Convergence of Geodesic and Euclidean distances on an Embedded Riemannian Manifold

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1 Introduction

Let $(\mathcal{M}, g)$ be a complete Riemannian manifold of dimension m embedded in $\mathbb{R}^n$. One conventional example is a 2-dimensional hypersphere $\mathbb{S}^2$ in 3-dimensional Euclidean space, as in Figure 1. For $A, B \in \mathbb{S}^2$, an easy way to connect two points would be a **straight line** in $\mathbb{R}^3$. However, if the domain is restricted to $\mathbb{S}^2$ (surface of the sphere), **geodesic** is the shortest path while preserving geometry of the underlying manifold.

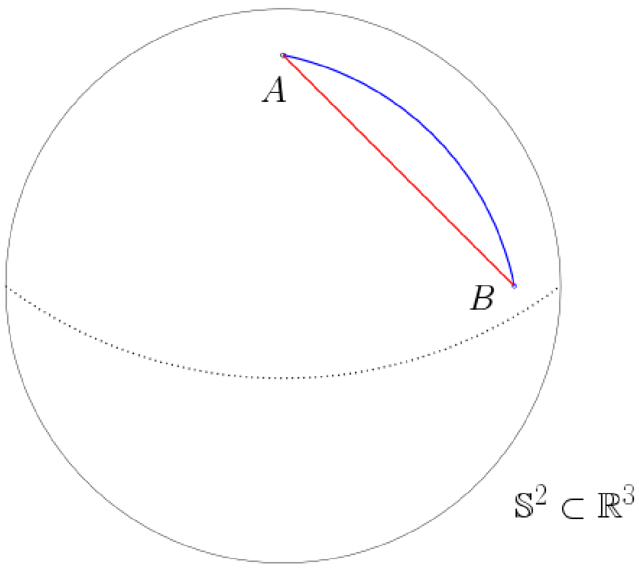

Figure 1: connecting two points $A, B \in \mathcal{M} = \mathbb{S}^2$ with **Euclidean** and **Geodesic** lines.

It is well known that two distances in an ambient space $\mathbb{R}^n$ (Euclidean) and on $\mathcal{M}$ (geodesic) converge in the limiting sense, i.e.,

$$\lim_{x \to y} \frac{d_{\mathcal{M}}(x, y)}{\|x - y\|} = 1 \tag{1}$$

where $\| \cdot \|$ is a norm in $\mathbb{R}^n$. Even though this is a conventional example in many textbooks, it will be detailed in the following.

2 Main Part

First, note that the geodesic distance is defined as $d_{\mathcal{M}}(x, y) = \int_I \|\gamma'(t)\|dt$ for $\gamma : I \to \mathcal{M}$ the geodesic curve connection x and y. Since the curve connecting two points is straight line, we have

$$d_{\mathcal{M}}(x, y) \geq \|x - y\| \iff \frac{d_{\mathcal{M}}(x, y)}{\|x - y\|} \geq 1 \text{ for all } x, y \in \mathcal{M}. \tag{2}$$

Thus, we only need to show the opposite direction of an inequality in Equation (2).

Without loss of generality, assume $x = 0 \in \mathcal{M}$ [1] and

https://math.stackexchange.com/questions/3263466/geodesic-distance-and-euclidean-distance-of-an-embedded-riemannian-manifold

References

[1] Leonid I. Rudin, Stanley Osher, and Emad Fatemi. Nonlinear total variation based noise removal algorithms. *Physica D: Nonlinear Phenomena*, 60(1-4):259–268, November 1992.

[1] This can be achieved by simply translating $\mathcal{M}$ to contain 0.