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Dynamic Programming and Optimal Control

Fall 2021

Problem Set:
Deterministic Continuous-Time Optimal Control

Notes:

- Problems marked with BERTSEKAS are taken from the book *Dynamic Programming and Optimal Control by Dimitri P. Bertsekas, Vol. I, 3rd edition, 2005, 558 pages, hardcover.*
- The solutions were derived by teaching assistants in the previous classes. Please report any error that you may find on Piazza forum.

Problem Set 4

Problem 1 (LQR)

In the LQR problem discussed in class we assumed that

1. the optimal cost to go is of the form $x^T K(t) x$,
2. the matrix $K(t)$ is symmetric.

To rigorously show that (1) is true a-priori is not trivial, and is beyond the scope of the class. We will tackle (2): prove that if the optimal cost to go is of the form $x^T K(t) x$, then one can assume, without loss of generality, that $K(t)$ is symmetric.

Problem 2 (BERTSEKAS, p. 143, exercise 3.2)

A young investor has earned in the stock market a large amount of money S and plans to spend it so as to maximize his enjoyment through the rest of his life without working. He estimates that he will live exactly T more years and that his capital $x(t)$ should be reduced to zero at time T, i.e., $x(T) = 0$. Also he models the evolution of his capital by the differential equation

$$\frac{dx(t)}{dt} = \alpha x(t) - u(t),$$

where $x(0) = S$ is his initial capital, $\alpha > 0$ is a given interest rate, and $u(t) \geq 0$ is his rate of expenditure. The total enjoyment he will obtain is given by

$$\int_0^T e^{-\beta t} \sqrt{u(t)}\, dt.$$

Here β is some positive scalar, which serves to discount future enjoyment. Find the optimal $\big\{u(t) \mid t \in [0, T]\big\}$.

Problem 3 (Isoperimetric Problem, BERTSEKAS, p. 144, exercise 3.5)

Analyze the problem of finding a curve $\big\{x(t) \mid t \in [0, T]\big\}$ that maximizes the area under x,

$$\int_0^T x(t) dt,$$

subject to the constraints

$$x(0) = a, \quad x(T) = b, \quad \int_0^T \sqrt{1 + \big(\dot{x}(t)\big)^2}\, dt = L,$$

where a, b, and L are given positive scalars. The last constraint is known as an isoperimetric constraint; it requires that the length of the curve be L. *Hint*: Introduce the system $\dot{x}_1 = u$, $\dot{x}_2 = \sqrt{1 + u^2}$, and view the problem as a fixed terminal state problem. Show that the sine of the optimal $u^*(t)$ depends linearly on t.[1] Under some assumptions on a, b and L, the optimal curve is a circular arc.

[1] This is partly misleading. It should read: Show that the sine of the slope angle ϕ, defined by $\tan(\phi) = \frac{dx}{dt}$, is affine linear in t, i.e. $ct + d$ with constants c and d.

Problem 4 (BERTSEKAS, p. 145, exercise 3.7)

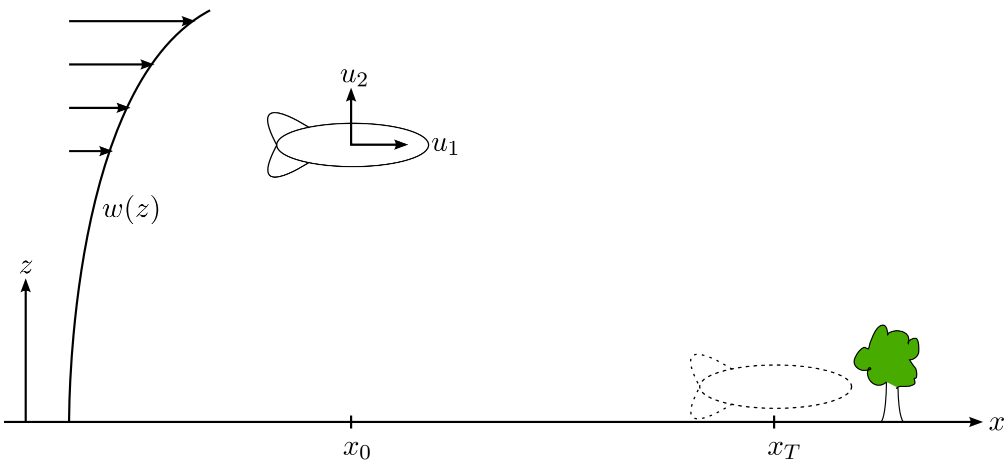
A boat moves with constant unit velocity in a stream moving at constant velocity s. The problem is to find the steering angle $u(t)$, $0 \le t \le T$, which minimizes the time T required for the boat to move between the point $(0,0)$ to a given point (a,b). The equations of motion are

$$\dot{x}_1(t) = s + \cos u(t), \qquad \dot{x}_2(t) = \sin u(t),$$

where $x_1(t)$ and $x_2(t)$ are the positions of the boat parallel and perpendicular to the stream velocity, respectively. Show that the optimal solution is to steer at a constant angle.

Problem 5 (Matlab Programming Exercise)

The goal of this programming exercise is to solve an optimal control problem, which has no closed-form solution, numerically.



Consider the landing maneuver of a blimp. The blimp can move in a vertical plane and its position is denoted by (x, z). A wind, of which the velocity is given by $w(z) = \beta z^2$, is blowing in x-direction. The blimp is controlled by means of two thrusters u_1 and u_2 pointing in x- and z-direction, respectively. Furthermore, a linear drag force acts on the blimp in each direction, resulting in the following system dynamics:

$$\ddot{x} = -\alpha_x \left(\dot{x} - w(z)\right) + u_1,$$
$$\ddot{z} = -\alpha_z \dot{z} + u_2,$$

where α_x and α_z denote the drag coefficients in x- and y-direction, respectively. The problem is to find the horizontal and vertical thrust input $u_1(t)$ and $u_2(t)$, $0 \le t \le T$, which minimize the the control effort

$$\int_0^T \frac{1}{2}\left(u_1(t)^2 + u_2(t)^2\right) dt$$

required for the blimp to move from an arbitrary initial state to a given landing point in time T.

1. For the state vector s being defined as

   $$s = (x, \dot{x}, z, \dot{z}),$$

   write down the Hamiltonian function of the above optimal control problem as a function of $s \in \mathbb{R}^4$, $p \in \mathbb{R}^4$ and $u \in \mathbb{R}^2$:

   $$H(s, p, u) = g(s, u) + p^T f(s, u).$$

2. Recall that the control u^* that minimize the above optimal control problem can be obtained by minimizing the Hamiltonian, i.e.

$$u^*(s,p) = \arg\min_u H(s,p,u).$$

Compute the optimal u^* by setting the gradient of the Hamiltonian with respect to u to zero, i.e. solve

$$\frac{\partial H}{\partial u}(p,s,u^*) = 0$$

for $u^* = (u_1^*, u_2^*)$.

3. Write down the adjoint equations

$$\dot{p}(t) = -\nabla_s H(s,p,u^*(s,p)).$$

4. Collect the blimp state s and the costate p in a new vector $y = (s,p)$. The evolution of y is then described by the first-order differential equation

$$y = \tilde{f}(y) = \begin{pmatrix} f(s, u^*(s,p)) \\ -\nabla_s H(s,p,u^*(s,p)) \end{pmatrix}.$$

The boundary value problem (BVP) that needs to be solved is given by

$$s(0) = s_0 = \begin{pmatrix} x_0 \\ \dot{x}_0 \\ z_0 \\ \dot{z}_0 \end{pmatrix},$$

$$s(T) = s_T = \begin{pmatrix} x_T \\ \dot{x}_T \\ z_T \\ \dot{z}_T \end{pmatrix},$$

$$y = \tilde{f}(y), \qquad 0 \leq t \leq T.$$

There are no constraints on the adjoints. We will solve the BVP by single shooting and using *fminsearch*. The first step is to solve the initial value problem

$$y(0) = y_0,$$
$$\dot{y}(t) = \tilde{f}(y(t)), \qquad 0 \leq t \leq T.$$

Note that $y_0 = (s_0, p_0)$. As the initial value for the blimp state is fixed to s_0, we only need to find the correct initial value for the adjoints, p_0. Write a Matlab function $F(p_0)$, using *ode45*, that takes p_0 as an input and returns the final blimp state $s(T)$, $F : \mathbb{R}^4 \to \mathbb{R}^4$. Use the following numerical values:

Time horizon T	1[h]	Drag coefficient α_x	5 [1/h]
Wind speed constant β	8 [h/km]	Drag coefficient α_z	10 [1/h]
Initial position x_0	-40 [km]	Final position x_T	0 [km]
Initial position z_0	2 [km]	Final position z_T	0 [km]
Initial velocity $\dot{x}_0$	20 [km/h]	Final velocity $\dot{x}_T$	0 [km/h]
Initial velocity $\dot{z}_T$	0 [km/h]	Final velocity $\dot{z}_T$	0 [km/h]

5. The solution of the BVP is found if we have found p_0^* such that $F(p_0^*) = s_T$. Use *fminsearch* to find p_0^* by minimizing $\|F(p_0) - s_T\|_2$.
Plot the optimal control input and blimp trajectory and comment the results.

Hint: Set the *options* in *fminsearch* to *optimset('TolFun'*,1e-7,*'TolX'*,1e-7,*'MaxFunEvals'*,1e5,*'MaxIter'*,1e5*)*.