The Total Area Statistic for Fuss-Catalan Paths

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2 Fuss-Catalan Preliminaries

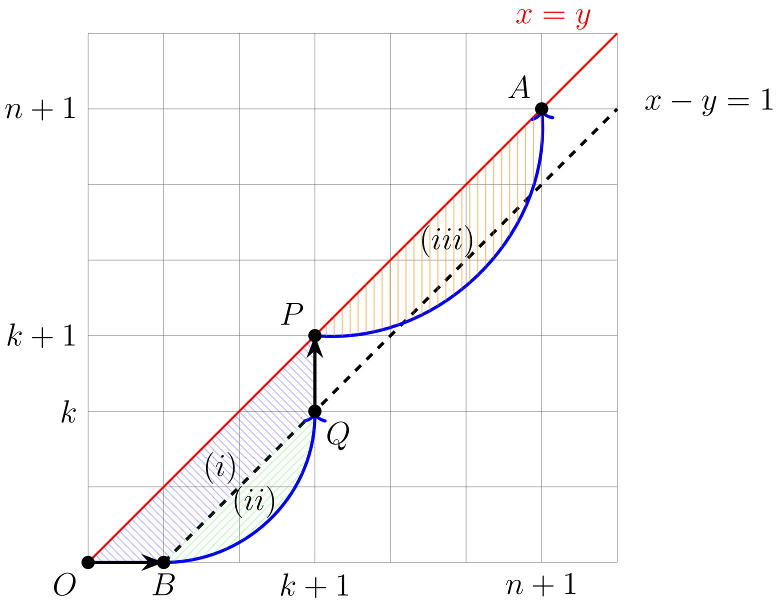
$$F_m(n,k) = \frac{k}{mn+k}\binom{mn+k}{n}$$

$$F_m(n) = F_m(n,1) = \frac{1}{mn+1}\binom{mn+1}{n} = \frac{1}{(m-1)n+1}\binom{mn}{n}$$

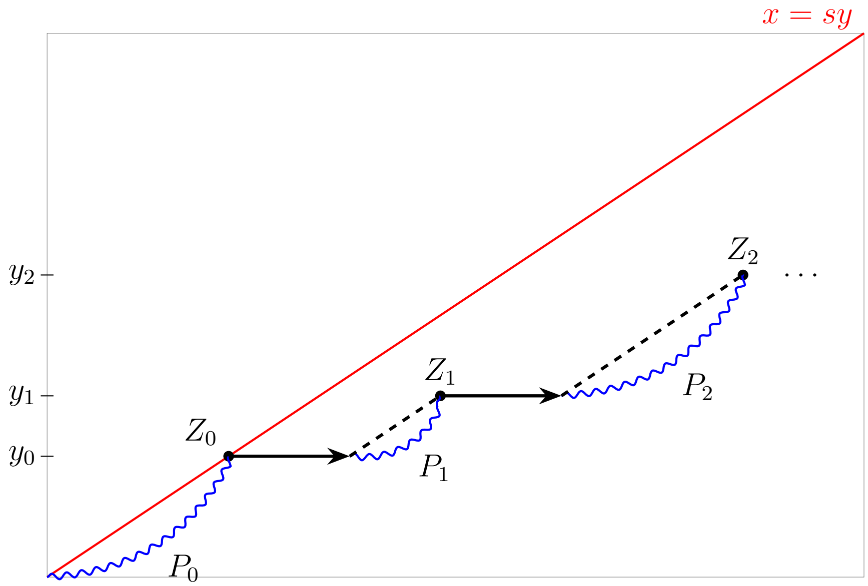
$$F_2(n) = F_2(n,1) = c_n$$

$$\mathcal{D}_n^s := L((0,0)\to(sn,n) : x \geq sy), \quad |\mathcal{D}_n^s| = F_{s+1}(n) = \frac{1}{sn+1}\binom{(s+1)n}{n} = \frac{1}{(s+1)n+1}\binom{(s+1)n+1}{n}$$

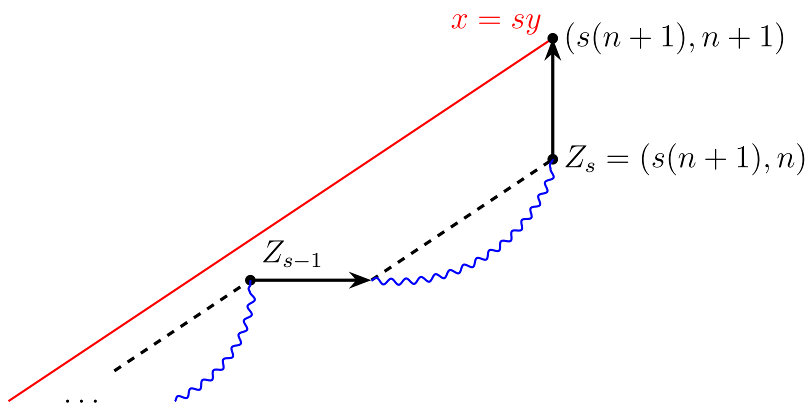
$$F_m(z) = \sum_{n\geq 0} F_m(n,1)z^n, \quad (F_m(z))^k = \sum_{n\geq 0} F_m(n,k)z^n, \quad F_m(z) = 1 + z(F_m(z))^m$$



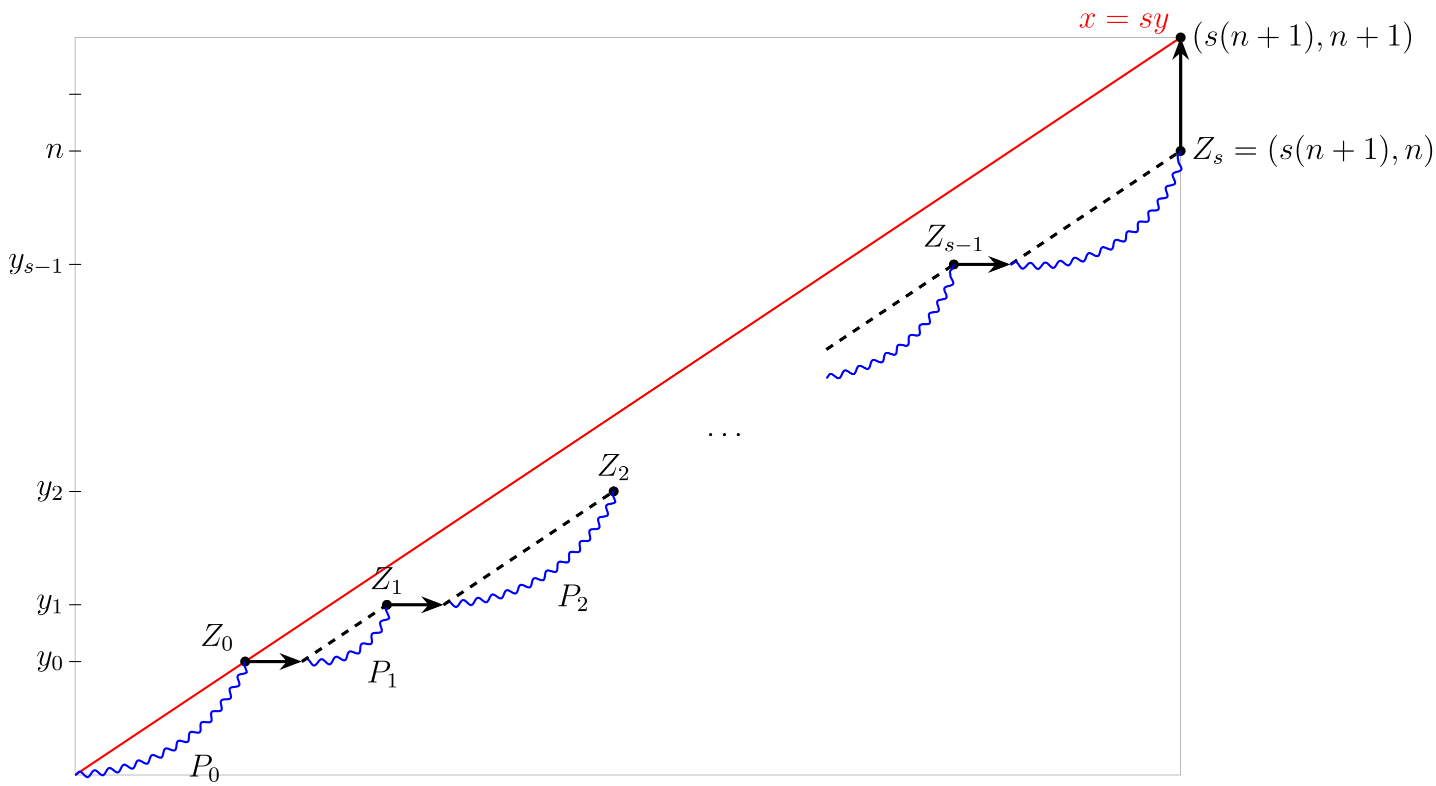
$x=y$
$x-y=1$
$n+1$
$k+1$
k
A
P
Q
O
B
(i)
(ii)
(iii)
$k+1$
$n+1$



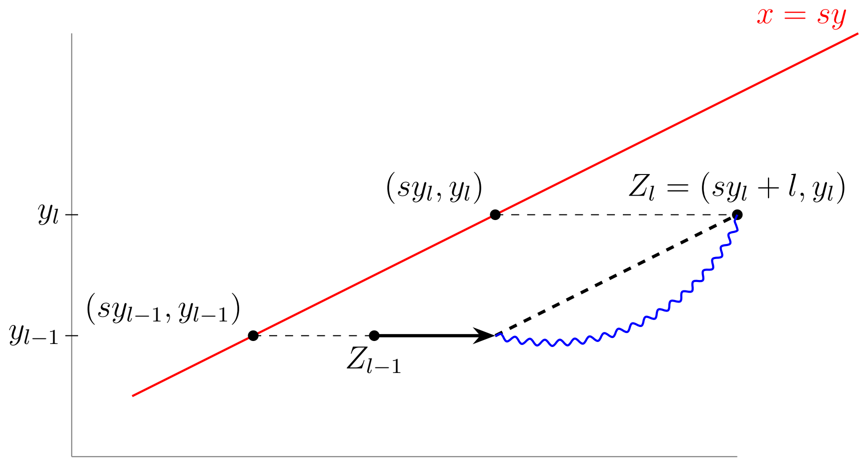
$x = sy$
y_2
y_1
y_0
Z_0
Z_1
Z_2
P_0
P_1
P_2
$\cdots$



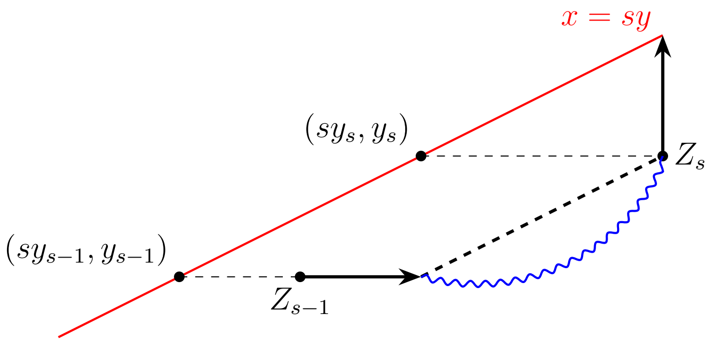
$x = sy$
$(s(n+1), n+1)$
$Z_s = (s(n+1), n)$
Z_{s-1}
$\dots$



$x = sy$
$(s(n+1), n+1)$
$Z_s = (s(n+1), n)$
n
y_{s-1}
Z_{s-1}
$\dots$
y_2
Z_2
y_1
Z_1
P_2
y_0
Z_0
P_1
P_0



$x = sy$
(sy_l, y_l)
$Z_l = (sy_l + l, y_l)$
y_l
(sy_{l-1}, y_{l-1})
y_{l-1}
Z_{l-1}



3 Recursive Decomposition

Let

$$A_n = \sum_{P \in \mathcal{D}_n^s} \text{area}(P)$$

$$\begin{aligned}
A_{n+1} &= \sum_{\substack{0 \le y_0 \le \cdots \le y_{s-1} \le n \\ y_s = n}} \sum_{\substack{P \in \mathcal{D}_{n+1}^s \\ \text{of type } y_0, \ldots, y_{s-1}}} \text{area}(P) \\
&= \sum_{\substack{0 \le y_0 \le \cdots \le y_{s-1} \le n \\ y_s = n}} \sum_{\substack{P \in \mathcal{D}_{n+1}^s \\ \text{of type } y_0, \ldots, y_{s-1}}} \left(\sum_{l=0}^{s} \text{area}(P_l) + \sum_{k=1}^{s} I_l \right) \\
&= \sum_{\substack{0 \le y_0 \le \cdots \le y_{s-1} \le n \\ y_s = n}} \sum_{\substack{P \in \mathcal{D}_{n+1}^s \\ \text{of type } y_0, \ldots, y_{s-1}}} \left(\sum_{l=1}^{s} \frac{1}{s}(l - \frac{1}{2}) + l(y_l - y_{l-1}) + \sum_{l=0}^{s} \text{area}(P_l) \right) \qquad (1)
\end{aligned}$$

3.1 Non-recursive Summands

The non-recursive part of equation (1) is

$$\sum_{\substack{0 \le y_0 \le \cdots \le y_{s-1} \le n \\ y_s = n}} \sum_{\substack{P \in \mathcal{D}_{n+1}^s \\ \text{of type } y_0, \ldots, y_{s-1}}} \left(\sum_{l=1}^{s} \frac{1}{s}(l - \frac{1}{2}) + l(y_l - y_{l-1}) \right).$$

As an intermediate computation, we note that

$$\sum_{l=1}^{s} \frac{1}{s} \left(l - \frac{1}{2} \right) = \frac{1}{s} \cdot \frac{s(s+1)}{2} - \frac{1}{s} \cdot \frac{s}{2} = \frac{s}{2}.$$

We treat the above sum in two parts. The first part is given by

$$\begin{aligned}
\sum_{\substack{0\le y_0\le\cdots\le y_{s-1}\le n\\ y_s=n}} \sum_{\substack{P\in\mathcal{D}^s_{n+1}\\ \text{of type } y_0,\ldots,y_{s-1}}} \sum_{l=1}^{s} \frac{1}{s}(l-\frac{1}{2}) &= \sum_{\substack{0\le y_0\le\cdots\le y_{s-1}\le n\\ y_s=n}} \sum_{\substack{P\in\mathcal{D}^s_{n+1}\\ \text{of type } y_0,\ldots,y_{s-1}}} \frac{s}{2} \\
&= \sum_{\substack{0\le y_0\le\cdots\le y_{s-1}\le n\\ y_s=n}} \frac{s}{2}\cdot|\{P\in\mathcal{D}^s_{n+1} \text{ of type } y_0,\ldots,y_{s-1}\}| \\
&= \frac{s}{2} \sum_{\substack{0\le y_0\le\cdots\le y_{s-1}\le n\\ y_s=n}} F_{s+1}(y_0)\cdot \prod_{1\le i\le s} F_{s+1}(y_i-y_{i-1}) \\
&\overset{\substack{k=y_0\\ d_i=y_i-y_{i-1}}}{=} \frac{s}{2}\sum_{k=0}^{n} F_{s+1}(k) \sum_{d_1+\cdots+d_s=n-k} F_{s+1}(d_1)\cdots F_{s+1}(d_s) \\
&= \frac{s}{2}\sum_{k=0}^{n} F_{s+1}(k)[F(z)^s]_{n-k} = \frac{s}{2}[F(z)^{s+1}]_n
\end{aligned}$$

The second part of the sum utilizes a similar trick and evaluates as

$$\begin{aligned}
\sum_{\substack{0\le y_0\le\cdots\le y_{s-1}\le n\\ y_s=n}} \sum_{\substack{P\in\mathcal{D}^s_{n+1}\\ \text{of type } y_0,\ldots,y_{s-1}}} \sum_{l=1}^{s} l(y_l-y_{l-1}) &= \sum_{l=1}^{s} l \sum_{\substack{0\le y_0\le\cdots\le y_{s-1}\le n\\ y_s=n}} \sum_{\substack{P\in\mathcal{D}^s_{n+1}\\ \text{of type } y_0,\ldots,y_{s-1}}} (y_l-y_{l-1}) \\
&= \sum_{l=1}^{s} l \sum_{\substack{0\le y_0\le\cdots\le y_{s-1}\le n\\ y_s=n}} (y_l-y_{l-1})F_{s+1}(y_0)\cdot\prod_{1\le i\le s} F_{s+1}(y_i-y_{i-1}) \\
&= \sum_{l=1}^{s} l \sum_{k=0}^{n} F_{s+1}(k) \sum_{d_1+\cdots+d_s=n-k} d_l\cdot \prod_{1\le i\le s} F_{s+1}(d_i) \\
&= \sum_{l=1}^{s} l \sum_{k=0}^{n} F_{s+1}(k) \sum_{r=0}^{n-k} rF_{s+1}(r) \sum_{k_1+\cdots+k_{s-1}=n-k-r} \prod_{1\le i\le s-1} F_{s+1}(k_i) \\
&= \sum_{l=1}^{s} l \sum_{k=0}^{n} F_{s+1}(k) \sum_{r=0}^{n-k} [zF'(z)]_r[F(z)^{s-1}]_{n-k-r} \\
&= \sum_{l=1}^{s} l \sum_{k=0}^{n} F_{s+1}(k)[zF'(z)F(z)^{s-1}]_{n-k} \\
&= \sum_{l=1}^{s} l\cdot[zF'(z)F(z)^s]_n = \frac{s(s+1)}{2}[zF'(z)F(z)^s]_n
\end{aligned}$$

3.2 Recursive Summands

The last piece of the main expression to deal with is the area recursive sum. Observe that

$$\sum_{\substack{0\le y_0\le\cdots\le y_{s-1}\le n\\ y_s=n}} \sum_{\substack{P\in\mathcal{D}^s_{n+1}\\ \text{of type } y_0,\ldots,y_{s-1}}} \sum_{l=0}^{s} \operatorname{area}(P_l) = \sum_{l=0}^{s} \sum_{\substack{0\le y_0\le\cdots\le y_{s-1}\le n\\ y_s=n}} \sum_{\substack{P\in\mathcal{D}^s_{n+1}\\ \text{of type } y_0,\ldots,y_{s-1}}} \operatorname{area}(P_l)$$

$$= \sum_{\substack{0\le y_0\le\cdots\le y_{s-1}\le n\\ y_s=n}} \sum_{\substack{P\in\mathcal{D}^s_{n+1}\\ \text{of type } y_0,\ldots,y_{s-1}}} \operatorname{area}(P_0) + \sum_{l=1}^{s} \sum_{\substack{0\le y_0\le\cdots\le y_{s-1}\le n\\ y_s=n}} \sum_{\substack{P\in\mathcal{D}^s_{n+1}\\ \text{of type } y_0,\ldots,y_{s-1}}} \operatorname{area}(P_l)$$

The first part of this sum we compute as

$$\begin{aligned}
\sum_{\substack{0\le y_0\le\cdots\le y_{s-1}\le n\\ y_s=n}} \sum_{\substack{P\in\mathcal{D}^s_{n+1}\\ \text{of type } y_0,\ldots,y_{s-1}}} \operatorname{area}(P_0) &= \sum_{\substack{0\le y_0\le\cdots\le y_{s-1}\le n\\ y_s=n}} \sum_{\gamma\in\mathcal{D}^s_{y_0}} \sum_{\substack{P\in\mathcal{D}^s_{n+1}\\ \text{of type } y_0,\ldots,y_{s-1}\\ \text{s.t. } P_0=\gamma}} \operatorname{area}(\gamma)\\
&= \sum_{\substack{0\le y_0\le\cdots\le y_{s-1}\le n\\ y_s=n}} \sum_{\gamma\in\mathcal{D}^s_{y_0}} \operatorname{area}(\gamma)\cdot |\{P\in\mathcal{D}^s_{n+1} \text{ of type } y_0,\ldots,y_{s-1} \text{ s.t. } P_0=\gamma\}|\\
&= \sum_{\substack{0\le y_0\le\cdots\le y_{s-1}\le n\\ y_s=n}} \sum_{\gamma\in\mathcal{D}^s_{y_0}} \operatorname{area}(\gamma)\cdot \prod_{1\le i\le s} F_{s+1}(y_i-y_{i-1})\\
&= \sum_{k=0}^{n} \sum_{d_1+\cdots+d_s=n-k} \sum_{\gamma\in\mathcal{D}^s_k} \operatorname{area}(\gamma)\cdot \prod_{1\le i\le s} F_{s+1}(d_i)\\
&= \sum_{k=0}^{n} A_k \sum_{d_1+\cdots+d_s=n-k} \prod_{1\le i\le s} F_{s+1}(d_i)\\
&= \sum_{k=0}^{n} A_k [F(z)^s]_{n-k} = [A(z)F(z)^s]_n
\end{aligned}$$

We now treat the second part of the aforementioned sum. For brevity, we write $F(z) = F_{s+1}(z) = \sum_{n\ge 0} \frac{1}{sn+1}\binom{(s+1)n}{n} z^n$. (might be big pageskip cuz big computation below)

$$\sum_{l=1}^{s}\sum_{\substack{0\le y_0\le\cdots\le y_{s-1}\le n\\ y_s=n}}\sum_{\substack{P\in\mathcal{D}^s_{n+1}\\ \text{of type } y_0,\ldots,y_{s-1}}}\operatorname{area}(P_l)=\sum_{\substack{0\le y_0\le\cdots\le y_{s-1}\le n\\ y_s=n}}\sum_{\gamma\in\mathcal{D}^s_{y_l-y_{l-1}}}\sum_{\substack{P\in\mathcal{D}^s_{n+1}\\ \text{of type } y_0,\ldots,y_{s-1}\\ P_l=\gamma}}\operatorname{area}(\gamma)$$

$$=\sum_{l=1}^{s}\sum_{\substack{0\le y_0\le\cdots\le y_{s-1}\le n\\ y_s=n}}\sum_{\gamma\in\mathcal{D}^s_{y_l-y_{l-1}}}\operatorname{area}(\gamma)\cdot|\{P\in\mathcal{D}^s_{n+1}\text{of type }(y_0,\ldots,y_{s-1}),\text{ with }P_l=\gamma\}|$$

$$=\sum_{l=1}^{s}\sum_{\substack{0\le y_0\le\cdots\le y_{s-1}\le n\\ y_s=n}}\sum_{\gamma\in\mathcal{D}^s_{y_l-y_{l-1}}}\operatorname{area}(\gamma)\cdot F_{s+1}(y_0)\prod_{\substack{1\le i\le s\\ i\ne l}}|F_{s+1}(y_i-y_{i-1})|$$

$$=\sum_{l=1}^{s}\sum_{\substack{0\le y_0\le\cdots\le y_{s-1}\le n\\ y_s=n}}A_{y_l-y_{l-l}}\cdot F_{s+1}(y_0)\left(\prod_{\substack{1\le i\le s\\ i\ne l}}|F_{s+1}(y_i-y_{i-1})|\right)$$

$$=\sum_{l=1}^{s}\sum_{k=0}^{n}\sum_{d_1+\cdots+d_s=n-k}A_{d_l}\cdot F_{s+1}(k)\left(\prod_{\substack{1\le i\le s\\ i\ne l}}|F_{s+1}(d_i)|\right)$$

$$=\sum_{l=1}^{s}\sum_{k=0}^{n}F_{s+1}(k)\sum_{r=0}^{n-k}A_r\sum_{k_1+\cdots+k_{s-1}=n-k-r}F_{s+1}(k_1)\cdots F_{s+1}(k_{s-1})$$

$$=\sum_{l=1}^{s}\sum_{k=0}^{n}F_{s+1}(k)\sum_{r=0}^{n-k}[A(z)]_r\cdot[F(z)^{s-1}]_{n-k-r}=\sum_{l=1}^{s}\sum_{k=0}^{n}[F(z)]_k\cdot[A(z)F(z)^{s-1}]_{n-k}$$

$$=s\cdot\sum_{l=1}^{s}[A(z)F(z)^s]_n=s[A(z)F(z)^s]_n$$

Lastly, we substitute all of these results back into the expression for A_{n+1} to find

$$A_{n+1}=\frac{s}{2}[F(z)^{s+1}]_n+\frac{s(s+1)}{2}[zF'(z)F(z)^s]_n+\sum_{\substack{0\le y_0\le\cdots\le y_{s-1}\le n\\ y_s=n}}\sum_{\substack{P\in\mathcal{D}^s_{n+1}\\ \text{of type } y_0,\ldots,y_{s-1}}}\sum_{l=0}^{s}\operatorname{area}(P_l)$$

$$=\frac{s}{2}[F(z)^{s+1}]_n+\frac{s(s+1)}{2}[zF'(z)F(z)^s]_n+[A(z)F(z)^s]_n+s[A(z)F(z)^s]_n$$

Hence, we finally obtain a functional equation for the generating function $A(z)$.

$$\frac{A(z)}{z}=\frac{s}{2}F(z)^{s+1}+\frac{s(s+1)}{2}zF'(z)F(z)^s+(s+1)A(z)F(z)^s \tag{2}$$

$$A=\frac{s}{2}zF^{s+1}+\binom{s+1}{2}z^2F'F^s+(s+1)zAF^s\implies A(1-(s+1)zF^s)=\frac{s}{2}zF^{s+1}+\binom{s+1}{2}z^2F'F^s$$

$$\implies A=\left(\frac{s}{2}zF^{s+1}+\binom{s+1}{2}z^2F'F^s\right)\cdot\frac{1}{1-(s+1)zF^s}$$

$$
\begin{aligned}
A &= \left(\frac{s}{2}zF^{s+1} + \binom{s+1}{2}z^2F'F^s\right)\cdot\frac{1}{1-(s+1)zF^s} \\
&= \frac{s}{2}\frac{zF^{s+1}}{1-(s+1)zF^s} + \binom{s+1}{2}zF'\frac{zF^s}{1-(s+1)zF^s} \\
&= \frac{s}{2}zF' + \binom{s+1}{2}\frac{1}{F}(zF')^2
\end{aligned}
$$

4 Asymptotics

$$
f_n := \frac{1}{sn+1}\binom{(s+1)n}{n} \sim \frac{1}{\sqrt{2\pi}}\frac{\sqrt{s+1}}{s^{3/2}}\left(\frac{(s+1)^{s+1}}{s^s}\right)^n n^{-3/2}
$$

$$
A_n = \frac{s}{2}nf_n + \binom{s+1}{2}\sum_{k=0}^{n} k(n-k)f_kf_{n-k} - \binom{s+1}{2}\sum_{j=0}^{n-1}\frac{1}{n-j}\binom{(s+1)(n-j)-2}{n-j-1}\sum_{i=0}^{j} i(j-i)f_if_{i-j}
$$

$$
\begin{aligned}
\frac{1}{f(n)}\sum_{k=\delta n}^{(1-\delta)n} k(n-k)f(k)f(n-k) &\geq (1-\varepsilon)^2\frac{1}{\sqrt{2\pi}}\frac{\sqrt{s+1}}{s^{3/2}}\sum_{k=\delta n}^{(1-\delta)n} k(n-k)\frac{k^{-3/2}(n-k)^{-3/2}}{n^{-3/2}} \\
&= (1-\varepsilon)^2\frac{1}{\sqrt{2\pi}}\frac{\sqrt{s+1}}{s^{3/2}}\sum_{k=\delta n}^{(1-\delta)n}\frac{n^{3/2}}{\sqrt{k(n-k)}} \\
&= (1-\varepsilon)^2\frac{1}{\sqrt{2\pi}}\frac{\sqrt{s+1}}{s^{3/2}}\left(\frac{1}{n}\sum_{k=\delta n}^{(1-\delta)n}\frac{n^{3/2}}{\sqrt{\left(\frac{k}{n}\right)\left(1-\frac{k}{n}\right)}}\right) \\
&\geq_{\text{error}\xrightarrow{n\to\infty}0} (1-\varepsilon)^2\frac{1}{\sqrt{2\pi}}\frac{\sqrt{s+1}}{s^{3/2}}n^{3/2}\int_{\delta}^{1-\delta}\frac{1}{\sqrt{\lambda(1-\lambda)}}\,d\lambda \\
&= (1-\varepsilon)^2\frac{1}{\sqrt{2\pi}}\frac{\sqrt{s+1}}{s^{3/2}}n^{3/2}\left(2\arcsin\left(\sqrt{1-\delta}\right) - 2\arcsin\left(\sqrt{\delta}\right)\right) \\
&\geq_{\text{error}\xrightarrow{\delta\to 0}0} (1-\varepsilon)^2\sqrt{\frac{\pi}{2}}\frac{\sqrt{s+1}}{s^{3/2}}n^{3/2}
\end{aligned}
$$

which when multiplying by $\binom{s+1}{2}$, we get a term of

$$
\frac{s(s+1)}{2}\sqrt{\frac{\pi}{2}}\frac{\sqrt{s+1}}{s^{3/2}}n^{3/2} = \frac{1}{2}\sqrt{\frac{\pi}{2}}\frac{(s+1)^{3/2}}{\sqrt{s}}n^{3/2}
$$

Section 3 for Paper

Instead of using a single centerline as in the (n,n) case, we define s lines

$$L_i := \{(x,y) : x = sy + i\}, \quad 0 \le i < s$$

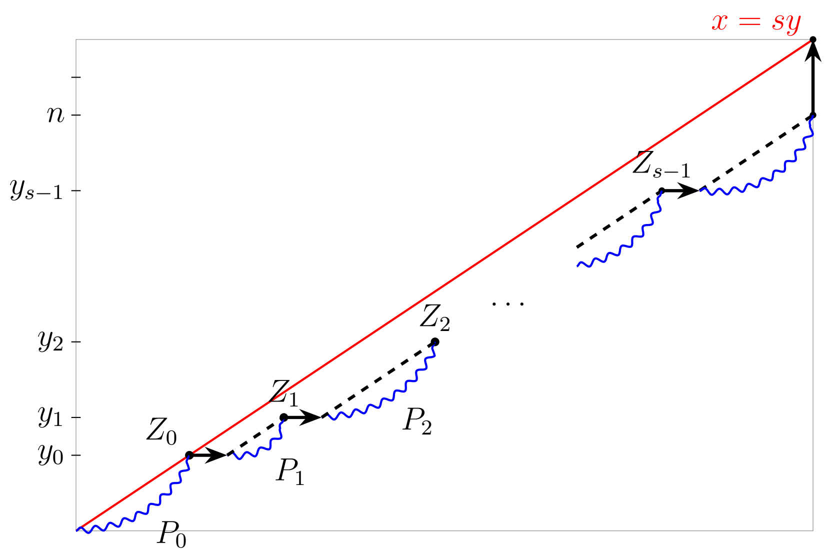
and use these lines to perform a touchpoint decomposition. Fix a path $P \in D^s_{n+1}$ and consider the following procedure. The path P will meet the line L_0 somewhere for the last time (excluding $(s(n+1), n+1)$). Denote this point $Z_0 = (sy_0, y_0)$. Let P_0 be the subpath of P from $(0,0)$ to Z_0. The next point in P is forced to be $Z_0 + (1,0) \in L_1$.

Now, for $1 \le k < s$,

- P meets L_k for the last time at $Z_k = (sy_k, y_k)$,
- P_k is the subpath of P from $Z_{k-1} + (1,0) \in L_k$ to Z_k,
- the next point in P after P_k is forced to be $Z_k + (1,0) \in L_{k+1}$.

Finally, let P_s be the subpath of P from $Z_{s-1} + (1,0)$ to $Z_s := (s(n+1), n)$ and note that the last move in the path is to $(s(n+1), n+1)$.

Definition 4.0.1. The vector $\vec{y} = (y_0, \dots, y_{s-1})$ is called the *type* of the path P. We denote by $D^s_n(\vec{y})$ the set of all paths in D^s_n of type $\vec{y}$.



4.1 Total Area Using Touchpoints

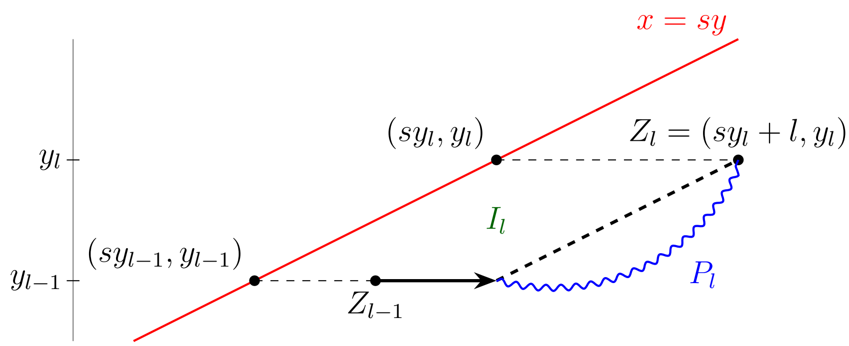
We are interested in studying the total area of all (sn, n)-Dyck paths i.e. the quantity

$$A_n := \sum_{P \in D^s_n} \text{area}(P).$$

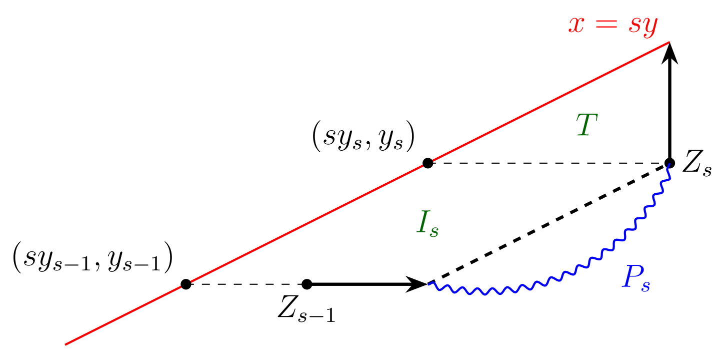
In this section, we derive a recursive formula for A_{n+1} using the touchpoint decomposition above. Note that the data for the decomposition of a path is entirely encoded in its type vector $\vec{y}$. Hence, in order to iterate over all possible paths, we will first iterate over all possible paths of a given type and then iterate over all possible types $0 \le y_0 \le y_1 \le \cdots \le y_{s-1} \le n$. For simplicity, set $y_s = n$.

To begin, fix a type $\vec{y} = (y_0, \ldots, y_{s-1})$ and a path $P \in D^s_{n+1}(\vec{y})$. The area of P decomposes into the contributions from the portions corresponding to the subpaths P_l for $0 \le l \le s$. The contribution of P_0 to the total area is precisely area(P_0).

<u>Contribution of $0 < l < s$</u>: The parallelogram I_l in diagram INSERT contributes $l(y_l - y_{l-1})$, whilst P_l contributes area(P_l).



<u>Contribution of $l = s$</u>: Again, the parallelogram I_s in diagram INSERT contributes $s(y_s - y_{s-1}) = s(n - y_{s-1})$ and P_s contributes area(P_s). Additionally, the triangle T contributes an area of $\frac{s}{2}$.



Combining these contributions gives

$$\text{area}(P) = \frac{s}{2} + \sum_{l=1}^{s} l(y_l - y_{l-1}) + \sum_{l=0}^{s} \text{area}(P_l).$$

Finally, summing over all possible types $\vec{y}$ and paths $P \in D^s_{n+1}(\vec{y})$ gives three large sums to evaluate. Since $\frac{s}{2}$ is constant, its corresponding sum evaluates to

$$\sum_{P \in D^s_{n+1}} \frac{s}{2} = \frac{s}{2} F_{n+1} = \frac{s}{2}[F(z)^{s+1}]_n$$

The sum from the parallelogram contributions (see APPENDIX REFERENCES for details) evaluates to

$$\sum_{\substack{0 \le y_0 \le \cdots \le y_{s-1} \le n \\ y_s = n}} \; \sum_{\substack{P \in \mathcal{D}^s_{n+1} \\ \text{of type } y_0, \ldots, y_{s-1}}} \; \sum_{l=1}^{s} l(y_l - y_{l-1}) = \frac{s(s+1)}{2}[zF'(z)F(z)^s]_n. \tag{3}$$

The recursive area sum (see APPENDIX REFERENCES for details) evaluates to

$$\sum_{\substack{0 \le y_0 \le \cdots \le y_{s-1} \le n \\ y_s = n}} \sum_{\substack{P \in \mathcal{D}^s_{n+1} \\ \text{of type } y_0, \ldots, y_{s-1}}} \sum_{l=0}^{s} \text{area}(P_l) = (s+1)[A(z)F(z)^s]_n \tag{4}$$

Combining these sums gives our desired recursive total area formula

$$A_{n+1} = \frac{s}{2}[F(z)^{s+1}]_n + \frac{s(s+1)}{2}[zF'(z)F(z)^s]_n + (s+1)[A(z)F(z)^s]_n. \tag{5}$$