Intro Math Research Hw3

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1 Reading Comments

QUOMODOCUMQUE Article

- I quite like the point about avoiding undermining your own work. Most advice I have heard for math talks have been about humility and being very specific about how much attention you put on your own work, but this warns against swinging the pendulum too far in the other direction.
- I entirely agree with the statement about timing; I pretty much unconsciously stop listening to a talk the moment the speaker goes over time.

Tao Article

- Tao's blogposts often have really poignant quotes, this one is no exception.
- I like that he expresses the importance of always giving a sense of global locality within a talk/paper, as this is often the main challenge for people who can already locally verify details.
- It's very interesting the point that one should sit on a paper for a while before releasing to, say, the arXiv.

Vakil Article

Note: I very much already admire Vakil as an expositor for his *Rising Sea*. It's where I first learned category theory ages ago and I continue to reference it whenever I need to think about schemes.

- When I initially began attending seminars, I found myself taking notes in a more "traditional sense" as Vakil puts it. I definitely agree with him now that this sapped from my focus on the talk and nowadays I take much less notes and only write key ideas.
- I find it quite interesting the idea of, after having written three things to take away down, cutting one of them to make room for another instead of simply adding to the list. This is quite cool as it forces you to distill the essence of the talk.

Contents

2 Project Topics

Assignment: Look over the Project Topics document and start searching for readable resources. List the two topics which look most interesting to you and give two references for each of these topics (different from the one I gave). Keep in mind the remark of § 2.2

Topic 1: Schur-Weyl Duality and its Analogues

- In [FH91, Ch.4, 6], Fulton and Harris provide a gentle introduction to Schur-Weyl duality and the various identities which arise as a consequence.
- In [DDH08], Dipper, Doty, and Hu obtain an analogue of Schur-Weyl duality for symplectic groups using the Brauer algebra.

Topic 2: Quasisymmetric Functions

- The book *An Introduction to Quasisymmetric Schur Functions* [LMW13] by Luoto, Mykytiuk and van Willigenburg gives a broad overview of the machinery behind quasisymmetric functions (namely Hopf algebras) as well as a description of the basis of quasisymmetric Schur functions for the quasisymmetric function algebra.
- In [Sta95], Stanley contstructs a symmetric function from a finite graph. This construction encodes the coloring data in a way that generalizes the graph chromatic polynomial. Nearly two decades later, Shareshian and Wachs in [SW12] develop a quasisymmetric version of Stanley's chromatic function and establish connections to cohomological invariants coming from algebraic geometry. Three years later, Athanasiadis produced a wonderful elementary account ([Ath15]) of the development of both the symmetric and quasisymmetric chromatic function .

3 Symmetric Polynomials/Functions Exposition

Preliminary Considerations

Throughout this article, fix a (unital) commutative ring R and a field k. For simplicity, we work over vector spaces instead of general modules.

Notation. Let X be a set such that $X = \{x_i\}_{i \in I}$ for some indexing set I. By $k[X]$ and $k[\![X]\!]$, we denote the rings of (commutative) polynomials and power series (resp.) in indeterminates $\{x_i\}$. We utilize multi-index notation throughout. Hence for $\alpha_{\bullet} : I \to \mathbb{N} \cup \{0\}$ finitely supported, we write $x_\alpha = \Pi_{i \in I} x_i^{\alpha_i}$ (where $x_i^0 := 1$ formally).

3.1 Algebraic Background

Often in algebra, elements of a given object may be decomposed into a sum of simpler elements which are, in a sense, "homogenous." For example, any polynomial in n-variable may be decomposed into a sum of simpler polynomials each of which are futher sums of monomials of the same total degree. In this way, a polynomial is split into a sum of homogenous parts. This behavior is codified in the notion of *grading*.

Definition 3.1.1. A *graded k-algebra* is a k-algebra A together with a direct sum decomposition

$$A = \bigoplus_{i=0}^{\infty} A_i$$

with $A_0, A_1, \ldots$ vector spaces such that $A_i \cdot A_j \subseteq A_{i+j}$ for all $i, j \in \mathbb{N} \cup \{0\}$. For fixed i, elements of A_i are called *homogenous*. The choice of such a direct sum decomposition is a *grading* for A.

Key Example. As before, for $X = \{x_i\}_{i \in I}$, we may give the ring $k[X]$ a canonical grading by declaring $A_0 := k$ and

$$A_n := \mathrm{Span}_k\{x_\alpha : \alpha \text{ multi-index such that } \sum_{i \in I} \alpha_i = n\}.$$

The reader is cautioned that not every k-algebra has a nontrivial grading. In fact, it can be shown that the ring of formal power series $k[\![x]\!]$ does not have a nontrivial grading.

3.2 Symmetric Polynomials

Definition 3.2.1. The permutation group S_n acts naturally on the polynomial ring $k[x_1, \ldots, x_n]$ by defining $\sigma \cdot x_{i_1}^{\alpha_1} \cdots x_{i_l}^{\alpha_l} := x_{\sigma(i_1)}^{\alpha_1} \cdots x_{\sigma(i_l)}^{\alpha_l}$ and extending by linearity. The ring of *symmetric polynomials* in n indeterminates is the fixed points of this action, namely $k[x_1, \ldots, x_n]^{S_n}$.

3.3 Partitions and Compositions

Definition 3.3.1.

- A *partition* of $n \in \mathbb{N}$ is a finite sequence $\alpha = (\alpha_1, \ldots, \alpha_l)$ of weakly decreasing positive integers which sum to n. We denote the set of partitions of n by $\mathrm{Par}(n)$. We denote the statement $[\lambda \in \mathrm{Par}(n)]$ by $\lambda \vdash n$. Also, we write $\mathrm{Par} := \bigcup_{n \geq 0} \mathrm{Par}(n)$.
- A *weak composition* of $n \in \mathbb{N}$ is a (finitely supported) sequence $\alpha = (\alpha_i)_{i=1}^{\infty} \in (\mathbb{N} \cup \{0\})^{\mathbb{N}}$ such that $\sum_i \alpha_i = n$. The length of a weak composition α is given by

$$l(\alpha) := \max\{i \in \mathbb{N} : \alpha_i \neq 0\}.$$

Example 3.3.1. For $n = 5$, the sequences $\alpha = (1, 0, 2, 2, 0, 0, \ldots)$ and $\beta = (2, 0, 1, 2, 0, 0, \ldots)$ are distinct weak compositions but neither are valid partitions of 5 due to the presence of a 0 between positive entries. On the other hand, $\lambda = (2, 2, 1)$ is a partition of 5.

3.4 Symmetric Functions

Definition 3.4.1 (pg. 308 in [Sta24])**.** The ring Λ_k of symmetric functions over a field k is the subring of all $f \in k[\![x_1, x_2, \ldots]\!]$ such that

$$f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots) = f(x_1, x_2, \ldots) \text{ for all } \sigma \in \mathrm{Sym}(\mathbb{N}).$$

Remark 3.4.1. For the algebraically-minded, there is a more natural construction of Λ_k by viewing the ring as the colimit of a certain directed system of injections of symmetric polynomial rings

$$k[x_1, \ldots, x_n]^{S_n} \xhookrightarrow{\varphi_n} k[x_1, \ldots, x_{n+1}]^{S_{n+1}}.$$

The construction of these maps φ_n is somewhat involved. This does justify the intuition that a symmetric function is simply taking a symmetric polynomial and adding more data, as any element of a direct limit of inclusions is faithfully represented by an element of one of the constituent objects.

Definition 3.4.2. A symmetric function $f \in \Lambda_k$ is homogenous of degree n if

$$f(x) = \sum_{\alpha \text{ weak composition of } n} c_\alpha x^\alpha,$$

where the c_α are elements of k. The set of degree n homogenous symmetric functions is denoted Λ_k^n. these subspaces give Λ_k the structure of a graded k-algebra, namely:

- Each Λ_k^n is a k-vector space,
- $\Lambda_k^i \Lambda_k^j \subseteq \Lambda_k^{i+j}$,
- $\Lambda_k = \bigoplus_{n=0}^{\infty} \Lambda_k^n$ as k-vector spaces.

The first interesting basis of Λ_k is the *monomial symmetric functions*. Given $\lambda \vdash n$, define $m_\lambda \in \Lambda_k^n$ by

$$m_\lambda := \sum_\alpha x^\alpha$$

where the sum is over all distinct permutations of the entries of λ. The set $\{m_\lambda : \lambda \vdash n\}$ forms a basis for Λ_k^n, whence $\bigcup_{n \geq 0} \{m_\lambda : \lambda \vdash n\} = \{m_\lambda : \lambda \in \mathrm{Par}\}$ forms a basis for Λ_k.

3.5 Complete Homogenous Symmetric Functions

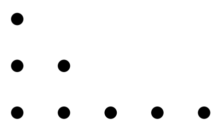
From the monomial symmetric functions, we may form another interesting basis for Λ_k called the *complete homogenous symmetric functions* h_λ by setting

$$h_\lambda := \prod_{i=1}^{\infty} \sum_{\nu \vdash \lambda_i} m_\nu.$$

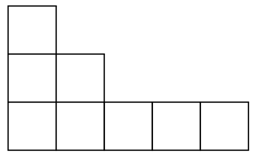
where $\lambda = (\lambda_1, \lambda_2, \ldots)$. Again, the set $\{h_\lambda : \lambda \vdash n\}$ is a basis for Λ_k^n and the set $\{h_\lambda : \lambda \in \mathrm{Par}\}$ is a basis for Λ_k.

3.6 Schur Functions

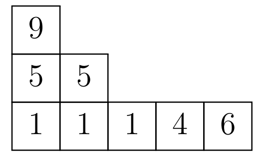
Definition 3.6.1. Given $\lambda \vdash n$, the *Ferrers diagram of shape* λ is the set $\{(i,j) \in \mathbb{N}^2 : j \in \mathbb{N}, 1 \le i \le \lambda_j\}$ depicted as points in $\mathbb{R}^2$. The *Young diagram of shape* λ is depicted identically to the Ferrers diagram except the points are replaced with squares. The *size* of the diagram is the number of entries, namely n. We depict the case $(5,2,1) \vdash 8$ below.



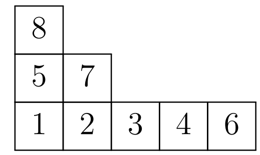
Ferrers Diagram for the partition (5,2,1)



Young Diagram for the partition (5,2,1)



Semi-standard Young tableau for the partition (5,2,1)



Standard Young tableau for the partition (5,2,1)

Definition 3.6.2. Given $\lambda \vdash n$ and a Young diagram of shape λ, a *semi-standard Young tableau of shape* λ is a filling of the boxes of the Young diagram with positive integers such that

- the entries are weakly increasing along rows,
- the entries are strictly increasing up columns.

A semi-standard Young tableau of size n is said to be *standard* if the elements of $\{1, \dots, n\}$ each appear exactly once in the tableau. We write $SSYT(\lambda)$ and $SYT(\lambda)$ for the sets of semi-standard and standard Young tableaux of shape λ. Given a semi-standard Young tableaux $\mathcal{T}$, the *weight* of $\mathcal{T}$ is a function $\alpha = \alpha_{\mathcal{T}} : \mathbb{N} \to \mathbb{N}$ given by

$$\alpha(i) := \text{number of times } i \text{ appears in } \mathcal{T}.$$

Note that $\alpha(i) = 0$ for sufficiently large i, so we may write $x^\alpha = x_1^{\alpha(1)} x_2^{\alpha(2)} \cdots$ and obtain a valid monomial.

Definition 3.6.3. Given $\lambda \vdash n$, the *Schur function* indexed by λ is defined by

$$s_\lambda := \sum_{\mathcal{T} \in SSYT(\lambda)} x^{\alpha_{\mathcal{T}}}$$

where $\alpha_{\mathcal{T}}$ denotes the weight of the Young tableau α.