MATH 7410 Homework 6

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Problem 1

Let G be a finitely generated group with finite generation set S. Suppose that S is symmetric and contains the identity. We let

$$B_S(n) = \{s_1 \cdots s_n : s_i \in S, i = 1, \dots, n\}.$$

Suppose that G has *subexponential growth*, namely $\limsup_{n\to\infty} |B_S(n)|^{1/n} = 1$ (note that this implies that the limit itself is 1). Show that there is a subsequence $n_1 < n_2 < \cdots$ of natural numbers so that $(B_S(n_k))_{k=1}^{\infty}$ is a Folner sequence.

$$\liminf_{n\to\infty} \frac{a_n}{a_{n-k}} \le \liminf_{n\to\infty} a_n^{k/n}.$$

Proof. For $g \in G$, let $l_S(g)$ be the reduced length of g when written as an S-word omitting occurrences of the identity and set $l_S(e) = 0$. Since S is fixed, for brevity we write $B(n) = B_S(n)$. Note that, for $g \in G$ we have that $gB(n), B(n) \subseteq B(n + l_S(g))$, so

$$\begin{aligned}
\frac{|gB(n) \,\Delta\, B(n)|}{|B(n)|} &= \frac{|gB(n) \setminus B(n)|}{|B(n)|} + \frac{|B(n) \setminus gB(n)|}{|B(n)|} \\
&\le \frac{|B(n + l_S(g)) \setminus B(n)|}{|B(n)|} + \frac{|B(n + l_S(g)) \setminus gB(n)|}{|B(n)|} \\
&\le 2\frac{|B(n + l_S(g)) \setminus B(n)|}{|B(n)|} = 2\frac{|B(n + l_S(g))|}{|B(n)|} - 2.
\end{aligned}$$

Note that, for $k \in \mathbb{N}$,

$$\liminf_{n\to\infty} \frac{|B(n+k)|}{|B(n)|} \le \liminf_{n\to\infty} |B(n+k)|^{\frac{k}{n+k}} = 1.$$

Choose a subsequence $(n_k)_{k=1}^{\infty}$ as follows: choose n_1 such that $\frac{|B(n_2+1)|}{|B(n_1)|} \le 1 + \frac{1}{1}$. Having chosen $n_1 < \ldots < n_{k-1}$, choose $n_k > n_{k-1}$ such that $\frac{|B(n_k+k)|}{|B(n_k)|} \le 1 + \frac{1}{k}$. Then

$$\limsup_{k\to\infty} \frac{|B(n_k + k)|}{|B(n_k)|} \le 1.$$

Hence, for $g \in G$,

$$\limsup_{k\to\infty} \frac{|gB(n_k)\Delta B(n_k)|}{|B(n_k)|} \le 2\limsup_{k\to\infty} \frac{|B(n_k + l_S(g))|}{|B(n_k)|} - 2 \le 2\limsup_{k\to\infty} \frac{|B(n_k + k)|}{|B(n_k)|} - 2 \le 0$$

□

Problem 2

Let G be a countable, discrete group. For $p \in [1, \infty)$ we say $(f_n)_{n=1}^{\infty}$ in $l^p(G)$ are almost invariant vectors if $\|f_n\|_p = 1$ and if

$$\|\lambda_g f_n - f_n\|_p \xrightarrow{n\to\infty} 0 \text{ for all } g \in G.$$

(a): For $p \in [1, +\infty)$ and $f \in l^p(G)$ prove that $\|\lambda_g|f| - |f|\|_p \le \|\lambda_g f - f\|_p$ for all $g \in G$.

Proof. By the reverse triangle inequality, we have that $|\lambda_g|f| - |f|| \le |\lambda_g f - f|$ pointwise. Now,

$$\|\lambda_g|f| - |f|\|_p^p = \int |\lambda_g|f| - |f||^p \, d\mu \le \int |\lambda_g f - f|^p \, d\mu \le \|\lambda_g f - f\|_p^p,$$

whence the result follows. □

(b): For $a, b \in [0, +\infty)$ and $p \in [1, +\infty)$ prove that $|a^{1/p} - b^{1/p}| \le |a - b|^{1/p}$ and

$$|a^p - b^p| \le p|a - b| \max(a^{p-1}, b^{p-1}) \le p|a - b|(a^{p-1} + b^{p-1}).$$

Proof. The first inequality follows from homework 1 problem 1 part (a). Note that the second inequality is trivial if a or b is zero or if $p = 1$, so assume $a, b > 0$ and $p > 1$.

Consider the polynomial $f(x) = x^p + p(1 - x) - 1$ on the interval $(0, 1]$. Computing $f'(x) = px^{p-1} - p$, the only critical points for f are at $x = 1$ whence $f(x) = 0$. As $f' < 0$ for all $x \in (0, 1)$ and $f(0) = p - 1 > 0$, it follows that $f(x) \ge f(1) = 0$ for all $x \in (0, 1]$.

Without loss of generality, assume $b \le a$. Consider $x = \frac{b}{a} \le 1$. By the nonegativity of the above polynomial,

$$1 - \frac{b^p}{a^p} = 1 - x^p \le p(1 - x) = p\frac{(a - b)a^{p-1}}{a}$$

whence the second inequality follows. □

(c): Suppose $p \in [1, +\infty)$. Prove that there are almost invariant vectors in $l^p(G)$ if and only if G is amenable.

Proof.
$\underline{\Longrightarrow}$: Suppose $(f_n)_{n=1}^{\infty}$ is a sequence of almost invariant unit vectors in $l^p(G)$, and fix $g \in G$. Let $\mu_n := |f_n|^p$ and note that $\mu_n \in \text{Prob}(G) \subseteq l^1(G)$. As $\|f_n\| = 1$, $|f_n| \ge 0$, and G is discrete, it follows that $|f_n| \le \|f_n\| = 1$. By part (a), it follows that

$$\|\lambda_g|f_n| - |f_n|\|_p \le \|\lambda_g f_n - f_n\|_p \xrightarrow{n\to\infty} 0.$$

Now observe that, applying Holder's inequality with conjugate exponents $p, \frac{p}{p-1}$,

$$\begin{aligned}
\|\lambda_g \mu_n - \mu_n\|_1 = \int ||\lambda_g f_n|^p - |f_n|^p| \, d\mu &\le p \int ||\lambda_g f_n| - |f_n|| \cdot \max\left\{|\lambda_g f_n|^{p-1}, |f_n|^{p-1}\right\} d\mu \\
&\le p\|\lambda_g|f_n| - |f_n|\|_p \cdot \left\|\max\left\{|\lambda_g f_n|^{p-1}, |f_n|^{p-1}\right\}\right\|_{\frac{p}{p-1}} \\
&\le p\|\lambda_g|f_n| - |f_n|\|_p \cdot \left(\int \max\left\{|\lambda_g f_n|^{p-1}, |f_n|^{p-1}\right\}^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \\
&\le p\|\lambda_g|f_n| - |f_n|\|_p \cdot \left(\int \max\left\{|\lambda_g f_n|^{p}, |f_n|^{p}\right\}\right)^{\frac{p-1}{p}} \\
&\le p\|\lambda_g|f_n| - |f_n|\|_p \cdot \left(\int |\lambda_g f_n|^p + |f_n|^p\right)^{\frac{p-1}{p}} \\
&\le 2^{\frac{p-1}{p}} p\|\lambda_g|f_n| - |f_n|\|_p \xrightarrow{n\to\infty} 0
\end{aligned}$$

$\underline{\Longleftarrow}$: Suppose that G is amenable and $p \in [1, +\infty)$. Choose a sequence $(\mu_n)_{n=1}^{\infty}$ of almost invariant probability measures for G. Set $f_n = \mu_n^{1/p}$. Then $f_n \in l^p(G)$ and $\|f_n\|_p = 1$. So, we compute that

$$\|\lambda_g f_n - f_n\|_p^p = \int \left|\lambda_g \mu_n^{1/p} - \mu_n^{1/p}\right|^p d\mu \leq \int |\lambda_g \mu_n - \mu_n| \, d\mu \xrightarrow{n\to\infty} 0$$

$\square$