Intro Math Research Hw4

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1 Abstracts

Talk Abstract

"Groups, as men, shall be know by their actions" -Guillermo Moreno. In this talk, we introduce the field of representation theory and its connection to combinatorics. Through examples, we hint towards a deep connection between the combinatorics of partitions and the actions of symmetric groups on vector spaces. The only background required will be linear algebra and some knowledge of group theory.

Paper Abstract

In this paper we exposit one of the fundamental results linking representation theory and algebraic combinatorics called Schur-Weyl duality. It provides a dictionary between the representation theory of finite symmetric groups and the representation theory of the general linear group of a finite dimensional complex vector space. Through this dictionary, we obtain representation theoretic constructions of many aspects of symmetric function theory, including Schur functions and Littlewood-Richardson coefficients.

2 Outlines

2.1 Talk Outline

- Run through the standard example $\pi : D_{2n} \to O(3)$
- Introduce definition of representations of finite groups.
- Get relevant defs to say $V \otimes V \cong Sym^2 \oplus \Lambda^2 V$.
- State $V \otimes V \otimes V \cong Sym^3 V \oplus \Lambda^3 V \oplus$ something else .
- Hint how that something else is related to the partition $(2, 1)$ of 3.

2.2 Paper Outline

- Set up relevant preliminary representation theoretic definitions (group algebra stuff etc.)
- Talk about the generic representation theory of S_n
- Define Young symmetrizers and Specht Modules
- Talk about the commuting left and right actions of $GL(V)$ and S_n respectively.
- State and cite the double centralizer theorem to prove Schur-Weyl duality.
- Obtain relations to Schur functions, Littlewood-Richardson coefficients, plethyism, etc.

3 Symmetric Polynomials/Functions Exposition

Preliminary Considerations

Throughout this article, fix a (unital) commutative ring R and a field k. For simplicity, we work over vector spaces instead of general modules.

Notation. Let X be a set such that $X = \{x_i\}_{i \in I}$ for some indexing set I. By $k[X]$ and $k[\![X]\!]$, we denote the rings of (commutative) polynomials and power series (resp.) in indeterminates $\{x_i\}$. We utilize multi-index notation throughout. Hence for $\alpha_\bullet : I \to \mathbb{N} \cup \{0\}$ finitely supported, we write $x_\alpha = \Pi_{i \in I} x_i^{\alpha_i}$ (where $x_i^0 := 1$ formally).

3.1 Algebraic Background

Often in algebra, elements of a given object may be decomposed into a sum of simpler elements which are, in a sense, "homogenous." For example, any polynomial in n-variable may be decomposed into a sum of simpler polynomials each of which are futher sums of monomials of the same total degree. In this way, a polynomial is split into a sum of homogenous parts. This behavior is codified in the notion of *grading*.

Definition 3.1.1. A *graded k-algebra* is a k-algebra A together with a direct sum decomposition

$$A = \bigoplus_{i=0}^{\infty} A_i$$

with $A_0, A_1, \ldots$ vector spaces such that $A_i \cdot A_j \subseteq A_{i+j}$ for all $i, j \in \mathbb{N} \cup \{0\}$. For fixed i, elements of A_i are called *homogenous*. The choice of such a direct sum decomposition is a *grading* for A.

Key Example. As before, for $X = \{x_i\}_{i \in I}$, we may give the ring $k[X]$ a canonical grading by declaring $A_0 := k$ and

$$A_n := \mathrm{Span}_k\{x_\alpha : \alpha \text{ multi-index such that } \sum_{i \in I} \alpha_i = n\}.$$

The reader is cautioned that not every k-algebra has a nontrivial grading. In fact, it can be shown that the ring of formal power series $k[\![x]\!]$ does not have a nontrivial grading.

3.2 Symmetric Polynomials

Definition 3.2.1. The permutation group S_n acts naturally on the polynomial ring $k[x_1, \ldots, x_n]$ by defining $\sigma \cdot x_{i_1}^{\alpha_1} \cdots x_{i_l}^{\alpha_l} := x_{\sigma(i_1)}^{\alpha_1} \cdots x_{\sigma(i_l)}^{\alpha_l}$ and extending by linearity. The ring of *symmetric polynomials* in n indeterminates is the fixed points of this action, namely $k[x_1, \ldots, x_n]^{S_n}$.

3.3 Partitions and Compositions

Definition 3.3.1.

- A *partition* of $n \in \mathbb{N}$ is a finite sequence $\alpha = (\alpha_1, \ldots, \alpha_l)$ of weakly decreasing positive integers which sum to n. We denote the set of partitions of n by $\mathrm{Par}(n)$. We denote the statement $[\lambda \in \mathrm{Par}(n)]$ by $\lambda \vdash n$. Also, we write $\mathrm{Par} := \bigcup_{n \geq 0} \mathrm{Par}(n)$.

- A *weak composition* of $n \in \mathbb{N}$ is a (finitely supported) sequence $\alpha = (\alpha_i)_{i=1}^{\infty} \in (\mathbb{N} \cup \{0\})^{\mathbb{N}}$ such that $\sum_i \alpha_i = n$. The length of a weak composition α is given by

$$l(\alpha) := \max\{i \in \mathbb{N} : \alpha_i \neq 0\}.$$

Example 3.3.1. For $n = 5$, the sequences $\alpha = (1, 0, 2, 2, 0, 0, \dots)$ and $\beta = (2, 0, 1, 2, 0, 0, \dots)$ are distinct weak compositions but neither are valid partitions of 5 due to the presence of a 0 between positive entries. On the other hand, $\lambda = (2, 2, 1)$ is a partition of 5.

3.4 Symmetric Functions

Definition 3.4.1 (pg. 308 in [Sta24])**.** The ring Λ_k of symmetric functions over a field k is the subring of all $f \in k[\![x_1, x_2, \dots]\!]$ such that

$$f(x_{\sigma(1)}, x_{\sigma(2)}, \dots) = f(x_1, x_2, \dots) \text{ for all } \sigma \in \mathrm{Sym}(\mathbb{N}).$$

Remark 3.4.1. For the algebraically-minded, there is a more natural construction of Λ_k by viewing the ring as the colimit of a certain directed system of injections of symmetric polynomial rings

$$k[x_1, \dots, x_n]^{S_n} \xhookrightarrow{\varphi_n} k[x_1, \dots, x_{n+1}]^{S_{n+1}}.$$

The construction of these maps φ_n is somewhat involved. This does justify the intuition that a symmetric function is simply taking a symmetric polynomial and adding more data, as any element of a direct limit of inclusions is faithfully represented by an element of one of the constituent objects.

Definition 3.4.2. A symmetric function $f \in \Lambda_k$ is homogenous of degree n if

$$f(x) = \sum_{\alpha \text{ weak composition of } n} c_\alpha x^\alpha,$$

where the c_α are elements of k. The set of degree n homogenous symmetric functions is denoted Λ_k^n. these subspaces give Λ_k the structure of a graded k-algebra, namely:

- Each Λ_k^n is a k-vector space,
- $\Lambda_k^i \Lambda_k^j \subseteq \Lambda_k^{i+j}$,
- $\Lambda_k = \bigoplus_{n=0}^\infty \Lambda_k^n$ as k-vector spaces.

The first interesting basis of Λ_k is the *monomial symmetric functions*. Given $\lambda \vdash n$, define $m_\lambda \in \Lambda_k^n$ by

$$m_\lambda := \sum_\alpha x^\alpha$$

where the sum is over all distinct permutations of the entries of λ. The set $\{m_\lambda : \lambda \vdash n\}$ forms a basis for Λ_k^n, whence $\bigcup_{n \geq 0} \{m_\lambda : \lambda \vdash n\} = \{m_\lambda : \lambda \in \mathrm{Par}\}$ forms a basis for Λ_k.

3.5 Complete Homogenous Symmetric Functions

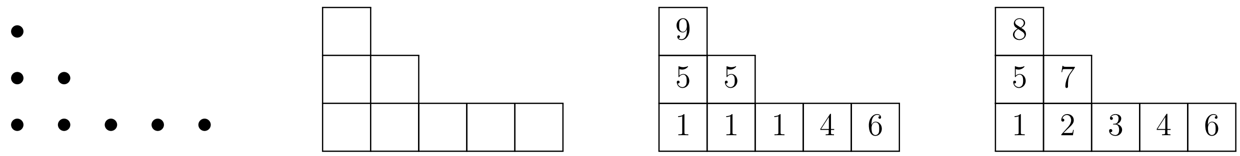
From the monomial symmetric functions, we may form another interesting basis for Λ_k called the *complete homogenous symmetric functions* h_λ by setting

$$h_\lambda := \prod_{i=1}^\infty \sum_{\nu \vdash \lambda_i} m_\nu.$$

where $\lambda = (\lambda_1, \lambda_2, \dots)$. Again, the set $\{h_\lambda : \lambda \vdash n\}$ is a basis for Λ_k^n and the set $\{h_\lambda : \lambda \in \mathrm{Par}\}$ is a basis for Λ_k.

3.6 Schur Functions

Definition 3.6.1. Given $\lambda \vdash n$, the *Ferrers diagram of shape* λ is the set $\{(i,j) \in \mathbb{N}^2 : j \in \mathbb{N}, 1 \le i \le \lambda_j\}$ depicted as points in $\mathbb{R}^2$. The *Young diagram of shape* λ is depicted identically to the Ferrers diagram except the points are replaced with squares. The *size* of the diagram is the number of entries, namely n. We depict the case $(5,2,1) \vdash 8$ below.



Ferrers Diagram for the partition (5,2,1)

Young Diagram for the partition (5,2,1)

Semi-standard Young tableau for the partition (5,2,1)

Standard Young tableau for the partition (5,2,1)

Definition 3.6.2. Given $\lambda \vdash n$ and a Young diagram of shape λ, a *semi-standard Young tableau of shape* λ is a filling of the boxes of the Young diagram with positive integers such that

- the entries are weakly increasing along rows,
- the entries are strictly increasing up columns.

A semi-standard Young tableau of size n is said to be *standard* if the elements of $\{1, \dots, n\}$ each appear exactly once in the tableau. We write $SSYT(\lambda)$ and $SYT(\lambda)$ for the sets of semi-standard and standard Young tableaux of shape λ. Given a semi-standard Young tableaux $\mathcal{T}$, the *weight* of $\mathcal{T}$ is a function $\alpha = \alpha_{\mathcal{T}} : \mathbb{N} \to \mathbb{N}$ given by

$$\alpha(i) := \text{number of times } i \text{ appears in } \mathcal{T}.$$

Note that $\alpha(i) = 0$ for sufficiently large i, so we may write $x^\alpha = x_1^{\alpha(1)} x_2^{\alpha(2)} \cdots$ and obtain a valid monomial. We write $SSYT(\lambda, \alpha)$ for the set of semi-standard Young tableaux of shape λ and weight α.

Definition 3.6.3. Given $\lambda \vdash n$, the *Schur function* indexed by λ is defined by

$$s_\lambda := \sum_{\mathcal{T} \in SSYT(\lambda)} x^{\alpha_{\mathcal{T}}} = \sum_\alpha \sum_{\mathcal{T} \in SSYT(\lambda,\alpha)} x^\alpha$$

where $\alpha_{\mathcal{T}}$ denotes the weight of the Young tableau α.

Proposition 3.6.1. *The Schur function s_λ is a symmetric function.*

Proof. Let $\sigma \in S_\infty$. Then by definition of the action of S_∞ we have

$$\sigma \cdot s_\lambda = \sum_\alpha |SSYT(\lambda,\alpha)| \cdot \sigma \cdot x^\alpha = \sum_\alpha |SSYT(\lambda,\alpha)| x^{\sigma\alpha} = \sum_\alpha |SSYT(\lambda, \sigma\alpha)| x^\alpha.$$

By the Bender-Knuth involution, we have that for fixed weak composition α, $|SSYT(\lambda,\alpha)| = |SSYT(\lambda,\sigma\alpha)|$, whence

$$\sigma s_\lambda = \sum_\alpha |SSYT(\lambda,\sigma\alpha)| x^\alpha = \sum_\alpha |SSYT(\lambda,\alpha)| x^\alpha = s_\lambda.$$

$\square$