# Fullness Notes

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**Lemma 0.0.1.** *Let $N$ be a tracial von Neumann algebra. Suppose that $P \subseteq p(M \otimes N)p$ is a von Neumann subalgebra such that*

- *$P$ is strongly nonamenable relative to $1 \otimes N$,*
- *$P_0 := P' \cap p(M \otimes N)p \prec_{M\otimes N} A \otimes N$ .*

*Assume there is some $s \in (0,1)$ and $\delta > 0$ such that*

$$\tau(b^*(\alpha_t \otimes id)(b)) \geq \delta \quad \textit{for all } b \in U(P_0).$$

*Let $Q_0 = \mathrm{q}\mathcal{N}_{p(M\otimes N)p}(P)''$. Then one of the following holds*

1. *$P_0 \prec 1 \otimes N$,*
2. *$Q_0 \prec A \otimes N$,*
3. *there is a partial isometry $v \in p(M \otimes N)p$ such that $vv^* \in Z(P_0)$ and*

$$v^* P_0 v \subseteq \{u_\sigma : \sigma \in [\mathcal{G}]\}'' \otimes N.$$

**Theorem 0.0.1.** *Let $N$ be a tracial von Neumann algebra. Suppose that $P \subseteq p(M \otimes N)p$ is a von Neumann subalgebra such that*

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- *$P_0 := P' \cap p(M \otimes N)p \prec_{M\otimes N} A \otimes N$ .*

*Then actually $P_0 \prec_{M\otimes N} 1 \otimes N$.*

*Proof.* We have

$$_{M\otimes N}L^2(\widetilde{M} \otimes N) \ominus L^2(M \otimes N)_{M\otimes N} \prec \ _{M\otimes N}L^2(M \otimes N) \otimes_{1\otimes N} L^2(M \otimes N)_{M\otimes N}$$

$$\forall z \in Z(P_0), \quad _{M\otimes N}L^2(M \otimes N)_{Pz} \not\prec \ _{M\otimes N}L^2(\widetilde{M} \otimes N) \ominus L^2(M \otimes N)_{M\otimes N}$$

Fix $\varepsilon > 0$. Then there is $F \subseteq U(P)$ finite and $\delta > 0$ such that, for $x \in (p(\widetilde{M} \otimes N)p)_1$,

$$\max_{u\in F} \|[u, x]\|_2 \leq \delta \implies \|x - \mathbb{E}_{M\otimes N}(x)\|_2 \leq \varepsilon.$$

Choose $s = 2^{-k}$ such that $\max_{u\in F} \|\alpha_s(u) - u\|_2 \leq \delta/2$. Then for $x \in P_0$, we have that $\|[\alpha_s(x), u]\|_2 \leq \delta$. Hence, for $x \in (P_0)_1$,

$$\|\alpha_{2s}(x) - x\|_2 \leq 2\|\alpha_s(x) - \mathbb{E}_{M\otimes N}(\alpha_s(x))\|_2 \leq 2\varepsilon. \tag{1}$$

Consequently,

$$\lim_{t\to 0}\sup_{x\in(P_0)_1}\|\alpha_t(x)-x\|_2=0,$$

so $P_0$ is $(\alpha_t\otimes id)$-rigid.

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Choose $\delta_0\in(0,1)$ and $t$ such that $\sup_{x\in(P_0)_1}\|(\alpha_t\otimes id)(x)-x\|_2<\sqrt{2(1-\delta_0)}$. Then we have that

$$|\tau(b^*(\alpha_t\otimes id)(b))|>\delta\quad\text{for } b\in U(P_0).$$

Suppose, for the sake of contradiction, that $P_0\not\prec 1\otimes N$. Then either part (2) or part (3) of the above lemma hold.

**(Case 1):** Suppose $Q_0\prec A\otimes N$. Then as $P\vee P_0\subseteq Q_0$, we have that $P,P_0\prec A\otimes N$.

By IPV section 2.4, as $A\otimes N$ is amenable relative to $1\otimes N$, there exists a non-zero projection $f\in P'\cap p(M\otimes N)p$ such that $Pf$ is amenable relative to $1\otimes N$. This contradicts the assumption that $P$ is strongly nonamenable relative to $1\otimes N$.

**(Case 2):** There is a partial isometry $v\in p(M\otimes N)p$ such that $vv^*\in Z(P_0)$ and

$$v^*P_0v\subseteq\{u_\sigma:\sigma\in[\mathcal{G}]\}''\otimes N.$$

NOTE: Maybe I can get this to be a unitary.

Let $S:=\{u_\sigma:\sigma\in[\mathcal{G}]\}''$. We need to build a contradiction from

- $P_0\prec A\otimes N$,
- $v^*P_0v\subseteq S\otimes N$.

We have assumed that $P_0\not\prec 1\otimes N$, so there is some sequence $(u_n)_n$ in $U(P_0)$ such that

$$\|\mathbb{E}_{1\otimes N}(xu_ny)\|_2\to 0\quad\text{for } x,y\in M\otimes N.$$

Then the intertwining into $A\otimes N$ implies that there is some $\varepsilon>0$ and $N\in\mathbb{N}$ such that

$$\|\mathbb{E}_{A\otimes N}(u_n)\|_2>\varepsilon\quad\text{for } n\geq N.$$

□

# 1 Weak Containment

## 1.1 Definitions

**Definition 1.1.1.** We say that $\pi$ is *weakly contained* in $\rho$, denoted $\pi\prec\rho$, if for any $\epsilon>0$, $\xi\in S(X*\mathcal{H})$, and $E\subset\mathcal{G}$ with $\mu_{\mathcal{G}}(E)<\infty$, there exists $\{\eta^1,\dots,\eta^m\}\subset S(X*\mathcal{K})$ with

$$\mu_{\mathcal{G}}\left(\{g\in E:|\langle\pi(g)\xi_{d(g)},\xi_{r(g)}\rangle-\sum_{i=1}^m\langle\rho(g)\eta^i_{d(g)},\eta^i_{r(g)}\rangle|\geq\epsilon\}\right)<\epsilon.$$

**Definition 1.1.2.** Given two $M$-$N$ bimodules $\mathcal{H}$ and $\mathcal{K}$, we say that $\mathcal{H}$ is *weakly contained* in $\mathcal{K}$ (denoted by $\mathcal{H} \prec \mathcal{K}$) if for every $\varepsilon > 0$ and finite subsets $E \subseteq M$, $F \subseteq N$, $\{\xi_1, \ldots, \xi_n\} \subseteq \mathcal{H}$, there exists $\{\eta_1, \ldots, \eta_n\} \subseteq \mathcal{K}$ such that

$$|\langle x\xi_i y, \xi_j\rangle - \langle x\eta_i y, \eta_j\rangle| < \varepsilon \quad \text{for all } x \in E,\ y \in F, \text{ and } 1 \leq i, j \leq n.$$

Note that any $M$-$N$ bimodule $\mathcal{H}$ gives rise to a $*$-homomorphism $\pi_{\mathcal{H}} : M \otimes N^{\mathrm{op}} \to B(\mathcal{H})$ given by $\pi_{\mathcal{H}}(x \otimes y)\xi = x\xi y$. In this language, we have that $\mathcal{H} \prec \mathcal{K}$ if and only if $\|\pi_H(T)\| \leq \|\pi_{\mathcal{K}}(T)\|$ for all $T \in M \otimes N^{\mathrm{op}}$.

**Definition 1.1.3.** Let $\pi_1, \pi_2$ be nondegenerate $*$-representations of a $C^*$-algebra $A$. We say that $\pi_1$ is *weakly contained* in $\pi_2$ and write $\pi_1 \prec \pi_2$ if $\|\pi_1(a)\| \leq \|\pi_2(a)\|$ for all $a \in A$.

Hence, ${}_M\mathcal{H}_N \prec {}_M\mathcal{K}_N$ if and only if $\pi_{\mathcal{H}} \prec \pi_{\mathcal{K}}$.

## 1.2 Ben Stuff, i.e. preliminary lemmas

For $\pi : A \to B(\mathcal{H})$ a $*$-representation of a $C^*$-algebra $A$ and $\xi \in \mathcal{H}$, define $\omega_\xi \in A^*$ by $\omega_\xi(a) = \langle \pi(a)\xi, \xi\rangle$.

**Theorem 1.2.1.** *Let $\pi_j : A \to B(\mathcal{H}_j)$ , $j = 1, 2$ be nondegenerate $*$-representations of a $C^*$-algebra $A$. Then the following are equivalent:*

1. $\pi_1 \prec \pi_2$,
2. $\ker(\pi_2) \subseteq \ker(\pi_1)$,
3. *For all $\xi \in \mathcal{H}_1$ with $\|\xi\| = 1$, we have*
$$\omega_\xi \in \overline{co}^{wk^*}\{\omega_\eta : \eta \in \mathcal{H}_2,\ \|\eta\| = 1\},$$
4. *For all $\xi \in \mathcal{H}_1$,*
$$\omega_\xi \in \overline{\mathrm{Span}}^{wk^*}\{\omega_\eta : \eta \in \mathcal{H}_2\}.$$

hello there

## 1.3 Main Argument

**Proposition 1.3.1.** *Consider two representations $\pi, \rho$ of $\mathcal{G}$ on bundles $\mathcal{H} = \{\mathcal{H}_x\}_{x\in X}$ and $\mathcal{K} = \{\mathcal{K}_x\}_{x\in X}$ respectively. Then $\pi \prec \rho$ implies that $B(\pi) \prec B(\rho)$.*

Fix $\varepsilon > 0$, $a \in A$, $\sigma \in [\mathcal{G}]$, $m \in (M)_1$, $\xi \in S_b(X * \mathcal{H})$, and $\varphi, \psi \in M$.

$$\begin{aligned}
\langle au_\sigma \cdot (\xi \otimes \widehat{\psi}) \cdot m, \xi \otimes \widehat{\psi}\rangle &= \langle \widetilde{\pi}(\sigma)\xi \otimes au_\sigma\widehat{\psi}m, \xi \otimes \widehat{\psi}\rangle \\
&= \langle \widetilde{\pi}(\sigma)\xi, \xi \cdot {}_A\langle au_\sigma\widehat{\psi}m, \widehat{\psi}\rangle\rangle \\
&= \langle \widetilde{\pi}(\sigma)\xi, \xi \cdot \mathbb{E}_A(au_\sigma\psi m\psi^*)\rangle \\
&= \int_X \mathbb{E}_A(au_\sigma\psi m\psi^*) \cdot \langle(\widetilde{\pi}(\sigma)\xi)_x, \xi_x\rangle \, d\mu(x)
\end{aligned}$$

We have $\overline{\mu}(\sigma) = 1 < +\infty$, so by assumption there exists $\{\eta^i\}_{i=1}^s \subseteq S_b(X * \mathcal{K})$ such that the set

$$A_\varepsilon^\sigma \coloneqq \{g \in \sigma \subseteq \mathcal{G} : |\langle \pi(g)\xi_{d(g)}, \xi_{r(g)}\rangle - \sum_i \langle \rho(g)\eta^i_{d(g)}, \eta^i_{r(g)}\rangle| \geq \varepsilon\}.$$

has measure $\overline{\mu}(A^\sigma_\varepsilon) < \varepsilon$. Taking the range of this set in the unit space $X$, we see

$$\begin{aligned}
r(A^\sigma_\varepsilon) &= \{r(g) \in X : g \in A^\sigma_\varepsilon \subseteq \sigma\} = \{x \in X : x\sigma \in A^\sigma_\varepsilon\} \\
&= \{x \in X : \left| \langle \pi(x\sigma)\xi_{d(x\sigma)}, \xi_x\rangle - \sum_i \langle \rho(x\sigma)\eta^i_{d(x\sigma)}, \eta^i_x\rangle \right| \geq \varepsilon\}
\end{aligned}$$

and compute that

$$\begin{aligned}
\overline{\mu}(A^\sigma_\varepsilon) &= \int_X \#\{g \in A^\sigma_\varepsilon : d(g) = x\}\, d\mu(x) \\
&= \int_X 1_{d(A^\sigma_\varepsilon)}(x)\, d\mu(x) = \mu(d(A^\sigma_\varepsilon)) = \mu(r(A^\sigma_\varepsilon)).
\end{aligned}$$

For brevity, write $f \coloneqq \mathbb{E}_A(au_\sigma\psi m\psi^*)$. Observe that

$$\begin{aligned}
&|\langle au_\sigma(\xi \otimes \widehat{\psi})m, \xi \otimes \widehat{\psi}\rangle - \sum_i \langle au_\sigma(\eta^i \otimes \widehat{\psi})m, \eta^i \otimes \widehat{\psi}\rangle| \\
&= \left| \int_X \mathbb{E}_A(au_\sigma\psi m\psi^*) \cdot \langle(\widetilde{\pi}(\sigma)\xi)_x, \xi_x\rangle\, d\mu(x) - \sum_{i=1}^s \int_X \mathbb{E}_A(au_\sigma\psi m\psi^*) \cdot \langle(\widetilde{\rho}(\sigma)\eta^i)_x, \eta^i_x\rangle\, d\mu(x) \right| \\
&\leq \int_X |f(x)| \cdot \left| \left\langle \pi(x\sigma)\xi_{d(x\sigma)},\, \xi_x \right\rangle - \sum_i \left\langle \rho(x\sigma)\eta^i_{d(x\sigma)},\, \eta^i_x \right\rangle \right| d\mu(x)
\end{aligned}$$

Let $C > 0$ such that

$$\|\xi_x\|, \left\|\eta^i_x\right\| \leq C \quad \text{for all } x \in X \text{ and } 1 \leq i \leq s.$$

By Cauchy-Schwartz,

$$\begin{aligned}
&\int_{r(A^\sigma_\varepsilon)} |f(x)| \cdot \left| \left\langle \pi(x\sigma)\xi_{d(x\sigma)},\, \xi_x \right\rangle - \sum_{i=1}^m \left\langle \rho(x\sigma)\eta^i_{d(x\sigma)},\, \eta^i_x \right\rangle \right| d\mu(x) \\
&\qquad \leq \|f\|_\infty \left[ \int_{r(A^\sigma_\varepsilon)} \left|\left\langle \pi(x\sigma)\xi_{d(x\sigma)},\, \xi_x \right\rangle\right| + \sum_{i=1}^m \int_{r(A^\sigma_\varepsilon)} \left|\left\langle \rho(x\sigma)\eta^i_{d(x\sigma)},\, \eta^i_x \right\rangle\right| d\mu(x) \right] \\
&\qquad \leq C^2\varepsilon(s+1)\|f\|_\infty
\end{aligned}$$

Substituting this bound back into BLANK, we compute

$$\begin{aligned}
&\int_X |\varphi(x)| \cdot \left| \left\langle \pi(x\sigma)\xi_{d(x\sigma)},\, \xi_x \right\rangle - \sum_i \left\langle \rho(x\sigma)\eta^i_{d(x\sigma)},\, \eta^i_x \right\rangle \right| d\mu(x) \\
&\qquad < C^2\varepsilon(s+1)\|f\|_\infty + \int_{X \setminus r(A^\sigma_\varepsilon)} |f(x)| \cdot \left| \left\langle \pi(x\sigma)\xi_{d(x\sigma)},\, \xi_x \right\rangle - \sum_i \left\langle \rho(x\sigma)\eta^i_{d(x\sigma)},\, \eta^i_x \right\rangle \right| d\mu(x) \\
&\qquad < C^2\varepsilon(s+1)\|f\|_\infty + \|f\|_\infty \int_{X \setminus r(A^\sigma_\varepsilon)} \varepsilon\, d\mu(x) \leq \varepsilon \cdot (C^2(s+1)\|f\|_\infty + 1)
\end{aligned}$$

# 2 Mixing Proof

**Definition 2.0.1.** A representation $\pi$ on $X * \mathcal{H}$ is called *mixing* or $c_0$ if for every $\epsilon, \delta > 0$ and every pair of normalized sections $\xi, \eta \in S(X * \mathcal{H})$, there is $E \subset X$ with $\mu(X \setminus E) < \delta$ such that

$$\left|\{g \in (\mathcal{G}|_E)_x : |\langle \pi(g)\xi_x, \eta_{r(g)}\rangle| > \epsilon\}\right| < \infty \quad \text{for } \mu\text{-a.e. } x \in E.$$

**Definition 2.0.2.** An $M$-$M$ bimodule ${}_M\mathcal{H}_M$ is *mixing relative to* $A \subseteq M$ if any net $(u_i)_{i\in I}$ in $(M)_1$ which satisfies $\|\mathbb{E}_A(xu_ny)\|_2 \to 0$ for all $x, y \in M$, satisfies

$$\lim_i \sup_{y\in(M)_1} |\langle u_n\xi y, \eta\rangle| = \lim_i \sup_{y\in(M)_1} |\langle y\xi u_n, \eta\rangle| = 0 \quad \text{for all } \xi, \eta \in \mathcal{H}$$

**Proposition 2.0.1.** *If $\pi$ is mixing then ${}_M\mathcal{B}(\pi)_M$ is mixing relative to $A$.*

*Proof.* Fix $\xi \in S_1(X * \mathcal{H})$ and $\varepsilon > 0$. The map $\varphi : \mathcal{G} \to \mathbb{C}$ given by $\varphi(g) = \langle \pi(g)\xi_{d(g)}, \xi_{r(g)}\rangle$ is then a unital positive definite function on $\mathcal{G}$ (see KIDA, CLAIRE, JOLISSAINT etc). By LUPINI, CLAIRE, there is a unique unital, completely positive, $\mathbb{E}_A$-preserving, $A$-$A$ bimodular map $\Phi : M \to M$ such that

$$\Phi(u_\sigma) = \varphi(\cdot\,\sigma)u_\sigma \quad \text{for all } \sigma \in [\mathcal{G}].$$

Such a map extends to a unique contraction $\widehat{\Phi} : L^2(M) \to L^2(M)$; moreover, the expectation-preservation and $A$-$A$-bimodularity imply that $\widehat{\Phi} \in \langle M, A\rangle \cap A'$.

Appealing to the mixingness of $\pi$, for each $k \in \mathbb{N}$ there is some measureable $E_k \subseteq X$ with $\mu(X \setminus E_k) < 2^{-k}$ such that

$$\mu_{\mathcal{G}}\left(\{g \in \mathcal{G}|_{E_k} : |\varphi(g)| > \varepsilon\}\right) < +\infty.$$

Let $F_1 = E_1$ and $F_{k+1} = E_{k+1} \setminus \bigsqcup_{i=1}^k F_k$, so $1 = \mu\left(\bigcup_{k=1}^\infty E_k\right) = \mu\left(\bigsqcup_{k=1}^\infty F_k\right)$. Consider the projections $p_i = 1_{F_i}$, $q_i = 1_{E_i}$ in $A$. These satisfy the following:

- $q_k = \sum_{i=1}^k p_i$,
- $\tau(1 - q_k) \xrightarrow{k\to\infty} 0$,
- $\tau(p_k) < 2^{-k}$

Consider the map $\Phi_k : q_kMq_k \to q_kMq_k$ given by $\Phi_k(\cdot) = \Phi(q_k \cdot q_k)$. By CLAIRE, we have that $\widehat{\Phi}_k \in \mathcal{K}(\langle q_kMq_k, q_kAq_k\rangle)$.

Suppose that $(u_n)_{n=1}^\infty$ is a sequence in $(M)_1$ such that $\|\mathbb{E}_A(m_1u_nm_2)\|_2 \to 0$ for all $m_1, m_2 \in M$.

Fix $k \in \mathbb{N}$ such that $\|1 - q_k\|_2 < \varepsilon/6$. Then, for all $n \in \mathbb{N}$, we have

$$\begin{aligned}\|u_n\|_2 &= \|(1-q_k)u_n(1-q_k)\|_2 + \|(1-q_k)u_nq_k\|_2 + \|q_ku_n(1-q_k)\|_2 + \|q_ku_nq_k\|_2 \\ &\le 3\|1-q_k\|_2 + \|q_ku_nq_k\|_2 < \frac{\varepsilon}{2} + \|q_ku_nq_k\|_2.\end{aligned}$$

Note that this choice of $k$ is independent of $n \in \mathbb{N}$, so the above is essentially a uniform integrability estimate.

As the sequence $(q_ku_nq_k)_{n=1}^\infty$ in $(q_kMq_k)_1$ satisfies the hypotheses of POPA 1.3.3.5, the compactness of $\widehat{\Phi}_k$ gives that

$$\|\Phi_k(q_ku_nq_k)\|_2 \xrightarrow{n\to\infty} 0.$$

Choose $N \in \mathbb{N}$ such that $\|\Phi_k(q_ku_nq_k)\|_2 < \varepsilon/2$ for all $n \ge N$. Then we have

$$\|\Phi(u_n)\|_2 \le \frac{\varepsilon}{2} + \|\Phi_k(q_ku_nq_k)\|_2 < \varepsilon$$

for all $n \ge N$, so $\|\Phi(u_n)\|_2 \to 0$.

Now fix a countable subset $\Gamma \subseteq [\mathcal{G}]$ which generates $\mathcal{G}$ and write $u_n = \sum_{\sigma\in\Gamma} a_\sigma^n u_\sigma$. For any $m, y \in (M)_1$, observe that

$$\begin{aligned}
\langle u_n \cdot (\xi \otimes \widehat{m}) \cdot y, \xi \otimes \widehat{m}\rangle &= \sum_{\sigma\in\Gamma} \langle a_\sigma^n (\xi \otimes \widehat{m}) \cdot y, \xi \otimes \widehat{m}\rangle \\
&= \sum_{\sigma\in\Gamma} \int_X \mathbb{E}_A(a_\sigma^n u_\sigma m y m^*) \langle \pi(x\sigma)\xi_{d(x\sigma)}, \xi_x\rangle \, d\mu(x) \\
&= \int_X \mathbb{E}_A \left( \sum_{\sigma\in\Gamma} \langle \pi(x\sigma)\xi_{d(x\sigma)}, \xi_x\rangle a_\sigma^n u_\sigma m y m^* \right) d\mu(x) \\
&= \int_X \mathbb{E}_A(\Phi(u_n) m y m^*) \, d\mu(x) = \langle \Phi(u_n), m y^* m^*\rangle
\end{aligned}$$

whence

$$\sup_{y\in(M)_1} |\langle u_n \cdot (\xi \otimes \widehat{m}) \cdot y, \xi \otimes \widehat{y}\rangle| \leq \|\Phi(u_n)\|_2 \xrightarrow{n\to\infty} 0.$$

We may then upgrade this result to any vectors $v, w \in \mathcal{B}(\pi)$ by applying the technique in TBOOK applied to the set

$$V = \{v \in \mathcal{B}(\pi) : \lim_{n\to\infty} \sup_{y\in(M)_1} |\langle u_n v y, v\rangle| = 0 \text{ for any } (u_n)_{n=1}^\infty \text{ with } \|\mathbb{E}_A(m_1 u_n m_2)\|_2 \to 0 \text{ for all } m_1, m_2 \in M\}.$$

One shows that this set is closed under scalar multiplication, the action of $M \odot M^{\text{op}}$, addition, and is norm closed, whence it follows that $V = \mathcal{B}(\pi)$. Lastly, one uses polarization to upgrade to pairs of vectors.

□

# 3 Weak Containment: Ben Fix

**Proposition 3.0.1.** *Suppose $\pi$, $\rho$ on $\{\mathcal{H}_x\}_{x\in X}$, $\{\mathcal{K}_x\}_{x\in X}$ respectively. Set $D = \mathrm{Span}\{au_\sigma : a \in A, \sigma \in [\mathcal{G}]\}$. Suppose that, for any $\xi \in \int_X^\oplus \mathcal{H}_x \, d\mu(x)$ and $\psi \in M$, we have*

*there exists $C_\xi > 0$ such that for all $\delta > 0$, $F \subseteq D$, and $E \subseteq M$ finite subsets, there are $\eta_1, \ldots, \eta_l \in \int_X^\oplus \mathcal{K}_x \, d\mu(x)$ such that $\sum_{i=1}^l \langle \eta_i, \eta_i\rangle \leq C_\xi$ and*

$$\left| \omega_{\xi\otimes\widehat{\psi}}(x' \otimes y) - \sum_{i=1}^l \omega_{\eta_i\otimes\widehat{\psi}}(x' \otimes y) \right| < \delta \quad \text{for all } x' \in F,\, y \in E.$$

*Then, when considered as an element of $(M \otimes_{max} M^{\text{op}})^*$, we have that*

$$\omega_{\xi\otimes\widehat{\psi}} \in \overline{co}^{wk^*} \left\{ \omega_{\eta\otimes\widehat{\psi}} : \eta \in \int_X^\oplus \mathcal{K}_x \, d\mu(x) \, , \, \|\eta\|_2 \leq C_\xi \right\}.$$