220A Homework 3

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Problem 1

Let G be an open subset of $\mathbb{C}$ and P a polygon in G from a to b. Use Theorems 5.15 and 5.17 to show that there is a polygon $Q \subseteq G$ from a to b which is composed of line segments which are parallel to either the real or imaginary axes.

Proof. Without loss of generality, assume that P is non self-intersecting (removing violating portions still results in a polygon from a to b). Write $P = \bigcup_{k=0}^{n-1}[a_k, a_{k+1}]$ where $a_0 = a$ and $a_n = 0$. Let $p_k : [0,1] \to \mathbb{C}$ be given by $p_k(t) := (1-t)a_k + ta_{k+1}$, so $p_k([0,1]) = [a_k, a_{k+1}] \subseteq G$. By theorem 5.15, as $[0,1]$ is compact, p_k is uniformly continuous.

As $[0,1]$ is compact, each $p_k([0,1]) = [a_k, a_{k+1}]$ is compact whence P is compact. As G is open, G^c is closed and $G^c \cap P = \emptyset$ by assumption, so theorem 5.17 implies

$$\varepsilon := d(G^c, P) > 0.$$

Fix $k \in \{0, \ldots, n-1\}$, and for brevity write $p = p_k$. As p is uniformly continuous, there is some $\delta > 0$ such that $|t - s| < \delta$ implies $|p(t) - p(s)| < \varepsilon$. Now choose $m \in \mathbb{N}$ and $0 = t_0 < t_1 < \cdots < t_m = 1$ such that $|t_{l+1} - t_l| < \delta$ for all $l \in \{0, \ldots, m-1\}$, whence

$$[0,1] = \bigcup_{l=0}^{m-1} [t_l, t_{l+1}].$$

Fix $l \in \{0, \ldots, m-1\}$, and set $z_l := \operatorname{Re}(p(t_{l+1}) + i\operatorname{Im}(p(t_l))$. Let $t \in [0,1]$ and observe that both

$$\begin{aligned}
|(1-t)p(t_l) + tz_l - p(t_l)| &= |t \cdot (z_l - p(t_l))| \\
&= t \cdot |\operatorname{Re}(p(t_{l+1}) - p(t_l))| \\
&\le t \cdot |p(t_{l+1}) - p(t_l)| < t\varepsilon \le \varepsilon,
\end{aligned}$$

and

$$\begin{aligned}
|(1-t)p(t_{l+1}) + tz_l - p(t_{l+1})| &= |t \cdot (z_l - p(t_{l+1}))| \\
&= t \cdot |i\operatorname{Im}(p(t_l - p(t_{l+1})| \\
&\le t \cdot |p(t_l) - p(t_{l+1})| < t\varepsilon \le \varepsilon.
\end{aligned}$$

As $\varepsilon = d(G^c, P)$ and $p(t_l), p(t_{l+1}) \in P$, it follows for $t \in [0,1]$ that $(1-t)p(t_l) + tz_l \in G$ and $(1-t)p(t_{l+1}) + tz_l \in G$, or equivalently that

$$[p(t_l), z_l],\ [z_l, p(t_{l+1})] \subseteq G.$$

Noting that these paths are parallel to the real and imaginary axes respectively, and their union is a polygon from $p(t_l)$ to $p(t_{l+1})$, it follows that we may replace each segment $[a_k, a_{k+1}]$ with a finite sequence of horizontal and vertical paths inside G, whence we may do the same for the whole path P. $\square$

Problem 2

Let (f_n) be a sequence of uniformly continuous functions from (X, d) into (Ω, ρ) and suppose that $f = \text{unif-}\lim f_n$ exists. Prove that f is uniformly continuous. If each f_n is Lipschitz with constant M_n and $\sup_{n\in\mathbb{N}} M_n < +\infty$, show that f is a Lipschitz function. If $\sup_{n\in\mathbb{N}} M_n = +\infty$, show that f may fail to be Lipschitz.

Proof. Let $\varepsilon > 0$. As $f_n \to f$ uniformly, there is some $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\sup_{x\in X} \rho(f(x), f_n(x)) < \frac{\varepsilon}{3}.$$

Fix $n \geq N$, and note that as f_n is uniformly continuous, there is some $\delta > 0$ such that for all $x, y \in X$ with $d(x, y) < \delta$, we have $\rho(f_n(x), f_n(y)) < \frac{\varepsilon}{3}$. Then for any $x, y \in X$ with $d(x, y) < \delta$, we have

$$\rho(f(x), f(y)) \leq \rho(f(x), f_n(x)) + \rho(f_n(x), f_n(y)) + \rho(f_n(y), f(y)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Thus f is uniformly continuous.

Now suppose that each f_n is also Lipschitz with constant M_n and $M := \sup_{n\in\mathbb{N}} M_n < +\infty$. Fix $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$\sup_{x\in X} \rho(f(x), f_n(x)) < \frac{\varepsilon}{2}.$$

Fixing $x, y \in X$ and $n \geq N$, then we have

$$\begin{aligned}\rho(f(x), f(y)) &\leq \rho(f(x), f_n(x)) + \rho(f_n(x), f_n(y)) + \rho(f_n(y), f(y)) \\ &\leq \frac{\varepsilon}{2} + M_n d(x, y) + \frac{\varepsilon}{2} < \varepsilon + M d(x, y).\end{aligned}$$

As $\varepsilon > 0$ was arbitrary, it follows that

$$\rho(f(x), f(y)) \leq M d(x, y),$$

so f is Lipschitz.

Unbounded Lipschitz constant counterexample: Write $L(f)$ for the Lipschitz constant of f, namely

$$L(f) = \sup_{x\neq y} \frac{\rho(f(x), f(y))}{d(x, y)}.$$

Let $(X, d) = ([0, 1], |\cdot|)$ and $(\Omega, \rho) = (\mathbb{R}, |\cdot|)$. So we are considering the Banach space $B := C([0, 1])$ with the supremum norm. Note that as $[0, 1]$ is compact, all elements of B are uniformly continuous. Let $f \in B$ be given by $f(x) = \sqrt{x}$.

Suppose, for the sake of contradiction, that $L(f) < +\infty$. Then, for all $x \in (0, 1]$, we have

$$\frac{1}{\sqrt{x}} = \frac{|\sqrt{x} - 0|}{|x - 0|} \leq L(f),$$

which is absurd as $\frac{1}{\sqrt{x}} \to +\infty$ as $x \to 0$. Thus f is not Lipschitz continuous, but is uniformly continuous as $[0, 1]$ is compact.

Let $P \subseteq B$ be the set of polynomial functions on $[0,1]$. By the Weierstrass approximation theorem, $\overline{P}^{\|\cdot\|_{\sup}} = B$, so there is a sequence of polynomial functions $f_n \in B$ such that $\|f_n - f\|_{\sup} \xrightarrow{n\to\infty} 0$, i.e. $f_n \to f$ uniformly.

As polynomial functions are Lipschitz on bounded subsets of $\mathbb{R}$, each f_n is Lipschitz. Suppose, for the sake of contradiction, that $\sup_{n\in\mathbb{N}} L(f_n) < +\infty$. Then by the statement we have proven above, noting that each f_n is uniformly continuous, we have $L(f) < +\infty$, which contradicts what we have shown. Thus $\sup_{n\in\mathbb{N}} L(f_n) = +\infty$ $\square$

Problem 3

If (a_n) is a convergent sequence in $\mathbb{R}$ and $a = \lim_{n\to\infty} a_n$, show that $\liminf a_n = \limsup a_n$.

Proof. We will show that $\limsup a_n = a$ and $\liminf a_n = a$. We are operating under the definitions

$$\limsup_{n\to\infty} a_n := \lim_{n\to\infty}\left(\sup_{k\geq n} a_k\right)$$
$$\liminf_{n\to\infty} a_n := \lim_{n\to\infty}\left(\inf_{k\geq n} a_k\right).$$

Fix $\varepsilon > 0$. As $a = \lim_{n\to\infty} a_n$, there is some $N \in \mathbb{N}$ such that for $n \geq N$ we have

$$|a_n - a| < \varepsilon.$$

Equivalently, we may write this for $n \geq N$ as

$$a - \varepsilon < a_n < a + \varepsilon.$$

Fixing $n \geq N$ for the moment, $k \geq n$ implies $k \geq N$, whence by definitions of sup and inf we have that

$$a - \varepsilon < \sup_{k\geq n} a_k \leq a + \varepsilon$$
$$a - \varepsilon \leq \inf_{k\geq n} a_k < a + \varepsilon.$$

Equivalently, we may write these inequalities as

$$\left|a - \sup_{k\geq n} a_k\right| \leq \varepsilon$$
$$\left|a - \inf_{k\geq n} a_k\right| \leq \varepsilon$$

whence, as $\varepsilon > 0$ was arbitrary,

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty}\left(\sup_{k\geq n} a_k\right) = a = \lim_{n\to\infty}\left(\inf_{k\geq n} a_k\right) = \liminf_{n\to\infty} a_n.$$

$\square$

Problem 4

Show that the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^{n(n+1)}$$

is 1, and discuss convergence for $z = 1$, -1, and i.

Solution.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^{n^2+n}$$

$$a_n = \{$$

Symbolically (i.e. inside $\mathbb{C}[\![x]\!]$), there is some $(a_n)_{n=0}^{\infty}$ such that

$$\sum_{k=0}^{\infty} a_k z^k = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^{n^2+n}.$$

As no z^0 term appears in the latter series, $a_0 = 0$. Fix $k \in \mathbb{N}$. If there exists some $n \in \mathbb{N}$ such that $k = n^2+n$, then such an n is unique as the function $x \mapsto x^2 + x$ is monotone increasing on $x > 0$, whence $a_k = \frac{(-1)^n}{n}$. If there is no such $n \in \mathbb{N}$, then $a_k = 0$. Concisely, we have

$$a_k = \begin{cases} \frac{(-1)^n}{n} & \text{if } k = n^2 + n \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Let $0 \leq R \leq \infty$ be the radius of convergence of the given power series. Then by definition,

$$\frac{1}{R} = \limsup_{k \to \infty} |a_k|^{\frac{1}{k}}.$$

To show that $R = 1$, it suffices to show $\frac{1}{R} = 1$, so we shall show the corresponding quantity is 1. Consider the subsequence $(k_n)_{n=1}^{\infty}$ given by $k_n := n^2 + n$. Then observe that

$$\lim_{n \to \infty} |a_{k_n}|^{\frac{1}{k_n}} = \lim_{n \to \infty} \left(\frac{1}{n}\right)^{\frac{1}{k_n}}.$$

We have shown in problem 6 that $\lim_{n \to \infty} n^{\frac{1}{n}} = 1$. By continuity of $x \mapsto \log(x)$ on $(0, \infty)$, it follows that

$$0 = \log(1) = \lim_{n \to \infty} \log\left(n^{\frac{1}{n}}\right) = \lim_{n \to \infty} \frac{1}{n} \log(n).$$

Noting that $\lim_{n \to \infty} \frac{1}{n+1} = 0$ and the sequence $\left(\frac{1}{n+1}\right)_{n=1}^{\infty}$ is bounded, it follows that

$$0 = \lim_{n \to \infty} \left(\frac{1}{n+1}\right) \cdot \left(\frac{1}{n} \log(n)\right) = \lim_{n \to \infty} \frac{1}{n^2 + n} \log(n) = \lim_{n \to \infty} \log\left(n^{\frac{1}{n^2+n}}\right).$$

Now appealing to the continuity of $x \mapsto e^x$, it follows that

$$1 = e^0 = \lim_{n \to \infty} e^{\log\left(n^{\frac{1}{n^2+n}}\right)} = \lim_{n \to \infty} n^{\frac{1}{n^2+n}}.$$

Thus, as $k_n = n^2 + n > n$ for all $n \in \mathbb{N}$, it follows that

$$\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \left(\sup_{k\geq n} |a_k|^{\frac{1}{k}} \right) \geq \lim_{n\to\infty} |a_{k_n}|^{\frac{1}{k_n}} = 1.$$

Suppose, for the sake of contradiction, that there is some $\varepsilon > 0$ such that $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} > 1 + \varepsilon$. Then, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$\sup_{k\geq n} |a_k|^{\frac{1}{k}} > 1 + \varepsilon.$$

Hence, for $n \geq N$, there is some $k_n \geq n$ such that $|a_{k_n}|^{\frac{1}{k_n}} > 1 + \varepsilon$. This implies that $a_{k_n} \neq 0$, whence by definition there is some $m_n \in \mathbb{N}$ such that $k_n = m_n^2 + m_n$ and

$$\left(\frac{1}{m_n} \right)^{\frac{1}{m_n^2 + m_n}} = |a_{k_n}|^{\frac{1}{k_n}} > 1 + \varepsilon.$$

Applying the quadratic formula and recalling that $k_n \geq n$, we see

$$m_n = \frac{-1 + \sqrt{1 + 4k_n}}{2} \geq \frac{-1 + \sqrt{1 + 4n}}{2} \xrightarrow{n\to\infty} \infty.$$

Thus $(m_n)_{n=1}^{\infty}$ is a monotone increasing subsequence of $(n)_{n=1}^{\infty}$, whence

$$\lim_{n\to\infty} \left(\frac{1}{m_n} \right)^{\frac{1}{m_n^2 + m_n}} = \lim_{n\to\infty} \left(\frac{1}{n} \right)^{\frac{1}{n^2 + n}} = 1,$$

which contradicts that $\left(\frac{1}{m_n} \right)^{\frac{1}{m_n^2 + m_n}} > 1 + \varepsilon$ for all $n \geq N$.

Thus we have shown that $\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = 1$, so the radius of convergence of the series is 1.

<u>($z = 1$ Case)</u>: If $z = 1$, then we have the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

which converges by the alternating series test as $\left(\frac{1}{n}\right)_{n=1}^{\infty}$ is a monotone decreasing sequence. Note however that this convergence is only conditional as the harmonic series diverges.

<u>($z = -1$ Case)</u>: If $z = -1$, then we have the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} (-1)^{n^2+n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n+n^2}}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n^2}}{n}.$$

As $n^2 \equiv n \pmod 2$, $(-1)^{n^2} = (-1)^n$ for all $n \in \mathbb{N}$, whence we have the same series as in the $z = 1$ case.

<u>($z = i$ Case)</u>: If $z = i$, then we have the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} (i)^{n^2+n} = \sum_{n=1}^{\infty} \frac{1}{n} (i)^{n^2+3n}.$$

We have the following congruences:

$$\begin{aligned}
n \equiv 0 \pmod 4 \quad &\implies \quad n(n+3) \equiv 0 \pmod 4 \implies i^{n(n+3)} = 1 \\
n \equiv 1 \pmod 4 \quad &\implies \quad n(n+3) \equiv 0 \pmod 4 \implies i^{n(n+3)} = 1 \\
n \equiv 2 \pmod 4 \quad &\implies \quad n(n+3) \equiv 2 \pmod 4 \implies i^{n(n+3)} = -1 \\
n \equiv 3 \pmod 4 \quad &\implies \quad n(n+3) \equiv 2 \pmod 4 \implies i^{n(n+3)} = -1.
\end{aligned}$$

So we may reindex our series (without reordering) to observe

$$\sum_{n=1}^{\infty} \frac{1}{n}(i)^{n^2+3n} = 1 + \sum_{k=1}^{\infty} \left(\frac{1}{2k} + \frac{1}{2k+1}\right) \cdot (-1)^k.$$

Noting that the sequence $\left(\frac{1}{2k} + \frac{1}{2k+1}\right)_{k=1}^{\infty}$ is monotone decreasing, it follows by the alternating series test that the above series converges.

$\square$

Problem 5

Show that $f(z) = |z|^2 = x^2 + y^2$ has a derivative only at the origin.

Proof. First we show that the derivative at the origin exists. Fix $\varepsilon > 0$. Using $\delta := \varepsilon$, for $z \in \mathbb{C}$ with $|z| < \delta$, we have

$$\left| \frac{f(z) - f(0)}{z - 0} - 0 \right| = \left| \frac{|z|^2}{z} \right| = |z| < \delta = \varepsilon.$$

As $\varepsilon > 0$, it follows that $f'(0)$ exists and is equal to 0.

Now we show that the derivative away from the origin does not exist. Fix $z \in \mathbb{C} \setminus \{0\}$, and considering $h \in \mathbb{C} \setminus \{0\}$ we compute

$$\begin{aligned}
\frac{f(z+h) - f(z)}{h} = \frac{\overline{(z+h)}(z+h) - \overline{z}z}{h} &= \frac{\overline{z}z + \overline{h}z + h\overline{z} + \overline{h}h - \overline{z}z}{h} \\
&= \frac{\overline{h}z + h\overline{z} + \overline{h}h}{h} = \frac{\overline{h}}{h}z + \overline{z} + \overline{h}.
\end{aligned}$$

Noting that $\lim_{h\to 0} \overline{h} = 0$ and $z \neq 0$ is a fixed constant, the above computation shows that the limit as $h \to 0$ of $\frac{f(z+h)-f(z)}{h}$ exists if and only the limit as $h \to 0$ of $\frac{\overline{h}}{h}$ exists. Suppose, for the sake of contradiction, that there is some $w \in \mathbb{C}$ such that $\lim_{h\to 0} \frac{\overline{h}}{h} = w$.
Fix $\varepsilon > 0$. Then there is some $\delta_0 > 0$ such that $0 < |h| < \delta_0$ implies $\left|\frac{\overline{h}}{h} - w\right| < \varepsilon$. Fix $\delta > 0$ such that $\delta < \delta_0$. Then $|i\delta| < \delta_0$ and $|\delta| < \delta_0$, whence

$$\begin{aligned}
\varepsilon &> \left| \frac{\overline{\delta}}{\delta} - w \right| = |1 - w| \\
\varepsilon &> \left| \frac{\overline{i\delta}}{i\delta} - w \right| = |-1 - w| = |1 + w|.
\end{aligned}$$

As $\varepsilon > 0$ was arbitrary, it follows that $|1 - w| = 0 = |1 + w|$, whence $1 = w = -1$, which is absurd. Thus, the proposed limit does not exist, whence the proposed derivative does not exist.

$\square$

Problem 6

Show that $\lim_{n\to\infty} n^{1/n} = 1$.

Proof. Note first that for all $n \in \mathbb{N}$, $\log(n) \geq 0$, whence

$$\log\left(n^{\frac{1}{n}}\right) = \frac{1}{n}\log(n) \geq 0 \qquad \Longrightarrow \qquad n^{\frac{1}{n}} \geq 1.$$

Fix $\varepsilon > 0$. We will show that there is some $N \in \mathbb{N}$ such that for all $n \geq N$, $n^{\frac{1}{n}} < 1+\varepsilon$.

$$\begin{aligned} n^{\frac{1}{n}} < 1+\varepsilon \qquad &\Longleftrightarrow \qquad n < (1+\varepsilon)^n = \sum_{k=0}^{n} \binom{n}{k} \varepsilon^k \\ &\Longleftrightarrow \qquad 1 < \frac{1}{n}(1+\varepsilon)^n = \frac{1}{n}\sum_{k=0}^{n} \binom{n}{k} \varepsilon^k \end{aligned}$$

As all terms present are positive, it suffices to show that a single term is greater than 1, so consider the third term. Then we must show that

$$1 < \frac{1}{n} \cdot \binom{n}{2}\varepsilon^2 = \frac{n-1}{2}\varepsilon^2.$$

Now we begin the proof itself. By the Archimedean principle, there is some $N \in \mathbb{N}$ such that $0 < \sqrt{\frac{2}{N-1}} < \varepsilon$. As the function $x \mapsto \sqrt{\frac{2}{x-1}}$ is monotone decreasing for $x > 1$, it follows that for all $n \geq N$,

$$0 < \sqrt{\frac{2}{n-1}} < \varepsilon.$$

Then for $n \geq N$,

$$\varepsilon > \sqrt{\frac{2}{n-1}} \qquad \Longrightarrow \qquad 1 < \frac{n-1}{2}\varepsilon^2 = \frac{1}{n} \cdot \binom{n}{2}\varepsilon^2,$$

whence we note that

$$\frac{1}{n}(1+\varepsilon)^n = \frac{1}{n}\sum_{k=0}^{n} \binom{n}{k}\varepsilon^k \geq \frac{1}{n}\binom{n}{2}\varepsilon^2 > 1.$$

Upon rearranging the above inequality, we see

$$(1+\varepsilon)^n > n \qquad \Longrightarrow \qquad 1+\varepsilon > n^{\frac{1}{n}}.$$

Thus for $n \geq N$, as $n^{\frac{1}{n}} \geq 1$, we have

$$|n^{\frac{1}{n}} - 1| = n^{\frac{1}{n}} - 1 < \varepsilon.$$

Hence $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$ as desired. $\square$

Problem 7

Let G be a region and define $G^* := \{z : \overline{z} \in G\}$. If $f : G \to \mathbb{C}$ is analytic, show that $f^* : G^* \to \mathbb{C}$, defined by $f^*(z) = \overline{f(\overline{z})}$, is also analytic.

Proof. The definition of analytic we are assuming in this course is continuously differentiable, i.e. f' exists for all points in the region and is continuous.

Fix $z \in G^*$ and $\varepsilon > 0$. Fix $\gamma > 0$ small enough such that $B_\gamma(z) \subseteq G^*$. By definition, $\overline{z} \in G$, whence $f'(\overline{z})$ exists. Thus, there is some $\delta > 0$ such that $0 < |k| < \delta$ implies that $\overline{z} + k \in G$ and

$$\left| \frac{f(\overline{z}+k) - f(\overline{z})}{k} - f'(\overline{z}) \right| < \varepsilon.$$

Suppose that $0 < |h| < \min\{\delta, \gamma\}$, so $z + h \in G^*$. Then $\overline{z} + \overline{h} = \overline{z+h} \in G$, $|\overline{h}| < \delta$, and $\overline{z} + \overline{h} \in G$, so

$$\begin{aligned}
\varepsilon &> \left| \frac{f(\overline{z}+\overline{h}) - f(\overline{z})}{\overline{h}} - f'(\overline{z}) \right| \\
&= \left| \frac{f(\overline{z+h}) - f(\overline{z})}{\overline{h}} - f'(\overline{z}) \right| \\
&= \left| \frac{\overline{f(\overline{z+h})} - \overline{f(\overline{z})}}{h} - \overline{f'(\overline{z})} \right| = \left| \frac{f^*(z+h) - f^*(z)}{h} - \overline{f'(\overline{z})} \right|.
\end{aligned}$$

As $\varepsilon > 0$ was arbitrary, it follows that f^* is differentiable at z and $(f^*)'(z) = \overline{f'(\overline{z})}$.

Now for analyticity, it remains to show continuity of $\frac{df'}{dz}$. Suppose that $z \in G^*$ and $(z_n)_{n=1}^\infty$ is a sequence (suffices to show for sequences and not nets as $\mathbb{C}$ is a metric space) in G' such that $|z_n - z| \xrightarrow{n\to\infty} 0$. Noting that $\overline{z} \in G$ and $\overline{z_n} \in G$ for all $n \in \mathbb{N}$ and

$$|\overline{z_n} - \overline{z}| = |z_n - z| \xrightarrow{n\to\infty} 0,$$

it follows by analyticity of f' that $|f'(\overline{z_n}) - f'(\overline{z})| \to 0$. Then, observe that

$$|(f^*)'(z_n) - (f^*)'(z)| = |\overline{f'(\overline{z_n})} - \overline{f'(\overline{z})}| = |f'(\overline{z_n}) - f'(\overline{z})| \xrightarrow{n\to\infty} 0,$$

whence $(f^*)'$ is continuous and thus f^* is analytic. $\square$