6316 Homework 3

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1 Problem 1

Show the following monotonicity property of VC-dimension: For every two hypothesis calsses, if $\mathcal{H}' \subseteq \mathcal{H}$ then $\text{VCdim}(\mathcal{H}') \leq \text{VCdim}(\mathcal{H})$.

Proof. Note that, if $C \subseteq X$ is any subset, then

$$\mathcal{H}'_C = \{(h(c_1), \dots, h(c_{\text{VCdim}(\mathcal{H}')})) : h \in \mathcal{H}'\} \overset{\mathcal{H}' \subseteq \mathcal{H}}{\subseteq} \{(h(c_1), \dots, h(c_{\text{VCdim}(\mathcal{H}')})) : h \in \mathcal{H}\} = \mathcal{H}_C \tag{1}$$

Suppose first that $\text{VCdim}(\mathcal{H}') < +\infty$. By definition, there exists a finite set $C \subseteq X$ of size $\text{VCdim}(\mathcal{H}')$ which is shattered by $\mathcal{H}'$, i.e. $\mathcal{H}'_C = \{0,1\}^C$. Appealing to equation (1), we find

$$\{0,1\}^C = \mathcal{H}'_C \overset{(1)}{\subseteq} \mathcal{H}_C \subseteq \{0,1\}^C,$$

whence $\mathcal{H}_C = \{0,1\}^C$. Thus, as $\mathcal{H}$ shatters C which is a set of size $\text{VCdim}(\mathcal{H}')$, it follows by definition that $\text{VCdim}(\mathcal{H}) \geq \text{VCdim}(\mathcal{H}')$ as desired.

Now assume that $\text{VCdim}(\mathcal{H}') = +\infty$. Fix arbitrary $N \in \mathbb{N}$. Since $\mathcal{H}'$ has infinite VC-dimension, there exists a set $C \subseteq X$ of size $|C| \geq N$ which is shattered by $\mathcal{H}'$. By equation (1), we have again that $\mathcal{H}_C = \{0,1\}^C$ whence $\mathcal{H}$ shatters C. Thus, $\mathcal{H}$ shatters sets of arbitrarily large size, whence $\text{VCdim}(\mathcal{H}) = +\infty$ and trivially $\text{VCdim}(\mathcal{H}') = +\infty \leq +\infty = \text{VCdim}(\mathcal{H})$. □

2 Problem 2

Let $\mathcal{H}$ be the class of signed intervals, that is, $\mathcal{H} = \{h_{a,b,s} : a \leq b, s \in \{\pm 1\}\}$ where

$$h_{a,b,s} = \begin{cases} s & \text{if } x \in [a,b] \\ -s & \text{otherwise} \end{cases}$$

Calculate $\text{VCdim}(\mathcal{H})$.

Solution. We claim that $\text{VCdim}(\mathcal{H}) = 3$. First consider the set $C = \{1,3,5\}$. Consider the hypotheses $h_{0,2,\pm 1}, h_{2,4,\pm 1}, h_{4,6,\pm 1}, h_{0,6,\pm 1}$. We claim these classes restricted to C witness all of $\{\pm 1\}^C$. This is clear from the computations below.

$$h_{0,2,\pm 1}(1) = \pm 1,\ h_{0,2,\pm 1}(3) = \mp 1,\ h_{0,2,\pm 1}(5) = \mp 1$$

$$h_{2,4,\pm 1}(1) = \mp 1,\ h_{2,4,\pm 1}(3) = \pm 1,\ h_{2,4,\pm 1}(5) = \mp 1$$

$$h_{4,6,\pm 1}(1) = \mp 1,\ h_{4,6,\pm 1}(3) = \mp 1,\ h_{4,6,\pm 1}(5) = \pm 1$$

$$h_{0,6,\pm 1}(1) = \pm 1,\ h_{0,6,\pm 1}(3) = \pm 1,\ h_{0,6,\pm 1}(5) = \pm 1$$

Hence $\mathcal{H}$ shatters C, so $\mathrm{VCdim}(\mathcal{H}) \geq 3$.

Now suppose that $C \subseteq \mathbb{R}$ with $|C| = 4$. Write $C = \{a, b, c, d\}$ with $a < b < c < d$. We claim that $(+1, -1, +1, -1) \notin \mathcal{H}_C$. Suppose, for the sake of contradiction, that there is some $h = h_{\alpha,\beta,s} \in \mathcal{H}$ such that $(+1, -1, +1, -1) = (h(a), h(b), h(c), h(d))$.

For the first case, assume $s = 1$. As $h(a), h(c) = +1$, we have $a, c \in [\alpha, \beta]$. But then $b \in [a, c] \subseteq [\alpha, \beta]$, whence $h(b) = +1$ contradicting the assumption that $h(b) = -1$.

For the second case, assume $s = -1$. As $h(b), h(d) = -1$, we have $b, d \in [\alpha, \beta]$. But then $c \in [b, d] \subseteq [\alpha, \beta]$, whence $h(c) = -1$ contradicting the assumption that $h(c) = +1$.

In both cases, we reach a contradiction, so the claim follows and thus $\mathcal{H}$ cannot shatter any set of size 4. Hence $\mathrm{VCdim}(\mathcal{H}) = 3$.

$\square$