MATH 7410 Homework 4

James Harbour

January 4, 2023

Problem 1

Let X be a normed space and x_n a sequence in X such that $x_n \to x$ weakly. Show that there is a sequence y_n such that $y_n \in co\{x_1, \ldots, x_n\}$ and $\|y_n - x\| \to 0$.

Proof. Let $C_n = co\{x_1, \ldots, x_n\}$ and $C = \bigcup_{n=1}^{\infty} C_n$. Note that $C = co\{x_i : i \in \mathbb{N}\}$ is convex. Thus $x \in \overline{C}^{wk} = \overline{C}^{\|\cdot\|}$, whence there is some sequence $(z_n)_{n=1}^{\infty}$ in C such that $\|z_n - x\| \to 0$. Let $k_n \in \mathbb{N}$ be such that $z_n \in C_{k_n}$.

Note that a sequence a_n in X converges to 0 in norm if and only if for every $\varepsilon > 0$, the set $\{n \in \mathbb{N} : \|a_n\| \geq \varepsilon\}$ is finite. This condition is invariant under rearrangements, so without loss of generality we may take the sequence $(k_n)_{n\in\mathbb{N}}$ to be nondecreasing. Construct a new sequence $(y_m)_{m\in\mathbb{N}}$ as follows. For $m < k_1$, set $y_m = 0$. For $n \in \mathbb{N}$ and $k_n \leq m < k_{n+1}$, set $y_m = z_n$.

The sequence $y_m - x$ still has the condition that for all $\varepsilon > 0$ the set of indices whose corresponding elements have norm at least ε is finite, as we have only added a finite number of elements to this set. Thus $y_n \in C_n$ for all $n \in \mathbb{N}$ and $\|y_n - x\| \to 0$. □

Problem 2

If $\mathcal{H}$ is a Hilbert space and h_n is a sequence in $\mathcal{H}$ such that $h_n \to h$ weakly and $\|h_n\| \to \|h\|$, show that $\|h_n - h\| \to 0$.

Proof. By weak convergence, we have that $\langle h_n, h\rangle \to \langle h, h\rangle = \|h\|^2$, whence

$$\|h_n - h\|^2 = \|h_n\|^2 + \|h\|^2 - 2\operatorname{Re}(\langle h_n, h\rangle) \xrightarrow{n\to\infty} \|h\|^2 + \|h\|^2 - 2\|h\|^2 = 0$$

□

Problem 3

If X, Y are Banach spaces and $B \in B(Y^*, X^*)$, then $B = A^*$ for some $A \in B(X, Y)$ if and only if B is wk^*-continuous.

Proof.
$(\implies)$: Suppose that $B = A^*$ for some $A \in B(X, Y)$ and let $(\psi_\alpha)_{\alpha\in I}$ be a net in Y^* such that $\psi_\alpha \to \psi \in Y^*$ weak*. Fix $x \in X$. Then

$$B(\psi_\alpha)(x) = A^*(\psi_\alpha)(x) = \psi_\alpha(Ax) \xrightarrow{\alpha\in I} \psi(Ax) = B(\psi)(x),$$

so $B(\psi_\alpha) \to B(\psi)$ weak*, i.e. B is weak*-continuous.

($\Longleftarrow$): Suppose that B is wk^*-continuous. Let ι_X, ι_Y be the canonical injections into the corresponding double-duals. For shorthand, we may write $\hat{x} := \iota_X(x)$ and similarly for Y. Noting that $B^* \in B(X^{**}, Y^{**})$ we investigate what occurs when B^* is restricted to the image of X inside its double dual.

Fix $x \in X$. We claim that $B^*(\hat{x}) \in (Y^*, wk^*)^*$. To this end, let $(\phi_\alpha)_{\alpha \in I}$ be net in Y^* such that $\phi_\alpha \to \phi \in Y^*$ weak*. Then,

$$B^*(\hat{x})(\phi_\alpha) = \hat{x}(B(\phi_\alpha)) = B(\phi_\alpha)(x) \to B(\phi)(x) = B^*(\hat{x})(\phi),$$

so $B^*(\hat{x})$ is weak*-continuous, whence there exists some $y \in Y$ such that $B^*(\hat{x}) = \hat{y}$. Define $A : X \to Y$ by $Ax = y$.

- (Uniqueness of y): Suppose that $y_0 \in Y$ also has that $\widehat{y_0} = B^*(\widehat{x}) = \widehat{y}$. Then by injectivity of ι_Y, it follows that $y_0 = y$.
- (A is a bounded operator): Suppose that $\|x\| \leq 1$. Then

$$\|Ax\| = \|y\| = \|\widehat{y}\| = \|B^*(\widehat{x})\| \leq \|B^*\| = \|B\| < +\infty.$$

- ($B = A^*$): Let $\phi \in Y^*$, $x \in X$. Then we compute

$$A^*(\phi)(x) = \phi(Ax) = \widehat{Ax}(\phi) = B^*(\widehat{x})(\phi) = \widehat{x}(B(\phi)) = B(\phi)(x).$$

□

Problem 4

Let X, Y be Banach spaces over $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$. For $C \subseteq B(X,Y)$ convex and $F \subseteq X$ finite, set $C_F = \{(Tx)_{x \in F} : T \in C\} \subseteq Y^{\oplus F}$. Equip $Y^{\oplus F}$ with the norm

$$\|(y_x)_{x \in F}\| = \sum_{x \in F} \|y_x\|.$$

(a): Let $C \subseteq B(X,Y)$ be convex. Show that $T \in \overline{C}^{SOT}$ if and only if for every $F \subseteq X$ finite, we have that $(Tx)_{x \in F} \in \overline{C_F}^{\|\cdot\|}$. Show that $T \in \overline{C}^{WOT}$ if and only if for every $F \subseteq X$ finite, we have that $(Tx)_{x \in F} \in \overline{C_F}^{weak}$.

Proof.
$\Longrightarrow$: Suppose $T \in \overline{C}^{SOT}$, so there is some net $(T_\alpha)_{\alpha \in I}$ in C such that $T_\alpha \to T$ SOT. Let $F \subseteq X$ finite. Then $\|T_\alpha x - Tx\| \to 0$ for every $x \in F$. Since F is finite,

$$\|(T_\alpha x)_{x \in F} - (Tx)_{x \in F}\| = \sum_{x \in F} \|T_\alpha x - Tx\| \xrightarrow{\alpha \in I} 0$$

so $T \in \overline{C}^{\|\cdot\|}$.

$\Longleftarrow$: By assumption, for all $\varepsilon > 0$ and finite $F \subseteq X$, there exists some $T^{F,\varepsilon} \in C$ such that $\sum_{x \in F} \left\|Tx - T^{F,\varepsilon}x\right\| < \varepsilon$. Define an ordering on $P(X)_{fin} \times (0, +\infty)$ by

$$(F, \varepsilon) \leq (F', \varepsilon') \iff F \subseteq F' \text{ and } \varepsilon' \leq \varepsilon.$$

Note that this defines a directed set. Fix $x \in X$ and consider the net $(T^{F,\varepsilon}x)_{(F,\varepsilon)}$ in Y. Fix $\varepsilon > 0$. Then for all $(F,\delta) \geq (\{x\},\varepsilon)$, it follows that

$$\left\|Tx - T^{F,\delta}x\right\| < \delta \leq \varepsilon.$$

Thus the net $T^{F,\varepsilon}x \to Tx$, so by definition $T \in \overline{C}^{SOT}$.

$\underline{\Longrightarrow}$: Suppose that $T \in \overline{C}^{WOT}$. Then there is some net $(T_\alpha)_{\alpha\in I}$ in C such that $T_\alpha \to T$ WOT. Let $F \subseteq X$ finite. Then $|\phi(T_\alpha x) - \phi(Tx)| \to 0$ for every $x \in F$ and $\phi \in Y^*$. Suppose $\psi = \sum_{x\in F} \phi_x \in (Y^{\oplus F})^*$ where $\phi_x \in Y^*$. Then

$$|\psi((T_\alpha x)_{x\in F}) - \psi((Tx)_{x\in F})| \leq \sum_{x\in F} |\phi_x(T_\alpha x) - \phi_x(Tx)| \xrightarrow{\alpha\in I} 0$$

so $T \in \overline{C_F}^{weak}$.

$\underline{\Longleftarrow}$: Define an ordering on $P(X)_{fin} \times \mathscr{T}_{weak}$ by

$$(F,U) \leq (F',U') \iff F \subseteq F' \text{ and } U' \subseteq U \times B(X,Y)^{\oplus F'\setminus F}.$$

Then for all $F \subseteq X$ finite and U weak-neighborhood of $(Tx)_{x\in F}$, there is some $T^{F,U} \in C$ such that $(T^{F,U}x)_{x\in F} \in U$. This gives a net. Now, letting V be an SOT neighborhood of T, there is some finite set F and weakly open U such that $T^{F',U'} \in V$ for all $(F',U') \geq (F,U)$.

□

(b): Suppose that $C \subseteq B(X,Y)$ is convex. Show that $\overline{C}^{WOT} = \overline{C}^{SOT}$.

Proof. Fix $F \subseteq X$ finite. Since C is convex, it follows that C_F is convex, whence $C_F^{\|\cdot\|} = C_F^{weak}$. Thus by part (a), the result follows. □

(c): If $\phi : B(X,Y) \to \mathbb{F}$ is linear, show that ϕ is WOT-continuous if and only if it is SOT-continuous.

Proof. Note that $\ker(\phi)$ is convex by linearity. By part (b), we have the following equivalences

$$\phi \text{ WOT-continuous} \iff \ker(\phi) \text{ WOT-closed} \iff \ker(\phi) \text{ SOT-closed} \iff \phi \text{ SOT-continuous}.$$

□

Problem 5

Let I be a set and $\mathcal{M}$ be the set of all $m \in l^\infty(I)^*$ such that :

- $m(f) \geq 0$ for all $f \geq 0$,
- $m(1) = 1$.

Identify $\text{Prob}(I)$ with $\{f \in l^1(I) : f \geq 0, \|f\|_1 = 1\}$ and view $l^1(I) \subseteq l^\infty(I)^*$ by $f \mapsto \phi_f$ where $\phi_f(g) = \sum_{i\in I} f(i)g(i)$. Show that $\text{Prob}(I)$ is weak*-dense in $\mathcal{M}$.

Proof. Suppose, for the sake of contradiction, that there is some $m \in \mathcal{M} \setminus \overline{\text{Prob}(I)}^{wk^*}$.

By separating Hahn-Banach, there is some wk^*-continuous linear functional $L : l^\infty(I)^* \to \mathbb{F}$ and $\alpha < \beta$ such that for all $\mu \in \text{Prob}(I)$

$$\text{Re}(L(\mu)) \leq \alpha < \beta \leq \text{Re}(L(m)).$$

As $(l^\infty(I)^*, wk^*)^* = l^\infty(I)$, there is some $g \in l^\infty(I)$ such that $L = ev_g$. Since $m \geq 0$ and linear, for any $f \in l^\infty(I)$, by writing $\mathrm{Re}(f)$ as a difference of positive functions we see that $m(\mathrm{Re}(f)) = \mathrm{Re}(m(f))$.

For $i \in I$, note that

$$L(\delta_i) = L(\phi_{\delta_i}) = \sum_{j \in I} g(j)\delta_i(j) = g(i),$$

so it folows that $\mathrm{Re}(g(i)) = \mathrm{Re}(L(\delta_i)) \leq \alpha$ pointwise. On the other hand, by positivity of m,

$$\beta \leq \mathrm{Re}(L(m)) = \mathrm{Re}(m(g)) = m(\mathrm{Re}(g)) \leq \alpha m(1) = \alpha,$$

which is absurd. □

Problem 6

Let X be a compact, Hausdorff space and let μ be a Borel probability measure on X.

(a): Show that $C(X)$ is wk^*-dense in $L^\infty(X, \mu)$.

Proof. Let $\phi : L^\infty(X, \mu) \to \mathbb{C}$ be weak-star continuous with $\|\phi\| = 1$ and $\phi|_{C(X)} = 0$. Note that $\mu \in C(X)^* = M(X)$. Moreover, since μ is a probability measure, $L^\infty(X, \mu) \hookrightarrow M(X)$. □

(b): Show that $\{f \in C(X) : 0 \leq f \leq 1\}$ is wk^*-dense in $\{f \in L^\infty(X, \mu) : 0 \leq f \leq 1 \text{ almost everywhere}\}$.