# Specht Modules and Schur Weyl Duality

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**Abstract**

In this paper we exposit one of the fundamental results linking representation theory and algebraic combinatorics called Schur-Weyl duality. It provides a dictionary between the representation theory of finite symmetric groups and the representation theory of the general linear group of a finite dimensional complex vector space. Through this dictionary, we obtain representation theoretic constructions of some aspects of symmetric function theory, including Schur functions, and internal/external products on the symmetric function ring.

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# 1 Representation Theory Background

## 1.1 Group Representations

**Definition 1.1.1.** A *representation* of a group $G$ is a pair $(\pi, V)$ where $V$ is a $\mathbb{C}$-vector space and $\pi : G \to \mathrm{GL}(V)$ is a group homomorphism.

**Definition 1.1.2.** A morphism between two representations $(\pi, V)$ and $(\rho, W)$ of a group $G$ is a linear map $T : V \to W$ such that $T\pi(g) = \rho(g)T$ for all $g \in G$. A morphism between representations is an isomorphism if it is an isomorphism of vector spaces.

We write $\mathrm{Hom}_G(V, W)$ for the set of morphisms between $(\pi, V)$ and $(\rho, W)$.

**Definition 1.1.3.** Fix a representation $(\pi, V)$ of a group $G$.

- A *subrepresentation* of $(\pi, V)$ is a subspace $W \subseteq V$ such that $\pi(g)w \in W$ for all $w \in W$.
- We say $(\pi, V)$ is *irreducible* if its only subrepresentations are $V$ and $0$.

The interplay between irreducible representations and morphisms of representations is encapsulated in the following fundamental result.

**Lemma 1.1.1** (Schur's Lemma). *Let $(\pi, V)$, $(\rho, W)$ be irreducible representations of a group $G$. Then $\mathrm{Hom}_G(V, W) \cong \mathbb{C}$ if $(\pi, V) \cong (\rho, W)$ and is $0$ otherwise.*

*Proof.* Suppose $\mathrm{Hom}_G(V, W) \neq 0$. Let $T \in \mathrm{Hom}_G(V, W) \setminus \{0\}$. Since $\ker(T) \neq V$, irreduciblity implies $\ker(T) = 0$. Likewise, as $\mathrm{Im}(T) \neq 0$, irreducibility implies $\mathrm{Im}(T) = W$. Hence $T$ is an isomorphism, so without loss of generality assume $(\pi, V) = (\rho, W)$. Let $\alpha \in \mathbb{C}$ be an eigenvalue of $T$ with eigenvector $v$ and observe

$$T\pi(g)v = \pi(g)Tv = \pi(g)\alpha v = \alpha\pi(g)v.$$

As $v \neq 0$, irreducibility implies $\pi(G)v = V$, whence $T = \alpha I$ on all of $V$. Thus every element of $\mathrm{Hom}_G(V, V)$ is a multiple of the identity. $\square$

**Definition 1.1.4.** Given a representation $(\pi, V)$ of a group $G$, we define the corresponding *dual respresentation* to be $(\pi^*, V^*)$ where $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ is the dual vector space and $\pi^*(g)f(v) = f(\pi(g^{-1})v)$ for $f \in V^*$, $g \in G$, and $v \in V$.

## 1.2 Character Theory

Whilst we will not need much character theory in the following, we mention some of the main definitions and results for the reader's enlightenment.

**Definition 1.2.1.** Given a representation $(\pi, V)$ of a group $G$, the corresponding character of the representation is the funciton $\chi_\pi : G \to \mathbb{C}$ given by

$$\chi_\pi(g) = Tr(\pi(g))$$

Note that as $Tr$ is conjugacy invariant, so is $\chi_\pi$.

An incredibly suprising result of finite group representation theory is that the characters of representations are enough to entirely determine the representation. This is encapsulated in the following theorem.

**Theorem 1.2.1.** *Let $(\pi, V), (\rho, W)$ be (complex) representations of a finite group $G$. Then $(\pi, V) \cong (\rho, W)$ if and only if $\chi_\pi = \chi_\rho$.*

One application we will need this theorem for is the self-duality of representations of symmetric groups.

**Proposition 1.2.1.** *Let $(\pi, V)$ be a representation of $S_n$. Then $V \cong V^*$ as representations.*

*Proof.* One may compute that $\chi_{V^*}(g) = \chi_V(g^{-1})$. In $S_n$, the group elements $g$ and $g^{-1}$ are conjugate, so $\chi_V(g^{-1}) = \chi_V(g)$, whence by the above theorem, since $V$ and $V^*$ have the same characters, it follows that they are isomorphic as representations. $\square$

# 2 Representations of $S_n$

## 2.1 Partitions, Young Diagrams, and Tabloids

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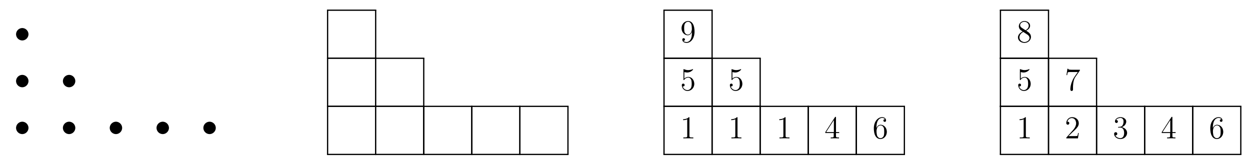
**Definition 2.1.1.** Given $\lambda \vdash n$, the *Ferrers diagram of shape* $\lambda$ is the set $\{(i,j) \in \mathbb{N}^2 : j \in \mathbb{N}, 1 \leq i \leq \lambda_j\}$ depicted as points in $\mathbb{R}^2$. The *Young diagram of shape* $\lambda$ is depicted identically to the Ferrers diagram except the points are replaced with squares. The *size* of the diagram is the number of entries, namely $n$. We depict the case $(5,2,1) \vdash 8$ below.




*Ferrers Diagram* for the partition (5,2,1)

*Young Diagram* for the partition (5,2,1)

*Semi-standard Young tableau* for the partition (5,2,1)

*Standard Young tableau* for the partition (5,2,1)

**Definition 2.1.2.** Given $\lambda \vdash n$ and a Young diagram of shape $\lambda$, a *semi-standard Young tableau of shape* $\lambda$ is a filling of the boxes of the Young diagram with positive integers such that

- the entries are weakly increasing along rows,
- the entries are strictly increasing up columns.

A semi-standard Young tableau of size $n$ is said to be *standard* if the elements of $\{1, \ldots, n\}$ each appear exactly once in the tableau. We write $SSYT(\lambda)$ and $SYT(\lambda)$ for the sets of semi-standard and standard Young tableaux of shape $\lambda$. Given a semi-standard Young tableau $\mathcal{T}$, the *weight* of $\mathcal{T}$ is a function $\alpha = \alpha_{\mathcal{T}} : \mathbb{N} \to \mathbb{N}$ given by

$$\alpha(i) := \text{number of times } i \text{ appears in } \mathcal{T}.$$

Note that $\alpha(i) = 0$ for sufficiently large $i$, so we may write $x^\alpha = x_1^{\alpha(1)} x_2^{\alpha(2)} \cdots$ and obtain a valid monomial. We write $SSYT(\lambda, \alpha)$ for the set of semi-standard Young tableaux of shape $\lambda$ and weight $\alpha$.

**Definition 2.1.3.** Given $\lambda \vdash n$, a $\lambda$-tableau is simply a filling of the boxes of the Young diagram of shape $\lambda$ with the elements of $\{1, \ldots, n\}$ without repetition (and no other restrictions). Denote the set of $\lambda$-tableaux by $YT(\lambda)$. Note that $S_n \curvearrowright YT(\lambda)$ by permuting labels.

**Definition 2.1.4.** Given $\lambda \vdash n$, define an equivalence relation $\sim$ on $YT(\lambda)$ by $\mathcal{T} \sim \mathcal{T}'$ if and only if $\mathcal{T}'$ can be obtained from $\mathcal{T}$ by permuting the entries of each row. An equivalence class with respect to this relation is called a $\lambda$-*tabloid*. If $\mathcal{T}$ is a $\lambda$-tableau, we write $\{\mathcal{T}\}$ for the corresponding $\lambda$-tabloid. Finally, we write $Tab(\lambda) := YT(\lambda)/\sim$ for the set of $\lambda$-tabloids. Note that the action of $S_n$ on $\lambda$-tableaux descends to an action on $\lambda$-tabloids.

## 2.2 Construction of Specht Modules

Young diagrams will give projection operators $P_\lambda : \mathbb{C}[S_n] \to \mathbb{C}[S_n]$ which commute with the action of $S_n$, whence the image $P_\lambda(\mathbb{C}[S_n])$ gives a subrepresentation of the regular representation. These subrepresentations will end up being precisely the irreducible representations of $S_n$. Throughout this section, $\lambda \vdash n$ will be fixed.

**Definition 2.2.1.** Given a $\lambda$-tableau $\mathcal{T}$, define the *row group* $R_{\mathcal{T}}$ to be the subgroup of $S_n$ which permutes only the labels in the rows of $\mathcal{T}$ and the *column group* $C_{\mathcal{T}}$ as the subgroup which permutes only the labels in the columns of $\mathcal{T}$.

Now we may define the *Young row and column symmetrizers* in $\mathbb{C}[S_n]$ by

$$a_{\mathcal{T}} := \sum_{\sigma \in R_{\mathcal{T}}} \sigma, \qquad b_{\mathcal{T}} := \sum_{\sigma \in C_{\mathcal{T}}} \operatorname{sgn}(\sigma)\sigma. \tag{1}$$

Note that for $\mathcal{T} \in YT(\lambda)$, the corresponding tabloid is precisely the orbit of $\mathcal{T}$ under its row group, i.e.

$$\{\mathcal{T}\} = R_{\mathcal{T}}\mathcal{T} = \{\sigma\mathcal{T} \in YT(\lambda) : \sigma \in R_{\mathcal{T}}\}.$$

Now let $M^\lambda$ be the free $\mathbb{C}$-vector space over the set of $\lambda$-tabloids. Extending the action $S_n \curvearrowright Tab(\lambda)$ linearly to all of $M^\lambda$, we obtain a $\mathbb{C}[S_n]$-module structure on $M^\lambda$. For $\mathcal{T} \in YT(\lambda)$, the element $e_{\mathcal{T}} \in M^\lambda$ given by

$$e_{\mathcal{T}} := b_{\mathcal{T}} \cdot \{\mathcal{T}\} = \sum_{\sigma \in C_{\mathcal{T}}} \operatorname{sgn}(\sigma)\{\sigma\mathcal{T}\}$$

is called the *polytabloid associated to* $\mathcal{T}$. Let $S^\lambda$ be the subspace of $M^\lambda$ generated by all polytabloids, namely

$$S^\lambda := \operatorname{Span}_{\mathbb{C}}\{e_{\mathcal{T}} : \mathcal{T} \in YT(\lambda)\}.$$

*Claim.* $S^\lambda$ is a $\mathbb{C}[S_n]$-submodule of $M^\lambda$.

*Proof of Claim.* Fix $\sigma \in S_n$. We first show that $C_{\sigma\mathcal{T}} = \sigma C_{\mathcal{T}}\sigma^{-1}$. Indeed, if $T_i$ is the set of entries for the $i$th column of $\mathcal{T}$, then $\sigma(T_i)$ is the entries for the $i$th column of $\sigma\mathcal{T}$. Now it suffices to note that $\tau \in S_n$ stabilizes $T_i$ if and only if $\sigma\tau\sigma^{-1}$ stabilizes $\sigma(T_i)$. Using this identity, we compute

$$\sigma b_{\mathcal{T}} = \sum_{\gamma \in C_{\mathcal{T}}} \operatorname{sgn}(\gamma)\sigma\gamma \overset{\tau = \sigma\gamma\sigma^{-1}}{=} \sum_{\tau \in \sigma C_{\mathcal{T}}\sigma^{-1}} \operatorname{sgn}(\sigma^{-1}\tau\sigma)\tau\sigma = \sum_{\tau \in C_{\sigma\mathcal{T}}} \operatorname{sgn}(\tau)\tau\sigma = b_{\sigma\mathcal{T}}\sigma.$$

Now we apply $\sigma$ to the generators of $S^\lambda$ and find

$$\sigma \cdot e_{\mathcal{T}} = \sigma \cdot (b_{\mathcal{T}} \cdot \{\mathcal{T}\}) = (\sigma b_{\mathcal{T}}) \cdot \{\mathcal{T}\} = b_{\sigma\mathcal{T}}\{\sigma\mathcal{T}\} = e_{\sigma\mathcal{T}}.$$

As $S_n$ stabilizes $S^\lambda$, the claim follows. □

**Definition 2.2.2.** The $\mathbb{C}[S_n]$-module $S^\lambda$ as defined above is the *Specht module corresponding to* $\lambda$.

**Example 2.2.1** (Sign Representation)**.** Consider the partition $\lambda = (1, 1, \ldots, 1)$ of $n$. Since each row of $\lambda$ has one element, the $\lambda$-tabloids are the same as $\lambda$-tableaux.

Let $\mathcal{T}$ be a $\lambda$-tableau. As $\mathcal{T}$ has only one column, $C_{\mathcal{T}} = S_n$, whence $b_{\mathcal{T}} = \sum_{\gamma \in S_n} \operatorname{sgn}(\gamma)\gamma$ and consequently

$$\sigma e_{\mathcal{T}} = \sum_{\gamma \in S_n} \operatorname{sgn}(\gamma)\sigma\gamma\{\mathcal{T}\} = \sum_{\tau \in S_n} \operatorname{sgn}(\sigma^{-1}\tau)\tau\{\mathcal{T}\} = \operatorname{sgn}(\sigma)e_{\mathcal{T}} \quad \text{for all } \sigma \in S_n.$$

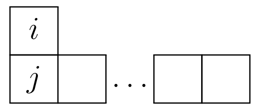
On the other hand, we know that $\sigma e_{\mathcal{T}} = e_{\sigma\mathcal{T}}$, so it follows that $S^\lambda = \mathbb{C}e_{\mathcal{T}}$ is the one-dimensional sgn representation.

**Example 2.2.2** (Trivial Representation)**.** Consider the partition $\lambda = (n)$ of $n$. Since there is one row of $\lambda$, all $\lambda$-tableaux are equivalent so there is only one $\lambda$-tabloid. Fix a $\lambda$-tableau $\mathcal{S}$.

Each $e_{\mathcal{T}} = \{T\} = \{S\}$, so $S^\lambda = \mathbb{C}e_{\mathcal{S}}$ is one-dimensional. The action of $\sigma$ is given by $\sigma e_{\mathcal{T}} = e_{\sigma\mathcal{T}} = e_{\mathcal{T}}$, so $S^\lambda$ is the trivial representation of $S_n$.

**Example 2.2.3** (Augmentation Subrepresentation)**.** Consider the partition $\lambda = (n-1, 1)$ of $n$. Observe that there are $n$ distinct $\lambda$-tabloids, each corresponding to the integer in singular box on the 2nd row. Denote the tabloid with $i$ in the $2nd$ row by $t_i$, so $Tab(\lambda) = \{t_1, \ldots, t_n\}$.

Let $V = \mathbb{C}\{v_1, \ldots, v_n\}$ be the standard representation of $S_n$ (i.e. $\sigma v_i = v_{\sigma(i)}$). Observe that the map $L : V \to M^\lambda$ given by $L(v_i) = t_i$ is an isomorphism of $\mathbb{C}[S_n]$-modules. The *augmentation subrepresentation* $W$ of $V$ is given by $W := \{\sum_{i=1}^n \alpha_i v_i : \sum_i \alpha_i = 0\}$. We claim that $S^\lambda \cong W$ as $\mathbb{C}[S_n]$-modules. Fix $i \in \{1, \ldots, n\}$ and let $\mathcal{T}$ be a $\lambda$-tableau such that $t_i = \{\mathcal{T}\}$. Let $j$ be the integer below $i$ on the tableau. Then the column



General form of $\mathcal{T}$ when $t_i = \{\mathcal{T}\}$

group $C_\mathcal{T}$ is then of order 2 generated by the transposition $(i\ j)$.

$$e_\mathcal{T} = \sum_{\gamma \in C_\mathcal{T}} \operatorname{sgn}(\gamma)\gamma t_i = t_i - t_j.$$

Hence, one checks

$$S^\lambda = \operatorname{Span}\{t_i - t_j : 1 \leq i, j \leq n, i \neq j\} = \operatorname{Span}\{t_i - t_{i+1} : 1 \leq i \leq n-1\}.$$

Moreover, $\{t_i - t_{i+1} : 1 \leq i \leq n-1\}$ gives a basis for $S^\lambda$. The restriction of $L$ to $W$ gives a vector space isomorphism $L : W \to S^\lambda$ as $\{v_i - v_{i+1}\}_{1 \leq i \leq n-1}$ gives a basis for $W$, so a basis gets mapped to a basis. Moreover, this map intertwines the $S_n$-action, so it produces $\mathbb{C}[S_n]$-module isomorphism.

## 2.3 Alternative Construction

Fix a $\lambda$-tableau $\mathcal{S}$ throughout this section, say the canonical one (increasing across rows and then moving up rows). Recall the row and column symmetrizers $a_\lambda := a_\mathcal{S}$, $b_\lambda := b_\mathcal{S}$ and define the Young symmetrizer

$$c_\lambda := a_\lambda \cdot b_\lambda \in \mathbb{C}[S_n].$$

Set $V_\lambda := \mathbb{C}[S_n]c_\lambda$. Define a map $T : \mathbb{C}[S_n]a_\lambda \to M^\lambda$ by $T(\sigma a_\lambda) = \{\sigma\mathcal{S}\}$.

*Claim.* The map $T$ is an isomorphism of $\mathbb{C}[S_n]$-modules.

*Proof of Claim.* We first show this map is well defined. If $\sigma a_\lambda = \tau a_\lambda$, then $\tau^{-1}\sigma$ fixes $a_\lambda$, whence $\tau^{-1}\sigma \in R_\mathcal{S}$ and consequently $\sigma\{\mathcal{S}\} = \tau\{\mathcal{S}\}$.

Since the action of $S_n$ on $\lambda$-tableau is transitive, it follows that the map $T$ is onto. On the other hand, suppose $\sum_\sigma \alpha_\sigma \sigma a_\lambda \in \ker(T)$. Then

$$0 = T(\sum_\sigma \alpha_\sigma \sigma a_\lambda) = \sum_\sigma \alpha_\sigma\{\sigma\mathcal{S}\}.$$

Since $M^\lambda$ is a free $\mathbb{C}$-module, it follows that $\sum_\sigma \alpha_\sigma \sigma = 0$. Lastly, if $\sigma, \gamma \in S_n$, then

$$\sigma T(\gamma a_\lambda) = \sigma\{\gamma\mathcal{S}\} = \{\sigma\gamma\mathcal{S}\} = T(\sigma\gamma s_\lambda).$$

□

*Claim.* The map $T$ restricted to the submodule $\mathbb{C}[S_n]b_\lambda a_\lambda$ gives a $\mathbb{C}[S_n]$-module isomorphism $\mathbb{C}[S_n]b_\lambda a_\lambda \cong S^\lambda$.

*Proof of Claim.* For $\sigma \in S_n$, we compute

$$\begin{aligned} T(\sigma b_\lambda a_\lambda) &= \sum_{\tau \in C_{\mathcal{S}}} \operatorname{sgn}(\tau) T(\sigma\tau a_\lambda) = \sum_{\tau \in C_{\mathcal{S}}} \operatorname{sgn}(\tau)\{\sigma\tau\mathcal{S}\} \\ &= \sigma \sum_{\tau \in C_{\mathcal{S}}} \operatorname{sgn}(\tau)\{\tau\mathcal{S}\} = \sigma e_{\mathcal{S}} = e_{\sigma\mathcal{S}} \end{aligned}$$

Since $S_n$ acts transitively on $\lambda$-tableaux, it follows that

$$T(\mathbb{C}[S_n]b_\lambda a_\lambda) = \operatorname{Span}_{\mathbb{C}}\{e_{\sigma\mathcal{S}} : \sigma \in S_n\} = S^\lambda$$

By the proof of the previous claim, $T$ is injective and intertwines the action of $S_n$, whence $T|_{\mathbb{C}[S_n]b_\lambda a_\lambda}$ furnishes an isomorphism of $\mathbb{C}[S_n]$-modules as desired. $\square$

**Proposition 2.3.1.** $\mathbb{C}[S_n]b_\lambda a_\lambda \cong \mathbb{C}[S_n]a_\lambda b_\lambda$.

## 2.4 Results on Specht Modules

Having obtained a few examples of Specht modules, we note that $\{S^\lambda : \lambda \vdash n\}$ forms a complete set of non-isomorphic, irreducible representations of $S_n$. This is established by the combining the following three theorems, which we leave unproven and cite standard references [FH91], [Sta24].

**Theorem 2.4.1.** *Given $\lambda \vdash n$, the Specht module $S^\lambda$ is irreducible as a $\mathbb{C}[S_n]$-module (i.e. an irreducible representation of $S_n$).*

**Theorem 2.4.2.** *If $\lambda, \mu \vdash n$ and $\lambda \neq \mu$, then $S^\lambda \not\cong S^\mu$ as $\mathbb{C}[S_n]$-modules.*

**Theorem 2.4.3.** *Every irreducible representation of $S_n$ is ismorphic to $S^\lambda$ for some $\lambda \vdash n$.*

# 3 Representations of $\mathrm{GL}(V)$

## 3.1 Schur Functors

Let $V$ be a finite dimensional complex vector space and consider the space $V^{\otimes n}$. We have a natural (right) action of $S_n$ on $V^{\otimes n}$ given for $\sigma \in S_n$ by

$$(v_1 \otimes \cdots \otimes v_n)\sigma := v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}.$$

We also have a natural (left) action of $\mathrm{GL}(V)$ on $V^{\otimes n}$ given for $T \in \mathrm{GL}(V)$ by

$$T(v_1 \otimes \cdots \otimes v_n) := Tv_1 \otimes \cdots \otimes Tv_n.$$

Moreover, these actions commute with each other.

**Definition 3.1.1.** Fix $\lambda \vdash n$. The *Schur functor of shape* $\lambda$ is the functor $\mathbb{S}_\lambda : Vect_{\mathbb{C}} \to Vect$ given, for a finite-dimensional vector space $V$, by

$$\mathbb{S}_\lambda(V) := \operatorname{Hom}_{S_n}(S^\lambda, V^{\otimes n})$$

Moreover, $\mathbb{S}_\lambda(V)$ is a representation of $\mathrm{GL}(V)$ under the natural action $[T\varphi](x) = T\varphi(x)$ for $\varphi \in \mathbb{S}_\lambda(V)$, $T \in \mathrm{GL}(V)$, and $x \in S^\lambda$.

With the notation of section 2.3, we note the following alternative construction of $\mathbb{S}_\lambda(V)$ by computing

$$\begin{aligned}\mathbb{S}_\lambda(V) = \operatorname{Hom}_{S_n}(S^\lambda, V^{\otimes n}) &\cong V^{\otimes n} \otimes_{\mathbb{C}[S_n]} (S^\lambda)^* \\ &\cong V^{\otimes n} \otimes_{\mathbb{C}[S_n]} S^\lambda \\ &\cong V^{\otimes n} \otimes_{\mathbb{C}[S_n]} \rho(c_\lambda)\mathbb{C}[S_n] \cong V^{\otimes n} c_\lambda \otimes_{\mathbb{C}[S_n]} \mathbb{C}[S_n] \cong V^{\otimes n} c_\lambda.\end{aligned}$$

Hence, the Schur functor may also be described as the image of the action of the Young symmetrizer when restricted to $V^{\otimes n}$.

## 3.2 Schur-Weyl Duality

**Theorem 3.2.1** (Schur-Weyl Duality). *Let $V$ be a finite-dimensional complex vector space and regard $V^{\otimes n}$ as a representation of $\mathrm{GL}(V) \times S_n$ as described above. Then, as representations,*

$$V^{\otimes n} \cong \bigoplus_{\lambda \vdash n} S^\lambda \otimes_{\mathbb{C}} \mathbb{S}_\lambda(V).$$

**Lemma 3.2.1.** *The symmetric tensor power $\operatorname{Sym}^n(V)$ is spanned by $v \otimes \cdots \otimes v$ for $v \in V$.*

**Lemma 3.2.2.** *A subspace of $V^{\otimes n}$ is an $\operatorname{End}_{S_n}(V^{\otimes n})$-submodule if and only if it is a $\mathrm{GL}(V)$-submodule.*

*Proof.* Consider the inclusion

$$\operatorname{End}(V) \overset{\iota}{\hookrightarrow} \operatorname{End}(V^{\otimes n}) \cong \operatorname{End}(V)^{\otimes n}$$

under the map $T \overset{\iota}{\mapsto} T \otimes \cdots \otimes T$. By Lemma 3.2.1,

$$\operatorname{Span}_{\mathbb{C}}(\iota(\operatorname{End}(V)) = \operatorname{Sym}^n(\operatorname{End}(V)) = \operatorname{End}_{S_n}(V^{\otimes n}).$$

Suppose $W \subseteq V^{\otimes n}$ is an $\operatorname{End}_{S_n}(V^{\otimes n})$-submodule. Let $T \in \mathrm{GL}(V)$. By definition, the action of $T$ on $V^{\otimes d}$ is given by $\iota(T)$, which is in $\operatorname{End}_{S_n}(V^{\otimes n})$ by above so $TW \subseteq T$.

On the other hand, suppose $W \subseteq V^{\otimes n}$ is a $\mathrm{GL}(V)$-submodule. Let $L \in \operatorname{End}_{S_n}(V^{\otimes n})$. By above, $L \in \operatorname{Span}_{\mathbb{C}}(\iota(\operatorname{End}(V)))$ so there is some $L_1, \dots, L_r \in \operatorname{End}(V)$ and $\alpha_1 \dots, \alpha_r \in \mathbb{C}$ such that

$$L = \sum_{i=1}^{r} \alpha_i L_i$$

Recall that $\mathrm{GL}(V)$ is dense in $\operatorname{End}(V)$ in the Euclidean operator topology, so we may choose $T_{ij} \in \mathrm{GL}(V)$ such that $\|L_i - T_{ij}\|_2 \xrightarrow{j\to\infty} 0$ for $i \in \{1, \dots, r\}$. Since $L$ is expressed as a finite sum of the $L_i$s, we observe that

$$\left\| L - \sum_{i=1}^{r} \alpha_i T_{ij} \right\|_2 \xrightarrow{j\to\infty} 0$$

Let $w \in W$. Identifying $W \subseteq V^{\otimes n} \cong \mathbb{C}^l$ for some $l$ with the standard topology, it follows by operator continuity that $\sum_{i=1}^r \alpha_i T_{ij} w \xrightarrow{j\to\infty} Lw$, so $Lw \in \overline{W}$ as each $\sum_{i=1}^r \alpha_i T_{ij} w \in W$. As finite dimensional topological vector spaces are closed, $\overline{W} = W$, so $W$ is an $\operatorname{End}_{S_n}(V^{\otimes n})$-submodule.

□

**Lemma 3.2.3.** *If $(W, \pi)$ is an irreducible representation of $S_n$, then $V^{\otimes n} \otimes_{\mathbb{C}[S_n]} W$ is a simple left $\operatorname{End}_{S_n}(V^{\otimes n})$-module.*

*Proof.* First decompose $V^{\otimes d}$ into a direct sum of irreducible $S_n$-representations by

$$V^{\otimes n} = \bigoplus_{i=1}^{l} V_i^{\oplus m_i},$$

so by Schur's lemma $\mathrm{End}_{S_n}(V^{\otimes n}) \cong \bigoplus_{i=1}^{l} M_{m_i}(\mathbb{C})$. Pick $s$ such that $V_s \cong W$. Applying self-duality of $S_n$-representations and Schur's lemma, we find

$$V_i \otimes_{\mathbb{C}[S_n]} V_s \cong (V_i)^* \otimes_{\mathbb{C}[S_n]} V_s \cong \mathrm{Hom}_{S_n}(V_i, V_s) = \begin{cases} \mathbb{C} & \text{if } i = s \\ 0 & \text{otherwise} \end{cases}$$

So, we compute

$$V^{\otimes n} \otimes_{\mathbb{C}[S_n]} W \cong \bigoplus_{i=1}^{l} V_i^{\oplus m_i} \otimes_{\mathbb{C}[S_n]} W \cong \bigoplus_{i=1}^{l} (V_i \otimes_{\mathbb{C}[S_n]} W)^{\oplus m_i} \cong \mathbb{C}^{\oplus m_s}$$

which is most definitely irreducible under the action of $\bigoplus_{i=1}^{l} M_{m_i}(\mathbb{C}) = \mathrm{End}_{S_n}(V^{\otimes n})$. $\square$

By the above lemma, each Schur functor $\mathbb{S}_\lambda(V)$ is an irreducible $\mathrm{GL}(V)$-representation. Applying the above lemmas and the decomposition of $\mathbb{C}[S_n]$, we obtain the theorem:

$$\begin{aligned} V^{\otimes n} = V^{\otimes n} \otimes_{\mathbb{C}[S_n]} \mathbb{C}[S_n] &= V^{\otimes n} \otimes_{\mathbb{C}[S_n]} \bigoplus_{\lambda \vdash n} (S^\lambda)^{\oplus \dim S^\lambda} \\ &\cong \bigoplus_{\lambda \vdash n} (V^{\otimes n} \otimes_{\mathbb{C}[S_n]} S^\lambda)^{\oplus \dim S^\lambda} \\ &\cong \bigoplus_{\lambda \vdash n} \mathbb{S}_\lambda(V)^{\oplus \dim S^\lambda} \cong \bigoplus_{\lambda \vdash n} S^\lambda \otimes_{\mathbb{C}} \mathbb{S}_\lambda(V). \end{aligned}$$

# 4 Applications

## 4.1 Kronecker Multiplication

The first step is to work over only with representations over $\mathbb{C}$, since then isomorphism classes of representations are determined entirely by their characters. For a finite group $G$, we may consider the representation ring of $G$ over $\mathbb{C}$, $R_{\mathbb{C}}(G)$. If $V_1, \dots, V_r$ are the irreducible complex representations of $G$, then the representation ring of $G$

$$R_{\mathbb{C}}(G) = \bigoplus_{i=1}^{r} \mathbb{Z} V_i = \bigoplus_{i=1}^{r} \mathbb{Z} \chi_{V_i}$$

i.e. it is a free $\mathbb{Z}$ module of rank $r$ generated by the (isomorphism classes of the) irreducible representations (or their characters since we are over $\mathbb{C}$).

By the Specht module construction, we know that the irreducible representations of $S_n$ are the Specht modules, so $R_{\mathbb{C}}(S_n) \cong \bigoplus_{\lambda \vdash n} \mathbb{Z} S_\lambda$. The product structure in $\mathbb{R}_{\mathbb{C}}(S_n)$ is determined by its generators, so consider two partitions $\lambda, \mu \vdash n$. Since this ring is free, there exist $g^{\nu}_{\lambda,\mu} \in \mathbb{Z}$ such that

$$S_\lambda \otimes S_\mu = \sum_{\nu \vdash n} g^{\nu}_{\lambda,\mu} S_\nu.$$

These coefficients $g_{\lambda,\mu}^{\nu}$ are called Kronecker coefficients and they are incredibly difficult to understand, hence ordinary multiplication in $\mathbb{R}_{\mathbb{C}}(S_n)$ is difficult to understand. At the level of symmetric functions, this does induce a new product called the *Kronecker product*

$$s_\lambda \star s_\mu = \sum_\nu g_{\lambda,\mu}^{\nu} s_\nu.$$

One difficulty with this approach is that we are only looking at one graded piece of the ring of symmetric functions and trying to stay within this piece—a philosophy counter to that of the notion of grading.

## 4.2 Frobenius Characteristic Map

Hence, we will incorporate all of $S_n$-representations as $n$ ranges into one ring. Consider the graded abelian group

$$R = \bigoplus_{n\geq 0} R_{\mathbb{C}}(S_n) = \bigoplus_{n\geq 0} \bigoplus_{\lambda\in\mathrm{Par}(n)} \mathbb{Z}S_\lambda = \bigoplus_{\lambda\in\mathrm{Par}} \mathbb{Z}S_\lambda$$

with grading $R_n = R_{\mathbb{C}}(S_n)$. We define a graded ring structure on $R$ as follows. Let $n, m \geq 0$ and $\lambda \vdash n$, $\mu \vdash m$. The product of $S_\lambda$ and $S_\mu$ needs to be a representation of $S_{n+m}$. Since $S_\lambda$ and $S_\mu$ are a priori representations of different groups, we only have access to an external tensor product $S_\lambda \boxtimes S_\mu$. The problem now is that this is a representation of $S_n \times S_m$, not $S_{n+m}$.

To fix this, note that we have a canonical inclusion $S_n \times S_m \hookrightarrow S_{n+m}$. All we have to do now is consider the induced representation under this inclusion to obtain a representation of $S_{n+m}$. Hence, we define the product in $R$ by

$$S_\lambda \bullet S_\mu := \mathrm{Ind}_{S_n\times S_m}^{S_{n+m}}(S_\lambda \otimes S_\mu) \in R_{\mathbb{C}}(S_{n+m})$$

With some work, one can show that this in fact induces the ordinary product on the ring of symmetric functions. Moreover, via the map that one utilizes to show this fact, one can also show that the characters of $S_\lambda(V)$ are precisely the Schur functions evaluated at the eigenvalues of the input matrix.

# References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