Algebraic Topology II Homework 2

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1 Problem 1

Assuming as known the cup product structure on the 2-torus $\mathbb{T}^2 = S^1 \times S^1$, compute the cup product structure in $H^*(M_g)$ for M_g the closed orientable surface of genus g using the quotient map from M_g to a wedge sum of g tori (see photo).

Proof. We take as known that the cohomology ring of $\mathbb{T}^2$ with coefficients in a ring R is given by

$$H^*(\mathbb{T}^2; R) = \Lambda_R[\alpha_1, \alpha_2]$$

$$\begin{tikzcd}
H^1(\bigvee_g \mathbb{T}^2) \oplus H^1(\bigvee_g \mathbb{T}^2) \arrow[r] \arrow[d] & H^2(\bigvee_g \mathbb{T}^2) \arrow[d] \\
H^1(M_g) \oplus H^1(M_g) \arrow[r] & H^2(M_g)
\end{tikzcd}$$

□

2 Problem 3

(a): Using the cup product structure, show that there is no map $\mathbb{R}P^n \to \mathbb{R}P^m$ inducing a nontrivial map $H^1(\mathbb{R}P^m; \mathbb{Z}_2) \to H^1(\mathbb{R}P^n; \mathbb{Z}_2)$ if $n > m$. What is corresponding result for maps $\mathbb{C}P^n \to \mathbb{C}P^m$.

(b): Prove the Borsuk-Ulam theorem.

3 Problem 7

Use cup products to show that $\mathbb{R}P^3$ is not homotopy equivalent to $\mathbb{R}P^2 \vee S^1$.

4 Problem 8

Let X be $\mathbb{C}P^2$ with a cell e^3 attached by a map $S^2 \to \mathbb{C}P^1 \subseteq \mathbb{C}P^2$ of degree p, and let $Y := M(\mathbb{Z}_p, 2)) \vee S^4$. Thus X and Y have the smae 3-skeleton but differ in the way their 4-cells are attached. Show that X and Y have isomorphic cohomology rings with coefficients in $\mathbb{Z}$, but not with $\mathbb{Z}_p$ coefficients.

5 Problem 9

Show that if $H_n(X;\mathbb{Z})$ free for each n, then $H^*(X;\mathbb{Z}_p)$ and $H^*(X;\mathbb{Z}) \otimes \mathbb{Z}_p$ are isomorphic as rings, so in particular the ring structure with $\mathbb{Z}$ coefficients determines the ring structure with $\mathbb{Z}_p$ coefficients.