Talk 1 and Paper 1 Abstracts and Outlines

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1 Abstracts

Talk Abstract

"Groups, as men, shall be know by their actions" -Guillermo Moreno. In this talk, we give an example based exploration of the field of representation theory and its relations to combinatorics. We hint towards a deep connection between the combinatorics of partitions and the actions of symmetric groups on vector spaces. The only background required will be linear algebra and some knowledge of group theory.

Paper Abstract

In this paper we exposit one of the fundamental results linking representation theory and algebraic combinatorics called Schur-Weyl duality. It provides a dictionary between the representation theory of finite symmetric groups and the representation theory of the general linear group of a finite dimensional complex vector space. Through this dictionary, we obtain representation theoretic constructions of many aspects of symmetric function theory, including Schur functions, Kostka numbers, and internal/external products on the symmetric function ring.

2 Outlines

2.1 Talk Outline

- Run through the standard example $\pi : D_{2n} \to O(3)$
- Introduce definition of representations of finite groups.
- Get relevant defs to say $V \otimes V \cong Sym^2 \oplus \Lambda^2 V$.
- State $V \otimes V \otimes V \cong Sym^3 V \oplus \Lambda^3 V \oplus$ something else .
- Hint how that something else is related to the partition $(2, 1)$ of 3.

2.2 Paper Outline

- Set up relevant preliminary representation theoretic definitions (group algebra stuff etc.
- Talk about the generic representation theory of S_n
- Define Young symmetrizers and Specht Modules
- Talk about the commuting left and right actions of $GL(V)$ and S_n respectively.
- State and cite the double centralizer theorem to prove Schur-Weyl duality.
- Obtain relations to Schur functions,
  - Schur functions as characters of Schur functors (hints towards categorification). Namely, if $g \in \mathrm{GL}(V)$ has eigenvalues $\alpha_1, \dots, \alpha_n$, then

    $$\chi_{\mathbb{S}_\lambda(V)}(g) = s_\lambda(\alpha_1, \alpha_2, \dots, \alpha_n, 0, 0, \dots)$$

  - Isometric isomorphism between the representation ring $\mathcal{R}(S_n)$ (viewed as the $\mathbb{Z}$-module generated by irreducible complex characters with multiplication defined by inducing representations along the inclusion $S_m \times S_n \hookrightarrow S_{m+n}$) and $\mathbb{Z}[x_1, \dots, x_n]^{S_n}$ under the Frobenius characteristic map

    $$\mathrm{ch}(f) = \frac{1}{n!} \sum_{\lambda \vdash n} \frac{f(\lambda)}{z_\lambda} p_\lambda$$

    where p_λ denotes a power sum symmetric function and $z_\lambda = \frac{n!}{K_\lambda}$ where K_λ is the size of the conjugacy class of S_n determined by λ.