Geometric Measure Theory Notes

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1 Notation

Throughout this document, Ω denotes an open set in $\mathbb{R}^n$.
For $u : \mathbb{R}^m \to \mathbb{R}^n$, we write $u = (u^1, \dots, u^n)$ where $u^i = \pi_i \circ u$.

2 Functions of Bounded Variation.

Definition 1. Given a function $u \in L^1(\Omega)$, define the *total variation of* u to be the quantity

$$V(u, \Omega) = \sup \left\{ \int_\Omega u \operatorname{div} \varphi \, dx : \varphi \in [C_c^1(\Omega, \mathbb{R})]^n, \|\varphi\|_\infty \leq 1 \right\}.$$

If $V(u, \Omega)$ is finite, then we say that u is of *bounded variation* and write $u \in BV(\Omega)$.

Definition 2. Similarly, given $u \in L^1_{loc}(\Omega)$ and $U \Subset \Omega$, define the *local variation of* u *in* U by

$$V(u, U) = \sup \left\{ \int_U u \operatorname{div} \varphi \, dx : \varphi \in [C_c^1(U, \mathbb{R})]^n, \|\varphi\|_\infty \leq 1 \right\}.$$

We define the set of functions of *locally bounded variation* to be

$$BV_{loc}(\Omega) = \{u \in L^1_{loc}(\Omega) : V(u, U) < +\infty \text{ for all } U \Subset \Omega\}.$$

An equivalent, and admittedly more transparent, characterization of BV_{loc} functions can be given as follows.

Proposition 1 (Characterization of BV_{loc})**.** *Suppose $u \in BV_{loc}(\Omega)$. Then there exists a Radon measure μ on Ω and a μ-measurable $\sigma : \Omega \to \mathbb{R}^n$ with $|\sigma| = 1$ μ-a.e. and*

$$\int_\Omega u \operatorname{div} \varphi \, dx = -\int_\Omega \varphi \cdot \sigma \, d\mu \;\; \textit{for all } \varphi \in C_c^1(\Omega, \mathbb{R}^n).$$

Proof. This is a routine application of the Riesz–Markov–Kakutani representation theorem. To this end, define a linear functional $L : C_c^1(\Omega, \mathbb{R}^n) \to \mathbb{R}$ by $L(\varphi) = -\int_\Omega u \operatorname{div} \varphi \, dx$.

For open $U \Subset \Omega$, the quantity $c(U) := \sup\{L(\varphi) : \varphi \in C_c^1(U, \mathbb{R}^n), \|\varphi\|_\infty \leq 1\}$ is finite by assumption, whence

$$|L(\varphi)| \leq c(U)\|\varphi\|_\infty \text{ for all } \varphi \in C_c^1(U, \mathbb{R}^n).$$

Let $K \subseteq \Omega$ be a fixed compact set, and choose open $U \Subset \Omega$ containing K. Then for $\varphi \in C_c(\Omega, \mathbb{R}^n)$ with $\operatorname{supp}(\varphi) \subseteq K$, there exists a sequence $(\varphi_k)_k$ in $C_c^1(U, \mathbb{R}^n)$ such that $\varphi_k \to \varphi$ uniformly on U.

Define an extension $\widetilde{L} : C_c(\Omega, \mathbb{R}^n) \to \mathbb{R}$ of L by $\widetilde{L}(\varphi) = \lim_{k\to\infty} L(\varphi_k)$, which exists and is well-defined by the above inequality. Applying the Riesz Representation Theorem to $\widetilde{L}$ gives the conclusion. □

Definition 3. For $u \in BV_{loc}(\Omega)$, we will write $\|Du\|$ for the measure μ and

$$d[Du] := \sigma \, d\|Du\| \,, \text{ i.e } \int \cdot \, d[Du] = \int \langle \cdot, \sigma \rangle \, d\|Du\| \,.$$

Then the conclusion of Proposition 1 can be rewritten as

$$\int u \operatorname{div} \varphi \, dx = - \int \varphi \cdot \sigma \, d\|Du\| = - \int \varphi \cdot d[Du] \text{ for all } \varphi \in C_c^1(\Omega, \mathbb{R}^n).$$

Write $\varphi = (\varphi^1, \ldots, \varphi^n) \in C_c^1(\Omega, \mathbb{R}^n)$.

$$[Du] = [Du]_{ac} + [Du]_s$$

3 Caccioppoli Sets (i.e. Sets of Locally Finite Perimeter)

Definition 4. Given a set $E \subseteq \mathbb{R}^n$, we say that E is of *locally finite perimeter in* Ω if $\chi_E \in BV_{loc}(\Omega)$.

Norms

$\|\cdot\|_{W^{k,p}(\Omega)}$	Sobolev-Norm
$\|\cdot\|_{\mathrm{BV}}$	BV-Norm