A Poisson Boundary Theory for Groupoids

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Goal: Develop a theory of Poisson boundary for groupoids using Das-Peterson's noncommutative PB framework from [DP22].

1 Setup

1.1 Groupoid Side

Let $(\mathcal{G}, \mu)$ be a discrete pmp groupoid and $X = \mathcal{G}^0$. Assume we have a collection $\{\pi_x\}_{x\in X}$ such that $\pi_x \in \mathrm{Prob}(\mathcal{G}^x)$ for all $x \in X$. Extend each π_x by zero to be defined on of $\mathcal{G}$. For $g \in \mathcal{G}$, set

$$\pi_g = g_*\pi_{s(g)}$$

In the framework of [Kai05], $\{\pi_g\}$ give the transition probabilites for a Markov operator on $\mathcal{G}$. Such a Markov operator is called *invariant* if $g_*\pi_h = \pi_{gh}$ for all $(g, h) \in \mathcal{G}^{(2)}$.

Definition 1.1.1 ([Kai05])**.** A family $\pi = \{\pi_g\}$ of transition probabilities is called Borel if for every non-negative Borel function f, the function $\pi(f) : \mathcal{G} \to \mathbb{C}$ given by $\pi(f)(g) = \int_{\mathcal{G}} f\, d\pi_g$ is Borel.

Given such a Borel family π, we then get an induced Markov operator $P : \mathrm{Bor}(\mathcal{G}) \to \mathrm{Bor}(\mathcal{G})$ given by $Pf = \pi(f)$. The corresponding *dual operator* $\widetilde{P} : M_+(\mathcal{G}) \to M_+(\mathcal{G})$ is then given by

$$\widetilde{P}(\theta) = \int_{\mathcal{G}} \pi_g\, d\theta(g) \text{ for all } \theta \in M_+(\mathcal{G}).$$

Now by definition of the vector-valued integral,

$$\begin{aligned}\langle \theta, Pf\rangle = \int_{\mathcal{G}} Pf(g)\, d\theta(g) &= \int_{\mathcal{G}} \left(\int_{\mathcal{G}} f\, d\pi_g\right) d\theta(g) \\ &= \int_{\mathcal{G}} f\, d\widetilde{P}\theta = \langle \widetilde{P}\theta, f\rangle\end{aligned}$$

1.2 Von Neumann Algebras Side

Fix a tracial von Neumann algebra (M, τ) and an embedding $M \subseteq \mathcal{A}$ into a C^* algebra $\mathcal{A}$.

$$S_\tau\mathcal{A}) := \{\varphi \in S(\mathcal{A}) : \varphi|_M = \tau\}.$$

Fixing $\varphi \in S_\tau(\mathcal{A})$ gives an inclusion $L^2(M, \tau) \subseteq L^2(\mathcal{A}, \varphi)$. Let $e_M = Proj_{L^2(M,\tau)} \in B(L^2(\mathcal{A}, \varphi))$. Define a u.c.p. map $\mathcal{P}_\varphi : \mathcal{A} \to B(L^2(M, \tau))$, by

$$\mathcal{P}_\varphi(T) := e_M T e_m \text{ for } T \in \mathcal{A}$$

For $x \in M$, $\mathcal{P}_\varphi(x) = x$. The map $\mathcal{P}_\varphi$ is the *Poisson transform* of the inclusion $M \subseteq \mathcal{A}$.

References

[DP22] Sayan Das and Jesse Peterson. "Poisson boundaries of II1 factors". In: *Compos. Math.* 158.8 (2022), pp. 1746–1776.

[Kai05] Vadim A. Kaimanovich. "Amenability and the Liouville property". In: vol. 149. Probability in mathematics. 2005, pp. 45–85.