The Total Area Statistic for Fuss-Catalan Paths

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Contents

1   Preliminaries

2   Fuss-Catalan Preliminaries

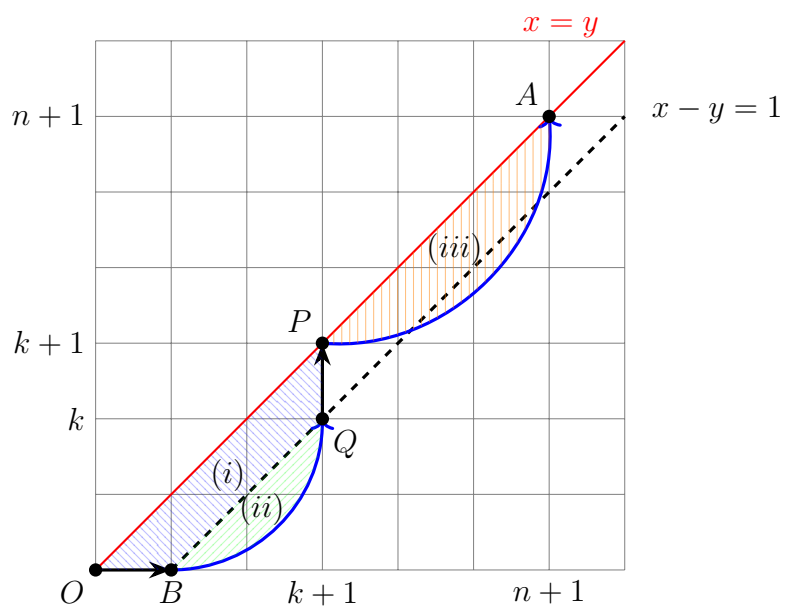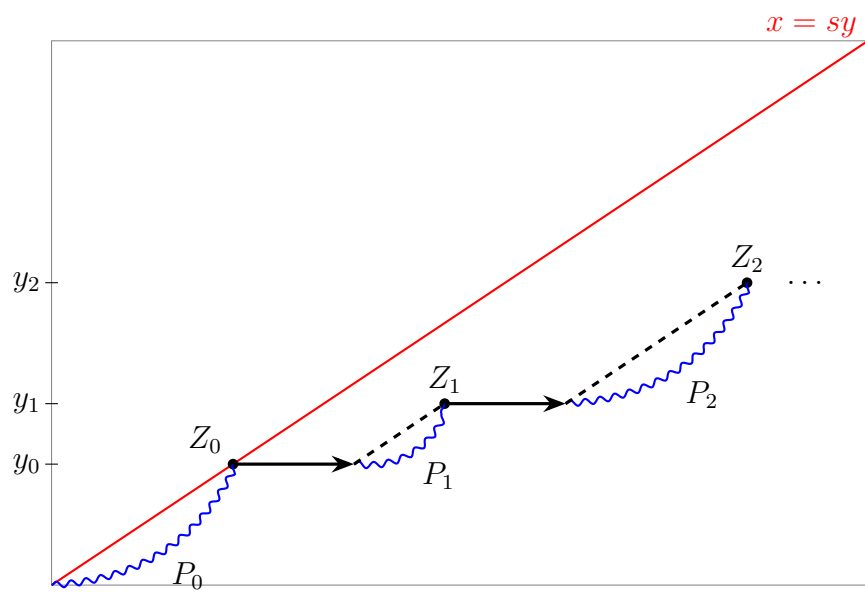$$F_m(n,k) = \frac{k}{mn+k}\binom{mn+k}{n}$$

$$F_m(n) = F_m(n,1) = \frac{1}{mn+1}\binom{mn+1}{n} = \frac{1}{(m-1)n+1}\binom{mn}{n}$$
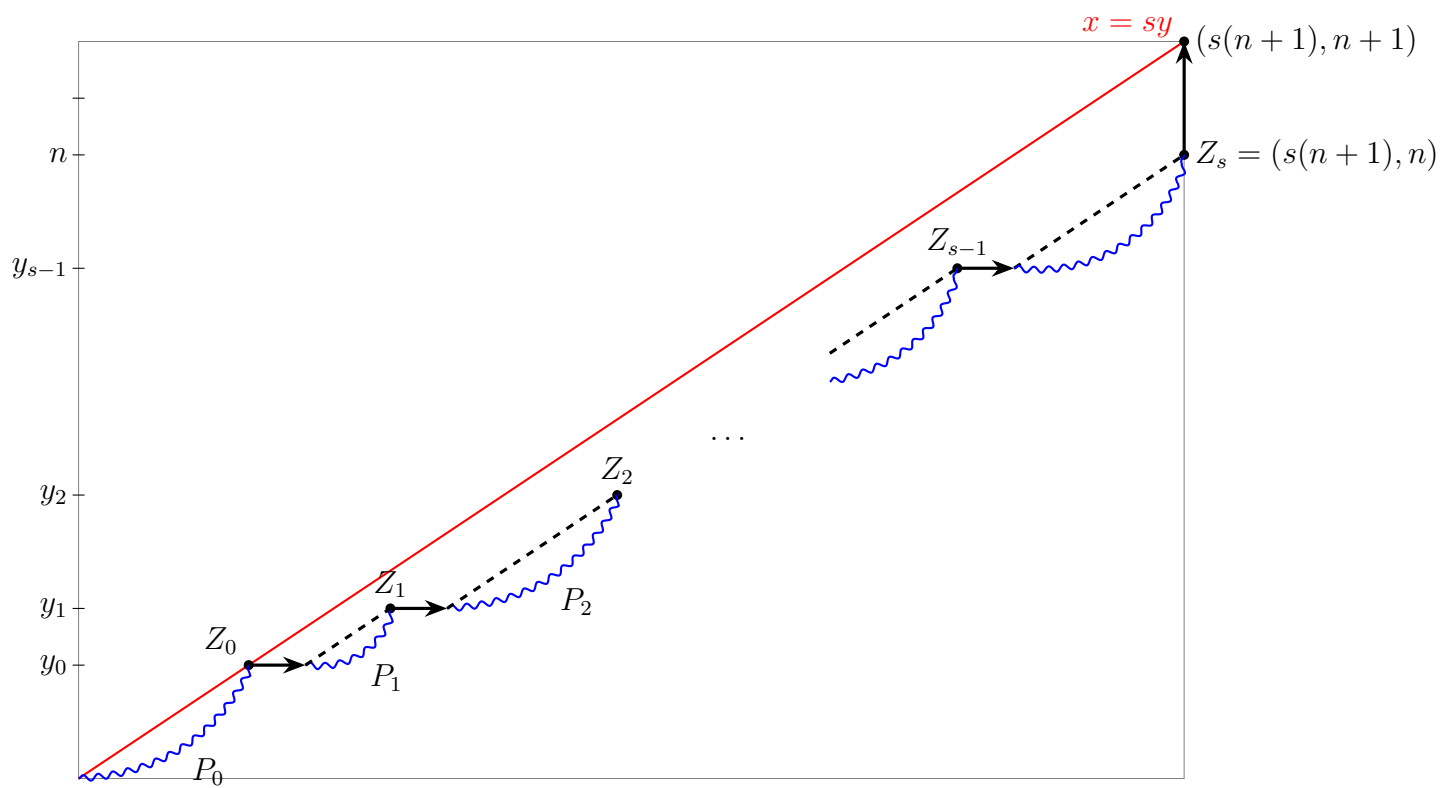
$$F_2(n) = F_2(n,1) = c_n$$

$$\mathcal{D}_n^s := L((0,0) \to (sn,n) : x \geq sy), \quad |\mathcal{D}_n^s| = F_{s+1}(n) = \frac{1}{sn+1}\binom{(s+1)n}{n} = \frac{1}{(s+1)n+1}\binom{(s+1)n+1}{n}$$

$$F_m(z) = \sum_{n\geq 0} F_m(n,1)z^n, \quad (F_m(z))^k = \sum_{n\geq 0} F_m(n,k)z^n, \quad F_m(z) = 1 + z(F_m(z))^m$$

$x = sy$

$(s(n+1), n+1)$

$Z_s = (s(n+1), n)$

Z_{s-1}

$\ldots$

$x = sy$

$(s(n+1), n+1)$

$Z_s = (s(n+1), n)$

n

y_{s-1}

Z_{s-1}

$\ldots$

y_2

Z_2

y_1

Z_1

y_0

Z_0

P_1

P_2

P_0
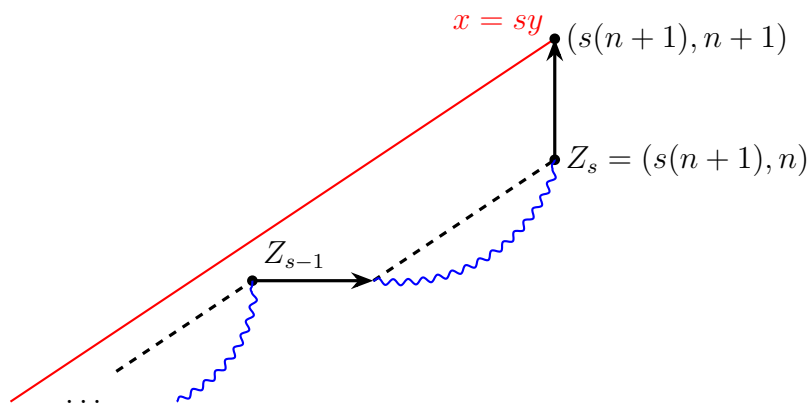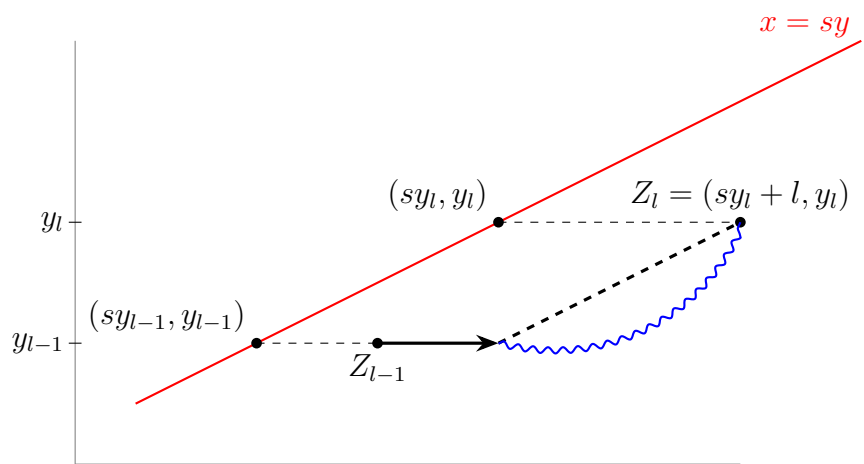
3 Recursive Decomposition

Let

$$A_n = \sum_{P \in \mathcal{D}_n^s} \text{area}(P)$$

$$
\begin{aligned}
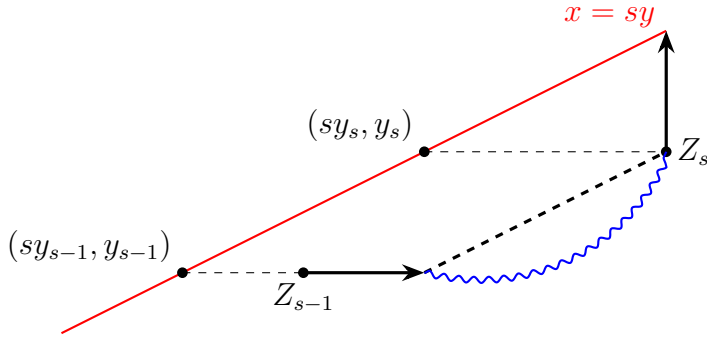A_{n+1} &= \sum_{\substack{0 \le y_0 \le \cdots \le y_{s-1} \le n \\ y_s = n}} \quad \sum_{\substack{P \in \mathcal{D}_{n+1}^s \\ \text{of type } y_0, \ldots, y_{s-1}}} \text{area}(P) \\
&= \sum_{\substack{0 \le y_0 \le \cdots \le y_{s-1} \le n \\ y_s = n}} \quad \sum_{\substack{P \in \mathcal{D}_{n+1}^s \\ \text{of type } y_0, \ldots, y_{s-1}}} \left(\sum_{l=0}^{s} \text{area}(P_l) + \sum_{k=1}^{s} I_l \right) \\
&= \sum_{\substack{0 \le y_0 \le \cdots \le y_{s-1} \le n \\ y_s = n}} \quad \sum_{\substack{P \in \mathcal{D}_{n+1}^s \\ \text{of type } y_0, \ldots, y_{s-1}}} \left(\sum_{l=1}^{s} \frac{1}{s}\left(l - \frac{1}{2}\right) + l(y_l - y_{l-1}) + \sum_{l=0}^{s} \text{area}(P_l) \right)
\end{aligned}
$$

(1)

3.1 Non-recursive Summands

The non-recursive part of equation (1) is

$$\sum_{\substack{0 \le y_0 \le \cdots \le y_{s-1} \le n \\ y_s = n}} \quad \sum_{\substack{P \in \mathcal{D}_{n+1}^s \\ \text{of type } y_0, \ldots, y_{s-1}}} \left(\sum_{l=1}^{s} \frac{1}{s}\left(l - \frac{1}{2}\right) + l(y_l - y_{l-1}) \right).$$

As an intermediate computation, we note that

$$\sum_{l=1}^{s} \frac{1}{s}\left(l - \frac{1}{2}\right) = \frac{1}{s} \cdot \frac{s(s+1)}{2} - \frac{1}{s} \cdot \frac{s}{2} = \frac{s}{2}.$$

We treat the above sum in two parts. The first part is given by

$$\sum_{\substack{0\le y_0\le\cdots\le y_{s-1}\le n\\ y_s=n}}\ \sum_{\substack{P\in\mathcal{D}^s_{n+1}\\ \text{of type } y_0,\ldots,y_{s-1}}}\ \sum_{l=1}^{s}\frac{1}{s}\Big(l-\frac{1}{2}\Big)=\sum_{\substack{0\le y_0\le\cdots\le y_{s-1}\le n\\ y_s=n}}\ \sum_{\substack{P\in\mathcal{D}^s_{n+1}\\ \text{of type } y_0,\ldots,y_{s-1}}}\frac{s}{2}$$

$$=\sum_{\substack{0\le y_0\le\cdots\le y_{s-1}\le n\\ y_s=n}}\frac{s}{2}\cdot|\{P\in\mathcal{D}^s_{n+1}\text{ of type } y_0,\ldots,y_{s-1}\}|$$

$$=\frac{s}{2}\sum_{\substack{0\le y_0\le\cdots\le y_{s-1}\le n\\ y_s=n}}F_{s+1}(y_0)\cdot\prod_{1\le i\le s}F_{s+1}(y_i-y_{i-1})$$

$$\overset{\substack{k=y_0\\ d_i=y_i-y_{i-1}}}{=}\frac{s}{2}\sum_{k=0}^{n}F_{s+1}(k)\sum_{d_1+\cdots+d_s=n-k}F_{s+1}(d_1)\cdots F_{s+1}(d_s)$$

$$=\frac{s}{2}\sum_{k=0}^{n}F_{s+1}(k)[F(z)^s]_{n-k}=\frac{s}{2}[F(z)^{s+1}]_n.$$

The second part of the sum utilizes a similar trick and evaluates as

$$\sum_{\substack{0\le y_0\le\cdots\le y_{s-1}\le n\\ y_s=n}}\ \sum_{\substack{P\in\mathcal{D}^s_{n+1}\\ \text{of type } y_0,\ldots,y_{s-1}}}\ \sum_{l=1}^{s}l(y_l-y_{l-1})=\sum_{l=1}^{s}l\sum_{\substack{0\le y_0\le\cdots\le y_{s-1}\le n\\ y_s=n}}\ \sum_{\substack{P\in\mathcal{D}^s_{n+1}\\ \text{of type } y_0,\ldots,y_{s-1}}}(y_l-y_{l-1})$$

$$=\sum_{l=1}^{s}l\sum_{\substack{0\le y_0\le\cdots\le y_{s-1}\le n\\ y_s=n}}(y_l-y_{l-1})F_{s+1}(y_0)\cdot\prod_{1\le i\le s}F_{s+1}(y_i-y_{i-1})$$

$$=\sum_{l=1}^{s}l\sum_{k=0}^{n}F_{s+1}(k)\sum_{d_1+\cdots+d_s=n-k}d_l\cdot\prod_{1\le i\le s}F_{s+1}(d_i)$$

$$=\sum_{l=1}^{s}l\sum_{k=0}^{n}F_{s+1}(k)\sum_{r=0}^{n-k}rF_{s+1}(r)\sum_{k_1+\cdots+k_{s-1}=n-k-r}\ \prod_{1\le i\le s-1}F_{s+1}(k_i)$$

$$=\sum_{l=1}^{s}l\sum_{k=0}^{n}F_{s+1}(k)\sum_{r=0}^{n-k}[zF'(z)]_r[F(z)^{s-1}]_{n-k-r}$$

$$=\sum_{l=1}^{s}l\sum_{k=0}^{n}F_{s+1}(k)[zF'(z)F(z)^{s-1}]_{n-k}$$

$$=\sum_{l=1}^{s}l\cdot[zF'(z)F(z)^s]_n=\frac{s(s+1)}{2}[zF'(z)F(z)^s]_n.$$

3.2 Recursive Summands

The last piece of the main expression to deal with is the area recursive sum. Observe that

$$\sum_{\substack{0\leq y_0\leq\cdots\leq y_{s-1}\leq n\\ y_s=n}}\sum_{\substack{P\in\mathcal{D}^s_{n+1}\\ \text{of type } y_0,\ldots,y_{s-1}}}\sum_{l=0}^{s}\text{area}(P_l) = \sum_{l=0}^{s}\sum_{\substack{0\leq y_0\leq\cdots\leq y_{s-1}\leq n\\ y_s=n}}\sum_{\substack{P\in\mathcal{D}^s_{n+1}\\ \text{of type } y_0,\ldots,y_{s-1}}}\text{area}(P_l)$$

$$= \sum_{\substack{0\leq y_0\leq\cdots\leq y_{s-1}\leq n\\ y_s=n}}\sum_{\substack{P\in\mathcal{D}^s_{n+1}\\ \text{of type } y_0,\ldots,y_{s-1}}}\text{area}(P_0) + \sum_{l=1}^{s}\sum_{\substack{0\leq y_0\leq\cdots\leq y_{s-1}\leq n\\ y_s=n}}\sum_{\substack{P\in\mathcal{D}^s_{n+1}\\ \text{of type } y_0,\ldots,y_{s-1}}}\text{area}(P_l)$$

The first part of this sum we compute as

$$\sum_{\substack{0\leq y_0\leq\cdots\leq y_{s-1}\leq n\\ y_s=n}}\sum_{\substack{P\in\mathcal{D}^s_{n+1}\\ \text{of type } y_0,\ldots,y_{s-1}}}\text{area}(P_0) = \sum_{\substack{0\leq y_0\leq\cdots\leq y_{s-1}\leq n\\ y_s=n}}\sum_{\gamma\in\mathcal{D}^s_{y_0}}\sum_{\substack{P\in\mathcal{D}^s_{n+1}\\ \text{of type } y_0,\ldots,y_{s-1}\\ \text{s.t. } P_0=\gamma}}\text{area}(\gamma)$$

$$= \sum_{\substack{0\leq y_0\leq\cdots\leq y_{s-1}\leq n\\ y_s=n}}\sum_{\gamma\in\mathcal{D}^s_{y_0}}\text{area}(\gamma)\cdot|\{P\in\mathcal{D}^s_{n+1}\text{ of type } y_0,\ldots,y_{s-1}\text{ s.t. } P_0=\gamma\}|$$

$$= \sum_{\substack{0\leq y_0\leq\cdots\leq y_{s-1}\leq n\\ y_s=n}}\sum_{\gamma\in\mathcal{D}^s_{y_0}}\text{area}(\gamma)\cdot\prod_{1\leq i\leq s}F_{s+1}(y_i-y_{i-1})$$

$$= \sum_{k=0}^{n}\sum_{d_1+\cdots+d_s=n-k}\sum_{\gamma\in\mathcal{D}^s_k}\text{area}(\gamma)\cdot\prod_{1\leq i\leq s}F_{s+1}(d_i)$$

$$= \sum_{k=0}^{n}A_k\sum_{d_1+\cdots+d_s=n-k}\prod_{1\leq i\leq s}F_{s+1}(d_i)$$

$$= \sum_{k=0}^{n}A_k[F(z)^s]_{n-k} = [A(z)F(z)^s]_n$$

We now treat the second part of the aforementioned sum. For brevity, we write $F(z) = F_{s+1}(z) = \sum_{n\geq 0}\frac{1}{sn+1}\binom{(s+1)n}{n}z^n$. (might be big pageskip cuz big computation below)

$$\sum_{l=1}^{s} \sum_{\substack{0 \le y_0 \le \cdots \le y_{s-1} \le n \\ y_s = n}} \sum_{\substack{P \in \mathcal{D}_{n+1}^s \\ \text{of type } y_0,\ldots,y_{s-1}}} \text{area}(P_l) = \sum_{\substack{0 \le y_0 \le \cdots \le y_{s-1} \le n \\ y_s = n}} \sum_{\gamma \in \mathcal{D}_{y_l - y_{l-1}}^s} \sum_{\substack{P \in \mathcal{D}_{n+1}^s \\ \text{of type } y_0,\ldots,y_{s-1} \\ P_l = \gamma}} \text{area}(\gamma)$$

$$= \sum_{l=1}^{s} \sum_{\substack{0 \le y_0 \le \cdots \le y_{s-1} \le n \\ y_s = n}} \sum_{\gamma \in \mathcal{D}_{y_l - y_{l-1}}^s} \text{area}(\gamma) \cdot |\{P \in \mathcal{D}_{n+1}^s \text{of type } (y_0, \ldots, y_{s-1}), \text{ with } P_l = \gamma\}|$$

$$= \sum_{l=1}^{s} \sum_{\substack{0 \le y_0 \le \cdots \le y_{s-1} \le n \\ y_s = n}} \sum_{\gamma \in \mathcal{D}_{y_l - y_{l-1}}^s} \text{area}(\gamma) \cdot F_{s+1}(y_0) \prod_{\substack{1 \le i \le s \\ i \ne l}} |F_{s+1}(y_i - y_{i-1})|$$

$$= \sum_{l=1}^{s} \sum_{\substack{0 \le y_0 \le \cdots \le y_{s-1} \le n \\ y_s = n}} A_{y_l - y_{l-1}} \cdot F_{s+1}(y_0) \left(\prod_{\substack{1 \le i \le s \\ i \ne l}} |F_{s+1}(y_i - y_{i-1})| \right)$$

$$= \sum_{l=1}^{s} \sum_{k=0}^{n} \sum_{d_1 + \cdots + d_s = n - k} A_{d_l} \cdot F_{s+1}(k) \left(\prod_{\substack{1 \le i \le s \\ i \ne l}} |F_{s+1}(d_i)| \right)$$

$$= \sum_{l=1}^{s} \sum_{k=0}^{n} F_{s+1}(k) \sum_{r=0}^{n-k} A_r \sum_{k_1 + \cdots + k_{s-1} = n - k - r} F_{s+1}(k_1) \cdots F_{s+1}(k_{s-1})$$

$$= \sum_{l=1}^{s} \sum_{k=0}^{n} F_{s+1}(k) \sum_{r=0}^{n-k} [A(z)]_r \cdot [F(z)^{s-1}]_{n-k-r} = \sum_{l=1}^{s} \sum_{k=0}^{n} [F(z)]_k \cdot [A(z) F(z)^{s-1}]_{n-k}$$

$$= s \cdot \sum_{l=1}^{s} [A(z) F(z)^s]_n = s [A(z) F(z)^s]_n$$

Lastly, we substitute all of these results back into the expression for A_{n+1} to find

$$A_{n+1} = \frac{s}{2} [F(z)^{s+1}]_n + \frac{s(s+1)}{2} [z F'(z) F(z)^s]_n + \sum_{\substack{0 \le y_0 \le \cdots \le y_{s-1} \le n \\ y_s = n}} \sum_{\substack{P \in \mathcal{D}_{n+1}^s \\ \text{of type } y_0,\ldots,y_{s-1}}} \sum_{l=0}^{s} \text{area}(P_l)$$

$$= \frac{s}{2} [F(z)^{s+1}]_n + \frac{s(s+1)}{2} [z F'(z) F(z)^s]_n + [A(z) F(z)^s]_n + s [A(z) F(z)^s]_n$$

Hence, we finally obtain a functional equation for the generating function $A(z)$.

$$\frac{A(z)}{z} = \frac{s}{2} F(z)^{s+1} + \frac{s(s+1)}{2} z F'(z) F(z)^s + (s+1) A(z) F(z)^s \tag{2}$$

$$A = \frac{s}{2} z F^{s+1} + \binom{s+1}{2} z^2 F' F^s + (s+1) z A F^s \implies A(1 - (s+1) z F^s) = \frac{s}{2} z F^{s+1} + \binom{s+1}{2} z^2 F' F^s$$

$$\implies A = \left(\frac{s}{2} z F^{s+1} + \binom{s+1}{2} z^2 F' F^s \right) \cdot \frac{1}{1 - (s+1) z F^s}$$

$$A = \left(\frac{s}{2}zF^{s+1} + \binom{s+1}{2}z^2F'F^s\right) \cdot \frac{1}{1-(s+1)zF^s}$$

$$= \frac{s}{2}\frac{zF^{s+1}}{1-(s+1)zF^s} + \binom{s+1}{2}zF'\frac{zF^s}{1-(s+1)zF^s}$$

$$= \frac{s}{2}zF' + \binom{s+1}{2}\frac{1}{F}(zF')^2$$

4 Asymptotics

$$f_n := \frac{1}{sn+1}\binom{(s+1)n}{n} \sim \frac{1}{\sqrt{2\pi}}\frac{\sqrt{s+1}}{s^{3/2}}\left(\frac{(s+1)^{s+1}}{s^s}\right)^n n^{-3/2}$$

$$A_n = \frac{s}{2}nf_n + \binom{s+1}{2}\sum_{k=0}^{n}k(n-k)f_kf_{n-k} - \binom{s+1}{2}\sum_{j=0}^{n-1}\frac{1}{n-j}\binom{(s+1)(n-j)-2}{n-j-1}\sum_{i=0}^{j}i(j-i)f_if_{i-j}$$

$$\frac{1}{f(n)}\sum_{k=\delta n}^{(1-\delta)n}k(n-k)f(k)f(n-k) \geq (1-\varepsilon)^2\frac{1}{\sqrt{2\pi}}\frac{\sqrt{s+1}}{s^{3/2}}\sum_{k=\delta n}^{(1-\delta)n}k(n-k)\frac{k^{-3/2}(n-k)^{-3/2}}{n^{-3/2}}$$

$$= (1-\varepsilon)^2\frac{1}{\sqrt{2\pi}}\frac{\sqrt{s+1}}{s^{3/2}}\sum_{k=\delta n}^{(1-\delta)n}\frac{n^{3/2}}{\sqrt{k(n-k)}}$$

$$= (1-\varepsilon)^2\frac{1}{\sqrt{2\pi}}\frac{\sqrt{s+1}}{s^{3/2}}\left(\frac{1}{n}\sum_{k=\delta n}^{(1-\delta)n}\frac{n^{3/2}}{\sqrt{\left(\frac{k}{n}\right)\left(1-\frac{k}{n}\right)}}\right)$$

$$\underset{\text{error}}{\geq}\xrightarrow{n\to\infty}0\;(1-\varepsilon)^2\frac{1}{\sqrt{2\pi}}\frac{\sqrt{s+1}}{s^{3/2}}n^{3/2}\int_{\delta}^{1-\delta}\frac{1}{\sqrt{\lambda(1-\lambda)}}d\lambda$$

$$= (1-\varepsilon)^2\frac{1}{\sqrt{2\pi}}\frac{\sqrt{s+1}}{s^{3/2}}n^{3/2}\left(2\arcsin\left(\sqrt{1-\delta}\right) - 2\arcsin\left(\sqrt{\delta}\right)\right)$$

$$\underset{\text{error}}{\geq}\xrightarrow{\delta\to0}0\;(1-\varepsilon)^2\sqrt{\frac{\pi}{2}}\frac{\sqrt{s+1}}{s^{3/2}}n^{3/2}$$

which when multiplying by $\binom{s+1}{2}$, we get a term of

$$\frac{s(s+1)}{2}\sqrt{\frac{\pi}{2}}\frac{\sqrt{s+1}}{s^{3/2}}n^{3/2} = \frac{1}{2}\sqrt{\frac{\pi}{2}}\frac{(s+1)^{3/2}}{\sqrt{s}}n^{3/2}$$