ASYMPTOTICS OF FRACTIONAL SOBOLEV NORMS AND s-PERIMETER

JAMES HARBOUR

Contents

1. Introduction

First introduced by Caffarelli in 2011 [5, 6], the study of the fractional perimeter (and more generally nonlocal perimeters) has devloped into a highly active area of research. We mention the 2019 monograph by Mazón [14] for an overview of the development of the field.

The quantity known as fractional perimeter initally arose as the energy functional for a nonlocal version of motion by mean curvature. We will give a much more elementary motivation in this paper by appealing to physical intuition.

Recall Newton's law of gravitation which states that the size of the gravitational interaction force between two point masses is inversely proportional to the square of the distance between them. Likewise, Coulomb's law states that the size of the electrostatic force between two point charges is inversely proportional to the square of the distance between them.

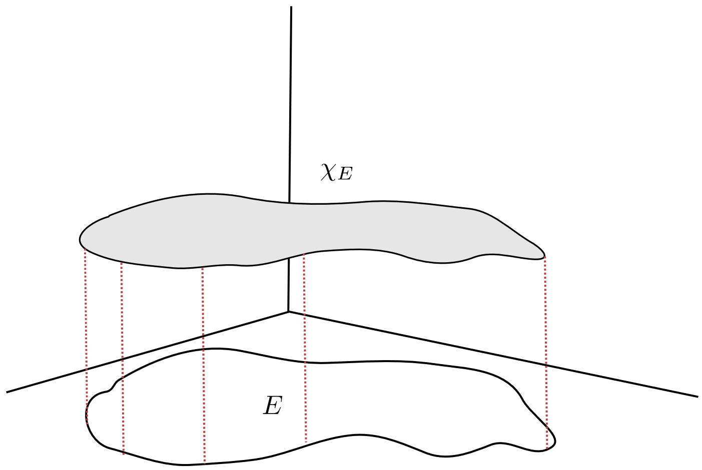
In both of these examples, we have some "interaction force" between two particles $\mathbf{r}, \mathbf{r'}$ given by $\frac{1}{|\mathbf{r}-\mathbf{r'}|^2}$. In sense, the power of 2 in the denominator controls how strongly this force quantity weights distant interactions.

Theorem 1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. If $E \subseteq \mathbb{R}^n$ is a Caccioppoli set, then*

$$\lim_{s\to 1}(1-s)Per_s(E,\Omega) = \omega_{n-1}Per(E,\overline{\Omega}). \tag{1.1}$$

Theorem 2. *Let Ω be either a bounded open set with Lipschitz boundary or all of $\mathbb{R}^n$. Let $E \subseteq \Omega$ be a Caccioppoli set. Then*

$$\lim_{s\to 0} sPer_s(E,\Omega) = \frac{\omega_{n-1}}{2}\mathcal{L}^n(E) \tag{1.2}$$



2.1. **The Spaces BV and BV_{loc}.** Our story begins with the classical Gauss-Green formula. Recall, when $\Omega \subseteq \mathbb{R}^n$ is open, bounded with C^1 boundary, we have for every smooth $\mathbb{R}^n$-valued function Φ that

$$\int_\Omega \operatorname{div} \Phi \, d\mathcal{H}^n = \int_{\partial\Omega} \Phi \cdot \nu \, d\mathcal{H}^{n-1} \,, \tag{2.1}$$

where $\nu \in C^1(\partial\Omega, \mathbb{R}^n)$ is the outward pointing unit normal vector field on $\partial\Omega$.

De Giorgi's program in the 1950s revolved around trying to make sense of (2.1) when the topological boundary of Ω is no longer smooth. The following section will outline his work in this area. De Giorgi begins with the following idea. Suppose that now we have a set E which is not necessarily smooth. As the characteristic function χ_E is locally integrable in $\mathbb{R}^n$, we can consider χ_E as a distribution via integration against χ_E. Thus, it makes sense to talk about the distributional derivatives $D_i\chi_E$ of χ_E.

Assume that each distribution $D_i\chi_E$ is in fact represented by some Radon measure, which by abuse of notation we also write $D_i\chi_E$. Then the distributional gradient is in fact represented by the vector-valued Radon measure $D\chi_E = (D_1\chi_E, \ldots, D_n\chi_E)$.

Following this discussion, we then compute for smooth vector fields $\Phi = (\Phi^1, \ldots, \Phi^n) \in [\mathcal{D}(\mathbb{R}^n)]^n$,

$$\begin{aligned} \int_{\mathbb{R}^n} \chi_E \operatorname{div} \Phi \, dx &= \sum_{i=1}^n \int_{\mathbb{R}^n} \chi_E \frac{\partial \Phi^i}{\partial x_i} \, dx = \sum_{i=1}^n -\langle D_i\chi_E, \Phi^i \rangle \\ &= \sum_{i=1}^n - \int_{\mathbb{R}^n} \Phi^i \, dD_i\chi_E = -\int_{\mathbb{R}^n} \Phi \cdot dD\chi_E \,. \end{aligned}$$

Now let $|D\chi_E|$ denote the total variation measure of $D\chi_E$. Then by Radon-Nikodym, there exists a μ-measurable function $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ with $|\sigma| = 1$ $|D\chi_E|$-a.e. such that

$$dD\chi_E = \sigma \cdot d|D\chi_E| \,.$$

Then the above equation becomes

$$\int_E \operatorname{div} \Phi \, dx = \int_{\mathbb{R}^n} \chi_E \operatorname{div} \Phi \, dx = -\int_{\mathbb{R}^n} \Phi \cdot \sigma \, d|D\chi_E| \,.$$

The latter integral being over all of $\mathbb{R}^n$ is quite unsatisfactory, so it is a natural question to ask where the measure $|D\chi_E|$ is supported. Our intuition would point to $|D\chi_E|$ being supported on the boundary, since the quantity is some kind of "gradient" of χ_E which is constant everywhere else. Moreover, as there are vertical jumps on ∂E, it would make sense for the "gradient" at these jumps to be vertical of "infinite length" (that is, a dirac delta at every point on the boundary). Of course, this is just intuition and not rigorous mathematics, but it is not bad intuition (see Section 2.2).

Claim. $\operatorname{supp}(D\chi_E) \subseteq \partial E$.

Proof. Suppose $z \in \mathbb{R}^n \setminus \partial E$. Then there is some open, bounded neighborhood U of z with smooth boundary such that $U \subseteq (\mathbb{R}^n \setminus \partial E)^o$. Thus U is either in the interior of E or the interior of $\mathbb{R}^n \setminus E$.

If $U \subseteq (\mathbb{R}^n \setminus E)^o$, then for $\Phi \in [\mathcal{D}(\mathbb{R}^n)]^n$ with $\operatorname{supp}(\Phi) \subseteq U$, we have

$$\int_{\mathbb{R}^n} \Phi \cdot dD\chi_E = -\int_{\mathbb{R}^n} \chi_E \operatorname{div} \Phi \, dx = -\int_U \chi_E \operatorname{div} \Phi \, dx = 0.$$

If $U \subseteq E^o$, then for smooth vector fields supported within U we have by (2.1) that

$$\int_{\mathbb{R}^n} \Phi \cdot dD\chi_E = -\int_{\mathbb{R}^n} \chi_E \operatorname{div} \Phi \, dx = -\int_U \operatorname{div} \Phi \, dx = -\int_{\partial U} \Phi \cdot \nu_U \, d\mathcal{H}^{n-1} = 0.$$

By density, these formulae actually hold for all $\Phi \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$ with $\operatorname{supp}(\Phi) \subseteq U$. Hence, $D\chi_E|_U \equiv 0$, so $z \notin \operatorname{supp}(D\chi_E)$. □

Hence, setting $\nu = -\sigma$ (so ν is like a generalized outward normal vector field), we recover a statement which looks like Gauss Green:

$$\int_E \operatorname{div} \Phi \, d\mathcal{H}^n = \int_{\partial E} \Phi \cdot \nu \, d|D\chi_E|$$

We obtained such a formula by considering sets E such that the distributional gradient of χ_E is represented by a vector-valued Radon measure. More generally, we can consider integrable (or locally integrable) functions f whose distributional gradient is represented by a vector-valued Radon measure. This line of thought leads to the notion of *functions of bounded variation.*

Definition 2.1.1. Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $u \in L^1(\Omega)$. We say that u is of *bounded variation in* Ω, written $u \in BV(\Omega)$, if the the distributional derivatives $D_i u$ are represented by finite Radon measures. We then form the vector-valued measure $Du := (D_1 u, \ldots, D_n u)$. Given $u \in BV(\Omega)$, the $BV(\Omega)$-seminorm of u is the quantity

$$[u]_{BV(\Omega)} := |Du|(\Omega).$$

We note that $BV(\Omega)$ is in fact a Banach space with norm

$$\|u\|_{BV(\Omega)} := \|u\|_{L^1(\Omega)} + [u]_{BV(\Omega)}.$$

Example 2.1.1. If $u \in W^{1,1}(\Omega)$, then $u \in BV(\Omega)$ and $dD\chi_E = Du \, d\mathcal{L}^n$. This follows by definition of the distributional derivative.

We would like a quantitative way to test whether a given function $u \in L^1(\Omega)$ is of bounded variation. The above definition of $BV(\Omega)$-seminorm is not satisfactory as it requires a function to have distributional derivative represented by a measure to be intelligible. *A priori*, the distributional gradient is just a continuous linear functional on the locally convex space $C_c^\infty(\Omega, \mathbb{R}^n)$. It is not true that such a functional is in general given by a integration against an $\mathbb{R}^n$-valued Radon measure.

Thankfully, the Riesz(-Markov-Kakutani) representation theorem gives us conditions upon which which linear functionals are given by integration against measures. We state the vector-valued version of the theorem, as this form may not be as familiar to the reader.

Theorem 2.1.1 (Riesz Representation Theorem)**.** *Let X be a locally compact, Hausdorff space. Suppose $L : C_0(X, \mathbb{R}^n) \to \mathbb{R}$ is a linear functional which is bounded, i.e.*

$$\|L\| := \sup\{L(\varphi) : \varphi \in C_0(X, \mathbb{R}^n), |\varphi| \leq 1\} < +\infty.$$

Then there is a unique finite $\mathbb{R}^n$-valued Radon measure $\mu = (\mu_1, \ldots, \mu_n)$ on X such that

$$L(u) = \int_X \varphi \cdot d\mu = \sum_{i=1}^n \int_X \varphi_i \, d\mu_i \, .$$

Moreover, we have $\|L\| = |\mu|(X)$*.*

Motivated by the norm-boundedness condition in the above theorem, we make the following definition.

Definition 2.1.2. Given a function $u \in L^1(\Omega)$, define the *total variation of* u to be the quantity

$$V(u,\Omega) = \sup\left\{ \int_\Omega u \operatorname{div} \varphi \, dx : \varphi \in C_c^1(\Omega, \mathbb{R}^n), |\varphi| \le 1 \right\}.$$

Note that this is simply saying that the distributional gradient is bounded when considered as a linear functional on $C_c^1(\Omega,\mathbb{R}^n)$. This does not automatically satisfy the conditions of Theorem 2.1.1 as L is not defined on all of $C_0(\Omega,\mathbb{R}^n)$; however, a density argument rectifies this problem.

Proposition 2.1.1 (Characterization of BV)**.** *Suppose* $u \in L^1(\Omega)$ *has finite total variation. Then there exists a finite Radon measure* μ *on* Ω *and a* μ*-measurable* $\sigma : \Omega \to \mathbb{R}^n$ *with* $|\sigma| = 1$ μ*-a.e. and*

$$\int_\Omega u \operatorname{div} \varphi \, dx = -\int_\Omega \varphi \cdot \sigma \, d\mu \ \textit{ for all } \varphi \in C_c^1(\Omega,\mathbb{R}^n).$$

Moreover, $V(u,\Omega) = |\mu|(\Omega)$*.*

Proof. Define a linear functional $L : C_c^1(\Omega,\mathbb{R}^n) \to \mathbb{R}$ by $L(\varphi) := -\int_\Omega u \operatorname{div} \varphi \, dx$. The boundedness of L combined with linearity implies

$$|L(\varphi)| \le V(u,\Omega)\|\varphi\|_\infty \text{ for all } \varphi \in C_c^1(\Omega,\mathbb{R}^n).$$

Fix $\varphi \in C_c(\Omega,\mathbb{R}^n)$. Since φ is continuous and compactly supported, we can choose a sequence $(\varphi_i)_i$ in $C_c(\Omega,\mathbb{R}^n)$ such that $\operatorname{supp}(\varphi) \subseteq \operatorname{supp}(\varphi_i)$ and $\varphi_i \to \varphi$ uniformly in a fixed compactly contained neighborhood of $\operatorname{supp}(\varphi)$.

Define an extension $\widetilde{L} : C_c(\Omega,\mathbb{R}^n) \to \mathbb{R}$ of L by $\widetilde{L}(\varphi) = \lim_{k\to\infty} L(\varphi_k)$, which exists and is well-defined by the above inequality. As $C_c(\Omega,\mathbb{R}^n)$ is dense in $C_0(\Omega,\mathbb{R}^n)$, continuity guarantees a unique continuous linear extension to the whole of $C_0(\Omega,\mathbb{R}^n)$. Applying the Riesz Representation Theorem to this extension gives the conclusion. □

The above proposition allows us to extend our definition of $BV(\Omega)$-seminorm to all of $L^1(\Omega)$.

Definition 2.1.3. Given a function $u \in L^1(\Omega)$, define the $BV(\Omega)$*-seminorm of* u to be the quantity $[u]_{BV(\Omega)} := V(u,\Omega)$. Note that for $u \in L^{1(\Omega)}$, we have $u \in BV(\Omega) \iff [u]_{BV(\Omega)} < +\infty$.

Remark 2.1.1. Although not used in this paper, we remark that there are local versions of both Definition 2.1.1 and Proposition 2.1.1.

2.2. **Classical Perimeter.** As alluded to earlier, the entire discussion around functions of (locally) bounded variation and Gauss-Green formula leads to properties of the boundary of a set. One of the first such properties one becomes intuitively "familiar" with is the notion of *perimeter*. From a young age, we are led to believe that perimeter is an "obvious" concept.

It might surprise the reader that the notion of perimeter was not systematically (and definitively) treated until the 1950s. Equipped with the Hausdorff measure, one initially jumps to thinking of the perimeter of a set in $\mathbb{R}^n$ as the $(n-1)$-dimensional Hausdorff measure of its boundary. Indeed, for nice enough boundaries (say C^1), this definition is good enough. Moreover, one can show using just the Gauss-Green formula that the "perimeter" of a set is the total variation of its indicator function.

Fact/Exercise 2.2.1. *Let* $E \subseteq \mathbb{R}^n$ *be a bounded open set with* C^2 *boundary. Then* $\mathcal{H}^{n-1}(\partial E) = V(\chi_E,\mathbb{R}^n)$*.*

Definition 2.2.1. Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $E \subseteq \mathbb{R}^n$ measurable. We define the *perimeter of* E *in* Ω to be the quantity

$$Per(E,\Omega) := V(\chi_E,\Omega).$$

The set E is a *Caccioppoli set* or has *locally finite perimeter* if $Per(E,\Omega) < +\infty$ for every bounded open subset Ω. Define the *global perimeter of* E to be the quantity $Per(E,\mathbb{R}^n)$.

To build intuition, suppose E is open and has both finite measure and finite (global) perimeter in $\mathbb{R}^n$. Then by Proposition 2.1.1, $\chi_E \in BV(\mathbb{R}^n)$ and $Per(E, \mathbb{R}^n) = |D\chi_E|(\mathbb{R}^n) = \int_{\mathbb{R}^n} |D\chi_E|$. Recall from Section 2.1 that the mesaure $|D\chi_E|$ gives us a rigorous way to treat the intuition that the "gradient" of χ_E should a dirac delta at every point of ∂E and 0 everywhere else. Moreover, because the distributional gradient $|D\chi_E|$ is a measure, we can "integrate" it (that is, take its total mass). Intuitively, integrating dirac deltas on the boundary of a set "should" extract the perimeter of that set, so the definition matches our intuition.

3. Sobolev Space Preliminaries

The theory behind fractional perimeter is written in the language of fractional Sobolev spaces. As such, we will motivate and define these function spaces as well as discuss their most relevant properties. For brevity, the majority of proofs are either omitted or relegated to Appendix B. There are plenty of good monographs on the subject. We mention the recent book by Leoni [12], the book by Adams [1], and the Hitchhikers Guide [8]. For an overview of the theory of classical Sobolev spaces, see the books by Evans [9, 10].

3.1. Fractional Sobolev Spaces.

Like classical Sobolev spaces, fractional Sobolev spaces attempt to capture the regularity (that is, (weak) differentiability), decay, and oscillation of functions. Unlike classical Sobolev spaces, this regularity parameter is no longer assumed to be an integer. There are two main ways to introduce fractional Sobolev spaces:

- *Explicitly*: using explicit norms which are analogues of those of the Hölder spaces C^α. Proceeding with the intuition that the difference quotients
$$\mathcal{Q}_s u(x,h) := \frac{u(x+h)-u(x)}{|h|^s}, \quad x, h \in \mathbb{R}^n$$
are our "s"-fractional derivatives. We then would capture L^p-decay and s-regularity with the norm
$$\left(\int_{\mathbb{R}^n} \|\mathcal{Q}_s u(\bullet, h)\|^p_{L^p_x(\mathbb{R}^n)} \frac{dh}{|h|^n}\right)^{\frac{1}{p}}.$$
The presence of the singular kernel $|h|^{-n}$ ensures that the norm also captures oscillatory behavoir. This is the approach we will take throughout the paper.

- *Abstractly*: as interpolation spaces between $W^{1,p}$ and L^p. Using the interpolation methods first developed by Lions and Peetre in the 1960s (that is, the K- and J-methods), one can abstractly construct the spaces $W^{s,p}$ as the ones which arise from *real interpolation*. For an introduction to interpolation methods, see Bergh's book [2] and Tice's notes [**<empty citation>**]

  Although we will not take this approach in the following paper, it does provide some context for why we are able to recover $W^{1,p}$-norms from $W^{s,p}$-norms. Moreover, the fact that the interpolation spaces between $W^{1,1}$ and L^1 are the same as those between BV and L^1 explains why we are able to recover BV-norms from $W^{s,1}$-norms. See [15] for details.

Definition 3.1.1. Fix $1 \le p < +\infty$ and let $s \in (0,1)$ be a fractional exponent. For $u \in L^p(\Omega)$, define the *Gagliardo (semi)norm* of u to be the quantity
$$[u]_{W^{s,p}(\Omega)} := \left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dx\, dy\right)^{\frac{1}{p}}.$$
and define the *fractional Sobolev space* $W^{s,p}(\Omega) := \{u \in L^p(\Omega) : [u]_{W^{s,p}(\Omega)} < +\infty\}$. This is a Banach space with the natural norm
$$\|u\|_{W^{s,p}(\Omega)} := \|u\|_{L^p(\Omega)} + [u]_{W^{s,p}(\Omega)}.$$
We remark that $C_c^\infty(\Omega) \subseteq W^{s,p}(\Omega)$ and we write $W_0^{s,p}(\Omega)$ for the closure of $C_c^\infty(\Omega)$ inside $W^{s,p}(\Omega)$. It is a fact that when $\Omega = \mathbb{R}^n$, these two spaces are equal; however, this is not necessarily true for general Ω.

In the case $p = 2$, $W^{s,2}(\Omega)$ is in fact a Hilbert space with inner product given by

$$\langle u, v\rangle_{H^s(\Omega)} := \int_\Omega u(x)v(x)\,dx + \int_\Omega\int_\Omega \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dx\,dy\,.$$

Althought these spaces and (semi)norms seem somewhat natural from the viewpoint of being an analogue of the Hölder condition for L^p spaces instead of L^∞, when presented as above they are ultimately quite artificial. Where would one find such spaces appearing in nature?

If one takes for granted that integer sobolev spaces "appear in nature," then the answer to the previous question is that *fractional sobolev spaces are the correct image of the trace operator* (see TODOINSERT THIS for background on the trace operator).

Proposition 3.1.1. *Suppose $\Omega \subseteq \mathbb{R}^n$ is a nice domain (see definition 3.2.1) and $k \in \mathbb{N}$. Then there is a split exact sequence of Hilbert spaces*

$$0 \longrightarrow W_0^{k,2}(\Omega) \hookrightarrow W^{k,2}(\Omega) \xrightarrow{T} W^{k-\frac{1}{2},2}(\partial\Omega) \longrightarrow 0$$

where $T : W^{k,2}(\Omega) \to W^{k-\frac{1}{2},2}(\partial\Omega)$ is the trace operator.

When thinking of the fractional parameter s as measuring some notion of regularity, one would hope that having a fixed regularity s implies having all of the regularities below s (e.g. functions that are k-differentiable are automatically $k-1$-differentiable). When comparing two fractional parameters of regularity, this inuition generalizes unconditionally.

Proposition 3.1.2. *Let $p \in [1, +\infty)$ and $0 < s \leq s' < 1$. Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $u : \Omega \to \mathbb{R}$ a measurable function. Then there exists a constant $C \geq 1$ depending only on n,s, p, such that*

$$\|u\|_{W^{s,p}(\Omega)} \leq C\|u\|_{W^{s',p}(\Omega)}$$

Hence, there is a continuous inclusion $W^{s',p}(\Omega) \subseteq W^{s,p}(\Omega)$.

3.2. **Extension Domains.** One difficulty that occurs with the Gagliardo definition of fractional sobolev spaces is how different the seminorm is to the classical Sobolev seminorms. This manifests itself whenever one tries to compare (say for inclusions) classical Sobolev spaces and fractional sobolev spaces. Oftentimes well-behavedness of the domain Ω can quell this friction between definitions. We begin with a definition.

Definition 3.2.1. An open set $\Omega \subseteq \mathbb{R}^n$ is an *extension domain for $W^{s,p}$* if there exists a constant $C = C(s, p, n, \Omega) > 0$ such that for every $u \in W^{s,p}(\Omega)$, there exists a $\widetilde{u} \in W^{s,p}(\mathbb{R}^n)$ such that

$$\widetilde{u}|_\Omega \equiv u \quad \text{and} \quad \|\widetilde{u}\|_{W^{s,p}(\mathbb{R}^n} \leq C\|u\|_{W^{s,p}(\Omega)}.$$

We remark that any bounded open set with Lipschitz boundary is a $W^{s,1}$ extension domain for all $s \in (0, 1)$ (see Hitchhiker's Guide [8] for details) and a $W^{1,p}$ extension domain for all $1 \leq p < \infty$ (see Gilbarg and Trudinger [11, Thm. 7.25]). This fact somewhat explains why in both 2 and 1 we restrict to bounded open sets with Lipschitz boundary. One might imagine extending these results to general extension domains,

The general principle with fractional Sobolev spaces is that, on an extension domain, statements that hold for classical integer Sobolev spaces likely generalize to the fractional setting. Indeed, on an extension domain we can replace one of the fractional spaces in Proposition 3.1.2 with a classical Sobolev space.

Proposition 3.2.1 ([8, Prop. 2.2])**.** *Suppose $s \in (0, 1)$, $1 \leq p < \infty$, and let $\Omega \subseteq \mathbb{R}^n$ be a bounded $W^{1,p}$-extension domain. Then the identity map is a continuous embedding*

$$W^{1,p}(\Omega) \hookrightarrow W^{s,p}(\Omega).$$

Moreover, as is the case with classical Sobolev spaces, $W^{1,1}(\Omega)$ embeds continuously into $W^{s,1}(\Omega)$. Hence, one is lead to wonder if the same is true for BV instead of $W^{1,1}(\Omega)$. Under the assumption that Ω is a $W^{s,1}$-extension domain, this is true.

Proposition 3.2.2 ([13, Prop. 2.1]). *Suppose that $\Omega \subseteq \mathbb{R}^n$ is an extension domain. Then for $s \in (0,1)$ we have a continuous embedding $BV(\Omega) \hookrightarrow W^{s,1}(\Omega)$.*

Proof. Suppose $u \in BV(\Omega)$. By mollification, there exists a sequence $u_i \in C^\infty(\Omega) \cap BV(\Omega)$ such that

- $u_n \xrightarrow{L^1(\Omega)} u$
- $\|Du_i\|_{L^1(\Omega)} \le |Du|(\Omega)$ for all $i \in \mathbb{N}$,
- $\|Du_i\|_{L^1(\Omega)} \to |Du|(\Omega)$

Since Ω is a $W^{1,1}$ extension domain, by Proposition 3.2.1 the identity map is a continuous embedding $W^{1,1}(\Omega) \hookrightarrow W^{s,1}(\Omega)$.

$$[u_i]_{W^{s,1}(\Omega)} \le \|u_i\|_{W^{s,1}(\Omega)} \le C\|u_i\|_{W^{1,1}(\Omega)} = C\|u_i\|_{BV(\Omega)}.$$

Now, appealing to Fatou's lemma, we find

$$[u]_{W^{s,1}(\Omega)} \le \liminf_{i\to\infty} \|u_i\|_{W^{s,1}(\Omega)} \le \liminf_{i\to\infty} C\|u_i\|_{BV(\Omega)} = C\|u\|_{BV(\Omega)}.$$

Hence,

$$\|u\|_{W^{s,1}(\Omega)} = \|u\|_{L^1(\Omega)} + [u]_{W^{s,1}(\Omega)} \le (C+1)\|u\|_{L^1(\Omega)} + C[u]_{BV(\Omega)} \le (C+1)\|u\|_{BV(\Omega)}.$$

□

3.3. **H^s: An Alternative approach to fractional Sobolev spaces using $\mathscr{F}$.** Another annoyance with the Gagliardo approach to fractional Sobolev spaces is that the they are defined in terms of singular integrals, and thus often require quite delicate care.

When $p = 2$ and $\Omega = \mathbb{R}^n$, there is another approach. In this realm, we have access to the Fourier transform (denoted $\mathscr{F}(\cdot)$ or $\widehat{\cdot}$ throughout) and the Plancherel identity. These tools allow us to work purely in frequency space and avoid much (but not all) of the headache of singular integrals.
Recall, for a Schwartz function $u \in \mathscr{S}$, we have $\mathscr{F}\partial_j u = -i\xi_j \mathscr{F}u$. Hence, by the Plancherel identity

$$\int_{\mathbb{R}^d} |\partial_j u|^2\, dx = \int_{\mathbb{R}^d} |\xi_j|^2 \cdot |\mathscr{F}u|^2\, d\xi\,.$$

Summing in $1 \le j \le n$, we see

$$\int_{\mathbb{R}^n} |Du|^2\, dx = \sum_{j=1}^n \int |\partial_j u|^2\, dx = \sum_{j=1}^n \int_{\mathbb{R}^n} |\xi_j|^2 \cdot |\mathscr{F}u|^2\, dx = \int_{\mathbb{R}^n} |\xi|^2 \cdot |\mathscr{F}u|^2\, d\xi$$

More generally, one can show that

$$\int_{\mathbb{R}^n} \sum_{|\alpha|=l} |\partial^\alpha u|^2\, dx = \int_{\mathbb{R}^n} |\xi|^{2l} \cdot |\widehat{u}|^2\, dx$$

whence summing in $0 \le l \le k$ gives an expression for the classical $W^{k,2}$-Sobolev norm

$$\|u\|^2_{W^{k,2}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \left(\sum_{l=0}^k |\xi|^{2l} \right) \cdot |\mathscr{F}u|^2\, d\xi\,.$$

We seek to rewrite the right hand side of the above equation as the L^2-norm of a single function. Noting that the quantity $\sum_{l=0}^k |\xi|^{2l}$ is comparable (up to dimensional constants) to $(1+|\xi|^2)^k$, we find that the norm

$$\left\| (1+|\xi|^2)^{k/2} \mathscr{F}u \right\|_{L^2(\mathbb{R}^n)} \tag{3.1}$$

is equivalent to the classical $W^{k,2}$ norm of u. By limiting arguments, we can extend this result to all $W^{k,2}$-functions. In summary, we have demonstrated

$$W^{k,2}(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) : (1+|\xi|^2)^{\frac{k}{2}} \mathscr{F}u \in L^2(\mathbb{R}^n)\}.$$

Observe that 3.1 makes sense for noninteger k, hence we use this expression to model a Fourier-based fractional Sobolev space.

Definition 3.3.1. Let $s \in (0,1)$. Define the Bessel potential space as

$$H^s(\mathbb{R}^n) := \{u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1+|\xi|^2)^s |\mathscr{F}u|^2 \, d\xi < +\infty\}$$

equipped with the norm

$$\|u\|^2_{H^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (1+|\xi|^2)^s |\mathscr{F}u|^2 \, d\xi \, .$$

Define the H^s-seminorm to be $[u]^2_{H^s(\mathbb{R}^n)} := \int_{\mathbb{R}^n} |\xi|^{2s} |\mathscr{F}u|^2 \, d\xi$.

A priori, it is not at all clear how Bessel potential spaces relate to Gagliardo's fractional Sobolev spaces. Indeed, there is work to be done here. We take a brief detour to introduce an important operator which will ultimately clarify the relationship between the two definitions.

Let $(-\Delta)^k := -\sum_{j=1}^n \frac{\partial^{2k}}{\partial x_j^{2k}}$ and note that for a Schwartz function $u \in \mathscr{S}$, we have $(-\Delta)^k u = \mathscr{F}^{-1}(|\xi|^{2k}(\mathscr{F}u))$. Again, the right-hand side of this expression makes sense for noninteger k, so we make the following definition.

Definition 3.3.2. Fix $s \in (0,1)$. Define the *fractional Laplacian* $(-\Delta)^s : \mathscr{S} \to L^2(\mathbb{R}^n)$ as a Fourier multiplier given by

$$(-\Delta)^s u = \mathscr{F}^{-1}(|\xi|^{2s}(\mathscr{F}u)).$$

In order to relate the Bessel and Gagliardo spaces, we must somewhere pass from ordinary integration in frequency space (Bessel norm) to a singular integral in physical space (Gagliardo seminorm). This is where the fractional Laplacian comes in, as it has the following singular integral operatior representation.

Proposition 3.3.1 ([12, Thms. 14.2, 14.8])**.** *For $s \in (0,1)$ and $u \in \mathscr{S}$, we have that*

$$(-\Delta)^s u(x) = C(n,s) \, P.V. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}} \, dy \tag{3.2}$$

where $C(n,s)$ is the constant

$$C(n,s) := \left(\int_{\mathbb{R}^n} \frac{1-\cos(\zeta_1)}{|\zeta|^{n+2s}} \, d\zeta \right)^{-1} . \tag{3.3}$$

Moreover, we can remove the P.V. from the above expression.

Proof. Let $\Lambda_s : \mathscr{S} \to L^2(\mathbb{R}^n)$ denote the operator $\Lambda_s u(x) := C(n,s) \, P.V. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}} \, dy$. After applying the ansatzes $y \rightsquigarrow x+h$ and $y \rightsquigarrow x-h$, we have the second order difference quotient representation

$$\Lambda_s u(x) = -\frac{1}{2} C(n,s) \, P.V. \int_{\mathbb{R}^n} \frac{u(x+h)+u(x-h)-2u(x)}{|h|^{n+2s}} \, dh \, . \tag{3.4}$$

Fix $u \in \mathscr{S}$ and $x \in \mathbb{R}^n$. Consider the second-order Taylor expansion of u about x given for h small by

$$u(x+h) = u(x) + Du(x) \cdot h + \frac{1}{2} h^T \cdot D^2 u(x) \cdot h + R(h) \tag{3.5}$$

where $R(h) \in o(|h|^2)$. Then

$$u(x+h) + u(x-h) - 2u(x) = h^T \cdot D^2 u(x) \cdot h + R(h) + R(-h).$$

and

$$|h^T D^2 u(x) h| \le |h| \cdot |D^2 u(x) h| \le \left\| D^2 u(x) \right\|_{op} |h|^2,$$

leading to a bound on the integral kernel

$$\frac{u(x+h)+u(x-h)-2u(x)}{|h|^{n+2s}} \leq \frac{\left\|D^2u(x)\right\|_{op}}{|h|^{n+2s-2}} + \frac{1}{|h|^{n+2s-2}} \cdot \frac{R(h)+R(-h)}{|h|^2}. \tag{3.6}$$

Note that 3.6 is integrable in h within a bounded neighborhood of 0, so in fact the equation 3.4 holds true even without the "PV." Moreover, one may refine the estimate 3.6 slightly further to justify an application of Fubini-Tonelli and find that, for $\xi \in \mathbb{R}^n$,

$$\begin{aligned}
\mathscr{F}_x(\Lambda_s u)(\xi) &= -\frac{1}{2}C(n,s)\int_{\mathbb{R}^n} \frac{\mathscr{F}\{u(\bullet+h)+u(\bullet-h)-2u(\bullet)\}(\xi)}{|h|^{n+2s}}\,dh \\
&= -\frac{1}{2}C(n,s)\mathscr{F}u(\xi)\int_{\mathbb{R}^n} \frac{e^{-ih\xi}+e^{ih\xi}-2}{|h|^{n+2s}}\,dh \\
&= C(n,s)\mathscr{F}u(\xi)\int_{\mathbb{R}^n} \frac{1-\cos(\xi\cdot h)}{|h|^{n+2s}}\,dh
\end{aligned}$$

Consider the quantity $I(\xi) := \int_{\mathbb{R}^n} \frac{1-\cos(\xi\cdot h)}{|h|^{n+2s}}\,dh$. Fix $\xi \in \mathbb{R}^n$ and choose a rotation $R \in SO(n)$ such that $R(|\xi|e_1) = \xi$ (where $e_1 = (1,0,\ldots,0)$ as usual). Breifly using the notation $\langle\cdot,\cdot\rangle$ to denote the Euclidean dot product for clarity, we see

$$\begin{aligned}
I(\xi) &= \int_{\mathbb{R}^n} \frac{1-\cos\langle R(|\xi|e_1),h\rangle}{|h|^{n+2s}}\,dh = \int_{\mathbb{R}^n} \frac{1-\cos\langle |\xi|e_1, R^T h\rangle}{|h|^{n+2s}}\,dh \\
&\overset{\widetilde{h}\rightsquigarrow R^T h}{=} \int_{\mathbb{R}^n} \frac{1-\cos\langle |\xi|e_1,\widetilde{h}\rangle}{|\widetilde{h}|^{n+2s}}\,dh = I(|\xi|e_1).
\end{aligned}$$

Thus, the quantity $I(\xi) = I(|\xi|e_1)$ is rotation invariant. Finally, applying the substitution $h = \frac{\widetilde{h}}{|\xi|}$ we compute

$$I(\xi) = I(|\xi|e_1|) = \int_{\mathbb{R}^n} \frac{1-\cos\langle e_1,|\xi|h\rangle}{|h|^{n+2s}}\,dh = \int_{\mathbb{R}^n} \frac{1-\cos\langle e_1,\widetilde{h}\rangle}{|\frac{\widetilde{h}}{|\xi|}|^{n+2s}}\frac{d\widetilde{h}}{|\xi|^n} = |\xi|^{2s}I(e_1) = |\xi|^{2s}C(n,s)^{-1}.$$

In summary for all $\xi \in \mathbb{R}^n$ we have,

$$\int_{\mathbb{R}^n} \frac{1-\cos(\xi\cdot h)}{|h|^{n+2s}}\,dh = |\xi|^{2s}C(n,s)^{-1}, \tag{3.7}$$

whence

$$\mathscr{F}(\Lambda_s u)(\xi) = |\xi|^{2s}\mathscr{F}u(\xi) \implies \Lambda_s u(x) = (-\Delta)^s u(x).$$

□

This singular integral representation immediately allows us to express the Gagliardo $W^{s,2}$-seminorm in terms of the H^s-seminorm.

Proposition 3.3.2 ([8, Prop. 3.4])**.** *Let $s \in (0,1)$ and $n \in \mathbb{N}$. Then for $u \in W^{s,2}(\mathbb{R}^n)$, we have*

$$[u]^2_{W^{s,2}(\mathbb{R}^n)} = \frac{2}{C(n,s)}\left\|(-\Delta)^{\frac{s}{2}}u\right\|^2_{L^2(\mathbb{R}^n)} = \frac{2}{C(n,s)}\int_{\mathbb{R}^n}|\xi|^{2s}|\mathscr{F}u|^2\,dx = \frac{2}{C(n,s)}[u]_{H^s(\mathbb{R}^n)} \tag{3.8}$$

Proof. By density, it suffices to show 3.8 for $u \in \mathscr{S}$. Using the ansatz $x \rightsquigarrow z+y$, we compute

$$\begin{aligned}
\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy &\overset{x\rightsquigarrow z+y}{=} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(z+y)-u(y)|^2}{|z|^{n+2s}}\,dx\,dy \overset{\text{fubini}}{=} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(z+y)-u(y)|^2}{|z|^{n+2s}}\,dy\,dx \\
&= \int_{\mathbb{R}^n} \left\| \frac{u(z+\bullet)-u(\bullet)}{|z|^{\frac{n}{2}+s}} \right\|^2_{L^2_y(\mathbb{R}^n)} dz \overset{\text{Plancherel}}{=} \int_{\mathbb{R}^n} \left\| \mathscr{F}_y\left(\frac{u(z+\bullet)-u(\bullet)}{|z|^{\frac{n}{2}+s}} \right) \right\|^2_{L^2_\xi(\mathbb{R}^n)} dz \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|e^{i\xi\cdot z}-1|^2}{|z|^{n+2s}} |\mathscr{F}u(\xi)|^2\,d\xi\,dz \overset{\text{Fubini}}{=} 2\int_{\mathbb{R}^n} |\mathscr{F}u(\xi)|^2 \left(\int_{\mathbb{R}^n} \frac{1-\cos(z\cdot\xi)}{|z|^{n+2s}}\,dz \right) d\xi \\
&\overset{3.7}{=} \frac{2}{C(n,s)} \int_{\mathbb{R}^n} |\xi|^{2s} |\mathscr{F}u(\xi)|^2\,d\xi = \frac{2}{C(n,s)} [u]_{H^s(\mathbb{R}^n)}.
\end{aligned}$$

On the other hand, by the definition of $(-\Delta)^s$ and the Plancherel identity, we see

$$\left\|(-\Delta)^{\frac{s}{2}} u\right\|^2_{L^2(\mathbb{R}^n)} = \left\|\mathscr{F}^{-1}(|\xi|^s(\mathscr{F}u))\right\| \overset{\text{Plancherel}}{=} \||\xi|^s \mathscr{F}u\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\xi|^{2s} |\mathscr{F}u|^2\,d\xi\,.$$

□

Finally, we will need information about the asymptotics of $C(n,s)$ as $s \to 0^+, 1^-$. These are important as we are interested in asymptotically recovering the H^1 and L^2 norms from the continuous scale of H^s norms. We state the result without proof as the lengthy polar coordinate computations are not particularly enlightening.

Proposition 3.3.3 ([8, Prop. 4.1, Cor. 4.2])**.** *For $n > 1$, we have the following asymptotitcs.*

(1) $\displaystyle\lim_{s\to 1^-} \frac{C(n,s)}{s(1-s)} = \frac{4n}{\omega_{n-1}}$

(2) $\displaystyle\lim_{s\to 0^+} \frac{C(n,s)}{s(1-s)} = \frac{2}{\omega_{n-1}}$

4. Detour: the Bourgain-Brezis-Mironescu Formula

The main technical tool to prove Theorem 1 is the celebrated *Bourgain-Brezis-Mironescu (BBM) Formula*. In [3, 4], Bourgain, Brezis, and Mironescu sought to gain new characterizations of Sobolev spaces whilst also contributing to the push to define a degree theory for discontinuous maps. The outcropping of this work was a new formula for Sobolev norms which contextualized a large amount of previously interpolation-theoretic literature.

Definition 4.0.1. From now on, $(\rho_i)_{i=1}^\infty$ will denote a sequence of *radial mollifiers* in the sense that

$$\rho_i \in L^1_{loc}((0,+\infty)), \quad \rho_i \geq 0 \tag{4.1}$$

$$\lim_{i\to\infty} \int_\delta^\infty \rho_i(r) r^{n-1}\,dr = 0 \text{ for all } \delta > 0 \tag{4.2}$$

$$\int_0^\infty \rho_i(r) r^{n-1}\,dr = 1 \text{ for all } i \in \mathbb{N}. \tag{4.3}$$

Throughout this section, Ω is either a bounded Lipschitz domain or all of $\mathbb{R}^n$. Define a dimensional constant $K_{n,p}$ by

$$K_{n,p} := \int_{S^{n-1}} |e\cdot\sigma|^p\,d\mathcal{H}^{n-1}(\sigma)$$

where $e \in S^{n-1}$ is arbitrary.

The original BBM formula is as follows.

Theorem 4.0.1 ([3, 4], BBM Formula)**.** *Suppose* $1 < p < +\infty$*.For* $u \in L^1_{loc}(\Omega)$*,*

$$\lim_{i\to\infty} \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^p} \rho_i(|x-y|)\, dx\, dy = K_{n,p} \int_\Omega |Du|^p\, dx$$

when $Du \in L^p(\Omega)$ *and* $K_{n,p}$ *is a constant given by .*

Note that the above result does not treat the $p = 1$ case. In their original 2001 paper [3], Bourgain, Brezis, and Mironescu conjectured that their formula holds in the $p = 1$ case after replacing $W^{1,1}$ with BV, but they were only able to obtain the following partial result.

Theorem 4.0.2 ([3, Cor. 5])**.** *For* $u \in L^1(\Omega)$*, there exist constants* $C_1, C_2 > 0$ *such that*

$$\begin{aligned} C_1[u]_{BV(\Omega)} &\le \liminf_{i\to\infty} \int_\Omega \int_\Omega \frac{|u(x)-u(y)|}{|x-y|} \rho_i(|x-y|)\, dx\, dy \\ &\le \limsup_{i\to\infty} \int_\Omega \int_\Omega \frac{|u(x)-u(y)|}{|x-y|} \rho_i(|x-y|)\, dx\, dy \le C_2[u]_{BV(\Omega)}. \end{aligned}$$

In 2002, Hïam Brezis' student Juan Dávila [7] answered their conjecture in the affirmative

Theorem 4.0.3 ([7], BBM-Dávila Formula)**.** *Let* $\Omega \subseteq \mathbb{R}^n$ *be a bounded Lipschitz domain or all of* $\mathbb{R}^n$*. Suppose* $u \in BV(\Omega)$*. Then*

$$\lim_{i\to\infty} \int_\Omega \int_\Omega \frac{|u(x)-u(y)|}{|x-y|} \rho_i(|x-y|)\, dx\, dy = K_{n,1} \int_\Omega |Du|$$

where Du *is the finite Radon measure corresponding to* u *and* $\int_\Omega |Du|$ *denotes the quantity* $|Du|(\Omega)$*.*

We will use Theorem 4.0.3 to obtain a characterization of $BV(\Omega)$-norms in terms of a renormalized limit of $W^{s,1}(\Omega)$-norms.

5. Fractional s-Perimeter

5.1. **Motivation and definition of the fractional perimeter.** First introduced by Caffarelli in 2011 [5, 6], the study of the fractional perimeter (and more generally nonlocal perimeters) has devloped into a highly active area of research. We mention the 2019 monograph by Mazón [14] for an overview of the development of the field.

The quantity known as fractional perimeter initally arose as the energy functional for a nonlocal version of motion by mean curvature. We will give a much more elementary motivation in this paper.

Fix a region Ω. Define a sequence of regions $\Omega_1, \Omega_2, \ldots$ and surfaces $S_1, S_2, \ldots$ as follows:

Definition 5.1.1. Let $\Omega \subseteq \mathbb{R}^n$ be a smooth bounded domain, $s \in (0,1)$, and $E \subseteq \mathbb{R}^n$ measurable. The *fractional* s*-perimeter of* E *in* Ω is the quantity

$$P_s(E,\Omega) : = \frac{1}{2}[\chi_E]_{W^{s,1}(\Omega)} + \int_\Omega \int_{\mathbb{R}^n\setminus\Omega} \frac{|\chi_E(x)-\chi_E(y)|}{|x-y|^{n+s}}\, dx\, dy$$

where $L(A,B)$ denotes the following interaction energy integral

$$L(A,B) := \int_A \int_B \frac{1}{|x-y|^{n+s}}\, dx\, dy \ \text{ for all } A, B \subseteq \mathbb{R}^n$$

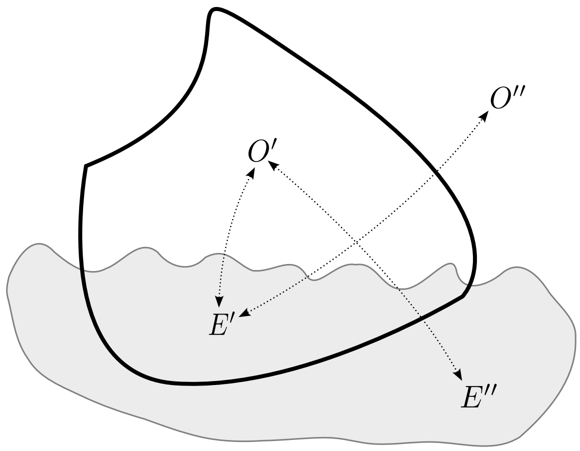
with the convention that $L(A,B) = 0$ if either A or B is empty. We denote the local and nonlocal energy terms by

$$P_s^L(E,\Omega) := \frac{1}{2}[\chi_E]_{W^{s,1}(\Omega)} \qquad P_s^{NL}(E,\Omega) := \int_\Omega \int_{\mathbb{R}^n\setminus\Omega} \frac{|\chi_E(x)-\chi_E(y)|}{|x-y|^{n+s}}\, dx\, dy$$

When $\Omega = \mathbb{R}^n$, we write $P_s(E) := P_s(E,\mathbb{R}^n)$.

To make this definition more transparent, consider the following reformulation:

$$\begin{aligned}
P_s(E,\Omega) &= \frac{1}{2}\int_\Omega\int_\Omega \frac{|\chi_E(x)-\chi_E(y)|}{|x-y|^{n+s}}\,dx\,dy + \int_\Omega\int_{\mathbb{R}^n\setminus\Omega} \frac{|\chi_E(x)-\chi_E(y)|}{|x-y|^{n+s}}\,dx\,dy \\
&= \frac{1}{2}\int_\Omega\int_\Omega \frac{\chi_E(x)\chi_{E^c}(y)+\chi_{E^c}(x)\chi_E(y)}{|x-y|^{n+s}}\,dx\,dy + \int_\Omega\int_{\mathbb{R}^n\setminus\Omega} \frac{\chi_E(x)\chi_{E^c}(y)+\chi_{E^c}(x)\chi_E(y)}{|x-y|^{n+s}}\,dx\,dy \\
&= L(E\cap\Omega, E^c\cap\Omega) + L(E^c\cap\Omega, E\cap\Omega^c) + L(E\cap\Omega, E^c\cap\Omega^c).
\end{aligned}$$



6. Asymptotics of $P_s(E,\Omega)$ as $s\to 0^+$

Proof of Theorem 2. Suppose first that $\Omega=\mathbb{R}^n$. Note that then $P_s^{NL}(E,\mathbb{R}^n)=0$, so we are left with only a local term. Noting that $|\chi_E(x)-\chi_E(y)| = |\chi_E(x)-\chi_E(y)|^2$ and appealing to Proposition 3.3.2, we find

$$[\chi_E]_{W^{s,1}(\mathbb{R}^n)} = [\chi_E]^2_{W^{\frac{s}{2},2}(\mathbb{R}^n)} = \frac{2}{C(n,s)}\int_{\mathbb{R}^n} |\xi|^{2s}|\mathscr{F}u|^2\,d\xi\,.$$

Hence, after normalizing by s (TODO APPEAL TO ASYMPTOTICS OF $C(n,s)$) and applying the monotone convergence theorem, we compute

$$\begin{aligned}
\lim_{s\to 0} sP_s(E,\mathbb{R}^n) &= \lim_{s\to 0^+}\frac{s}{2}[\chi_E]_{W^{s,1}(\mathbb{R}^n)} = \lim_{s\to 0^+}\frac{s}{C(n,s)}\int_{\mathbb{R}^n}|\xi|^{2s}|\mathscr{F}\chi_E|^2\,d\xi \\
&= \frac{\omega_{n-1}}{2}\|\mathscr{F}\chi_E\|^2_{L^2(\mathbb{R}^n)} = \frac{\omega_{n-1}}{2}\mathcal{L}^n(E)
\end{aligned}$$

Now we look at the case where $\Omega\subseteq\mathbb{R}^n$ is a bounded domain with Lipschitz boundary. As $E\subseteq\Omega$, the interactions in Definition 5.1.1 simplify to

$$P_s(E,\Omega) = L(E,\Omega\setminus E) + L(E,\Omega^c) = L(E,E^c) = \frac{1}{2}[\chi_E]_{W^{s,1}(\mathbb{R}^n)},$$

whence as before,

$$\lim_{s\to 0^+} sP_s(E,\Omega) = \lim_{s\to 0^+} sL(E,E^c) = \lim_{s\to 0^+}\frac{1}{2}[\chi_E]_{W^{s,1}(\mathbb{R}^n)} = \frac{\omega_{n-1}}{2}\mathcal{L}^n(E).$$

7. Asymptotics of $P_s(E,\Omega)$ as $s\to 1^-$

7.1. **Local Contribution to $P_s(E,\Omega)$.** In this subsection, we estimate the contribution of P_s^L to the s-perimeter as $s\to 1^-$. Our main tool is the Bourgain-Brezis-Mironescu-Davila Formula 4.0.3.

Recall that $P_s^L(E,\Omega) = \frac{1}{2}[\chi_E]_{W^{s,1}(\Omega)}$. Let $(s_i)_{i=1}^{\infty}$ be a sequence of integers with $s_i \to 1^-$. Choose $R >> 0$ such that $\Omega \subseteq B_{\frac{R}{2}}(0)$. Define mollifiers $\rho_i : (0,+\infty) \to [0,+\infty)$ as in 4.0.3 by

$$\rho_i(t) := a_{s_i} \frac{1-s_i}{t^{n+s_i-1}} \chi_{(0,R)}(t) \tag{7.1}$$

Where $a_{s_i} := \frac{1}{\omega_{n-1} R^{1-s_i}}$ is chosen such that $\int_{\mathbb{R}^n} \rho_i(|x|)\, dx = 1$. Using polar coordinates, one may check that the mollifiers $(\rho_i)_i$ satisfy the conditions of 4.0.3, namely that

$$\text{for all } \delta > 0, \quad \lim_{i\to\infty} \int_\delta^\infty \rho_i(r) r^{n-1}\, dr = \lim_{i\to\infty} \frac{1}{\omega_{n-1}} \left(1 - \left(\frac{\delta}{R}\right)^{1-s_i}\right) = 0.$$

For $u \in W^{s,1}(\Omega)$, as $\Omega - \Omega \subseteq B_R(0)$, we expand

$$\begin{aligned}\int_\Omega \int_\Omega \frac{|u(x)-u(y)|}{|x-y|} \rho_i(|x-y|)\, dx\, dy &= a_{s_i}(1-s_i) \int_\Omega \int_\Omega \frac{|u(x)-u(y)|}{|x-y|^{n+s_i}}\, dx\, dy \\ &= a_{s_i}(1-s_i)[u]_{W^{s_i,1}(\Omega)}.\end{aligned}$$

Then for $u \in BV(\Omega)$,

$$\begin{aligned}\lim_{i\to\infty} (1-s_i)[u]_{W^{s_i,1}(\Omega)} &= \lim_{i\to\infty} \frac{1}{a_{s_i}} \int_\Omega \int_\Omega \frac{|u(x)-u(y)|}{|x-y|} \rho_i(|x-y|)\, dx\, dy \\ &= \omega_{n-1} K_{1,n} [u]_{BV(\Omega)}.\end{aligned}$$

As the sequence (s_i) was arbitrary, we conclude that $\lim_{s\to 1^-}(1-s)[u]_{W^{s,1}(\Omega)} = \omega_{n-1} K_{n,1} [u]_{BV(\Omega)}$. In summary, we have shown the following proposition.

Proposition 7.1.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. If $E \subseteq \mathbb{R}^n$ has finite perimeter inside Ω, then*

$$\lim_{s\to 1^-} (1-s) P_s^L(E,\Omega) = \frac{\omega_{n-1}}{2} K_{n,1} Per(E,\Omega).$$

7.2. **Nonlocal Contribution to $P_s(E,\Omega)$.** The aim of this subsection is to show that the limiting behavior of the nonlocal contribution to s-perimeter, $P_s^{NL}(E,\Omega)$, is controlled by how much the set E fails to intersect $\partial\Omega$ "transversally." Namely, we will demonstrate that

$$\limsup_{s\to 1^-} (1-s) P_s^{NL}(E,\Omega) \preceq \lim_{\delta\to 0^+} Per(E, N_\delta(\partial\Omega))$$

where $N_\delta(\partial\Omega)$ denotes the δ-tubular neighborhood of $\partial\Omega$. The results and methods in this subsection are new developments due to Lombardini [13].

Recall the following fact from point-set topology.

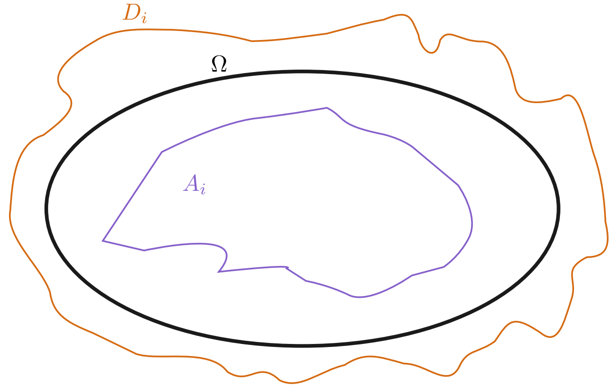
Fact 7.2.1. *If $X \subseteq \mathbb{R}^n$ is nonempty and $y \in \mathbb{R}^n \setminus \operatorname{Int} X$, then $d(y,X) = d(y,\partial X) = d(y,\partial X^c)$.*

Theorem 7.2.1 ([13, Prop. 2.5])**.** *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Suppose that $E \subseteq \mathbb{R}^n$ has finite perimeter inside Ω and $Per(E, \partial\Omega) = 0$. Then*

$$\lim_{s \to 1^-} (1-s) P_s^{NL}(E, \Omega) = 0$$

We strictly approximate Ω from inside and outside by bounded open sets with Lipschitz boundary, i.e. consider sets $(A_i)_i, (D_i)_i$ such that

- $A_i \subseteq A_{i+1} \Subset \Omega$ and $\Omega = \bigcup_{i \in \mathbb{N}} A_i$;
- $D_i \supseteq D_{i+1} \Supset \overline{\Omega}$ and $\overline{\Omega} = \bigcap_{i \in \mathbb{N}} D_i$.



We make a couple auxiliary definitions.

- $\Omega_i^+ := D_i \setminus \overline{\Omega}$, the outer excess portion;
- $\Omega_i^- := \Omega \setminus \overline{A_i}$, the inner excess portion;
- $T_i := \Omega_i^+ \cup \Omega_i^- \cup \partial\Omega$, the total excess;
- $d_i := \min\{d(A_i, \partial\Omega), d(\partial D_i, \Omega)\}$ minimum distance from approximation to the boundary of Ω.

As $|D\chi_E|$ is a finite Radon measure on D_1 and all other relevant sets are nested inside D_1, continuity from above implies that

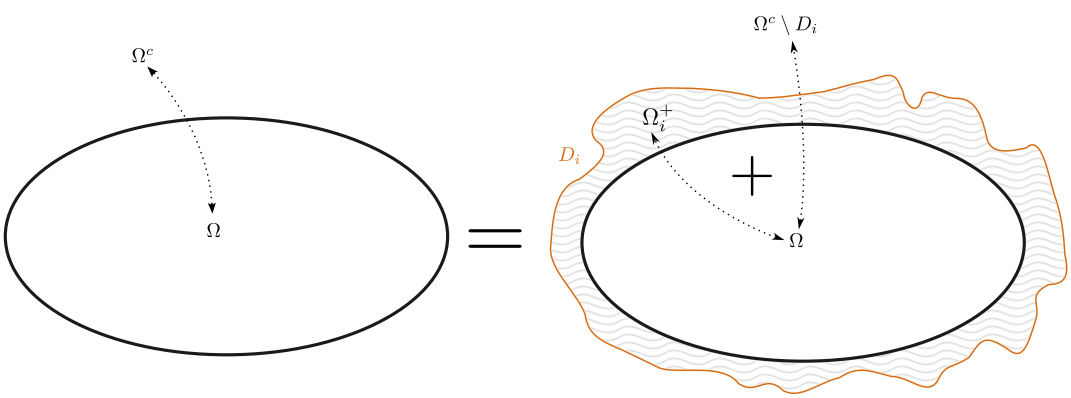
$$Per(E, \partial\Omega) = |D\chi_E|(\cap_i T_i) = \lim_{i\to\infty} |D\chi_E|(T_i) = \lim_{i\to\infty} Per(E, T_i).$$

Hence, in order to control the nonlocal contribution ot the s-perimeter in terms of $Per(E, \partial\Omega)$, it suffices to obtain control in terms of $Per(E, T_i)$ for fixed i. First, recall the two interactions involved in P_s^{NL} are given by

$$P_s^{NL}(E, \Omega) = L(E \cap \Omega, E^c \cap \Omega^c) + L(E^c \cap \Omega, E \cap \Omega^c) \tag{7.2}$$

As $P_s^{NL}(E, \Omega) = P_s^{NL}(E^c, \Omega)$ and $Per(E, T_i) = Per(E^c, T_i)$, it suffices to estimate only the first interaction term as an estimate for the second term would follow from this via the replacement $E \rightsquigarrow E^c$. We proceed by decomposing the interaction as

$$L(E \cap \Omega, E^c \setminus \Omega) = L(E \cap \Omega, E^c \cap \Omega_i^+) + L(E \cap \Omega, E^c \cap (\Omega^c \setminus D_i)). \tag{7.3}$$



To estimate these interactions, we need the following claim.

Claim. If $y \in \Omega$ and $\rho_y := \sup\{\rho > 0 : B_\rho(y) \subseteq D_i\}$, then $\rho_y \geq d_i$ and thus $B_{d_i}(y) \subseteq D_i$.

This claim follows by taking $X = D_i^c$ in Fact 7.2.1 and then noting that $B_{d(y, D_i^c)}(y) \subseteq D_i$ holds by definition.

Now for $y \in \Omega$, by the following claim we estimate

$$\int_{\Omega^c \setminus D_i} \frac{1}{|x-y|^{n+s}}\,dx \leq \int_{\mathbb{R}^n \setminus B_{d_i}(y)} \frac{1}{|x-y|^{n+s}}\,dx = \frac{\omega_{n-1}}{s} d_i^{-s}. \tag{7.4}$$

Hence, after integrating over Ω,

$$L(E \cap \Omega, E^c \cap (\Omega^c \setminus D_i)) \leq L(\Omega, \Omega^c \setminus D_i) = \int_\Omega \int_{\Omega^c \setminus D_i} \frac{1}{|x-y|^{n+s}} \, dx \, dy \leq \omega_{n-1} \frac{\mathcal{L}^n(\Omega)}{s} d_i^{-s} \tag{7.5}$$

giving us a bound on the second term in 7.3. To deal with the first term, we use a similar technique as in 7.3. We decompose the interaction with respect to the inner approximation $A_i \nearrow \Omega$ and its inner excess Ω_i^- to obtain

$$L(E \cap \Omega, E^c \cap \Omega_i^+) = L(E \cap \Omega_i^-, E^c \cap \Omega_i^+) + L(E \cap A_i, E^c \cap \Omega_i^+). \tag{7.6}$$

We analyze the second term in 7.6 and use similar logic to the estimate in 7.4 to find

$$L(E \cap A_i, E^c \cap \Omega_i^+) \leq L(A_i, \Omega_i^+) \leq L(A_i, \Omega^c) = \int_{A_i} \int_{\Omega^c} \frac{1}{|x-y|^{n+s}} \, dx \, dy \leq \omega_{n-1} \frac{\mathcal{L}^n(A_i)}{s} d_i^{-s}. \tag{7.7}$$

To deal with the first interaction term in 7.6, we note that both Ω_i^- and Ω_i^+ are contained in T_k. Hence by definition, we see

$$L(E \cap \Omega_i^-, E^c \cap \Omega_i^+) \leq L(E \cap T_i, E^c \cap T_i) = P_s^L(E, T_i) \tag{7.8}$$

which, in the context of Proposition 7.1.1, hints towards an asymptotic connection between the energy in question and the classical perimeters of E relative to the T_is. Now, combining estimates 7.5, 7.7, and 7.8, we have

$$L(E \cap \Omega, E^c \cap \Omega^c) \leq P_s^L(E, T_i) + 2\omega_{n-1} \frac{\mathcal{L}^n(\Omega)}{s} d_i^{-s}.$$

As discussed previously, the same estimate holds for the other term in 7.2, so our final energy bound becomes

$$P_s^{NL}(E, \Omega) \leq 2P_s^L(E, T_i) + 4\omega_{n-1} \frac{\mathcal{L}^n(\Omega)}{s} d_i^{-s}. \tag{7.9}$$

By Proposition 7.1.1,

$$\lim_{s \to 1^-} (1-s) P_s^L(E, T_i) = \frac{\omega_{n-1}}{2} K_{n,1} Per(E, T_i)$$

hence 7.9 implies

$$\limsup_{s \to 1^-} (1-s) P_s^{NL}(E, \Omega) \leq \omega_{n-1} K_{n,1} Per(E, T_i).$$

As this holds for all i, it follows that

$$\limsup_{s \to 1^-} (1-s) P_s^{NL}(E, \Omega) \leq \lim_{i \to \infty} \omega_{n-1} K_{n,1} Per(E, T_i) = \omega_{n-1} K_{n,1} Per(E, \partial\Omega).$$

When the RHS is zero, the claim follows.

Appendix A. Symmetric Decreasing Rearrangement

Appendix B. Miscellaneous Fractional Proofs

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