# A Poisson Boundary Theory for Groupoids

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**Goal**: Develop a theory of Poisson boundary for groupoids using Das-Peterson's noncommutative PB framework from [DP22].

# 1 Setup

## 1.1 Groupoid Side

Let $(\mathcal{G}, \mu)$ be a discrete pmp groupoid and $X = \mathcal{G}^{(0)}$. Assume we have a collection $\{\pi_x\}_{x\in X}$ such that $\pi_x \in \mathrm{Prob}(\mathcal{G}^x)$ for all $x \in X$. Extend each $\pi_x$ by zero to be defined on of $\mathcal{G}$. For $g \in \mathcal{G}$, set

$$\pi_g = g_* \pi_{s(g)}$$

In the framework of [Kai05], $\{\pi_g\}$ give the transition probabilites for a Markov operator on $\mathcal{G}$. Such a Markov operator is called *invariant* if $g_*\pi_h = \pi_{gh}$ for all $(g, h) \in \mathcal{G}^{(2)}$.

**Definition 1.1.1** ([Kai05])**.** A family $\pi = \{\pi_g\}$ of transition probabilities is called Borel if for every non-negative Borel function $f$, the function $\pi(f) : \mathcal{G} \to \mathbb{C}$ given by $\pi(f)(g) = \int_{\mathcal{G}} f \, d\pi_g$ is Borel.

Given such a Borel family $\pi$, we then get an induced Markov operator $P : \mathrm{Bor}(\mathcal{G}) \to \mathrm{Bor}(\mathcal{G})$ given by $Pf = \pi(f)$. The corresponding *dual operator* $\widetilde{P} : M_+(\mathcal{G}) \to M_+(\mathcal{G})$ is then given by

$$\widetilde{P}(\theta) = \int_{\mathcal{G}} \pi_g \, d\theta(g) \ \text{ for all } \theta \in M_+(\mathcal{G}).$$

Now by definition of the vector-valued integral,

$$\begin{aligned}\langle \theta, Pf\rangle = \int_{\mathcal{G}} Pf(g)\, d\theta(g) &= \int_{\mathcal{G}} \left( \int_{\mathcal{G}} f \, d\pi_g \right) d\theta(g) \\ &= \int_{\mathcal{G}} f \, d\widetilde{P}\theta = \langle \widetilde{P}\theta, f\rangle\end{aligned}$$

## 1.2 Von Neumann Algebras Side

Fix a tracial von Neumann algebra $(M, \tau)$ and an embedding $M \subseteq \mathcal{A}$ into a $C^*$ algebra $\mathcal{A}$.

$$S_\tau\mathcal{A}) := \{\varphi \in S(\mathcal{A}) : \varphi|_M = \tau\}.$$

Fixing $\varphi \in S_\tau(\mathcal{A})$ gives an inclusion $L^2(M, \tau) \subseteq L^2(\mathcal{A}, \varphi)$. Let $e_M = Proj_{L^2(M,\tau)} \in B(L^2(\mathcal{A}, \varphi))$. Define a u.c.p. map $\mathcal{P}_\varphi : \mathcal{A} \to B(L^2(M, \tau))$, by

$$\mathcal{P}_\varphi(T) := e_M T e_m \text{ for } T \in \mathcal{A}$$

For $x \in M$, $\mathcal{P}_\varphi(x) = x$. The map $\mathcal{P}_\varphi$ is the *Poisson transform* of the inclusion $M \subseteq \mathcal{A}$.

# 2 Ideas moving forward

- Similar to Remi and Boutonnet in this paper, study intermediate von Neumann algebras

$$L(\mathcal{G}) \subseteq M \subseteq L(\mathcal{G} \curvearrowright \mathcal{B})$$

where $\mathcal{G} \curvearrowright \mathcal{B}$ is the Poisson boundary action. This framework can be utilized to possible study the question of whether

$$(L(G) \subseteq L^\infty \ltimes G) \cong (L(H) \subseteq L^\infty(Y) \ltimes H) \stackrel{?}{\iff} G \curvearrowright (X, \mu) \cong H \curvearrowright (Y, \nu)$$

in the same vein as the fact the inclusions of the $L^\infty$-space determine action up to orbit equivalence.

- Recent paper of Sartini in [SS24] studying Poisson boundaries of ergodic groupoids in the vain of Kaimonovich. Note that this doesn't run through any von Neumann algebraic framework.

# 3 Weak Containment

Write $\mathscr{S} := \{au_\sigma : a \in A,\, \sigma \in [\mathcal{G}]\}$. Suppose $\xi \in S_1(X * \mathcal{H})$, $\psi \in M$, and both $E \subseteq \mathscr{S}$, $F \subseteq M$ are finite subsets. For $\gamma = au_\sigma \in E$ and $\varphi \in F$, writing $f_{\gamma,\varphi} := \mathbb{E}_A(\gamma\psi\varphi\psi^*)$, we have

$$\langle \gamma \cdot (\xi \otimes \widehat{\psi}) \cdot \varphi, \xi \otimes \widehat{\psi} \rangle = \int_X f_{\gamma,\varphi}(x) \cdot \langle \pi(x\sigma)\xi_{s(x\sigma)}, \xi_x \rangle \, d\mu(x)$$

Fix $\varepsilon' > 0$ and set $\varepsilon := \varepsilon' \cdot (4 \max_{(\gamma,\varphi) \in E \times F}\{\|f_{\gamma,\varphi}\|_\infty\})^{-1}$. By weak containment, there exist sections $\eta^1, \ldots, \eta^s \in S(X * \mathcal{K})$ such that the set

$$A_\varepsilon := \{g \in \mathcal{G} : \left| \langle \pi(g)\xi_{s(g)}, \xi_{r(g)} \rangle - \sum_{i=1}^{s} \langle \rho(g)\eta^i_{s(g)}, \eta^i_{r(g)} \rangle \right| \geq \varepsilon\}$$

has measure $\mu_{\mathcal{G}}(A_\varepsilon) < \varepsilon$. Observe that then

$$A_\varepsilon \cap X = \{x \in X : |1 - \sum_{i=1}^{s} \left\|\eta^i_x\right\|^2| \geq \varepsilon\}$$

has measure $\mu(A_\varepsilon \cap X) = \mu_{\mathcal{G}}(A_\varepsilon \cap X) < \varepsilon$. Consider

$$S_\varepsilon := \{x \in X : |1 - \sum_{i=1}^{s} \left\|\eta^i_x\right\|^2| < \varepsilon\} = X \setminus (A_\varepsilon \cap X).$$

This set has measure $\mu(S_\varepsilon) > 1 - \varepsilon$ and for $x \in S_\varepsilon$ we have that

$$\sum_{i=1}^{s} \left\|\eta^i_x\right\|^2 < 1 + \varepsilon \leq 2,$$

whence the sections given by $\widetilde{\eta}^i_x := 1_{S_\varepsilon}(x)\eta^i_x$ are bounded.

Now, fix $\gamma = au_\sigma \in E$ and $\varphi \in F$. On one hand we have

$$\begin{aligned} &\left| \int_{r(A_\varepsilon)} f(x) \cdot \left( \langle \pi(x\sigma)\xi_{s(x\sigma)}, \xi_x \rangle - \sum_{i=1}^{s} \langle \rho(g)\widetilde{\eta}^i_{s(x\sigma)}, \widetilde{\eta}^i_x \rangle \right) d\mu(x) \right| \\ &\qquad \leq \int_{r(A_\varepsilon)} |f(x)| \left( 1 + \sum_{i=1}^{s} \left\|\widetilde{\eta}^i_x\right\|^2 \right) d\mu(x) \leq 3\varepsilon \|f\|_\infty. \end{aligned}$$

On the other hand, since $X \setminus r(A_\varepsilon) \subseteq S_\varepsilon$, the definition of $S_\varepsilon$ implies that

$$\begin{aligned} &\left| \int_{X \setminus r(A_\varepsilon)} f(x) \cdot \left( \langle \pi(x\sigma)\xi_{s(x\sigma)}, \xi_x \rangle - \sum_{i=1}^{s} \langle \rho(g)\widetilde{\eta}^i_{s(x\sigma)}, \widetilde{\eta}^i_x \rangle \right) d\mu(x) \right| \\ &\qquad \leq \mu(X \setminus r(A_\varepsilon))\varepsilon \|f\|_\infty \leq \varepsilon \|f\|_\infty. \end{aligned}$$

Combining these estimates, we obtain

$$|\langle au_\sigma \cdot (\xi \otimes \widehat{\psi}) \cdot \varphi, \xi \otimes \widehat{\psi} \rangle - \sum_{i=1}^{s} \langle au_\sigma \cdot (\widetilde{\eta}^i \otimes \widehat{\psi}) \cdot \varphi, \widetilde{\eta}^i \otimes \widehat{\psi} \rangle| < 4\varepsilon \|f\|_\infty = \varepsilon'$$

as desired.

# References


[DP22] Sayan Das and Jesse Peterson. “Poisson boundaries of II1 factors”. In: *Compos. Math.* 158.8 (2022), pp. 1746–1776.

[Kai05] Vadim A. Kaimanovich. “Amenability and the Liouville property”. In: vol. 149. Probability in mathematics. 2005, pp. 45–85.

[SS24] Filippo Sarti and Alessio Savini. *Boundaries and equivariant maps for ergodic groupoids*. 2024.