

# A Poisson Boundary Theory for Groupoids

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March 20, 2024

**Goal:** Develop a theory of Poisson boundary for groupoids using Das-Peterson's noncommutative PB framework from [DP22].

## 1 Setup

### 1.1 Groupoid Side

Let  $(\mathcal{G}, \mu)$  be a discrete pmp groupoid and  $X = \mathcal{G}^{(0)}$ . Assume we have a collection  $\{\pi_x\}_{x \in X}$  such that  $\pi_x \in \text{Prob}(\mathcal{G}^x)$  for all  $x \in X$ . Extend each  $\pi_x$  by zero to be defined on  $\mathcal{G}$ . For  $g \in \mathcal{G}$ , set

$$\pi_g = g_* \pi_{s(g)}$$

In the framework of [Kai05],  $\{\pi_g\}$  give the transition probabilities for a Markov operator on  $\mathcal{G}$ . Such a Markov operator is called *invariant* if  $g_* \pi_h = \pi_{gh}$  for all  $(g, h) \in \mathcal{G}^{(2)}$ .

**Definition 1.1.1** ([Kai05]). A family  $\pi = \{\pi_g\}$  of transition probabilities is called Borel if for every non-negative Borel function  $f$ , the function  $\pi(f) : \mathcal{G} \rightarrow \mathbb{C}$  given by  $\pi(f)(g) = \int_{\mathcal{G}} f d\pi_g$  is Borel.

Given such a Borel family  $\pi$ , we then get an induced Markov operator  $P : \text{Bor}(\mathcal{G}) \rightarrow \text{Bor}(\mathcal{G})$  given by  $Pf = \pi(f)$ . The corresponding *dual operator*  $\tilde{P} : M_+(\mathcal{G}) \rightarrow M_+(\mathcal{G})$  is then given by

$$\tilde{P}(\theta) = \int_{\mathcal{G}} \pi_g d\theta(g) \text{ for all } \theta \in M_+(\mathcal{G}).$$

Now by definition of the vector-valued integral,

$$\begin{aligned} \langle \theta, Pf \rangle &= \int_{\mathcal{G}} Pf(g) d\theta(g) = \int_{\mathcal{G}} \left( \int_{\mathcal{G}} f d\pi_g \right) d\theta(g) \\ &= \int_{\mathcal{G}} f d\tilde{P}\theta = \langle \tilde{P}\theta, f \rangle \end{aligned}$$

### 1.2 Von Neumann Algebras Side

Fix a tracial von Neumann algebra  $(M, \tau)$  and an embedding  $M \subseteq \mathcal{A}$  into a  $C^*$  algebra  $\mathcal{A}$ .

$$S_{\tau}(\mathcal{A}) := \{\varphi \in S(\mathcal{A}) : \varphi|_M = \tau\}.$$

Fixing  $\varphi \in S_{\tau}(\mathcal{A})$  gives an inclusion  $L^2(M, \tau) \subseteq L^2(\mathcal{A}, \varphi)$ . Let  $e_M = \text{Proj}_{L^2(M, \tau)} \in B(L^2(\mathcal{A}, \varphi))$ . Define a u.c.p. map  $\mathcal{P}_{\varphi} : \mathcal{A} \rightarrow B(L^2(M, \tau))$ , by

$$\mathcal{P}_{\varphi}(T) := e_M T e_M \text{ for } T \in \mathcal{A}$$

For  $x \in M$ ,  $\mathcal{P}_{\varphi}(x) = x$ . The map  $\mathcal{P}_{\varphi}$  is the *Poisson transform* of the inclusion  $M \subseteq \mathcal{A}$ .

## 2 Ideas moving forward

- Similar to Remi and Boutonnet in [this paper](#), study intermediate von Neumann algebras

$$L(\mathcal{G}) \subseteq M \subseteq L(\mathcal{G} \curvearrowright \mathcal{B})$$

where  $\mathcal{G} \curvearrowright \mathcal{B}$  is the Poisson boundary action. This framework can be utilized to possibly study the question of whether

$$(L(G) \subseteq L^\infty \rtimes G) \cong (L(H) \subseteq L^\infty(Y) \rtimes H) \stackrel{?}{\Longleftrightarrow} G \curvearrowright (X, \mu) \cong H \curvearrowright (Y, \nu)$$

in the same vein as the fact the inclusions of the  $L^\infty$ -space determine action up to orbit equivalence.

- Recent paper of Sartini in [\[SS24\]](#) studying Poisson boundaries of ergodic groupoids in the vein of Kaimonovich. Note that this doesn't run through any von Neumann algebraic framework.

## References

- [DP22] Sayan Das and Jesse Peterson. “Poisson boundaries of II1 factors”. In: *Compos. Math.* 158.8 (2022), pp. 1746–1776.
- [Kai05] Vadim A. Kaimanovich. “Amenability and the Liouville property”. In: vol. 149. *Probability in mathematics*. 2005, pp. 45–85.
- [SS24] Filippo Sarti and Alessio Savini. *Boundaries and equivariant maps for ergodic groupoids*. 2024.