Example 0.1 (problem)

When $\cos(x) = \frac{4}{5}$ and $0 \leq x \leq \frac{\pi}{2}$, $\sum_{k=0}^{\infty} \frac{\cos(kx)}{5^k} = \frac{a}{b}$ for some coprime positive integers a and b. Find $\frac{a}{b}$.

Utilizing Euler's formula, we find a complex definition for $\cos(x)$.

$$e^{ix} \overset{\text{Euler}}{=} \cos(x) + i\sin(x) \tag{1}$$

Substituting $-x$, we find that

$$\begin{aligned} e^{-ix} &\overset{\text{Euler}}{=} \cos(-x) + i\sin(-x) \\ &\overset{\text{def}}{=} \cos(x) - i\sin(x) \end{aligned} \tag{2}$$

Combining (1) and (2), we see that

$$\begin{aligned} 2\cos(x) &= e^{ix} + e^{-ix} \\ \cos(x) &= \frac{e^{ix} + e^{-ix}}{2} \end{aligned} \tag{3}$$

Thus, using (3), the original problem becomes

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{\cos(kx)}{5^k} &= \sum_{k=0}^{\infty} \frac{e^{ikx} + e^{-ikx}}{2 \cdot 5^k} \\ &= \frac{1}{2}\sum_{k=0}^{\infty} \frac{e^{ikx}}{5^k} + \frac{1}{2}\sum_{k=0}^{\infty} \frac{e^{-ikx}}{5^k} \\ &= \frac{1}{2}\sum_{k=0}^{\infty} \left(\frac{e^{ix}}{5}\right)^k + \frac{1}{2}\sum_{k=0}^{\infty} \left(\frac{e^{-ix}}{5}\right)^k \\ &= \frac{1}{2}\left(\frac{1}{1 - \frac{e^{ix}}{5}} + \frac{1}{1 - \frac{e^{-ix}}{5}}\right) \\ &= \frac{1}{2}\left(\frac{5}{5 - e^{ix}} + \frac{5}{5 - e^{-ix}}\right) \end{aligned} \tag{4}$$

We are given that $\cos(x) = \frac{4}{5}$. It follows then, by the fundamental theorem of trig, that $\sin(x) = \frac{3}{5}$. Thus we can apply euler's formula oncemore to find the values of e^{ix} and e^{-ix}

$$\begin{aligned} e^{ix} &= \cos(x) + i\sin(x) \\ &= \frac{4}{5} + \frac{3}{5}i \end{aligned} \tag{5}$$

$$
\begin{aligned}
e^{-ix} &= \cos(-x) + i\sin(-x) \\
&= \cos(x) - i\sin(x) \\
&= \frac{4}{5} - \frac{3}{5}i
\end{aligned} \tag{6}
$$

Finally, substituting (5) and (6) into equation (4), we find that

$$
\begin{aligned}
\sum_{k=0}^{\infty} \frac{\cos(kx)}{5^k} &= \frac{1}{2}\left(\frac{5}{5-e^{ix}} + \frac{5}{5-e^{-ix}}\right) \\
&= \frac{1}{2}\left(\frac{5}{5-\left(\frac{4}{5}+\frac{3}{5}i\right)} + \frac{5}{5-\left(\frac{4}{5}-\frac{3}{5}i\right)}\right) \\
&= \frac{1}{2}\left(\frac{25}{25-(4+3i)} + \frac{25}{25-(4-3i)}\right) \\
&= \frac{1}{2}\left(\frac{25}{21+3i} + \frac{25}{21-3i}\right) \\
&= \frac{1}{2}\left(\frac{25\,(21-3i)}{21^2+3^2} + \frac{25\,(21+3i)}{21^2+3^2}\right) \\
&= \frac{1}{2}\left(\frac{25(21) - 25(3)i + 25(3)i + 25(21)}{21^2+3^2}\right) \\
&= \frac{1}{2}\left(\frac{2(25)(21)}{21^2+3^2}\right) \\
&= \frac{7}{6}
\end{aligned}
$$