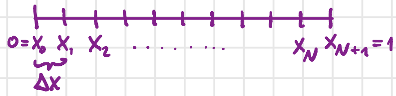
Finite Diff. for poisson equation (1D)

$$\boxed{-\Delta u(x) = f(x)} \quad \text{Poisson eq.}$$

1D: $-u''(x) = f(x) \quad \forall x \in \Omega = (0,1)$

with: $u(0) = u(1) = 0$

1. Discretize space



$x_0 = 0$
$x_{N+1} = 1$
$x_j = j\,\Delta x$

2. Discretize derivatives

$u_j \approx u(x_j)$
$f_j = f(x_j)$

$$u_j' \approx \frac{u_{j+1} - u_j}{\Delta x} \quad \text{(Forward euler)}$$

$$u_j'' \approx \frac{u_j' - u_{j-1}'}{\Delta x} \quad \text{(Backward euler)}$$

$$= \frac{1}{\Delta x}\left[\underbrace{\frac{u_{j+1} - u_j}{\Delta x}}_{u_j'} - \underbrace{\frac{u_j - u_{j-1}}{\Delta x}}_{u_{j-1}'}\right]$$

$$\boxed{u_j'' \approx \frac{1}{\Delta x^2}\left(u_{j+1} - 2u_j + u_{j-1}\right)} \quad \text{2nd order central diff.}$$

$$\Rightarrow -u_j'' = f_j$$

$$-(u_{j+1} - 2u_j + u_{j-1}) = \Delta x^2 f_j$$

3. Write as linear system

Note: boundaries u_0 / u_{N+1} missing (we already know them!)

$$\underline{\underline{A}}\,\underline{u} = \underline{F}\ ; \quad \underline{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix}, \quad F = \Delta x^2 \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{bmatrix}$$

$$\left.\begin{array}{ll} j=1: & -u_2 + 2u_1 - \cancelto{=0}{u_0} = \Delta x^2 f_1 \\ j=2: & -u_3 + 2u_2 - u_1 = \Delta x^2 f_2 \\ \vdots & \\ j=N: & -\cancelto{0}{u_{N+1}} + 2u_N - u_{N-1} = \Delta x^2 f_N \end{array}\right\} \Rightarrow$$

$$\underbrace{\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \\ & & & & & & -1 \\ 0 & 0 & \cdots & \cdots & 0 & -1 & 2 \end{bmatrix}}_{\underline{\underline{A}}} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix}}_{\underline{u}} = \underbrace{\Delta x^2 \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{N-1} \\ f_N \end{bmatrix}}_{\underline{F}}$$

CMEA S3 Theory

Finite Diff. for poisson equation (2D)

$$\boxed{-\Delta u = f}$$ Poisson eq.

2D: $-(u_{xx}(x,y) + u_{yy}(x,y)) = f(x,y)$ in $\Omega := (0,1)^2$

$u = 0$ on $\partial\Omega$ — boundary of region Ω

$\Updownarrow$

$$\begin{cases} u(x,0) = u(x,1) = 0 \quad \forall x \\ u(0,y) = u(1,y) = 0 \quad \forall y \end{cases}$$

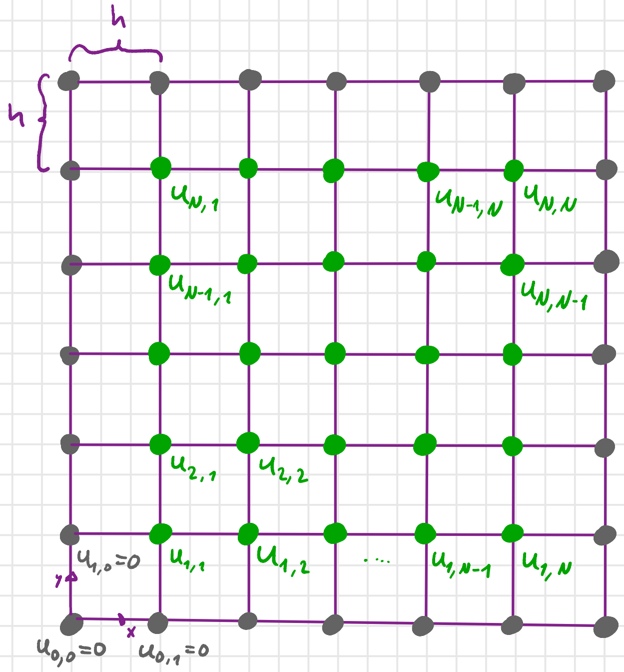
1. Discretize space

$x_0 = y_0 = 0$

$x_{N+1} = y_{N+1} = 1$

$x_i = i \cdot h$

$y_j = j \cdot h$



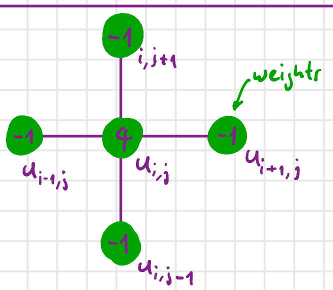
$u(x_i, y_j) \approx u_{i,j}$

$f(x_i, y_j) = f_{i,j}$

2. Discretize derivative with a stencil

$$\boxed{\Delta u = u''_{i,j}}$$

$-u''_{i,j} \cdot h^2 =$



$$\Rightarrow \boxed{-u''_{i,j} = \frac{1}{h^2}\left(4u_{i,j} - u_{i-1,j} - u_{i+1,j} - u_{i,j-1} - u_{i,j+1}\right)}$$

$$u''_{i,j} = \underbrace{\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h}}_{\substack{\text{central diff} \\ \text{in } y\text{-direction} \\ \frac{\partial u_{i,j}}{\partial y}}} + \underbrace{\frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h}}_{\substack{\text{central diff} \\ \text{in } x\text{-direction} \\ \frac{\partial u_{i,j}}{\partial x}}}$$

3. Write as linear system

$$-u''_{i,j} = h^2 f_{i,j}$$

$$\underbrace{4u_{i,j} - u_{i-1,j} - u_{i+1,j} - u_{i,j-1} - u_{i,j+1}}_{\underline{\underline{A}} \cdot \underline{u}} = \underbrace{h^2 f_{i,j}}_{\underline{F}}$$

Stack $u_{i,j}$ line-by-line up to vector: $\underline{u} = \begin{bmatrix} u_{1,1} \\ u_{1,2} \\ \vdots \\ u_{1,N} \\ u_{2,1} \\ \vdots \\ u_{2,N} \\ u_{3,1} \\ \vdots \\ u_{3,N} \\ \vdots \\ u_{N,N} \end{bmatrix}$ → analog $f_{i,j}$: $\underline{F} = h^2 \begin{bmatrix} f_{1,1} \\ \vdots \\ f_{1,N} \\ \vdots \\ f_{N,N} \end{bmatrix}$

$\underline{u}, \underline{F} \in \mathbb{R}^{N^2}$

CMEA S3 Theory

$$\underline{u} = \begin{array}{l} \left[\begin{array}{c} \boxed{\begin{array}{c} u_{1,1} \\ u_{1,2} \\ \vdots \\ u_{1,N} \end{array}} \\ \boxed{\begin{array}{c} u_{2,1} \\ \vdots \\ u_{2,N} \end{array}} \\ \boxed{\begin{array}{c} u_{3,1} \\ \vdots \\ u_{3,N} \end{array}} \\ \vdots \\ u_{N,N} \end{array}\right] \end{array} \begin{array}{l} \text{line below} \\ \\ \text{current line} \\ \\ \text{line above} \\ \\ \end{array}$$

for $i = 2$: $\quad -u''_{2,j} = \underbrace{\boxed{-u_{1,j}}}_{\text{line below}} \underbrace{\boxed{-u_{2,j-1} + 4u_{2,j} - u_{2,j+1}}}_{\text{current line}} \underbrace{\boxed{-u_{3,j}}}_{\text{line above}}$

current line

line above

$$\underline{\underline{A}} = \begin{bmatrix} \underline{\underline{B}} & -\underline{\underline{I}} & \underline{\underline{0}} & \underline{\underline{0}} & \cdots & \underline{\underline{0}} \\ \boxed{-\underline{\underline{I}}} & \boxed{\underline{\underline{B}}} & \boxed{-\underline{\underline{I}}} & \underline{\underline{0}} & \cdots & \underline{\underline{0}} \\ \underline{\underline{0}} & \ddots & \ddots & \ddots & \ddots & -\underline{\underline{I}} \\ \vdots & & \ddots & \ddots & \ddots & \\ \underline{\underline{0}} & \cdots & & \underline{\underline{0}} & -\underline{\underline{I}} & \underline{\underline{B}} \end{bmatrix} \in \mathbb{R}^{N^2 \times N^2}$$

line below

$$\underline{\underline{B}} = \begin{bmatrix} 4 & -1 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & 0 \\ & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & -1 \\ \vdots & & \ddots & \ddots & \\ 0 & \cdots & 0 & -1 & 4 \end{bmatrix} \in \mathbb{R}^{N \times N}$$

$$\underline{\underline{I}} \in \mathbb{R}^{N \times N}$$