220A Homework 5

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Problem 1

Define $\gamma(t) = e^{it}$ for $0 \le t \le 2\pi$ and find $\int_\gamma z^n \, dz$ for every integer $n \in \mathbb{Z}$.

Solution. Suppose first that $n \neq -1$. Note that $\gamma'(t) = ie^{it}$, so

$$\begin{aligned}\int_\gamma z^n \, dz = \int_0^{2\pi} (e^{it})^n ie^{it} \, dt = i \int_0^{2\pi} e^{it(n+1)} \, dt \\ = \frac{i}{i(n+1)} e^{it(n+1)} \Big|_0^{2\pi} = 0.\end{aligned}$$

On the other hand, suppose that $n = -1$. Then

$$\int_\gamma \frac{1}{z} \, dz = \int_0^{2\pi} \frac{1}{e^{it}} ie^{it} \, dt = \int_0^{2\pi} i \, dt = 2\pi i.$$

□

Problem 2

Let $I(r) = \displaystyle\int_\gamma \frac{e^{iz}}{z} \, dz$ where $\gamma : [0, \pi] \to \mathbb{C}$ is defined by $\gamma(t) = re^{it}$. Show that $\displaystyle\lim_{r \to \infty} I(r) = 0$.

Proof. For $z \in \mathbb{C}$, observe that

$$|e^{iz}| = |e^{i(\mathrm{Re}(z) + i\,\mathrm{Im}(z))}| = |e^{i\,\mathrm{Re}(z)}| \cdot |e^{-\mathrm{Im}(z)}| = |e^{-\mathrm{Im}(z)}|.$$

Hence, for $z \in \gamma$, we have that $z = re^{it}$ for some $t \in [0, \pi]$, whence

$$|e^{iz}| = |e^{-\mathrm{Im}(z)}| = |e^{-r\sin(t)}|$$

Combining this with $z \in \gamma$ implying that $|z| = r$, we estimate

$$\begin{aligned}\left| \int_\gamma \frac{e^{iz}}{z} \, dz \right| \le \int_\gamma \left| \frac{e^{iz}}{z} \right| |\, dz \,| = \int_\gamma \frac{e^{-\mathrm{Im}(z)}}{r} |\, dz \,| \\ = \int_0^\pi e^{-r\sin(t)} \, dt \,.\end{aligned}$$

As $x \mapsto \sin(x)$ is symmetric in $[0, \pi]$ about the line $x = \pi/2$, it follows that

$$\int_0^\pi e^{-r\sin(t)}\, dt = 2\int_0^{\pi/2} e^{-r\sin(t)}\, dt\,.$$

On $[0, \pi/2]$, the function $x \mapsto \sin(x)$ is concave down, whence it lies entirely above the secant line between its endpoints, namely $(0, 0)$ and $(\pi/2, 1)$. This line is given by $y = \frac{2}{\pi}x$, so for $x \in [0, \pi/2]$ we have $\sin(x) \geq \frac{2}{\pi}x$. Hence $e^{-r\sin(x)} \leq e^{-r\cdot\frac{2}{\pi}x}$, so

$$\begin{aligned} I(r) = 2\int_0^{\pi/2} e^{-r\sin(t)}\, dt \leq 2\int_0^{\pi/2} e^{-\frac{2r}{\pi}x}\, dt &= 2\left(-\frac{\pi}{2r}\cdot e^{-r} + \frac{\pi}{2r}\cdot 1\right) \\ &= \frac{\pi}{r}(1 - e^{-r}) \xrightarrow{r\to\infty} 0 \end{aligned}$$

as desired □

Problem 3

Show that if F_1 and F_2 are primitives for $f : G \to \mathbb{C}$ and G is connected, then there is a constant c such that $F_1(z) = c + F_2(z)$ for each $z \in G$.

Proof. The question is not clear whether we assume G to be open (I believe so since we cannot define the derivatives of F_1 and F_2 otherwise, but such is Conway).
Suppose we are assuming G is open. Define $F : G \to \mathbb{C}$ by $F := F_1 - F_2$. Then F is analytic as F_1 and F_2 are, and $F' = F_1' - F_2' = f - f = 0$ on all of G. Thus $F'^{(z)} = 0$ for all $z \in G$ where G is an open, connected set, whence by Proposition 2.10 in Conway, F is constant.

Thus there is some $c \in \mathbb{C}$ such that $F = c$, whence $F_1 = c + F_2$.
TODO maybe see if G not open still works (idk how). □

Problem 4

Prove the following analogue of Leibniz's rule. Let G be an open set and γ a rectifiable curve in $\mathbb{C}$. Suppose that $\varphi : \{\gamma\} \times G \to \mathbb{C}$ is a continuous function and define $g : G \to \mathbb{C}$ by

$$g(z) := \int_\gamma \varphi(w, z)\, dw\,.$$

Prove that g is continuous. If $\frac{\partial\varphi}{\partial z}$ exists for each $(w, z) \in \{\gamma\} \times G$ and is continuous, prove that g is analytic and

$$g'(z) = \int_\gamma \frac{\partial\varphi}{\partial z}(w, z)\, dw\,.$$

Proof. As γ is a continuous function on a compact set (namely an interval), it follows that the trace $\{\gamma\} \subseteq \mathbb{C}$ is compact. Suppose that $(z_n)_{n=1}^\infty$ is a sequence in G, $z \in G$, and that $z_n \xrightarrow{n\to\infty} z$. Without loss of generality, assume that $z_n \in B_r(z)$ for all $n \in \mathbb{N}$ where $r > 0$ is chosen so that $\overline{B_r(z)} \subseteq B_{2r}(z) \subseteq G$.

As $\{\gamma\}$ is compact, it follows that $\{\gamma\} \times \overline{B_r(z)}$ is compact in the product topology. Thus $\varphi|_{\{\gamma\}\times\overline{B_r(z)}}$ is uniformly continuous. Fixing $\varepsilon > 0$, it follows that there is some $\delta > 0$ such that if $|w - w'| < \delta$ and $|u - u'| < \delta$ for some $w, w' \in \{\gamma\}$ and $\xi, \xi' \in \overline{B_r(z)}$, then

$$|\varphi(w, \xi) - \varphi(w', \xi')| < \frac{\varepsilon}{V(\gamma) + 1}.$$

Choose $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|z - z_n| < \delta$. Then, for all $n \geq N$,

$$
\begin{aligned}
|g(z) - g(z_n)| &\leq \int_\gamma |\varphi(w,z) - \varphi(w,z_n)| \cdot |\, dz\,| \\
&\leq V(\gamma) \cdot \sup_{w \in \{\gamma\}} |\varphi(w,z) - \varphi(w,z_n)| \\
&\leq \frac{V(\gamma)}{V(\gamma)+1}\varepsilon < \varepsilon.
\end{aligned}
$$

Thus $g(z_n) \xrightarrow{n\to\infty} g(z)$, whence g is continuous.

Now suppose that $\frac{\partial\varphi}{\partial z}$ exists for each point in $\{\gamma\} \times G$ and is continuous. Fix $z_0 \in G$ and $r > 0$ small enough such that $\overline{B_r(z_0)} \subseteq G$. Then for $0 < |h| < r$, we have

$$
\frac{g(z_0+h) - g(z_0)}{h} = \frac{1}{h}\int_\gamma (\varphi(w, z_0 + h) - \varphi(w, z_0))\, dw
$$

As $\frac{\partial\varphi}{\partial z}$ is continuous, it follows by compactness that $\frac{\partial\varphi}{\partial z}|_{\{\gamma\}\times\overline{B_r(z)}}$ is uniformly continuous. Fix $\varepsilon > 0$, and choose $0 < \delta < r$ such that if $|w - w'| < \delta$ and $|\xi - \xi'| < \delta$ for $w, w' \in \{\gamma\}$ and $\xi, \xi' \in \overline{B_r(z_0)}$, then

$$
\left|\frac{\partial\varphi}{\partial z}(w,\xi) - \frac{\partial\varphi}{\partial z}(w',\xi')\right| < \frac{\varepsilon}{V(\gamma)+1}.
$$

Note also that for fixed $w \in \{\gamma\}$, letting $\sigma_h := [z_0, z_0 + h]$ denote the line segment between the two points, we have that

$$
\frac{1}{h}\int_{\sigma_h} \frac{\partial\varphi}{\partial z}(w, z_0)\, dz = \frac{\partial\varphi}{\partial z}(w, z_0)
$$

as we are simply integrating a constant. On the other hand, we see that for fixed $w \in \{\gamma\}$,

$$
\varphi(w, z_0 + h) - \varphi(w, z_0) = \int_{\sigma_h} \frac{\partial\varphi}{\partial z}(w, z)\, dz\,.
$$

Then, we estimate for $0 < |h| < \delta < r$,

$$
\begin{aligned}
\left|\frac{g(z_0+h) - g(z_0)}{h} - \int_\gamma \frac{\partial\varphi}{\partial z}(w,z_0)\, dw\right| &= \left|\int_\gamma \frac{1}{h}\int_{\sigma_h} \frac{\partial\varphi}{\partial z}(w,z)\, dz\, dw - \int_\gamma \frac{1}{h}\int_{\sigma_h} \frac{\partial\varphi}{\partial z}(w,z_0)\, dz\, dw\right| \\
&= \left|\frac{1}{h}\int_\gamma\int_{\sigma_h} \frac{\partial\varphi}{\partial z}(w,z) - \frac{\partial\varphi}{\partial z}(w,z_0)\, dz\, dw\right| \\
&\leq \frac{1}{|h|}\int_\gamma\int_{\sigma_h} \left|\frac{\partial\varphi}{\partial z}(w,z) - \frac{\partial\varphi}{\partial z}(w,z_0)\right| |\, dz\,||\, dw\,| \\
&\leq \frac{1}{|h|}\int_\gamma\int_{\sigma_h} \frac{\varepsilon}{V(\gamma)+1} |\, dz\,||\, dw\,| \\
&\leq \frac{1}{|h|} \cdot |h| \cdot \frac{V(\gamma)}{V(\gamma)+1}\varepsilon < \varepsilon.
\end{aligned}
$$

Thus g' exists at z_0 and is given by the desired expression. To see that g is analytic, it remains to show that g' is continuous on G. Let $(z_n)_{n=1}^\infty$ be a sequence in G and $z \in G$ such that $z_n \xrightarrow{n\to\infty} z$. Without loss of generality, assume there is some $r > 0$ such that $z_n \in B_r(z)$ for all $n \in \mathbb{N}$ and that $\overline{B_r(z)} \subseteq G$. Then by compactness $\frac{\partial\varphi}{\partial z}|_{\{\gamma\}\times\overline{B_r(z)}}$ is uniformly continuous, whence for $\varepsilon > 0$ there is some $0 < \delta < r$ such that

$$
\left|\frac{\partial\varphi}{\partial z}(w,\xi) - \frac{\partial\varphi}{\partial z}(w,\xi')\right| < \frac{\varepsilon}{V(\gamma)+1}
$$

for all $w \in \{\gamma\}$ and $\xi, \xi' \in \overline{B_r(z)}$ with $|\xi - \xi'| < \delta$. Choose $N \in \mathbb{N}$ such that for $n \geq N$ we have $|z_n - z| < \delta$. Then for $n \geq N$,

$$|g'(z_n) - g'(z)| \leq V(\gamma) \cdot \sup_{w \in \{\gamma\}} \left| \frac{\partial \varphi}{\partial z}(w, z_n) - \frac{\partial \varphi}{\partial z}(w, z) \right| < \frac{V(\gamma}{V(\gamma) + 1}\varepsilon < \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, it follows that $g'(z_n) \xrightarrow{n \to \infty} g'(z)$, so g is analytic. □

Problem 5

Suppose that γ is a piecewise smooth curve in $\mathbb{C}$ and φ is defined and continuous on $\{\gamma\}$. Use the previous exercise to show that

$$g(z) := \int_\gamma \frac{\varphi(w)}{w - z}\, dw$$

is analytic on $\mathbb{C} \setminus \{\gamma\}$ and

$$g^{(n)}(z) = n! \int_\gamma \frac{\varphi(w)}{(w - z)^{n+1}}\, dw\,.$$

Proof. □

Problem 6

(a): The following is Abel's Theorem. Let $\sum a_n (z-a)^n$ have radius of convergence 1 and suppose that $\sum a_n$ converges to A. Prove that

$$\lim_{r \to 1^-} \sum_{n=0}^{\infty} a_n r^n = A.$$

(b): Use Abel's Theorem to prove that $\log(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$.

Problem 7

Use Corollary 2.13 to evaluate the following integrals:

(a):

$$\int_\gamma \frac{e^z - e^{-z}}{z^n}\, dz$$

where n is a positive integer and $\gamma(t) = e^{it}$, $0 \leq t \leq 2\pi$.

(b):

$$\int_\gamma \frac{dz}{\left(z - \frac{1}{2}\right)^n}$$

where n is a positive integer and $\gamma(t) = \frac{1}{2} + e^{it}$, $0 \leq t \leq 2\pi$.

(c):

$$\int_\gamma \frac{dz}{z^2 + 1}$$

where $\gamma(t) = 2e^{it}$, $0 \leq t \leq 2\pi$. *Hint.* Expand $(z^2 + 1)^{-1}$ by means of partial fractions.

(d):

$$\int_\gamma \frac{\sin z}{z}\,dz$$

where $\gamma(t) = e^{it}$, $0 \le t \le 2\pi$.

(e):

$$\int_\gamma \frac{z^{1/m}}{(z-1)^n}\,dz$$

where $\gamma(t) = 1 + \frac{1}{2}e^{it}$, $0 \le t \le 2\pi$.