Intro Math Research Hw2

James Harbour

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Preliminary Considerations

Throughout this article, fix a (unital) commutative ring R and a field k. For simplicity, we work over vector spaces instead of general modules.

Notation. Let X be a set such that $X = \{x_i\}_{i\in I}$ for some indexing set I. By $k[X]$ and $k[\![X]\!]$, we denote the rings of (commutative) polynomials and power series (resp.) in indeterminates $\{x_i\}$. We utilize multi-index notation throughout. Hence for $\alpha_\cdot : I \to \mathbb{N} \cup \{0\}$ finitely supported, we write $x_\alpha = \Pi_{i\in I} x_i^{\alpha_i}$ (where $x_i^0 := 1$ formally).

2.1 Algebraic Background

Many common algebraic objects possess both a vector space structure and an internal product structure. For example, the set of $n \times n$ matrices over k, denoted $M_n(k)$, has a product given by matrix multiplication and is also a vector space under addition and scalar multiplication by k.

Definition 2.1.1. Let A be a k-vector space equipped with a map $\cdot : A \times A \to A$ (written $(x, y) \mapsto x \cdot y$). The pair $(A, \cdot)$ is a *k-algebra* if, for $x, y, z \in A$ and $a, b \in k$, the following hold:

- $(x + y) \cdot z = x \cdot z + y \cdot z$,
- $z \cdot (x + y) = z \cdot x + z \cdot y$,
- $(ax) \cdot (by) = (ab)(x \cdot y)$.

Key Example. For $X = \{x_i\}_{i\in I}$, the rings $k[X]$ and $k[\![X]\!]$ form the prototypical example of k-algebras.

Often in algebra, elements of a given object may be decomposed into a sum of simpler elements which are, in a sense, "homogenous." For example, any polynomial in n-variable may be decomposed into a sum of simpler polynomials each of which are futher sums of monomials of the same total degree. In this way, a polynomial is split into a sum of homogenous parts. This behavior is codified in the notion of *grading*.

Definition 2.1.2. A *graded k-algebra* is a k-algebra A together with a direct sum decomposition

$$A = \bigoplus_{i=0}^{\infty} A_i$$

with $A_0, A_1, \ldots$ vector spaces such that $A_i \cdot A_j \subseteq A_{i+j}$ for all $i, j \in \mathbb{N} \cup \{0\}$. For fixed i, elements of A_i are called *homogenous*. The choice of such a direct sum decomposition is a *grading* for A.

Key Example. As before, for $X = \{x_i\}_{i\in I}$, we may give the ring $k[X]$ a canonical grading by declaring $A_0 := k$ and

$$A_n := \mathrm{Span}_k\{x_\alpha : \alpha \text{ multi-index such that } \sum_{i\in I} \alpha_i = n\}.$$

The reader is cautioned that not every k-algebra has a nontrivial grading. In fact, it can be shown that the ring of formal power series $k[\![x]\!]$ does not have a nontrivial grading.

2.2 Symmetric Polynomials

Definition 2.2.1. The permutation group S_n acts naturally on the polynomial ring $k[x_1, \ldots, x_n]$ by defining $\sigma \cdot x_{i_1}^{\alpha_1} \cdots x_{i_l}^{\alpha_l} := x_{\sigma(i_1)}^{\alpha_1} \cdots x_{\sigma(i_l)}^{\alpha_l}$ and extending by linearity. The ring of *symmetric polynomials* in n indeterminates is the fixed points of this action, namely $k[x_1, \ldots, x_n]^{S_n}$.

2.3 Compositions and Partitions

Definition 2.3.1.

- A *partition* of $n \in \mathbb{N}$ is a set $\alpha = \{\alpha_1, \ldots, \alpha_l\}$ of positive integers which sum to n. We denote the set of partitions of n by $\mathrm{Par}(n)$. We denote the statement $[\lambda \in \mathrm{Par}(n)]$ by $\lambda \vdash n$. Also, we write $\mathrm{Par} := \bigcup_{n \geq 0} \mathrm{Par}(n)$.

- A *weak composition* of $n \in \mathbb{N}$ is a (finitely supported) sequence $\alpha = (\alpha_i)_{i=1}^{\infty} \in (\mathbb{N} \cup \{0\})^{\mathbb{N}}$ such that $\sum_i \alpha_i = n$. The length of a weak composition α is given by

$$l(\alpha) := \max\{i \in \mathbb{N} : \alpha_i \neq 0\}.$$

2.4 Symmetric Functions

Definition 2.4.1 (pg. 308 in [Sta24])**.** The ring Λ_k of symmetric functions over a field k is the subring of all $f \in k[\![x_1, x_2, \ldots]\!]$ such that

$$f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots) = f(x_1, x_2, \ldots) \text{ for all } \sigma \in \mathrm{Sym}(\mathbb{N}).$$

Remark 2.4.1. For the algebraically-minded, there is a more natural construction of Λ_k by viewing the ring as the colimit of a certain directed system of injections of symmetric polynomial rings

$$k[x_1, \ldots, x_n]^{S_n} \overset{\varphi_n}{\hookrightarrow} k[x_1, \ldots, x_{n+1}]^{S_{n+1}}.$$

The construction of these maps φ_n is somewhat involved. This does justify the intuition that a symmetric function is simply taking a symmetric polynomial and adding more data, as any element of a direct limit of inclusions is faithfully represented by an element of one of the constituent objects.

Definition 2.4.2. A symmetric function $f \in \Lambda_k$ is homogenous of degree n if

$$f(x) = \sum_{\alpha \text{ weak composition of } n} c_\alpha x^\alpha,$$

where the c_α are elements of k. The set of degree n homogenous symmetric functions is denoted Λ_k^n. these subspaces give Λ_k the structure of a graded k-algebra, namely:

- Each Λ_k^n is a k-vector space,
- $\Lambda_k^i \Lambda_k^j \subseteq \Lambda_k^{i+j}$,
- $\Lambda_k = \bigoplus_{n=0}^{\infty} \Lambda_k^n$ as k-vector spaces.

The first interesting basis of Λ_k is the *monomial symmetric functions*. Given $\lambda \vdash n$, define $m_\lambda \in \Lambda_k^n$ by

$$m_\lambda := \sum_\alpha x^\alpha$$

where the sum is over all distinct permutations of the entries of λ. The set $\{m_\lambda : \lambda \vdash n\}$ forms a basis for Λ_k^n, whence $\bigcup_{n \geq 0} \{m_\lambda : \lambda \vdash n\} = \{m_\lambda : \lambda \in \mathrm{Par}\}$ forms a basis for Λ_k.

2.5 Complete Homogenous Symmetric Functions

From the monomial symmetric functions, we may form another interesting basis for Λ_k called the *complete homogenous symmetric functions* h_λ by setting

$$h_\lambda := \prod_{i=1}^{\infty} \sum_{\nu \vdash \lambda_i} m_\nu.$$

where $\lambda = (\lambda_1, \lambda_2, \ldots)$. Again, the set $\{h_\lambda : \lambda \vdash n\}$ is a basis for Λ_k^n and the set $\{h_\lambda : \lambda \in \mathrm{Par}\}$ is a basis for Λ_k.