ASYMPTOTICS OF FRACTIONAL SOBOLEV NORMS AND s-PERIMETER

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Contents

1. Introduction

Theorem 1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. If $E \subseteq \mathbb{R}^n$ is a Caccioppoli set, then*

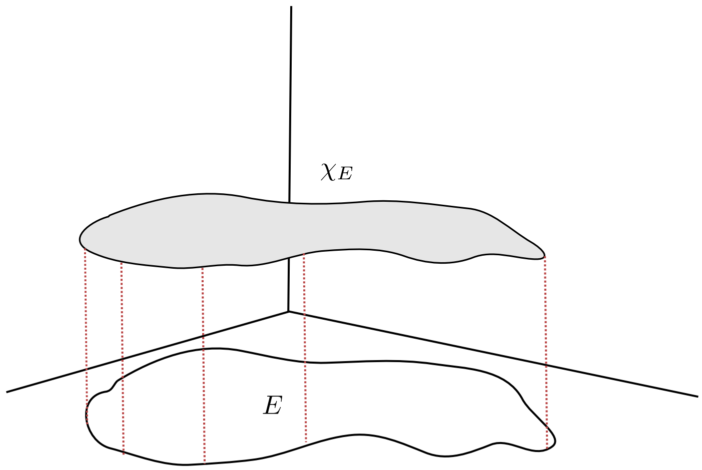
$$\lim_{s \to 1} (1-s) Per_s(E, \Omega) = \omega_{n-1} Per(E, \overline{\Omega}). \tag{1.1}$$

Theorem 2. *Let Ω be either a bounded open set with Lipschitz boundary or all of $\mathbb{R}^n$. Let $E \subseteq \Omega$ be a Caccioppoli set. Then*

$$\lim_{s \to 0} s Per_s(E, \Omega) = \frac{\omega_{n-1}}{2} \mathcal{L}^n(E) \tag{1.2}$$

2. Preliminary Definitions

3. Bounded Variation and Cacciopoli Sets



3.1. The Spaces BV and BV_{loc}. Our story begins with the classical Gauss-Green formula. Recall, when $\Omega \subseteq \mathbb{R}^n$ is open, bounded with smooth boundary, we have for every smooth $\mathbb{R}^n$-valued function Φ that

$$\int_{\Omega} \operatorname{div} \Phi \, d\mathcal{H}^n = \int_{\partial\Omega} \Phi \cdot \nu \, d\mathcal{H}^{n-1} \,, \tag{3.1}$$

where $\nu \in C^1(\partial\Omega, \mathbb{R}^n)$ is the outward pointing unit normal vector field on $\partial\Omega$.

De Giorgi's program in the 1950s revolved around trying to make sense of (3.1) when the topological boundary of Ω is no longer smooth. The following section will outline his work in this area. De Giorgi begins with the following idea. Suppose that now we have a set E which is not necessarily smooth. As the characteristic function χ_E is locally integrable in $\mathbb{R}^n$, we can consider χ_E as a distribution via integration against χ_E. Thus, it makes sense to talk about the distributional derivatives $D_i\chi_E$ of χ_E.

Assume that each distribution $D_i\chi_E$ is in fact represented by some Radon measure, which by abuse of notation we also write $D_i\chi_E$. Then the distributional gradient is in fact represented by the vector-valued Radon measure $D\chi_E = (D_1\chi_E, \ldots, D_n\chi_E)$.

Following this discussion, we then compute for smooth vector fields $\Phi = (\Phi^1, \ldots, \Phi^n) \in [\mathcal{D}(\mathbb{R}^n)]^n$,

$$\begin{aligned}\int_{\mathbb{R}^n} \chi_E \operatorname{div} \Phi \, dx &= \sum_{i=1}^n \int_{\mathbb{R}^n} \chi_E \frac{\partial \Phi^i}{\partial x_i} \, dx = \sum_{i=1}^n -\langle D_i\chi_E, \Phi^i \rangle \\ &= \sum_{i=1}^n - \int_{\mathbb{R}^n} \Phi^i \, dD_i\chi_E = - \int_{\mathbb{R}^n} \Phi \cdot dD\chi_E \,.\end{aligned}$$

Now let $|D\chi_E|$ denote the total variation measure of $D\chi_E$. Then by Radon-Nikodym, there exists a μ-measurable function $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ with $|\sigma| = 1$ $|D\chi_E|$-a.e. such that

$$dD\chi_E = \sigma \cdot d|D\chi_E| \,.$$

Then the above equation becomes

$$\int_E \operatorname{div} \Phi \, dx = \int_{\mathbb{R}^n} \chi_E \operatorname{div} \Phi \, dx = - \int_{\mathbb{R}^n} \Phi \cdot \sigma \, d|D\chi_E| \,.$$

The latter integral being over all of $\mathbb{R}^n$ is quite unsatisfactory, so it is a natural question to ask where the measure $|D\chi_E|$ is supported. Our intuition would point to $|D\chi_E|$ being supported on the boundary, since the quantity is some kind of "gradient" of χ_E which is constant everywhere else. Moreover, as there are vertical jumps on ∂E, it would make sense for the "gradient" at these jumps to be vertical of "infinite length" (that is, a dirac delta at every point on the boundary). Of course, this is just intuition and not rigorous mathematics, but it is not bad intuition.

Claim. $\text{supp}(D\chi_E) \subseteq \partial E$.

Proof. Suppose $z \in \mathbb{R}^n \setminus \partial E$. Then there is some open, bounded neighborhood U of z with smooth boundary such that $U \subseteq (\mathbb{R}^n \setminus \partial E)^o$. Thus U is either in the interior of E or the interior of $\mathbb{R}^n \setminus E$.

If $U \subseteq (\mathbb{R}^n \setminus E)^o$, then for $\Phi \in [\mathcal{D}(\mathbb{R}^n)]^n$ with $\text{supp}(\Phi) \subseteq U$, we have

$$\int_{\mathbb{R}^n} \Phi \cdot dD\chi_E = -\int_{\mathbb{R}^n} \chi_E \operatorname{div} \Phi \, dx = -\int_U \chi_E \operatorname{div} \Phi \, dx = 0.$$

If $U \subseteq E^o$, then for smooth vector fields supported within U we have by (3.1) that

$$\int_{\mathbb{R}^n} \Phi \cdot dD\chi_E = -\int_{\mathbb{R}^n} \chi_E \operatorname{div} \Phi \, dx = -\int_U \operatorname{div} \Phi \, dx = -\int_{\partial U} \Phi \cdot \nu_U \, d\mathcal{H}^{n-1} = 0.$$

By density, these formulae actually hold for all $\Phi \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$ with $\text{supp}(\Phi) \subseteq U$. Hence, $D\chi_E|_U \equiv 0$, so $z \notin \text{supp}(D\chi_E)$. □

Hence, setting $\nu = -\sigma$ (so ν is like a generalized outward normal vector field), we recover a statement which looks like Gauss Green:

$$\int_E \operatorname{div} \Phi \, d\mathcal{H}^n = \int_{\partial E} \Phi \cdot \nu \, d\|D\chi_E\|$$

We obtained such a formula by considering sets E such that the distributional gradient of χ_E is represented by a vector-valued Radon measure. More generally, we can consider integrable (or locally integrable) functions f whose distributional gradient is represented by a vector-valued Radon measure. This line of thought leads to the notion of *functions of bounded variation*.

Definition 3.1.1. Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $u \in L^1(\Omega)$. We say that u is of *bounded variation in* Ω, written $u \in BV(\Omega)$, if the the distributional derivatives $D_i u$ are represented by finite Radon measures, (that is, the distributional gradient Du is represened by a finite vector-valued Radon measure).

Given $u \in BV(\Omega)$, the $BV(\Omega)$-seminorm of u is the quantity

$$[u]_{BV(\Omega)} := |Du|(\Omega)$$

We would like a quantitative way to test whether a given function $u \in L^1(\Omega)$ is of bounded variation. The above definition of $BV(\Omega)$-seminorm is not satisfactory as it requires a function to have distributional derivative represented by a measure to be intelligible. We need a definition which makes no reference to the distributional derivative.

Definition 3.1.2. Given a function $u \in L^1(\Omega)$, define the *total variation of* u to be the quantity

$$V(u, \Omega) = \sup \left\{ \int_\Omega u \operatorname{div} \varphi \, dx : \varphi \in [C_c^1(\Omega, \mathbb{R})]^n, \|\varphi\|_\infty \leq 1 \right\}.$$

If $V(u, \Omega)$ is finite, then we say that u is of *bounded variation* and write $u \in BV(\Omega)$.

Definition 3.1.3. Similarly, given $u \in L^1_{loc}(\Omega)$ and $U \Subset \Omega$, define the *local variation of* u *in* U by

$$V(u, U) = \sup \left\{ \int_U u \operatorname{div} \varphi \, dx : \varphi \in [C_c^1(U, \mathbb{R})]^n, \|\varphi\|_\infty \leq 1 \right\}.$$

We define the set of functions of *locally bounded variation* to be

$$BV_{loc}(\Omega) = \{u \in L^1_{loc}(\Omega) : V(u, U) < +\infty \text{ for all } U \Subset \Omega\}.$$

An equivalent, and admittedly more transparent, characterization of BV_{loc} functions can be given as follows.

Proposition 3.1.1 (Characterization of BV_{loc})**.** *Suppose $u \in BV_{loc}(\Omega)$. Then there exists a Radon measure μ on Ω and a μ-measurable $\sigma : \Omega \to \mathbb{R}^n$ with $|\sigma| = 1$ μ-a.e. and*

$$\int_\Omega u \operatorname{div} \varphi \, dx = -\int_\Omega \varphi \cdot \sigma \, d\mu \ \text{ for all } \varphi \in C_c^1(\Omega, \mathbb{R}^n).$$

Proof. This is a routine application of the Riesz–Markov–Kakutani representation theorem. To this end, define a linear functional $L : C_c^1(\Omega, \mathbb{R}^n) \to \mathbb{R}$ by $L(\varphi) = -\int_\Omega u \operatorname{div} \varphi \, dx$.

For open $U \Subset \Omega$, the quantity $c(U) := \sup\{L(\varphi) : \varphi \in C_c^1(U, \mathbb{R}^n), \|\varphi\|_\infty \leq 1\}$ is finite by assumption, whence

$$|L(\varphi)| \leq c(U)\|\varphi\|_\infty \text{ for all } \varphi \in C_c^1(U, \mathbb{R}^n).$$

Let $K \subseteq \Omega$ be a fixed compact set, and choose open $U \Subset \Omega$ containing K. Then for $\varphi \in C_c(\Omega, \mathbb{R}^n)$ with $\operatorname{supp}(\varphi) \subseteq K$, there exists a sequence $(\varphi_k)_k$ in $C_c^1(U, \mathbb{R}^n)$ such that $\varphi_k \to \varphi$ uniformly on U.

Define an extension $\widetilde{L} : C_c(\Omega, \mathbb{R}^n) \to \mathbb{R}$ of L by $\widetilde{L}(\varphi) = \lim_{k \to \infty} L(\varphi_k)$, which exists and is well-defined by the above inequality. Applying the Riesz Representation Theorem to $\widetilde{L}$ gives the conclusion. ☐

Definition 3.1.4. For $u \in BV_{loc}(\Omega)$, we will write $\|Du\|$ for the measure μ and

$$d[Du] := \sigma \, d\|Du\| \, , \ \text{i.e} \int \cdot \, d[Du] = \int \langle \cdot, \sigma \rangle \, d\|Du\| \, .$$

Then the conclusion of Proposition 3.1.1 can be rewritten as

$$\int u \operatorname{div} \varphi \, dx = -\int \varphi \cdot \sigma \, d\|Du\| = -\int \varphi \cdot d[Du] \ \text{ for all } \varphi \in C_c^1(\Omega, \mathbb{R}^n).$$

4. Sobolev Space Preliminaries

The theory behind fractional perimeter is written in the language of fractional Sobolev spaces. As such, we will motivate and define these function spaces as well as discuss their most relevant properties.

4.1. **Fractional Sobolev Spaces.**

Definition 4.1.1. Fix $1 \leq p < +\infty$ and let $s \in (0, 1)$ be a fractional exponent. For $u \in L^p(\Omega)$, define the *Gagliardo (semi)norm* of u to be the quantity

$$[u]_{W^{s,p}(\Omega)} := \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dx \, dy \right)^{\frac{1}{p}} .$$

and define the *fractional Sobolev space* $W^{s,p}(\Omega) := \{u \in L^p(\Omega) : [u]_{W^{s,p}(\Omega)} < +\infty\}$. This is a Banach space with the natural norm

$$\|u\|_{W^{s,p}(\Omega)} := \|u\|_{L^p(\Omega)} + [u]_{W^{s,p}(\Omega)}.$$

We remark that $C_c^\infty(\Omega) \subseteq W^{s,p}(\Omega)$ and we write $W_0^{s,p}(\Omega)$ for the closure of $C_c^\infty(\Omega)$ inside $W^{s,p}(\Omega)$. It is a fact that when $\Omega = \mathbb{R}^n$, these two spaces are equal; however, this is not necessarily true for general Ω.

In the case $p = 2$, $W^{s,2}(\Omega)$ is in fact a Hilbert space with inner product given by

$$\langle u, v \rangle_{H^s(\Omega)} := \int_\Omega u(x)v(x) \, dx + \int_\Omega \int_\Omega \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} \, dx \, dy \, .$$

In a somewhat precise sense (real interpolation), the fractional sobolev spaces $W^{s,p}(\Omega)$ are intermediary spaces between $L^p(\Omega)$ and the classical Sobolev space $W^{1,p}(\Omega)$.

Althought these spaces and (semi)norms seem somewhat natural from the viewpoint of being an analogue of the Hölder condition for L^p spaces instead of L^∞, when presented as above they are ultimately quite artificial. Where would one find such spaces appearing in nature?

If one takes for granted that integer sobolev spaces "appear in nature," then the answer to the previous question is that *fractional sobolev spaces are the correct image of the trace operator* (see TODOINSERT THIS for background on the trace operator).

Proposition 4.1.1. *Suppose $\Omega \subseteq \mathbb{R}^n$ is a nice domain (see definition 4.2.1) and $k \in \mathbb{N}$. Then there is a split exact sequence of Hilbert spaces*

$$0 \longrightarrow W_0^{k,2}(\Omega) \hookrightarrow W^{k,2}(\Omega) \xrightarrow{T} W^{k-\frac{1}{2},2}(\partial\Omega) \longrightarrow 0$$

where $T : W^{k,2}(\Omega) \to W^{k-\frac{1}{2},2}(\partial\Omega)$ is the trace operator.

Proposition 4.1.2. *Let $p \in [1, +\infty)$ and $0 < s \leq s' < 1$. Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $u : \Omega \to \mathbb{R}$ a measurable function. Then there exists a constant $C \geq 1$ depending only on n,s, p, such that*

$$\|u\|_{W^{s,p}(\Omega)} \leq C\|u\|_{W^{s',p}(\Omega)}$$

Hence, there is a continuous inclusion $W^{s',p}(\Omega) \subseteq W^{s,p}(\Omega)$.

4.2. **Extension Domains.**

Definition 4.2.1. An open set $\Omega \subseteq \mathbb{R}^n$ is an *extension domain for $W^{s,p}$* if there exists a constant $C = C(s, p, n, \Omega) > 0$ such that for every $u \in W^{s,p}(\Omega)$, there exists a $\widetilde{u} \in W^{s,p}(\mathbb{R}^n)$ such that

$$\widetilde{u}|_\Omega \equiv u \quad \text{and} \quad \|\widetilde{u}\|_{W^{s,p}(\mathbb{R}^n} \leq C\|u\|_{W^{s,p}(\Omega)}.$$

We remark that any bounded open set with Lipschitz boundary is a $W^{s,1}$ extension domain for all $s \in (0, 1)$ (see Hitchhiker's Guide [4] for details) and a $W^{1,p}$ extension domain for all $1 \leq p < \infty$ (see Gilbarg and Trudinger [**<empty citation>**]). This fact somewhat explains why in both 2 and 1 we restrict to bounded open sets with Lipschitz boundary. One might imagine extending these results to general extension domains,

The general principle with fractional Sobolev spaces is that, on an extension domain, statements that for classical integer Sobolev spaces likely generalize to the fractional setting. One example of this is the intuitive notion that functions with high regularity automatically lie in all spaces requiring only lower regularity (that is, functions that are $k + 1$-differentiable are automatically k-differentiable). When Ω is a $W^{s,p}$ extension domain this intuition does not break.

Proposition 4.2.1 ([4, Prop. 2.2])**.** *Suppose $s \in (0, 1)$, $1 \leq p < \infty$, and let $\Omega \subseteq \mathbb{R}^n$ be a bounded $W^{1,p}$-extension domain. Then the identity map is a continuous embedding*

$$W^{1,p}(\Omega) \hookrightarrow W^{s,p}(\Omega).$$

Proof. □

We have seen that $BV(\Omega)$ is in some sense a replacement for $W^{1,1}(\Omega)$ with better compactness properties. Moreover, as is the case with classical Sobolev spaces, $W^{1,1}(\Omega)$ embeds continuously into $W^{s,1}(\Omega)$. Hence, one is lead to wonder if the same is true for BV instead of $W^{1,1}(\Omega)$. Under the assumption that Ω is a $W^{s,1}$-extension domain, this is true.

Proposition 4.2.2 ([5, Prop. 2.1])**.** *Suppose that $\Omega \subseteq \mathbb{R}^n$ is an extension domain. Then for $s \in (0, 1)$ we have a continuous embedding $BV(\Omega) \hookrightarrow W^{s,1}(\Omega)$.*

Proof. Suppose $u \in BV(\Omega)$. By mollification, there exists a sequence $u_i \in C^\infty(\Omega) \cap BV(\Omega)$ such that

- $u_n \xrightarrow{L^1(\Omega)} u$
- $\|Du_i\|_{L^1(\Omega)} \leq |Du|(\Omega)$ for all $i \in \mathbb{N}$,
- $\|Du_i\|_{L^1(\Omega)} \to |Du|(\Omega)$

Since Ω is a $W^{1,1}$ extension domain, by Proposition 4.2.1 the identity map is a continuous embedding $W^{1,1}(\Omega) \hookrightarrow W^{s,1}(\Omega)$.

$$[u_i]_{W^{s,1}(\Omega)} \leq \|u_i\|_{W^{s,1}(\Omega)} \leq C\|u_i\|_{W^{1,1}(\Omega)} = C\|u_i\|_{BV(\Omega)}.$$

Now, appealing to Fatou's lemma, we find

$$[u]_{W^{s,1}(\Omega)} \leq \liminf_{i\to\infty} \|u_i\|_{W^{s,1}(\Omega)} \leq \liminf_{i\to\infty} C\|u_i\|_{BV(\Omega)} = C\|u\|_{BV(\Omega)}.$$

Hence,

$$\|u\|_{W^{s,1}(\Omega)} = \|u\|_{L^1(\Omega)} + [u]_{W^{s,1}(\Omega)} \leq (C+1)\|u\|_{L^1(\Omega)} + C[u]_{BV(\Omega)} \leq (C+1)\|u\|_{BV(\Omega)}.$$

□

4.3. **The Fractional Laplacian.**

Definition 4.3.1. Fix $s \in (0,1)$. We define the fractional Laplacian $(-\Delta)^s : \mathscr{S} \to L^2(\mathbb{R}^n)$ as a Fourier multiplier given by

$$(-\Delta)^s u = \mathscr{F}^{-1}(|\xi|^{2s}(\mathscr{F}u)).$$

Proposition 4.3.1. *Fix $s \in (0,1)$ and let $C(n,s)$ be the constant*

$$C(n,s) := \left(\int_{\mathbb{R}^n} \frac{1-\cos(\zeta_1)}{|\zeta|^{n+2s}}\, d\zeta\right)^{-1}. \tag{4.1}$$

Then for $u \in \mathscr{S}$, we have that

$$(-\Delta)^s u(x) = C(n,s)\, P.V. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy \tag{4.2}$$

Proof. See Appendix B for the proof. □

4.4. H^s**: An Alternative approach to fractional Sobolev spaces using** $\mathscr{F}$**.**

Definition 4.4.1. Let $s \in (0,1)$. Consider the space

$$H^s(\mathbb{R}^n) := \{u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1+|\xi|^2)^s |\mathscr{F}u|^2\, d\xi < +\infty\}$$

equipped with the norm

$$[u]^2_{H^s} = \int_{\mathbb{R}^n} (1+|\xi|^2)^s |\mathscr{F}u|^2\, d\xi$$

Proposition 4.4.1 ([4, Prop. 3.4])**.** *Let $s \in (0,1)$ and $n \in \mathbb{N}$. Then for $u \in W^{s,2}(\mathbb{R}^n)$, we have*

$$[u]^2_{W^{s,2}(\mathbb{R}^n)} = \frac{2}{C(n,s)} \left\|(-\Delta)^{\frac{s}{2}} u\right\|^2_{L^2(\mathbb{R}^n)} = \frac{2}{C(n,s)} \int_{\mathbb{R}^n} |\xi|^{2s} |\mathscr{F}u|^2\, dx \tag{4.3}$$

Proof. This is an immediate corollary of Proposition 4.3.1. □

5. Detour: the Bourgain-Brezis-Mironescu Formula

Definition 5.0.1. Fron now on, $(\rho_i)_{i=1}^{\infty}$ will denote a sequence of *radial mollifiers* in the sense that

$$\rho_i \in L^1_{loc}((0,+\infty)), \quad \rho_i \geq 0 \tag{5.1}$$

$$\lim_{i\to\infty} \int_\delta^\infty \rho_i(r) r^{n-1}\, dr = 0 \text{ for all } \delta > 0 \tag{5.2}$$

$$\int_0^\infty \rho_i(r) r^{n-1}\, dr = 1 \text{ for all } i \in \mathbb{N}. \tag{5.3}$$

Theorem 5.0.1 ([1, 2], BBM Formula)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain. For $u \in L^1_{loc}(\Omega)$,*

$$\lim_{i\to\infty} \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^p} \rho_i(|x-y|)\, dx\, dy = \gamma_{n,p} \int_\Omega |Du|\, dx$$

when $Du \in L^p(\Omega)$ and $\gamma_{n,p}$ is a constant given by .

Theorem 5.0.2 ([3], BBM-Davila Formula)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain. Suppose $u \in BV(\Omega)$. Then*

$$\lim_{i\to\infty} \int_\Omega \int_\Omega \frac{|u(x)-u(y)|}{|x-y|} \rho_i(|x-y|)\, dx\, dy = \gamma_{n,1} \int_\Omega |Du|\, dx$$

where Du is the finite Radon measure corresponding to u and $\int_\Omega |Du|$ denotes the quantity $|Du|(\Omega)$.

The constant $\gamma_{n,p}$ above is given by

$$\gamma_{n,p} := \int_{S^{n-1}} |e \cdot \sigma|^p \, d\mathcal{H}^{n-1}(\sigma)$$

where $e \in S^{n-1}$ is arbitrary.

We will outline the proof of Theorem 5.0.2. This proof is the main technical novelty of this paper and we will connect it to asymptotics of fractional perimeter.

Consider the Radon measures μ_i given by

$$d\mu_i := \left(\int_\Omega \frac{|u(x)-u(y)|}{|x-y|} \rho_i(|x-y|)\, dx \right) dy$$

Davila proves an extension theorem analagous to but with an extra condition that, in addition to the BV norm in $\mathbb{R}^n$ being controlled, we also have control over the BV measure in neighborhoods of $\partial\Omega$.

Proposition 5.0.1 (Dávila [3])**.** *Existence of bounded extension operator $\mathcal{E} : BV(\Omega) \to BV(\mathbb{R}^n)$ for nice $\Omega \subseteq \mathbb{R}^n$.*

6. Fractional s-Perimeter

6.1. Why study the Asymptotics.

Fact 6.1.1. *For any measurable subset $E \subseteq \mathbb{R}^n$ of finite positive measure, the characteristic function χ_E is not an element of $H^{\frac{1}{2}}(\mathbb{R}^n)$.*

Proof. By assumption $\chi_E \in L^2(\mathbb{R}^n)$, so it suffices to show that χ_E is not in the corresponding homogenous sobolev space $\dot{H}^{\frac{1}{2}}(\mathbb{R}^n)$.

TODO INSERT SYMMETRIC REARRANGEMENT THINGS

Let B be the ball centered at 0 with the same (finite) measure as E, i.e. the symmetric decreasing rearrangement of the set E. (INSERT EXPOSITION ABOUT THIS)$\chi_E^* = \chi^B$ and we have that

$$\|\chi_E\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^n)} \geq \|\chi_E^*\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^n)} = \|\chi_B\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^n)}$$

$$\|\chi_B\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (1+|\xi|^2)^{\frac{1}{2}} |\widehat{\xi_B}(\xi)|^2 \, d\xi$$

To estimate this "symmetrized" Gagliardo seminorm, we must compute the fourier transform of the characteristic function of a ball. Let R be the radius of B and note that, as χ_B is rotationally symmetric, so is its Fourier transform. Hence we evaluate at the point $\xi = (0, 0, \ldots, 0, \rho)$ using polar coordinates

$$\begin{aligned} \widehat{\xi_{B_R(0)}}(\xi) &= \frac{1}{(2\pi)^{n/2}} \int_{B_R(0)} e^{-ix\cdot\xi}\, dx = \frac{1}{(2\pi)^{n/2}} \int_{B_R(0)} e^{-i\rho x_n}\, dx \\ &= \frac{1}{(2\pi)^{n/2}} \int_{S^{n-1}} \int_0^R e^{-i\rho x_n}\, dx \end{aligned}$$

$$\widehat{\chi_{B_R(0)}}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \chi_{B_R(0)}(x) e^{-ix\cdot\xi}\, dx = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \chi_{B_R(0)}(x) e^{-i\rho x_n}\, dx$$

$\square$

Fact 6.1.2. *For any measurable subset $E \subseteq \mathbb{R}^n$ of finite positive measure, the characteristic function χ_E is not an element of $W^{1,1}(\mathbb{R}^n)$.*

6.2. **Motivation for the definition of fractional perimeter.**

Definition 6.2.1. Let $\Omega \subseteq \mathbb{R}^n$ be a smooth bounded domain, $s \in (0,1)$, and $E \subseteq \mathbb{R}^n$ measurable. The *fractional s-perimeter of E in Ω* is the quantity

$$P_s(E,\Omega) := \frac{1}{2}[\chi_E]_{W^{s,1}(\Omega)} + \int_\Omega \int_{\mathbb{R}^n\setminus\Omega} \frac{|\chi_E(x)-\chi_E(y)|}{|x-y|^{n+s}}\, dx\, dy$$

where $L(A,B)$ denotes the following interaction energy integral

$$L(A,B) := \int_A \int_B \frac{1}{|x-y|^{n+s}}\, dx\, dy \text{ for all } A,B \subseteq \mathbb{R}^n$$

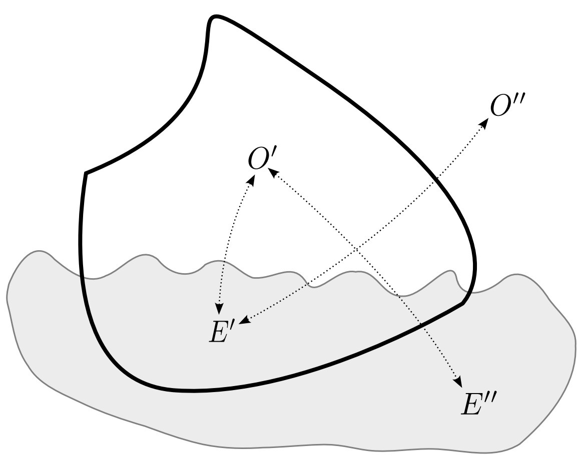
with the convention that $L(A,B) = 0$ if either A or B is empty. We denote the local and nonlocal energy terms by

$$P_s^L(E,\Omega) := \frac{1}{2}[\chi_E]_{W^{s,1}(\Omega)} \qquad P_s^{NL}(E,\Omega) := \int_\Omega \int_{\mathbb{R}^n\setminus\Omega} \frac{|\chi_E(x)-\chi_E(y)|}{|x-y|^{n+s}}\, dx\, dy$$

When $\Omega = \mathbb{R}^n$, we write $P_s(E) := P_s(E,\mathbb{R}^n)$.

To make this definition more transparent, consider the following reformulation:

$$\begin{aligned}
P_s(E,\Omega) &= \frac{1}{2}\int_\Omega\int_\Omega \frac{|\chi_E(x)-\chi_E(y)|}{|x-y|^{n+s}}\, dx\, dy + \int_\Omega \int_{\mathbb{R}^n\setminus\Omega} \frac{|\chi_E(x)-\chi_E(y)|}{|x-y|^{n+s}}\, dx\, dy \\
&= \frac{1}{2}\int_\Omega\int_\Omega \frac{\chi_E(x)\chi_{E^c}(y)+\chi_{E^c}(x)\chi_E(y)}{|x-y|^{n+s}}\, dx\, dy + \int_\Omega \int_{\mathbb{R}^n\setminus\Omega} \frac{\chi_E(x)\chi_{E^c}(y)+\chi_{E^c}(x)\chi_E(y)}{|x-y|^{n+s}}\, dx\, dy \\
&= L(E\cap\Omega, E^c\cap\Omega) + L(E^c\cap\Omega, E\cap\Omega^c) + L(E\cap\Omega, E^c\cap\Omega^c).
\end{aligned}$$



6.3. **Supplementary Observations.**

Lemma 6.3.1. *Let $A,B \subseteq \mathbb{R}^n$ be bounded measurable sets such that $dist(A,B) \geq C > 0$ for some $C > 0$. Then* $\lim_{s\to 0} sL(A,B) = 0$.

Proof.

$$sL(A,B) = s\int_A\int_B \frac{dx\,dy}{|x-y|^{n+s}} \le s\int_A\int_B \frac{1}{C^{n+s}}\,dx\,dy = \frac{s|A||B|}{C^{n+s}} \xrightarrow{s\to 0} 0$$

□

7. Asymptotics of $P_s(E,\Omega)$ as $s\to 0^+$

Proof of Theorem 2. Suppose first that $\Omega = \mathbb{R}^n$. Note that then $P_s^{NL}(E,\mathbb{R}^n) = 0$, so we are left with only a local term. Noting that $|\chi_E(x) - \chi_E(y)| = |\chi_E(x)-\chi_E(y)|^2$ and appealing to Proposition 4.4.1, we find

$$[\chi_E]_{W^{s,1}(\mathbb{R}^n)} = [\chi_E]^2_{W^{\frac{s}{2},2}(\mathbb{R}^n)} = \frac{2}{C(n,s)}\int_{\mathbb{R}^n} |\xi|^{2s}|\mathscr{F}u|^2\,d\xi\,.$$

Hence, after normalizing by s (TODO APPEAL TO ASYMPTOTICS OF $C(n,s)$) and applying the monotone convergence theorem, we compute

$$\begin{aligned}\lim_{s\to 0} sP_s(E,\mathbb{R}^n) &= \lim_{s\to 0^+}\frac{s}{2}[\chi_E]_{W^{s,1}(\mathbb{R}^n)} = \lim_{s\to 0^+}\frac{s}{C(n,s)}\int_{\mathbb{R}^n}|\xi|^{2s}|\mathscr{F}\chi_E|^2\,d\xi\\ &= \frac{\omega_{n-1}}{2}\|\mathscr{F}\chi_E\|^2_{L^2(\mathbb{R}^n)} = \frac{\omega_{n-1}}{2}\mathcal{L}^n(E)\end{aligned}$$

Now we look at the case where $\Omega\subseteq\mathbb{R}^n$ is a bounded domain with Lipschitz boundary. As $E\subseteq\Omega$, the interactions in Definition 6.2.1 simplify to

$$P_s(E,\Omega) = L(E,\Omega\setminus E) + L(E,\Omega^c) = L(E,E^c) = \frac{1}{2}[\chi_E]_{W^{s,1}(\mathbb{R}^n)},$$

whence as before,

$$\lim_{s\to 0^+} sP_s(E,\Omega) = \lim_{s\to 0^+} sL(E,E^c) = \lim_{s\to 0^+}\frac{1}{2}[\chi_E]_{W^{s,1}(\mathbb{R}^n)} = \frac{\omega_{n-1}}{2}\mathcal{L}^n(E).$$

8. Asymptotics of $P_s(E,\Omega)$ as $s\to 1^-$

8.1. Local Contribution to $P_s(E,\Omega)$.

In this subsection, we estimate the contribution of P_s^L to the s-perimeter as $s\to 1^-$. Our main tool is the Bourgain-Brezis-Mironescu-Davila Formula 5.0.2.

Recall that $P_s^L(E,\Omega) = \frac{1}{2}[\chi_E]_{W^{s,1}(\Omega)}$. Let $(s_i)_{i=1}^\infty$ be a sequence of integers with $s_i\to 1^-$. Choose $R >> 0$ such that $\Omega\subseteq B_{\frac{R}{2}}(0)$. Define mollifiers $\rho_i : (0,+\infty)\to[0,+\infty)$ as in 5.0.2 by

$$\rho_i(t) := a_{s_i}\frac{1-s_i}{t^{n+s_i-1}}\chi_{(0,R)}(t) \tag{8.1}$$

Where $a_{s_i} := \frac{1}{\omega_{n-1}R^{1-s_i}}$ is chosen such that $\int_{\mathbb{R}^n}\rho_i(|x|)\,dx = 1$. Using polar coordinates, one may check that the mollifiers $(\rho_i)_i$ satisfy the conditions of 5.0.2, namely that

$$\text{for all } \delta>0,\quad \lim_{i\to\infty}\int_\delta^\infty \rho_i(r)r^{n-1}\,dr = \lim_{i\to\infty}\frac{1}{\omega_{n-1}}\left(1-\left(\frac{\delta}{R}\right)^{1-s_i}\right) = 0.$$

For $u\in W^{s,1}(\Omega)$, as $\Omega-\Omega\subseteq B_R(0)$, we expand

$$\begin{aligned}\int_\Omega\int_\Omega \frac{|u(x)-u(y)|}{|x-y|}\rho_i(|x-y|)\,dx\,dy &= a_{s_i}(1-s_i)\int_\Omega\int_\Omega\frac{|u(x)-u(y)|}{|x-y|^{n+s_i}}\,dx\,dy\\ &= a_{s_i}(1-s_i)[u]_{W^{s_i,1}(\Omega)}.\end{aligned}$$

Then for $u \in BV(\Omega)$,

$$\begin{aligned}\lim_{i\to\infty}(1-s_i)[u]_{W^{s_i,1}(\Omega)} &= \lim_{i\to\infty}\frac{1}{a_{s_i}}\int_\Omega\int_\Omega \frac{|u(x)-u(y)|}{|x-y|}\rho_i(|x-y|)\,dx\,dy \\ &= \omega_{n-1}K_{1,n}[u]_{BV(\Omega)}.\end{aligned}$$

As the sequence (s_i) was arbitrary, we conclude that $\lim_{s\to1^-}(1-s)[u]_{W^{s,1}(\Omega)} = \omega_{n-1}K_{n,1}[u]_{BV(\Omega)}$. In summary, we have shown the following proposition.

Proposition 8.1.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. If $E \subseteq \mathbb{R}^n$ has finite perimeter inside Ω, then*

$$\lim_{s\to1^-}(1-s)P_s^L(E,\Omega) = \frac{\omega_{n-1}}{2}K_{n,1}Per(E,\Omega).$$

8.2. **Nonlocal Contribution to $P_s(E,\Omega)$.** The aim of this subsection is to show that the limiting behavior of the nonlocal contribution to s-perimeter, $P_s^{NL}(E,\Omega)$, is controlled by how much the set E fails to intersect $\partial\Omega$ "transversally." Namely, we will demonstrate that

$$\limsup_{s\to1^-}(1-s)P_s^{NL}(E,\Omega) \preceq \lim_{\delta\to0^+} Per(E, N_\delta(\partial\Omega))$$

where $N_\delta(\partial\Omega)$ denotes the δ-tubular neighborhood of $\partial\Omega$. The results and methods in this subsection are new developments due to Lombardini [5].

Recall the following fact from point-set topology.

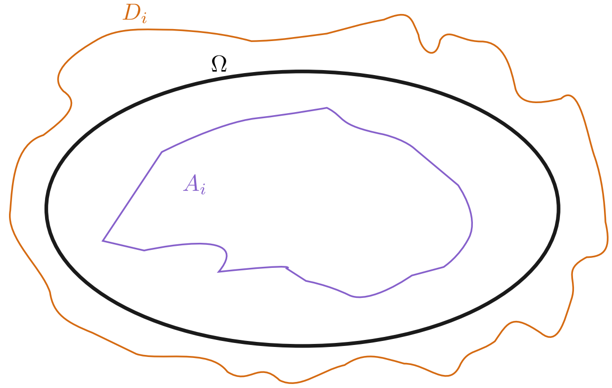
Fact 8.2.1. *If $X \subseteq \mathbb{R}^n$ is nonempty and $y \in \mathbb{R}^n \setminus \operatorname{Int} X$, then $d(y,X) = d(y,\partial X) = d(y,\partial X^c)$.*

Theorem 8.2.1 ([5, Prop. 2.5])**.** *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Suppose that $E \subseteq \mathbb{R}^n$ has finite perimeter inside Ω and $Per(E, \partial\Omega) = 0$. Then*

$$\lim_{s \to 1^-} (1-s) P_s^{NL}(E, \Omega) = 0$$

We strictly approximate Ω from inside and outside by bounded open sets with Lipschitz boundary, i.e. consider sets $(A_i)_i, (D_i)_i$ such that

- $A_i \subseteq A_{i+1} \Subset \Omega$ and $\Omega = \bigcup_{i \in \mathbb{N}} A_i$;
- $D_i \supseteq D_{i+1} \Supset \overline{\Omega}$ and $\overline{\Omega} = \bigcap_{i \in \mathbb{N}} D_i$.



We make a couple auxiliary definitions.

- $\Omega_i^+ := D_i \setminus \overline{\Omega}$, the outer excess portion;
- $\Omega_i^- := \Omega \setminus \overline{A_i}$, the inner excess portion;
- $T_i := \Omega_i^+ \cup \Omega_i^- \cup \partial\Omega$, the total excess;
- $d_i := \min\{d(A_i, \partial\Omega), d(\partial D_i, \Omega)\}$ minimum distance from approximation to the boundary of Ω.

As $|D\chi_E|$ is a finite Radon measure on D_1 and all other relevant sets are nested inside D_1, continuity from above implies that

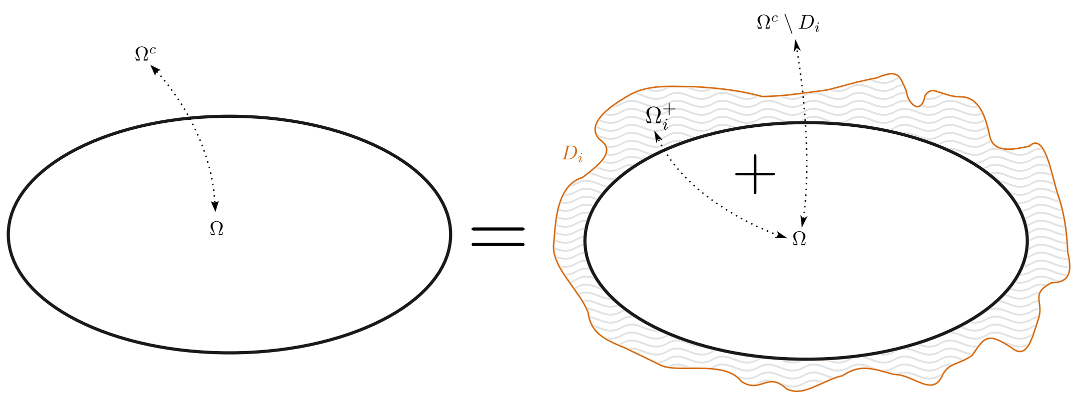
$$Per(E, \partial\Omega) = |D\chi_E|(\cap_i T_i) = \lim_{i \to \infty} |D\chi_E|(T_i) = \lim_{i \to \infty} Per(E, T_i).$$

Hence, in order to control the nonlocal contribution ot the s-perimeter in terms of $Per(E, \partial\Omega)$, it suffices to obtain control in terms of $Per(E, T_i)$ for fixed i. First, recall the two interactions involved in P_s^{NL} are given by

$$P_s^{NL}(E, \Omega) = L(E \cap \Omega, E^c \cap \Omega^c) + L(E^c \cap \Omega, E \cap \Omega^c) \tag{8.2}$$

As $P_s^{NL}(E, \Omega) = P_s^{NL}(E^c, \Omega)$ and $Per(E, T_i) = Per(E^c, T_i)$, it suffices to estimate only the first interaction term as an estimate for the second term would follow from this via the replacement $E \rightsquigarrow E^c$. We proceed by decomposing the interaction as

$$L(E \cap \Omega, E^c \setminus \Omega) = L(E \cap \Omega, E^c \cap \Omega_i^+) + L(E \cap \Omega, E^c \cap (\Omega^c \setminus D_i)). \tag{8.3}$$



To estimate these interactions, we need the following claim.

Claim. If $y \in \Omega$ and $\rho_y := \sup\{\rho > 0 : B_\rho(y) \subseteq D_i\}$, then $\rho_y \geq d_i$ and thus $B_{d_i}(y) \subseteq D_i$.

This claim follows by taking $X = D_i^c$ in Fact 8.2.1 and then noting that $B_{d(y, D_i^c)}(y) \subseteq D_i$ holds by definition.

Now for $y \in \Omega$, by the following claim we estimate

$$\int_{\Omega^c \setminus D_i} \frac{1}{|x-y|^{n+s}}\, dx \leq \int_{\mathbb{R}^n \setminus B_{d_i}(y)} \frac{1}{|x-y|^{n+s}}\, dx = \frac{\omega_{n-1}}{s} d_i^{-s}. \tag{8.4}$$

Hence, after integrating over Ω,

$$L(E\cap\Omega, E^c\cap(\Omega^c\setminus D_i)) \le L(\Omega,\Omega^c\setminus D_i) = \int_\Omega\int_{\Omega^c\setminus D_i}\frac{1}{|x-y|^{n+s}}\,dx\,dy \le \omega_{n-1}\frac{\mathcal{L}^n(\Omega)}{s}d_i^{-s} \tag{8.5}$$

giving us a bound on the second term in 8.3. To deal with the first term, we use a similar technique as in 8.3. We decompose the interaction with respect to the inner approximation $A_i \nearrow \Omega$ and its inner excess Ω_i^- to obtain

$$L(E\cap\Omega, E^c\cap\Omega_i^+) = L(E\cap\Omega_i^-, E^c\cap\Omega_i^+) + L(E\cap A_i, E^c\cap\Omega_i^+). \tag{8.6}$$

We analyze the second term in 8.6 and use similar logic to the estimate in 8.4 to find

$$L(E\cap A_i, E^c\cap\Omega_i^+) \le L(A_i,\Omega_i^+) \le L(A_i,\Omega^c) = \int_{A_i}\int_{\Omega^c}\frac{1}{|x-y|^{n+s}}\,dx\,dy \le \omega_{n-1}\frac{\mathcal{L}^n(A_i)}{s}d_i^{-s}. \tag{8.7}$$

To deal with the first interaction term in 8.6, we note that both Ω_i^- and Ω_i^+ are contained in T_k. Hence by definition, we see

$$L(E\cap\Omega_i^-, E^c\cap\Omega_i^+) \le L(E\cap T_i, E^c\cap T_i) = P_s^L(E,T_i) \tag{8.8}$$

which, in the context of Proposition 8.1.1, hints towards an asymptotic connection between the energy in question and the classical perimeters of E relative to the T_is. Now, combining estimates 8.5, 8.7, and 8.8, we have

$$L(E\cap\Omega, E^c\cap\Omega^c) \le P_s^L(E,T_i) + 2\omega_{n-1}\frac{\mathcal{L}^n(\Omega)}{s}d_i^{-s}.$$

As discussed previously, the same estimate holds for the other term in 8.2, so our final energy bound becomes

$$P_s^{NL}(E,\Omega) \le 2P_s^L(E,T_i) + 4\omega_{n-1}\frac{\mathcal{L}^n(\Omega)}{s}d_i^{-s}. \tag{8.9}$$

By Proposition 8.1.1,

$$\lim_{s\to1^-}(1-s)P_s^L(E,T_i) = \frac{\omega_{n-1}}{2}K_{n,1}Per(E,T_i)$$

hence 8.9 implies

$$\limsup_{s\to1^-}(1-s)P_s^{NL}(E,\Omega) \le \omega_{n-1}K_{n,1}Per(E,T_i).$$

As this holds for all i, it follows that

$$\limsup_{s\to1^-}(1-s)P_s^{NL}(E,\Omega) \le \lim_{i\to\infty}\omega_{n-1}K_{n,1}Per(E,T_i) = \omega_{n-1}K_{n,1}Per(E,\partial\Omega).$$

When the RHS is zero, the claim follows.

Appendix A. Symmetric Decreasing Rearrangement

Appendix B. Miscellaneous Fractional Proofs

Proof of Proposition 4.3.1. Let $\Lambda_s : \mathscr{S} \to L^2(\mathbb{R}^n)$ denote the operator $\Lambda_s u(x) := C(n,s)\,P.V.\int_{\mathbb{R}^n}\frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy$. After applying the ansatzes $y \rightsquigarrow x+h$ and $y \rightsquigarrow x-h$, we have the second order difference quotient representation

$$\Lambda_s u(x) = -\frac{1}{2}C(n,s)\,P.V.\int_{\mathbb{R}^n}\frac{u(x+h)+u(x-h)-2u(x)}{|h|^{n+2s}}\,dh\,. \tag{B.1}$$

Fix $u\in\mathscr{S}$ and $x\in\mathbb{R}^n$. Consider the second-order Taylor expansion of u about x given for h small by

$$u(x+h) = u(x) + Du(x)\cdot h + \frac{1}{2}h^T\cdot D^2u(x)\cdot h + R(h) \tag{B.2}$$

where $R(h)\in o(|h|^2)$. Then

$$u(x+h)+u(x-h)-2u(x) = h^T\cdot D^2u(x)\cdot h + R(h) + R(-h).$$

and

$$|h^T D^2 u(x) h| \le |h| \cdot |D^2 u(x) h| \le \left\| D^2 u(x) \right\|_{op} |h|^2,$$

leading to a bound on the integral kernel

$$\frac{u(x+h)+u(x-h)-2u(x)}{|h|^{n+2s}} \le \frac{\left\| D^2 u(x) \right\|_{op}}{|h|^{n+2s-2}} + \frac{1}{|h|^{n+2s-2}} \cdot \frac{R(h)+R(-h)}{|h|^2}. \tag{B.3}$$

Note that B.3 is integrable in h within a bounded neighborhood of 0, so in fact the equation B.1 holds true even without the "PV." Moreover, one may refine the estimate B.3 slightly further to justify an application of Fubini-Tonelli and find that, for $\xi \in \mathbb{R}^n$,

$$\begin{aligned}
\mathscr{F}_x(\Lambda_s u)(\xi) &= -\frac{1}{2} C(n,s) \int_{\mathbb{R}^n} \frac{\mathscr{F}\{u(\bullet + h) + u(\bullet - h) - 2u(\bullet)\}(\xi)}{|h|^{n+2s}} \, dh \\
&= -\frac{1}{2} C(n,s) \mathscr{F} u(\xi) \int_{\mathbb{R}^n} \frac{e^{-ih\xi} + e^{ih\xi} - 2}{|h|^{n+2s}} \, dh \\
&= C(n,s) \mathscr{F} u(\xi) \int_{\mathbb{R}^n} \frac{1-\cos(h \cdot \xi)}{|h|^{n+2s}} \, dh
\end{aligned}$$

It is an exercise to the reader to show that the quantity $I(\xi) := \int_{\mathbb{R}^n} \frac{1-\cos(h\cdot\xi)}{|h|^{n+2s}} \, dh$ is rotation invariant and $I(\xi) = |\xi|^{2s} I(e_1) = |\xi|^{2s} C(n,s)^{-1}$, whence

$$\mathscr{F}(\Lambda_s u)(\xi) = |\xi|^{2s} \mathscr{F} u(\xi) \implies \Lambda_s u(x) = (-\Delta)^s u(x).$$

$\square$

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