Truncation Error

- error if exact solution inserted in the numerical methods' consistent form

Recap: Consistent Form Explicit Euler

$$\frac{u^{n+1} - u^n}{\Delta t} - \mathcal{F}(t^n, u^n) = 0$$

1st write the scheme in its consistent form i.e. $Q(u^{n+1}, u^n) = 0$

2nd insert the exact solutions i.e. $u(t^{n+1}), u(t^n)$

$\Rightarrow Q_{new}(u(t^{n+1}), u(t^n)) = T_n$ i.e. expand $u(t^{n+1})$ around t^n

One-Step Error

- error if exact solution inserted in the update form for a single step

Recap: Update Form Explicit Euler

$$u^{n+1} - u^n - \Delta t\, \mathcal{F}(t^n, u^n) = 0$$

1st write scheme in its update form i.e. $Q(u^{n+1}, u^n) = 0$

2nd insert the exact solutions i.e. $u(t^{n+1}), u(t^n)$

$\Rightarrow Q_{new}(u(t^{n+1}), u(t^n)) = L_n$

Global Error

- error between final approximated solution u^N and the true solution $u(t^N)$

i.e. the sum of all errors made at each time level

$$E_N := u(t^N) - u^N \approx \sum_{i=0}^{N-1} L_i$$

→ think about that concept as well as the order of the error w.r.t. Δt

i.e. for non-linear ODEs an amplification of errors across time steps is likely

Taylor Expansion

$$f(x_1, x_2) = f(x_{1,0}, x_{2,0}) + \frac{\partial f(x_{1,0}, x_{2,0})}{\partial x_1}(x_1 - x_{1,0}) + \frac{\partial f(x_{1,0}, x_{2,0})}{\partial x_2}(x_2 - x_{2,0})$$
$$+ \frac{\partial^2 f(x_{1,0}, x_{2,0})}{\partial x_1^2} \cdot \frac{\Delta x_1^2}{2!} + \frac{\partial^2 f(x_{1,0}, x_{2,0})}{\partial x_2^2} \cdot \frac{\Delta x_2^2}{2!} + 2 \cdot \underbrace{\frac{\partial^2 f(..)}{\partial x_1 \partial x_2}}_{\text{order does not matter}} \cdot \frac{\Delta x_1 \Delta x_2}{2!} + \dots$$

Higher-Order Methods for ODEs

→ refers to higher-order accuracy

Runge-Kutta Methods:

'multi-stage' numerical schemes for approximating solutions to

$$\begin{cases} u'(t) = \mathcal{F}(t, u(t)) \\ u(0) = u_0 \end{cases}$$

where $u_0 \in \mathbb{R}^m$ is const., $u: [0,T] \to \mathbb{R}^m$ and $\mathcal{F}: [0,T] \times \mathbb{R}^m \to \mathbb{R}^m$

Let $s \in \mathbb{N}$ be an int, let $\{a_{ij}\}_{i,j=1}^{s}$, $\{b_i\}_{i=1}^{s}$, $\{c_i\}_{i=1}^{s}$ be real numbers and for all time levels $\{t^n\}_{n=0}^{N-1}$; where $N = \frac{T}{\Delta t}$

we define:

$$Y_1 = U^n + \Delta t \sum_{j=1}^{s} a_{1j} \mathcal{F}(t^n + c_j \Delta t, Y_j)$$

$$Y_2 = U^n + \Delta t \sum_{j=1}^{s} a_{2j} \mathcal{F}(t^n + c_j \Delta t, Y_j)$$

$$\vdots$$

(i.e. s-stages) $$Y_s = U^n + \Delta t \sum_{j=1}^{s} a_{sj} \mathcal{F}(t^n + c_j \Delta t, Y_j)$$

$$U^{n+1} = U^n + \Delta t \sum_{j=1}^{s} b_j \mathcal{F}(t^n + c_j \Delta t, Y_j)$$

$$U^0 = u_0$$

→ a general s-stage RK-Method

which can be represented in tabular format as a s.c.

Butcher Tableau

$$\begin{array}{c|ccc} c_1 & a_{11} & \cdots & a_{1s} \\ \vdots & \vdots & \ddots & \vdots \\ c_s & a_{s1} & & a_{ss} \\ \hline & b_1 & \cdots & b_s \end{array}$$

Consistency Conditions for RK-Methods

i.e. consistent s.t. there is at least one set of values for our unknowns that satisfies our equations

For the general case of the s-stage RK-Method we impose the following conditions to ensure consistency:

$\forall i \in \{1, .., s\}$ it must hold that

$$\sum_{j=1}^{s} a_{ij} = c_i$$

and

$$\sum_{j=1}^{s} b_j = 1$$

Explicit RK-Methods

if all diagonal and upper-diagonal terms of the Butcher Tableau are trivial

i.e. $a_{ij} = 0 \quad \forall j \geq i$; where $A = (a_{ij}) \in \mathbb{R}^{s \times s}$

→ could be implemented as a time-marching scheme

Diagonally Implicit RK-Methods known as DIRK

if all upper-diagonal terms are trivial and at least one diagonal term is non-trivial

i.e. $a_{ij} = 0 \quad \forall j > i$ and some $a_{ii} \neq 0$

→ Newton Method or some other method to approximate the stage (while t^n is fix)

First Properties

- Every explicit q-stage RK-Method needs q function evaluations to perform one time-step
  → for implicit methods it depends on the function
  (solve one or several non-lin eq'ns per time step i.e. employing Newton)

Order of Accuracy

- for higher order we require further conditions in addition to consistency to be satisfied, namely

  2^{nd} order accuracy $\quad \sum_{j=1}^{s} b_j c_j = \frac{1}{2}$

  3^{rd} order accuracy $\quad \sum_{j=1}^{s} b_j c_j^2 = \frac{1}{3}$

  and $\quad \sum_{j=1}^{s} \sum_{i=1}^{s} b_i a_{ij} c_j = \frac{1}{6}$

  higher orders require add. conditions

  i.e. γ-order accuracy $(\gamma < 5) \to$ at least γ stages necessary

  $(\gamma \geq 5) \to$ strictly more than γ stages