Specht Modules and Schur Weyl Duality

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Abstract

In this paper we exposit one of the fundamental results linking representation theory and algebraic combinatorics called Schur-Weyl duality. It provides a dictionary between the representation theory of finite symmetric groups and the representation theory of the general linear group of a finite dimensional complex vector space. Through this dictionary, we obtain representation theoretic constructions of some aspects of symmetric function theory, including Schur functions, and internal/external products on the symmetric function ring.

Contents

1 Representation Theory Background

Before the introduction of the abstract group concept by Emmy Nöether, groups were only studied in relation to their actions. As the old adage goes,"*groups, as men, shall be known by their actions.*" This philosophy takes its strongest modern form in the notion of representation theory, which studies groups through the lens of their actions on vector spaces. As an illustrative example, consider the dihedral group of order $2n$. This group has the abstract presentation

$$D_{2n} = \langle r, s | r^n = s^2 = e, rsr = s^{-1} \rangle;$$

however, this presentation is not how we intuitively think about D_{2n}. This group is always introduced as the symmetries of an n-gon, which suggests some type of action on $\mathbb{R}^2$. Consider the map $\pi : D_{2n} \to \mathrm{GL}_2(\mathbb{R})$ given by

$$\begin{aligned} r &\mapsto \begin{pmatrix} \cos\left(\frac{2\pi}{n}\right) & -\sin\left(\frac{2\pi}{n}\right) \\ \sin\left(\frac{2\pi}{n}\right) & \cos\left(\frac{2\pi}{n}\right) \end{pmatrix} \\ s &\mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

The map π makes concrete our intuition about the dihedral group and encodes its information via an action on the vector space $\mathbb{R}^2$. This leads to the notion of a representation.

1.1 Group Representations

Definition 1.1.1. A *representation* of a group G is a pair (π, V) where V is a $\mathbb{C}$-vector space and $\pi : G \to \mathrm{GL}(V)$ is a group homomorphism.

Example 1.1.1. Let V have basis $e_1, \ldots, e_n$ and consider the map $\pi : S_n \to \mathrm{GL}(V)$ given by $\pi(\sigma)e_i = e_{\sigma(i)}$ extended by linearity.

Definition 1.1.2. A morphism between two representations (π, V) and (ρ, W) of a group G is a linear map $T : V \to W$ such that $T\pi(g) = \rho(g)T$ for all $g \in G$. A morphism between representations is an isomorphism if it is an isomorphism of vector spaces.

We write $\mathrm{Hom}_G(V, W)$ for the set of morphisms between (π, V) and (ρ, W).

Definition 1.1.3. Fix a representation (π, V) of a group G.

- A *subrepresentation* of (π, V) is a subspace $W \subseteq V$ such that $\pi(g)w \in W$ for all $w \in W$.
- We say (π, V) is *irreducible* if its only subrepresentations are V and 0.

Example 1.1.2. Let $\pi : S_n \to \mathrm{GL}(V)$ be the representation above and consider the subspace $W := \{\sum \alpha_i e_i \in V : \sum_i \alpha_i = 0\}$. Note that if $\sum_i \alpha_i = 0$, then $\sum_i \alpha_{\sigma(i)} = 0$, so $\pi(\sigma)W \subseteq W$ for all $\sigma \in S_n$. The representation $(W, \pi|_W)$ is called the *standard representation* of S_n.

Exercise 1.1.1. *Prove that $(W, \pi|_W)$ is an irreducible representation of S_n.*

The interplay between irreducible representations and morphisms of representations is encapsulated in the following fundamental result.

Lemma 1.1.1 (Schur's Lemma). *Let (π, V), (ρ, W) be irreducible representations of a group G. Then $\mathrm{Hom}_G(V, W) \cong \mathbb{C}$ if $(\pi, V) \cong (\rho, W)$ and is 0 otherwise.*

Proof. Suppose $\mathrm{Hom}_G(V, W) \neq 0$. Let $T \in \mathrm{Hom}_G(V, W) \setminus \{0\}$. Since $\ker(T) \neq V$, irreduciblity implies $\ker(T) = 0$. Likewise, as $\mathrm{Im}(T) \neq 0$, irreducibility implies $\mathrm{Im}(T) = W$. Hence T is an isomorphism, so without loss of generality assume $(\pi, V) = (\rho, W)$. Let $\alpha \in \mathbb{C}$ be an eigenvalue of T with eigenvector v and observe

$$T\pi(g)v = \pi(g)Tv = \pi(g)\alpha v = \alpha\pi(g)v.$$

As $v \neq 0$, irreducibility implies $\pi(G)v = V$, whence $T = \alpha I$ on all of V. Thus every element of $\mathrm{Hom}_G(V, V)$ is a multiple of the identity. $\square$

Definition 1.1.4. Given a representation (π, V) of a group G, we define the corresponding *dual respresentation* to be (π^*, V^*) where $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ is the dual vector space and $\pi^*(g)f(v) = f(\pi(g^{-1})v)$ for $f \in V^*$, $g \in G$, and $v \in V$.

1.2 Character Theory

Character theory studies a very refined invariant of representation theory which contains a surprising amoun of information. Whilst we will not need much character theory in the following, we mention some of the main definitions and results for the reader's enlightenment.

Definition 1.2.1. Given a representation (π, V) of a group G, the corresponding character of the representation is the function $\chi_\pi : G \to \mathbb{C}$ given by

$$\chi_\pi(g) = Tr(\pi(g))$$

Note that as Tr is conjugacy invariant, so is χ_π.

Exercise 1.2.1. *Compute the character of the standard representation of S_n.*

An incredibly suprising result of finite group representation theory is that the characters of representations are enough to entirely determine the representation. This is encapsulated in the following theorem.

Theorem 1.2.1. *Let $(\pi, V), (\rho, W)$ be (complex) representations of a finite group G. Then $(\pi, V) \cong (\rho, W)$ if and only if $\chi_\pi = \chi_\rho$.*

One application of this theorem is the self-duality of representations of symmetric groups.

Proposition 1.2.1. *Let (π, V) be a representation of S_n. Then $V \cong V^*$ as representations.*

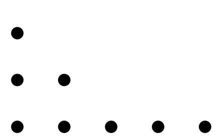
Proof. One may compute that $\chi_{V^*}(\pi(g)) = \chi_V(\pi(g^{-1}))$. In S_n, the group elements g and g^{-1} are conjugate, so $\chi_V(\pi(g^{-1})) = \chi_V(\pi(g))$, whence by the above theorem, since V and V^* have the same characters, it follows that they are isomorphic as representations. $\square$

2 Representations of S_n

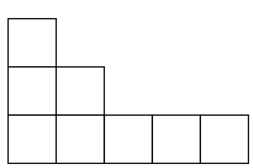
2.1 Partitions, Young Diagrams, and Tabloids

To begin discussing the representation theory of S_n, we first lay out some combinatorial groundwork with which to build the theory upon.

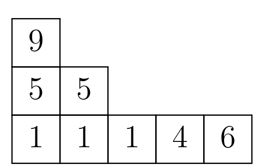
Definition 2.1.1. Given $\lambda \vdash n$, the *Ferrers diagram of shape* λ is the set $\{(i, j) \in \mathbb{N}^2 : j \in \mathbb{N}, 1 \leq i \leq \lambda_j\}$ depicted as points in $\mathbb{R}^2$. The *Young diagram of shape* λ is depicted identically to the Ferrers diagram except the points are replaced with squares. The *size* of the diagram is the number of entries, namely n. We depict the case $(5, 2, 1) \vdash 8$ below.



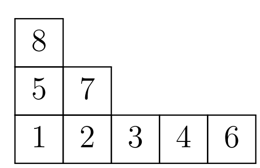
Ferrers Diagram for the partition (5,2,1)



Young Diagram for the partition (5,2,1)



Semi-standard Young tableau for the partition (5,2,1)



Standard Young tableau for the partition (5,2,1)

Definition 2.1.2. Given $\lambda \vdash n$, a λ-tableau is simply a filling of the boxes of the Young diagram of shape λ with the elements of $\{1, \ldots, n\}$ without repetition (and no other restrictions). Denote the set of λ-tableaux by $YT(\lambda)$. Note that $S_n \curvearrowright YT(\lambda)$ by permuting labels.

Definition 2.1.3. Given $\lambda \vdash n$, define an equivalence relation $\sim$ on $YT(\lambda)$ by $\mathcal{T} \sim \mathcal{T}'$ if and only if $\mathcal{T}'$ can be obtained from $\mathcal{T}$ by permuting the entries of each row. An equivalence class with respect to this relation is called a λ-*tabloid*. If $\mathcal{T}$ is a λ-tableau, we write $\{\mathcal{T}\}$ for the corresponding λ-tabloid. Finally, we write $Tab(\lambda) := YT(\lambda)/\sim$ for the set of λ-tabloids. Note that the action of S_n on λ-tableaux descends to an action on λ-tabloids.

Example 2.1.1. Below, we show two equivalent and inequivalent $(5,2,1)$-tableaux. The first two tableau are equal as tabloids, whereas the last is not equal to the first two.

$$\begin{array}{ccccc} 8 & & & & \\ 5 & 7 & & & \\ 1 & 2 & 3 & 4 & 6 \end{array} \sim \begin{array}{ccccc} 8 & & & & \\ 7 & 5 & & & \\ 1 & 3 & 2 & 6 & 4 \end{array} \not\sim \begin{array}{ccccc} 8 & & & & \\ 1 & 7 & & & \\ 5 & 2 & 3 & 4 & 6 \end{array}$$

Definition 2.1.4. Given $\lambda \vdash n$ and a Young diagram of shape λ, a *semi-standard Young tableau of shape* λ is a filling of the boxes of the Young diagram with positive integers such that

- the entries are weakly increasing along rows,
- the entries are strictly increasing up columns.

A semi-standard Young tableau of size n is said to be *standard* if the elements of $\{1,\ldots,n\}$ each appear exactly once in the tableau. We write $SSYT(\lambda)$ and $SYT(\lambda)$ for the sets of semi-standard and standard Young tableaux of shape λ. Given a semi-standard Young tableau $\mathcal{T}$, the *weight* of $\mathcal{T}$ is a function $\alpha = \alpha_{\mathcal{T}} : \mathbb{N} \to \mathbb{N}$ given by

$$\alpha(i) := \text{number of times } i \text{ appears in } \mathcal{T}.$$

Note that $\alpha(i) = 0$ for sufficiently large i, so we may write $x^\alpha = x_1^{\alpha(1)} x_2^{\alpha(2)} \cdots$ and obtain a valid monomial. We write $SSYT(\lambda, \alpha)$ for the set of semi-standard Young tableaux of shape λ and weight α.

2.2 Construction of Specht Modules

Young diagrams will give projection operators $P_\lambda : \mathbb{C}[S_n] \to \mathbb{C}[S_n]$ which commute with the action of S_n, whence the image $P_\lambda(\mathbb{C}[S_n])$ gives a subrepresentation of the regular representation. These subrepresentations will end up being precisely the irreducible representations of S_n. Throughout this section, $\lambda \vdash n$ will be fixed.

Definition 2.2.1. Given a λ-tableau $\mathcal{T}$, define the *row group* $R_{\mathcal{T}}$ to be the subgroup of S_n which permutes only the labels in the rows of $\mathcal{T}$ and the *column group* $C_{\mathcal{T}}$ as the subgroup which permutes only the labels in the columns of $\mathcal{T}$.

As an illustrative example for these groups, note that the equality below implies $(57)(23)(64) \in R_{\mathcal{T}}$. Now we may define the *Young row and column symmetrizers* in $\mathbb{C}[S_n]$ by

$$a_{\mathcal{T}} := \sum_{\sigma \in R_{\mathcal{T}}} \sigma, \qquad b_{\mathcal{T}} := \sum_{\sigma \in C_{\mathcal{T}}} \operatorname{sgn}(\sigma)\sigma. \tag{1}$$

Note that for $\mathcal{T} \in YT(\lambda)$, the corresponding tabloid is precisely the orbit of $\mathcal{T}$ under its row group, i.e.

$$\{\mathcal{T}\} = R_{\mathcal{T}}\mathcal{T} = \{\sigma\mathcal{T} \in YT(\lambda) : \sigma \in R_{\mathcal{T}}\}.$$

$$(57)(23)(64)\cdot \begin{array}{|c|c|c|c|c|}\cline{1-1} 8 \\ \cline{1-2} 5 & 7 \\ \hline 1 & 2 & 3 & 4 & 6 \\ \hline \end{array} \quad = \quad \begin{array}{|c|c|c|c|c|}\cline{1-1} 8 \\ \cline{1-2} 7 & 5 \\ \hline 1 & 3 & 2 & 6 & 4 \\ \hline \end{array}$$

Now let M^λ be the free $\mathbb{C}$-vector space over the set of λ-tabloids. Extending the action $S_n \curvearrowright Tab(\lambda)$ linearly to all of M^λ, we obtain a $\mathbb{C}[S_n]$-module structure on M^λ. For $\mathcal{T} \in YT(\lambda)$, the element $e_\mathcal{T} \in M^\lambda$ given by

$$e_\mathcal{T} := b_\mathcal{T} \cdot \{\mathcal{T}\} = \sum_{\sigma \in C_\mathcal{T}} \operatorname{sgn}(\sigma)\{\sigma\mathcal{T}\}$$

is called the *polytabloid associated to* $\mathcal{T}$. Let S^λ be the subspace of M^λ generated by all polytabloids, namely

$$S^\lambda := \operatorname{Span}_\mathbb{C}\{e_\mathcal{T} : \mathcal{T} \in YT(\lambda)\}.$$

Claim. S^λ is a $\mathbb{C}[S_n]$-submodule of M^λ.

Proof of Claim. Fix $\sigma \in S_n$. We first show that $C_{\sigma\mathcal{T}} = \sigma C_\mathcal{T}\sigma^{-1}$. Indeed, if T_i is the set of entries for the ith column of $\mathcal{T}$, then $\sigma(T_i)$ is the entries for the ith column of $\sigma\mathcal{T}$. Now it suffices to note that $\tau \in S_n$ stabilizes T_i if and only if $\sigma\tau\sigma^{-1}$ stabilizes $\sigma(T_i)$. Using this identity, we compute

$$\sigma b_\mathcal{T} = \sum_{\gamma \in C_\mathcal{T}} \operatorname{sgn}(\gamma)\sigma\gamma \overset{\tau=\sigma\gamma\sigma^{-1}}{=} \sum_{\tau \in \sigma C_\mathcal{T}\sigma^{-1}} \operatorname{sgn}(\sigma^{-1}\tau\sigma)\tau\sigma = \sum_{\tau \in C_{\sigma\mathcal{T}}} \operatorname{sgn}(\tau)\tau\sigma = b_{\sigma\mathcal{T}}\sigma.$$

Now we apply σ to the generators of S^λ and find

$$\sigma \cdot e_\mathcal{T} = \sigma \cdot (b_\mathcal{T} \cdot \{\mathcal{T}\}) = (\sigma b_\mathcal{T}) \cdot \{\mathcal{T}\} = b_{\sigma\mathcal{T}}\{\sigma\mathcal{T}\} = e_{\sigma\mathcal{T}}.$$

As S_n stabilizes S^λ, the claim follows. □

Definition 2.2.2. The $\mathbb{C}[S_n]$-module S^λ as defined above is the *Specht module corresponding to* λ.

Example 2.2.1 (Sign Representation)**.** Consider the partition $\lambda = (1, 1, \ldots, 1)$ of n. Since each row of λ has one element, the λ-tabloids are the same as λ-tableaux.

Let $\mathcal{T}$ be a λ-tableau. As $\mathcal{T}$ has only one column, $C_\mathcal{T} = S_n$, whence $b_\mathcal{T} = \sum_{\gamma \in S_n} \operatorname{sgn}(\gamma)\gamma$ and consequently

$$\sigma e_\mathcal{T} = \sum_{\gamma \in S_n} \operatorname{sgn}(\gamma)\sigma\gamma\{\mathcal{T}\} = \sum_{\tau \in S_n} \operatorname{sgn}(\sigma^{-1}\tau)\tau\{\mathcal{T}\} = \operatorname{sgn}(\sigma)e_\mathcal{T} \quad \text{for all } \sigma \in S_n.$$

On the other hand, we know that $\sigma e_\mathcal{T} = e_{\sigma\mathcal{T}}$, so it follows that $S^\lambda = \mathbb{C}e_\mathcal{T}$ is the one-dimensional sgn representation.

Example 2.2.2 (Trivial Representation)**.** Consider the partition $\lambda = (n)$ of n. Since there is one row of λ, all λ-tableaux are equivalent so there is only one λ-tabloid. Fix a λ-tableau $\mathcal{S}$.

Each $e_\mathcal{T} = \{T\} = \{S\}$, so $S^\lambda = \mathbb{C}e_\mathcal{S}$ is one-dimensional. The action of σ is given by $\sigma e_\mathcal{T} = e_{\sigma\mathcal{T}} = e_\mathcal{T}$, so S^λ is the trivial representation of S_n.

$$\begin{array}{|c|c|c|c|c|}
\cline{1-1}
i \\
\hline
j & & \cdots & & \\
\hline
\end{array}$$

General form of $\mathcal{T}$ when $t_i = \{\mathcal{T}\}$

Example 2.2.3 (Augmentation Subrepresentation)**.** Consider the partition $\lambda = (n-1, 1)$ of n. Observe that there are n distinct λ-tabloids, each corresponding to the integer in singular box on the 2nd row. Denote the tabloid with i in the $2nd$ row by t_i, so $Tab(\lambda) = \{t_1, \ldots, t_n\}$.

Let $V = \mathbb{C}\{v_1, \ldots, v_n\}$ be the standard representation of S_n (i.e. $\sigma v_i = v_{\sigma(i)}$). Observe that the map $L : V \to M^\lambda$ given by $L(v_i) = t_i$ is an isomorphism of $\mathbb{C}[S_n]$-modules. The *augmentation subrepresentation* W of V is given by $W := \{\sum_{i=1}^n \alpha_i v_i : \sum_i \alpha_i = 0\}$. We claim that $S^\lambda \cong W$ as $\mathbb{C}[S_n]$-modules. Fix $i \in \{1, \ldots, n\}$ and let $\mathcal{T}$ be a λ-tableau such that $t_i = \{\mathcal{T}\}$. Let j be the integer below i on the tableau. Then the column group $C_\mathcal{T}$ is then of order 2 generated by the transposition $(i\ j)$.

$$e_\mathcal{T} = \sum_{\gamma \in C_\mathcal{T}} \operatorname{sgn}(\gamma)\gamma t_i = t_i - t_j.$$

Hence, one checks

$$S^\lambda = \operatorname{Span}\{t_i - t_j : 1 \leq i, j \leq n, i \neq j\} = \operatorname{Span}\{t_i - t_{i+1} : 1 \leq i \leq n-1\}.$$

Moreover, $\{t_i - t_{i+1} : 1 \leq i \leq n-1\}$ gives a basis for S^λ. The restriction of L to W gives a vector space isomorphism $L : W \to S^\lambda$ as $\{v_i - v_{i+1}\}_{1 \leq i \leq n-1}$ gives a basis for W, so a basis gets mapped to a basis. Moreover, this map intertwines the S_n-action, so it produces $\mathbb{C}[S_n]$-module isomorphism.

2.3 Alternative Construction

More algebraically-minded sources on the representation theory of S_n will take an alternative approach to the construction of Specht modules which may elucidate some details to the intended audience and obfuscate some details from others. We present this alternative construction for personal edification as well as to appeal to the former audience.

Fix a λ-tableau $\mathcal{S}$ throughout this section, say the canonical one (increasing across rows and then moving up rows). Recall the row and column symmetrizers $a_\lambda := a_\mathcal{S}$, $b_\lambda := b_\mathcal{S}$ and define the Young symmetrizer

$$c_\lambda := a_\lambda \cdot b_\lambda \in \mathbb{C}[S_n].$$

Set $V_\lambda := \mathbb{C}[S_n]c_\lambda$. Define a map $T : \mathbb{C}[S_n]a_\lambda \to M^\lambda$ by $T(\sigma a_\lambda) = \{\sigma\mathcal{S}\}$.

Claim. The map T is an isomorphism of $\mathbb{C}[S_n]$-modules.

Proof of Claim. We first show this map is well defined. If $\sigma a_\lambda = \tau a_\lambda$, then $\tau^{-1}\sigma$ fixes a_λ, whence $\tau^{-1}\sigma \in R_\mathcal{S}$ and consequently $\sigma\{\mathcal{S}\} = \tau\{\mathcal{S}\}$.

Since the action of S_n on λ-tableau is transitive, it follows that the map T is onto. On the other hand, suppose $\sum_\sigma \alpha_\sigma \sigma a_\lambda \in \ker(T)$. Then

$$0 = T(\sum_\sigma \alpha_\sigma \sigma a_\lambda) = \sum_\sigma \alpha_\sigma \{\sigma\mathcal{S}\}.$$

Since M^λ is a free $\mathbb{C}$-module, it follows that $\sum_\sigma \alpha_\sigma \sigma = 0$. Lastly, if $\sigma, \gamma \in S_n$, then

$$\sigma T(\gamma a_\lambda) = \sigma\{\gamma\mathcal{S}\} = \{\sigma\gamma\mathcal{S}\} = T(\sigma\gamma s_\lambda).$$

$\square$

Claim. The map T restricted to the submodule $\mathbb{C}[S_n]b_\lambda a_\lambda$ gives a $\mathbb{C}[S_n]$-module isomorphism $\mathbb{C}[S_n]b_\lambda a_\lambda \cong S^\lambda$.

Proof of Claim. For $\sigma \in S_n$, we compute

$$\begin{aligned} T(\sigma b_\lambda a_\lambda) &= \sum_{\tau \in C_{\mathcal{S}}} \operatorname{sgn}(\tau) T(\sigma\tau a_\lambda) = \sum_{\tau \in C_{\mathcal{S}}} \operatorname{sgn}(\tau)\{\sigma\tau\mathcal{S}\} \\ &= \sigma \sum_{\tau \in C_{\mathcal{S}}} \operatorname{sgn}(\tau)\{\tau\mathcal{S}\} = \sigma e_{\mathcal{S}} = e_{\sigma\mathcal{S}} \end{aligned}$$

Since S_n acts transitively on λ-tableaux, it follows that

$$T(\mathbb{C}[S_n]b_\lambda a_\lambda) = \operatorname{Span}_{\mathbb{C}}\{e_{\sigma\mathcal{S}} : \sigma \in S_n\} = S^\lambda$$

By the proof of the previous claim, T is injective and intertwines the action of S_n, whence $T|_{\mathbb{C}[S_n]b_\lambda a_\lambda}$ furnishes an isomorphism of $\mathbb{C}[S_n]$-modules as desired. $\square$

Proposition 2.3.1. $\mathbb{C}[S_n]b_\lambda a_\lambda \cong \mathbb{C}[S_n]a_\lambda b_\lambda$.

2.4 Results on Specht Modules

Having obtained a few examples of Specht modules, we note that $\{S^\lambda : \lambda \vdash n\}$ forms a complete set of non-isomorphic, irreducible representations of S_n. This is established by the combining the following three theorems, which we leave unproven and cite standard references [FH91], [Sta24].

Theorem 2.4.1. *Given $\lambda \vdash n$, the Specht module S^λ is irreducible as a $\mathbb{C}[S_n]$-module (i.e. an irreducible representation of S_n).*

Theorem 2.4.2. *If $\lambda, \mu \vdash n$ and $\lambda \neq \mu$, then $S^\lambda \ncong S^\mu$ as $\mathbb{C}[S_n]$-modules.*

Theorem 2.4.3. *Every irreducible representation of S_n is ismorphic to S^λ for some $\lambda \vdash n$.*

3 Representations of $\mathrm{GL}(V)$

Having discussed for some time the representation theory of S_n, we pivot sharply and discuss the representation theory of $\mathrm{GL}(V)$ (an infinite group!) and peruse through a surprising relation between these two.

3.1 Schur Functors

Let V be a finite dimensional complex vector space and consider the space $V^{\otimes n}$. We have a natural (right) action of S_n on $V^{\otimes n}$ given for $\sigma \in S_n$ by

$$(v_1 \otimes \cdots \otimes v_n)\sigma := v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}.$$

We also have a natural (left) action of $\mathrm{GL}(V)$ on $V^{\otimes n}$ given for $T \in \mathrm{GL}(V)$ by

$$T(v_1 \otimes \cdots \otimes v_n) := Tv_1 \otimes \cdots \otimes Tv_n.$$

Moreover, these actions commute with each other.

Definition 3.1.1. Fix $\lambda \vdash n$. The *Schur functor of shape* λ is the functor $\mathbb{S}_\lambda : Vect_{\mathbb{C}} \to Vect$ given, for a finite-dimensional vector space V, by

$$\mathbb{S}_\lambda(V) := \mathrm{Hom}_{S_n}(S^\lambda, V^{\otimes n})$$

Moreover, $\mathbb{S}_\lambda(V)$ is a representation of $\mathrm{GL}(V)$ under the natural action $[T\varphi](x) = T\varphi(x)$ for $\varphi \in \mathbb{S}_\lambda(V)$, $T \in \mathrm{GL}(V)$, and $x \in S^\lambda$.

With the notation of section 2.3, we note the following alternative construction of $\mathbb{S}_\lambda(V)$ by computing

$$\begin{aligned}\mathbb{S}_\lambda(V) = \mathrm{Hom}_{S_n}(S^\lambda, V^{\otimes n}) &\cong V^{\otimes n} \otimes_{\mathbb{C}[S_n]} (S^\lambda)^* \\ &\cong V^{\otimes n} \otimes_{\mathbb{C}[S_n]} S^\lambda \\ &\cong V^{\otimes n} \otimes_{\mathbb{C}[S_n]} \rho(c_\lambda)\mathbb{C}[S_n] \cong V^{\otimes n} c_\lambda \otimes_{\mathbb{C}[S_n]} \mathbb{C}[S_n] \cong V^{\otimes n} c_\lambda.\end{aligned}$$

Hence, the Schur functor may also be described as the image of the action of the Young symmetrizer when restricted to $V^{\otimes n}$.

3.2 Schur-Weyl Duality

Theorem 3.2.1 (Schur-Weyl Duality)**.** *Let V be a finite-dimensional complex vector space and regard $V^{\otimes n}$ as a representation of $\mathrm{GL}(V) \times S_n$ as described above. Then, as representations,*

$$V^{\otimes n} \cong \bigoplus_{\lambda \vdash n} S^\lambda \otimes_{\mathbb{C}} \mathbb{S}_\lambda(V).$$

Lemma 3.2.1. *The symmetric tensor power $\mathrm{Sym}^n(V)$ is spanned by $v \otimes \cdots \otimes v$ for $v \in V$.*

Lemma 3.2.2. *A subspace of $V^{\otimes n}$ is an $\mathrm{End}_{S_n}(V^{\otimes n})$-submodule if and only if it is a $\mathrm{GL}(V)$-submodule.*

Proof. Consider the inclusion

$$\mathrm{End}(V) \overset{\iota}{\hookrightarrow} \mathrm{End}(V^{\otimes n}) \cong \mathrm{End}(V)^{\otimes n}$$

under the map $T \overset{\iota}{\mapsto} T \otimes \cdots \otimes T$. By Lemma 3.2.1,

$$\mathrm{Span}_{\mathbb{C}}(\iota(\mathrm{End}(V)) = \mathrm{Sym}^n(\mathrm{End}(V)) = \mathrm{End}_{S_n}(V^{\otimes n}).$$

Suppose $W \subseteq V^{\otimes n}$ is an $\mathrm{End}_{S_n}(V^{\otimes n})$-submodule. Let $T \in \mathrm{GL}(V)$. By definition, the action of T on $V^{\otimes d}$ is given by $\iota(T)$, which is in $\mathrm{End}_{S_n}(V^{\otimes n})$ by above so $TW \subseteq T$.

On the other hand, suppose $W \subseteq V^{\otimes n}$ is a $\mathrm{GL}(V)$-submodule. Let $L \in \mathrm{End}_{S_n}(V^{\otimes n})$. By above, $L \in \mathrm{Span}_{\mathbb{C}}(\iota(\mathrm{End}(V)))$ so there is some $L_1, \ldots, L_r \in \mathrm{End}(V)$ and $\alpha_1 \ldots, \alpha_r \in \mathbb{C}$ such that

$$L = \sum_{i=1}^{r} \alpha_i L_i$$

Recall that $\mathrm{GL}(V)$ is dense in $\mathrm{End}(V)$ in the Euclidean operator topology, so we may choose $T_{ij} \in \mathrm{GL}(V)$ such that $\|L_i - T_{ij}\|_2 \xrightarrow{j \to \infty} 0$ for $i \in \{1, \ldots, r\}$. Since L is expressed as a finite sum of the L_is, we observe that

$$\left\| L - \sum_{i=1}^{r} \alpha_i T_{ij} \right\|_2 \xrightarrow{j \to \infty} 0$$

Let $w \in W$. Identifying $W \subseteq V^{\otimes n} \cong \mathbb{C}^l$ for some l with the standard topology, it follows by operator continuity that $\sum_{i=1}^r \alpha_i T_{ij} w \xrightarrow{j \to \infty} Lw$, so $Lw \in \overline{W}$ as each $\sum_{i=1}^r \alpha_i T_{ij} w \in W$. As finite dimensional topological vector spaces are closed, $\overline{W} = W$, so W is an $\mathrm{End}_{S_n}(V^{\otimes n})$-submodule.

∎

Lemma 3.2.3. *If (W, π) is an irreducible representation of S_n, then $V^{\otimes n} \otimes_{\mathbb{C}[S_n]} W$ is a simple left $\mathrm{End}_{S_n}(V^{\otimes n})$-module.*

Proof. First decompose $V^{\otimes d}$ into a direct sum of irreducible S_n-representations by

$$V^{\otimes n} = \bigoplus_{i=1}^{l} V_i^{\oplus m_i},$$

so by Schur's lemma $\mathrm{End}_{S_n}(V^{\otimes n}) \cong \bigoplus_{i=1}^{l} M_{m_i}(\mathbb{C})$. Pick s such that $V_s \cong W$. Applying self-duality of S_n-representations and Schur's lemma, we find

$$V_i \otimes_{\mathbb{C}[S_n]} V_s \cong (V_i)^* \otimes_{\mathbb{C}[S_n]} V_s \cong \mathrm{Hom}_{S_n}(V_i, V_s) = \begin{cases} \mathbb{C} & \text{if } i = s \\ 0 & \text{otherwise} \end{cases}$$

So, we compute

$$V^{\otimes n} \otimes_{\mathbb{C}[S_n]} W \cong \bigoplus_{i=1}^{l} V_i^{\oplus m_i} \otimes_{\mathbb{C}[S_n]} W \cong \bigoplus_{i=1}^{l} (V_i \otimes_{\mathbb{C}[S_n]} W)^{\oplus m_i} \cong \mathbb{C}^{\oplus m_s}$$

which is most definitely irreducible under the action of $\bigoplus_{i=1}^{l} M_{m_i}(\mathbb{C}) = \mathrm{End}_{S_n}(V^{\otimes n})$. $\square$

By the above lemma, each Schur functor $\mathbb{S}_\lambda(V)$ is an irreducible $\mathrm{GL}(V)$-representation. Applying the above lemmas and the decomposition of $\mathbb{C}[S_n]$, we obtain the theorem:

$$\begin{aligned} V^{\otimes n} = V^{\otimes n} \otimes_{\mathbb{C}[S_n]} \mathbb{C}[S_n] &= V^{\otimes n} \otimes_{\mathbb{C}[S_n]} \bigoplus_{\lambda \vdash n} (S^\lambda)^{\oplus \dim S^\lambda} \\ &\cong \bigoplus_{\lambda \vdash n} (V^{\otimes n} \otimes_{\mathbb{C}[S_n]} S^\lambda)^{\oplus \dim S^\lambda} \\ &\cong \bigoplus_{\lambda \vdash n} \mathbb{S}_\lambda(V)^{\oplus \dim S^\lambda} \cong \bigoplus_{\lambda \vdash n} S^\lambda \otimes_{\mathbb{C}} \mathbb{S}_\lambda(V). \end{aligned}$$

4 Applications

4.1 Kronecker Multiplication

The first step is to work over only with representations over $\mathbb{C}$, since then isomorphism classes of representations are determined entirely by their characters. For a finite group G, we may consider the representation ring of G over $\mathbb{C}$, $R_{\mathbb{C}}(G)$. If $V_1, \ldots, V_r$ are the irreducible complex representations of G, then the representation ring of G

$$R_{\mathbb{C}}(G) = \bigoplus_{i=1}^{r} \mathbb{Z} V_i = \bigoplus_{i=1}^{r} \mathbb{Z} \chi_{V_i}$$

i.e. it is a free $\mathbb{Z}$ module of rank r generated by the (isomorphism classes of the) irreducible representations (or their characters since we are over $\mathbb{C}$).

By the Specht module construction, we know that the irreducible representations of S_n are the Specht modules, so $R_{\mathbb{C}}(S_n) \cong \bigoplus_{\lambda \vdash n} \mathbb{Z} S_\lambda$. The product structure in $\mathbb{R}_{\mathbb{C}}(S_n)$ is determined by its generators, so consider two partitions $\lambda, \mu \vdash n$. Since this ring is free, there exist $g_{\lambda,\mu}^{\nu} \in \mathbb{Z}$ such that

$$S_\lambda \otimes S_\mu = \sum_{\nu \vdash n} g_{\lambda,\mu}^{\nu} S_\nu.$$

These coefficients $g^{\nu}_{\lambda,\mu}$ are called Kronecker coefficients and they are incredibly difficult to understand, hence ordinary multiplication in $\mathbb{R}_{\mathbb{C}}(S_n)$ is difficult to understand. At the level of symmetric functions, this does induce a new product called the *Kronecker product*

$$s_\lambda \star s_\mu = \sum_\nu g^{\nu}_{\lambda,\mu} s_\nu.$$

One difficulty with this approach is that we are only looking at one graded piece of the ring of symmetric functions and trying to stay within this piece—a philosophy counter to that of the notion of grading.

4.2 Frobenius Characteristic Map

The Frobenius characteristic map is what arises when one considers the entire graded ring of symmetric functions globally under this framework. It is a shadow of the behavior of the full *graded* representation ring of all the groups $\{S_n\}_{n=1}^{\infty}$. Consider the graded abelian group

$$R = \bigoplus_{n\geq 0} R_{\mathbb{C}}(S_n) = \bigoplus_{n\geq 0} \bigoplus_{\lambda\in \mathrm{Par}(n)} \mathbb{Z}S_\lambda = \bigoplus_{\lambda\in\mathrm{Par}} \mathbb{Z}S_\lambda$$

with grading $R_n = R_{\mathbb{C}}(S_n)$. We define a graded ring structure on R as follows. Let $n, m \geq 0$ and $\lambda \vdash n$, $\mu \vdash m$. The product of S_λ and S_μ needs to be a representation of S_{n+m}. Since S_λ and S_μ are a priori representations of different groups, we only have access to an external tensor product $S_\lambda \boxtimes S_\mu$. The problem now is that this is a representation of $S_n \times S_m$, not S_{n+m}.

To fix this, note that we have a canonical inclusion $S_n \times S_m \hookrightarrow S_{n+m}$. All we have to do now is consider the induced representation under this inclusion to obtain a representation of S_{n+m}. Hence, we define the product in R by

$$S_\lambda \bullet S_\mu := \mathrm{Ind}_{S_n\times S_m}^{S_{n+m}}(S_\lambda \otimes S_\mu) \in R_{\mathbb{C}}(S_{n+m})$$

With some work, one can show that this in fact induces the ordinary product on the ring of symmetric functions. Moreover, via the map that one utilizes to show this fact, one can also show that the characters of $S_\lambda(V)$ are precisely the Schur functions evaluated at the eigenvalues of the input matrix.

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