200A Homework 2

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Problem 1

Suppose G is a simple group and it has a subgroup H of index n where n is an integer more than 1. Prove that G can be embedded into the symmetric group S_n.

Proof. Let G act by left multiplication on the set X of left cosets of H in G. This furnishes a homomorphism $\theta : G \to \mathrm{Sym}(X) \cong S_n$. Then $\ker(\theta)$ is a normal subgroup of G. As $[G : H] > 1$, G acts nontrivially on X whence $\ker(\theta) \neq G$. Thus by simplicity of G, $\ker(\theta) = \{e\}$ so θ is an embedding. □

Problem 2

For a group G, let $\mathrm{Aut}(G)$ be the group of automorphisms of G. Let

$$c : G \to \mathrm{Aut}(G), \qquad c(g) := c_g, \ \text{ where } c_g(x) := gxg^{-1} \ \text{ for every } x \in G.$$

(a): Prove that c_g is an automorphism of G and c is a group homomorphism.

Proof. Fix $g \in G$. Let $x, y \in G$. Then

$$c_g(xy) = gxyg^{-1} = gxg^{-1}gyg^{-1} = c_g(x)c_g(y),$$

so c_g is a group homomorphism.

For $h \in G$, $c_g(g^{-1}hg) = h$, so c_g is surjective. If $ghg^{-1} = c_g(h) = e$, then $h = e$, so $\ker(c_g) = \{e\}$ whence c_g is injective. Thus $c_g \in Aut(G)$.

Fix $g, h \in G$ and $x \in G$. Then

$$c_g c_h(x) = c_g(hxh^{-1}) = ghxh^{-1}g^{-1} = ghx(gh)^{-1} = c_{gh}(x),$$

So $c_{gh} = c_g c_h$, whence c is a group homomorphism. □

(b): Prove that $\ker c$ is the center $Z(G)$ of G.

Proof. Suppose $g \in \ker(c)$ so $c_g = id_G$. Then for all $x \in G$, $x = c_g(x) = gxg^{-1}$, so $g \in Z(G)$. Now suppose $g \in Z(G)$. Then for $x \in G$, $c_g(x) = gxg^{-1} = xgg^{-1} = x$, so $c_g = id_G$ whence $g \in \ker(c)$. □

(c): The image of c is called the group of inner automorphisms of G, and it is denoted by $\mathrm{Inn}(G)$. Prove that $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.

Proof. Fix $\varphi \in \mathrm{Aut}(G)$ and let $g \in G$. Set $h := \varphi(g)$. For $x \in G$

$$\varphi c_g \varphi^{-1}(x) = \varphi c_g(\varphi^{-1}(x)) = \varphi(\varphi^{-1}(h)\varphi^{-1}(x)\varphi^{-1}(h^{-1})) = \varphi(\varphi^{-1}(hxh^{-1})) = hxh^{-1} = c_{\varphi(g)}(x),$$

so $\varphi c_g \varphi^{-1} = c_{\varphi(g)} \in \mathrm{Inn}(G)$. Thus $\mathrm{Inn}(G) \triangleleft \mathrm{Aut}(G)$. $\square$

(d): Prove that $|Z(\mathrm{Aut}(G))| \leq |\mathrm{Hom}(G, Z(G))|$; in particular, if either $Z(G) = 1$ or G is perfect (that is, $G = [G, G]$), then $Z(\mathrm{Aut}(G)) = \{1\}$.

Proof. As the function c is a homomorphism with image $\mathrm{Inn}(G)$, it follows that $G/Z(G) \cong \mathrm{Inn}(G)$. For $\varphi \in Z(\mathrm{Aut}(G))$ and $g \in G$, we have that $c_g = c_{\varphi(g)}$, so under the isomorphism we infer $gZ(g) = \varphi(g)Z(g)$. Hence there is some $\eta(g) \in Z(G)$ such that $\varphi(g) = g\eta(g)$.
For $g, h \in G$,

$$gh\eta(gh) = \varphi(gh) = \varphi(g)\varphi(h) = g\eta(g)h\eta(h) = gh\eta(g)\eta(h)$$

whence $\eta(gh) = \eta(g)\eta(h)$, so η is a group homomorphism.
Now for $\varphi \in Z(\mathrm{Aut}(G))$, let η_φ be the corresponding homomorphism obtained above. Suppose $\varphi, \psi \in Z(\mathrm{Aut}(G))$ are such that $\eta_\varphi = \eta_\psi$. Then for $g \in G$,

$$\varphi(g) = g\eta_\varphi(g) = g\eta_\psi(g) = \psi(g),$$

so $\varphi = \psi$. Hence the assignment $\varphi \mapsto \eta_\varphi$ is injective, whence $|Z(\mathrm{Aut}(G))| = |\{\eta_\varphi : \varphi \in Z(\mathrm{Aut}(G))\}| \leq |\mathrm{Hom}(G, Z(G))|$. $\square$

Problem 3

Let $SL_2(\mathbb{R})$ be the set of real 2×2 matrices with determinant 1. For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ and $z \in \mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$, let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az + b}{cz + d}.$$

(a): Prove that

$$\mathrm{Im}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z\right) = \frac{\mathrm{Im}(z)}{|cz + d|^2}.$$

(b): Prove that $\cdot$ is an action of $SL_2(\mathbb{R})$ on $\mathbb{H}$.

Problem 4

Suppose G is a finite group, $C \subseteq \mathbb{R}^n$ is a convex subset (that is, for all two points P, Q in C, the segment PQ is a subset of C). Suppose G acts on C by affine transformations; that means

$$\forall P, Q \in C,\ \forall t \in [0, 1],\ \forall g \in G, \qquad g \cdot (tP + (1 - t)Q) = t\, g \cdot P + (1 - t)\, g \cdot Q.$$

Prove that G has a fixed point; that is, there exists $x \in C$ such that for all $g \in G$, $g \cdot x = x$.

Proof. By definition, for $c_1, c_2 \in C$, we have that $\frac{c_1 + c_2}{2} \in C$. Suppose that we have shown that for any $c_1, \ldots, c_n \in C$, we have $\frac{c_1 + \cdots + c_n}{n} \in C$. Observe that then

$$\frac{c_1 + \cdots + c_{n+1}}{n + 1} = \frac{n}{n + 1}\left(\frac{c_1 + \cdots + c_n}{n}\right) + \left(1 - \frac{n}{n + 1}\right) c_{n+1} \in C.$$

Fix $y \in C$. Then $A_G(y) = \frac{1}{|G|} \sum_{h \in G} h \cdot y \in C$. For $g \in G$, we have

$$g \cdot A_G(y) = \frac{1}{|G|} \sum_{h \in G} gh \cdot y = \frac{1}{|G|} \sum_{h \in G} h \cdot y = A_G(y),$$

so $A_G(y)$ is a fixed point of the affine action $G \curvearrowright C$.

$\square$

Problem 5

Suppose G is a finite subgroup of the group $GL_n(\mathbb{R})$ of $n \times n$ invertible real matrices. Prove that there is a G-invariant inner product on $\mathbb{R}^n$.

Proof. Define

$$\langle v, w \rangle_G := \frac{1}{|G|} \sum_{g \in G} \langle g \cdot v, g \cdot w \rangle$$

$\square$

Problem 6

Suppose H is a subgroup of G. Let

$$C_G(H) := \{ x \in G \mid \forall h \in H,\ xh = hx \}$$

be the centralizer of H in G, and

$$N_G(H) := \{ x \in G \mid xHx^{-1} = H \}$$

be the normalizer of H in G. Both of these are subgroups of G and clearly $C_G(H) \subseteq N_G(H)$. Prove that $N_G(H)/C_G(H)$ can be embedded into $\mathrm{Aut}(H)$.

Problem 7

Suppose N is a finite cyclic normal subgroup of G. Prove that every subgroup of N is normal in G.

Proof. Let $K \leq N$ have order k. Then for $g \in G$, $gKg^{-1} \leq N$ has order k, but cyclic groups have unique subgroups of each given order, so $gKg^{-1} = K$. $\square$