Laplace's Method to Approximate Integrals

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1 Introduction

Laplace's method [1] is an approximation technique for integrals of the form

$$\int_a^b e^{Nf(x)} dx \tag{1}$$

where $f(x)$ is a smooth (or at least twice-differentiable) function. This approximation scheme is one of the oldest yet frequently appearing in the context of statistics. In this note, we introduce it's derivation, variants, multivariate extension in a bit of details. We also elaborate several examples from Stirling's formula to Bayesian inference.

2 Formula

Say we have univariate smooth functions [1] $f, g : \mathbb{R} \to \mathbb{R}$ and assume $f(x)$ attains local maximum at x_0. That means, $f'(x_0) = 0$ and $f''(x_0) < 0$. Since the domain of integral is an interval $[a, b]$, we assume $x_0 \neq a$ and $x_0 \neq b$. Then using Taylor's expansion, we have Laplace's approximation in two versions

Version 1.

$$\int_a^b e^{Nf(x)} dx \approx e^{Nf(x_0)} \sqrt{\frac{2\pi}{N|f''(x_0)|}} \tag{2}$$

Version 2.

$$\int_a^b e^{Nf(x)} g(x) dx \approx e^{Nf(x_0)} g(x_0) \sqrt{\frac{2\pi}{N|f''(x_0)|}} \tag{3}$$

For multiavariate functions $f, g : \mathbb{R}^n \to \mathbb{R}$ with local maximum x_0, we have $\nabla f(x_0) = 0$ and $\nabla\nabla^\top f(x_0) = H_f(x_0) < 0$ for first and second-order conditions in multivariate calculus. Similar to univariate cases, we have following approximations,

- **Version 3.**

$$ver3 \tag{4}$$

[1] In fact, twice differentiability is only required.

- **Version 4.**

$$ver4 \tag{5}$$

Derivation of aforementioned approximation formula will be covered in Section 4.

3 Examples

1. Stirling's formula

For large $M \in \mathbb{Z}_+$, $M! = M \cdot (M-1) \cdots 2 \cdot 1$ can be approximated by $\sqrt{2\pi M} M^M e^{-M}$ using Gamma function representation. It is a well-known fact that $M! = \Gamma(M+1) = \int_0^\infty e^{-x} x^M dx$ in that the integral can be approximated by re-writing the integrand and applying Laplace's method.

$$\begin{aligned} M! &= \int_0^\infty e^{-x} x^M dx = \int_0^\infty e^{-x} e^{M \log x} \\ &= \int_0^\infty \exp\left(M \log x - x\right) dx. \end{aligned} \tag{6}$$

From Equation (2), set $N = 1$ and $f(x) = M \log x - x$ then we have

$$f'(x) = \frac{M}{x} - 1 \quad \text{and} \quad f''(x) = -\frac{M}{x^2}$$

in that f attains local extremum at $x_0 = M$ and it turns out to be a maximum since $f''(x_0) = -1/M < 0$. Therefore, continuing from Equation (6),

$$\begin{aligned} M! &= \int_0^\infty \exp(M \log x - x) dx \\ &= e^{f''(x_0)} \sqrt{\frac{2\pi}{|f''(x_0)|}} = e^{M \log M - M} \sqrt{\frac{2\pi}{\frac{1}{M}}} \\ &= \sqrt{2\pi M} M^M e^{-M}. \end{aligned}$$

4 Derivation

Version 1.

$$\int_a^b e^{Nf(x)} dx \approx e^{Nf(x_0)} \sqrt{\frac{2\pi}{N|f''(x_0)|}}$$

First, compute Taylor's expanson for $f(x)$ at $x = x_0$

$$
\begin{aligned}
f(x) &= f(x_0) + f'(x_0)(x - x_0) + f''(x_0)\frac{(x - x_0)^2}{2} + \epsilon \\
&\approx f(x_0) + f''(x_0)\frac{(x - x_0)^2}{2} \quad \text{since } f'(x_0) = 0 \text{ in that} \\
\int_a^b e^{Nf(x)}dx &\approx \int_a^b \exp(Nf(x_0))\exp\left(-\frac{N|f''(x_0)|(x - x_0)^2}{2}\right)dx \\
&\approx e^{Nf(x_0)}\int_a^b \exp\left(-\frac{N|f''(x_0)|(x - x_0)^2}{2}\right)dx \\
&\approx e^{Nf(x_0)}\int_\infty^\infty \exp\left(-\frac{N|f''(x_0)|(x - x_0)^2}{2}\right)dx
\end{aligned}
$$

where change of domain to $(-\infty, \infty)$ is justified by fast decay of the integrand outside of $[a, b]$

$$
\begin{aligned}
&= e^{Nf(x_0)}\sqrt{\frac{2\pi}{N|f''(x_0)|}}\sqrt{\frac{N|f''(x_0)|}{2\pi}}\int_{\mathbb{R}} \exp\left(-\frac{N|f''(x_0)|(x - x_0)^2}{2}\right)dx \\
&= e^{Nf(x_0)}\sqrt{\frac{2\pi}{N|f''(x_0)|}}
\end{aligned}
$$

where the last equation is from the fact that re-arranging the terms in integrand leads to somehow similar form of normal distribution by setting $\sigma^2 = 1/\{N|f''(x_0)|\}$ so that by the following

$$
\frac{1}{\sqrt{2\pi\sigma^2}}\int_{\mathbb{R}} \exp\left(-\frac{(x - x_0)^2}{2\sigma^2}\right) = 1
$$

do we have the simplified form as above.

Version 2.

$$
\int_a^b e^{Nf(x)}g(x)dx \approx e^{Nf(x_0)}g(x_0)\sqrt{\frac{2\pi}{N|f''(x_0)|}}
$$

Derivation for this version is very similar to that of the above by adding one more step of Taylor's expansion for $g(x)$ up to first order, i.e., $g(x) = g(x_0) + g'(x)(x - x_0)$.

$$
\begin{aligned}
\int_a^b e^{Nf(x)}g(x)dx &\approx e^{Nf(x_0)}\int_a^b g(x)\exp\left(-\frac{N|f''(x_0)|(x - x_0)^2}{2}\right)dx \\
&\approx e^{Nf(x_0)}\int_a^b \{g(x_0) + g'(x_0)(x - x_0)\}\exp\left(-\frac{N|f''(x_0)|(x - x_0)^2}{2}\right)dx \\
&\approx e^{Nf(x_0)}g(x_0)\int_{\mathbb{R}} \exp\left(-\frac{N|f''(x_0)|(x - x_0)^2}{2}\right)dx \\
&\quad + e^{Nf(x_0)}g'(x_0)\int_{\mathbb{R}} (x - x_0)\exp\left(-\frac{N|f''(x_0)|(x - x_0)^2}{2}\right)dx \qquad (7) \\
&\approx e^{Nf(x_0)}g(x_0)\sqrt{\frac{2\pi}{N|f''(x_0)|}} + 0 \\
&= e^{Nf(x_0)}g(x_0)\sqrt{\frac{2\pi}{N|f''(x_0)|}}.
\end{aligned}
$$

Equation (7) vanishes and we can check this by replacing $x - x_0$ with y, which leads to the following,

$$\begin{aligned}
\int_{\mathbb{R}} y \exp\left(-\frac{N|f''(x_0)|y^2}{2}\right) dy &= \int_{\mathbb{R}} -\frac{1}{N|f''(x_0)|} \cdot -N|f''(x_0)|y \exp\left(-\frac{N|f''(x_0)|y^2}{2}\right) dy \\
&= -\frac{1}{N|f''(x_0)|} \int_{\mathbb{R}} -N|f''(x_0)|y \exp\left(-\frac{N|f''(x_0)|y^2}{2}\right) dy \\
&= -\frac{1}{N|f''(x_0)|} \cdot \exp\left(-\frac{N|f''(x_0)|y^2}{2}\right)\Bigg|_{-\infty}^{\infty} \\
&= -\frac{1}{N|f''(x_0)|} \cdot (0 - 0) = 0.
\end{aligned}$$

5 online notes

Jordan's note

Lecture Slide for Multivariate Case

References

[1] Pierre Simon Laplace. Memoir on the probability of the causes of events. *Statistical Science*, 1(3):364–378, 1986.