Last Week:

- Going from strong formulation of Poisson's equation:

$$\begin{cases} -u''(x) = f(x), & x \in (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

to a weak formulation (variational formulation (VF)):

Find $u \in H_0^1$, s.t. $\forall\, v \in H_0^1$

$$\int_0^1 u'(x)\, v'(x)\, dx = \int_0^1 v(x)\, f(x)\, dx$$

or:

Find $u \in H_0^1$, s.t. $\forall\, v \in H_0^1$

$$(u, v)_{H_0^1} = (v, f)_{L^2}$$

with:

$$H_0^1([0,1]) = \{ w : [0,1] \to \mathbb{R} : w(0) = w(1) = 0,\ w' \text{ exists},\ \int_0^1 |w'(x)|^2\, dx < \infty \}$$

$$\|u\|_{H_0^1([0,1])} := \left(\int_0^1 |u'(x)|^2\, dx \right)^{1/2}$$

$$(u,v)_{H_0^1([0,1])} := \int_0^1 u'(x)\, v'(x)\, dx = (u', v')_{L^2([0,1])}$$

(Sobolev space)

and:

$$L^2([0,1]) = \{ w : [0,1] \to \mathbb{R} : \int |u(x)|\, dx < \infty \}$$

$$\|u\|_{L^2([0,1])} := \left(\int_0^1 |u(x)|^2\, dx \right)^{1/2}$$

$$(u,v)_{L^2([0,1])} := \int_0^1 u(x)\, v(x)\, dx$$

- Using the VF to solve problem using 1D FEM:
  - Some notation: $N \in \mathbb{N}$, $h := \frac{1}{N+1}$, $x_i := i \cdot h$, $i \in \{0, 1, \dots, N\}$
  - Discretization by replacing H_0^1 by V^h, where:

  $$V^h = \{ w : [0,1] \to \mathbb{R} : w(0) = w(1) = 0,\ w \text{ cont.},\ w|_{[x_i, x_{i+1}]} \text{ is linear} \}$$

  - Find good basis $\{\phi_i\}_{i=1}^N$ of $V^h \subset H_0^1$:

  $$\phi_i(x_j) = \begin{Bmatrix} 0 & , i \neq j \\ 1 & , i = j \end{Bmatrix} = \delta_{ij}.$$

  - Resulting in a discrete variational formulation (DVF):

  Find $u_h \in V^h$: $\forall\, v \in V^h$

  $$(u_h, v)_{H_0^1} = (v, f)_{L^2}$$

  - Using $u_h(x) = \sum_{i=1}^N c_i\, \phi_i(x)$ and using $\{\phi_i\}_{i=1}^N$ as the test function v as well, we get a lin. sys. of equations

  $$AU = F,$$

  where U contains the coefficients that define u_h in terms of $\{\phi_i\}_{i=1}^N$.

This Week: Going over to 2D Problems (2D FEM)

- From a notation point of view:

$$[0,1] \longrightarrow \begin{Bmatrix} \Omega \subset \mathbb{R}^2, \text{ open}, \mu(\Omega) < \infty \\ \partial\Omega \text{ is "piecewise smooth"} \end{Bmatrix}$$

- Strong formulation

$$\begin{Bmatrix} -\Delta u(x) = f(x), & x \in \Omega \\ u|_{\partial\Omega} \equiv 0 \end{Bmatrix}$$

- Weak formulation

Find $u \in H_0^1(\Omega)$: $\forall v \in H_0^1(\Omega)$

$$(u,v)_{H_0^1(\Omega)} = (v, f)_{L^2(\Omega)}$$

- With the spaces having changed accordingly as well:

$$H_0^1(\Omega) = \{ w : \Omega \to \mathbb{R} : \nabla w \text{ exists}, \int_\Omega \|\nabla w(x)\|^2 dx < \infty \}$$

$$\|w\|_{H_0^1(\Omega)} := \left(\int_\Omega \|\nabla w(x)\|^2 dx\right)^{1/2}$$

$$(u,v)_{H_0^1(\Omega)} := \int_\Omega \langle \nabla u(x), v(x) \rangle \, dx$$

(Sobolev space)

and

$$L^2(\Omega) = \{ w : \Omega \to \mathbb{R} : \int_\Omega w^2(x) \, dx < \infty \}$$

$$\|u\|_{L^2} := \left(\int_\Omega u^2(x) dx\right)^{1/2}$$

$$(u,v)_{L^2(\Omega)} := \int_\Omega u(x) \, v(x) \, dx$$

- Regarding the derivation:

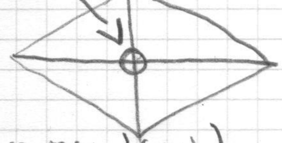
1D	2D
- Integration by parts	- 1st Green Identity
- Fundamental Theorem of calculus	- Divergence Thm.

- Mesh: Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain
  ($\Rightarrow \partial\Omega$ is p.w. linear except for finite # points)

  $\rightsquigarrow$ Triangulation $T^h = \{K_i\}_{i=1}^N$
  - K_i is triangle, $K_i \subset \Omega \; \forall i \in \{0, \dots, N\}$
  - $\bigcup_{i=1}^N \bar{K}_i = \bar{\Omega}$ (cover)
  - $\mathring{K}_i \cap \mathring{K}_j = \emptyset \quad \forall i \neq j$ (only overlap at edges)
  - No nodes on interior of sides:

No:

Yes:



$\left(h := \max_{K \in T^h} \operatorname{diam}(K), \quad \operatorname{diam}(K) := \max_{x,y \in K} d(x,y) \right)$

- Discretize space : Using our mesh defined by T^h we can now define a finite dimensional space $V^h \subset H_0^1(\Omega)$ in order to arrive at a DVF:

$$V^h := \{ w : \Omega \to \mathbb{R} : w \text{ cont.}, \quad w|_K \text{ is lin. } \forall K \in T^h, \; w|_{\partial\Omega} = 0 \}$$

- Discrete Variational Formulation

Find $u_h \in V^h$ s.t. $\forall v \in V^h : (u_h, v)_{H_0^1(\Omega)} = (f, v)_{L^2(\Omega)}$

- Basis functions : - Recall from 1D:

hat fct. $\phi_i \in V^h : \phi_i(x_j) = \delta_{ij} \quad \forall i \in \{1, \dots, N\}$

$$\rightsquigarrow \phi_i(x) = \begin{cases} 0 & (0, x_{i-1}) \\ \frac{x_i - x_{i-1}}{h} & (x_{i-1}, x_i) \\ \frac{x_{i+1} - x_i}{h} & (x_i, x_{i+1}) \\ 0 & (x_{i+1}, 1) \end{cases}$$

- In 2D

hat fct. $\phi_i \in V^h : \phi_i(x)(\mathcal{N}_j) = \delta_{ij}$

where : $\mathcal{N} = \{\mathcal{N}_i\}_{i=1}^N$ is the set of interior nodes of T^h.

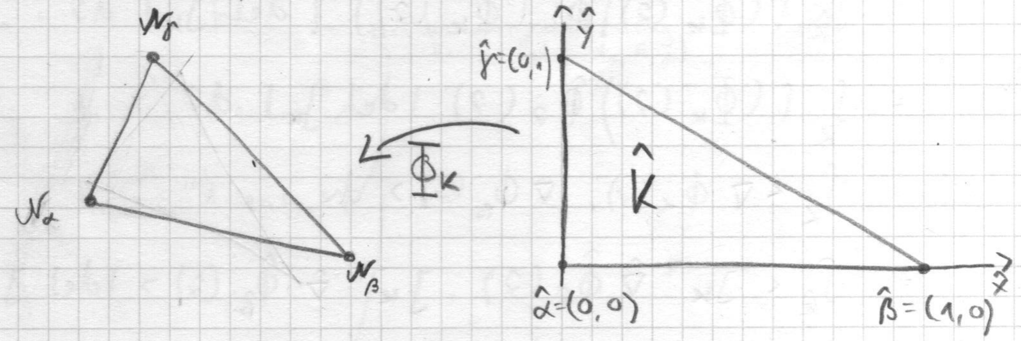
$\rightsquigarrow \phi_i(x) = \{ \dots$, not very nice

FEM Algorithm

1. Generate a mesh ⇝ usually supplied by library, understand provided interface
2. Compute "element stiffness matrix" & "element load vectors"
3. Assemble A, F
4. Solve $AU = F$

We want to know how exactly step 2 works:

Important: • How to apply quadrature rule to each individual K?
⇝ Use parametrized elements in order to have a unified & simplified approach.



(To see how exactly this can be helpful, check series 4, exercise 3 for an example.)

• Now it is easy to find our hat fcts.:

$$\hat{\phi}_{\hat{\alpha}}(\hat{x},\hat{y}) = 1-\hat{x}-\hat{y} \quad \Rightarrow \quad \hat{\nabla}\hat{\phi}_{\hat{\alpha}} = (-1,-1)$$
$$\hat{\phi}_{\hat{\beta}}(\hat{x},\hat{y}) = \hat{x} \quad \Rightarrow \quad \hat{\nabla}\hat{\phi}_{\hat{\beta}} = (1,0)$$
$$\hat{\phi}_{\hat{\gamma}}(\hat{x},\hat{y}) = \hat{y} \quad \Rightarrow \quad \hat{\nabla}\phi_{\hat{\gamma}} = (0,1)$$

• Defining Φ_K:

$$\Phi_K\left(\begin{bmatrix}\hat{x}\\ \hat{y}\end{bmatrix}\right) = \left(N_\beta - N_\alpha \,\middle|\, N_\gamma - N_\alpha\right)\begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix} + \left(N_\alpha\right)$$
$$= \begin{pmatrix}\beta_1-\alpha_1 & \gamma_1-\alpha_1\\ \beta_2-\alpha_2 & \gamma_2-\alpha_2\end{pmatrix}\begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix} + \begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}$$

$$\Rightarrow J_K := J\Phi_K = \begin{pmatrix}\beta_1-\alpha_1 & \gamma_1-\alpha_1\\ \beta_2-\alpha_2 & \gamma_2-\alpha_2\end{pmatrix}$$

Why do we care?

- Change of variables theorem: $(*)$

  Let $A \subset \mathbb{R}^k$, $\Phi : A \to \Phi(A)$ injective.

  $\Phi \in C^1$, $J\Phi(z) \neq 0 \quad \forall z \in A$

  Let $f : \Phi(A) \to \mathbb{R}$ integrable. Then:

  $$\int_{\Phi(A)} f(x)\, dx = \int_A f(\Phi(z)) \cdot |\det J\Phi(z)|\, dz.$$

- With $K = \Phi_K(\hat{K})$ and Φ_K we meet all conditions.

  $\Rightarrow$ We can use $(*)$ to compute our element stiffness matrix and the element load vector:

$$F_\alpha^K := \int_K f(x)\, \phi_\alpha(x)\, dx$$

$$= \int_{\hat{K}} f(\Phi_K(\hat{x}))\, \phi_\alpha(\Phi_K(\hat{x}))\, |\det J_K|\, d\hat{x}$$

$$= \int_{\hat{K}} f(\Phi_K(\hat{x}))\, \hat{\phi}_\alpha(\hat{x})\, |\det J_K|\, d\hat{x} \qquad (1)$$

$$A_{\alpha\beta}^K := \int_K \langle \nabla \phi_\alpha(x), \nabla \phi_\beta(x) \rangle\, dx$$

$$= \int_{\hat{K}} \langle J_K^{-T} \hat{\nabla} \hat{\phi}_\alpha(\hat{x}), J_K^{-T} \hat{\nabla} \hat{\phi}_\beta(\hat{x}) \rangle\, |\det J_K|\, d\hat{x} \qquad (2)$$

(Check series 4, exercise 4 for a more detailed guide on how to arrive at these expressions)

- Each of the computations we have to do now are on a known, fixed domain $\hat{K}$.

  $\rightsquigarrow$ Find QR and apply it to (1) & (2)

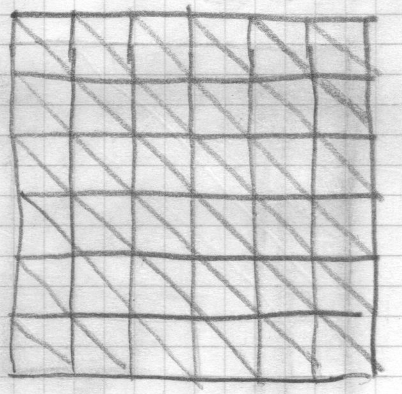
Assembling A & F

- Algorithm:
  - $A = \text{zeros}(N, N)$
  - $F = \text{zeros}(N)$ } initialize
  - for $m = 1, \dots, M$ (pick $K_m \in T^h$)

```
for m = 1,...,M
    for α = 1,...,3
        F(T(α,m)) += F_α^{K_m}
        for β = 1,...,3
            A(T(α,m), T(β,m)) += A_{α,β}^{K_m}
        end
    end
end
```

Computed as in (1), (2)

- Visualization:



"Regular Triangles"

$$\Rightarrow A := \begin{pmatrix} B & -I & & 0 \\ -I & B & \ddots & \\ & \ddots & \ddots & -I \\ 0 & & -I & B \end{pmatrix}, \quad B := \begin{pmatrix} 4 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 4 \end{pmatrix}$$

$\leadsto$ The same as 2D FD,
not for other meshes though.

- Choosing non overlapping basis fcts. helps a lot!
  > Have to compute fewer entries
  > Have sparse matrix for further computation.

- Next week:
  - Maybe some more matrix / vector assembly
  - "Other FEMs", choose different discretized spaces, eg. linear → polynomial