220A Homework 2

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Problem 1

Prove the following statements.
(a): A set is closed if and only if it contains all its limit points.

Proof. Here we use the definition of closedness as being the complement of an open set.

($\implies$): Suppose that $A \subseteq X$ is closed, so $X \setminus A$ is open. Suppose, for the sake of contradiction, that $x \in X \setminus A$ is a limit point of A. Then for all $\varepsilon > 0$, $(B_\varepsilon(x) \setminus \{x\}) \cap A \neq \emptyset$, whence $B_\varepsilon(x) \not\subseteq X \setminus A$. Thus no ball around x can be contained in $X \setminus A$, contradicting that $X \setminus A$ is open.

($\impliedby$): We proceed by contraposition. Suppose that $A \subseteq X$ is not closed. Then by definition $X \setminus A$ is not open, so there exists some $x \in X \setminus A$ such that $B_\varepsilon(x) \not\subseteq X \setminus A$ for all $\varepsilon > 0$. So for all $\varepsilon > 0$, there is some $x_\varepsilon \in B_\varepsilon(x)$ with $x_\varepsilon \notin X \setminus A$, i.e. $x_\varepsilon \in A$. Thus $x \in X \setminus A$ is a limit point of A by definition, whence we have found a limit point of A which is not in A. □

(b): If $A \subseteq X$ then $\overline{A} = A \cup \{x : x \text{ is a limit point of } A\}$.

Proof. By definition

$$\overline{A} = \bigcap_{\substack{C \supseteq A \\ C \text{ closed}}} C.$$

As this family is nonempty (since X is closed and $X \supseteq A$), it follows that $A \subseteq \overline{A}$. Suppose that x is a limit point of A. Let $C \subseteq X$ be a closed set such that $A \subseteq C$. As x is a limit point of A, it is also a limit point of C, whence by part (a) we have that $x \in C$. As C was arbitrary, it follows that $x \in \overline{A}$. Thus $A \cup \{x : x \text{ is a limit point of } A\} \subseteq \overline{A}$.

On the other hand suppose $x \in \overline{A} \setminus A$. We wish to show that x is a limit point of A. Let $\varepsilon > 0$. It suffices to show that $A \cap (B_\varepsilon(x) \setminus \{x\}) \neq \emptyset$. As $x \notin A$, this is equivalent to showing that $A \cap B_\varepsilon(x) \neq \emptyset$.

Suppose, for the sake of contradiction, that $A \cap B_\varepsilon(x) = \emptyset$. Then $A \subseteq X \setminus B_\varepsilon(x)$, which is a closed subset of X, so by definition $\overline{A} \subseteq X \setminus B_\varepsilon(x)$. By assumption, $x \in \overline{A}$, but $x \notin X \setminus B_\varepsilon(x)$, which gives a contradiction. □

Problem 2

Let z_n, z be points in $\mathbb{C}$ and let d be the metric on $\mathbb{C}_\infty$. Show that $|z_n - z| \to 0$ if and only if $d(z_n, z) \to 0$. Also show that if $|z_n| \to \infty$ then $\{z_n\}$ is Cauchy in $\mathbb{C}_\infty$. (Must $\{z_n\}$ converge in $\mathbb{C}_\infty$?)

Proof.

$$d(z_n, z) = \frac{2|z_n - z|}{\sqrt{(1+|z_n|^2)(1+|z|^2)}}$$

Suppose that $|z_n - z| \xrightarrow{n\to\infty} 0$. Noting that for all $w \in \mathbb{C}$, $\sqrt{1+|w|^2} \geq 1$, it follows that

$$d(z_n, z) = \frac{2|z_n - z|}{\sqrt{(1+|z_n|^2)(1+|z|^2)}} \leq 2|z_n - z| \xrightarrow{n\to\infty} 0.$$

Now suppose that $d(z_n, z) \xrightarrow{n\to\infty} 0$. Let $Z_n, Z \in \mathbb{R}^3$ be the points in the Riemann sphere corresponding to z_n, z respectively. Then

$$\|Z_n - Z\|_2 = d(z_n, z) \xrightarrow{n\to\infty} 0.$$

Letting $N = (0,0,1)$ be the north pole in the Riemann sphere, as $z \in \mathbb{C}$ we have $\|Z - N\|_2 > 0$. Choose $M \in \mathbb{N}$ such that for $n \geq M$, we have $\|Z_n - Z\|_2 < \frac{1}{2}\|Z - N\|$. Then for $n \geq M$, by the reverse triangle inequality we have

$$\|Z_n - N\|_2 \geq \|Z - N\|_2 - \|Z_n - Z\|_2 > \frac{1}{2}\|Z - N\|_2 > 0.$$

Let $b := \frac{1}{2}\|Z - N\|_2 > 0$. Then observe that for $n \geq M$,

$$b < \|Z_n - N\|_2 = d(z_n, \infty) = \frac{2}{\sqrt{1+|z_n|^2}},$$

whence it follows that

$$\begin{aligned} d(z_n, z) = \frac{2|z_n - z|}{\sqrt{1+|z_n|^2}\sqrt{1+|z|^2}} &= \frac{2}{\sqrt{1+|z_n|^2}} \cdot \frac{1}{\sqrt{1+|z|^2}} \cdot |z_n - z| \\ &> \frac{b}{\sqrt{1+|z|^2}} \cdot |z_n - z| \end{aligned}$$

Fix $\varepsilon > 0$ and choose $L \in \mathbb{N}$ such that for all $n \geq L$,

$$d(z_n, z) < \frac{b\varepsilon}{\sqrt{1+|z|^2}}.$$

Then for $n \geq \max\{M, L\}$, we have

$$|z_n - z| < \frac{\sqrt{1+|z|^2}}{b} \cdot d(z_n, z) < \varepsilon,$$

thus $|z_n - z| \xrightarrow{n\to\infty} 0$. □

Problem 3

Suppose $\{x_n\}$ is a Cauchy sequence and $\{x_{n_k}\}$ is a subsequence that is convergent. Show that $\{x_n\}$ must be convergent.

Proof. Let $x_{n_k} \xrightarrow{k\to\infty} x$ in the metric d. Fix $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that for all $r, s \geq N$, $d(x_r, x_s) < \varepsilon/2$. Chooes $M \in \mathbb{N}$ such that for all $m \geq M$, $d(x_{n_m}, x) < \varepsilon/2$. Then for all $k \geq N$, choosing r such that $r \geq M$ and $n_r \geq N$, we have

$$d(x_k, x) \leq d(x_k, x_{n_r}) + d(x_{n_r}, x) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus $x_n \xrightarrow{n\to\infty} x$ in the metric d. □

Problem 4

Prove the converse direction in the following statement. A set $K \subseteq X$ is compact if and only if every collection $\mathscr{F}$ of closed subsets of K with the finite intersection property has $\bigcap\{F : F \in \mathscr{F}\} \neq \emptyset$.

Proof. Suppose that every collection of closed subsets of K with the finite intersection property has nonempty total intersection. Let $\{U_i\}_{i\in I}$ be an open cover of K where I is some index set. Suppose, for the sake of contradiction, that for every $J \subseteq I$ finite, we have $K \not\subseteq \bigcup_{j\in J} U_j$.

Consider the collection $\mathscr{F} := \{F_i\}_{i\in I}$ where $F_i := K \cap (X \setminus U_i)$. Each F_i is a closed subset of K. For any subset $J \subseteq I$, using De Morgan's law we compute

$$\begin{aligned}\bigcap_{j\in J} F_j &= \bigcap_{j\in J} K \cap (X \setminus U_j) \\ &= K \cap \bigcap_{j\in J}(X \setminus U_j) = K \cap \left(X \setminus \bigcap_{j\in J} U_j\right).\end{aligned}$$

For finite subsets $J \subseteq I$, by assumption we have $K \not\subseteq \bigcup_{j\in J} U_j$ which is equivalent to the statement that $\bigcap_{j\in J} F_j = K \cap \left(X \setminus \bigcap_{j\in J} U_j\right) \neq \emptyset$. Thus the collection $\mathscr{F}$ has the finite intersection property, whence by assumption

$$\emptyset \neq \bigcap_{i\in I} F_i = K \cap \left(X \setminus \bigcap_{i\in I} U_i\right),$$

which is equivalent to the statement that $K \not\subseteq \bigcup_{i\in I} U_i$, contradicting the assumption that $\{U_i\}_{i\in I}$ is an open cover of K. Thus $\{U_i\}_{i\in I}$ has a finite subcover of K. As the choice of cover was arbitrary, it follows by definition that K is compact. $\square$

Problem 5

Show that the union of a finite number of compact sets is compact.

Proof. Suppose $K_1, \ldots, K_n$ are compact subsets of X and consider the set $K := \bigcup_{r=1}^n K_r$. Let $\mathcal{U}$ be an open cover of K. For $1 \leq i \leq n$, we have that $\mathcal{U}$ is also an open cover of K_i, so there is some finite subcover $\widetilde{\mathcal{U}}^{(i)} \subseteq \mathcal{U}$ of K_i.

Consider the collection $\widetilde{\mathcal{U}} := \bigcup_{i=1}^n \widetilde{\mathcal{U}}^{(i)}$. Note that

$$|\widetilde{\mathcal{U}}| \leq \sum_{i=1}^n |\widetilde{\mathcal{U}}^{(i)}| < +\infty$$

so $\widetilde{\mathcal{U}}$ is a finite subcollection of $\mathcal{U}$. Moreover, we note that

$$\bigcup_{U\in\widetilde{\mathcal{U}}} U = \bigcup_{i=1}^n \bigcup_{U\in\widetilde{\mathcal{U}}^{(i)}} U \supseteq \bigcup_{i=1}^n K_i = K,$$

whence $\widetilde{\mathcal{U}} \subseteq \mathcal{U}$ is in fact a finite subcover of K. $\square$

Problem 6

Show that the closure of a totally bounded set is totally bounded.

Proof. Recall that a set $A \subseteq X$ is *totally bounded* if for every $\varepsilon > 0$, there is some $n \in \mathbb{N}$ and points $x_1, \ldots, x_n \in X$ such that $A \subseteq \bigcup_{i=1}^n B_\varepsilon(x_i)$.

Suppose that $A \subseteq X$ is totally bounded, and fix $\varepsilon > 0$. Let $x_1, \ldots, x_n \in X$ be such that $A \subseteq \bigcup_{i=1}^n B_{\varepsilon/2}(x_i)$. Let $p \in \overline{A}$. If $p \in A$, then we are done, so suppose that $p \notin A$. Then by problem 1(b), p is a limit point of A, whence by definition $A \cap (B_{\varepsilon/2}(p) \setminus \{p\}) \neq \emptyset$.

Let $x \in A \cap (B_{\varepsilon/2}(p) \setminus \{p\})$. As $x \in A$, there is some $1 \leq k \leq n$ such that $x \in B_{\varepsilon/2}(x_k)$. Finally, we estimate

$$d(p, x_k) \leq d(p, x) + d(x, x_k) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

so $p \in B_\varepsilon(x_k)$. Hence

$$\{p : p \text{ is a limit point of } A\} \subseteq \bigcup_{i=1}^n B_\varepsilon(x_i).$$

By assumption,

$$A \subseteq \bigcup_{i=1}^n B_{\varepsilon/2}(x_i) \subseteq \bigcup_{i=1}^n B_\varepsilon(x_i),$$

whence by problem 1(b),

$$\overline{A} = A \cup \{p : p \text{ is a limit point of } A\} \subseteq \bigcup_{i=1}^n B_\varepsilon(x_i).$$

□

Problem 7

Suppose that $f : X \to \Omega$ is uniformly continuous; show that if $\{x_n\}$ is a Cauchy sequence in X then $\{f(x_n)\}$ is a Cauchy sequence in Ω. Is this still true if we only assume f is continuous?

Proof. Let d, ρ be the metrics topologizing X and Ω respectively. Suppose that $(x_n)_{n=1}^\infty$ is a Cauchy sequence in X. Fix $\varepsilon > 0$. By uniform continuity, there is some $\delta > 0$ such that, for $x, y \in X$, we have $\rho(f(x), f(y)) < \varepsilon$ whenever $d(x, y) < \delta$.

As $(x_n)_{n=1}^\infty$ is Cauchy in X, there is some $N \in \mathbb{N}$ such that for $r, s \geq N$, we have $d(x_r, x_s) < \delta$. Hence, it follows that for $r, s \geq N$, we have $\rho(f(x_r), f(x_s)) < \varepsilon$. Hence $(f(x_n))_{n=1}^\infty$ is a Cauchy sequence in Ω.

<u>(Counterexample for just continuity)</u>: We claim that this is not necessarily true when f is just continuous. Let $X = (0, \infty)$ with the restriction of the absolute value metric on $\mathbb{R}$, and $\Omega = \mathbb{R}$ with the absolute value metric.

Let $f : (0, \infty) \to \mathbb{R}$ be given by $f(x) = \frac{1}{x^2}$. This function is continuous. Consider $x_n := \frac{1}{n}$ for $n \in \mathbb{N}$. To see that this sequence is Cauchy, fix $\varepsilon > 0$. By the archimedean principle, there is some $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\varepsilon}{2}$. Then for $n, m \geq N$, we have

$$|x_n - x_m| = \left|\frac{1}{n} - \frac{1}{m}\right| \leq \left|\frac{1}{n}\right| + \left|\frac{1}{m}\right| \leq 2\left|\frac{1}{N}\right| < \varepsilon.$$

Now note that $f(x_n) = n^2$. For $n \in \mathbb{N}$, observe that

$$|f(x_{n+1}) - f(x_n)| = |(n+1)^2 - n^2| = |2n+1| \geq 1,$$

so $(f(x_n))_{n=1}^{\infty}$ cannot be Cauchy. $\square$