MATH 8851 Homework 1

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Problem 1

Prove the Schreier Subgroup Lemma (the statement is recalled below) without the extra assumption $1 \in T$.
Note: You just need to slightly adjust the proof from class (where we assumed that $1 \in T$).

Proof. Let $H \leq G$, $S \subseteq G$ a generating set for G, and $T \subseteq G$ a right transversal for H inside G. Set

$$U = U(S,T) = \{ts \cdot \overline{ts}^{-1} : s \in S,\, t \in T\}.$$

For $s \in S, t \in T$, we have that

$$Hts = H\overline{ts} \implies Hts \cdot \overline{ts}^{-1} = H \implies ts \cdot \overline{ts}^{-1} \in H,$$

so $U \subseteq H$ whence $\langle U \rangle \leq H$. Thus, it suffices to show the reverse inclusion. First, let $t = \overline{1} \in T \cap H$. Then Choose s_1
Choose $s_1, \ldots, s_n \in S \cup S^{-1}$ such that $t = \prod_{i=1}^{n} s_i$, and set $t_k := \overline{\prod_{i=1}^{k} s_i}$, $t_0 = t$. Then we have

$$t = t_0^{-1}\left(\prod_{i=0}^{n-1} t_i s_{i+1} t_{i+1}^{-1}\right) t_n = t^{-1}\left(\prod_{i=0}^{n-1} t_i s_{i+1} t_{i+1}^{-1}\right) t_n \implies t = \prod_{i=0}^{n-1} t_i s_{i+1} t_{i+1}^{-1}$$

□

Problem 2

Let $F = F(X)$ for some set X and H a subgroup of F. Prove that H always has a Schreier transversal in F (with respect to X) in two different ways as follows:

(a) Using Zorn's lemma

(b) Using suitable total order on F.

Hint for (b): Choose an arbitrary total order on $X \sqcup X^{-1}$ and consider the corresponding lexicographical order on F: given two elements $f \neq f' \in F$, put $f < f'$ if one of the following holds:

(i) $l(f) < l(f')$, where $l(\cdot)$ is the word length

(ii) $l(f) = l(f')$, and if f and f' first differ in k^{th} position, then the k^{th} symbol in f is smaller than the k^{th} symbol in f'.

Then form a transversal by choosing the smallest element in each right coset of H.

Proof (a). Consider the poset

$$\mathscr{S} = \{T \subseteq F : T \text{ is a Schreier subset, } Ht \neq Ht' \ \forall t \neq t' \text{ in } T\}$$

ordered by inclusion. This poset is nonempty as it contains $\{1\}$. Consider any linear chain $(T_\alpha)_{\alpha \in I}$ in $\mathscr{S}$ and set $T = \bigcup_{\alpha \in I} T_\alpha$. Suppose $t_1, t_2 \in T$ have $t_1 \neq t_2$. By linearity of the chain, we may choose an $\alpha \in I$ such that $t_1, t_2 \in T_\alpha$ whence by assumption, $Ht_1 \neq Ht_2$. Now, suppose that w is a reduced word in T. Again by linearity, there exists some $\alpha \in I$ such that all of the symbols from T appearing in the reduced word decomposition for w actually lie inside T_α. Since T_α is Schreier, every initial segment of w also lies in T_α and thus T. So, $T \in \mathscr{S}$ is an upper bound for the chain inside $\mathscr{S}$.

By Zorn's lemma, there exists a maximal element T of $\mathscr{S}$. We claim that T is a right transversal for H. To this end, suppose for the sake of contradiction that T is not a right transversal for H. Then there is some $x \in F$ such that $Hx \neq Ht$ for all $t \in T$.

Consider the set S consisting of all initial segments of reduced words in $T \sqcup \{x\}$. By construction, $S \supseteq T$ is a Schreier subset of F. We claim that S is in fact an element of T.

□

Proof (b). □

Problem 3

Let $F = F(x, y)$ be the free group on two generators. Consider the following two subgroups of F:

(a) $H = [F, F]$, the commutator subgroup of F

(b) $H = \ker \pi$ where π is the epimorphism from F onto S_3 (symmetric group on 3 letters) which sends x to (12) and y to (23).

For each of these subgroups do the following:

(i) Find a Schreier transversal T for H (with respect to $X = \{x, y\}$).

(ii) Draw the Schreier graph $Sch(H \setminus F, X)$ and the maximal tree $\mathcal{T}$ in $Sch(H \setminus F, X)$ corresponding to T (we will define the natural bijection between the Schreier transversals and maximal trees in class on Monday, Jan 29)

(iii) Use the strong Nielsen-Schreier Theorem (the statement is recalled below) to find a free generating set for H.

Problem 4

Prove the *Schreier index formula*: If F is a free group of finite rank and H a subgroup of F of finite index, then

$$\text{rk}(H) - 1 = (\text{rk}(F) - 1) \cdot [F : H].$$

Hint: Count the number of vertices and edges in the Schreier graph $Sch(H \setminus F, X)$ and use the fact that $H \cong \pi_1(Sch(H \setminus F, X))$.

Proof. Let $X = \{x_1, \ldots, x_n\}$ be a free generating set for F and assume without loss of generality that $F = F(X)$. □

Problem 5

Use the Schreier Subgroup lemma to find a generating set with 2 elements for the alternating group A_n.

Solution. Consider the generating set $S = \{(12), (12\cdots n)\}$ for S_n. The set $T = \{e, (12)\}$ is a right transversal for A_n inside S_n. Note that then, for a given permuation $\sigma \in S_n$,

$$\overline{\sigma} = \begin{cases} e & \sigma \text{ is even} \\ (12) & \sigma \text{ is odd.} \end{cases}$$

If n is odd, then $U(S,T)$ consists of

$$\begin{aligned}
e(12)\overline{e(12)}^{-1} &= e \\
(12)(12)\overline{(12)(12)}^{-1} &= e \\
e(12\cdots n)\overline{e(12\cdots n)}^{-1} &= (12\cdots n) \\
(12)(12\cdots n)\overline{(12)(12\cdots n)}^{-1} &= (12)(12\cdots n)(12).
\end{aligned}$$

If n is even, then $U(S,T)$ consists of

$$\begin{aligned}
e(12)\overline{e(12)}^{-1} &= e \\
(12)(12)\overline{(12)(12)}^{-1} &= e \\
e(12\cdots n)\overline{e(12\cdots n)}^{-1} &= (12\cdots n)(12) \\
(12)(12\cdots n)\overline{(12)(12\cdots n)}^{-1} &= (12)(12\cdots n).
\end{aligned}$$

Thus, Schreier Subgroup lemma produces the following 2-element generating sets for A_n

$$U(S,T) = \begin{cases} \{(12\cdots n), (12)(12\cdots n)(12)\} & n \text{ is odd} \\ \{(12\cdots n)(12), (12)(12\cdots n)\} & n \text{ is even} \end{cases}$$

□

Lemma 1 (Schreier Subgroup Lemma). *Let G be a group, H a subgroup of G, S a generating set for G and T a right transversal for H in G. Then H is generated by the set*

$$U = U(S,T) = \{ts \cdot \overline{ts}^{-1} : s \in S, t \in T\}$$

where $\overline{g}$ is the unique element of T such that $H\overline{g} = Hg$.

Theorem 1 (Strong Nielsen-Schreier Theorem). *Let H a subgroup of $F(X)$, and let T be a (right) Schreier transversal for H (with respect to X). For every $x \in X$ and $t \in T$ let $h_{x,t} = xt \cdot \overline{xt}^{-1}$. Let*

$$I = \{(x,t) \in X \times T : h_{x,t} \neq 1\}.$$

Then the elements $\{h_{x,t} : (x,t) \in I\}$ are all distinct and form a free generating set for H.