Schur Weyl Duality and The Frobenius Formula

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Abstract

In this paper we exposit one of the fundamental results linking representation theory and algebraic combinatorics called Schur-Weyl duality. It provides a dictionary between the representation theory of finite symmetric groups and the representation theory of the general linear group of a finite dimensional complex vector space. Through this dictionary, we obtain representation theoretic constructions of many aspects of symmetric function theory, including Schur functions, Kostka numbers, and internal/external products on the symmetric function ring.

Contents

1 Representation Theory Background **1**
1.1 Group Representations 1
1.2 Character Theory and Orthogonality Relations 1
1.3 Fundamental Examples 1

2 Representations of S_n **1**
2.1 Partitions, Young Diagrams, and Tabloids 1
2.2 Construction of Specht Modules 2
2.3 Alternative Construction 4
2.4 Results on Specht Modules 5

1 Representation Theory Background

1.1 Group Representations

1.2 Character Theory and Orthogonality Relations

1.3 Fundamental Examples

2 Representations of S_n

[FH91]

2.1 Partitions, Young Diagrams, and Tabloids

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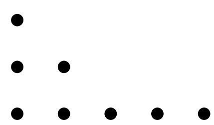
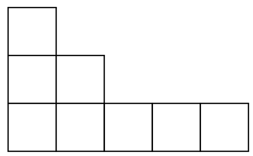
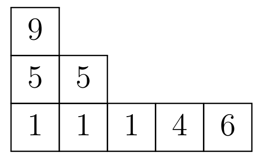
Definition 2.1.1. Given $\lambda \vdash n$, the *Ferrers diagram of shape* λ is the set $\{(i,j) \in \mathbb{N}^2 : j \in \mathbb{N}, 1 \leq i \leq \lambda_j\}$ depicted as points in $\mathbb{R}^2$. The *Young diagram of shape* λ is depicted identically to the Ferrers diagram except the points are replaced with squares. The *size* of the diagram is the number of entries, namely n. We depict the case $(5,2,1) \vdash 8$ below.



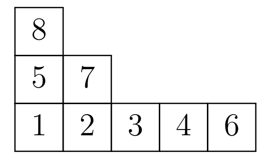
Ferrers Diagram for the partition (5,2,1)



Young Diagram for the partition (5,2,1)



Semi-standard Young tableau for the partition (5,2,1)



Standard Young tableau for the partition (5,2,1)

Definition 2.1.2. Given $\lambda \vdash n$ and a Young diagram of shape λ, a *semi-standard Young tableau of shape* λ is a filling of the boxes of the Young diagram with positive integers such that

- the entries are weakly increasing along rows,
- the entries are strictly increasing up columns.

A semi-standard Young tableau of size n is said to be *standard* if the elements of $\{1, \ldots, n\}$ each appear exactly once in the tableau. We write $SSYT(\lambda)$ and $SYT(\lambda)$ for the sets of semi-standard and standard Young tableaux of shape λ. Given a semi-standard Young tableau $\mathcal{T}$, the *weight* of $\mathcal{T}$ is a function $\alpha = \alpha_{\mathcal{T}} : \mathbb{N} \to \mathbb{N}$ given by

$$\alpha(i) := \text{number of times } i \text{ appears in } \mathcal{T}.$$

Note that $\alpha(i) = 0$ for sufficiently large i, so we may write $x^\alpha = x_1^{\alpha(1)} x_2^{\alpha(2)} \cdots$ and obtain a valid monomial. We write $SSYT(\lambda, \alpha)$ for the set of semi-standard Young tableaux of shape λ and weight α.

Definition 2.1.3. Given $\lambda \vdash n$, a λ-tableau is simply a filling of the boxes of the Young diagram of shape λ with the elements of $\{1, \ldots, n\}$ without repetition (and no other restrictions). Denote the set of λ-tableaux by $YT(\lambda)$. Note that $S_n \curvearrowright YT(\lambda)$ by permuting labels.

Definition 2.1.4. Given $\lambda \vdash n$, define an equivalence relation $\sim$ on $YT(\lambda)$ by $\mathcal{T} \sim \mathcal{T}'$ if and only if $\mathcal{T}'$ can be obtained from $\mathcal{T}$ by permuting the entries of each row. An equivalence class with respect to this relation is called a λ*-tabloid*. If $\mathcal{T}$ is a λ-tableau, we write $\{\mathcal{T}\}$ for the corresponding λ-tabloid. Finally, we write $Tab(\lambda) := YT(\lambda)/\sim$ for the set of λ-tabloids. Note that the action of S_n on λ-tableaux descends to an action on λ-tabloids.

2.2 Construction of Specht Modules

Young diagrams will give projection operators $P_\lambda : \mathbb{C}[S_n] \to \mathbb{C}[S_n]$ which commute with the action of S_n, whence the image $P_\lambda(\mathbb{C}[S_n])$ gives a subrepresentation of the regular representation. These subrepresentations will end up being precisely the irreducible representations of S_n. Throughout this section, $\lambda \vdash n$ will be fixed.

Definition 2.2.1. Given a λ-tableau $\mathcal{T}$, define the *row group* $R_{\mathcal{T}}$ to be the subgroup of S_n which permutes only the labels in the rows of $\mathcal{T}$ and the *column group* $C_{\mathcal{T}}$ as the subgroup which permutes only the labels in the columns of $\mathcal{T}$.

Now we may define the *Young row and column symmetrizers* in $\mathbb{C}[S_n]$ by

$$a_{\mathcal{T}} := \sum_{\sigma \in R_{\mathcal{T}}} \sigma, \qquad b_{\mathcal{T}} := \sum_{\sigma \in C_{\mathcal{T}}} \operatorname{sgn}(\sigma)\sigma. \tag{1}$$

Note that for $\mathcal{T} \in YT(\lambda)$, the corresponding tabloid is precisely the orbit of $\mathcal{T}$ under its row group, i.e.

$$\{\mathcal{T}\} = R_{\mathcal{T}}\mathcal{T} = \{\sigma\mathcal{T} \in YT(\lambda) : \sigma \in R_{\mathcal{T}}\}.$$

Now let M^λ be the free $\mathbb{C}$-vector space over the set of λ-tabloids. Extending the action $S_n \curvearrowright Tab(\lambda)$ linearly to all of M^λ, we obtain a $\mathbb{C}[S_n]$-module structure on M^λ. For $\mathcal{T} \in YT(\lambda)$, the element $e_{\mathcal{T}} \in M^\lambda$ given by

$$e_{\mathcal{T}} := b_{\mathcal{T}} \cdot \{\mathcal{T}\} = \sum_{\sigma \in C_{\mathcal{T}}} \operatorname{sgn}(\sigma)\{\sigma\mathcal{T}\}$$

is called the *polytabloid associated to* $\mathcal{T}$. Let S^λ be the subspace of M^λ generated by all polytabloids, namely

$$S^\lambda := \operatorname{Span}_{\mathbb{C}}\{e_{\mathcal{T}} : \mathcal{T} \in YT(\lambda)\}.$$

Claim. S^λ is a $\mathbb{C}[S_n]$-submodule of M^λ.

Proof of Claim. Fix $\sigma \in S_n$. We first show that $C_{\sigma\mathcal{T}} = \sigma C_{\mathcal{T}}\sigma^{-1}$. Indeed, if T_i is the set of entries for the ith column of $\mathcal{T}$, then $\sigma(T_i)$ is the entries for the ith column of $\sigma\mathcal{T}$. Now it suffices to note that $\tau \in S_n$ stabilizes T_i if and only if $\sigma\tau\sigma^{-1}$ stabilizes $\sigma(T_i)$. Using this identity, we compute

$$\sigma b_{\mathcal{T}} = \sum_{\gamma \in C_{\mathcal{T}}} \operatorname{sgn}(\gamma)\sigma\gamma \overset{\tau=\sigma\gamma\sigma^{-1}}{=} \sum_{\tau \in \sigma C_{\mathcal{T}}\sigma^{-1}} \operatorname{sgn}(\sigma^{-1}\tau\sigma)\tau\sigma = \sum_{\tau \in C_{\sigma\mathcal{T}}} \operatorname{sgn}(\tau)\tau\sigma = b_{\sigma\mathcal{T}}\sigma.$$

Now we apply σ to the generators of S^λ and find

$$\sigma \cdot e_{\mathcal{T}} = \sigma \cdot (b_{\mathcal{T}} \cdot \{\mathcal{T}\}) = (\sigma b_{\mathcal{T}}) \cdot \{\mathcal{T}\} = b_{\sigma\mathcal{T}}\{\sigma\mathcal{T}\} = e_{\sigma\mathcal{T}}.$$

As S_n stabilizes S^λ, the claim follows. □

Definition 2.2.2. The $\mathbb{C}[S_n]$-module S^λ as defined above is the *Specht module corresponding to* λ.

Example 2.2.1 (Sign Representation)**.** Consider the partition $\lambda = (1, 1, \ldots, 1)$ of n. Since each row of λ has one element, the λ-tabloids are the same as λ-tableaux.

Let $\mathcal{T}$ be a λ-tableau. As $\mathcal{T}$ has only one column, $C_{\mathcal{T}} = S_n$, whence $b_{\mathcal{T}} = \sum_{\gamma \in S_n} \operatorname{sgn}(\gamma)\gamma$ and consequently

$$\sigma e_{\mathcal{T}} = \sum_{\gamma \in S_n} \operatorname{sgn}(\gamma)\sigma\gamma\{\mathcal{T}\} = \sum_{\tau \in S_n} \operatorname{sgn}(\sigma^{-1}\tau)\tau\{\mathcal{T}\} = \operatorname{sgn}(\sigma)e_{\mathcal{T}} \quad \text{for all } \sigma \in S_n.$$

On the other hand, we know that $\sigma e_{\mathcal{T}} = e_{\sigma\mathcal{T}}$, so it follows that $S^\lambda = \mathbb{C}e_{\mathcal{T}}$ is the one-dimensional sgn representation.

Example 2.2.2 (Trivial Representation)**.** Consider the partition $\lambda = (n)$ of n. Since there is one row of λ, all λ-tableaux are equivalent so there is only one λ-tabloid. Fix a λ-tableau $\mathcal{S}$.

Each $e_{\mathcal{T}} = \{T\} = \{S\}$, so $S^\lambda = \mathbb{C}e_{\mathcal{S}}$ is one-dimensional. The action of σ is given by $\sigma e_{\mathcal{T}} = e_{\sigma\mathcal{T}} = e_{\mathcal{T}}$, so S^λ is the trivial representation of S_n.

$$\begin{array}{|c|c|c|c|c|}\cline{1-1} i & \multicolumn{4}{c}{} \\ \hline j & & \cdots & & \\ \hline \end{array}$$

General form of $\mathcal{T}$ when $t_i = \{\mathcal{T}\}$

Example 2.2.3 (Augmentation Subrepresentation)**.** Consider the partition $\lambda = (n-1, 1)$ of n. Observe that there are n distinct λ-tabloids, each corresponding to the integer in singular box on the 2nd row. Denote the tabloid with i in the $2nd$ row by t_i, so $Tab(\lambda) = \{t_1, \dots, t_n\}$.

Let $V = \mathbb{C}\{v_1, \dots, v_n\}$ be the standard representation of S_n (i.e. $\sigma v_i = v_{\sigma(i)}$). Observe that the map $L : V \to M^\lambda$ given by $L(v_i) = t_i$ is an isomorphism of $\mathbb{C}[S_n]$-modules. The *augmentation subrepresentation* W of V is given by $W := \{\sum_{i=1}^n \alpha_i v_i : \sum_i \alpha_i = 0\}$. We claim that $S^\lambda \cong W$ as $\mathbb{C}[S_n]$-modules. Fix $i \in \{1, \dots, n\}$ and let $\mathcal{T}$ be a λ-tableau such that $t_i = \{\mathcal{T}\}$. Let j be the integer below i on the tableau. Then the column group $C_\mathcal{T}$ is then of order 2 generated by the transposition $(i\ j)$.

$$e_\mathcal{T} = \sum_{\gamma \in C_\mathcal{T}} \operatorname{sgn}(\gamma) \gamma t_i = t_i - t_j.$$

Hence, one checks

$$S^\lambda = \operatorname{Span}\{t_i - t_j : 1 \leq i, j \leq n, i \neq j\} = \operatorname{Span}\{t_i - t_{i+1} : 1 \leq i \leq n-1\}.$$

Moreover, $\{t_i - t_{i+1} : 1 \leq i \leq n-1\}$ gives a basis for S^λ. The restriction of L to W gives a vector space isomorphism $L : W \to S^\lambda$ as $\{v_i - v_{i+1}\}_{1 \leq i \leq n-1}$ gives a basis for W, so a basis gets mapped to a basis. Moreover, this map intertwines the S_n-action, so it produces $\mathbb{C}[S_n]$-module isomorphism.

2.3 Alternative Construction

Fix a λ-tableau $\mathcal{S}$ throughout this section, say the canonical one (increasing across rows and then moving up rows). Recall the row and column symmetrizers $a_\lambda := a_\mathcal{S}$, $b_\lambda := b_\mathcal{S}$ and define the Young symmetrizer

$$c_\lambda := a_\lambda \cdot b_\lambda \in \mathbb{C}[S_n].$$

Set $V_\lambda := \mathbb{C}[S_n]c_\lambda$. Define a map $T : \mathbb{C}[S_n]a_\lambda \to M^\lambda$ by $T(\sigma a_\lambda) = \{\sigma\mathcal{S}\}$.

Claim. The map T is an isomorphism of $\mathbb{C}[S_n]$-modules.

Proof of Claim. We first show this map is well defined. If $\sigma a_\lambda = \tau a_\lambda$, then $\tau^{-1}\sigma$ fixes a_λ, whence $\tau^{-1}\sigma \in R_\mathcal{S}$ and consequently $\sigma\{\mathcal{S}\} = \tau\{\mathcal{S}\}$.

Since the action of S_n on λ-tableau is transitive, it follows that the map T is onto. On the other hand, suppose $\sum_\sigma \alpha_\sigma \sigma a_\lambda \in \ker(T)$. Then

$$0 = T(\sum_\sigma \alpha_\sigma \sigma a_\lambda) = \sum_\sigma \alpha_\sigma \{\sigma\mathcal{S}\}.$$

Since M^λ is a free $\mathbb{C}$-module, it follows that $\sum_\sigma \alpha_\sigma \sigma = 0$. Lastly, if $\sigma, \gamma \in S_n$, then

$$\sigma T(\gamma a_\lambda) = \sigma\{\gamma\mathcal{S}\} = \{\sigma\gamma\mathcal{S}\} = T(\sigma\gamma s_\lambda).$$

□

Claim. The map T restricted to the submodule $\mathbb{C}[S_n]b_\lambda a_\lambda$ gives a $\mathbb{C}[S_n]$-module isomorphism $\mathbb{C}[S_n]b_\lambda a_\lambda \cong S^\lambda$.

Proof of Claim. For $\sigma \in S_n$, we compute

$$\begin{aligned} T(\sigma b_\lambda a_\lambda) &= \sum_{\tau \in C_{\mathcal{S}}} \operatorname{sgn}(\tau) T(\sigma\tau a_\lambda) = \sum_{\tau \in C_{\mathcal{S}}} \operatorname{sgn}(\tau)\{\sigma\tau\mathcal{S}\} \\ &= \sigma \sum_{\tau \in C_{\mathcal{S}}} \operatorname{sgn}(\tau)\{\tau\mathcal{S}\} = \sigma e_{\mathcal{S}} = e_{\sigma\mathcal{S}} \end{aligned}$$

Since S_n acts transitively on λ-tableaux, it follows that

$$T(\mathbb{C}[S_n]b_\lambda a_\lambda) = \operatorname{Span}_{\mathbb{C}}\{e_{\sigma\mathcal{S}} : \sigma \in S_n\} = S^\lambda$$

By the proof of the previous claim, T is injective and intertwines the action of S_n, whence $T|_{\mathbb{C}[S_n]b_\lambda a_\lambda}$ furnishes an isomorphism of $\mathbb{C}[S_n]$-modules as desired. □

Proposition 2.3.1. *Set $A = \mathbb{C}[S_n]$, so $V_\lambda = Aa_\lambda b_\lambda = Ac_\lambda$.*

1. *$V_\lambda \cong Ab_\lambda a_\lambda$.*
2. *V_λ is the image of the map from Aa_λ to Ab_λ given by right multiplication by b_λ.*

2.4 Results on Specht Modules

Having obtained a few examples of Specht modules, we now show that $\{S^\lambda : \lambda \vdash n\}$ forms a complete set of non-isomorphic, irreducible representations of S_n. This is established by the combining the following three theorems.

Theorem 2.4.1. *Given $\lambda \vdash n$, the Specht module S^λ is irreducible as a $\mathbb{C}[S_n]$-module (i.e. an irreducible representation of S_n).*

Theorem 2.4.2. *If $\lambda, \mu \vdash n$ and $\lambda \neq \mu$, then $S^\lambda \ncong S^\mu$ as $\mathbb{C}[S_n]$-modules.*

Theorem 2.4.3. *Every irreducible representation of S_n is ismorphic to S^λ for some $\lambda \vdash n$.*

References

[FH91] William Fulton and Joe Harris. *Representation theory*. Vol. 129. Graduate Texts in Mathematics. A first course, Readings in Mathematics. Springer-Verlag, New York, 1991, pp. xvi+551.