Self-study for Continuous Variable Quantum Information

Introductory Quantum Optics (Chapter 3: Coherent States)

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December 1, 2025

• Introduction

Photon number state $|n\rangle$: uniform phase distribution over the range $[0, 2\pi)$.

→ no well defined phase.

Expectation of field operator : vanishing number state.

The classical limit of the quantized field. → # number of photons become very large.

: number operator goes classical limit. in Not single mode ($\langle n|\hat{E}(r,t)|n\rangle = 0$),

can only be "similarly classical field" on multi mode.

"Coherent state" : most classical quantum states of a harmonic oscillator.

→ give rise to a sensible classical limit.

3.1. Eigenstates of an annihilation operator and minimum uncertainty principle.

To have a non-zero expectation of the E-field operator, we are required to have superposition.

$$|\psi\rangle = C_n|n\rangle + C_{n+1}|n+1\rangle \quad : \text{normalized.}$$

Eigenstate of the annihilation operator for $|\psi\rangle \to |\alpha\rangle$.

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle. \quad (\alpha : \text{complex}, \hat{a} : \text{non-Hermitian}).$$

$$\langle\alpha|\hat{a}^{+} = \alpha^{*}\langle\alpha|$$

where $|\alpha\rangle = \sum_{n=0}^{\infty} C_n|n\rangle$ (complete set of $|n\rangle$).

Acting with $\hat{a}$ on each term of the expansion.

$$\hat{a}|\alpha\rangle = \sum_{n=1}^{\infty} C_n\sqrt{n}|n-1\rangle = \alpha\sum_{n=0}^{\infty} C_n|n\rangle$$

$$C_n\sqrt{n} = \alpha C_{n-1} \quad \text{or} \quad C_n = \frac{1}{\sqrt{n}}\alpha C_{n-1} = \frac{\alpha^2}{\sqrt{n(n-1)}}C_{n-2} = \cdots = \frac{\alpha^n}{\sqrt{n!}}C_0.$$

so, $|\alpha\rangle = C_0\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$.

Normalized, $\langle\alpha|\alpha\rangle = 1 = |C_0|^2\sum_{n,n'}\frac{(\alpha^{*})^{n}\alpha^{n'}}{\sqrt{n!n'!}}\langle n|n'\rangle = |C_0|^2\sum_{n}\frac{\alpha^{2n}}{n!} = |C_0|^2 e^{|\alpha|^2}$

where $C_0 = \exp\left(-\frac{1}{2}e^{|\alpha|^2}\right)$.

$$|\alpha\rangle = \exp\left(-\frac{1}{2}e^{|\alpha|^2}\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle.$$

Expectation value of the single mode E-field $\hat{E}(r,t) = \hat{x}\hat{E}_x(r,t)$.

$$\hat{E}_x(r,t) = i\left(\frac{\hbar\omega}{2\varepsilon_0 V}\right)^{1/2}\left[\hat{a}e^{i(k\cdot r-\omega t)} - \hat{a}^{+}e^{-i(k\cdot r-\omega t)}\right].$$

$$\langle\alpha|\hat{E}_x(r,t)|\alpha\rangle = i\left(\frac{\hbar\omega}{2\varepsilon_0 V}\right)^{1/2}\left[\alpha e^{i(k\cdot r-\omega t)} - \alpha^{*}e^{-i(k\cdot r-\omega t)}\right].$$

$\hat{a}|\alpha\rangle$

$\alpha|\alpha\rangle$

In polar form $\alpha = |\alpha|e^{i\theta}$:

$$\langle\alpha|\hat{E}_x(r,t)|\alpha\rangle = 2|\alpha|\left(\frac{\hbar\omega}{2\varepsilon_0 V}\right)^{1/2}\sin(\omega t - k\cdot r - \theta).$$ (looks like classical field).

$$\hat{E}_x^2(r,t) = -\left(\frac{\hbar\omega}{2\varepsilon_0 V}\right)\left[\hat{a}e^{i(k\cdot r-\omega t)} - \hat{a}^{+}e^{-i(k\cdot r-\omega t)}\right]^2$$

for Boson mode. $[\hat{a},\hat{a}^{+}] = \hat{a}\hat{a}^{+} - \hat{a}^{+}\hat{a} = 1$.

$$= -\left(\frac{\hbar\omega}{2\varepsilon_0 V}\right)\left[\hat{a}^2 e^{i2(k\cdot r-\omega t)} + (\hat{a}^{+})^2 e^{-i2(k\cdot r-\omega t)} - \hat{a}\hat{a}^{+} - \hat{a}^{+}\hat{a}\right].$$

$$\langle\alpha|\hat{E}_x^2(r,t)|\alpha\rangle = -\left(\frac{\hbar\omega}{2\varepsilon_0 V}\right)\left[\alpha^2 e^{i2(k\cdot r-\omega t)} + (\alpha^{*})^2 e^{-i2(k\cdot r-\omega t)} - |\alpha|^2 - (|\alpha|^2+1)\right]$$

$$= -\left(\frac{\hbar\omega}{2\varepsilon_0 V}\right)\left[|\alpha|^2 e^{i2(k\cdot r-\omega t+\theta)} + |\alpha|^2 e^{-i2(k\cdot r-\omega t+\theta)} - 2|\alpha|^2 - 1\right).$$

$$= -\left(\frac{\hbar\omega}{2\varepsilon_0 V}\right)\left[2|\alpha|^2\cos(2(k\cdot r-\omega t+\theta)) - 2|\alpha|^2 - 1\right)$$

$$= \left(\frac{\hbar\omega}{2\varepsilon_0 V}\right)\left(1 + 4|\alpha|^2\sin^2(\omega t - k\cdot r - \theta)\right)$$

Thus the fluctuations in $\hat{E}_x(x,t)$.

$$\Delta E_x = \langle(\Delta\hat{E}_x)^2\rangle^{\frac{1}{2}} = \boxed{\left(\frac{\hbar\omega}{2\varepsilon_0 V}\right)^{1/2}}$$ (identical to a vacuum state).

noise vacuum.

Coherent state is nearly a classical state.

Introducing Quadrature operator.

$$\hat{X}_1 = \frac{1}{2}(\hat{a}+\hat{a}^{+}),\quad \hat{X}_2 = \frac{1}{2i}(\hat{a}-\hat{a}^{+})$$

$$\hat{X}_1^2 = \frac{1}{4}\left(\hat{a}^2 + (\hat{a}^{+})^2 + \hat{a}\hat{a}^{+} + \hat{a}^{+}\hat{a}\right),\quad \hat{X}_2^2 = \frac{1}{-4}\left(\hat{a}^2 + (\hat{a}^{+})^2 - \hat{a}\hat{a}^{+} - \hat{a}^{+}\hat{a}\right)$$

$\langle\alpha|\hat{X}_1|\alpha\rangle = \frac{1}{2}(\alpha+\alpha^*) = |\alpha|\cos\theta$

$\langle\alpha|\hat{X}_1^2|\alpha\rangle = \frac{1}{4}(|\alpha|^2 e^{i2\theta} + |\alpha|^2 e^{-i2\theta} + 2|\alpha|^2 + 1) = \frac{1}{4}(1+2|\alpha|^2(1+\cos 2\theta))$

$= \frac{1}{4}(1+4|\alpha|^2\cos^2\theta)$

$\langle(\Delta\hat{X}_1)^2\rangle^{1/2} = \frac{1}{4}$.

$\langle\alpha|\hat{X}_2|\alpha\rangle = \frac{1}{2i}(\alpha-\alpha^*) = |\alpha|\sin\theta$.

$\langle\alpha|\hat{X}_2^2|\alpha\rangle = -\frac{1}{4}(|\alpha|^2 e^{i2\theta} + |\alpha|^2 e^{-i2\theta} - 2|\alpha|^2 - 1)$

$= -\frac{1}{4}(|\alpha|^2(\cos 2\theta - 1) - 1) = \frac{1}{4}(1+4|\alpha|^2\sin^2\theta)$.

$\langle(\Delta\hat{X}_2)^2\rangle^{1/2} = \frac{1}{4}$. $\longrightarrow$ fluctuation at vacuum.

$[\hat{A},\hat{B}] = i\hat{C}$ with

$\langle(\Delta\hat{A})^2\rangle\langle(\Delta\hat{B})^2\rangle \geq \frac{1}{4}\langle\hat{C}^2\rangle$

Eigenvalue equation, $\left[\hat{A} + \frac{i\langle\hat{C}\rangle}{2\langle(\Delta\hat{B})^2\rangle}\hat{B}\right]|\psi\rangle = \left[\langle\hat{A}\rangle + \frac{i\langle\hat{C}\rangle}{2\langle(\Delta\hat{B})^2\rangle}\langle\hat{B}\rangle\right]|\psi\rangle$.

For intelligent state,

$\langle(\Delta A)^2\rangle = \langle(\Delta B)^2\rangle = \frac{1}{4}\langle(\Delta C)^2\rangle$

$[\hat{A} + i\hat{B}]|\psi\rangle = [\langle\hat{A}\rangle + i\langle\hat{B}\rangle]|\psi\rangle$.

$\longrightarrow$ In this case, $[\hat{X}_1 + i\hat{X}_2]|\psi\rangle = [\langle\hat{X}_1\rangle + i\langle\hat{X}_2\rangle]|\psi\rangle$, $\alpha = \langle\hat{X}_1\rangle + i\langle\hat{X}_2\rangle$.

* Physical meaning of the complex parameter α.

$|\alpha|$: related to the amplitude of the field.

The average of the photon number $\hat{n} = \hat{a}^\dagger\hat{a}$.

$\bar{n} = \langle\alpha|\hat{n}|\alpha\rangle = |\alpha|^2$: average photon number of the field.

$$\langle\alpha|\hat n^2|\alpha\rangle = \langle\alpha|\hat a^\dagger\hat a\hat a^\dagger\hat a|\alpha\rangle$$
$$= \langle\alpha|\hat a^\dagger(\hat a^\dagger\hat a+1)\hat a|\alpha\rangle = \langle\alpha|(\hat a^\dagger\hat a^\dagger\hat a\hat a+\hat a^\dagger\hat a)|\alpha\rangle.$$
$$= |\alpha|^4+|\alpha|^2 = \bar n^2+\bar n$$

thus $\Delta n = \sqrt{\langle\hat n^2\rangle - \langle\hat n\rangle^2} = \bar n^{1/2}$: characteristic of a Poisson process.

The probability of detecting n photon

$$P_n = |\langle n|\alpha\rangle|^2 = e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!} = e^{-\bar n}\frac{\bar n^n}{n!}$$: Poisson distribution with mean of $\bar n$.

The fractional uncertainty of photon number,

$\Delta n/\bar n = 1/\sqrt{\bar n}$: which decreases with increasing $\bar n$.

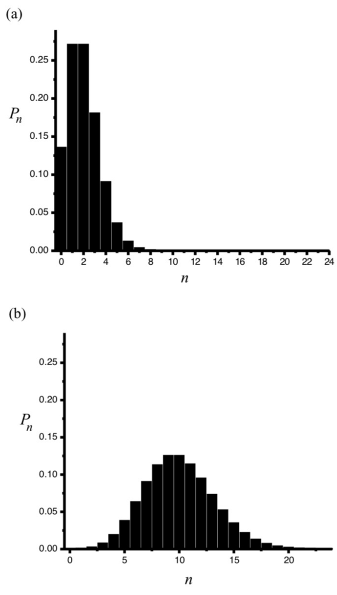


Fig. 3.1. Coherent state photon number probability distributions for (a) $\bar n = 2$ and (b) $\bar n = 10$.

Phase distribution of the coherent state.

For a coherent state $|\alpha\rangle$ with $\alpha = |\alpha|e^{i\theta}$.

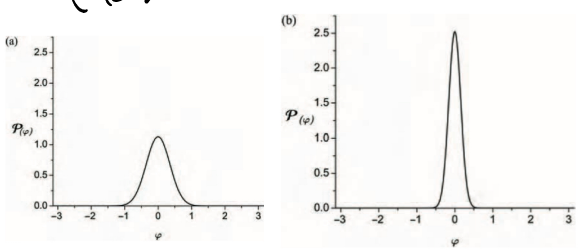
$$P(\varphi) = \frac{1}{2\pi}|\langle\varphi|\alpha\rangle|^2$$
$$= \frac{1}{2\pi}e^{-|\alpha|^2}\left|\sum_{n=0}^{\infty} e^{in(\varphi-\theta)}\frac{|\alpha|^n}{\sqrt{n!}}\right|^2.$$

For large $|\alpha|^2$, the Poisson distribution may be approximated as a Gaussian.

$$e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!} = (2\pi|\alpha|^2)^{-1/2}\exp\left(-\frac{(n-|\alpha|^2)^2}{2|\alpha|^2}\right).$$

$$\overset{FT}{\Longleftrightarrow} P(\varphi) \propto \left(\frac{2|\alpha|^2}{\pi}\right)^{1/2}\exp\left(-2|\alpha|^2(\varphi-\theta)^2\right).$$ $\bar n = |\alpha|^2$: narrow.

Fig. 3.2. Phase distributions for coherent states with $\theta = 0$ for (a) $\bar n = 2$ and (b) $\bar n = 10$.



The reason why the coherent state is nearly close to classical state.

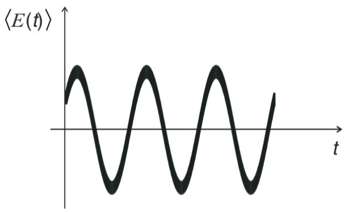
(i) the expectation value of the E-field = form of the classical expression.

(ii) the fluctuations in the E-field = same as for a vacuum.

(iii) the fluctuations in the fractional uncertainty for the photon number,

⇒ decrease with the increasing average photon number.

(iv) the states become well localized in phase with increasing average photon number



quantum fluctuation superimposed.

Fig. 3.3. Coherent state expectation value of the electric field as a function of time for a fixed position showing the quantum fluctuations. The fluctuations of the field are the same at all times such that the field is as close to a classical field as is possible for any quantum state.

3.2 Displaced vacuum state.

Defining coherent state. (1) eigenstates of the annihilation operator.

(2) states that minimize the uncertainty relation for the two orthogonal field quadrature with equal uncertainties. (identical to vacuum)

(3) Displacement of the vacuum : generating the coherent states from classical current.

Displacement operator $\hat{D}(\alpha)$

$$\hat{D}(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$$

the coherent state are given as

$$|\alpha\rangle = \hat{D}(\alpha)|0\rangle.$$

Disentangling theorem,

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}e^{-\frac{1}{2}[\hat{A},\hat{B}]} = e^{\hat{B}}e^{\hat{A}}e^{\frac{1}{2}[\hat{A},\hat{B}]}.$$ valid if $[\hat{A},\hat{B}] \neq 0$,

but where $[\hat{A},[\hat{A},\hat{B}]] = [\hat{B},[\hat{A},\hat{B}]] = 0$.

with $\hat{A} = \alpha\hat{a}^\dagger$, $\hat{B} = -\alpha^*\hat{a}$, $[\hat{A},\hat{B}] = |\alpha|^2$.

thus displacement operator, $\hat{D}(\alpha) = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}} = e^{-\frac{1}{2}|\alpha|^2}e^{\alpha\hat{a}^\dagger}e^{-\alpha^*\hat{a}}$.

expanding $\exp(-\alpha^*\hat{a})$, $\quad \exp(-\alpha^*\hat{a})|0\rangle = \sum_{\ell=0}^{\infty}\frac{(-\alpha^*\hat{a})^\ell}{\ell!}|0\rangle = |0\rangle$. $\quad \hat{a}|0\rangle = 0$ for $\ell \neq 0$.

$$e^{\alpha\hat{a}^\dagger}|0\rangle = \sum_{n=0}^{\infty}\frac{\alpha^n}{n!}(\hat{a}^\dagger)^n|0\rangle = \sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \quad \text{where} \quad (\hat{a}^\dagger)^n|0\rangle = \sqrt{n!}\,|n\rangle$$

$$|\alpha\rangle = \hat{D}(\alpha)|0\rangle.$$

$$= e^{-\frac{1}{2}|\alpha|^2}e^{\alpha\hat{a}^\dagger}e^{-\alpha^*\hat{a}}|0\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle.$$

Displacement operator $\hat{D}$ is a unitary operator.

$$\hat{D}^\dagger(\alpha) = \hat{D}(-\alpha) = e^{-\frac{1}{2}|\alpha|^2}e^{-\alpha\hat{a}^\dagger}e^{\alpha^*\hat{a}}.$$

$$\hat{D}(\alpha)\hat{D}^\dagger(\alpha) = \hat{D}^\dagger(\alpha)\hat{D}(\alpha) = 1. \text{ for unitary.}$$

Displacement operator obeys the semigroup relation.

$$\hat{D}(\alpha)\hat{D}(\beta) = \hat{D}(\alpha+\beta) \quad \text{where} \quad \hat{A} = \alpha\hat{a}^\dagger - \alpha^*\hat{a}, \quad \hat{B} = \beta\hat{a}^\dagger - \beta^*\hat{a}.$$

$$[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} = (\alpha^*\beta - \alpha\beta^*)\hat{a}^\dagger\hat{a} + (\beta^*\alpha - \alpha^*\beta)\hat{a}\hat{a}^\dagger, \quad \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1$$

$$= \alpha\beta^* - \alpha^*\beta = 2i\,\mathrm{Im}(\alpha\beta^*)$$

Using disentanglement theorem.

$$\hat{D}(\alpha)\hat{D}(\beta) = e^{\hat{A}}e^{\hat{B}} = \exp[(\alpha+\beta)\hat{a}^\dagger - (\alpha^* + \beta^*)\hat{a}]\exp(i\,\mathrm{Im}(\alpha\beta^*))$$

$$= \exp(i\,\mathrm{Im}(\alpha\beta^*))\,\hat{D}(\alpha+\beta)$$

Applied to the vacuum state,

$$\hat{D}(\alpha)\hat{D}(\beta)|0\rangle = \hat{D}(\alpha)|\beta\rangle = \underline{\exp(i\,\mathrm{Im}(\alpha\beta^*))}\,|\alpha+\beta\rangle.$$

Phase factor (physically irrelevant).

By the decomposition of the displacement operator,

$$\langle m|\hat{D}(\alpha)|n\rangle = e^{-\frac{1}{2}|\alpha|^2}\langle m|e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}|n\rangle$$

Expand the right-most exponential to obtain,

$$e^{-\alpha^*\hat{a}}|n\rangle = \sum_{p=0}^{\infty}\frac{(-\alpha^*\hat{a})^p}{p!}|n\rangle = \sum_{p=0}^{n}\frac{(-\alpha^*)^p}{p!}\sqrt{\frac{n!}{(n-p)!}}\,|n-p\rangle.$$

with $n-p=l$,

$$e^{-\alpha^*\hat{a}}|n\rangle = \sum_{l=0}^{n} \frac{(-\alpha^*)^{n-l}}{(n-l)!}\sqrt{\frac{n!}{l!}}|l\rangle.$$

In the same manner, $(m-j=k)$

$$\langle m|e^{\alpha\hat{a}^\dagger} = \sum_{j=0}^{\infty}\frac{(\alpha\hat{a}^\dagger)^j}{j!}\langle m| = \sum_{j=0}^{m}\frac{\alpha^j}{j!}\sqrt{\frac{m!}{(m-j)!}}\langle m-j|.$$

$$= \sum_{k=0}^{m}\frac{\alpha^{m-k}}{(m-k)!}\sqrt{\frac{m!}{k!}}\langle k|$$

It follows that,

$$\langle m|\hat{D}(\alpha)|n\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{k=0}^{n}\sum_{l=0}^{m}\sqrt{\frac{m!n!}{k!l!}}\frac{\alpha^{m-k}(-\alpha^*)^{n-l}}{(m-k)!(n-l)!}\delta_{kl}.$$

$$= (m!n!)^{1/2}e^{-\frac{1}{2}|\alpha|^2}\sum_{l=0}^{\min(m,n)}\frac{\alpha^{m-l}(-\alpha^*)^{n-l}}{l!(m-l)!(n-l)!}$$

3.3 Wave packets and time evolution.

- Position operator. $\hat{q} = \sqrt{\frac{\hbar}{2\omega}}(\hat{a}+\hat{a}^\dagger) = \sqrt{\frac{2\hbar}{\omega}}\hat{X}_1$
- eigenstate for $\hat{q}$, $\hat{q}|q\rangle = q|q\rangle$.
- wavefunction for the number state.

$$\psi_n(q) = \langle q|n\rangle = (2^n n!)^{-1/2}\left(\frac{\omega}{\pi\hbar}\right)^{1/4}\exp(-y^2/2)H_n(y).$$

where $y = q\sqrt{\omega/\hbar}$ and the $H_n(y)$ are Hermite Polynomials.

- wavefunction for the coherent state.

$$\psi_\alpha(q) = \langle q|\alpha\rangle = \left(\frac{\omega}{\pi\hbar}\right)^{1/4}e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{(\alpha/\sqrt{2})^n}{n!}H_n(y).$$

Generating function for Hermite polynomial, $e^{2xt-t^2} = \sum_{n=0}^{\infty}H_n(x)\frac{t^n}{n!}$.

$$\exp(\sqrt{2}\alpha y - \alpha^2/2) = \exp(y^2 - (y-\alpha/\sqrt{2})^2)$$

$$\psi_\alpha(q) = \left(\frac{\omega}{\pi\hbar}\right)^{1/4}e^{-|\alpha|^2/2}e^{-y^2/2}\left[e^{y^2}e^{-(y-\alpha/\sqrt{2})^2}\right] : \text{A gausian wave function.}$$

$$= \left(\frac{\omega}{\pi\hbar}\right)^{1/4}e^{-|\alpha|^2/2}\,e^{y^2/2}e^{-(y-\alpha/\sqrt{2})^2}$$

The probability distribution over the 'position' variable q,

$$P(q) = |\psi_\alpha(q)|^2 .$$

Time evolution of a coherent state for a single mode free field with $\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})$.

For a time evolution coherent state.

$$|\alpha, t\rangle \equiv \exp(-i\hat{H}t/\hbar)|\alpha\rangle = \exp(-i\omega t/2)\exp(-i\omega\hat{n})|\alpha\rangle$$

$$= \exp(-i\omega t/2)\,|\alpha e^{-i\omega t}\rangle .$$

remains a coherent state under free-field evolution.

$$\psi_\alpha(q,t) = \left(\frac{\omega}{\pi\hbar}\right)^{1/4} e^{-|\alpha|^2/2} e^{y^2/2} e^{-(y - \alpha e^{-i\omega t}/\sqrt{2})^2}.$$

- A Gaussian whose shape does not change with time.
- centroid follows the motion of a classical point particle in H.O. potential.

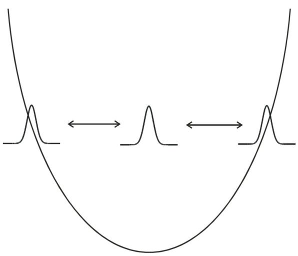


Fig. 3.4. A coherent-state wave function moves through the harmonic oscillator potential, between the classical turning points, without dispersion.

the motion of a coherent state wave packet.

The motion state engineered to have minimum uncertainty.

The stability of the wave packet. : because of integer spaced energy level.

(But Coulomb problem, formulating coherent states is a bit of a challenge).

3.4 Generation of coherent state.

A coherent state generating by classical oscillating current.

vector potential for a field interacting with a classical current.

$$\hat{V}(t) = -\int d^3r\, j(r,t)\cdot A(r,t)$$

For a single field in the interacting picture.

$$\hat{A}(r,t) = \hat{e}\left(\frac{\hbar}{2\omega\epsilon_0 V}\right)^{1/2}\left[\hat{a}e^{i(k\cdot r - \omega t)} + \hat{a}^\dagger e^{-i(k\cdot r - \omega t)}\right]$$

where $\hat{a} = \hat{a}(0)$

$$\hat{V}(t) = -\left(\frac{\hbar}{2\omega\varepsilon_0 V}\right)^{1/2} \int d^3r\, j(r,t) \cdot \left(\hat{e}\hat{a}e^{i(k\cdot r-\omega t)} + \hat{e}\hat{a}^{\dagger}e^{-i(k\cdot r-\omega t)}\right)$$

using $J(k,t) = \int d^3r\, j(r,t)\, e^{ik\cdot r}$

$$\hat{V}(t) = -\left(\frac{\hbar}{2\omega\varepsilon_0 V}\right)^{1/2} \left[\hat{a}\hat{e}\cdot J(k,t)e^{-i\omega t} + \hat{a}^{\dagger}\hat{e}\cdot J^*(k,t)e^{i\omega t}\right].$$

Since $V(t)$ depends on time, the associated evolution operator is a time-ordered product.

From t to $t+\delta t$, $\hat{U}(t+\delta t, t) \approx \exp(-i\hat{V}(t)\delta t/\hbar)$

let $u(t) = \exp\left[i\left(\frac{1}{2\hbar\omega\varepsilon_0 V}\right)^{1/2}\hat{e}\cdot J^*(k,t)e^{i\omega t}\right]$

$$\hat{U}(t+\delta t, t) \approx \exp\left((\hat{a}^{\dagger}u(t) + \hat{a}u(t))\delta t\right) = \hat{D}[u(t)\delta t].$$

For a finite time time interval $(0,T)$, the evolution operator,

$$\hat{U}(T,0) = \lim_{\delta t\to 0} \hat{T} \prod_{l=0}^{T/\delta t} \hat{D}[u(t_l)\delta t] \quad \text{where } \hat{T}\text{: time-ordering operator}$$

using disentanglement theorem, → Accumulated overall phase. ($J(k,t)$ apart from the irrelevant overall phase).

$$\hat{U}(T,0) = \lim_{\delta t\to 0} e^{i\boxed{\Phi}}\hat{D}\left[\sum_{l=0}^{T/\delta t} u(t_l)\delta t\right] = e^{i\Phi}\hat{D}[\alpha(T)]$$

where $\alpha(T) = \lim_{\delta t\to 0}\sum_{l=0}^{T/\delta t} u(t_l)\delta t = \int_0^T u(t')dt'$.

With initial state the vacuum, the state at T is "coherent state" $|\alpha(T)\rangle$.

3.5 More on the properties of coherent state.

The number states are orthonormal, complete set $\sum_{n=0}^{\infty}|n\rangle\langle n| = I$

$|\psi\rangle = \sum_n C_n|n\rangle$. where $C_n = \langle n|\psi\rangle$.

The coherent states themselves are not orthogonal. $|\alpha\rangle, |\beta\rangle$.

$$\langle\alpha|\beta\rangle = e^{-\frac{1}{2}|\alpha|^2-\frac{1}{2}|\beta|^2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\beta^*)^n\alpha^m}{\sqrt{n!m!}}\langle n|m\rangle = e^{-\frac{1}{2}|\alpha|^2-\frac{1}{2}|\beta|^2}\sum_{n=0}^{\infty}\frac{(\beta^*\alpha)^n}{n!} = e^{-\frac{1}{2}|\alpha|^2-\frac{1}{2}|\beta|^2+(\beta^*\alpha)}$$

$$= \exp\left(-\tfrac{1}{2}|\beta-\alpha|^2\right)\underline{\exp\left(\tfrac{1}{2}(\beta^*\alpha-\beta\alpha^*)\right)} \text{ global phase. } \neq 0.$$

if $|\beta-\alpha|^2$ is large, they are nearly orthogonal.

The completeness relation for the coherent state over the α-plane

$$\int |\alpha\rangle\langle\alpha| \frac{d^2\alpha}{\pi} = 1 \quad \text{where } d^2\alpha = d\mathrm{Re}(\alpha)\, d\mathrm{Im}(\alpha),$$

$$\int |\alpha\rangle\langle\alpha| d^2\alpha = \int e^{-|\alpha|^2} \sum_n \sum_m \frac{\alpha^n (\alpha^*)^m}{\sqrt{n!m!}} |n\rangle\langle m| d^2\alpha.$$

Polar form, $\alpha = re^{i\theta}$, $d\alpha^2 = r\,dr\,d\theta$.

$$\int |\alpha\rangle\langle\alpha| d^2\alpha = \sum_{n,m} \frac{|n\rangle\langle m|}{\sqrt{n!m!}} \int_0^\infty dr\, e^{-r^2} r^{m+n+1} \underbrace{\int_0^{2\pi} d\theta\, e^{i(n-m)\theta}}_{2\pi\delta_{nm}}$$

let $r^2 = y$, $2r\,dr = dy$,

$$\int |\alpha\rangle\langle\alpha| d\alpha^2 = \pi \sum_{n=0}^\infty \frac{|n\rangle\langle n|}{n!} \underbrace{\int_0^\infty dy\, e^{-y} y^n}_{n!} = \pi \sum_{n=0}^\infty |n\rangle\langle n| = \pi.$$

Any state vector $|\psi\rangle$ in the Hilbert space of the quantized single-mode field,

$$|\psi\rangle = \int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha|\psi\rangle$$

The state $|\psi\rangle$ itself is in the coherent state $|\beta\rangle$. $\langle\alpha|\beta\rangle$ roles Dirac delta function

$$|\beta\rangle = \int \frac{d\alpha^2}{\pi} |\alpha\rangle\langle\alpha|\beta\rangle = \int \frac{d^2\alpha}{\pi} |\alpha\rangle \boxed{\exp\left(\tfrac{1}{2}(\beta^*\alpha - \beta\alpha^*)\right) \exp\left(-\tfrac{1}{2}|\beta-\alpha|^2\right)}$$: fts a reproducing kernel.

→ "The coherent states are not linearly independent". ∴ overcomplete.

For arbitrary state $|\psi\rangle$,

$$\langle\alpha|\psi\rangle = \exp\left(-\tfrac{1}{2}|\alpha|^2\right) \sum_{n=0}^\infty \langle n|\psi\rangle \frac{(\alpha^*)^n}{\sqrt{n!}} = \exp\left(-\tfrac{1}{2}|\alpha|^2\right) \sum_{n=0}^\infty \psi_n \frac{(\alpha^*)^n}{\sqrt{n!}} = \exp\left(-\tfrac{1}{2}|\alpha|^2\right) \psi(\alpha^*)$$

where $\psi(z) = \sum_{n=0}^\infty \psi_n \frac{z^n}{\sqrt{n!}}$ ∵ absolutely convergent on complex z-plane.

By Cauchy–Schwarz, $\sum_{n=0}^\infty \left|\psi_n \frac{z^n}{\sqrt{n!}}\right| \le \left(\sum_{n=0}^\infty |\psi_n|^2\right)^{1/2} \left(\sum_{n=0}^\infty \left|\frac{z^{2n}}{n!}\right|\right)^{1/2} = 1 \times e^{|z|^2/2} < \infty$.

$$\left(\left[\sum_{n=0}^\infty a_n b_n\right]^2 \le \left(\sum_{n=0}^\infty a_n^2\right)\left(\sum_{n=0}^\infty b_n^2\right)\right)$$

$|\psi\rangle = \sum_{n=0}^\infty c_n |n\rangle$.

∴ $|\psi\rangle$ is an entire function since $|\langle\psi|\psi\rangle|^2 = \sum_n |\langle n|\psi\rangle|^2 = 1$

$\psi(z)$ constitute the Segal–Bargmann space of entire function.

$|\psi\rangle$ is number state $|n\rangle$, $\psi_n(z) = z^n/\sqrt{n!}$

→ These functions form an orthonormal basis on the Segal-Bargmann space.

$\hat{F}$: An operator given as a function of $\hat{a}, \hat{a}^\dagger$.

$\hat{F} = F(\hat{a}, \hat{a}^\dagger)$

With the Fock state,

$\hat{F} = \sum_{n,m} |m\rangle\langle m|\hat{F}|n\rangle\langle n| = \sum_{n,m} |m\rangle F_{mn} \langle n|$, F_{mn} are the matrix element.

With the coherent state,

$\hat{F} = \frac{1}{\pi^2}\int d\beta^2 \int d\alpha^2 |\beta\rangle\langle\beta|\hat{F}|\alpha\rangle\langle\alpha|$

let $F(\beta^*, \alpha)$

$\langle\beta|\hat{F}|\alpha\rangle = \sum_{m,n} F_{mn} \langle\beta|m\rangle\langle n|\alpha\rangle = \exp(-\frac{1}{2}|\beta|^2 - \frac{1}{2}|\alpha|^2) \boxed{\sum_{mn} F_{mn} \frac{(\beta^*)^m (\alpha)^n}{\sqrt{m!n!}}}$

Thus, $\hat{F} = \frac{1}{\pi^2}\int d\beta^2 \int d\alpha^2 \exp(-\frac{1}{2}|\beta|^2 - \frac{1}{2}|\alpha|^2) F(\beta^*, \alpha) |\beta\rangle\langle\alpha|$.

Suppose Hermitian operator $\hat{F}$ with eigenstates $|\lambda\rangle$

$\hat{F} = \sum_\lambda \lambda |\lambda\rangle\langle\lambda|$

Fock state $|n\rangle$ in Hilbert space

$\updownarrow$

$\langle m|\hat{F}|n\rangle = \sum_\lambda \lambda \langle m|\lambda\rangle\langle\lambda|n\rangle$ entire function $f_n(z) = z^n/n!$ in Segal Bargmann space.

By triangle inequality, $|\langle m|\hat{F}|n\rangle| \le \sum_\lambda \lambda |\langle m|\lambda\rangle\langle\lambda|n\rangle|$.

$\le \sum_\lambda \lambda = \mathrm{Tr}\, F$.

$|F(\beta^*, \alpha)| \le \sum_{mn} |F_{mn}| \left|\frac{(\beta^*)^m \alpha^n}{\sqrt{m!n!}}\right| \le \sum_{mn} (\mathrm{Tr}\,\hat{F}) \frac{|(\beta^*)^m \alpha^n|}{\sqrt{m!n!}} \le (\mathrm{Tr}\,\hat{F}) \sum_m \frac{|\beta^*|^m}{\sqrt{m!}} \sum_n \frac{|\alpha|^n}{\sqrt{n!}}$.

convergent.

Absolutely convergent.

entire = expanding power series with convergence on complex plane.

$\therefore$ function $F(\beta^*, \alpha)$ is a entire function in both β^* and α.

The diagonal elements of an operator $\hat{F}$ in a coherent state basis,

$\rightarrow$ completely determine the operator.

$\langle\alpha|\hat{F}|\alpha\rangle = \exp(-|\alpha|^2) \sum_{m,n} \frac{(\alpha^*)^m (\alpha)^n}{\sqrt{m!n!}} \langle m|\hat{F}|n\rangle$.

$\langle\alpha|\hat{F}|\alpha\rangle e^{\alpha^*\alpha} = \sum_{m,n} \frac{(\alpha^*)^m \alpha^n}{\sqrt{m!n!}} \langle m|\hat{F}|n\rangle$

$f(x,y) = \sum_{mn} C_{mn} x^m y^n$, $\left.\frac{\partial^{m+n} f}{\partial x^m \partial y^n}\right|_{x=0, y=0} = m!n!$

Treating α and α^* as independent variable,

$$\frac{\partial^{m+n}}{\partial\alpha^{*m}\partial\alpha^{n}}\left(\langle\alpha|\hat{F}|\alpha\rangle e^{\alpha^*\alpha}\right) = (m!n!)\frac{1}{\sqrt{m!n!}}\langle m|\hat{F}|n\rangle .$$

$$\frac{1}{\sqrt{m!n!}}\frac{\partial^{m+n}}{\partial\alpha^{*m}\partial\alpha^{n}}\left(\langle\alpha|\hat{F}|\alpha\rangle e^{\alpha^*\alpha}\right) = \langle m|\hat{F}|n\rangle .$$

From diagonal coherent state matrix elements of $\hat{F}$,
we can obtain all the matrix elements of the operator in the number basis.

3.6 Phase-space pictures of coherent states.

- The concept of phase space in Q.M.
  : problematic due to incompatible (not commute) canonical variables $\hat{x}$ and $\hat{p}$.
  → Not well localized as a point in phase space.
- The coherent state minimizes the uncertainty relation for two quadrature operators. and uncertainties of the two quadratures are equal.

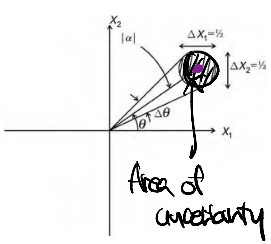
  $\hat{X}_1 = \frac{1}{2}(\hat{a}+\hat{a}^\dagger)$: dimensionless scaled position operator.

  $\hat{X}_2 = \frac{1}{2i}(\hat{a}-\hat{a}^\dagger)$: dimensionless scaled momentum operator.

  Coherent space expectation value.

  $\langle\hat{X}_1\rangle = \frac{1}{2}(\alpha+\alpha^*) = \mathrm{Re}\,\alpha$. → The complex α-plane = phase space.

  $\langle\hat{X}_2\rangle = \frac{1}{2i}(\alpha-\alpha^*) = \mathrm{Im}\,\alpha$.



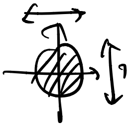
Area of uncertainty

Fig. 3.5. Phase-space portrait of a coherent state of amplitude $|\alpha|$ and phase angle θ. Note the error circle is the same for all coherent states. Note that as $|\alpha|$ increases, the phase uncertainty $\Delta\theta$ decreases, as would be expected in the "classical limit".

- $|\alpha\rangle$ with $\alpha = |\alpha|e^{i\theta}$.
- The fluctuation in all direction of phase.
- center of circle
  → $|\alpha| = \langle\hat{n}\rangle^{1/2}$

$\Delta\theta$: diminishing for increasing $|\alpha|$.

The fluctuations in X_1, X_2 are dependent of α. (identical to the vacuum).



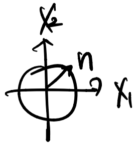
$\Delta X_1 = \frac{1}{2}$

$\Delta X_2 = \frac{1}{2}$.

for a vacuum $|\alpha| = 0$

$\rightarrow$ phase uncertainty $\uparrow$ (ASAP) : $\Delta\theta = 2\pi$

in vacuum state.



A number state $|n\rangle$ in phase space.

radius : n.

the uncertainty in n : 0, the uncertainty in θ : 2π.

Graphical way : good for visualization on distribution of noise.

- Phase-space diagram.

to illustrate the time evolution of quantum state for a non-interacting field.

For non-interacting field,

$|\alpha\rangle \mapsto |\alpha e^{-i\omega t}\rangle$: pictured as a "clockwise rotation of the error circle".

$$\langle \alpha e^{-i\omega t} | \hat{X}_1 | \alpha e^{-i\omega t} \rangle = \frac{1}{2}\left(\alpha e^{-i\omega t} + (\alpha e^{-i\omega t})^*\right) = \alpha \cos\omega t.$$

$$\langle \alpha e^{-i\omega t} | \hat{X}_2 | \alpha e^{-i\omega t} \rangle = \frac{1}{2i}\left(\alpha e^{-i\omega t} - (\alpha e^{-i\omega t})^*\right) = -\alpha \sin\omega t.$$

In the Schrödinger picture, the E-field operator is

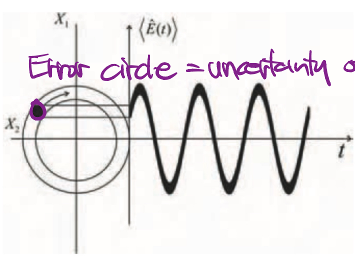
$$\hat{E}_x = 2\mathcal{E}_0 \sin(kz)\hat{X}_1.$$ projection on $\langle \hat{X}_1 \rangle$.

just factor

$$\langle \alpha e^{-i\omega t} | \hat{E}_x | \alpha e^{-i\omega t} \rangle = \boxed{2\mathcal{E}_0 \sin(kz)}\,\underline{\alpha\cos(\omega t)}.$$

the evolution of the E-field - projection of $\langle \hat{X}_1 \rangle$ points with error circle as a function of time.

Fig. 3.8. The error circle of a coherent state (the black dot) revolves about the origin of phase space at the oscillator angular frequency ω and the expectation value of the electric field is the projection onto an axis parallel with X_1.



Error circle = uncertainty of E-field. (quantum flesh, classical bone).

- The greater the excitation of the field $|\alpha\rangle$, the more classical the field appears (relatively classical).
  because the fluctuations are independent of α.

- The coherent state is the most classical of the quantum states.

- A number state is a very non-classical state. (representative points from its phase-space portrait).
  (field expectation is zero). Number state is not defined about phase information.
  → if selecting which phase, state seems to same.

Then, is it possible to be less than fluctuation of coherent state? → not oscillating on specific direction.
→ $\langle n|\hat{E}|n\rangle = 0$.

→ squeezed state. (Ch. 7).

3.7 Density operators and Phase-space probability distribution

the mixture (ensemble) of quantum state $|\psi_1\rangle, |\psi_2\rangle \cdots$

$\hat{\rho} = \sum_i P_i |\psi_i\rangle\langle\psi_i|$, P_i: the probability of finding the system in the i-th member of the ensemble.

$\mathrm{Tr}(\hat{\rho}) = \sum_i P_i = 1$.

The expectation of the operator,

$\langle\hat{O}\rangle = \mathrm{Tr}(\hat{O}\hat{\rho}) = \sum_i P_i \langle\psi_i|\hat{O}|\psi_i\rangle$

the density operator in terms of number state.

$\hat{\rho} = \sum_{n,m} |m\rangle \rho_{mn} \langle n|$ where $\rho_{mn} = \langle m|\hat{\rho}|n\rangle$.

The diagonal elements $P_n = \rho_{nn}$ = the probability of finding "n" photons.

For coherent state,

$$\hat{\rho} = \iint \langle\alpha'|\hat{\rho}|\alpha''\rangle |\alpha'\rangle\langle\alpha''| \frac{d^2\alpha' d^2\alpha''}{\pi^2}$$

Another way.

$\hat{\rho} = \int |\alpha\rangle P(\alpha) \langle\alpha| d^2\alpha$, $P(\alpha)$: weight function (Glauber Sudarshan P function).

→ phase-space distribution of statistical physics.

→ $P(\alpha)$ is real since $\hat{\rho}$ is Hermitian.

$$\mathrm{Tr}(\hat{\rho}) = \mathrm{Tr}\int |\alpha\rangle P(\alpha)\langle\alpha|\, d^2\alpha = \int \sum_n \langle n|\alpha\rangle P(\alpha)\langle\alpha|n\rangle d^2\alpha$$

$$= \int P(\alpha)\sum_n \langle\alpha|n\rangle\langle n|\alpha\rangle d^2\alpha = \int P(\alpha)\langle\alpha|\alpha\rangle d^2\alpha$$

$$= \int P(\alpha)\, d^2\alpha = 1.$$ (phase-space probability distribution).

For quantum state of the field, $P(\alpha)$ is quite unlike true probability distribution ($P(\alpha)\geq 0$)

$\rightarrow$ $P(\alpha)$ is negativity or highly singular. : nonclassical.

Define a "nonclassical state" : negative $P(\alpha)$ in phase space (α-plane) singular than a delta function.

positive or non-singular $P(\alpha)$ $\rightarrow$ classical light.

Aren't all states of light quantum mechanical?

Yes! All states of light are quantum mechanical

Coherent state (quasi-classical)

: close to classical oscillating coherent field. (P function : delta function).

Quadrature and amplitude (or number squeezing)

: P function $\rightarrow$ negative or highly singular. (nonclassical effect).

Calculate $P(\alpha)$, using coherent state $|u\rangle$ and $|-u\rangle$.

$$\langle -u|\hat{\rho}|u\rangle = \int \langle -u|\alpha\rangle P(\alpha)\langle\alpha|u\rangle d^2\alpha = \int P(\alpha)\left[e^{-\frac{1}{2}|\alpha|^2}e^{-\frac{1}{2}|u|^2}e^{-u^*\alpha}\right]\left[e^{-\frac{1}{2}|\alpha|^2}e^{-\frac{1}{2}|u|^2}e^{\alpha^* u}\right]d\alpha^2.$$

$$= e^{-|u|^2}\int P(\alpha)e^{-|\alpha|^2}e^{-u^*\alpha+\alpha^* u}\,d\alpha^2$$

let $\alpha = x+iy$, $u = x'+iy'$

$$-u^*\alpha+\alpha^* u = -(x'-iy')(x+iy)+(x-iy)(x'+iy') = 2i(xy'-x'y)$$

Define the Fourier transform in the complex plane. Fourier kernel.

$$g(u) = \int f(\alpha)\boxed{e^{\alpha^* u-\alpha u^*}}d^2\alpha \quad , \quad f(\alpha) = \frac{1}{\pi^2}\int g(u)\boxed{e^{u^*\alpha-u\alpha^*}}d^2u.$$

$$\langle -u|\hat{\rho}|u\rangle e^{-|u|^2} = \int P(\alpha)e^{-|\alpha|^2}e^{-u^*\alpha+\alpha^* u}\,d\alpha^2.$$

$\rightarrow$ $g(u) = e^{|u|^2}\langle -u|\hat{\rho}|u\rangle$ and $f(\alpha) = P(\alpha)e^{-|\alpha|^2}$.

$$P(\alpha) = f(\alpha)e^{|\alpha|^2} = \frac{e^{|\alpha|^2}}{\pi^2}\int e^{|u|^2}\langle -u|\hat{\rho}|u\rangle e^{u^*\alpha - u\alpha^*} d^2u.$$

In regard to the convergence of the integral since $e^{|u|^2} \to \infty$ as $|u| \to \infty$.

For the pure coherent state $|\beta\rangle$, where $\hat{\rho} = |\beta\rangle\langle\beta|$.

$$\langle -u|\hat{\rho}|u\rangle = \langle -u|\beta\rangle\langle\beta|u\rangle = e^{-|\beta|^2}e^{-|u|^2}e^{-u^*\beta + \beta^* u}.$$

$$P(\alpha) = e^{|\alpha|^2}e^{-|\beta|^2}\frac{1}{\pi^2}\int e^{u^*(\alpha-\beta) - u(\alpha^*-\beta^*)} d^2u.$$ → Fourier integral (Dirac-delta function)

$$\delta^2(\alpha-\beta) = \delta(Re[\alpha]-Re[\beta])\,\delta(Im[\alpha]-Im[\beta]) = \frac{1}{\pi^2}\int e^{u^*(\alpha-\beta)-u(\alpha^*-\beta^*)} d^2u.$$

$\longrightarrow P(\alpha) = \delta^2(\alpha-\beta)$ same as the distribution for classical H.O.

Coherent state : classical-like state. → number state cannot be described classically.

$$\langle -u|\hat{\rho}|u\rangle = \langle -u|n\rangle\langle n|u\rangle = \frac{(-u^*)^n}{\sqrt{n!}}e^{-|u|^2/2}\,\frac{u^n}{\sqrt{n!}}e^{-|u|^2/2} = e^{-|u|^2}\frac{(-u^*u)^n}{n!}.$$

$$P(\alpha) = \frac{e^{|\alpha|^2}}{n!\pi^2}\int e^{|u|^2}(-u^*u)^n e^{u^*\alpha - u\alpha^*} d^2u.$$: integral does not exists in terms of ordinary function.

Formally, $$P(\alpha) = \frac{e^{|\alpha|^2}}{n!}\frac{\partial^{2n}}{\partial\alpha^n\partial\alpha^{*n}}\frac{1}{\pi^2}\int e^{u^*\alpha - u\alpha^*} d^2u.$$

$$= \frac{e^{|\alpha|^2}}{n!}\underline{\frac{\partial^{2n}}{\partial\alpha^n\partial\alpha^{*n}}\delta^{(2)}(\alpha)}$$ tempered distribution.

The derivative of the delta function is more singular than a delta function.

$$\int F(\alpha,\alpha^*)\frac{\partial^{2n}}{\partial\alpha^n\partial\alpha^{*n}}\delta^{(2)}(\alpha)\,d^2\alpha = \left[\frac{\partial^{2n}F(\alpha,\alpha^*)}{\partial\alpha^n\partial\alpha^{*n}}\right]_{\alpha^*=0,\ \alpha=0}.$$

Introducing the <u>optical equivalence theorem</u> of Sudarshan.

mean of observable in Q.O.

$\hat{a}^\dagger\hat{a}$ (not $\hat{a}\hat{a}^\dagger$)

$G^{(N)}(\hat{a},\hat{a}^\dagger)$: "<u>normally ordered</u>" function of the operator $\hat{a},\hat{a}^\dagger$.

$$G^{(N)}(\hat{a},\hat{a}^\dagger) = \sum_{n,m} C_{nm}(\hat{a}^\dagger)^n\hat{a}^m$$

The average of this function,

$$\langle G^{(N)}(\hat{a},\hat{a}^\dagger)\rangle = \mathrm{Tr}[G^{(N)}(\hat{a},\hat{a}^\dagger)\hat{\rho}]$$ where $\mathrm{Tr}(\hat{\rho}) = \mathrm{Tr}\int P(\alpha)|\alpha\rangle\langle\alpha|d\alpha$.

$$= \mathrm{Tr}\int P(\alpha)\sum_{n,m} C_{nm}(\hat{a}^\dagger)^n\hat{a}^m|\alpha\rangle\langle\alpha|d^2\alpha.$$

$$= \int P(\alpha)\sum_{n,m} C_{n,m}\langle\alpha|(\hat{a}^\dagger)^n\hat{a}^m|\alpha\rangle d^2\alpha$$

$$= \int P(\alpha)\sum_{n,m} C_{n,m}(\alpha^*)^n\alpha^m d^2\alpha = \underline{\int P(\alpha)G^{(N)}(\alpha^*,\alpha)d\alpha^2}.$$

optical equivalent theorem.

The expectation value of a normally ordered operator.

= P function weighted average of the function ($\hat{a}\to\alpha$, $\hat{a}^\dagger\to\alpha^*$)

$:\hat{O}(\hat{a},\hat{a}^\dagger) = O^{(N)}(\hat{a},\hat{a}^\dagger)$ where the commutation relations are to be disregarded.

ex) number operator $\hat{n}=\hat{a}^\dagger\hat{a}$ is normally ordered.

$$\langle\hat{n}\rangle = \langle\hat{a}^\dagger\hat{a}\rangle = \int P(\alpha)|\alpha|^2 d^2\alpha$$

ex) $\hat{n}^2 = \hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}$ are not normally ordered.

$$:\hat{n}^2: = (\hat{a}^\dagger)^2\hat{a}^2$$

$$\langle:\hat{n}^2:\rangle = \langle(\hat{a}^\dagger)^2\hat{a}^2\rangle = \int P(\alpha)|\alpha|^4 d\alpha^2$$

Consider operator $\hat{B}$ in the Hilbert space of the single mode quantized field.

$$\hat{B} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|m\rangle\langle m|\hat{B}|n\rangle\langle n| = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|m\rangle B_{mn}\langle n|.$$

for coherent state,

$$\hat{B} = \iint|\alpha'\rangle\langle\alpha'|\hat{B}|\alpha\rangle\langle\alpha|\frac{d^2\alpha' d^2\alpha}{\pi^2} = \iint|\alpha'\rangle B(\alpha'^*,\alpha)\langle\alpha|\frac{d^2\alpha' d^2\alpha}{\pi^2}.$$

where $B(\alpha'^*,\alpha) = \langle\alpha'|\hat{B}|\alpha\rangle = \exp[-(|\alpha'|^2+|\alpha|^2)/2]\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}B_{mn}\frac{(\alpha'^*)^m\alpha^n}{\sqrt{m!n!}}.$

or introducing "diagonal" coherent state.

$$\hat{B} = \int |\alpha\rangle B_p(\alpha, \alpha^*) \langle\alpha| \frac{d^2\alpha}{\pi}$$ where $B_p(\alpha,\alpha^*)$ is P-representative of $\hat{B}$.

Average $\hat{B}$

$$\langle \hat{B} \rangle = \mathrm{Tr}(\hat{B}\hat{\rho}) = \sum_n \langle n| \int B_p(\alpha,\alpha^*) |\alpha\rangle\langle\alpha|\hat{\rho}|n\rangle \frac{d^2\alpha}{\pi^2} = \int B_p(\alpha,\alpha^*) \langle\alpha|\hat{\rho}|n\rangle \frac{d^2\alpha}{\pi^2}.$$

$\langle\alpha|\hat{\rho}|\alpha\rangle$: the role of phase-space distribution.

called the Q, or Husimi, function. $Q = \frac{1}{\pi}\langle\alpha|\hat{\rho}|\alpha\rangle$.

For $\hat{B}=\hat{I}$, normalization condition.

$$\int Q(\alpha) d^2\alpha = 1.$$

Unlike the P function, Q function is positive for all quantum state.

Define Q-representation of the operator $\hat{B}$ as the expectation value.
→ respect to the coherent state.

$$B_Q(\alpha,\alpha^*) \equiv \langle\alpha|\hat{B}|\alpha\rangle = e^{-|\alpha|^2} \sum_n \sum_m \frac{B_{nm}}{(n!\,m!)^{1/2}} (\alpha^*)^n \alpha^m .$$ where $B_{nm} = \langle n|\hat{B}|m\rangle$

In the P-representation,

$$\langle\hat{B}\rangle = \mathrm{Tr}(\hat{B}\hat{\rho}) = \mathrm{Tr}\int \hat{B}\, P(\alpha) |\alpha\rangle\langle\alpha| d^2\alpha$$

$$= \int P(\alpha) \langle\alpha|\hat{B}|\alpha\rangle d^2\alpha = \int P(\alpha) B_Q(\alpha,\alpha^*) d^2\alpha .$$

→ If we use the P-representation of $\hat{\rho}$ → we should need Q-representation of $\hat{B}$.

Q function : probability distribution with positivity.

P function : quasi probability distribution.

Wigner function : quasi-probability distribution over phase space.

for arbitrary density operator $\hat{\rho}$,

$$F_W(q,p) \equiv \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} \langle q+\tfrac{1}{2}x|\hat{\rho}|q-\tfrac{1}{2}x\rangle e^{-ipx/\hbar}dx$$

where $|q\pm\frac{1}{2}x\rangle$: eigenket of position operator.

for pure state $\hat{\rho} = |\psi\rangle\langle\psi|$,

$$F_W(q,p) = \frac{1}{2\pi\hbar}\psi^*(q-\tfrac{1}{2}x)\psi(q+\tfrac{1}{2}x)e^{-ipx/\hbar}dx.$$

where $\langle q+\frac{1}{2}x|\psi\rangle = \psi(q+\frac{1}{2}x)$.

Integrating over the momentum,

$$\int_{-\infty}^{\infty}F_W(q,p)dp = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\psi^*(q-\tfrac{1}{2}x)\psi(q+\tfrac{1}{2}x)\int_{-\infty}^{\infty}e^{-ipx/\hbar}dp\,dx.$$

$$= \int_{-\infty}^{\infty}\psi^*(q-\tfrac{1}{2}x)\psi(q+\tfrac{1}{2}x)\delta(x)dx. = \boxed{|\psi(q)|^2}$$

probability density for position variable "q".

Likewise, $\int_{-\infty}^{\infty}F_W(q,p)dq = |\varphi(p)|^2$.

in the momentum space

$\psi(q) \overset{FT}{\Longleftrightarrow} \varphi(p)$

probability density for momentum variable p

But $W(q,p)$ itself is not probability distribution.

$\longrightarrow$ for nonclassical state, it takes negative.

Wigner 함수 쓰는 동안 operator는 Weyl order로 저장 불는다.

Wigner function can be used to calculate average.

the operator $\hat{q},\hat{p}$ $\rightarrow$ averaged must be Weyl (symmetrically ordered).

ex) $qp \rightarrow \frac{1}{2}(\hat{q}\hat{p}+\hat{p}\hat{q})$

$$\langle\hat{q}\hat{p}+\hat{p}\hat{q}\rangle = \int(qp+pq)W(q,p)dpdq.$$

If $\{G(\hat{q},\hat{p})\}_W$: Weyl-ordered function,

$$\langle\{G(\hat{q},\hat{p})\}_W\rangle = \int\{G(\hat{q},\hat{p})\}_W F_W(q,p)dq\,dp.$$: phase-space average.

for section 3.d, $F_W(q,p) \equiv \frac{1}{\pi\hbar}\langle q+s|\hat{\rho}|q-s\rangle e^{-2ips/\hbar}ds$. from $x=2s$

3.8 The photon number parity operator and the Wigner function

Expectation value of the displaced photon number parity operator.

$\longrightarrow$ expressing Wigner function.

Define $\hat{\Pi}$

$$\hat{\Pi} = (-1)^{\hat{a}^\dagger \hat{a}} = \sum_{n=0}^{\infty} (-1)^n |n\rangle\langle n|$$: Dichotomic observable. (Hermitian operator).

- eigenvalue $+1$: even photon number,
  -1 : odd photon number.
- unusual quantum observable as it has no classical analogue.

$$\hat{\Pi} = \int_{-\infty}^{\infty} dx\, |-x\rangle\langle x|$$: integral form.

$$\hat{x}|x\rangle = x|x\rangle \quad -\infty < x < \infty$$

completeness relation

$$\hat{I} = \int_{-\infty}^{\infty} dx\, |x\rangle\langle x|$$

respect to number state $|n\rangle$.

$$\langle n|\hat{\Pi}|n\rangle = (-1)^n = \int_{-\infty}^{\infty} dx \langle n|-x\rangle \langle x|n\rangle.$$ well-known energy eigenstate of Q.H.O with unit mass

$$\langle x|n\rangle = \psi_n(x) = \frac{1}{\sqrt{2^n n!}} \left(\frac{\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{y^2}{2}} H_n(y), \quad y = x\left(\frac{\omega}{\hbar}\right)^{1/2}.$$

Hermite polynomial, $H_n(-y) = (-1)^n H_n(y) \longrightarrow \langle -x|n\rangle = (-1)^n \langle x|n\rangle$

$$\langle n|\hat{\Pi}|n\rangle = (-1)^n = \int_{-\infty}^{\infty} dx (-1)^n \langle n|x\rangle\langle x|n\rangle = (-1)^n \int_{-\infty}^{\infty} dx \langle n|x\rangle\langle x|n\rangle.$$

$\langle n|x\rangle = \langle x|n\rangle$ for real function.

and normalization condition $\int_{-\infty}^{\infty} dx \langle n|x\rangle\langle x|n\rangle = 1.$

The displacement operator $\hat{D}(\alpha) = \exp(\alpha \hat{a}^\dagger - \alpha^* \hat{a})$

set $\alpha = (2\hbar\omega)^{-1/2}(\omega q + ip)$, $\alpha^* = (2\hbar\omega)^{-1/2}(\omega q - ip)$

where q, p are c-number phase-space variable (not $\hat{q}, \hat{p}$)

and $\hat{a} = (2\hbar\omega)^{-1/2}(\omega\hat{q}+i\hat{p})$, $\hat{a}^{+} = (2\hbar\omega)^{-1/2}(\omega\hat{q}-i\hat{p})$

$$\hat{D}(\alpha) = \exp\left(\frac{1}{2\hbar\omega}\left[(\omega q+ip)(\omega\hat{q}-i\hat{p}) - (\omega q - ip)(\omega\hat{q}+i\hat{p})\right]\right)$$

$$= \exp\left[\frac{1}{2\hbar\omega}\left(-2i\omega q\hat{p} + 2ip\omega\hat{q}\right)\right] = \exp\left(\frac{i}{\hbar}(p\hat{q} - q\hat{p})\right).$$

using disentangling theorem, $e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}e^{-\frac{1}{2}[\hat{A},\hat{B}]}$.

$[p\hat{q}, -q\hat{p}] = -pqi$.

$$\hat{D}(\alpha) = e^{-\frac{i}{2\hbar}pq}\, e^{\frac{i}{\hbar}p\hat{q}}\, \boxed{e^{-\frac{i}{\hbar}q\hat{p}}} \longrightarrow$$ translation operator $\hat{T}(q) = \exp(-iq\hat{p}/\hbar)$

: $\hat{p}$ being generator of translation.

$\hat{T}(q)|x\rangle = |x+q\rangle$.

$$\hat{D}(\alpha)|-s\rangle = e^{-\frac{i}{2\hbar}pq}\, e^{\frac{i}{\hbar}p\hat{q}}\, \hat{T}(q)|-s\rangle = e^{-\frac{i}{2\hbar}pq}\, e^{\frac{i}{\hbar}p(q-s)}|q-s\rangle.$$

$$\hat{D}(\alpha)|+s\rangle = e^{-\frac{i}{2\hbar}pq}\, e^{\frac{i}{\hbar}p(q+s)}|q+s\rangle.$$

$$\langle q+s|\hat{\rho}|q-s\rangle = \underline{\langle s|\hat{D}^{+}(\alpha)\hat{\rho}\hat{D}(\alpha)}|-s\rangle.$$

Wigner function, $F_W(q,p) = \frac{1}{\pi\hbar}\int_{-\infty}^{\infty}\langle s|\hat{D}^{+}(\alpha)\hat{\rho}\hat{D}(\alpha)|-s\rangle ds$.

taking general density operator, $\hat{\rho} = \sum_{ij} C_{ij}|\psi_i\rangle\langle\psi_j|$.

$$F_W(q,p) = \frac{1}{\pi\hbar}\sum_{ij}C_{ij}\int_{-\infty}^{\infty}\langle s|\hat{D}^{+}(\alpha)|\psi_i\rangle\langle\psi_j|\hat{D}(\alpha)|-s\rangle ds$$

$$= \frac{1}{\pi\hbar}\sum_{ij}C_{ij}\int_{-\infty}^{\infty}\langle\psi_j|\hat{D}(\alpha)|-s\rangle\langle s|\hat{D}^{+}(\alpha)|\psi_i\rangle ds.$$

$\int_{-\infty}^{\infty}|-s\rangle\langle s|ds = \sum_{n=0}^{\infty}(-1)^n|n\rangle\langle n|$.

Integrating over s,

$$F_W(q,p) = \frac{1}{\pi\hbar}\sum_{ij}C_{ij}\sum_{n=0}^{\infty}\langle\psi_j|\hat{D}(\alpha)\underbrace{(-1)^n|n\rangle\langle n|}_{\Pi}\hat{D}^{+}(\alpha)|\psi_i\rangle ds$$

$$= \frac{1}{\pi\hbar}\langle\hat{D}(\alpha)\hat{\Pi}\hat{D}^{+}(\alpha)\rangle.$$ where $\hat{D}^{+}(\alpha) = \hat{D}(-\alpha)$.

$\therefore$ The Wigner function $\xLeftarrow[\text{expressed as}]{}$ the expectation value of displaced parity operator.

In the normalization condition,

$$\mathrm{Re}(\alpha) = \frac{1}{2}(\alpha + \alpha^*) = (2\hbar\omega)^{-1/2}\omega q. \quad \Rightarrow \quad dq = (2\hbar\omega)^{1/2}\frac{1}{\omega}\, d\mathrm{Re}(\alpha).$$

$$\mathrm{Im}(\alpha) = \frac{1}{2}(\alpha - \alpha^*) = (2\hbar\omega)^{-1/2} p \quad \Rightarrow \quad dp = (2\hbar\omega)^{1/2}\, d\mathrm{Im}(\alpha)$$

$$dq\,dp = 2\hbar\, d\mathrm{Re}(\alpha)\, d\mathrm{Im}(\alpha), = 2\hbar\, d^2\alpha.$$

$$\iint F_w(q,p)\, dq\, dp = \int W(\alpha)\, d^2\alpha. \quad \text{defined} \quad W(\alpha) \equiv \frac{2}{\pi}\langle \hat{D}(\alpha)\hat{\Pi}\hat{D}^\dagger(\alpha)\rangle.$$

"quantum optical" Wigner function.

Parity operator $\hat{\Pi} = (-1)^{\hat{a}^\dagger\hat{a}} = \exp(i\pi\hat{a}^\dagger\hat{a})$

$$\hat{D}(\alpha)\hat{\Pi}\hat{D}^\dagger(\alpha) = \hat{D}(\alpha)e^{i\pi\hat{a}^\dagger\hat{a}}\hat{D}(-\alpha)e^{-i\pi\hat{a}^\dagger\hat{a}}e^{i\pi\hat{a}^\dagger\hat{a}}.$$

the parity operator as a transformation operator = performs reflection.

$$e^{i\pi(\hat{a}^\dagger\hat{a})}\hat{D}(-\alpha)e^{-i\pi(\hat{a}^\dagger\hat{a})} = \hat{D}(\alpha).$$

$$\hat{D}(\alpha)\hat{\Pi}\hat{D}^\dagger(\alpha) = \hat{D}(\alpha)\hat{D}(\alpha)e^{i\pi\hat{a}^\dagger\hat{a}} = \hat{D}(2\alpha)\hat{\Pi}.$$

$$W(\alpha) = \frac{2}{\pi}\langle\hat{D}(2\alpha)\hat{\Pi}\rangle$$

for pure state of the form $|\psi\rangle = \sum_{n=0}^{\infty} c_n|n\rangle.$

$$W(\alpha) = \frac{2}{\pi}\sum_{n'=0}^{\infty}\sum_{n=0}^{\infty} c_{n'}^* c_n (-1)^n \langle n'|\hat{D}(2\alpha)|n\rangle.$$

3.8 Characteristic function.

A classical RV x.

$\rho(x)$: classical probability density with variable X.

$\rightarrow \rho(x) \geq 0$, $\int \rho(x) dx = 1$.

for nth moment,

$\langle x^n \rangle = \int dx \, x^n \rho(x)$. 특정 거듭의 평균값.

특정하는 또는 momentum 성분

All $\langle x^n \rangle$ is well-known $\Rightarrow$ $\rho(x)$: completely specified.

$$\boxed{C(k)} = \langle e^{ikx} \rangle = \int dx \, e^{ikx} \rho(x) = \int dx \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} x^n \rho(x) = \boxed{\sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \langle x^n \rangle}$$

$\Rightarrow$ the probability density is just the FT of the characteristic function.

$\rho(x) = \frac{1}{2\pi} \int dk \, e^{-ikx} C(k)$: All $\langle x^n \rangle$ well-known $\Rightarrow$ $\rho(x)$ is known.

$$\langle x^n \rangle = \frac{1}{i^n} \left. \frac{d^n C(k)}{dk^n} \right|_{k=0}.$$

$$\left. \frac{d^n C(k)}{dk^n} \right|_{k=0} = \left. \langle (ix)^n e^{ikx} \rangle \right|_{k=0} = \langle (ix)^n \rangle = i^n \langle x^n \rangle$$

Introducing quantum-mechanical characteristic function,

three function.

$$C_W(\lambda) = \mathrm{Tr}(\hat{\rho} e^{\lambda \hat{a}^\dagger - \lambda^* \hat{a}}) = \mathrm{Tr}(\hat{\rho} \hat{D}(\lambda)) \quad : \text{Wigner}.$$

$$C_N(\lambda) = \mathrm{Tr}(\hat{\rho} e^{\lambda \hat{a}^\dagger} e^{-\lambda^* \hat{a}}) \quad : \text{normally ordered}.$$

$$C_A(\lambda) = \mathrm{Tr}(\hat{\rho} e^{-\lambda^* \hat{a}} e^{\lambda \hat{a}^\dagger}) \quad : \text{antinormally ordered}.$$

related disentanglement theorem.

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{-\frac{1}{2}[\hat{A},\hat{B}]} = e^{\hat{B}} e^{\hat{A}} e^{\frac{1}{2}[\hat{A},\hat{B}]}$$

By disentanglement theorem,

$$e^{\lambda \hat{a}^\dagger - \lambda^* \hat{a}} = e^{\lambda \hat{a}^\dagger} e^{-\lambda^* \hat{a}} e^{-\frac{1}{2}[\lambda \hat{a}^\dagger, -\lambda^* \hat{a}]} = e^{\lambda \hat{a}^\dagger} e^{-\lambda^* \hat{a}} e^{-\frac{1}{2}|\lambda|^2}$$

$$e^{\lambda \hat{a}^\dagger - \lambda^* \hat{a}} = e^{-\lambda^* \hat{a}} e^{\lambda \hat{a}^\dagger} e^{\frac{1}{2}|\lambda|^2}$$

$$C_W(\lambda) = C_N(\lambda) e^{-\frac{1}{2}|\lambda|^2} = C_A(\lambda) e^{\frac{1}{2}|\lambda|^2}.$$

Furthermore, another way.

$$\langle (\hat{a}^\dagger)^m \hat{a}^n \rangle = \mathrm{Tr}[\hat{\rho} (\hat{a}^\dagger)^m \hat{a}^n] = \left. \frac{\partial^{(m+n)}}{\partial \lambda^m \partial (-\lambda^*)^n} C_N(\lambda) \right|_{\lambda=0}$$

$$\langle \hat{a}^m (\hat{a}^\dagger)^n \rangle = \mathrm{Tr}[\hat{\rho} \hat{a}^m (\hat{a}^\dagger)^n] = \left. \frac{\partial^{(m+n)}}{\partial \lambda^n \partial (-\lambda^*)^m} C_A(\lambda) \right|_{\lambda=0}.$$

$$\langle \{(\hat{a}^\dagger)^m \hat{a}^n\}_w \rangle = \mathrm{Tr}[\hat{\rho}\{(\hat{a}^\dagger)^m \hat{a}^n\}_w] = \frac{\partial^{(m+n)}}{\partial\lambda^m \partial(-\lambda^*)^n} C_w(\lambda)\Big|_{\lambda=0}.$$

Introducing s-parameterized function of Cahill and Glauber.

$$C(\lambda, s) = \mathrm{Tr}[\hat{\rho}\exp(\lambda\hat{a}^\dagger - \lambda^*\hat{a} + s|\lambda|^2/2)]$$

such that $C(\lambda,0) = C_w(\lambda)$, $C(\lambda,1) = C_N(\lambda)$, $C(\lambda,-1) = C_A(\lambda)$.

- The connection btw. these characteristic function
- The quasi-probability distribution.

for example, antinormally ordered characteristic function.

$$C_A(\lambda) = \mathrm{Tr}[\hat{\rho}e^{-\lambda^*\hat{a}}e^{\lambda\hat{a}^\dagger}] = \mathrm{Tr}[e^{\lambda\hat{a}^\dagger}\hat{\rho}e^{-\lambda^*\hat{a}}] = \frac{1}{\pi}\int d^2\alpha \langle\alpha|e^{\lambda\hat{a}^\dagger}\hat{\rho}e^{-\lambda^*\hat{a}}|\alpha\rangle.$$

$$= \int d^2\alpha\, Q(\alpha)e^{\lambda\alpha^*-\lambda^*\alpha} \quad \text{where } \frac{1}{\pi}\langle\alpha|\hat{\rho}|\alpha\rangle = Q(\alpha).$$

2D FT of Q function.

Normally ordered case,

$$C_N(\lambda) = \mathrm{Tr}(\hat{\rho}e^{\lambda\hat{a}^\dagger}e^{-\lambda\hat{a}}) = \int P(\alpha)\langle\alpha|e^{\lambda\hat{a}^\dagger}e^{-\lambda^*\hat{a}}|\alpha\rangle = \int P(\alpha)\,e^{\lambda\alpha^*-\lambda^*\alpha}\,d^2\alpha.$$

2D FT of P function

Inverse Fourier transform,

$$P(\alpha) = \frac{1}{\pi^2}\int e^{\lambda\alpha^*-\lambda^*\alpha}C_N(\lambda)\,d^2\lambda.$$

Wigner function = FT of Weyl ordered characteristic function.

$$W(\alpha) = \frac{1}{\pi^2}\int \exp(\lambda^*\alpha-\lambda\alpha^*)C_w(\lambda)\,d^2\alpha = \frac{1}{\pi^2}\int \exp(\lambda^*\alpha-\lambda\alpha^*)C_N(\lambda)e^{-|\lambda|^2/2}\,d^2\alpha.$$

* An application of the characteristic function.

P function $\rightarrow$ thermal, chaotic, state of the field.

recall thermal state (mixed state)

$$\hat{\rho}_{th} = \frac{1}{1+\bar{n}}\sum_{n=0}^{\infty}\left(\frac{\bar{n}}{1+\bar{n}}\right)^n |n\rangle\langle n|.$$

$$Q(\alpha) = \langle \alpha | \hat{\rho}_{th} | \alpha \rangle / \pi .$$

$$= \frac{1}{\pi} e^{-|\alpha|^2} \sum_{n,m} \langle m | \hat{\rho}_{th} | n \rangle \frac{(\alpha^*)^m \alpha^n}{(m!\,n!)^{1/2}}$$

$$= \frac{e^{-|\alpha|^2}}{\pi(1+\bar{n})} \sum_n \left(\frac{\bar{n}}{1+\bar{n}}\right)^n \frac{(\alpha^*\alpha)^n}{n!} = \frac{e^{-|\alpha|^2}}{\pi(1+\bar{n})} \exp\left(\left(\frac{\bar{n}}{1+\bar{n}}\right)|\alpha|^2 \right).$$

$$= \frac{1}{\pi(1+\bar{n})} \exp\left(-\frac{|\alpha|^2}{1+\bar{n}}\right)$$

$$C_A(\lambda) = \frac{1}{\pi(1+\bar{n})} \int d^2\alpha \exp\left(-\frac{|\alpha|^2}{1+\bar{n}}\right) e^{\lambda\alpha^* - \lambda^*\alpha}.$$

let $\alpha = (q+ip)/\sqrt{2}$ $\quad \lambda = (x+iy)/\sqrt{2}$ $\quad$ where $d^2\alpha = dq\,dp/2$.

$$C_A(x,y) = \frac{1}{2\pi(1+\bar{n})} \int dq\,dp \exp\left[-\frac{q^2+p^2}{2(1+\bar{n})}\right] \exp(i(yq - xp))$$

using gaussian integral $\qquad \int dq \exp\left(-\frac{q^2}{1+\bar{n}}\right) \exp(iyq)$

$$\int e^{-as^2} e^{\pm\beta s} ds = \sqrt{\frac{\pi}{a}}\, e^{\beta^2/4a}. \qquad a = \frac{1}{1+\bar{n}},\ \beta = iy.$$

$$C_A(x,y) = \left(\frac{1}{2\pi(1+\bar{n})}\right)\left(\frac{\pi}{\frac{1}{1+\bar{n}}}\right) \exp(-(1+\bar{n})|\lambda|^2) \qquad \begin{matrix} e^{y^2/2\,(1+\bar{n})} \\ e^{x^2/2\,(1+\bar{n})} \end{matrix}$$

$$= \exp(-(1+\bar{n})|\lambda|^2)$$

with $C_N(\lambda) = C_A(\lambda) \exp(|\lambda|^2)$

$$P(\alpha) = \frac{1}{\pi^2} \int \exp(-\bar{n}|\lambda|^2)\, e^{\lambda^*\alpha - \lambda\alpha^*} d^2\lambda . = \frac{1}{\pi^2}\left(\sqrt{\frac{\pi}{\bar{n}}}\right)^2 \exp\left(-2\frac{2|\alpha|^2}{4\bar{n}}\right) = \frac{1}{\pi\bar{n}} \exp\left(-\frac{|\alpha|^2}{\bar{n}}\right)$$

→ Gaussian as a true probability distribution.

∘ Examine Q function and Wigner function for most desired state. $\rho = |\beta\rangle\langle\beta|$

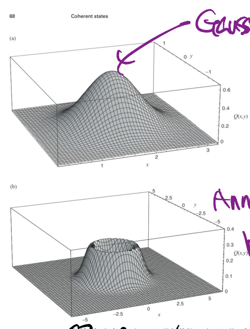
$$Q(\alpha) = \frac{1}{\pi} |\langle \alpha | \beta \rangle|^2 = \frac{1}{\pi} \exp(-|\alpha - \beta|^2)$$

number state $\hat{\rho} = |n\rangle\langle n|$

$$Q(\alpha) = \frac{1}{\pi} |\langle \alpha | n \rangle|^2 = \frac{1}{\pi} \exp(-|\alpha|^2) \frac{|\alpha|^{2n}}{n!}$$

setting $\alpha = x + iy$. $\longrightarrow$ plot.

Gaussian centered at β.



Annular form.

run.

Q function.

Always probability distribution

Wigner function for the coherent state $|\beta\rangle$.

$$W(\alpha) = \frac{2}{\pi} \exp\left(-2|\alpha-\beta|^2\right) = \frac{2}{\pi}(-1)^n \underbrace{L_n(4|\alpha|^2)}_{\text{Laguerre polynomial.}} \exp(-2|\alpha|^2)$$

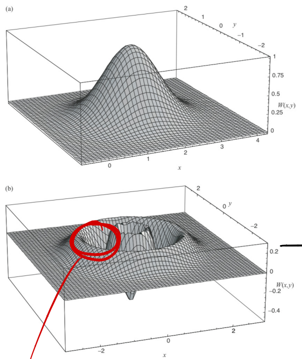


Fig. 3.10 Wigner function for (a) a coherent state with $\bar{n} = 10$, (b) a number state with $n = 3$.

coherent state → Q = Wigner (apart from an overall phase)

→ oscillates
- becomes negative over a wide range. → not pdf.
(Q function can never become negative)

Wigner function.

Wigner function takes on negative values ⇒ "non-classical".

(but nonclassical $\not\to$ negative Wigner function) ⇒ Nonclassical state yet have non-negative Wigner.
not necessarily true.

P function : negative or more singular than delta function on phase space.
⇒ nonclassical. (squeezed state, always positive)
↗ not primary.

⇒ mainstream : Wigner function.

Wigner function can display totally of interference effects associated with a quantum state.

→ off-diagonal elements of density matrix → FT → oscillating fringe term

QST → reconstructing the Wigner function from experimental data.