} = u^n + \Delta t\, F(t^{n+1}, u^{n+1})$$ $\rightarrow$ implicit (u at future time)

$$\Longleftrightarrow$$

$$u^{n+1} - \Delta t\, F(t^{n+1}, u^{n+1}) = u^n \qquad \underline{\text{Backward Euler}}$$

in general F is non-linear: solve numerically

in the linear case: $F(t, u) = A \cdot u$, $A \in \mathbb{R}^{m \times m}$

$$(\mathbb{I} - \Delta t A)\, u^{n+1} = u^n$$