Self-study for Continuous Variable Quantum Information

Introductory Quantum Optics (Chapter 5: Quantum Coherence Functions)

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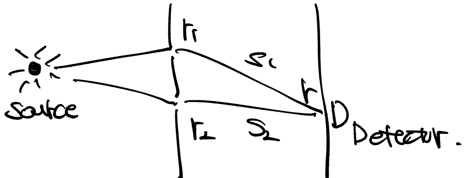
December 1, 2025

5.1 Classical coherence functions.



Young's double slit interference.

Under the certain condition, interference fringe will appear on the screen.

A bandwidth $\Delta\omega$ of source.

$\Delta s = |s_1 - s_2|$: path difference $\Rightarrow$ the interference will be occurs $\Delta s \leq c/\Delta\omega$.

$\Delta t_{coh} = \frac{\Delta s_{coh}}{c} = \frac{1}{\Delta\omega}$: coherence time.

Interference fringe will be visible for $\Delta t_{coh}\Delta\omega \approx 1$

The field on the screen, or the detector.

$\Rightarrow$ the linear combination of the fields at the earlier times.

$t_1 = t - s_1/c$
$t_2 = t - s_2/c$: $E(\vec{r},t) = \underbrace{k_1}E(\vec{r_1},t_1) + \underbrace{k_2}E(\vec{r_2},t_2)$

geometrical factor.

① Assuming the same polarization $\Rightarrow$ treating the field as a scalar.

② diffraction is ignored.

Optical light detectors have long response time and are capability only of measuring the average light intensity.

$I(t) = \langle |E(\vec{r},t)|^2 \rangle$ where $\langle \cdot \rangle = \lim_{T\to\infty}\frac{1}{T}\int_0^T f(t)dt$.

Assuming that this average is stationary

By ergodic hypothesis, the time average $\Rightarrow$ the ensemble average.

$$I(t) = \underbrace{|k_1|^2\langle |E(\vec{r_1},t_1)|^2\rangle}_{I_1} + \underbrace{|k_2|^2\langle |E(\vec{r_2},t_2)|^2\rangle}_{I_2} + \underbrace{2\mathrm{Re}[k_1^* k_2 \langle E_1^*(\vec{r_1},t_1)E_2^*(\vec{r_2},t_2)\rangle]}_{\text{gives rise to the interference}}.$$

Introduce the first order normalized mutual coherence function,

$$\gamma^{(1)}(x_1,x_2) = \frac{\langle E^*(x_1)E(x_2)\rangle}{\sqrt{\langle|E(x_1)|^2\rangle\langle|E(x_2)|^2\rangle}} \qquad x_i = \vec{r}_i, t_i.$$

A coherence term

$$I(\vec{r}) = I_1 + I_2 + 2\sqrt{I_1I_2}\,\mathrm{Re}[K_1K_2\gamma^{(1)}(x_1,x_2)].$$

If we now set $K_i = |K_i|\exp(i\psi_i)$

$$\gamma^{(1)}(x_1,x_2) = |\gamma^{(1)}(x_1,x_2)|\exp(i\Phi_{12})$$

$$\Rightarrow I(\vec{r}) = I_1 + I_2 + 2\sqrt{I_1I_2}\,|\gamma^{(1)}(x_1,x_2)|\cos(\Phi_{12}-\psi)$$

$\psi = \psi_1 - \psi_2$: path difference → phase difference. ($\leq$ Coherence length)

Three types of coherence.

$|\gamma^{(1)}(x_1,x_2)| = 1$ (complete coherence)

$0 \leq |\gamma^{(1)}(x_1,x_2)| < 1$ (partial coherence)

$|\gamma^{(1)}(x_1,x_2)| = 0$ (complete incoherence).

Introduce Rayleigh's definition of fringe visibility.

$V = (I_{max} - I_{min})/(I_{max} + I_{min})$, where $I_{\substack{max\\min}} = I_1 + I_2 \pm 2\sqrt{I_1I_2}\,|\gamma^{(1)}(x_1,x_2)|$.

visibility ⟺ mutual coherence function.

$$V = \frac{2\sqrt{I_1I_2}\,|\gamma^{(1)}(x_1,x_2)|}{I_1+I_2} \Rightarrow \begin{array}{l} \text{Max}: V = \dfrac{2\sqrt{I_1I_2}}{I_1+I_2} \\ \text{min}: V = 0. \end{array}$$

for complete optical coherence $|\gamma^{(1)}(x_1,x_2)| = 1$,

$$\langle E^*(x_1)E(x_2)\rangle = \sqrt{\langle|E(x_1)|^2\rangle}\sqrt{\langle|E(x_2)|^2\rangle}$$

⇒ give rise to a criterion for the complete optical coherence in Q.M. light field.

- A classical first order coherence function.
  : temporal coherence of stationary field at a fixed position.

The field propagating in the z-direction at times t, t+τ.

$$E(z,t) = E_0 e^{i(kz-\omega t)}$$

$$E(z,t+\tau) = E_0 e^{i(kz-\omega(t+\tau))}$$

$\langle E^*(z,t)E(z,t+\tau)\rangle = E_0^2 e^{-i\omega\tau}$: Autocorrelation function. (usually written $\langle E(t)E(t+\tau)\rangle$).

$$\gamma^{(1)}(x_1,x_2) = \gamma^{(1)}(\tau) = \gamma^{(1)*}(-\tau) = e^{-i\omega\tau}.$$

For $|\gamma^{(1)}(\tau)|=1$, complete temporal coherence. But perfectly monochromatic source ⇒ don't exist.

A more realistic model of a "monochromatic" source.

: random process by which light is emitted from the decay of each atom in the source.

: emitting light as wave trains of finite length.

separated by a discontinuous change of phase.

The average wave train time τ_0 : coherence time of the source.

$$\text{Coherence time} \propto \frac{1}{\text{natural line width of the spectral lines}}$$ of the radiating atoms in the source.

coherence length $l_{coh} = c\tau_0$.

the field : $E(z,t) = E_0 e^{i(kz-\omega t)} e^{i\psi(t)}$, $\psi(t)$: random step function $(0,2\pi]$ with period τ_0.

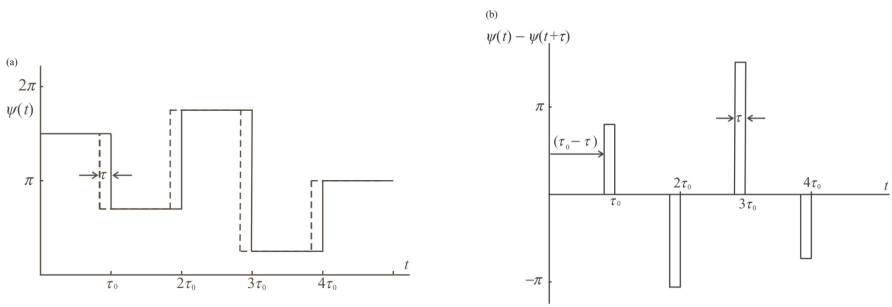


Fig. 5.2. (a) Random phase of period τ_0 $\psi(t)$ (solid line), $\psi(t+\tau)$ (dashed line), and (b) the phase difference $\psi(t) - \psi(t+\tau)$.

Autocorrelation function in this case,

$$\langle E^*(t)E(t+\tau)\rangle = E_0^2 e^{-i\omega\tau}\langle e^{i[\psi(t+\tau)-\psi(t)]}\rangle.$$

$$\gamma^{(1)}(\tau) = e^{-i\omega\tau}\lim_{T\to\infty}\frac{1}{T}\int_0^T e^{i[\psi(t+\tau)-\psi(t)]}\,dt.$$

For $0<t<\tau_0$, $\psi(t+\tau)-\psi(t)=0$ for $0<t<\tau_0-\tau$.

But $\tau_0-\tau<t<\tau_0$, $\psi(t+\tau)-\psi(t)$ is a random number in the range $(0,2\pi)$.

The same holds for subsequent coherent time intervals.

$$\gamma^{(1)}(\tau) = \left(1-\frac{\tau}{\tau_0}\right)e^{-i\omega\tau} \quad (\tau<\tau_0)$$
$$= 0 \quad (\tau\geq\tau_0)$$

So, $|\gamma^{(1)}(\tau)| = 1 - \frac{\tau}{\tau_0} \quad (\tau < \tau_0)$

$= 0 \quad (\tau \geq \tau_0)$

If the time delay τ is larger than the coherence time τ_0.

→ there is no coherence.

$x(t) = e^{-t/\tau} u(t)$

bell shape spectrum.

FT of exponentially decaying autocorrelation.

A more realistic model of collision-broadened light, for which the radiation has a $\boxed{\text{Lorentzian power spectrum}}$ centered at ω_0.

the autocorrelation function,

$$\langle E^*(t) E(t+\tau) \rangle = E_0^2 e^{-i\omega_0 \tau - |t|/\tau_0} \quad (\tau_0 \text{: average time between collision})$$

$$\gamma^{(1)}(\tau) = e^{-i\omega_0\tau - |\tau|/\tau_0}, \quad 0 \leq |\gamma^{(1)}(\tau)| \leq 1.$$

As $\tau \to \infty$, $|\gamma^{(1)}(\tau)| \to 0$.

the beam becomes increasingly incoherent with increased time delay.

$|\tau| \ll \tau_0$, complete coherence

If the average time between collision is shortened, ⇒ chaotic

the spectrum is broadened, $|\gamma^{(1)}(\tau)| \to 0$. (ex) gaussian).

5.2 Quantum coherence function.

The intensity of a light beam is measured by devices that actually attenuate the beam by absorbing photons.

The actual determination of the intensity of a beam.

⇒ the response of the absorbing system.

An ideal detector consisting of a single atom of small dimension compared to the wavelength of the light.

The absorption of light over a broad band of wavelength = ionization of the atom.

The subsequent detection of the photoelectron = the detection of the photon.

The single-atom detector couples to the quantized field through the dipole interaction.

$$\hat{H}^{(I)} = -\hat{d}\cdot\hat{E}(\vec{r},t)$$

E-field in the Heisenberg picture.

$$\hat{E}(\vec{r},t) = i\sum_{\vec{k},s}\left(\frac{\hbar\omega_k}{2\epsilon_0 V}\right)^{1/2}\hat{e}_{\vec{k}s}\left[\hat{a}_{\vec{k}s}(t)e^{i\vec{k}\cdot\vec{r}} - \hat{a}^{\dagger}_{\vec{k}s}(t)e^{-i\vec{k}\cdot\vec{r}}\right]$$

Because the wavelength of the light >> dimension of atoms.

$\Rightarrow |\vec{k}\cdot\vec{r}| \ll 1$ (the dipole approximation)

the field at the atomic detector,

$$\hat{E}(\vec{r},t) \approx i\sum_{\vec{k}s}\sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}}\,\hat{e}_{\vec{k}s}\left[\hat{a}_{\vec{k}s}(t) - \hat{a}^{\dagger}_{\vec{k}s}(t)\right]$$

Absorption → frequency part.

$$\hat{E}^{(+)}(\vec{r},t) = i\sum_{\vec{k}s}\left(\frac{\hbar\omega_k}{2\epsilon_0 V}\right)^{1/2}\hat{e}_{\vec{k}s}\boxed{\hat{a}_{\vec{k}s}(t)}$$

Assumption: initial state atom and field, $|g\rangle$, $|i\rangle$

$\int$ Absorption of radiation.

transition to $|e\rangle, |f\rangle$ ($|e\rangle$: ionized atomic state ~ free electron state)

⇒ highly idealized model of photodetection.

wave function (free electron state)

$$\langle\vec{r}|e\rangle = \frac{1}{\sqrt{V}}e^{i\vec{q}\cdot\vec{r}}.$$ ($\hbar\vec{q}$: the momentum of the ionized electron).

let $|I\rangle = |g\rangle|i\rangle$, $|F\rangle = |e\rangle|f\rangle$

For transition,

$$\langle F|\hat{H}^{(I)}|I\rangle = -\langle e|\hat{d}|g\rangle\langle f|\hat{E}^{(+)}(\vec{r},t)|i\rangle.$$

The transitional probability for the atom-field system.

$$P \propto |\langle F|\hat{H}^{(I)}|I\rangle|^2.$$

$|i\rangle \rightarrow |f\rangle$: the field undergoes transition. $|\langle f|\hat{E}^{(+)}(\vec{r},t)|i\rangle|^2$.

Our interest : not the field, but the final state of the detector.

⇒ sum over the possible final field.

For all practical purpose, the set of final state: complete set. (including unallowed transition)

$$\sum_f|\langle f|\hat{E}^{(+)}(\vec{r},t)|i\rangle|^2 = \sum_f|\langle i|\hat{E}^{(-)}(\vec{r},t)\underbrace{|f\rangle\langle f|}_{\text{complete set}}\hat{E}^{(+)}(\vec{r},t)|i\rangle|^2$$

$= \langle i | \hat{E}^{(-)}(\vec{r},t) \cdot \hat{E}^{(+)}(\vec{r},t) | i \rangle .$

Assumption: the field to be initially in a pure state.

If mixed state,

$\hat{\rho}_F = \sum_i P_i |i\rangle\langle i|$

The expectation value → ensemble average.

$\mathrm{Tr}\{\hat{\rho}_F \hat{E}^{(-)}(\vec{r},t) \cdot \hat{E}^{(+)}(\vec{r},t)\} = \sum_i P_i \langle i | \hat{E}^{(-)}(\vec{r},t) \cdot \hat{E}^{(+)}(\vec{r},t) | i \rangle .$

A consequence of the absorbing detector = the normal ordering of the operators.

for simplification, (1) the fields are the same polarization, (2) $x = (\vec{r}, t)$.

Define the function $G^{(1)}(x,x) = I(\vec{r},t)$,

$G^{(1)}(x,x) = \mathrm{Tr}\{\rho \hat{E}^{(-)}(x) \hat{E}^{(+)}(x)\}$: the quantum analog of the classical expression.

for Young's interference from two slits.

$\hat{E}^{(+)}(\vec{r},t) = K_1 \hat{E}^{(+)}(\vec{r}_1,t) + K_2 \hat{E}^{(+)}(\vec{r}_2,t)$

The intensity of light on the screen (the photodetector).

$$I(\vec{r},t) = \mathrm{Tr}\{\hat{\rho} \hat{E}^{(-)}(\vec{r},t) \hat{E}^{(+)}(\vec{r},t)\}$$
$$= |K_1|^2 G^{(1)}(x_1,x_1) + |K_2|^2 G^{(1)}(x_2,x_2) + 2\mathrm{Re}[K_1^* K_2 \boxed{G^{(1)}(x_1,x_2)}]$$

General first order correlation.

$G^{(1)}(x_i,x_i)$: the intensity at the detector of light arriving at x_i.

normalization on $G^{(1)}$

$$g^{(1)}(x_1,x_2) = \frac{G^{(1)}(x_1,x_2)}{[G^{(1)}(x_1,x_1) G^{(1)}(x_2,x_2)]^{1/2}}, \quad 0 \le |g^{(1)}(x_1,x_2)| \le 1.$$

Three types of coherence.

$|g^{(1)}(x_1,x_2)| = 1$ Complete coherence

$0 < |g^{(1)}(x_1,x_2)| < 1$ partial coherence.

$|g^{(1)}(x_1,x_2)| = 0$ incoherence.

From the definition of the correlation function,

$$G^{(1)}(x_1,x_2) = [G^{(1)}(x_2,x_1)]^*$$

$$G^{(1)}(x_i,x_i) \geq 0.$$

$$G^{(1)}(x_1,x_1)G^{(1)}(x_2,x_2) \geq |G^{(1)}(x_1,x_2)|^2.$$

Maximum fringe visibility, $|g^{(1)}(x_1,x_2)|=1$.

For a single mode plane wave quantized field. propagating with wavevector k.

$$\hat{E}^{(+)}(x) = i\mathcal{K}\hat{a}e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$ where $\mathcal{K}=\left(\frac{\hbar\omega}{2\epsilon_0 V}\right)^{1/2}$: electric field per photon.

If the field in a number state,

$$G^{(1)}(x,x) = \langle n|\hat{E}^{(-)}(x)\hat{E}^{(+)}(x)|n\rangle = \mathcal{K}^2 n$$

$$G^{(1)}(x_1,x_2) = \langle n|\hat{E}^{(-)}(x_1)\cdot\hat{E}^{(+)}(x_2)|n\rangle = \mathcal{K}^2 n\exp(i\vec{k}\cdot(\vec{r_1}-\vec{r_2})-\omega(t_2-t_1)).$$

$$\Rightarrow |g^{(1)}(x_1,x_2)|=1.$$

for a coherent state $|\alpha\rangle$,

$$G^{(1)}(x,x) = \langle\alpha|\hat{E}^{(-)}(x)\hat{E}^{(+)}(x)|\alpha\rangle = \mathcal{K}^2|\alpha|^2.$$

$$G^{(1)}(x_1,x_2) = \langle\alpha|\hat{E}^{(-)}(x_1)\hat{E}^{(+)}(x_2)|\alpha\rangle = \mathcal{K}^2|\alpha|^2\exp(i\vec{k}\cdot(\vec{r_1}-\vec{r_2})-\omega(t_2-t_1)).$$

deterministic complete optical coherence $\langle E^*(x_1)E(x_2)\rangle = \sqrt{\langle|E(x_1)|^2\rangle}\sqrt{\langle|E(x_2)|^2\rangle}$

As in the classical case, the key to first-order quantum coherence.

⇒ the factorization of the expectation value of the correlation function.

$$G^{(1)}(x_1,x_2) = \langle\hat{E}^{(-)}(x_1)\hat{E}^{(+)}(x_2)\rangle = \langle\hat{E}^{(-)}(x_1)\rangle\langle\hat{E}^{(+)}(x_2)\rangle.$$

→ satisfied for $|n\rangle$, $|\alpha\rangle$.

5.3 Young's interference

Assumption: (1) monochromatic light source.

(2) order : the source of light ~ wavelength of light.

(3) ignoring diffraction effects

(4) the pinhole = source of spherical wave

The field at the screen at position $\vec{r}$ at time t,

$$\hat{E}^{(+)}(\vec{r},t) = f(r)\left[\hat{a}_1 e^{iks_1} + \hat{a}_2 e^{iks_2}\right]e^{-i\omega t}$$

where $f(r) = i\left[\frac{\hbar\omega}{2\varepsilon_0(4\pi R)}\right]^{1/2}\frac{1}{r}$. R: radius of normalized volume

$s_1 \approx s_2 = r$, $k = |k_1| = |k_2|$, $r = |r|$

The field operator $\hat{a}_1, \hat{a}_2$: radial mode of the photon emitted from pinhole 1, 2.

The intensity

$$I(\vec{r},t) = \mathrm{Tr}\{\rho \hat{E}^{(-)}(\vec{r},t)\hat{E}^{(+)}(\vec{r},t)\} = |f(r)|^2\{\mathrm{Tr}(\hat{\rho}\hat{a}_1^{+}\hat{a}_1) + \mathrm{Tr}(\hat{\rho}\hat{a}_2^{+}\hat{a}_2)\} + 2|\mathrm{Tr}(\hat{\rho}\hat{a}_1^{+}\hat{a}_2)|\cos\Phi\}$$

$\mathrm{Tr}(\hat{\rho}\hat{a}_1^{+}\hat{a}_2) = |\mathrm{Tr}(\hat{\rho}\hat{a}_1^{+}\hat{a}_2)|e^{i\psi}$ and $\Phi = k(s_1 - s_2) + \psi$

Maximum visibility of the interference fringe : $\Phi = 2\pi m$

falls off as $1/r^2$ with increasing distance of the detector of the central fringe.

The beam falling onto the pinhole ~ plane wave mode. $(\hat{a})$

If pinholes are equal size and right behind of detectors

→ the incident photons has equally like propability.

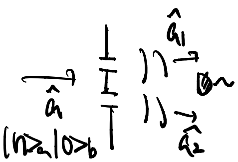
⇒ The two pinholes act to split a single beam into two beams.

$\hat{a} = \frac{1}{2}(\hat{a}_1 + \hat{a}_2)$, $[\hat{a}_i, \hat{a}_j^{+}] = \delta_{ij}$, $[\hat{a}, \hat{a}^{+}] = 1$.

↳ not unitary transformation matrix.

The "fictitious" mode to make it unitary

$\hat{b} = (\hat{a}_1 - \hat{a}_2)/\sqrt{2}$ where $[\hat{b}, \hat{b}^{+}] = 1$ always be in a vaccum state.



$$|n\rangle_a|0\rangle_b = \frac{1}{\sqrt{n!}}\hat{a}^{+n}|0\rangle_a|0\rangle_b$$

$$\rightarrow \frac{1}{\sqrt{n!}}\left(\frac{1}{\sqrt{2}}\right)^n(\hat{a}_1^{+} + \hat{a}_2^{+})^n|0\rangle_1|0\rangle_2$$

where $|0\rangle_1|0\rangle_2$: a_1 and a_2 mode vaccum states.

$$|1\rangle_a|0\rangle_b \rightarrow \frac{1}{\sqrt{2}}(|1\rangle_a|0\rangle_b + |0\rangle_a|1\rangle_b) = \frac{1}{\sqrt{2}}(|1,0\rangle + |0,1\rangle).$$

$\hat{b}\,|1\rangle_a|0\rangle_b = (\hat{a}_1 - \hat{a}_2)/\sqrt{2}\,|1\rangle_a|0\rangle_b = \frac{1}{\sqrt{2}}(\hat{a}_1 - \hat{a}_2)\frac{1}{\sqrt{2}}(|1,0\rangle + |0,1\rangle)$

$= \frac{1}{2}(|0,0\rangle - |0,0\rangle) = 0$

If the incident state is $|1\rangle_a|0\rangle_b$, the intensity in Young's experiment,

$$I(\vec{r},t) = |f(\vec{r})|^2\left\{\frac{1}{2}\langle 1,0|\hat{a}_1^\dagger\hat{a}_1|1,0\rangle + \frac{1}{2}\langle 0,1|\hat{a}_2^\dagger\hat{a}_2|0,1\rangle + \langle 1,0|\hat{a}_1^\dagger\hat{a}_2|0,1\rangle\cos\Phi\right\}$$

Dirac: "Each photon interferes only with itself, interference between different photons does not occur"

If a_1 and a_2 are truly independent, $|n_1\rangle_1|n_2\rangle_2$ doesn't give rise to interference fringe.

but coherent state from independent laser

$\Downarrow$

$$I(\vec{r},t) = |f(\vec{r})|^2(1+\cos\Phi)$$

$$I(\vec{r},t) = |f(\vec{r})|^2\left\{|\alpha_1|^2 + |\alpha_2|^2 + 2|\alpha_1^*\alpha_2|\cos\Phi\right\}$$

: exhibit interference fringe.

In case of two photons,

$$|2\rangle_a|0\rangle_b \to \frac{1}{2}(|2,0\rangle + \sqrt{2}|1,1\rangle + |0,2\rangle)$$

$$I(\vec{r},t) = 2|f(\vec{r})|^2|1+\cos\Phi|$$

General n-photon state.

$$I(\vec{r},t) = n|f(\vec{r})|^2|1+\cos\Phi|$$

$\hat{a} = \frac{1}{\sqrt{2}}(\hat{a}_1 + \hat{a}_2)$

For a coherent state incident on the pinhole,

$$|\alpha\rangle_a|0\rangle_b = \hat{D}_a(\alpha)|0\rangle_a|0\rangle_b = \hat{D}_1\left(\frac{\alpha}{\sqrt{2}}\right)\hat{D}_2\left(\frac{\alpha}{\sqrt{2}}\right)|0,0\rangle = \left|\frac{\alpha}{\sqrt{2}}\right\rangle_1\left|\frac{\alpha}{\sqrt{2}}\right\rangle_2.$$

where $D_i(\alpha/\sqrt{2}) = \exp\left(\frac{\alpha^*}{\sqrt{2}}\hat{a}_i - \frac{\alpha}{\sqrt{2}}\hat{a}_i^\dagger\right)$

The intensity, $I(\vec{r},t) = |\alpha|^2|f(\vec{r})|^2(1+\cos\Phi)$

Two interference modes $\Leftarrow$ same single mode on two holes.

: first order coherent giving rise to interference pattern.

Overall intensity : affected by photon number (average number)

5.4 Higher-order Coherence Functions.

First-order coherence in Young's interference.

⇒ As the result of factorization of the expectation values in the correlation function.

: the degree to which a light source is monochromatic, or to determine the coherence length

⇒ but! Say nothing about the statistical properties of the light.

⇒ ~~first-order~~ coherence exp. : distinguish not the lights between identical spectral distribution but quite different photon number distribution.

* Hanbury Brown and Twiss experiment.

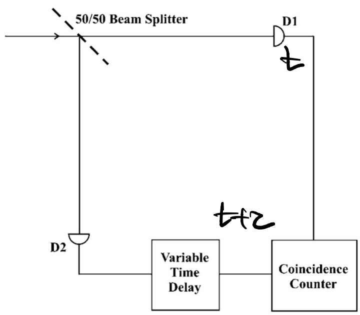


Fig. 5.3. Sketch of the setup for the Hanbury-Brown and Twiss experiment.

$\tau \ll \tau_0$ (coherence time)

The rate of coincident counts is proportional to times.

$$C(t, t+\tau) = \langle I(t) I(t+\tau) \rangle.$$

instantaneous intensity at the two detectors.

Assumption: the fields are stationary.

If the average intensity at each detector is $\langle I(t) \rangle$.

The probability of coincident count

$$\gamma^{(2)}(\tau) = \frac{\langle I(t) \rangle \langle I(t+\tau) \rangle}{\langle I(t) \rangle^2} = \frac{\langle E^*(t) E^*(t+\tau) E(t) E(t+\tau) \rangle}{\langle E^*(t) E(t) \rangle^2}$$

If the detectors are at difference distances, the second-order coherence function.

$$\gamma^{(2)}(x_1, x_2; x_2, x_1) = \frac{\langle I(x_1) I(x_2) \rangle}{\langle I(x_1) \rangle \langle I(x_2) \rangle} = \frac{\langle E^*(x_1) E^*(x_2) E(x_2) E(x_1) \rangle}{\langle |E(x_1)|^2 \rangle \langle |E(x_2)|^2 \rangle}$$

for $\gamma^{(2)}(x_1, x_2; x_2, x_1) = 1$, we need factorization

$$\langle E^*(x_1) E^*(x_2) E(x_2) E(x_1) \rangle = \langle |E(x_1)|^2 \rangle \langle |E(x_2)|^2 \rangle$$

For plane wave propagation in the z-direction.

$$\langle E^*(t) E^*(t+\tau) E(t+\tau) E(t) \rangle = E_0^4. \Rightarrow \gamma^{(2)}(\tau) = 1.$$

For nonfluctuating intensity, $I(t) = I(t+\tau) = I_0$, $\gamma^{(2)}(\tau) = 1$.

Second-order coherence function is not restricted to be unity or less

$$\gamma^{(2)}(0) = \frac{\langle I(t)^2 \rangle}{\langle I(t) \rangle^2} \quad \text{for zero-delay.}$$

For a sequence of N measurements,

$$\langle I(t) \rangle = \frac{I(t_1)+I(t_2)+\cdots+I(t_N)}{N}$$

$$\langle I(t)^2 \rangle = \frac{I(t_1)^2+I(t_2)^2+\cdots+I(t_N)^2}{N}$$

Cauchy's inequality applied to a pair of measurements.

$$2I(t_1)I(t_2) \leq I(t_1)^2 + I(t_2)^2.$$

In $\langle I(t) \rangle^2$,

$$\langle I(t)^2 \rangle \geq \langle I(t) \rangle^2 . \longrightarrow 1 \leq \gamma^{(2)}(0) \leq \infty .$$

For nonzero time delay, the positive of the intensity ensures $0 \leq \gamma^{(2)}(\tau) \leq \infty$.

From the inequality,

$$[I(t_1)I(t_1+\tau)\cdots I(t_N)I(t_N+\tau)]^2 \leq [I(t_1)^2+\cdots+I(t_N)^2][I(t_1+\tau)^2+\cdots+I(t_N+\tau)^2]$$

For long series of many measurement,

$$\langle I(t)I(t+\tau) \rangle \leq \langle I(t) \rangle^2 . \Rightarrow \gamma^{(2)}(\tau) \leq \gamma^{(2)}(0) \quad \text{limits for classical field light.}$$

Q.M. version violates this inequality.

For a light source containing a large number of independently radiating atoms undergoing collisional broadening,

$$\gamma^{(2)}(\tau) = 1 + |\gamma^{(1)}(\tau)|^2 . \quad : \text{holds for chaotic light.}$$

Since $0 \leq |\gamma^{(1)}(\tau)| \leq 1 \longrightarrow 1 \leq \gamma^{(2)}(\tau) \leq 2$.

With Lorentzian spectra, $\gamma^{(2)}(\tau) = 1 + e^{-2|\tau|/\tau_0}$.

$\tau \to \infty$, $\gamma^{(2)}(\tau) \to 1$. It is evident for zero time delay $\tau \to 0$, $\gamma^{(2)}(0) = 2$.

For any kind of chaotic light, $\gamma^{(2)}(0) = 2$.

If the lights are independent on detectors, there should be a uniform (independent on t) coincident rate.

$\Rightarrow$ HBT expected

They expected:

- Emitted independently by the source, the photon is splitted by BS,
- flat coincident rate.

The result:

- for zero time delay, twice the detection rate compared to the rate at long time delays.

⇒ they evidently arrive in pairs at zero time delay.

"Photon bunching". (exhibited by thermal light)

By measuring the coincidence counts at increasing delay times, it is possible to measure the coherence time τ_0 of the source.

Quantum-mechanically,

The first-order case to the detection of two photons by absorption.

The transition probability for the absorption of two photons

$$P \propto |\langle f|\hat{E}^{(+)}(\vec{r}_2, t_2)\hat{E}^{(+)}(\vec{r}_1, t_1)|i\rangle|^2 .$$

Summation on all the final state,

$$\langle i|\hat{E}^{(-)}(\vec{r}_1, t_1)\hat{E}^{(-)}(\vec{r}_2, t_2)\hat{E}^{(+)}(\vec{r}_2, t_2)\hat{E}^{(+)}(\vec{r}_1, t_1)|i\rangle$$

Cases of non-pure field states, second order quantum correlation function,

$$G^{(2)}(x_1, x_2 : x_2, x_1) = \mathrm{Tr}\{\hat{\rho}\,\hat{E}^{(-)}(x_1)\hat{E}^{(-)}(x_2)\hat{E}^{(+)}(x_2)\hat{E}^{(+)}(x_1)\}$$

→ interpreted as the ensemble average of $I(x_1)I(x_2)$

In the first-order case, the normal ordering of the field operator for absorptive detection.

(must be preserved)

$$g^{(2)}(x_1, x_2 : x_2, x_1) = \frac{G^{(2)}(x_1, x_2 ; x_2 : x_1)}{G^{(1)}(x_1, x_1)\, G^{(1)}(x_2, x_2)}$$

joint probability of detecting one photon at $(\vec{r}_1, t_1)$ and second at $(\vec{r}_2, t_2)$.

A quantum field is said to be "second order coherent" if $|g(x_1, x_2)| = 1$

and $g(x_1, x_2 ; x_2, x_1) = 1$.

It requires factorization,

$$G^{(2)}(x_1, x_2 ; x_2, x_1) = G^{(1)}(x_1, x_1)\, G^{(2)}(x_2, x_2).$$

At a fixed position,

$g^{(2)}$ depends only on the time difference $\tau = t_2 - t_1$

$$g^{(2)} = \frac{\langle \hat{E}^{(-)}(t)\hat{E}^{(-)}(t+\tau)\hat{E}^{(+)}(t+\tau)\hat{E}^{(+)}(t)\rangle}{\langle \hat{E}^{(-)}(t)\hat{E}^{(+)}(t)\rangle\langle \hat{E}^{(-)}(t+\tau)\hat{E}^{(+)}(t+\tau)\rangle}.$$

interpreted as the conditional probability if a photon is detected at time t_1, then one is also detected at time $t+\tau$.

For single-mode field of the form, $g^{(2)}(\tau)$ reduces to.

$$g^{(2)}(\tau) = \frac{\langle \hat{a}^{\dagger}\hat{a}^{\dagger}\hat{a}\hat{a}\rangle}{\langle \hat{a}^{\dagger}\hat{a}\rangle^2} = \frac{\langle \hat{n}(\hat{n}-1)\rangle}{\langle \hat{n}\rangle^2} = 1 + \frac{\langle(\Delta\hat{n})^2\rangle - \langle\hat{n}\rangle}{\langle\hat{n}\rangle^2}$$ independent on τ.

→ single mode field, $g^{(2)}(\tau) = g^{(2)}(0)$.

For a coherent state $|\alpha\rangle$,

$g^{(2)}(\tau) = g^{(2)}(0) = 1$. : second order coherent.

For a field in a single-mode thermal state (all other modes filters out),

$g^{(2)}(\tau) = g^{(2)}(0) = 2$. : higher probability of detecting coincident photons.

For a multi-mode (unfiltered) thermal state,

$g^{(2)}(\tau) = 1 + |g^{(1)}(\tau)|^2$. in the range $1 \leq g^{(2)}(\tau) \leq 2$.

For collision-broadened light with a Lorentzian spectrum, a first-order coherence function.

$g^{(1)}(\tau) = e^{-i\omega_0\tau - |\tau|/\tau_0}$. and $g^{(2)}(\tau) = 1 + e^{-2|\tau|/\tau_0}$.

For $|\tau| \ll \tau_0$, the probability getting two photon counts with time $|\tau|$ ↑↑

For zero delay, $g^{(2)}(0) = 2$, $g^{(2)}(\tau) < g^{(2)}(0)$. : this inequality characterizes photon bunching.

For a multimode coherent state, $g^{(2)}(\tau) = 1$.

⇒ the photons arrive randomly as per the Poisson distribution. (independent on τ).

Another case $g^{(2)}(0) < g^{(2)}(\tau)$: photon anti-bunching.

⇒ the probability of obtaining coincident photons in time interval τ is less than for a coherent state.

⇒ negative probability (meaningless for classical field).

Consider the single-mode field in a number state $|n\rangle$.

$$g^{(2)}(\tau) = g^{(2)}(0) = \begin{cases} 0 & n=0,1 \\ 1-\frac{1}{n} & n\geq 2. \end{cases}$$

classically forbidden $g^{(2)}(0)$: interpreted as a quantum-mechanical violation of Cauchy's inequality.

$g^{(2)}(0)<1 \Rightarrow \left(\langle(\Delta\hat{n})^2\rangle\right) < \left(\langle\hat{n}\rangle\right)$: sub-Poissonian. $Var(n) < \langle n\rangle$.

since $g^{(2)}(\tau)$ is constant for single-field mode, antibunching doesn't occur.

$\Rightarrow g^{(2)}(0) < g^{(2)}(\tau)$ for antibunching.

"Sub-Poissonian statistics and photon antibunching are different effects".

Photon statistics : Poissonian, super-Poissonian, sub-Poissonian.

$\Rightarrow$ calculation on Mandel's Q-parameter.

$$Q = \frac{\langle(\Delta\hat{n})^2\rangle - \langle\hat{n}\rangle}{\langle\hat{n}\rangle} = \langle\hat{n}\rangle\left(g^{(2)}(0)-1\right)$$

$Q=0$: Poissonian

$Q>0$: super-Poissonian

$-1<Q<0$: sub-Poissonian,) nonclassical state.

For $|\alpha\rangle$, $\langle\hat{n}\rangle = |\alpha|^2$, $\langle(\Delta n)^2\rangle = |\alpha|^2$. $\Rightarrow Q=0$.

n-th order coherence function.

$$G^{(n)}(x_1,\cdots,x_n;x_n,\cdots,x_1) = Tr\{\rho\, \hat{E}^{(-)}(x_1)\hat{E}^{(-)}(x_2)\cdots\hat{E}^{(+)}(x_n)\}.$$

$$g^{(n)}(x_1,\cdots,x_n;x_n,\cdots,x_1) = \frac{G^{(n)}(x_1,\cdots,x_n;x_n,\cdots,x_1)}{G^{(1)}(x_1,x_1)\cdots G^{(1)}(x_n,x_n)}$$

for fully coherent, it will be factorized.