ETH Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

*Prof. Dr. Jens Honore Walther*
*Dr. Julija Zavadlav*
*ETH Zentrum, CLT*
*CH-8092 Zürich*

# Set 8

Issued: 12.04.2019; Due: 21.04.2019

In this exercise, you will use the Newton-Cotes formulas to derive the Simpson's rule for numerical integration and implement trapezoidal and the Simpson's methods to solve an engineering problem.

## Question 1: Simpson's rule from Newton-Cotes formulas

a) Use the Newton-Cotes formulas for $n = 2$ to compute the coefficients:

$$C_k^n = \frac{1}{b-a}\int_a^b l_k^n(x)dx, \quad k = 0, \ldots, n, \tag{1}$$

where $l_k^n(x)$ are Lagrange polynomials in interval $[a, b]$ of degree $n$:

$$l_k^n(x) = \frac{(x-x_0)(x-x_1)\ldots(x-x_{k-1})(x-x_{k+1})\ldots(x-x_n)}{(x_k-x_0)(x_k-x_1)\ldots(x_k-x_{k-1})(x_k-x_{k+1})\ldots(x_k-x_n)}, \tag{2}$$

where $x_i$ are equidistant points in $[a, b]$. For $n = 2$ that is: $x_0 = a,\ x_1 = (a+b)/2,\ x_2 = b$.

b) Using the computed coefficients $C_k^n$ from (1), derive the resulting numerical integration rule using the Newton-Cotes formula:

$$I \approx (b-a)\sum_{k=0}^{n} C_k^n f(x_k). \tag{3}$$

Verify that you have obtained the so-called Simpson's rule:

$$I \approx \frac{f(a) + 4f((a+b)/2) + f(b)}{6}(b-a). \tag{4}$$

## Question 2: Thermal characteristics of disk brakes

To simulate the thermal characteristics of disk brakes (see Fig. 1), we need to numerically approximate the area-averaged lining temperature $\bar{T}$ of the brake pad from the equation:

$$\bar{T} = \frac{\int_{r_e}^{r_0} T(r) r\theta_p dr}{\int_{r_e}^{r_0} r\theta_p dr}, \tag{5}$$

| r (m) | T (C) | r (m) | T (C) | r (m) | T (C) |
|---|---|---|---|---|---|
| 0.095 | 344.824 | 0.115 | 550.413 | 0.135 | 643.941 |
| 0.100 | 413.923 | 0.120 | 580.481 | 0.140 | 658.861 |
| 0.105 | 469.207 | 0.125 | 605.451 | 0.145 | 671.585 |
| 0.110 | 513.912 | 0.130 | 626.342 | | |

Table 1: Values of temperature at different locations along the radial direction of the disk.

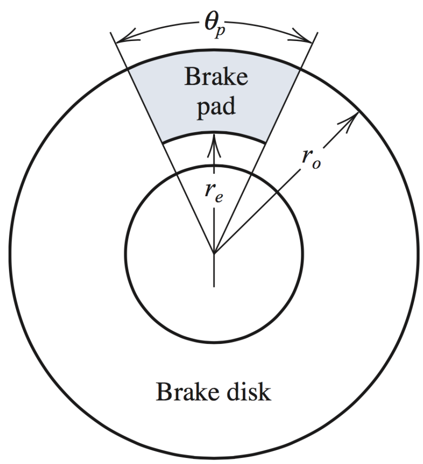


Figure 1: Brake disk schematic.

where $r_e$ represents the radius at which the pad-disk contact begins, $r_0$ represents the outside radius of the pad-disk contact, $\theta_p$ represents the angle subtended by the sector brake pads, and T(r) is the temperature at each point of the pad (obtained numerically from analyzing the heat equation). Take $r_e$ = 0.095 m, $r_0$ = 0.145 m, $\theta_p$ = 0.7051 radians. The temperatures corresponding to different radii are given in Table 1. Create a `.txt` file containing the input values, in which the first column will correspond to the radius and the second to the temperature. Follow the next steps to write a code to approximate $\bar{T}$ using the trapezoidal rule.

a) Write a code that reads in the data (radius and temperature) and stores them in two vectors. The data is stored in the `.txt` file you created. Test the code to make sure the data is read correctly.

b) Implement the trapezoidal rule to evaluate the two integrals in Eq. 5 and reports the "area-averaged lining temperature" $\bar{T}$.

   Keep in mind that with the trapezoidal rule, we assume that the integral range (here $[r_e, r_0]$) is split into $N$ subintervals (here given by the $N+1$ data points).

   From the lecture notes, we know that the integral of a function $f(x)$ in the range $[a, b]$ is split into subintervals $[x_i, x_{i+1}]$ with $x_0 = a$ and $x_N = b$ and the integral is approximated as:

$$I = \int_a^b f(x)\,dx = \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} f(x)\,dx \approx \sum_{i=0}^{N-1} I_{T_i}, \tag{6}$$

$$I_{T_i} = \frac{f(x_i) + f(x_{i+1})}{2}\Delta_i, \quad \text{with } \Delta_i = x_{i+1} - x_i. \tag{7}$$

Implement this method for the two integrals in Eq. 5 and report $\bar{T}$.

(*This question is adapted from the book 'Numerical Analysis' by R.L. Burden and J.D. Faires.*)

## Question 3: Comparison of Quadrature Rules

In the current exercise, we use an analytical function to approximate the temperature field that was given with discrete data points in the previous exercise. The function is defined as follows:

$$T(r) = ar^b + c\,, \tag{8}$$

where $a, b, c$ are constant parameters set to $a = -0.125$, $b = -3.455$ and $c = 770.3$. You are asked to:

a) Calculate the expression $\bar{T} = \frac{\int_{r_e}^{r_0} T(r) r \theta_p dr}{\int_{r_e}^{r_0} r \theta_p dr}$ ($r_e$ = 0.095 m, $r_0$ = 0.145 m, $\theta_p$ = 0.7051 radians) analytically to obtain the "exact" approximate value of $\bar{T}$.

b) Extend your code from the previous problem to approximate the integrals within the temperature $\bar{T}$ expression with the composite Simpson's rule (Eq. 4). Notice that Simpson's rule uses three points while trapezoidal rule uses two points (use the same number of data points in both cases).

c) Compare the approximation obtained by Trapezoidal and Simpson's rule with the exact value of the integrals. Which one of the two methods approximates the integral better?