So far
- IVP i.e. $u'(t) = \mathcal{F}(u(t))$
- Discretisation: i.e. time levels and derivatives

$$FE \quad \frac{u^{n+1} - u^n}{\Delta t} = \mathcal{F}(u^n)$$

→ Solve

We will approach the Poisson Eq'n similarly

Poisson Eq'n

$$-\Delta u = f \qquad \text{an elliptic PDE,}$$

where Δ denotes the Laplace operator

i.e. $\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$

→ In the lecture the Poisson Eq'n in 1D was motivated and its solution derived employing Green's function representations (not available for 2D, 3D Poisson Eq'n)
- Explicit solution evaluation limited to 'easy' load f'ns f
- Slight perturbation in form of eq'n i.e. introducing a first derivative can easily render us incapable of finding a solution.

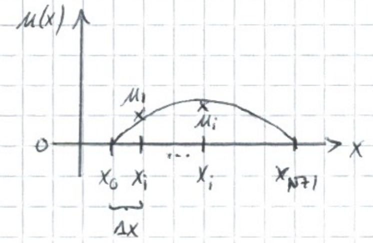
⇒ Numerical Methods to approx. solutions to the Poisson Eq'n

Finite Difference Methods

Poisson eq'n 1D
$$\begin{cases} -u''(x) = f(x) & ; \forall x \in \Omega = (0,1) \\ u|_{\partial\Omega} = u(0) = u(1) = 0 \end{cases}$$

where $f: \mathbb{R} \to \mathbb{R}$ is a cont. f'n.

1st Discretise the domain



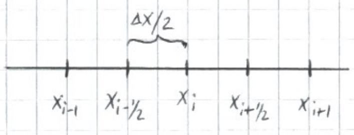
- divide domain into $N+1$ subintervals (equally spaced)

↳ $N+2$ grid points i.e. $x_0 = 0$; $x_{N+1} = 1$; $x_i = i \cdot \Delta x$ $\forall i \in \{1, \dots, N\}$, where $\Delta x = \frac{1}{N+1}$

- solutions $u_i \approx u(x_i)$, where $u_0 = u_{N+1} = 0$.

2^{nd} Discretising the derivatives

- central difference approx. (simple)



$$u''(x_i) \approx \frac{u'(x_{i+1/2}) - u'(x_{i-1/2})}{\Delta x}$$

We approx. again:

$$u'(x_{i+1/2}) \approx \frac{u(x_{i+1}) - u(x_i)}{\Delta x} \quad ; \quad u'(x_{i-1/2}) \approx \frac{u(x_i) - u(x_{i-1})}{\Delta x}$$

and combining yields

$$u''(x_i) \approx \frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{\Delta x^2}$$

$\rightarrow$ only using point values on the grid

3^{rd} Finite Difference Scheme

- Remember $u_i \approx u(x_i)$ (Notation $U_i \mathrel{\hat{=}} u_i$)
- similarly $f_i \approx f(x_i)$

approx. the Poisson Eq'n 1D (at a general pt. in the grid) for $i = 1, \dots, N$ (i in Ω)

$$-u_{i+1} + 2u_i - u_{i-1} = \Delta x^2 f_i$$

taking a closer look at boundary:

$\underline{i = 1}$

$$-u_2 + 2u_1 - u_0 = \Delta x^2 f_1 \quad ; \quad u_0 = 0$$

$$\Rightarrow -u_2 + 2u_1 = \Delta x^2 f_1$$

$\underline{i = N}$

$$\Rightarrow 2u_N - u_{N-1} = \Delta x^2 f_N \quad ; \quad u_{N+1} = 0$$

We get an eq'n for every i $\rightarrow$ system of lin eq'ns

$$\begin{aligned}
2u_1 - u_2 \qquad\qquad &= \Delta x^2 f_1 \\
-u_1 + 2u_2 - u_3 \qquad &= \Delta x^2 f_2 \\
-u_2 + 2u_3 - u_4 &= \Delta x^2 f_3 \\
\ddots \qquad &\ \ \vdots \\
-u_{N-1} + 2u_N &= \Delta x^2 f_N
\end{aligned}$$

Again introducing the vectors

$$\underline{\underline{u}} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix} \; ; \; \underline{F} = \Delta x^2 \begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix}$$; when u_i are "internal grid pts"

$\Rightarrow$ Matrix notation $\quad A u = F \quad , \; A \in \mathbb{R}^{N \times N}$

$$A = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & & \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{bmatrix}$$

In summary:

Assemble A $\quad \rightarrow$ sparse matrix: tridiag. + diag. dominant

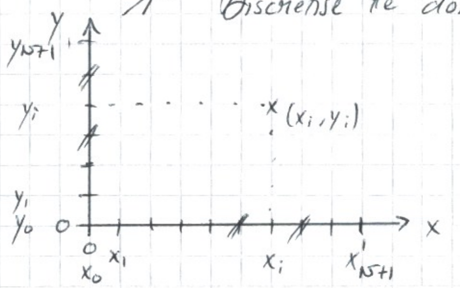
Assemble F (RHS)

Solve $AU = F$

Poisson Eq'n 2D $\quad \begin{cases} -u_{xx} - u_{yy} = f & \text{on } \Omega \\ \quad u|_{\partial\Omega} = g \end{cases}$

say for simplicity $\Omega = (0,1)^2$

1^{st} Discretise the domain



- discretising Ω into a set of $(N+2) \times (N+2)$ points s_j:
  $\Delta x = \Delta y = \frac{1}{N+1}$; $x_0 = 0$; $x_{N+1} = 1$
  $y_0 = 0$; $y_{N+1} = 1$
  $x_i = i\Delta x$; $y_i = i\Delta y \quad \forall \, 1 \le i \le N$
- solutions $u_{ij} \approx u(x_i, y_j)$; load vector $f_{ij} \approx f(x_i, y_j)$

2^{nd} Discretising the Laplacian (assume $\Delta x = \Delta y$)

$$u_{xx}(x_i, y_j) \approx \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} \; ; \; u_{yy}(x_i, y_j) \approx \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2}$$

3^{rd} Finite Difference Scheme (for general pt. (x_i, y_j))

$$-u_{xx} - u_{yy} \overset{\Delta x = \Delta y}{\approx} -\frac{u_{i+1,j} - 4u_{i,j} + u_{i-1,j} + u_{i,j-1} + u_{i,j+1}}{\Delta x^2} = f_{ij}$$

Let's take a closer look at the boundaries ($\partial\Omega$)

$$u|_{\partial\Omega} = g$$

hence for $i = j = 1$ our finite diff. scheme writes

$$-u_{2,1} + 4u_{1,1} - \underbrace{u_{0,1}}_{g_{0,1}} - \underbrace{u_{1,0}}_{g_{1,0}} - u_{1,2} = \Delta x^2 f_{1,1}$$

and consequently for a non-trivial boundary condition

$$-u_{2,1} + 4u_{1,1} - u_{1,2} = \Delta x^2 f_{1,1} + g_{0,1} + g_{1,0} := f_{1,1}^{RHS}$$

→ include in the RHS (since its given in the IVP)

Continuing similarly to the 1D scenario we get an eq'n for every pt. in the grid
→ System of Eq'ns

by introducing reasonable notation (for simplicity assume $g \equiv 0 \ \forall \partial\Omega$)

$$\underline{U} = \begin{pmatrix} u_{1,1} \\ \vdots \\ u_{N,1} \\ u_{1,2} \\ \vdots \\ u_{N,N} \end{pmatrix} ; \quad \underline{F} = \begin{pmatrix} f_{1,1}^{RHS} \\ \vdots \\ f_{N,1}^{RHS} \\ f_{1,2}^{RHS} \\ \vdots \\ f_{N,N}^{RHS} \end{pmatrix}$$

; U and $F \in \mathbb{R}^{N^2}$; u_{ij} internal grid pts

; f_{ij}^{RHS} including non-trivial contributions from boundary pts.

$$\Rightarrow AU = F \quad ; \quad A \in \mathbb{R}^{N^2 \times N^2}$$

$$A = \begin{bmatrix} B & -I & 0 & \cdots & 0 \\ -I & B & \ddots & & \vdots \\ 0 & \ddots & \ddots & & 0 \\ \vdots & & -I & B & -I \\ 0 & \cdots & 0 & -I & B \end{bmatrix} ; \text{ where } B = \begin{bmatrix} 4 & -1 & \cdots & 0 \\ -1 & 4 & \ddots & \vdots \\ \vdots & \ddots & 4 & -1 \\ 0 & \cdots & -1 & 4 \end{bmatrix} ; \quad B \in \mathbb{R}^{N \times N}$$

again A is a sparse matrix

→ Include the boundaries in your final solution vector (for Dirichlet B.C.)

↳ different B.C. s.a. Neumann require including the boundary in computation

→ Never implement a sparse matrix as a dense matrix

→ One can derive such finite difference scheme for variations of the Poisson Eq'n