The Total Area Statistic for Fuss-Catalan Paths

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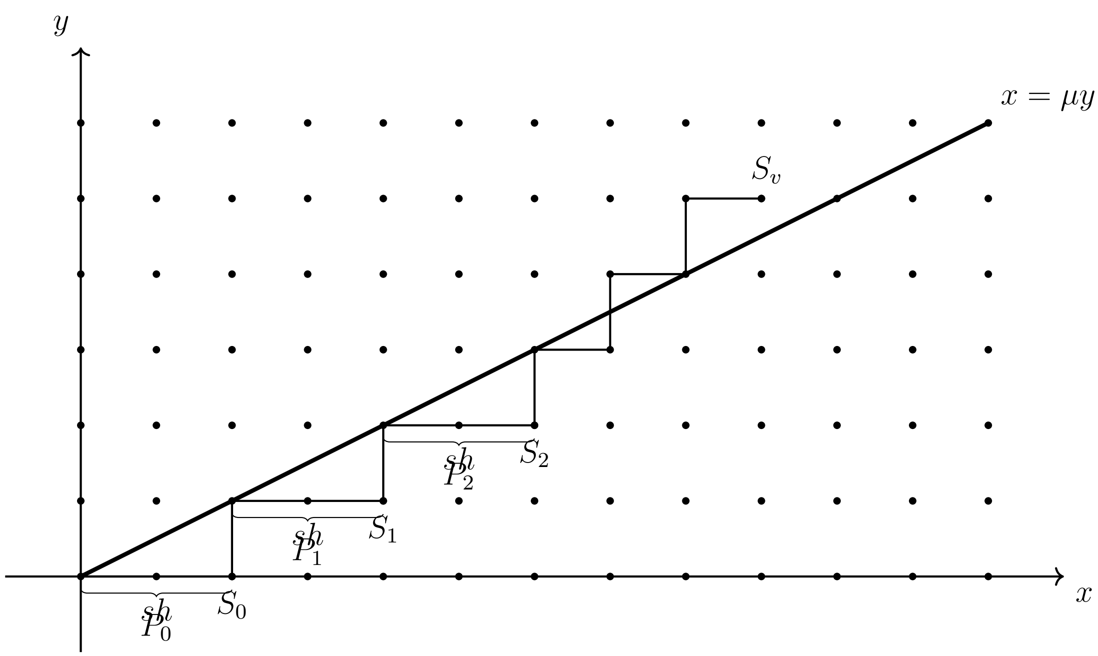
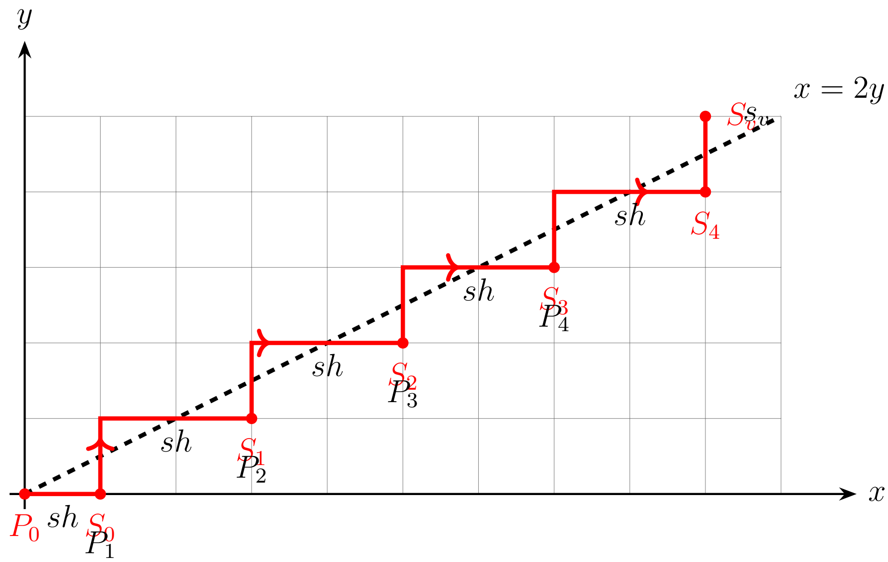
1 Preliminaries





2 Fuss-Catalan Preliminaries

$$F_m(n,k) = \frac{k}{mn+k}\binom{mn+k}{n}$$

$$F_m(n) = F_m(n,1) = \frac{1}{mn+1}\binom{mn+1}{n} = \frac{1}{(m-1)n+1}\binom{mn}{n}$$

$$F_2(n) = F_2(n,1) = c_n$$

$$\mathcal{D}_n^s := L((0,0) \to (sn,n) : x \geq sy), \quad |\mathcal{D}_n^s| = F_{s+1}(n) = \frac{1}{sn+1}\binom{(s+1)n}{n} = \frac{1}{(s+1)n+1}\binom{(s+1)n+1}{n}$$

$$F_m(z) = \sum_{n\geq 0} F_m(n,1)z^n, \quad (F_m(z))^k = \sum_{n\geq 0} F_m(n,k)z^n, \quad F_m(z) = 1 + z(F_m(z))^m$$

3 Recursive Decomposition

Let

$$A_n = \sum_{P\in\mathcal{D}_n^s} \operatorname{area}(P)$$

$$\begin{aligned}
A_{n+1} &= \sum_{\substack{0\le y_0\le\cdots\le y_{s-1}\le n\\ y_s=n}} \sum_{\substack{P\in\mathcal{D}^s_{n+1}\\ \text{of type } y_0,\dots,y_{s-1}}} \text{area}(P) \\
&= \sum_{\substack{0\le y_0\le\cdots\le y_{s-1}\le n\\ y_s=n}} \sum_{\substack{P\in\mathcal{D}^s_{n+1}\\ \text{of type } y_0,\dots,y_{s-1}}} \left(\sum_{l=0}^{s}\text{area}(P_l)+\sum_{k=1}^{s} I_l\right) \\
&= \sum_{\substack{0\le y_0\le\cdots\le y_{s-1}\le n\\ y_s=n}} \sum_{\substack{P\in\mathcal{D}^s_{n+1}\\ \text{of type } y_0,\dots,y_{s-1}}} \left(\sum_{l=1}^{s}\frac{s\sqrt{2}(y_l-y_{l-1})}{\sqrt{1+s^2}}+\sum_{l=0}^{s}\text{area}(P_l)\right) \\
&= \sum_{\substack{0\le y_0\le\cdots\le y_{s-1}\le n\\ y_s=n}} \sum_{\substack{P\in\mathcal{D}^s_{n+1}\\ \text{of type } y_0,\dots,y_{s-1}}} \left(\frac{s\sqrt{2}(y_s-y_0)}{\sqrt{1+s^2}}+\sum_{l=0}^{s}\text{area}(P_l)\right) \\
&= \frac{sn\sqrt{2}}{\sqrt{1+s^2}}\cdot F_{s+1}(n+1)-\frac{s\sqrt{2}}{\sqrt{1+s^2}}\sum_{k=0}^{n}\sum_{\substack{P\in\mathcal{D}^s_{n+1}\\ \text{with } y_0=k}} k+\sum_{\substack{0\le y_0\le\cdots\le y_{s-1}\le n\\ y_s=n}} \sum_{\substack{P\in\mathcal{D}^s_{n+1}\\ \text{of type } y_0,\dots,y_{s-1}}} \sum_{l=0}^{s}\text{area}(P_l)
\end{aligned}$$

Use an analogue of the generalized ballot problem to handle the sum below. See here

Lemma 3.0.1. *The number of paths in D_n^s which only touch $x = sy$ at $(0,0)$ and (sn,n) is given by*

$$\frac{1}{n}\binom{(s+1)n-2}{n-1}$$

As an intermediate computation, we note that

$$\begin{aligned}
\frac{1}{n+1}\binom{(s+1)(n+1)-2}{n} &= \frac{1}{n+1}\binom{(s+1)n+s-1}{n} \\
&= \frac{1}{n+1}\frac{(s+1)n+s-n}{(s+1)n+s}\binom{(s+1)n+s}{n} \\
&= \frac{1}{n+1}\frac{s(n+1)}{(s+1)n+s}\binom{(s+1)n+s}{n} \\
&= \frac{s}{(s+1)n+s}\binom{(s+1)n+s}{n} = F_{s+1}(n,s).
\end{aligned}$$

Now we may proceed to handle this sum by writing

$$\begin{aligned}
\sum_{k=0}^{n}\sum_{\substack{P\in\mathcal{D}^s_{n+1}\\ \text{with } y_0=k}} k &= \sum_{k=0}^{n} k\cdot|\{P\in D^s_{n+1} : y_0=k\}| \\
&= \sum_{k=0}^{n} k|\mathcal{D}^s_k|\cdot|\{P\in\mathcal{D}^s_{n+1-k} : P \text{ only touches } x=sy \text{ at extremities}\}| \\
&= \sum_{k=0}^{n} kF_{s+1}(k)\cdot\frac{1}{n+1-k}\binom{(s+1)(n+1-k)-2}{n-k} \\
&= \sum_{k=0}^{n} kF_{s+1}(k)\cdot F_{s+1}(n-k,s)
\end{aligned}$$

The last piece of the main expression to deal with is the area recursive sum. Observe that

$$\sum_{\substack{0\le y_0\le\cdots\le y_{s-1}\le n\\ y_s=n}}\ \sum_{\substack{P\in\mathcal{D}^s_{n+1}\\ \text{of type } y_0,\ldots,y_{s-1}}}\ \sum_{l=0}^{s}\text{area}(P_l)=\sum_{l=0}^{s}\ \sum_{\substack{0\le y_0\le\cdots\le y_{s-1}\le n\\ y_s=n}}\ \sum_{\substack{P\in\mathcal{D}^s_{n+1}\\ \text{of type } y_0,\ldots,y_{s-1}}}\text{area}(P_l)$$

$$=\sum_{\substack{0\le y_0\le\cdots\le y_{s-1}\le n\\ y_s=n}}\ \sum_{\substack{P\in\mathcal{D}^s_{n+1}\\ \text{of type } y_0,\ldots,y_{s-1}}}\text{area}(P_0)+\sum_{l=1}^{s}\ \sum_{\substack{0\le y_0\le\cdots\le y_{s-1}\le n\\ y_s=n}}\ \sum_{\substack{P\in\mathcal{D}^s_{n+1}\\ \text{of type } y_0,\ldots,y_{s-1}}}\text{area}(P_l)$$

The first part of this sum we compute as

$$\sum_{\substack{0\le y_0\le\cdots\le y_{s-1}\le n\\ y_s=n}}\ \sum_{\substack{P\in\mathcal{D}^s_{n+1}\\ \text{of type } y_0,\ldots,y_{s-1}}}\text{area}(P_0)=\sum_{k=0}^{n}\sum_{\gamma\in\mathcal{D}^s_k}\sum_{\substack{P\in\mathcal{D}^s_{n+1}\\ P_0=\gamma}}\text{area}(\gamma)$$

$$=\sum_{k=0}^{n}\sum_{\gamma\in\mathcal{D}^s_k}\text{area}(\gamma)\cdot|\{P\in\mathcal{D}^s_{n+1}:P_0=\gamma\}|$$

$$=\sum_{k=0}^{n}A_k\cdot F_s(n+1-k)$$

For the second part of this sum, we sum over the start of P_l and length of P_l. Assume first that $0<l<s$

$$\sum_{k=0}^{n}\sum_{\substack{P\in D^s_{n+1}\\ \text{with } y_l=k}}\text{area}(P_l)=\sum_{k=0}^{n}\sum_{m=k}^{n}\sum_{\gamma\in D^s_{m-k}}\ \sum_{\substack{P\in D^s_{n+1}\\ \text{s.t. } y_{l-1}=k,\, y_l=m\\ P_l=\gamma}}\text{area}(\gamma)$$

$$=\sum_{k=0}^{n}\sum_{m=k}^{n}\sum_{\gamma\in D^s_{m-k}}\text{area}(\gamma)\cdot|\{P\in D^s_{n+1}:y_{l-1}=k,\ y_l=m,\ P_l=\gamma\}|$$

$$|\{P\in\mathcal{D}^s_{n+1}\text{of type }(y_0,\ldots,y_{s-1}),\text{ with }P_l=\gamma\}|=\prod_{\substack{1\le i\le s\\ i\ne l}}|F_s(y_i-y_{i-1})|$$

Temporarily consider the sequence

$$B_k:=\frac{A_k}{F_s(k)}$$

Fix

$$\sum_{\substack{0\le y_0\le\cdots\le y_{s-1}\le n\\ y_s=n}}\ \sum_{\substack{P\in\mathcal{D}^s_{n+1}\\ \text{of type } y_0,\dots,y_{s-1}}} \operatorname{area}(P_l) = \sum_{\substack{0\le y_0\le\cdots\le y_{s-1}\le n\\ y_s=n}}\ \sum_{\gamma\in\mathcal{D}^s_{y_l-y_{l-1}}}\ \sum_{\substack{P\in\mathcal{D}^s_{n+1}\\ \text{of type } y_0,\dots,y_{s-1}\\ P_l=\gamma}} \operatorname{area}(\gamma)$$

$$= \sum_{\substack{0\le y_0\le\cdots\le y_{s-1}\le n\\ y_s=n}}\ \sum_{\gamma\in\mathcal{D}^s_{y_l-y_{l-1}}} \operatorname{area}(\gamma)\cdot|\{P\in\mathcal{D}^s_{n+1}\text{of type }(y_0,\dots,y_{s-1}),\text{ with } P_l=\gamma\}|$$

$$= \sum_{\substack{0\le y_0\le\cdots\le y_{s-1}\le n\\ y_s=n}}\ \sum_{\gamma\in\mathcal{D}^s_{y_l-y_{l-1}}} \operatorname{area}(\gamma)\cdot \prod_{\substack{1\le i\le s\\ i\ne l}} |F_s(y_i-y_{i-1})|$$

$$= \sum_{\substack{0\le y_0\le\cdots\le y_{s-1}\le n\\ y_s=n}} A_{y_l-y_{l-l}}\cdot\left(\prod_{\substack{1\le i\le s\\ i\ne l}} |F_s(y_i-y_{i-1})|\right)$$

$$= \sum_{d_1=0}^{n}\sum_{d_2=0}^{n-d_1}\cdots\sum_{d_{s-1}=0}^{n-\sum_{j=1}^{s-2}d_j}\ \sum_{\substack{y_{s-1}=n-\sum_{j=1}^{s-1}d_j\\ d_s=n-y_{s-1}}}^{n} A_{d_l}\cdot\left(\prod_{\substack{1\le i\le s\\ i\ne l}}|F_s(d_i)|\right)$$

$$= \sum_{d_1=0}^{n}F_s(d_1)\sum_{d_2=0}^{n-d_1}F_s(d_2)\cdots\sum_{d_l=0}^{n-\sum_{j=1}^{l-1}d_j}F_s(d_l)B_{d_l}\cdots\sum_{d_{s-1}=0}^{n-\sum_{j=1}^{s-2}d_j}F_s(d_{s-1})\sum_{k=n-\sum_{j=1}^{s-1}d_j}^{n}F_s(n-k)$$

Define generating functions $P_l(z)$ and $Q_l(z)$ by

$$P_l(z)=\frac{1}{(1-z)^l}=\sum_{n\ge0}\binom{n+l-l}{l-1}z^n,\quad Q_l(z)=\frac{1}{(1-z)^{s-l}}=\sum_{n\ge0}\binom{n+s-1-l}{s-1-l}z^n.$$

If $0<l<s$, then we compute

$$\sum_{\substack{0\le y_0\le\cdots\le y_{s-1}\le n\\ y_s=n}} A_{y_l-y_{l-l}} = \sum_{0\le a\le b\le n} A_{b-a}\cdot|\{0\le y_0\le\cdots\le y_{s-1}\le n: y_l=b,\ y_{l-1}=a\}|$$

$$= \sum_{0\le a\le b\le n} A_{b-a}\cdot|\{0\le y_0\le\cdots\le y_{l-2}\le a\}|\cdot|\{b\le y_{l+1}\le\cdots\le y_{s-1}\le n\}|$$

$$= \sum_{0\le a\le b\le n} A_{b-a}\cdot\binom{(a+1)+(l-1)-1}{l-1}\cdot\binom{(n-b+1)+(s-1-l)-1}{s-1-l}$$

$$= \sum_{0\le a\le b\le n} A_{b-a}\cdot\binom{a+l-1}{l-1}\cdot\binom{n-b+s-l-1}{s-1-l}$$

$$\overset{k=b-a}{=} \sum_{k=0}^{n}A_k\sum_{a=0}^{n-k}\binom{a+l-1}{l-1}\cdot\binom{(n-k)-a+s-l-1}{s-1-l}$$

$$= \sum_{k=0}^{n}A_k[P_l(z)Q_l(z)]_{n-k} = [A(z)P_l(z)Q_l(z)]_n$$

If $l = s$, then

$$\begin{aligned}
\sum_{\substack{0 \le y_0 \le \cdots \le y_{s-1} \le n \\ y_s = n}} A_{y_l - y_{l-l}} &= \sum_{a=0}^{n} A_{n-a} \cdot |\{0 \le y_0 \le \cdots \le y_{s-2} \le a\}| \\
&= \sum_{a=0}^{n} A_{n-a} \cdot \binom{(a+1)+(s-1)-1}{s-1} \\
&= \sum_{a=0}^{n} A_{n-a} \cdot \binom{a+s-1}{s-1} = [A(z)P_s(z)]_n
\end{aligned}$$

Note that if $l = s$, then $Q_l = 1$, whence in fact we have the formulae

$$\sum_{\substack{0 \le y_0 \le \cdots \le y_{s-1} \le n \\ y_s = n}} A_{y_l - y_{l-1}} = [A(z)P_l(z)Q_l(z)]_n$$

for all $0 < l \le s$ (maybe even works for $l = 0$ idk yet).

$$\begin{aligned}
A_{n+1} &= \frac{sn\sqrt{2}}{\sqrt{1+s^2}} \cdot F_{s+1}(n+1) - \frac{s\sqrt{2}}{\sqrt{1+s^2}} \sum_{k=0}^{n} \sum_{\substack{P \in \mathcal{D}_{n+1}^s \\ \text{with } y_0 = k}} k + \sum_{\substack{0 \le y_0 \le \cdots \le y_{s-1} \le n \\ y_s = n}} \sum_{\substack{P \in \mathcal{D}_{n+1}^s \\ \text{of type } y_0, \ldots, y_{s-1}}} \sum_{l=0}^{s} \text{area}(P_l) \\
&= \frac{sn\sqrt{2}}{\sqrt{1+s^2}} \cdot F_{s+1}(n+1) - \frac{s\sqrt{2}}{\sqrt{1+s^2}} \sum_{k=0}^{n} kF_{s+1}(k) \cdot F_{s+1}(n-k, s) + \sum_{k=0}^{n} A_k \cdot F_s(n+1-k) \\
&\quad + \sum_{l=1}^{s} \left(\prod_{\substack{1 \le i \le s \\ i \ne l}} |F_s(y_i - y_{i-1})| \right) \cdot [A(z)P_l(z)Q_l(z)]_n
\end{aligned}$$