Measured Equivalence Relations

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Preliminary Definitions

Definition 1 (Measured Equivalence Relations)**.** (X, μ) standard probability space. $\mathcal{R} \subseteq X \times X$ measurable equivalence relation .

- $[\mathcal{R}] = \{\phi \in \mathrm{Aut}(X) : \mathrm{graph}(\phi) \subseteq \mathcal{R}\}$
- $\mathcal{R}$ is *probability measure preserving (pmp)* if $\mu \circ \phi = \mu$ for all $\phi \in [\mathcal{R}]$
- A pmp $[\mathcal{R}]$ is *ergodic* if $\mu(E) \in \{0, 1\}$ whenever $\mu(E \setminus \phi(E)) = 0$ for all $\phi \in [\mathcal{R}]$.

Given a positive measure subset, let $\mathcal{R}|_E$ denote the measured equivalence relation on $(E, \mu/\mu(E))$ given by $\mathcal{R}|_E = \mathcal{R} \cap (E \times E)$

Measured equivalence relations $\mathcal{R}_i$ on (X_i, μ_i) for $i = 1, 2$ are *isomorphic* if there are full measure subsets $E_i \subseteq X_i$ which admit a measure space isomorphism $\phi : (E_1, \mu_1|_{E_1}) \to (E_2, \mu_2|_{E_2})$ such that

$$(x, y) \in \mathcal{R}_1|_{E_1} \iff (\phi(x), \phi(y)) \in \mathcal{R}_2|_{E_2}.$$

From now on, $\mathcal{R}$ is a countable pmp equivalence relation on a standard probability space (X, μ). Endow $\mathcal{R}$ with a measure m given by

$$m(E) = \int_X |\{y \in [x]_{\mathcal{R}} : (x, y) \in E\}| \, d\mu(x) \text{ for all measurable } E \subseteq \mathcal{R}$$

Definition 2 (Equivalence Relation vNas)**.**

$$\begin{aligned} g \in [\mathcal{R}] \rightsquigarrow u_g \in \mathcal{U}(L^2(\mathcal{R}, m)) \text{ by } [u_g f](x, y) = f(g^{-1}x, y)) \\ a \in L^\infty(X) \rightsquigarrow a \in B(L^2(\mathcal{R}, m)) \text{ by } [af](x, y) = a(x) f(x, y) \end{aligned}$$

The von Neumann algebra of the equivalence relation $\mathcal{R}$ is defined to be

$$L(\mathcal{R}) = (L^\infty(X) \cup \{u_g : g \in [\mathcal{R}]\})'' \subseteq B(L^2(\mathcal{R}, m))$$

$L(\mathcal{R})$ has a faithful normal trace given by $\tau(x) = \langle x\mathbb{1}_D, \mathbb{1}_D\rangle$ where $\mathbb{1}_D \in L^2(\mathcal{R}, m)$ is the indicator of the diagonal $D = \{(x, x) : x \in X\}$.

Let $Z^1(\mathcal{R}, S^1)$ denote the group of S^1-valued multiplicative 1-cocyles on $\mathcal{R}$, i.e. the group of measurable maps $c : \mathcal{R} \to S^1$ such that for μ-a.e. $x \in X$,

$$c(x, z) = c(x, y)c(y, z) \text{ for all } (x, y), (y, z) \in \mathcal{R}.$$

Given $c \in Z^1(\mathcal{R}, S^1)$ and $g \in [\mathcal{R}]$, let $f_{c,g} \in \mathcal{U}(L^\infty(X))$ be given by $f_{c,g}(x) = c(x, g^{-1}x)$. Can check that the formula

$$\psi_c(au_g) = f_{c,g}au_g \text{ for all } a \in L^\infty(X), g \in [\mathcal{R}]$$

gives rise to a well defined *-isomorphism $\psi_c \in \mathrm{Aut}(L(\mathcal{R}))$. Moreover, $c \mapsto \psi_c$ defines an action $\psi : Z^1(\mathcal{R}, S^1) \to \mathrm{Aut}(L(\mathcal{R}))$.

Definition 3 (Hilbert Bundles)**.** Given $\{\mathcal{H}_x\}_{x\in X}$ collection of Hilbert spaces, define the Hilbert bundle

$$X * \mathcal{H} = \{(x, \xi_x) : x \in X, \xi_x \in \mathcal{H}_x\}.$$

- A *section* ξ of the bundle $X * \mathcal{H}$ is a map $x \mapsto \xi_x \in \mathcal{H}_x$.
- *Fundamental sequence of sections* $\{\xi_n\}_{n=1}^\infty$ satisfy
  - $\mathcal{H}_x = \overline{\mathrm{Span}\{\xi_n(x)\}_{n=1}^\infty}$ for each $x \in X$, and
  - the maps $\{x \mapsto \|\xi_n(x)\|\}_{n=1}^\infty$ are measurable.
- *Orthonormal fundamental sequence of sections* $\{\xi_n\}_{n=1}^\infty$ is a fundamental sequence of sections such that
  - $\{\xi_n(x)\}_{n=1}^\infty$ is an ONB of $\mathcal{H}_x$ for $x \in X$ with $\dim(\mathcal{H}_x) = \infty$, and if $\dim(\mathcal{H}_x) < \infty$, the sequence $\{\xi_n(x)\}_{n=1}^{\dim(\mathcal{H}_x)}$ is an ONB and $\xi_n(x) = 0$ for $n > \dim(\mathcal{H}_x)$.

Now for measurable stuff

- *Measurable Hilbert bundle* $X * \mathcal{H}$ has σ-algebra generated by maps $\{(x, \xi_x) \mapsto \langle \xi(x), \xi_n(x)\rangle\}_{n=1}^\infty$.
- A *measurable section* of $X * \mathcal{H}$ is a section ξ such that $x \mapsto (x, \xi(x)) \in X * \mathcal{H}$ is a measurable map, or equivalently, such that the maps $\{x \mapsto \langle \xi(x), \xi_n(x)\rangle\}_{n=1}^\infty$ are measurable.
- $S(X * \mathcal{H})$ is the vector space of measurable sections up to μ-a.e. equivalence.
- The *direct integral*
$$\int_X^\oplus \mathcal{H}_x \, d\mu(x) = \{\xi \in S(X * \mathcal{H}) : \int_X \|\xi(x)\|^2 \, d\mu(x) < \infty\}$$
is a Hilbert space with inner product $\langle \xi, \eta\rangle = \int_X \langle \xi(x), \eta(x)\rangle \, d\mu(x)$.
- If $a \in A = L^\infty(X)$ and $\xi \in \int_X^\oplus \mathcal{H}_x \, d\mu(x)$ we denote by $a\xi$ or ξa the element of $\int_X^\oplus \mathcal{H}_x \, d\mu(x)$ given by $[a\xi](x) = [\xi a](x) = \xi(x)a(x)$.
- If $\{\xi_n\}_{n=1}^\infty$ orthonormal fundamental sequence of sections, any $\xi \in \int_X^\oplus \mathcal{H}_x \, d\mu(x)$ has an expansion $\xi = \sum_{n=1}^\infty a_n\xi_n$ where $a_n = \langle \xi(\cdot), \xi_n(\cdot)\rangle \in A$.

Representations

Definition 4 (Representations of Equivalence Relations)**.** A *unitary (resp. orthogonal)* representation of $\mathcal{R}$ on a complex (resp. real) measurable Hilbert bundle $X * \mathcal{H}$ is a map $(x, y) \mapsto \pi(x, y) \in \mathcal{U}(\mathcal{H}_y, \mathcal{H}_x)$ on $\mathcal{R}$ such that for μ-a.e. $x \in X$, we have

$$\pi(x, z) = \pi(x, y)\pi(y, z) \text{ for all } (x, y), (y, z) \in \mathcal{R},$$

and such that $(x, y) \mapsto \langle \pi(x, y)\xi(y), \eta(x)\rangle$ is a measurable map on $\mathcal{R}$ for all $\xi, \eta \in S(X * \mathcal{H})$.

- *Identity representation*: Given orthonormal fundamental sequence of sections $\mathcal{S} = \{\xi_n\}$, can form the *identity representation* $id_{\mathcal{S}}$ of $\mathcal{R}$ on $X * \mathcal{H}$ by

$$id_S(x, y)\xi_n(y) = \xi_n(x) \text{ for all } (x, y) \in \mathcal{R} \text{ and } \xi_n \in \mathcal{S}.$$

- *Regular representation*: Take $\mathcal{H}_x = l^2([x]_{\mathcal{R}})$ for each $x \in X$ and form $X * \mathcal{H}$ with fundamental sequence of sections $\{\xi_g\}_{g \in \Gamma}$ where
  - $\xi_g(x) = \mathbb{1}_{g^{-1}x}$ for all $x \in X$, and
  - Γ is a countable subgroup of $[\mathcal{R}]$ which generates $[\mathcal{R}]$ (FM75a showed this exists).

  The *regular representation* of $\mathcal{R}$ is the representation λ on $X * \mathcal{H}$ given by $\lambda(x, y) = id$ for all $(x, y) \in \mathcal{R}$.

We say representations π on $X * \mathcal{H}$ and ρ on $X * \mathcal{K}$ are *unitarily equivalent* if there is a family of unitaries $\{U_x \in \mathcal{U}(\mathcal{H}_x, \mathcal{K}_x)\}_{x \in X}$ with

$$U_x \pi(x, y) = \rho(x, y) U_y \text{ for all } (x, y) \in \mathcal{R},$$

and such that $x \mapsto U_x \xi(x)$ is in $S(X * \mathcal{K})$ for each $\xi \in S(X * \mathcal{H})$.

Cohomology

Definition 5 (1-cohomology)**.**

- A *1-cocycle* for a representation π on $X * \mathcal{H}$ is a map $(x, y) \mapsto b(x, y) \in \mathcal{H}_x$ on $\mathcal{R}$ such that for μ-a.e. $x \in X$,

$$b(x, z) = b(x, y) + \pi(x, y) b(y, z) \text{ for all } (x, y), (y, z) \in \mathcal{R},$$

  and such that $(x, y) \mapsto (x, b(x, y)) \in X * \mathcal{H}$ is measurable.

- A 1-cocycle b is a *coboundary* if there is a $\xi \in S(X * \mathcal{H})$ such that

$$b(x, y) = \xi(x) - \pi(x, y)\xi(y) \text{ for } m\text{-a.e. } (x, y) \in \mathcal{R}.$$

- A pair of 1-cocycles b, b' are *cohomologous* if $b - b'$ is a coboundary.

- A 1-cocycle is *bounded* if there exists a sequence of measurable subsets $(E_n)_{n=1}^{\infty}$ of X with

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = 1 \text{ and } \sup\{\|b(x, y)\| : (x, y) \in \mathcal{R}|_{E_n}\} < \infty \text{ for each } n \geq 1.$$

Lemma 1. *A 1-cocycle b for a representation π of $X * \mathcal{H}$ is a coboundary if and only if it is bounded.*

Lemma 2 (Characterization for unboundedness)**.** *A 1-cocycle b for a representation π of $X * \mathcal{H}$ is unbounded if and only if there is a $\delta > 0$ such that for any $R > 0$ there is a $g \in [\mathcal{R}]$ with $\mu(\{x \in X : \|b(x, g^{-1}x)\| > R\}) \geq \delta$.*

Orbit Equivalence Relations

Definition 6 (OE Relation)**.** Given countable group Γ and pmp action $\Gamma \curvearrowright X$, the *orbit equivalence relation* $\mathcal{R}(\Gamma \curvearrowright X)$ is defined by

$$(x, y) \in \mathcal{R}(\Gamma \curvearrowright X) \qquad \Longleftrightarrow \qquad y = gx \text{ for some } g \in \Gamma,$$

and two group actions are *orbit equivalent* (OE) if and only if they have isomorphic orbit equivalence relations.

Recall that $\Gamma \curvearrowright (X, \mu)$ is *free* if $\mu(\{x \in X : gx = x\}) = 0$ for each nonidentity $g \in \Gamma$.

If $\mathcal{R} = \mathcal{R}(\Gamma \curvearrowright X)$ for a free pmp action of a countable group Γ, then any group representation π gives rise to a representation $\pi_{\mathcal{R}}$ of $\mathcal{R}$, and any 1-cocyle b for π gives rise to a 1-cocycle $b_{\mathcal{R}}$ for $\pi_{\mathcal{R}}$. Note that

$$E_0 := \{x \in X : gx = x \text{ for some } g \in \Gamma \setminus \{e\}\} = \bigcup_{g \in \Gamma \setminus \{e\}} Stab_X(g)$$

is null since Γ is countable and the action is (essentially) free. Then the representation and cocycle are given as follows

- $\pi_{\mathcal{R}}(x, g^{-1}x) = \pi(g)$ for all $g \in \Gamma$
- $b_{\mathcal{R}}(x, g^{-1}x) = b(g)$ for all $g \in \Gamma$, $x \notin E_0$,

and since $\mu(E_0) = 0$, for $x \in E_0$ take $\pi(x, y) = id$ and $b(x, y) = 0$. Can check that $\pi_{\mathcal{R}}$ is mixing if and only if π is mixing and $b_{\mathcal{R}}$ is unbounded if and only if b is unbounded. This association also respects weak containment. When π is either left or right regular representation, $\pi_{\mathcal{R}}$ is unitarily equivalent to the regular representation λ.

Gaussian Construction

π orthogonal representation of $\mathcal{R}$ on a real Hilbert bundle $X * \mathcal{H}$, $\{\xi_n\}_{n=1}^{\infty}$ orthonormal fundamental sequence of sections.

$$(\Omega_x, \nu_x) := \prod_{i=1}^{\dim(\mathcal{H}_x)} \left(\mathbb{R}, \frac{1}{\sqrt{2\pi}} e^{-s^2/2}\, ds\right)$$

Define $\omega_x : \mathrm{Span}(\{\xi_i\}_{i=1}^{\dim(\mathcal{H}_x)}) \to \mathcal{U}(L^\infty(\Omega_x))$ by

$$\omega_x\left(\sum_{n=1}^{k} a_n \xi_{i_n}(x)\right) = \exp\left(i\sqrt{2}\sum_{n=1}^{k} a_n S_{i_n}^x\right),$$

where S_j^x is the j-th coordinate function for $j \leq \dim(\mathcal{H}_x)$. Then ω_x extends to $\omega_x : \mathcal{H}_x \to \mathcal{U}(L^\infty(\Omega_x))$ such that

- $\tau(\omega_x(\xi)) = e^{-\|\xi\|^2}$
- $\omega_x(\xi + \eta) = \omega_x(\xi)\omega_x(\eta)$
- $\omega_x(\xi)^* = \omega_x(-\xi)$.

$D_x := \mathrm{Span}_{\mathbb{C}}(\{\omega_x(\xi)\}_{\xi\in\mathcal{H}_x})$ has $D_x'' = \overline{D_x}^{WOT} = L^\infty(\Omega_x)$.

For $(x,y)\in\mathcal{R}$, define a *-homomorphism $\rho(x,y) : D_y \to L^\infty(\Omega_x)$ by

$$\rho(x,y)\omega_y(\xi) = \omega_x(\pi(x,y)\xi),$$

which is well defined and $\|\cdot\|_2$-isometric since

$$\tau(\omega_y(\eta)^*\omega_y(\xi)) = \tau(\omega_x(\pi(x,y)\eta)^*\omega_x(\pi(x,y)\xi))$$

as seen below.

$$\begin{aligned}\tau(\omega_x(\pi(x,y)\eta)^*\omega_x(\pi(x,y)\xi)) &= \tau(\omega_x(-\pi(x,y)\eta)\omega_x(\pi(x,y)\xi)) = \tau(\omega_x(-\pi(x,y)\eta+\pi(x,y)\xi))\\ &= \tau(\omega_x(\pi(x,y)(\xi-\eta)) = e^{-\|\pi(x,y)(\xi-\eta)\|^2} = e^{-\|\xi-\eta\|^2} = \cdots = \tau(\omega_y(\eta)^*\omega_y(\xi))\end{aligned}$$

Now $\rho(x,y)$ extends to a $*$-isomorphism $\rho(x,y) : L^\infty(\Omega_y)\to L^\infty(\Omega_x)$. Let $\theta_{(x,y)} : \Omega_y \to \Omega_x$ be the corresponding measure space isomorphism.
Consider the measurable bundle $X * \Omega$ with σ-algebra generated by the maps $(x,r)\mapsto \omega_x\left(\sum_{n\in I} a_n\xi_n(x)\right)(r)$ for all finite subsets $I\subseteq\mathbb{N}$ and $a_n\in\mathbb{R}$.
Define a measure $\mu * \nu$ on $X*\Omega$ by

$$(\mu*\nu)(E) := \int_X \nu_x(E_x)\mu(x)$$

where $E_x := \{s\in\Omega_x : (x,s)\in E\}$.

Definition 7. The *Gaussian* extension of $\mathcal{R}$ is the equivalence relation $\widetilde{\mathcal{R}}$ on $(X*\Omega,\mu*\nu)$ given by

$$((x,r),(y,s))\in\widetilde{\mathcal{R}} \iff (x,y)\in\mathcal{R} \text{ and } \theta_{(y,x)}(r) = s.$$

This is a countable pmp equivalence relation.

For $g\in[R]$, define $\widetilde{g}\in[\widetilde{\mathcal{R}}]$ by

$$\widetilde{g}(x,r) = (gx, \theta_{(gx,x)}(r))$$

Then we can embed $L(\mathcal{R})$ into $L(\widetilde{\mathcal{R}})$ by $au_g\mapsto au_{\widetilde{g}}$. From now on, identify u_g and $u_{\widetilde{g}}$.

Note $\widetilde{\mathcal{R}} = \bigcup_{g\in[\mathcal{R}]}\mathrm{graph}(\widetilde{g})$, whence

$$L(\widetilde{\mathcal{R}}) = (L^\infty(X*\Omega)\cup\{u_{\widetilde{g}} : g\in[\mathcal{R}]\})'' = (L^\infty(X*\Omega)\cup L(\mathcal{R}))'' \subseteq B(L^2(\widetilde{R}))$$

Gaussian Deformation

b a 1-cocycle for π on $X*\mathcal{H}$. Set $M := L(\mathcal{R})$, $\widetilde{M} := L(\widetilde{\mathcal{R}})$.

Consider

$$c_t((x,r),(y,s)) := \omega_x(tb(x,y))(r)$$

This defines a family $\{c_t\}_{t\in\mathbb{R}}\subseteq Z^1(\widetilde{\mathcal{R}},S^1)$.
Given $t\in\mathbb{R}$ and $g\in[R]$, let $f_{c_t,g}\in\mathcal{U}(L^\infty(X*\Omega))$ be given by

$$f_{c_t,g}(x,r) = c_t((x,r),\widetilde{g}^{-1}(x,r)) = c_t((x,r),(g^{-1}x,\theta_{(g^{-1}x,x)}(x))) = \omega_x(tb(x,g^{-1}x))(r).$$

Then consider $\psi_{c_t} \in \operatorname{Aut}(\widetilde{M})$ given by

$$\psi_{c_t}(au_{\widetilde{g}}) = f_{c_t,g} au_{\widetilde{g}} \text{ for all } a \in L^\infty(X * \Omega), g \in [\mathcal{R}].$$

Write $\alpha_t := \psi_{c_t} \in \operatorname{Aut}(\widetilde{M})$.

$$\begin{aligned}\tau(f_{c_t,g}) &= \int_{X*\Omega} f_{c_t,g} \, d\mu * \nu = \int_X \int_{\Omega_x} \omega_x(tb(x, g^{-1}x))(r) \, d\nu_x(r) \, d\mu(x) \\ &= \int_X \tau(\omega_x(tb(x, g^{-1}x))) \, d\mu(x) = \int_X e^{-\left\|tb(x,g^{-1}x)\right\|^2} \, d\mu(x)\end{aligned}$$

So,

$$\begin{aligned}\|\alpha_t(au_g) - au_g\|_2^2 = \|f_{c_t,g} au_g - au_g\|_2^2 \leq \|a\|^2 \|f_{c_t,g} - 1\|_{2^2} &= 2\|a\|^2(1 - \operatorname{Re}(\tau(f_{c_t,g})) \\ = 2\|a\|^2 \left(1 - \int_X e^{-\left\|tb(x,g^{-1}x)\right\|^2} \, d\mu(x)\right) &\xrightarrow{t \to 0} 0,\end{aligned}$$

whence α is a malleable deformation of $M \subseteq \widetilde{M}$.

Facts

Note that given a representation π of $\mathcal{R}$, we can get a group representation $\widetilde{\pi} : [\mathcal{R}] \to \mathcal{U}\left(\int_X^\oplus \mathcal{H}_x \, d\mu(x)\right)$ by

$$[\widetilde{\pi}(g)\xi](x) := \pi(x, g^{-1}x)\xi(g^{-1}x)$$

- According to `https://ncatlab.org/nlab/show/measurable+field+of+Hilbert+spaces`, the category of measurable Hilbert bundles on (X, Σ, N) is equivalent to the category of Hilbert $L^\infty(X, \Sigma, N)$-modules. I assume this is through the direct integral being an L^∞-module.
- TODO Try to encode representations of $L(\mathcal{R})$ in this framework.

Explorative arguments

Utilizing Hopf Algebra structure

Definition 8. Let $\mathcal{R}$ be a countable pmp equivlence relation on a standard probability space (X, μ). A subequivalence relation $\mathcal{S} \subseteq \mathcal{R}$ is called *full* if $(x, x) \in \mathcal{S}$ for all $x \in X$, i.e. $\mathcal{S}$ is also an equivalence relation over X.

Definition 9 (Thomas and Schneider [1])**.** A countable pmp equivalence relation $\mathcal{R}$ on a standard probability space (X, μ) is *free* if there exists a countable group Γ with a free pmp action $\Gamma \curvearrowright X$ such that $\mathcal{R} = \mathcal{R}(\Gamma \curvearrowright X)$.

Definition 10 (Thomas and Schneider [1])**.** A countable pmp equivalence relation $\mathcal{R}$ on a standard probability space (X, μ) is *essentially free* if there exists a free countable pmp equivalence relation $\mathcal{S}$ which is isomorphic to $\mathcal{R}$.

Lemma 3 (essentially free implies action of full group is essentially free)**.** *Let $\mathcal{R}$ be an essentially free countable pmp equivalence relation on a standard probability space (X, μ). Then for all $g \in [R] \setminus \{e\}$, we have that $\mu(\{x \in X : gx = x\}) = 0$.*

Proof. TODO prove this. $\square$

Proposition 1. *TODO adjust for only essentially free case since proof doesn't work in general. Let $\mathcal{R}$ be a countable pmp equivlence relation on a standard probability space (X, μ), and $N \subseteq L(\mathcal{R})$ a unital von Neumann subalgebra such that $L^\infty(X) \subseteq N$. Consider the relative coproduct on $L(\mathcal{R})$ given by*

$$\Delta : L(\mathcal{R}) \to L(R)\overline{\otimes}L(R)$$
$$au_g \mapsto au_g \otimes u_g.$$

Then $\Delta(N) \subseteq N\overline{\otimes}N$ if and only if there exists a (full?) subequivalence relation $\mathcal{S} \subseteq \mathcal{R}$ such that $N = L(\mathcal{S})$.

Proof. Note that for $g \in [R]$,

$$\begin{aligned}\tau(u_g) = \langle u_g \mathbb{1}_D, \mathbb{1}_D\rangle &= \int_{\mathcal{R}} (u_g \mathbb{1}_D)\overline{\mathbb{1}_D}\, dm(x,y) \\ &= \int_{\mathcal{R}} \mathbb{1}_D(g^{-1}x, y)\mathbb{1}_D(x,y)\, dm(x,y) = m(\{(x,y) : g^{-1}x = y \text{ and } x = y\}) = \mu(\{x : g^{-1}x = x\}).\end{aligned}$$

Suppose $n \in N$ and write $n = \sum_{h \in [\mathcal{R}]} a_h u_h$ where $a_h \in L^\infty(X)$ and all sums converge in $\|\cdot\|_2$-norm. Fix $g \in [\mathcal{R}]$. Then $\tau(nu_g^*) = a_g$ whence under the identification of $a_h u_h \otimes u_e = a_h u_h$, we have that

$$(id \otimes \tau u_g^*)\Delta(n) = \sum_{h \in [R]} a_h u_h \otimes \tau(u_g^* u_h)u_e = a_g u_g \otimes u_e = \tau(nu_g^*)u_g$$

$\underline{\implies}$: Suppose that $\Delta(N) \subseteq N\overline{\otimes}N$. Let

$$\mathscr{S} = \{g \in [\mathcal{R}] : \exists\, n \in N \text{ such that } \tau(nu_g^*) \neq 0\}.$$

If $g \in [\mathcal{R}]$ and $n \in N$, then again under the aforementioned identification,

$$\tau(nu_g^*)u_g = (id \otimes \tau u_g^*)\Delta(n) \in (id \otimes \tau u_g^*)(N \otimes N) \subseteq N.$$

Now, for $g \in \mathscr{S}$ there is some $n \in N$ such that $\tau(nu_g^*) \neq 0$, whence the above identity implies that $u_g \in N$. Conversely, if $u_g \in N$, then $\tau(u_g u_g^*) = 1 \neq 0$ so $g \in \mathscr{S}$. Thus for $g \in [\mathcal{R}]$ we have the following equivalence:

$$u_g \in N \iff g \in \mathscr{S}.$$

In other words, $\mathcal{U}(N) = \mathcal{U}(L^\infty(X)) \cup \{u_g : g \in \mathscr{S}\}$. Let $\mathcal{S}$ be the equivalence relation generated by $\bigcup_{g \in \mathscr{S}} \operatorname{graph}(g)$. Note that $\mathcal{S}$ is a full countable pmp measured subequivalence relation of $\mathcal{R}$ over X, so $L(\mathcal{S}) \subseteq L(\mathcal{R})$.

By construction, $\mathscr{S} \subseteq [\mathcal{S}]$, so $N \subseteq L(\mathcal{S})$. For equality, it suffices to show that $\mathscr{S} = [\mathcal{S}]$.

Suppose, for the sake of contradiction, that there is some $g \in [\mathcal{S}] \setminus \mathscr{S}$. Then $\operatorname{graph}(g) \subseteq \mathcal{S}$. Consider the equivalence relation $\mathcal{T}$ generated by $\operatorname{graph}(g) \cup \bigcup_{h \in \mathscr{S}} \operatorname{graph}(h)$. TODO finish this

$\underline{\impliedby}$: Suppose that $N = L(\mathcal{S})$ for some full subequivalence relation $\mathcal{S} \subseteq \mathcal{R}$. Then for $a \in L^\infty(X)$ and $g \in [S]$, $\Delta(au_g) = au_g \otimes u_g \in L(\mathcal{S})\overline{\otimes}L(\mathcal{S})$. Thus by linearity and continuity, $\Delta(L(\mathcal{S})) \subseteq L(\mathcal{S})\overline{\otimes}L(\mathcal{S})$.

$\square$

References

[1] Simon Thomas and Scott Schneider. *Countable Borel Equivalence Relations*. 2007. URL: `https://www.math.cmu.edu/~eschimme/Appalachian/ThomasNotes.pdf`.