

200A Homework 2

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Problem 1

Suppose G is a simple group and it has a subgroup H of index n where n is an integer more than 1. Prove that G can be embedded into the symmetric group S_n .

Proof. Let G act by left multiplication on the set X of left cosets of H in G . This furnishes a homomorphism $\theta : G \rightarrow \text{Sym}(X) \cong S_n$. Then $\ker(\theta)$ is a normal subgroup of G . As $[G : H] > 1$, G acts nontrivially on X whence $\ker(\theta) \neq G$. Thus by simplicity of G , $\ker(\theta) = \{e\}$ so θ is an embedding. \square

Problem 2

For a group G , let $\text{Aut}(G)$ be the group of automorphisms of G . Let

$$c : G \rightarrow \text{Aut}(G), \quad c(g) := c_g, \quad \text{where } c_g(x) := gxg^{-1} \text{ for every } x \in G.$$

(a): Prove that c_g is an automorphism of G and c is a group homomorphism.

Proof. Fix $g \in G$. Let $x, y \in G$. Then

$$c_g(xy) = gxyg^{-1} = gxg^{-1}gyg^{-1} = c_g(x)c_g(y),$$

so c_g is a group homomorphism.

For $h \in G$, $c_g(g^{-1}hg) = h$, so c_g is surjective. If $ghg^{-1} = c_g(h) = e$, then $h = e$, so $\ker(c_g) = \{e\}$ whence c_g is injective. Thus $c_g \in \text{Aut}(G)$.

Fix $g, h \in G$ and $x \in G$. Then

$$c_g c_h(x) = c_g(hxh^{-1}) = ghxh^{-1}g^{-1} = ghx(gh)^{-1} = c_{gh}(x),$$

So $c_{gh} = c_g c_h$, whence c is a group homomorphism. \square

(b): Prove that $\ker c$ is the center $Z(G)$ of G .

Proof. Suppose $g \in \ker(c)$ so $c_g = \text{id}_G$. Then for all $x \in G$, $x = c_g(x) = gxg^{-1}$, so $g \in Z(G)$. Now suppose $g \in Z(G)$. Then for $x \in G$, $c_g(x) = gxg^{-1} = xgg^{-1} = x$, so $c_g = \text{id}_G$ whence $g \in \ker(c)$. \square

(c): The image of c is called the group of inner automorphisms of G , and it is denoted by $\text{Inn}(G)$. Prove that $\text{Inn}(G)$ is a normal subgroup of $\text{Aut}(G)$.

Proof. Fix $\varphi \in \text{Aut}(G)$ and let $g \in G$. Set $h := \varphi(g)$. For $x \in G$

$$\varphi c_g \varphi^{-1}(x) = \varphi c_g(\varphi^{-1}(x)) = \varphi(\varphi^{-1}(h) \varphi^{-1}(x) \varphi^{-1}(h^{-1})) = \varphi(\varphi^{-1}(h x h^{-1})) = h x h^{-1} = c_{\varphi(g)}(x),$$

so $\varphi c_g \varphi^{-1} = c_{\varphi(g)} \in \text{Inn}(G)$. Thus $\text{Inn}(G) \triangleleft \text{Aut}(G)$. □

(d): Prove that $|Z(\text{Aut}(G))| \leq |\text{Hom}(G, Z(G))|$; in particular, if either $Z(G) = 1$ or G is perfect (that is, $G = [G, G]$), then $Z(\text{Aut}(G)) = \{1\}$.

Proof. As the function c is a homomorphism with image $\text{Inn}(G)$, it follows that $G/Z(G) \cong \text{Inn}(G)$. For $\varphi \in Z(\text{Aut}(G))$ and $g \in G$, we have that $c_g = c_{\varphi(g)}$, so under the isomorphism we infer $gZ(g) = \varphi(g)Z(g)$. Hence there is some $\eta(g) \in Z(G)$ such that $\varphi(g) = g\eta(g)$.

For $g, h \in G$,

$$gh\eta(gh) = \varphi(gh) = \varphi(g)\varphi(h) = g\eta(g)h\eta(h) = gh\eta(g)\eta(h)$$

whence $\eta(gh) = \eta(g)\eta(h)$, so η is a group homomorphism.

Now for $\varphi \in Z(\text{Aut}(G))$, let η_φ be the corresponding homomorphism obtained above. Suppose $\varphi, \psi \in Z(\text{Aut}(G))$ are such that $\eta_\varphi = \eta_\psi$. Then for $g \in G$,

$$\varphi(g) = g\eta_\varphi(g) = g\eta_\psi(g) = \psi(g),$$

so $\varphi = \psi$. Hence the assignment $\varphi \mapsto \eta_\varphi$ is injective, whence $|Z(\text{Aut}(G))| = |\{\eta_\varphi : \varphi \in Z(\text{Aut}(G))\}| \leq |\text{Hom}(G, Z(G))|$. □

Problem 3

Let $SL_2(\mathbb{R})$ be the set of real 2×2 matrices with determinant 1. For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ and $z \in \mathbb{H} := \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$, let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az + b}{cz + d}.$$

(a): Prove that

$$\text{Im}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z\right) = \frac{\text{Im}(z)}{|cz + d|^2}.$$

Proof.

$$\begin{aligned} \text{Im}\left(\frac{az + b}{cz + d}\right) &= \text{Im}\left(\frac{(az + b)(c\bar{z} + d)}{|cz + d|^2}\right) = \frac{1}{|cz + d|^2} \text{Im}(ac|z|^2 + adz + bc\bar{z} + bd) \\ &= \frac{1}{|cz + d|^2} \text{Im}(adz + bc\bar{z}) = \frac{1}{|cz + d|^2} (ad - bc) \text{Im}(z) = \frac{\text{Im}(z)}{|cd + d|^2} \end{aligned}$$

□

(b): Prove that \cdot is an action of $SL_2(\mathbb{R})$ on \mathbb{H} .

Proof. Let $z \in \mathbb{H}$. Then for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$,

$$\text{Im}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z\right) = \frac{\text{Im}(z)}{|cz + d|^2} > 0$$

so $\frac{az+b}{cz+d} \in \mathbb{H}$. Now let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL_2(\mathbb{R})$. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{pmatrix}$$

and the action computes to

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \left(\begin{pmatrix} p & q \\ r & s \end{pmatrix} \cdot z \right) &= \frac{a \frac{pz+q}{rz+s} + b}{c \frac{pz+q}{rz+s} + d} \\ &= \frac{a(pz+q) + b(rz+s)}{c(pz+q) + d(rz+s)} = \frac{(ap+br)z + (aq+bs)}{(cp+dr)z + (cq+ds)} \\ &= \begin{pmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{pmatrix} \cdot z \\ &= \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} \right) \cdot z, \end{aligned}$$

so we indeed have an action on the upper half plane. □

Problem 4

Suppose G is a finite group, $C \subseteq \mathbb{R}^n$ is a convex subset (that is, for all two points P, Q in C , the segment PQ is a subset of C). Suppose G acts on C by affine transformations; that means

$$\forall P, Q \in C, \forall t \in [0, 1], \forall g \in G, \quad g \cdot (tP + (1-t)Q) = t g \cdot P + (1-t) g \cdot Q.$$

Prove that G has a fixed point; that is, there exists $x \in C$ such that for all $g \in G$, $g \cdot x = x$.

Proof. By definition, for $c_1, c_2 \in C$, we have that $\frac{c_1+c_2}{2} \in C$. Suppose that we have shown that for any $c_1, \dots, c_n \in C$, we have $\frac{c_1+\dots+c_n}{n} \in C$. Observe that then

$$\frac{c_1 + \dots + c_{n+1}}{n+1} = \frac{n}{n+1} \left(\frac{c_1 + \dots + c_n}{n} \right) + \left(1 - \frac{n}{n+1} \right) c_{n+1} \in C.$$

Fix $y \in C$. Then $A_G(y) = \frac{1}{|G|} \sum_{h \in G} h \cdot y \in C$. For $g \in G$, we have

$$g \cdot A_G(y) = \frac{1}{|G|} \sum_{h \in G} gh \cdot y = \frac{1}{|G|} \sum_{h \in G} h \cdot y = A_G(y),$$

so $A_G(y)$ is a fixed point of the affine action $G \curvearrowright C$. □

Problem 5

Suppose G is a finite subgroup of the group $GL_n(\mathbb{R})$ of $n \times n$ invertible real matrices. Prove that there is a G -invariant inner product on \mathbb{R}^n .

Proof. Define

$$\langle v, w \rangle_G := \frac{1}{|G|} \sum_{g \in G} \langle g \cdot v, g \cdot w \rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the usual dot product in \mathbb{R}^n . Note that this form is symmetric as

$$\langle v, w \rangle_G = \frac{1}{|G|} \sum_{g \in G} \langle g \cdot v, g \cdot w \rangle = \frac{1}{|G|} \sum_{g \in G} \langle g \cdot w, g \cdot v \rangle = \langle w, v \rangle_G.$$

Moreover,

$$\begin{aligned} \langle av_1 + bv_2, w \rangle_G &= \frac{1}{|G|} \sum_{g \in G} \langle g \cdot (av_1 + bv_2), g \cdot w \rangle \\ &= \frac{1}{|G|} \sum_{g \in G} \langle a(g \cdot v_1) + b(g \cdot v_2), g \cdot w \rangle \\ &= \frac{1}{|G|} \sum_{g \in G} a \langle g \cdot v_1, g \cdot w \rangle + b \langle g \cdot v_2, g \cdot w \rangle \\ &= a \langle v_1, w \rangle_G + b \langle v_2, w \rangle_G. \end{aligned}$$

Symmetry gives linearity in the other coordinate, so it suffices to show positive definiteness. Noting that $\langle \cdot, \cdot \rangle$ is positive definite, for $v \neq 0$ we have

$$\langle v, v \rangle_G = \frac{1}{|G|} \sum_{g \in G} \langle g \cdot v, g \cdot v \rangle = \frac{1}{|G|} \sum_{g \in G} |g \cdot v|^2 \geq \frac{|v|^2}{|G|} > 0.$$

Lastly, for fixed $h \in G$, noting that summation over gh is the same as summation over g since left multiplication is a bijection, we have

$$\begin{aligned} \langle h \cdot v, h \cdot w \rangle_G &= \frac{1}{|G|} \sum_{g \in G} \langle g \cdot (h \cdot v), g \cdot (h \cdot w) \rangle \\ &= \frac{1}{|G|} \sum_{g \in G} \langle (gh) \cdot v, (gh) \cdot w \rangle \\ &= \frac{1}{|G|} \sum_{g \in G} \langle g \cdot v, g \cdot w \rangle = \langle v, w \rangle_G. \end{aligned}$$

Thus $\langle \cdot, \cdot \rangle_G$ is a G -invariant inner product. □

Problem 6

Suppose H is a subgroup of G . Let

$$C_G(H) := \{ x \in G \mid \forall h \in H, \ xh = hx \}$$

be the centralizer of H in G , and

$$N_G(H) := \{ x \in G \mid xHx^{-1} = H \}$$

be the normalizer of H in G . Both of these are subgroups of G and clearly $C_G(H) \subseteq N_G(H)$. Prove that $N_G(H)/C_G(H)$ can be embedded into $\text{Aut}(H)$.

Proof. Let $N_G(H) \curvearrowright H$ by conjugation. This action furnishes a homomorphism $\Phi : N_G(H) \rightarrow \text{Aut}(H)$. The kernel of this action is

$$\ker(\Phi) = \{x \in N_G(H) : \Phi(x) = \text{id}_H\}.$$

Suppose $x \in C_G(H)$. Then for all $h \in H$, $\Phi(x)(h) = xhx^{-1} = h$ whence $\Phi(x) = \text{id}_H$ so $x \in \ker(\Phi)$. On the other hand suppose $x \in N_G$ is such that $\Phi(x) = \text{id}_H$. For $h \in H$, we have

$$xhx^{-1} = \Phi(x)(h) = \text{id}_H(h) = h$$

whence $x \in C_G(H)$. Hence $\ker(\Phi) = C_G(H)$, so $N_G(H)/C_G(H) \cong \Phi(N_G(H)) \subseteq \text{Aut}(H)$. □

Problem 7

Suppose N is a finite cyclic normal subgroup of G . Prove that every subgroup of N is normal in G .

Proof. Let $K \leq N$ have order k . Then for $g \in G$, $gKg^{-1} \leq N$ has order k , but cyclic groups have unique subgroups of each given order, so $gKg^{-1} = K$. □