

Intro Math Research Hw2

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1 Reading Comments

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2 Symmetric Polynomials/Functions Exposition

Preliminary Considerations

Throughout this article, fix a (unital) commutative ring R and a field k . For simplicity, we work over vector spaces instead of general modules.

Notation. Let X be a set such that $X = \{x_i\}_{i \in I}$ for some indexing set I . By $k[X]$ and $k[[X]]$, we denote the rings of (commutative) polynomials and power series (resp.) in indeterminates $\{x_i\}$. We utilize multi-index notation throughout. Hence for $\alpha : I \rightarrow \mathbb{N} \cup \{0\}$ finitely supported, we write $x_\alpha = \prod_{i \in I} x_i^{\alpha_i}$ (where $x_i^0 := 1$ formally).

2.1 Algebraic Background

Many common algebraic objects possess both a vector space structure and an internal product structure. For example, the set of $n \times n$ matrices over k , denoted $M_n(k)$, has a product given by matrix multiplication and is also a vector space under addition and scalar multiplication by k .

Definition 2.1.1. Let A be a k -vector space equipped with a map $\cdot : A \times A \rightarrow A$ (written $(x, y) \mapsto x \cdot y$). The pair (A, \cdot) is a k -algebra if, for $x, y, z \in A$ and $a, b \in k$, the following hold:

- $(x + y) \cdot z = x \cdot z + y \cdot z$,
- $z \cdot (x + y) = z \cdot x + z \cdot y$,
- $(ax) \cdot (by) = (ab)(x \cdot y)$.

Key Example. For $X = \{x_i\}_{i \in I}$, the rings $k[X]$ and $k[[X]]$ form the prototypical example of k -algebras.

Often in algebra, elements of a given object may be decomposed into a sum of simpler elements which are, in a sense, “homogenous.” For example, any polynomial in n -variable may be decomposed into a sum of simpler polynomials each of which are further sums of monomials of the same total degree. In this way, a polynomial is split into a sum of homogenous parts. This behavior is codified in the notion of *grading*.

Definition 2.1.2. A *graded k -algebra* is a k -algebra A together with a direct sum decomposition

$$A = \bigoplus_{i=0}^{\infty} A_i$$

with A_0, A_1, \dots vector spaces such that $A_i \cdot A_j \subseteq A_{i+j}$ for all $i, j \in \mathbb{N} \cup \{0\}$. For fixed i , elements of A_i are called *homogenous*. The choice of such a direct sum decomposition is a *grading* for A .

Key Example. As before, for $X = \{x_i\}_{i \in I}$, we may give the ring $k[X]$ a canonical grading by declaring $A_0 := k$ and

$$A_n := \text{Span}_k \{x_\alpha : \alpha \text{ multi-index such that } \sum_{i \in I} \alpha_i = n\}.$$

The reader is cautioned that not every k -algebra has a nontrivial grading. In fact, it can be shown that the ring of formal power series $k[[x]]$ does not have a nontrivial grading.

2.2 Symmetric Polynomials

Definition 2.2.1. The permutation group S_n acts naturally on the polynomial ring $k[x_1, \dots, x_n]$ by defining $\sigma \cdot x_1^{\alpha_1} \cdots x_n^{\alpha_n} := x_{\sigma(i_1)}^{\alpha_1} \cdots x_{\sigma(i_n)}^{\alpha_n}$ and extending by linearity. The ring of *symmetric polynomials* in n indeterminates is the fixed points of this action, namely $k[x_1, \dots, x_n]^{S_n}$.

2.3 Compositions and Partitions

Definition 2.3.1.

- A *partition* of $n \in \mathbb{N}$ is a set $\alpha = \{\alpha_1, \dots, \alpha_l\}$ of positive integers which sum to n . We denote the set of partitions of n by $\text{Par}(n)$. We denote the statement $[\lambda \in \text{Par}(n)]$ by $\lambda \vdash n$. Also, we write $\text{Par} := \bigcup_{n \geq 0} \text{Par}(n)$.
- A *weak composition* of $n \in \mathbb{N}$ is a (finitely supported) sequence $\alpha = (\alpha_i)_{i=1}^\infty \in (\mathbb{N} \cup \{0\})^\mathbb{N}$ such that $\sum_i \alpha_i = n$. The length of a weak composition α is given by

$$l(\alpha) := \max\{i \in \mathbb{N} : \alpha_i \neq 0\}.$$

2.4 Symmetric Functions

Definition 2.4.1 (pg. 308 in [Sta24]). The ring Λ_k of symmetric functions over a field k is the subring of all $f \in k[[x_1, x_2, \dots]]$ such that

$$f(x_{\sigma(1)}, x_{\sigma(2)}, \dots) = f(x_1, x_2, \dots) \text{ for all } \sigma \in \text{Sym}(\mathbb{N}).$$

Remark 2.4.1. For the algebraically-minded, there is a more natural construction of Λ_k by viewing the ring as the colimit of a certain directed system of inclusions of symmetric polynomial rings

$$k[x_1, \dots, x_n]^{S_n} \xrightarrow{\varphi_n} k[x_1, \dots, x_{n+1}]^{S_{n+1}}.$$

The construction of these inclusions φ_n is somewhat involved. This does justify the intuition that a symmetric function is simply taking a symmetric polynomial and adding more data, as any element of a direct limit of inclusions is faithfully represented by an element of one of the constituent objects.

Definition 2.4.2. A symmetric function $f \in \Lambda_k$ is homogenous of degree n if

$$f(x) = \sum_{\alpha \text{ weak composition of } n} c_\alpha x^\alpha,$$

where the c_α are elements of k . The set of degree n homogenous symmetric functions is denoted Λ_k^n . these subspaces give Λ_k the structure of a graded k -algebra, namely:

- Each Λ_k^n is a k -vector space,
- $\Lambda_k^i \Lambda_k^j \subseteq \Lambda_k^{i+j}$,
- $\Lambda_k = \bigoplus_{n=0}^\infty \Lambda_k^n$ as k -vector spaces.

The first interesting basis of Λ_k is the *monomial symmetric functions*. Given $\lambda \vdash n$, define $m_\lambda \in \Lambda_k^n$ by

$$m_\lambda := \sum_{\alpha} x^\alpha$$

where the sum is over all distinct permutations of the entries of λ . The set $\{m_\lambda : \lambda \vdash n\}$ forms a basis for Λ_k^n , whence $\bigcup_{n \geq 0} \{m_\lambda : \lambda \vdash n\} = \{m_\lambda : \lambda \in \text{Par}\}$ forms a basis for Λ_k .

2.5 Complete Homogenous Symmetric Functions

From the monomial symmetric functions, we may form another interesting basis for Λ_k called the *complete homogenous symmetric functions* h_λ by setting

$$h_\lambda := \prod_{i=1}^\infty \sum_{\nu \vdash \lambda_i} m_\nu.$$

where $\lambda = (\lambda_1, \lambda_2, \dots)$. Again, the set $\{h_\lambda : \lambda \vdash n\}$ is a basis for Λ_k^n and the set $\{h_\lambda : \lambda \in \text{Par}\}$ is a basis for Λ_k .