220A Homework 1

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Problem 1

Let Λ be a circle lying in S. Then there is a unique plane P in $\mathbb{R}^3$ such that

$$P \cap S = \Lambda.$$

Recall from analytic geometry that

$$P = \{(x_1, x_2, x_3) : x_1\beta_1 + x_2\beta_2 + x_3\beta_3 = l\}$$

where $(\beta_1, \beta_2, \beta_3)$ is a vector orthogonal to P and l is some real number. It can be assumed that

$$\beta_1^2 + \beta_2^2 + \beta_3^2 = 1.$$

Use this information to show that if Λ contains the point N then its projection on C is a straight line. Otherwise, Λ projects onto a circle in C.

Proof. We denote the projection of (x_1, x_2, x_3) by $z = a+bi$ or $(a, b, 0)$. Let $t \in [0, 1]$ be such that $(x_1, x_2, x_3) = t(a, b, 0) + (1-t)N$. Then we obtain

$$(x_1, x_2, x_3 - 1) = t(a, b, -1)$$

This gives the relations $x_1 = ta$, $x_2 = tb$, $x_3 = 1 - t$. Moreover, upon substituting these relations and using that $x_1^2 + x_2^2 + x_3^2 = 1$, we obtain

$$|z|^2 = a^2 + b^2 = \frac{x_1^2 + x_2^2}{t^2} = \frac{1 - x_3^2}{(1 - x_3)^2} = \frac{1 + x_3}{1 - x_3} = \frac{2 - t}{t} = \frac{2}{t} - 1$$

which leads to $t = \frac{2}{|z|^2+1}$. Now substituting these relations into the equation for the plane P, we compute

$$\begin{aligned} l &= \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 \\ &= \beta_1 ta + \beta_2 tb + \beta_3(1 - t) \\ &= \frac{2\beta_1}{|z|^2 + 1} a + \frac{2\beta_2}{|z|^2 + 1} b + \beta_3 \left(1 - \frac{2}{|z|^2 + 1}\right) \\ &= \frac{2\beta_1}{|z|^2 + 1} a + \frac{2\beta_2}{|z|^2 + 1} b + \beta_3 \left(\frac{|z|^2 - 1}{|z|^2 + 1}\right). \end{aligned}$$

After moving over l and multiplying through by $|z|^2 + 1$, we obtain

$$\begin{aligned} 0 &= 2\beta_1 a + 2\beta_2 b + \beta_3(|z|^2 - 1) - l(|z|^2 + 1) \\ &= (\beta_3 - l)|z|^2 + 2\beta_1 a + 2\beta_2 b - \beta_3 - l \\ &= (\beta_3 - l)(a^2 + b^2) + 2\beta_1 a + 2\beta_2 b - \beta_3 - l \end{aligned}$$

Suppose first that $N \in \Lambda$. Then $(0,0,1)$ satisfies the equation for P which gives $\beta_3 = l$. Then the above equation simplifies to the following equation for a line in a and b

$$0 = 2\beta_1 a + 2\beta_2 b.$$

From now on, we assume that $N \notin \Lambda$, so $\beta_3 \neq l$. Completing the square for the quadratics in a and b in the previously obtained equation, we obtain

$$0 = (\beta_3 - l)\left(a + \frac{4\beta_1}{\beta_3 - l}\right)^2 - \frac{\beta_1^2}{\beta_3 - l} + (\beta_3 - l)\left(b + \frac{4\beta_2}{\beta_3 - l}\right) - \frac{\beta_2^2}{\beta_3 - l} - \beta_3 - l$$

which upon rearranging and dividing by $\beta_3 - l$ gives the following equation for a circle in a and b

$$1 + \frac{1}{(\beta_3 - l)^2}\left(\beta_1^2 + \beta_2^2\right) = \left(a + \frac{4\beta_1}{\beta_3 - l}\right)^2 + \left(b + \frac{4\beta_2}{\beta_3 - l}\right)^2.$$

□

Problem 2

Prove that G is open if and only if $X \setminus G$ is closed.

Proof.
In Conway, the definition of closed is being the complement of closed, so this question is silly. For the sake of writing something I will just use baby Rudin's definition of closed as a set which contains its limit points.

$(\implies)$: Suppose that G is open. Let $p \in X$ be a limit point of $X \setminus G$. Suppose, for the sake of contradiction, that $p \notin X \setminus G$. Then $p \in G$. As G is open, there is some $\varepsilon > 0$ such that $B_\varepsilon(p) \subseteq G$. As p is a limit point of $X \setminus G$, there is some point $q \in B_\varepsilon(p) \cap (X \setminus G)$, which contradicts $B_\varepsilon(p) \subseteq G$. Thus, $p \in X \setminus G$, so $X \setminus G$ contains all of its limit points and is thus closed.

$(\impliedby)$: We proceed by contraposition. Suppose that G is not open. Then by definition there is some point $p \in G$ such that $B_r(p) \nsubseteq G$ for all $r > 0$. Hence, for each $r > 0$, there is some point $q \in B_r(p) \cap (X \setminus G)$ with $q \neq p$ as $p \in G$. Thus by definition $p \in G$ is a limit point of $X \setminus G$, so we have found a limit point of $X \setminus G$ which is not in $X \setminus G$, so $X \setminus G$ is not closed.

□

Problem 3

Prove that $(\widehat{\mathbb{C}}, d)$ is a metric space. [NOTE I write $\widehat{\mathbb{C}}$ for the Riemann sphere].

$$d(z,w) := \frac{2|z-w|}{[(1+|z|^2)(1+|w|^2)]^{\frac{1}{2}}} \quad \text{for } z, w \in \mathbb{C}$$

$$d(\infty, z) := d(z, \infty) := \frac{2}{(1+|z|^2)^{\frac{1}{2}}} \quad \text{for } z \in \mathbb{C}$$

$$d(\infty, \infty) := 0$$

Proof. That d is nonnegative and symmetric is clear by the above expressions. It is also clear that $d(z,z) = 0$ for all $z \in \widehat{\mathbb{C}}$. Now suppose that $z, w \in \mathbb{C}$ are such that $d(z,w) = 0$. Then

$$0 = d(z,w) = \frac{2|z-w|}{[(1+|z|^2)(1+|w|^2)]^{\frac{1}{2}}}$$
$$\implies 0 = 2|z-w| \quad \implies z = w.$$

Let $z \in \widehat{\mathbb{C}}$ and suppose for the sake of contradiction that $d(z,\infty) = 0$ but $z \neq \infty$. Then

$$0 = d(z,\infty) = \frac{2}{(1+|z|^2)} \implies 0 = 2$$

which is absurd, thus $d(z,\infty) = 0$ implies that $z = \infty$. Lastly we need to check the triangle inequality. For $P = (P_1, P_2, P_3) \in \mathbb{R}^3$, let $\|P\|_2 = \sqrt{P_1^2 + P_2^2 + P_3^2}$ denote the euclidean norm in $\mathbb{R}^3$. By construction, if $z, w \in \widehat{C}$ and$Z, W \in \mathbb{R}^3$ are the corresponding points on the Riemann sphere in $\mathbb{R}^3$, then

$$d(z,w) = \|Z - W\|_2.$$

Suppose that $u, v, w \in \mathbb{C}$. Let $U, V, W \in \mathbb{R}^3$ be the corresponding points on the Riemann sphere in $\mathbb{R}^3$. Then by the triangle inequality in $\mathbb{R}^3$,

$$d(u,w) = \|U - W\|_2 \leq \|U - V\|_2 + \|V - W\|_2 = d(u,v) + d(v,w).$$

□

Problem 4

The purpose of this exercise is to show that a connected subset of $\mathbb{R}$ is an interval.

(a): Show that a set $A \subset \mathbb{R}$ is an interval iff for any two points a and b in A with $a < b$, the interval $[a,b] \subset A$.

Proof.
$(\implies)$: Let $\alpha \in \mathbb{R} \cup \{-\infty\}$ and $\beta \in \mathbb{R} \cup \{+\infty\}$.
Suppose $A = [\alpha, \beta]$. Then if $a, b \in A$ with $a < b$ and $x \in [a,b]$, then $\alpha \leq a \leq x \leq b \leq \beta$, so $x \in A$ whence $[a,b] \subseteq A$.
Suppose $A = (\alpha, \beta]$. Then if $a, b \in A$ with $a < b$ and $x \in [a,b]$, then $\alpha < a \leq x \leq b \leq \beta$, so $x \in A$ whence $[a,b] \subseteq A$.
Suppose $A = (\alpha, \beta)$. Then if $a, b \in A$ with $a < b$ and $x \in [a,b]$, then $\alpha < a \leq x \leq b < \beta$, so $x \in A$ whence $[a,b] \subseteq A$.
$(\impliedby)$: Singletons are intervals so suppose A is not a singleton. Let $M := \sup(A) \in \mathbb{R} \cup \{+\infty\}$ and $m := \inf(A) \in \mathbb{R} \cup \{-\infty\}$. Let $x \in (m, M)$. Then by definition of supremum and infimum, there exist $a, b \in A$ such that $m < a < x < b < M$. By the assumption, it follows that $x \in A$. Thus $(\inf(A), \sup(A)) \subseteq A$, whence $A = (m, M)$, $[m, M)$, or $[m, M]$.

□

(b): Use part (a) to show that if a set $A \subset \mathbb{R}$ is connected then it is an interval.

Proof. Suppose that $A \subseteq \mathbb{R}$ is not an interval. Then by (a) there are points $a, b \in A$ with $a < b$ and $x \in \mathbb{R} \setminus A$ such that $a < x < b$. Then in the subspace topology, the sets $A \cap (-\infty, x)$ and $A \cap (x, +\infty)$ are open, proper, and nonempty. Moreover

$$A \setminus (A \cap (-\infty, x)) = A \cap (x, +\infty),$$

so these sets are also closed. Thus A can be written as the union of two disjoint, proper, clopen sets, so A is not connected. □

Problem 5

Prove the following generalization of Lemma 2.6. If $\{D_j : j \in J\}$ is a collection of connected subsets of X and if for each j and k in J we have $D_j \cap D_k \neq \emptyset$ then

$$D = \bigcup_{j \in J} D_j$$

is connected.

Proof. Let A be a nonempty clopen subset of D. Then $D = A \sqcup (D \setminus A)$ so it suffices to show that $A = D$. By definition of the subspace topology, $A \cap D_i$ is clopen in D_i for all $i \in J$. As each D_i is connected, it follows that $A \cap D_i = D_i$ or $A \cap D_i = \emptyset$. As A is nonempty, there is some D_k with $A \cap D_k \neq \emptyset$, whence $A \cap D_k = D_k$. By assumption, for each $i \in J$ there is some $x_i \in D_i \cap D_k$. Hence, for fixed $i \in J$, $x_i \in A$ whence $x_i \in A \cap D_i$. This implies that $A \cap D_i$ is nonempty, so connectedness gives $A \cap D_i = D_i$. Hence, for all $i \in J$, $D_i \subseteq A$, so

$$A = A \cap D = A \cap \bigcup_{j \in J} D_j = \bigcup_{j \in J} A \cap D_j = \bigcup_{j \in J} D_j = D.$$

$\square$

Problem 6

Show that if $F \subset X$ is closed and connected then for every pair of points a, b in F and each $\varepsilon > 0$ there are points $z_0, z_1, \ldots, z_n$ in F with $z_0 = a$, $z_n = b$ and $d(z_{k-1}, z_k) < \varepsilon$ for $1 \leq k \leq n$. Is the hypothesis that F be closed needed? If F is a set which satisfies this property then F is not necessarily connected, even if F is closed. Give an example to illustrate this.

Proof. We construct an analgoue of connected components for this notion of ε-ball connectedness. We denote such an ε-bounded sequence of points by $(z_0, z_1, \ldots, z_n)$. Fix $a \in F$. Let

$$\begin{aligned}
G_0 &= F \cap B_\varepsilon(a) \\
G_1 &= F \cap \bigcup_{x \in G_0} B_\varepsilon(x) \\
&\vdots \\
G_{n+1} &= F \cap \bigcup_{x \in G_n} B_\varepsilon(x) \\
&\vdots
\end{aligned}$$

Note that each of the G_ns are open in the subspace topology of F. Let $C_a := \bigcup_{n=0}^{\infty} G_n$, the ε-ball component of the point $a \in F$. Suppose now $a, b \in F$, $a \neq b$, and assume that $C_a \cap C_b \neq \emptyset$. Let $p \in C_a \cap C_b$ and let the corresponding sequences of ε-close points be $(a, z_1, z_2, \ldots, z_n, p)$, $(b, w_1, w_2, \ldots, w_m, p)$. Fix $q \in C_a$ and let q have corresponding path $(a, u_1, u_2, \ldots, u_k, q)$. Then the concatenated path

$$(b, w_1, \ldots, w_m, p, z_n, z_{n-1}, \ldots, z_1, a, u_1, u_2, \ldots, u_k, q)$$

furnishes an ε-close sequence of points from b to q, whence $q \in C_b$. Hence $C_a \subseteq C_b$, so by the symmetry of a and b it follows that $C_a = C_b$. So the sets $\{C_a\}_{a \in F}$ partition F. Suppose, for the sake of contradiction, that $|\{C_a\}_{a \in F}| > 1$. Pick $a \in F$ and set

$$D = C_a, \quad E = \bigcup_{\substack{b \in F \\ C_b \neq C_a}} C_b.$$

These sets are clopen in the subspace topology as they are open, disjoint, and have $D \cup E = F$, which contradicts the connectedness of F. Thus $F = C_a = C_b$ for all $b \in F$, whence any two points may be reached by an ε-bounded sequence.

(Closedness): As all topological considerations dealt with the subspace topology for F, the closedness is not necessary.

(Counterexample for connectedness): The connectedness assumption is not implied. Consider $F := \{(x, \frac{1}{x^2}) : x \in \mathbb{R} \setminus \{(0,0)\}\}$. Points on both sides of the graph become arbitrarily close, so eventually given any ε a ball will intersect both sides of the curve furnishing the required paths.

$\square$