0.) ODE: $u'(t) = F(t, u(t))$

Theory Ex 1:

a)

3. - Autonomous: $F(u(t))$, F is <u>not</u> directly func. of time
   - Non-autonomous: $F(u(t), t)$ ← directly func. of time

2. - Linear: $F(u(t), t) = \underbrace{A(t)}_{\text{matrix}} \cdot \underbrace{u(t)}_{\text{vector}} + \underbrace{C(t)}_{\text{vector}}$
   - Non-Linear:
     u is non-linear e.g.: $u'(t) = \sin(u(t))$

1. - Scalar: just 1 eq.
   - System of ODEs: multiple eq.

4. <u>Trick Non-Autonomous to Autonomous:</u>

   e.g.: $F(u(t), t) = u'(t) = u(t) + t$

   1. introduce t as extra dimension: $\underline{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} u(t) \\ t \end{pmatrix}$

   2. rewrite ODE in terms of new $\underline{u}(t)$ vector:

   $$\underline{u}' = \begin{pmatrix} u_1'(t) \\ u_2'(t) \end{pmatrix} = \underline{\underline{\begin{pmatrix} u_1(t) + u_2(t) \\ 1 \end{pmatrix}}}$$

b)

<u>Time Discretization:</u>

Aprox. u at discrete time points $t^n = n \cdot \underbrace{\Delta t}$

$\Delta t = \frac{T}{N} = \frac{\text{full time}}{\text{number of points}}$

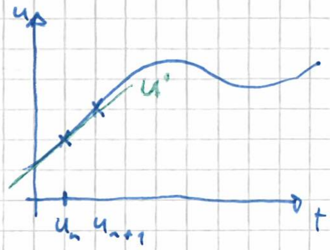
$\Rightarrow U_n \approx u(t^n)$

<u>Forward Euler:</u>

$$\frac{\Delta u}{\Delta x} = u'(t^n) \approx \frac{U_{n+1} - U_n}{\Delta t} = F(t^n, U^n)$$

$\Rightarrow$ solve for U_{n+1} yields <u>update form</u>:

$$\boxed{U_{n+1} = U_n + \Delta t \, F(t^n, U^n)}$$



e.x.: $u'(t) = 2u(t) = F(t, u)$

$$\underline{\underline{\to U_1 = U_0 + \Delta t \,(2U_0)}}$$

Backward Euler:

$$u'(t^{n+1}) \approx \frac{U_{n+1} - U_n}{\Delta t} = F(t^{n+1}, U_{n+1})$$

Solve for U_{n+1}:

$$U_{n+1} - \Delta t \, F(t^{n+1}, U_{n+1}) = U_n$$

$\rightarrow$ U_{n+1} is usually found numerically i.e. Newton's Method

e.x.: $u'(t) = 2u(t) = F(t, u)$

$$\rightarrow U_1 + \Delta t (2U_1) = U_0$$

$$U_1 (1 + 2\Delta t) = U_0$$

$$\Rightarrow U_1 = \frac{U_0}{1 + 2\Delta t}$$

Tipp for (1b) Express $U_0 - A$

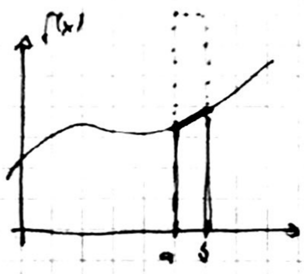
Trapezoidal Rule:

$$\int_{t^n}^{t^{n+1}} u'(s) \, ds = \Big[u(s) \Big]_{t^n}^{t^{n+1}}$$

$$\boxed{\underbrace{\int_{t^n}^{t^{n+1}} F(s) \, ds}_{} = u(t^{n+1}) - u(t^n)}$$

Integrate with Trapezoidal Rule:

$$\int_a^b f(x) \, dx \approx \underbrace{(b-a)}_{\Delta t} (f(a) + f(b)) \cdot \frac{1}{2}$$



$$\Rightarrow U_{n+1} - U_n = \frac{\Delta t}{2} \left[F(t^n, U_n) + F(t^{n+1}, U_{n+1}) \right]$$

e.x. $u'(t) = 2u(t) = F(t, u)$

$$\rightarrow U_1 = U_0 + \frac{\Delta t}{\cancel{2}} \left[\cancel{2} U_0 + \cancel{2} U_1 \right]$$

$$U_1 - \Delta t \, U_1 = U_0 + \Delta t \, U_0$$

$$\Rightarrow U_1 = U_0 \cdot \frac{1 + \Delta t}{1 - \Delta t}$$

One-Step Error

$$L_n = \underbrace{u(t^{n+1})}_{\text{exact}} - \underbrace{\text{update Form}}_{\text{approx.}}\left(\underbrace{u(t^{n+1})}_{\text{exact}}\right)$$

Taylor-Exp.: i.e. forward euler

$$f(t^{n+1}) = f(t^n + \Delta t) = f(t^n) + \Delta t\, f'(t^n) + \frac{\Delta t^2}{2} f''(t^n) + \frac{\Delta t^3}{6} f'''$$

i.e. Backward Eul.:

$$L_n = \underbrace{u(t^{n+1})}_{\text{Taylor}} - \left[u(t^n) + \Delta t\, \underbrace{F(t^{n+1}, u(t^{n+1}))}_{u'(t^{n+1}) \leftarrow \text{Taylor}}\right]$$

$$= \cancel{u(t^n)} + \cancel{\Delta t\, u'(t^n)} + \frac{\Delta t^2}{2} u''(t^n) + \frac{\Delta t^3}{6} u'''(t^n) + \mathcal{O}(\Delta t^4) - \cancel{u(t^n)}$$

$$+ \Delta t\left[\cancel{u'(t^n)} + \Delta t \cdot u''(t^n) + \frac{\Delta t^2}{2} u'''(t^n) + \mathcal{O}(\Delta t^3)\right]$$

$$= \frac{3}{2}\Delta t^2 u'' + \frac{4}{6}\Delta t^3 u''' + \mathcal{O}(\Delta t^4) \Rightarrow \underline{\underline{L_n = \mathcal{O}(\Delta t^2)}}$$

Tips:

1b) express $U_0 = A$

2c) - bring update equation in $\underline{\underline{G}}(x) = 0$ form

- $D\,\underline{\underline{G}}(\underline{x}) = \begin{bmatrix} \frac{\partial G_1}{\partial x_1} & \frac{\partial G_1}{\partial x_2} \\ \frac{\partial G_2}{\partial x_1} & \frac{\partial G_2}{\partial x_2} \end{bmatrix}$ (Jacobian Matrix)