Algebraic Actions Notes

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1 Preliminary Definitions

Definition 1.0.1. An *algebraic action* is an action of a countable, discrete group G on a compact, metrizable group X by continuous automorphisms. After giving X the Haar measure m_X, the action $G \curvearrowright (X, m_X)$ is measure-preserving.

Suppose $G \curvearrowright A$ and A is a discrete abelian group. Then A has a $\mathbb{Z}(G)$-module structure given by

$$\left(\sum_g a_g g\right) a := \sum_g a_g \alpha_g(a) = \sum_g a_g(ga) \quad \text{for } a \in A.$$

Note that $\mathbb{Z}(G)$-modules and actions of G on discrete abelian groups are equivalent by the above procedure. Fix $f \in \mathbb{Z}(G)$ and consider the principal $\mathbb{Z}(G)$-module $A := \mathbb{Z}(G)/\mathbb{Z}(G)f$ with $\mathbb{Z}(G)$-action given by left multiplication. Letting $X_f := \widehat{A}$ (the space of continuous homomorphisms from A to $\mathbb{R}/\mathbb{Z}$), we have an induced action $G \curvearrowright X_f$ given by

$$[g\chi](a) := \chi(g^{-1}a) \quad \text{for } \chi \in \widehat{A},\, a \in A.$$

This is an example of an algebraic action since X_f is a compact, metrizable group. Such algebraic actions are called *principal*.

Definition 1.0.2. An element $f = \sum a_g g \in \mathbb{Z}(G)$ is *semi-lopsided* if

$$\sum_{g \in G \setminus e} |a_g| \leq a_e,$$

and is *lopsided* if the above inequality is strict.

Definition 1.0.3. A *harmonic model* is a principal algebraic action X_f where $f \in \mathbb{Z}(G)$ is of the form

$$f = m\,(1 - p)$$

where $m \in \mathbb{Z}$, $p \in \mathbb{Q}(G)$ and $p = \sum_{g \in G} \mu_g g$ with $\mu \in Prob(G)$ is finitely supported.

Definition 1.0.4. Let Γ be a countable discrete group. Denote by $\mathrm{Ord}(\Gamma)$ (resp. p-$\mathrm{Ord}(\Gamma)$) the set of total (resp. partial) orders on Γ. We may view

$$\mathrm{Ord}(\Gamma) \subseteq \text{p-}\mathrm{Ord}(\Gamma) \subseteq \{0,1\}^{\Gamma \times \Gamma},$$

whence the set of total orders forms a closed subset of the set of partial orders on Γ. Then $\Gamma \times \Gamma \curvearrowright \mathrm{Ord}(\Gamma)$ by

$$[(\gamma, \delta) \cdot \rho](x, y) = \rho(\gamma^{-1} x \delta, \gamma^{-1} y \delta).$$

Viewing $\Gamma = \Gamma \times \{e\} \subseteq \Gamma \times \Gamma$, we get an action $\Gamma \curvearrowright \mathrm{Ord}(\Gamma)$. An *invariant random order* (IRO) on Γ is then a Γ-invariant Borel probability measure on $\mathrm{Ord}(\Gamma)$.