Central Difference Scheme 1st order

With Runge Kutta we have seen, that we can improve the accuracy (resp. the error) by combining different schemes

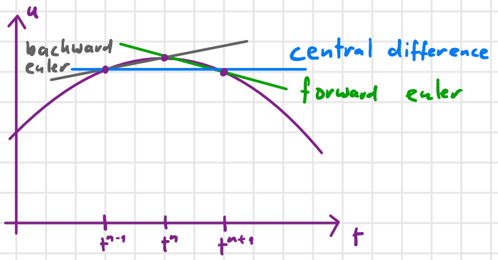
→ let's combine Forward and Backward euler:

Forward euler:

$$u'_{n-1} \approx \frac{u_n - u_{n-1}}{\Delta t}$$

Backward euler

$$u'_{n+1} \approx \frac{u_{n+1} - u_n}{\Delta t}$$



Average of u'_{n-1} and u'_{n+1}:

$$u'_n \approx \frac{u'_{n-1} + u'_{n+1}}{2}$$

$$= \frac{1}{2\Delta t}\left(\cancel{u_n} - u_{n-1} + u_{n+1} - \cancel{u_n}\right)$$

$$\boxed{u'_n \approx \frac{u_{n+1} - u_{n-1}}{2\Delta t}}$$

1st order central diff / mid-point rule

↳ Truncation error: $\mathcal{O}(\Delta t^2)$

Central Difference Scheme 2nd order

$$u'_j \approx \frac{u_{j+1} - u_j}{\Delta t} \quad \text{(Forward euler)}$$

$$u''_j \approx \frac{u'_j - u'_{j-1}}{\Delta t} \quad \text{(Backward euler)}$$

$$= \frac{1}{\Delta t}\left[\underbrace{\frac{u_{j+1} - u_j}{\Delta t}}_{u'_j} - \underbrace{\frac{u_j - u_{j-1}}{\Delta t}}_{u'_{j-1}}\right]$$

$$\boxed{u''_j \approx \frac{1}{\Delta t^2}\left(u_{j+1} - 2u_j + u_{j-1}\right)}$$

2nd order central diff. rule

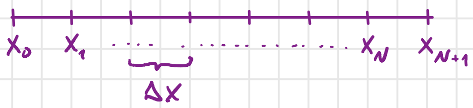
Finite Diff. for Poisson equation (1D) with Neumann boundary condition

$$\boxed{-\Delta u(x) = f(x)} \quad \text{Poisson eq.}$$

1D: $-u''(x) = f(x) \quad \forall x \in \Omega = (0,1)$

$$\text{s.t.} \begin{cases} u'(0) = d_1 \\ u'(1) = d_2 \end{cases}$$

1. Discretize space



$$\Delta x = \frac{1}{N+1}$$

$x_0 = 0$

$x_{N+1} = 1$

$x_j = j \cdot \Delta x$

2. Discretize derivatives

$u_j \approx u(x_j)$

$f_j = f(x_j)$

$$\boxed{u_j'' \approx \frac{1}{\Delta x^2}\left(u_{j+1} - 2u_j + u_{j-1}\right)} \quad \text{2nd order central diff.}$$

$$-u_j'' = f_j$$

$$-(u_{j+1} - 2u_j + u_{j-1}) = \Delta x^2 f_j$$

Boundary conditions

$$\left.\begin{aligned} u_0' &= d_1 \\ u_{N+1}' &= d_2 \end{aligned}\right\} \text{approx. with 1st order central diff}$$

$$\boxed{u_j' \approx \frac{u_{j+1} - u_{j-1}}{2\Delta x}}$$

ghost point

$$u_0' = \frac{u_1 - u_{-1}}{2\Delta x} \approx d_1 \quad \rightarrow \quad \boxed{u_{-1} = u_1 - 2\Delta x \cdot d_1}$$

analog: $\rightarrow \quad \boxed{u_{N+2} = u_N + 2\Delta x \cdot d_2}$

3. Write as linear system

$$\boxed{\underline{\underline{A}}\,\underline{u} = \underline{F}} \quad \text{with:} \quad \underline{u} = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \\ u_{N+1} \end{bmatrix}$$

$$\boxed{u_{-1} = u_1 - 2\Delta x \cdot d_1}$$

$$\left.\begin{aligned}
j=0: &\quad -u_{-1} + 2u_0 - u_1 = \Delta x^2 \cdot f_0 \\
j=1: &\quad -u_0 + 2u_1 - u_2 = \Delta x^2 f_1 \\
j=2: &\quad -u_1 + 2u_2 - u_3 = \Delta x^2 f_2 \\
&\quad \vdots \\
j=N: &\quad -u_{N-1} + 2u_N - u_{N+1} = \Delta x^2 f_N \\
j=N+1: &\quad -u_N + 2u_{N+1} - u_{N+2} = \Delta x^2 \cdot f_{N+1}
\end{aligned}\right\} \Rightarrow$$

$$\boxed{u_{N+2} = u_N + 2\Delta x \cdot d_2}$$

$$\underbrace{\begin{bmatrix}
2 & -2 & 0 & \cdots & & & & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & \cdots & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & & & \vdots \\
0 & \cdots & & 0 & -1 & 2 & -1 & 0 \\
0 & \cdots & & 0 & 0 & -1 & 2 & -1 \\
0 & \cdots & & 0 & 0 & 0 & -2 & 2
\end{bmatrix}}_{\underline{\underline{A}}}
\underbrace{\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \\ u_{N+1} \end{bmatrix}}_{\underline{u}}
= \underbrace{\Delta x^2 \begin{bmatrix} f_0 - \frac{2}{\Delta x} \cdot d_1 \\ f_1 \\ f_2 \\ \vdots \\ f_{N-1} \\ f_N \\ f_{N+1} + \frac{2}{\Delta x} d_2 \end{bmatrix}}_{\underline{F}}$$

Finite Diff. for poisson equation (2D)

$$\boxed{-\Delta u = f}$$ Poisson eq.

2D: $-(u_{xx}(x,y) + u_{yy}(x,y)) = f(x,y)$ in $\Omega := (0,1)^2$

$u = 0$ on $\partial\Omega$ — boundary of region Ω

$\Updownarrow$

$$\begin{cases} u(x,0) = u(x,1) = 0 & \forall x \\ u(0,y) = u(1,y) = 0 & \forall y \end{cases}$$

1. Discretize space

$h = \frac{1}{N+1}$

$x_0 = y_0 = 0$

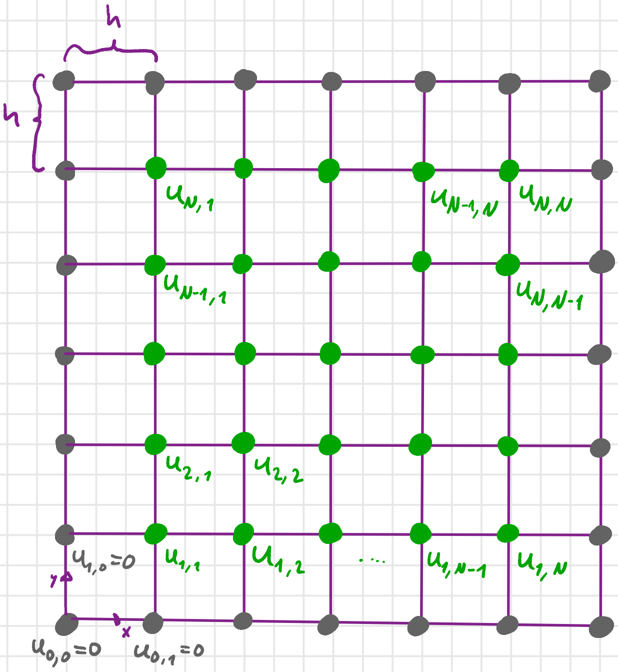
$x_{N+1} = y_{N+1} = 1$

$x_i = i \cdot h$

$y_j = j \cdot h$

$u(x_i, y_j) \approx u_{i,j}$

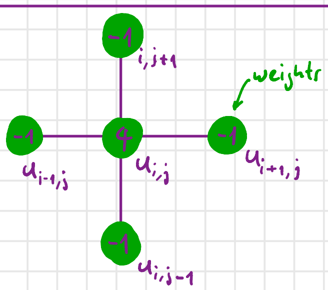
$f(x_i, y_j) = f_{i,j}$



2. Discretize derivative with a stencil

$$\boxed{\Delta u = u''_{i,j}}$$

$-u''_{i,j} \cdot h^2 =$



$$\Rightarrow \boxed{-u''_{i,j} = \frac{1}{h^2}\left(4u_{i,j} - u_{i-1,j} - u_{i+1,j} - u_{i,j-1} - u_{i,j+1}\right)}$$

$$u''_{i,j} = \underbrace{\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h}}_{\text{central diff in } y\text{-direction } \frac{\partial u_{i,j}}{\partial y}} + \underbrace{\frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h}}_{\text{central diff in } x\text{-direction } \frac{\partial u_{i,j}}{\partial x}}$$

3. Write as linear system

$$-u''_{i,j} = h^2 f_{i,j}$$

$$\underbrace{4u_{i,j} - u_{i-1,j} - u_{i+1,j} - u_{i,j-1} - u_{i,j+1}}_{\underline{A} \cdot \underline{u}} = \underbrace{h^2 f_{i,j}}_{\underline{F}}$$

Stack $u_{i,j}$ line-by-line up to vector: $\underline{u} = \begin{bmatrix} u_{1,1} \\ u_{1,2} \\ \vdots \\ u_{1,N} \\ u_{2,1} \\ \vdots \\ u_{2,N} \\ u_{3,1} \\ \vdots \\ u_{3,N} \\ \vdots \\ u_{N,N} \end{bmatrix}$ $\rightarrow$ analog $f_{i,j}$: $\underline{F} = h^2 \begin{bmatrix} f_{1,1} \\ \vdots \\ f_{1,N} \\ \vdots \\ f_{N,N} \end{bmatrix}$

$\underline{u}, \underline{F} \in \mathbb{R}^{N^2}$

for $i=2$: $$-u''_{2,j} = \underbrace{\boxed{-u_{1,j}}}_{\text{line below}} \underbrace{\boxed{-u_{2,j-1} + 4u_{2,j} - u_{2,j+1}}}_{\text{current line}} \underbrace{\boxed{-u_{3,j}}}_{\text{line above}}$$

$$\underline{u} = \begin{bmatrix} \left.\boxed{\begin{matrix} u_{1,1} \\ u_{1,2} \\ \vdots \\ u_{1,N} \end{matrix}}\right\} \text{line below} \\ \left.\boxed{\begin{matrix} u_{2,1} \\ \vdots \\ u_{2,N} \end{matrix}}\right\} \text{current line} \\ \left.\boxed{\begin{matrix} u_{3,1} \\ \vdots \\ u_{3,N} \end{matrix}}\right\} \text{line above} \\ \vdots \\ u_{N,N} \end{bmatrix}$$

current line, line above

$$\underline{\underline{A}} = \begin{bmatrix} \underline{\underline{B}} & -\underline{\underline{I}} & \underline{\underline{0}} & \underline{\underline{0}} & \cdots & \underline{\underline{0}} \\ \boxed{-\underline{\underline{I}}} & \boxed{\underline{\underline{B}}} & \boxed{-\underline{\underline{I}}} & \underline{\underline{0}} & \cdots & \underline{\underline{0}} \\ \underline{\underline{0}} & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & -\underline{\underline{I}} \\ \underline{\underline{0}} & \cdots & & \underline{\underline{0}} & -\underline{\underline{I}} & \underline{\underline{B}} \end{bmatrix} \in \mathbb{R}^{N^2 \times N^2}$$

line below

$$\underline{\underline{B}} = \begin{bmatrix} 4 & -1 & 0 & 0 & & 0 \\ -1 & 4 & -1 & 0 & & 0 \\ 0 & \ddots & \ddots & \ddots & & -1 \\ \vdots & & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & & -1 & 4 \end{bmatrix} \in \mathbb{R}^{N \times N}$$

$$\underline{\underline{I}} \in \mathbb{R}^{N \times N}$$