Algebraic Actions Notes

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We then note that

$$\begin{aligned}
\sum_{r=1}^{\delta n} \frac{1}{r}\binom{(s+1)r-2}{r-1} \cdot \left(\frac{s^s}{(s+1)^{(s+1)}}\right)^r &= \\
\sum_{r=1}^{\delta n} \frac{1}{(s+1)r-1}\binom{(s+1)r-1}{r} \cdot \left(\frac{s^s}{(s+1)^{(s+1)}}\right)^r &= \\
1+\sum_{r=0}^{\delta n} \frac{1}{(s+1)r-1}\binom{(s+1)r-1}{r} \cdot \left(\frac{s^s}{(s+1)^{(s+1)}}\right)^r &= \\
1-\sum_{r=0}^{\delta n} \frac{-1}{(s+1)r-1}\binom{(s+1)r-1}{r} \cdot \left(\frac{s^s}{(s+1)^{(s+1)}}\right)^r &
\end{aligned} \tag{1}$$

Let us note that if we apply Lemma **??** with $a=-1$, $b=s+1$, and $k=r$, we have

$$\sum_{r=0}^{\infty} \frac{-1}{(s+1)r-1}\binom{(s+1)r-1}{r} z^r = \frac{1}{x} \tag{2}$$

Provided that we meet the conditions

$$z = \frac{x-1}{x^{(s+1)}} \qquad\qquad |z| < \frac{s^s}{(s+1)^{(s+1)}} \tag{3}$$

We then note that for $\frac{s}{s+1} < \gamma < 1$, if we set $x = \gamma^{\frac{s+1}{s}}$, then

$$|z| = \left|\frac{\gamma^{\frac{s+1}{s}}-1}{(\gamma^{\frac{s+1}{s}})^{(s+1)}}\right| = \frac{1}{\gamma^s}\cdot\frac{s^s}{(s+1)^{(s+1)}}\cdot\left|\left(s\left(1-\frac{1}{\gamma}\right)+1\right)\right| < \frac{s^s}{(s+1)^{(s+1)}} \tag{4}$$

1 Therefore, if $x = \gamma^{\frac{s+1}{s}}$ and $z = \frac{x-1}{x^{(s+1)}}$, then we have

$$\sum_{r=0}^{\infty} \frac{-1}{(s+1)r-1}\binom{(s+1)r-1}{r} z^r = \frac{1}{\gamma}\frac{s}{s+1} \tag{5}$$

Consider the function $F : (0, +\infty) \to \mathbb{R}$ given by

$$F(\gamma) = \frac{\gamma^{\frac{s+1}{s}} - 1}{(\gamma^{\frac{s+1}{s}})^{(s+1)}} = \frac{1}{\gamma^s} \cdot \frac{s^s}{(s+1)^{(s+1)}} \cdot \left(s \left(1 - \frac{1}{\gamma} \right) + 1 \right)$$

This function is nonnegative and monotone increasing for $\gamma \in (\frac{s}{s+1}, 1)$. Moreover, in this range we have

$$\sum_{r=0}^{\infty} \frac{-1}{(s+1)r-1} \binom{(s+1)r-1}{r} F(\gamma)^r = \frac{1}{\gamma} \frac{s}{s+1}$$

whence

$$1 - \frac{1}{\gamma} \frac{s}{s+1} = \sum_{r=1}^{\infty} \frac{1}{(s+1)r-1} \binom{(s+1)r-1}{r} F(\gamma)^r$$

Let $(\varepsilon_k)_k$ be a sequence increasing to 1 with $\varepsilon_k > \frac{s}{s+1}$ for all $k \in \mathbb{N}$, and define

$$a_{r,k} = \frac{1}{(s+1)r-1} \binom{(s+1)r-1}{r} F(\varepsilon_k)^r.$$

As F is monotone increasing in the range of the ε_ks, we have $a_{r,k} \leq a_{r,k+1}$ for all $r, k \in \mathbb{N}$. By the monotone convergence theorem,

$$\sum_{r=1}^{\infty} \frac{1}{(s+1)r-1} \binom{(s+1)r-1}{r} \left(\frac{s^s}{(s+1)^{s+1}} \right)^r$$
$$= \sum_{r=1}^{\infty} \sup_k a_{r,k} = \sup_k \sum_{r=1}^{\infty} a_{r,k} = \sup_k 1 - \frac{1}{\varepsilon_k} \frac{s}{s+1} = 1 - \frac{s}{s+1}$$

or equivalently

$$\sum_{r=0}^{\infty} \frac{-1}{(s+1)r-1} \binom{(s+1)r-1}{r} \cdot \left(\frac{s^s}{(s+1)^{(s+1)}} \right)^r = \frac{s}{s+1}$$

Equivalently (plugging in z), we have

$$\sum_{r=0}^{\infty} \frac{-1}{(s+1)r-1} \binom{(s+1)r-1}{r} \cdot \left(\frac{s^s}{(s+1)^{(s+1)}} \right)^r \cdot (\gamma^s \cdot (1 - s(1-\gamma)))^r = \gamma \frac{s}{s+1} \tag{6}$$

TODO: insert justification for why this converges at the radius

We then have

$$1 - \sum_{r=0}^{\delta n} \frac{-1}{(s+1)r-1} \binom{(s+1)r-1}{r} \cdot \left(\frac{s^s}{(s+1)^{(s+1)}} \right)^r \sim \frac{1}{s+1} \tag{7}$$