Qauntum Optics Study

Christopher C. Gerry - Introductory Quantum Optics

CHAPTER 2.1 ~ 2.4

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* The quantization of the E-field

- the interpretation of the photon as an elementary excitation of a normal mode of the field.

- Single mode and multi mode of E-fields in free space.

Quantization of a single mode

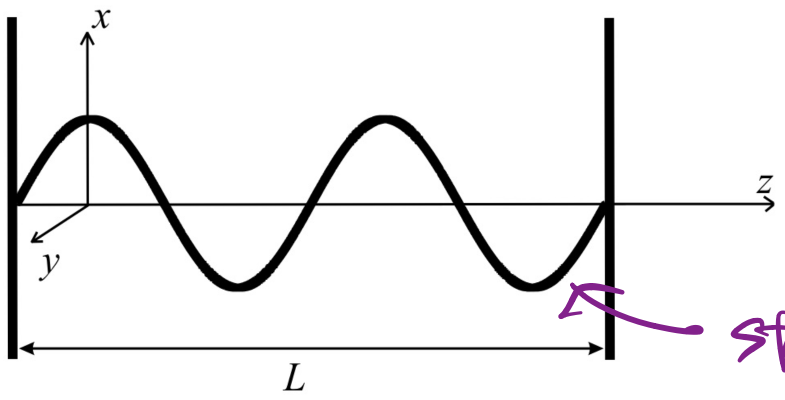

Fig. 2.1. Cavity with perfectly conducting walls located at $z = 0$ and $z = L$. The electric field is polarized along the x-direction.

boundary condition

standing wave

Assume that ① there are no sources of radiation

or

there are no dielectric media in the cavity

no source, no charge

② it is polarized along the x-direction.

$$\vec{E}(\vec{r},t) = \hat{e}_x E_x(z,t)$$

$\hat{e}_x$: polarization vector.

Maxwell equation without sources

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{B} = \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t} \qquad \vec{\nabla}\cdot\vec{E} = 0.$$

A single mode field,

$$E_x(z,t) = \left(\frac{2\omega^2}{V\epsilon_0}\right)^{1/2} q(t)\sin(kz),$$

dispersion relation

$$\omega = ck.$$

for BC, $\omega_m = c\left(\frac{m\pi}{L}\right)$ $m = 1, 2, 3, \cdots$

Assume that ω is one of these frequencies and ignore the rest.

- V = effective volume
- $q(t)$: time dependent factor that will act as a canonical position.

M-field in the cavity.

$$\nabla \times B = \mu_0 \epsilon_0 \frac{\partial E}{\partial t} \quad \text{(by Ampere's law)}$$

$$B_y(z,t) = \left(\frac{2\omega^2}{V\epsilon_0}\right)^{1/2} \left(\frac{\mu_0 \epsilon_0}{k}\right) \dot{q}(t) \cos(kz)$$

$\dot{q}(t)$: role of a canonical momentum for a "particle" of unit mass

$$p(t) = \dot{q}(t)$$

The classical field energy (Hamiltonian) of the single mode.

$$H = \frac{1}{2}\int dV \left[\epsilon_0 \vec{E}^2(\vec{r},t) + \frac{1}{\mu_0}\vec{B}^2(\vec{r},t) \right]$$

$$= \frac{1}{2}\int dV \left[\epsilon_0 E_x^2(z,t) + \frac{1}{\mu_0} B_y^2(z,t) \right]$$

$$H = \frac{1}{2}(p^2 + \omega^2 q^2) \quad \text{(formally equivalent to HO of unit mass)}.$$

roles of canonical position and momentum.

Approach that having identified the canonical variables q and p for the classical system.

$$(q, p) \xrightarrow{\text{Q.M.}} \text{operator } \hat{q}, \hat{p}$$

Canonical commutation relation

$$[\hat{q}, \hat{p}] = i\hbar \hat{I} \xrightarrow{\text{drop } \hat{I}} [\hat{q}, \hat{p}] = i\hbar$$

E-field, H-field operators,

$$\hat{E}_x(z,t) = \left(\frac{2\omega^2}{V\varepsilon_0}\right)^{1/2} \hat{q}(t) \sin(kz)$$

$$\hat{B}_y(z,t) = \left(\frac{\mu_0\varepsilon_0}{k}\right)\left(\frac{2\omega^2}{V\varepsilon_0}\right)^{1/2} \hat{p}(t) \cos(kz)$$

Then the Hamiltonian is,

$$\hat{H} = \frac{1}{2}(\hat{p}^2 + \omega^2 \hat{q}^2)$$

$\hat{p}, \hat{q}$ are Hermitian. (observable)

Introduce $\hat{a}^{+}, \hat{a}$ (creation, annihilation operator)

they are non-Hermitian (non-observable)

$$\frac{2\omega^2}{V\varepsilon_0}\frac{\hbar}{2\omega}$$

$$\hat{a} = (2\hbar\omega)^{-1/2}(\omega\hat{q} + i\hat{p})$$

$$\hat{a}^{+} = (2\hbar\omega)^{-1/2}(\omega\hat{q} - i\hat{p})$$

$$\longrightarrow \hat{q} = \frac{1}{2\omega}(2\hbar\omega)^{1/2}(\hat{a} + \hat{a}^{+}) = \left(\frac{\hbar}{2\omega}\right)^{1/2}(\hat{a} + \hat{a}^{+})$$

$$\hat{p} = \frac{1}{2i}(2\hbar\omega)^{1/2}(\hat{a} - \hat{a}^{+})$$

$$\hat{E}_x(z,t) = \left(\frac{\hbar\omega}{\varepsilon_0 V}\right)^{1/2}(\hat{a} + \hat{a}^{+})\sin kz$$

$$\hat{B}_y(z,t) = \frac{1}{i}\left(\frac{\mu_0\varepsilon_0}{k}\right)\left(\frac{2\omega^2}{V\varepsilon_0}\right)^{1/2}\left(\frac{\hbar\omega}{2}\right)^{1/2}(\hat{a} - \hat{a}^{+})\cos kz = \frac{1}{i}\left(\frac{\mu_0}{k}\right)\left(\frac{\varepsilon_0\hbar\omega^3}{V}\right)^{1/2} \sim$$

EM field operators,

$$\hat{E}_x(z,t) = E_0(\hat{a} + \hat{a}^{+})\sin kz$$

$$\hat{B}_y(z,t) = B_0\frac{1}{i}(\hat{a} - \hat{a}^{+})\cos kz.$$

EM fields "per photon".

the average of these fields for a definite number of photons is zero

$\rightarrow$ Nevertheless, they are useful measures of the fluctuations of the quantized field

Before measuring fluctuation, let's study $\hat{a}, \hat{a}^\dagger$.

Commutation relation, $[\hat{a}, \hat{a}^\dagger] = 1$

so, the Hamiltonian is, $\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})$

time dependence of $\hat{a}, \hat{a}^\dagger$.

for an arbitrary operator $\hat{O}$ having no explicit time dependence,

Heisenberg equations,

$$\frac{d\hat{O}}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{O}]$$

for annihilation operator,

$$\frac{d\hat{a}}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{a}] = \frac{i}{\hbar}\left[\hbar\omega(\hat{a}^\dagger\hat{a} + \tfrac{1}{2}), \hat{a}\right] = i\omega(\hat{a}^\dagger\hat{a}\hat{a} - \hat{a}\hat{a}^\dagger\hat{a})$$

$$= -i\omega[\hat{a}, \hat{a}^\dagger]\hat{a} = -i\omega\hat{a}$$

so, $\hat{a}(t) = \hat{a}(0)e^{-i\omega t}$. $\xrightarrow{\text{conjugate}}$ $\hat{a}^\dagger(t) = \hat{a}^\dagger(0)e^{i\omega t}$.

other solution is,

$$\hat{O}(t) = e^{i\hat{H}t/\hbar}\hat{O}(0)e^{-i\hat{H}t/\hbar}$$

By using Baker - Hausdorf lemma,

$$\hat{O}(t) = \hat{O}(0) + \frac{it}{\hbar}[\hat{H},\hat{O}(0)] + \frac{1}{2!}\left(\frac{it}{\hbar}\right)^2[\hat{H},[\hat{H},\hat{O}(0)]] + \cdots$$

$$\frac{1}{n!}[\hat{H},[\hat{H},[\hat{H}\cdots[\hat{H},\hat{O}]]]\cdots]$$

$$\hat{a}(t) = \hat{a}(0)\left(1 - i\omega t + \frac{\omega^2 t^2}{2!} + \cdots\right) = \hat{a}(0)e^{-i\omega t}$$

number operator $\hat{n} = \hat{a}^{\dagger}\hat{a}$

An eigenstate of the single mode field with eigen-energy E_n : $|n\rangle$

$$\hat{H}|n\rangle = \hbar\omega\left(\hat{a}^{\dagger}\hat{a} + \frac{1}{2}\right)|n\rangle = E_n|n\rangle$$

multiplying $\hat{a}^{+}$, $\hat{a}^{+}\hat{H}|n\rangle = E_n \hat{a}^{+}|n\rangle$

$\hbar\omega(\hat{a}^{+}\hat{a}^{+}\hat{a} + \frac{1}{2}\hat{a}^{+})|n\rangle = E_n \hat{a}^{+}|n\rangle$

$\hbar\omega(\hat{a}^{+}\hat{a}\hat{a}^{+} - \hat{a}^{+} + \frac{1}{2}\hat{a}^{+})|n\rangle = E_n \hat{a}^{+}|n\rangle$

$\hbar\omega(\hat{a}^{+}\hat{a} + \frac{1}{2})\hat{a}^{+}|n\rangle = \underbrace{(E_n + \hbar\omega)}_{\text{eigen-energy}}\underbrace{\hat{a}^{+}|n\rangle}_{\text{eigenstate}}$

so, $\hat{a}^{+}$: creation operator
create "$\hbar\omega$".
→ photon of "$\hbar\omega$" is created by $\hat{a}^{+}$

$\hat{H}(\hat{a}^{+}|n\rangle) = (E_n + \hbar\omega)\hat{a}^{+}|n\rangle$

$\hat{H}(\hat{a}|n\rangle) = (E_n - \hbar\omega)\hat{a}|n\rangle$ → $\hat{a}$ destroys one photon of "$\hbar\omega$".

Energy of HO must be positive ($E_n > 0$) → the lowest energy $E_0 > 0$.

$\hat{H}(\hat{a}|0\rangle) = (E_0 - \hbar\omega)(\hat{a}|0\rangle) = 0$.

so, $\hat{a}|0\rangle = 0$

$\hat{H}|0\rangle = \hbar\omega(\hat{a}^{+}\hat{a} + \frac{1}{2})|0\rangle = \boxed{\frac{1}{2}\hbar\omega}|0\rangle$

lowest energy
zero point energy

since $E_{n+1} = E_n + \hbar\omega$, eigen-energy,

$E_n = \hbar\omega(n+\frac{1}{2})$, $n = 0, 1, \dots$

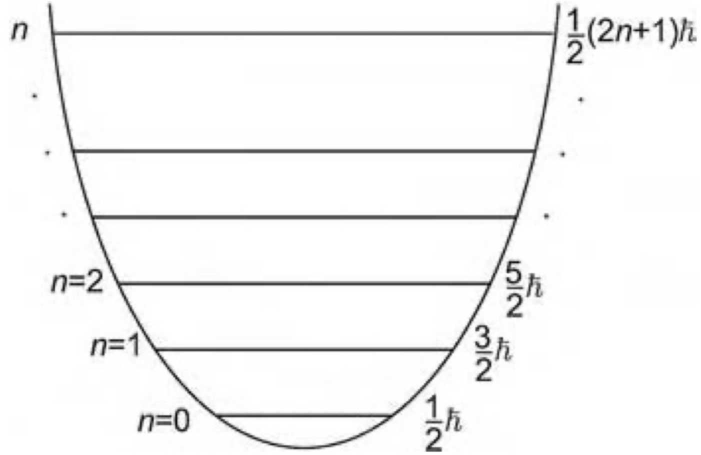

Fig. 2.2. The energy levels of a harmonic oscillator of frequency ω.

By number operator $\hat{n} = \hat{a}^{\dagger}\hat{a}$

$\hat{n}|n\rangle = n|n\rangle$

$\hat{a}|n\rangle = c_n|n-1\rangle$

$\langle n|\hat{a}^{\dagger}\hat{a}|n\rangle = |c_n|^2 = n \longrightarrow c_n = \sqrt{n}$.

$\hat{a}^{\dagger}|n\rangle = c_n|n+1\rangle$

$\langle n|\hat{a}\hat{a}^{\dagger}|n\rangle = |c_n|^2 = n+1 \longrightarrow c_n = \sqrt{n+1}$.

$\longrightarrow$ $\hat{a}|n\rangle = \sqrt{n}|n\rangle$

$\hat{a}^{\dagger}|n\rangle = \sqrt{n+1}\,|n\rangle$

$|n\rangle$ can be generated from the ground state $|0\rangle$

$$|n\rangle = \frac{(\hat{a}^{\dagger})^n}{\sqrt{n!}}|0\rangle$$

$\hat{H}$ and $\hat{n}^\dagger\hat{n}$ is Hermitian, so number states are orthogonal set $\langle n|n'\rangle = \delta_{nn'}$

the number states form a complete set

$$\sum_{n=0}^{\infty} |n\rangle\langle n| = 1$$

matrix elements of $\hat{a}, \hat{a}^\dagger$,

$$\langle n-1|\hat{a}|n\rangle = \sqrt{n}\langle n-1|n-1\rangle = \sqrt{n}$$

$$\langle n+1|\hat{a}^\dagger|n\rangle = \sqrt{n+1}\langle n+1|n+1\rangle = \sqrt{n+1}$$

* Quantum fluctuations of a single field

$|n\rangle$ is well-defined energy

But it is not a state of well-defined E-field.

the mean field $=0$

$$\langle n|\hat{E}_x(z,t)|n\rangle = E_0 \sin(kz)[\langle n|\hat{a}|n\rangle + \langle n|\hat{a}^\dagger|n\rangle] = 0.$$

the mean of squared field $\neq 0$

$$\langle n|\hat{E}_x^2(z,t)|n\rangle = E_0 \sin^2(kz)[\langle n|\hat{a}\hat{a}^\dagger|n\rangle + \langle n|\hat{a}^\dagger\hat{a}|n\rangle] = E_0 \sin^2(kz)(1+2n).$$

$$= 2E_0 \sin^2(kz)\left(n+\frac{1}{2}\right)$$

the fluctuations in E-field $\longrightarrow$ variance.

$$\langle(\Delta\hat{E}_x(z,t))^2\rangle = \langle\hat{E}_x^2(z,t)\rangle - \langle\hat{E}_x(z,t)\rangle^2$$

standard deviation (uncertainty of field)

$$\Delta E_x = \langle(\Delta\hat{E}_x(z,t))^2\rangle^{1/2} = \sqrt{2E_0}\sin(kz)\left(n+\frac{1}{2}\right)^{1/2}$$

even $n=0 \longrightarrow$ the field has fluctuation!

: vacuum fluctuation.

$|n\rangle \to \hat{n}$ $\longrightarrow$ representation "n photon". $\longrightarrow$ uncertainty principle. $[\hat{n}, \hat{E}] \neq 0$.

$\hat{E}$ $\longrightarrow$ $\langle \hat{E} \rangle = 0$. not commute.

$$[\hat{n}, \hat{E}] = E_0 \sin(kz)\left[\hat{a}^\dagger\hat{a}(\hat{a}+\hat{a}^\dagger) - (\hat{a}+\hat{a}^\dagger)\hat{a}^\dagger\hat{a}\right] = E_0 \sin(kz)\left(\hat{a}^\dagger - \hat{a}\right)$$

$$\hat{a}^\dagger\hat{a}\hat{a} + \hat{a}^\dagger\hat{a}\hat{a}^\dagger - \hat{a}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}^\dagger\hat{a} = \hat{a}^\dagger - \hat{a}$$

$$[\hat{A}, \hat{B}] = \hat{C}, \quad \Delta A \Delta B \geq \frac{1}{2}|\langle \hat{C} \rangle|$$

$$\Delta n \Delta E_x \geq \frac{1}{2} E_0 |\sin(kz)| |\langle \hat{a}^\dagger - \hat{a} \rangle|.$$

$\Delta n = 0$

for $|n\rangle$, accuracy known "n" $\longrightarrow$ field (phase) is never unknown.

accuracy known "E" $\longrightarrow$ # of photons is uncertain.

(phase of E-field)

Classically, the amplitude an phase of a field can be simultaneously well-defined

$\longrightarrow$ cannot in quantum-mechanically. phase $\Longleftrightarrow$ photon number

Complementary

number-phase uncertainty

$\Delta n \Delta \phi \geq 1$

well-defined phase $\longrightarrow$ photon number is uncertain.

well-defined photon number $\longrightarrow$ phase is uncertain

$\hookrightarrow$ randomly distributed $[0, 2\pi]$.

* Quadrature operators for a single mode field.

time dependence of the E-field operator. with $\hat{a}(0) \equiv \hat{a}$, $\hat{a}^\dagger(0) \equiv \hat{a}^\dagger$

$$\hat{E}_x = E_0 (\hat{a} e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}) \sin(kz)$$

Introduce Quadrature operator.

$$\hat{X}_1 = \frac{1}{2}(\hat{a} + \hat{a}^\dagger), \quad \hat{X}_2 = \frac{1}{2i}(\hat{a} - \hat{a}^\dagger)$$

$$\hat{E}_x = 2E_0 \sin(kz) \left[\hat{X}_1 \cos(\omega t) + \hat{X}_2 \sin(\omega t) \right]$$

$X_1, X_2 \rightarrow$ field amplitudes oscillating out of phase ($90°$)

$\downarrow$

the dimensionless position and momentum operators

Commutation relation.

$$[\hat{X}_1, \hat{X}_2] = \hat{X}_1 \hat{X}_2 - \hat{X}_2 \hat{X}_1 = \frac{1}{4i} \left[(\hat{a}^2 - \hat{a}^{\dagger 2} - \hat{a}\hat{a}^\dagger + \hat{a}^\dagger \hat{a}) - (\hat{a}^2 - \hat{a}^{\dagger 2} + \hat{a}^\dagger \hat{a} - \hat{a}\hat{a}^\dagger) \right].$$

$$= \frac{1}{2i}(\hat{a}^\dagger \hat{a} - \hat{a}\hat{a}^\dagger) = \frac{-1}{2i} = \frac{i}{2}$$

$$\rightarrow \langle (\Delta \hat{X}_1)^2 \rangle \langle (\Delta \hat{X}_2)^2 \rangle \geq \left(\frac{1}{4}\right)^2 = \frac{1}{16}$$

$$\langle n|\hat{X}_1|n\rangle = \langle n|\hat{X}_2|n\rangle = 0.$$

but $\langle n|\hat{X}_1^2|n\rangle = \frac{1}{4}(1+2n) = \langle n|\hat{X}_2^2|n\rangle$

- for number states, the uncertainties in both quadratures are the same,
- the vacuum state $(n=0)$ minimizes the uncertainty product.

$$\langle(\Delta\hat{X}_1)^2\rangle_{n=0} = \frac{1}{4} = \langle(\Delta\hat{X}_2)^2\rangle_{n=0}.$$

- The $\boxed{\text{quanta}}$ of the single mode cavity field are the excitations of energy in discrete amount of $\hbar\omega$.

- quanta (photons) are not localized particles (no position operator for photon)

  → they are spread out over the entire mode volume.

※ Multimode fields

Assuming that no sources of radiation and no charges.

EM radiation fields → vector potential $A(r,t)$.

wave equation, $\nabla^2 A - \frac{1}{c^2}\frac{\partial^2 A^2}{\partial t^2} = 0$.

Coulumb gauge condition, $\nabla \cdot A = 0$.

where $E(r,t) = -\frac{\partial A(r,t)}{\partial t}$, $B(r,t) = \nabla \times A(r,t)$.

free space can be modeled as a cubic cavity of length L with perfectly reflecting wall.

L: very large → the radiation could interact atoms.

$L \gg \lambda$ (field wavelength)

All results from model should be independent of the size of cavity. ($L \to \infty$)

cubical cavity → Boundary Condition.

$$e^{ik_x x} = e^{ik_x(x+L)}, \quad k_x = \left(\frac{2\pi}{L}\right) m_x, \quad m_x = 0, \pm 1, \pm 2$$

y, z direction are same, $k_y = k_z = \left(\frac{2\pi}{L}\right) m_{y,z}$

the wavevector, $k = \frac{2\pi}{L}(m_x, m_y, m_z) = \frac{\omega_k}{c}$

Integer (m_x, m_y, m_z) : a normal mode of the field.

# of modes : infinite but denumerable.

for simply dealing with the continuum of modes in free space.

total # of modes in the intervals → two independent polarizations.

$$\Delta n = \Delta n_x \Delta n_y \Delta n_z = \boxed{2}\left(\frac{L}{2\pi}\right)^3 \Delta k_x \Delta k_y \Delta k_z$$

Quasi-continuous limit

Assume that $\lambda \ll L$.

$$dn = 2\left(\frac{V}{8\pi^3}\right) dk_x dk_y dk_z$$

In k-space spherical coordinate system,

$$\vec{k} = k(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$$dn = 2\left(\frac{V}{8\pi^3}\right) k^2 dk\, d\Omega \quad (d\Omega = \sin\theta\, d\theta\, d\phi)$$

→ solid angle around the direction of k.

$$= 2\left(\frac{V}{8\pi^3}\right) \frac{\omega_k^2}{c^3} d\omega_k\, d\Omega$$

the numbers of modes in all direction $k \sim k+dk$ $\rightarrow \left(\frac{V}{4\pi^3}\right) k^2 dk \int d\Omega = \frac{V}{\pi^2} k^2 dk = V\rho_k dk$

$\rho_k dk$: mode density (# of modes per unit volume)

$\rho_k = \frac{k^2}{\pi^2}$

in the same fashion,

$$\left(\frac{V}{4\pi^3}\right) \frac{\omega_k^2}{c^3} d\omega_k (4\pi) = V\left(\frac{\omega_k^2}{\pi^2 c^3}\right) d\omega_k \equiv V \boxed{\rho(\omega_k)} d\omega_k$$

mode density.

Vector potential $\rightarrow$ superposition of plane waves

$$A(r,t) = \sum_{ks} \hat{e}_{ks} \left[A_{ks}(t) e^{ik\cdot r} + A^*_{ks}(t) e^{-ik\cdot r} \right]$$

A_{ks} : complex amplitude of the field.

$\hat{e}_{ks}$: reel polarization vector.

Sum over k : Sum over set of integers (m_x, m_y, m_z).

Sum over s : Sum over the two independent polarization.

polarization vector : $\hat{e}_{ks} \cdot \hat{e}_{ks'} = \delta_{ss'}$

gauge condition (transversality condition) : $k \cdot \hat{e}_{ks} = 0$.

$$\hat{e}_{k1} \times \hat{e}_{k2} = \frac{k}{|k|} = K$$

In free space, $\Sigma \to \int$.

$$\sum_k = \frac{V}{\pi^2}\int k^2 dk.$$

for obtaining A_{ks}, $\frac{d^2 A_{ks}}{dt^2} + \omega_k^2 A_{ks} = 0$.

$$A_{ks}(t) = A_{ks} e^{-i\omega t} \quad (A_{ks}(0) = A_{ks}).$$

then EM fields,

$$E(r,t) = -\frac{\partial A(r,t)}{\partial t} = i\sum_{ks}\omega_k \hat{e}_{ks}\left[A_{ks}e^{i(k\cdot r-\omega_k t)} - A_{ks}^* e^{-i(k\cdot r-\omega_k t)}\right]$$

$$B(r,t) = \nabla\times A(r,t) = \frac{i}{c}\sum_{ks}\omega_k(K\times\hat{e}_{ks})\left[A_{ks}e^{i(k\cdot r-\omega_k t)} - A_{ks}^* e^{-i(k\cdot r-\omega_k t)}\right]$$

The energy of the field.

$$H = \frac{1}{2}\int_V\left(\epsilon_0\vec{E}\cdot\vec{E} + \frac{1}{\mu_0}\vec{B}\cdot\vec{B}\right)dV$$

Periodic boundary condition results,

$$\int_0^L e^{\pm i k_x x} dx = \begin{cases} L & k_x = 0 \\ 0 & k_x \neq 0 \end{cases} \longrightarrow \int_V e^{i(k'-k)\cdot r} dV = \delta_{kk'} V.$$

$$E(r,t) = i \sum_{ks} \omega_k \hat{e}_{ks} \left[A_{ks}(t) e^{ik\cdot r} - A^*_{ks}(t) e^{-ik\cdot r} \right] \quad (\alpha^* - \beta^*)$$

$$\frac{1}{2} \int \varepsilon_0 E \cdot E \, dV = \varepsilon_0 V \sum_{ks} \omega_k^2 |A_{ks}(t)|^2 - R. \quad (k = k')$$

where $R = \frac{1}{2} \varepsilon_0 V \sum_{kss'} \omega_k^2 \hat{e}_{ks} \cdot (\hat{e}_{-ks'}) \left[A_{ks}(t) A_{-ks'}(t) + A^*_{ks}(t) A^*_{-ks'}(t) \right].$

To obtain magnetic contribution,

$$(A \times B) \cdot (C \times D) = (A \cdot C)(B \cdot D) - (A \cdot D)(B \cdot C)$$

$$(k \cdot \hat{e}_{ks}) \cdot (k \cdot \hat{e}_{ks'}) = \delta_{ss'}$$

$$(k \cdot \hat{e}_{ks}) \cdot (-k \cdot \hat{e}_{-ks'}) = - \hat{e}_{ks} \cdot \hat{e}_{-ks'} - 0.$$

So, M-field energy,

$$\frac{1}{2}\int \frac{1}{\mu_0} B \cdot B \, dV = \varepsilon_0 V \sum_{ks} \omega_k^2 |A_{ks}(t)|^2 + R. \quad (R \text{ is same})$$

The field energy,

$$H = 2\varepsilon_0 V \sum_{ks} \omega_k^2 |A_{ks}(t)|^2 = 2\varepsilon_0 V \sum_{ks} \omega_k^2 |A_{ks}|^2.$$

In order to quantize the field, canonical variables p_{ks}, q_{ks}.

$$A_{ks} = \frac{1}{2\omega_k(\varepsilon_0 V)^{1/2}} (\omega_k q_{ks} + i p_{ks})$$

$$A^*_{ks} = \frac{1}{2\omega_k(\varepsilon_0 V)^{1/2}} (\omega_k q_{ks} - i p_{ks})$$

$$\rightarrow H = 2\varepsilon_0 V \sum_{ks} \omega_k^2 \frac{1}{4\omega_k^2 \varepsilon_0 V} (\omega_k^2 q_{ks}^2 + p_{ks}^2)$$

$H = \frac{1}{2}\sum_{ks}(p_{ks}^2 + \omega_k^2 q_{ks}^2)$: HO of unit mass.

commutation relation.

$$[\hat{q}_{ks}, \hat{q}_{k's'}] = [\hat{p}_{ks}, \hat{p}_{k's'}] = 0.$$

$$[\hat{q}_{ks}, \hat{p}_{k's'}] = i\hbar\,\delta_{kk'}\delta_{ss'}$$

annihilation, creation operator for single mode.

$$\hat{a}_{ks} = \frac{1}{(2\hbar\omega_k)^{1/2}}(\omega_k \hat{q}_{ks} + i\hat{p}_{ks})$$

$$\hat{a}^{\dagger}_{ks} = \frac{1}{(2\hbar\omega_k)^{1/2}}(\omega_k \hat{q}_{ks} - i\hat{p}_{ks})$$

commutation relation

$$[\hat{a}_{ks}, \hat{a}_{k's'}] = [\hat{a}^{\dagger}_{ks}, \hat{a}^{\dagger}_{k's'}] = 0.$$

$$[\hat{a}_{ks}, \hat{a}^{\dagger}_{ks}] = \delta_{kk'}\delta_{ss'}$$

the energy of the field $\longrightarrow$ Hamiltonian operator.

$$\hat{H} = \sum_{ks} \hbar\omega_{ks}\left(\hat{a}^{+}_{ks}\hat{a}_{ks} + \frac{1}{2}\right) = \sum_{ks} \hbar\omega_{k}\left(\hat{n}_{ks} + \frac{1}{2}\right),$$

$\hat{n}_{ks} = \hat{a}^{+}_{ks}\hat{a}_{ks} \longrightarrow$ number operator for the mode ks.

Each of these modes $\longrightarrow$ independent of all the others.

For the jth modes,

$$\hat{a}_{k_j s_j} \equiv \hat{a}_j, \quad \hat{a}^{+}_{k_j s_j} \equiv \hat{a}^{+}_j \quad \rightarrow \quad \hat{H} = \sum_j \hbar\omega_j\left(\hat{n}_j + \frac{1}{2}\right).$$

- multimode photon number state = A product of number states.

$$|n_1\rangle|n_2\rangle\cdots = |n_1, n_2 \cdots\rangle = |\{n_j\}\rangle.$$

$$\hat{H}|\{n_j\}\rangle = E|\{n_j\}\rangle \quad \text{where} \quad E = \sum_j \hbar\omega_j\left(n_j + \frac{1}{2}\right)$$

number states are orthogonal

$$\langle n_1, n_2 \cdots | n_1', n_2' \cdots \rangle = \delta_{n_1 n_1'} \delta_{n_2 n_2'} \cdots \delta_{n_j n_j'}$$

Annihilation operator of the jth mode,

$$\hat{a}_j | n_1 n_2 \cdots n_j \rangle = \sqrt{n_j} \, | n_1 n_2 \cdots n_j - 1 \rangle$$

Creation operator of the jth mode,

$$\hat{a}_j^{+} | n_1 n_2 \cdots n_j \rangle = \sqrt{n_j + 1} \, | n_1 n_2 \cdots n_j + 1 \rangle$$

multi-mode vacuum state.

$$|\{0\}\rangle = |0_1 0_2 \cdots 0_j \cdots \rangle \quad , \quad \hat{a}_j |\{0\}\rangle = 0 .$$

All number states is generated from the vacuum.

$$|\{n_j\}\rangle = \prod_j \frac{(a_j^{+})^{n_j}}{\sqrt{n_j !}} |\{0\}\rangle .$$

Upon the quantized vector potential, A_{ks} must be operator.

$$\hat{A}_{ks} = \left(\frac{\hbar}{2\omega_k \varepsilon_0 V}\right)^{1/2} \hat{a}_{ks}$$

the quantized vector potential and quantized EM field operators.

$$\hat{A}(r,t) = \sum_{ks} \left(\frac{\hbar}{2\omega_k \varepsilon_0 V}\right)^{1/2} \hat{e}_{ks} \left[\hat{a}_{ks} e^{i(k\cdot r-\omega t)} + \hat{a}^{\dagger}_{ks} e^{-i(k\cdot r-\omega t)}\right],$$

$$\hat{E}(r,t) = i \sum_{ks} \left(\frac{\hbar\omega_k}{2\varepsilon_0 V}\right)^{1/2} \hat{e}_{ks} \left[\hat{a}_{ks} e^{i(k\cdot r-\omega t)} - \hat{a}^{\dagger}_{ks} e^{-i(k\cdot r-\omega t)}\right]$$

$$\vec{B}(r,t) = \frac{i}{c} \sum_{ks} (k\times\hat{e}_{ks}) \left(\frac{\hbar\omega_k}{2\varepsilon_0 V}\right)^{1/2} \left[\hat{a}_{ks} e^{i(k\cdot r-\omega t)} - \hat{a}^{\dagger}_{ks} e^{-i(k\cdot r-\omega t)}\right].$$

→ annihilation, creation operators as Heisenberg picture operators at $t=0$.

$$\hat{a}_{ks}(t) = \hat{a}_{ks}(0) e^{-i\omega_k t}$$

$$\hat{E}(r,t) = i\sum_{ks}\left(\frac{\hbar\omega_k}{2\epsilon_0 V}\right)^{1/2}\hat{e}_{ks}\left(\hat{a}_{ks}(t)e^{ik\cdot r} - \hat{a}^{\dagger}_{ks}(t)e^{-ik\cdot r}\right).$$

$$= \hat{E}^{(+)}(r,t) + \hat{E}^{(-)}(r,t). \qquad \left(\hat{E}^{(-)}(r,t) = \hat{E}^{(+)\dagger}(r,t)\right)$$

$\hat{E}^{(+)}$ = positive frequency part of the field; $e^{-i\omega t}$

→ collective annihilation operator.

$\hat{E}^{(-)}$: negative frequency part

→ collective creation operator.

vector potential, magnetic fields are same.

In most quantum optics situations,

the coupling of the field to matter.

→ E field interacting with a dipole moment

or

nonlinear type of interaction involving powers of E-field

M-field : weaker than E-field by a factor of $1/c$.

↳ field $\xleftrightarrow{\text{coupling}}$ spin magnetic moment. → negligible.

for single mode E-field,

$$\hat{E}(r,t) = i\left(\frac{\hbar\omega}{2\epsilon_0 V}\right)^{1/2} \hat{e}_x \left(\hat{a} e^{i(k\cdot r - \omega t)} - \hat{a}^\dagger e^{-i(k\cdot r - \omega t)}\right)$$

spatial variance of the field of atomic system → negligible.

for optical radiation, λ (~ several thousand ångströms)

$$\frac{\lambda}{2\pi} = \frac{1}{|k|} \gg |r|$$ size of atom,

so, $e^{\pm i k\cdot r} \sim 1 \pm i k\cdot r$

$$\hat{E}(r,t) \approx \hat{E}(t) = i\left(\frac{\hbar\omega}{2\varepsilon_0 V}\right)^{1/2} e_x \left[\hat{a}e^{-i\omega t} - \hat{a}^{+}e^{i\omega t}\right].$$

∴ dipole approximation.