Intro Math Research Hw3

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Contents

1 Reading Comments

QUOMODOCUMQUE Article

Tao Article

Vakil Article

2 Project Topics

Assignment: Look over the Project Topics document and start searching for readable resources. List the two topics which look most interesting to you and give two references for each of these topics (different from the one I gave). Keep in mind the remark of § 2.2

Topic 1: Schur-Weyl Duality and its Analogues

- In [FH91, Ch.4, 6], Fulton and Harris provide a gentle introduction to Schur-Weyl duality and the various identities which arise as a consequence.
- In [DDH08], Dipper, Doty, and Hu obtain an analogue of Schur-Weyl duality for symplectic groups using the Brauer algebra.

Topic 2: Quasisymmetric Functions

- The book *An Introduction to Quasisymmetric Schur Functions* [LMW13] by Luoto, Mykytiuk and van Willigenburg gives a broad overview of the machinery behind quasisymmetric functions (namely Hopf algebras) as well as a description of the basis of quasisymmetric Schur functions for the quasisymmetric function algebra.
- In [Sta95], Stanley contstructs a quasisymmetric function from a finite graph. This construction encodes the coloring data in a way that generalizes the graph chromatic polynomial.

3 Symmetric Polynomials/Functions Exposition

Assignment: Revise your write-up on symmetric polynomials, focusing on the extra tip in § 2.1 ("indispensable, interesting, illustrative.") If you did not write about the basis of homogeneous symmetric functions last time, this should be included in your two pages. Add a third page dedicated to the combinatorial definition of Schur functions and their symmetry.

Preliminary Considerations

Throughout this article, fix a (unital) commutative ring R and a field k. For simplicity, we work over vector spaces instead of general modules.

Notation. Let X be a set such that $X = \{x_i\}_{i \in I}$ for some indexing set I. By $k[X]$ and $k[\![X]\!]$, we denote the rings of (commutative) polynomials and power series (resp.) in indeterminates $\{x_i\}$. We utilize multi-index notation throughout. Hence for $\alpha_\bullet : I \to \mathbb{N} \cup \{0\}$ finitely supported, we write $x_\alpha = \Pi_{i \in I} x_i^{\alpha_i}$ (where $x_i^0 := 1$ formally).

3.1 Algebraic Background

Often in algebra, elements of a given object may be decomposed into a sum of simpler elements which are, in a sense, "homogenous." For example, any polynomial in n-variable may be decomposed into a sum of simpler polynomials each of which are futher sums of monomials of the same total degree. In this way, a polynomial is split into a sum of homogenous parts. This behavior is codified in the notion of *grading*.

Definition 3.1.1. A *graded k-algebra* is a k-algebra A together with a direct sum decomposition

$$A = \bigoplus_{i=0}^{\infty} A_i$$

with $A_0, A_1, \ldots$ vector spaces such that $A_i \cdot A_j \subseteq A_{i+j}$ for all $i, j \in \mathbb{N} \cup \{0\}$. For fixed i, elements of A_i are called *homogenous*. The choice of such a direct sum decomposition is a *grading* for A.

Key Example. As before, for $X = \{x_i\}_{i \in I}$, we may give the ring $k[X]$ a canonical grading by declaring $A_0 := k$ and

$$A_n := \operatorname{Span}_k\{x_\alpha : \alpha \text{ multi-index such that } \sum_{i \in I} \alpha_i = n\}.$$

The reader is cautioned that not every k-algebra has a nontrivial grading. In fact, it can be shown that the ring of formal power series $k[\![x]\!]$ does not have a nontrivial grading.

3.2 Symmetric Polynomials

Definition 3.2.1. The permutation group S_n acts naturally on the polynomial ring $k[x_1, \ldots, x_n]$ by defining $\sigma \cdot x_{i_1}^{\alpha_1} \cdots x_{i_l}^{\alpha_l} := x_{\sigma(i_1)}^{\alpha_1} \cdots x_{\sigma(i_l)}^{\alpha_l}$ and extending by linearity. The ring of *symmetric polynomials* in n indeterminates is the fixed points of this action, namely $k[x_1, \ldots, x_n]^{S_n}$.

3.3 Partitions and Compositions

Definition 3.3.1.

- A *partition* of $n \in \mathbb{N}$ is a finite sequence $\alpha = (\alpha_1, \ldots, \alpha_l)$ of weakly decreasing positive integers which sum to n. We denote the set of partitions of n by $\mathrm{Par}(n)$. We denote the statement $[\lambda \in \mathrm{Par}(n)]$ by $\lambda \vdash n$. Also, we write $\mathrm{Par} := \bigcup_{n \geq 0} \mathrm{Par}(n)$.
- A *weak composition* of $n \in \mathbb{N}$ is a (finitely supported) sequence $\alpha = (\alpha_i)_{i=1}^{\infty} \in (\mathbb{N} \cup \{0\})^{\mathbb{N}}$ such that $\sum_i \alpha_i = n$. The length of a weak composition α is given by

$$l(\alpha) := \max\{i \in \mathbb{N} : \alpha_i \neq 0\}.$$

3.4 Symmetric Functions

Definition 3.4.1 (pg. 308 in [Sta24])**.** The ring Λ_k of symmetric functions over a field k is the subring of all $f \in k[\![x_1, x_2, \ldots]\!]$ such that

$$f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots) = f(x_1, x_2, \ldots) \text{ for all } \sigma \in \mathrm{Sym}(\mathbb{N}).$$

Remark 3.4.1. For the algebraically-minded, there is a more natural construction of Λ_k by viewing the ring as the colimit of a certain directed system of injections of symmetric polynomial rings

$$k[x_1, \ldots, x_n]^{S_n} \xhookrightarrow{\varphi_n} k[x_1, \ldots, x_{n+1}]^{S_{n+1}}.$$

The construction of these maps φ_n is somewhat involved. This does justify the intuition that a symmetric function is simply taking a symmetric polynomial and adding more data, as any element of a direct limit of inclusions is faithfully represented by an element of one of the constituent objects.

Definition 3.4.2. A symmetric function $f \in \Lambda_k$ is homogenous of degree n if

$$f(x) = \sum_{\alpha \text{ weak composition of } n} c_\alpha x^\alpha,$$

where the c_α are elements of k. The set of degree n homogenous symmetric functions is denoted Λ_k^n. these subspaces give Λ_k the structure of a graded k-algebra, namely:

- Each Λ_k^n is a k-vector space,
- $\Lambda_k^i \Lambda_k^j \subseteq \Lambda_k^{i+j}$,
- $\Lambda_k = \bigoplus_{n=0}^{\infty} \Lambda_k^n$ as k-vector spaces.

The first interesting basis of Λ_k is the *monomial symmetric functions*. Given $\lambda \vdash n$, define $m_\lambda \in \Lambda_k^n$ by

$$m_\lambda := \sum_\alpha x^\alpha$$

where the sum is over all distinct permutations of the entries of λ. The set $\{m_\lambda : \lambda \vdash n\}$ forms a basis for Λ_k^n, whence $\bigcup_{n \geq 0} \{m_\lambda : \lambda \vdash n\} = \{m_\lambda : \lambda \in \mathrm{Par}\}$ forms a basis for Λ_k.

3.5 Complete Homogenous Symmetric Functions

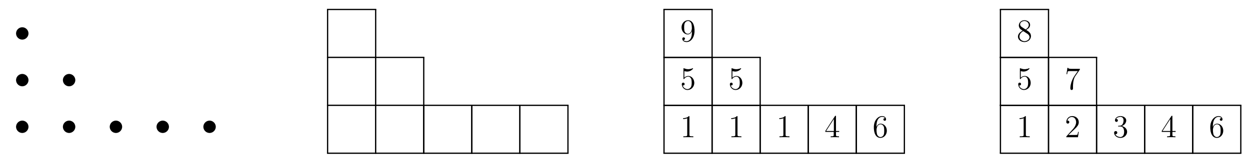
From the monomial symmetric functions, we may form another interesting basis for Λ_k called the *complete homogenous symmetric functions* h_λ by setting

$$h_\lambda := \prod_{i=1}^{\infty} \sum_{\nu \vdash \lambda_i} m_\nu.$$

where $\lambda = (\lambda_1, \lambda_2, \ldots)$. Again, the set $\{h_\lambda : \lambda \vdash n\}$ is a basis for Λ_k^n and the set $\{h_\lambda : \lambda \in \mathrm{Par}\}$ is a basis for Λ_k.

3.6 Schur Functions

Definition 3.6.1. Given $\lambda \vdash n$, the *Ferrers diagram of shape* λ is the set $\{(i,j) \in \mathbb{N}^2 : j \in \mathbb{N}, 1 \leq i \leq \lambda_j\}$ depicted as points in $\mathbb{R}^2$. The *Young diagram of shape* λ is depicted identically to the Ferrers diagram except the points are replaced with squares. The *size* of the diagram is the number of entries, namely n. We depict the case $(5,2,1) \vdash 8$ below.



Ferrers Diagram for the partition (5,2,1)

Young Diagram for the partition (5,2,1)

Semi-standard Young tableau for the partition (5,2,1)

Standard Young tableau for the partition (5,2,1)

Definition 3.6.2. Given $\lambda \vdash n$ and a Young diagram of shape λ, a *semi-standard Young tableau of shape* λ is a filling of the boxes of the Young diagram with positive integers such that

- the entries are weakly increasing along rows,
- the entries are strictly increasing up columns.

A semi-standard Young tableau of size n is said to be *standard* if the elements of $\{1, \ldots, n\}$ each appear exactly once in the tableau. We write $SSYT(\lambda)$ and $SYT(\lambda)$ for the sets of semi-standard and standard Young tableaux of shape λ. Given a semi-standard Young tableaux $\mathcal{T}$, the *weight* of $\mathcal{T}$ is a function $\alpha = \alpha_{\mathcal{T}} : \mathbb{N} \to \mathbb{N}$ given by

$$\alpha(i) := \text{number of times } i \text{ appears in } \mathcal{T}.$$

Note that $\alpha(i) = 0$ for sufficiently large i, so we may write $x^\alpha = x_1^{\alpha(1)} x_2^{\alpha(2)} \cdots$ and obtain a valid monomial.

Definition 3.6.3.

$$s_\lambda := \sum_{\mathcal{T} \in SSYT(\lambda)} x^{\alpha_{\mathcal{T}}}.$$