CS 6316 Homework 1

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1 Problem 1

Show that, given a training set $S = \{(x_i, f(x_i))\}_{i=1}^m \subseteq (\mathbb{R}^d \times \{0,1\})^m$, there exists a polynomial p_S such that $h_S(x) = 1$ if and only if $p_S(x) \geq 0$, where h_S is defined as

$$h_S(x) = \begin{cases} y_i & \text{if } x = x_i \text{ for some } i \\ 0 & \text{otherwise} \end{cases}$$

Solution. Consider the polynomial

$$p_S(x) = -\prod_{i=1}^m \|x - x_i\|_2^2 - (f(x) - 1)^2$$

where $\|\cdot\|_2$ denotes the standard Euclidean norm on $\mathbb{R}^d$. Note that, by squaring the $\|x - x_i\|_2$ terms above, we obtain a true polynomial in the coordinates of $x \in \mathbb{R}^d$.

Claim. $h_S(x) = 1$ if and only if $p_S(x) \geq 0$.

Proof of Claim. Note first that, for arbitrary x, both $(f(x)-1)^2 \geq 0$ and $\prod_{i=1}^m \|x - x_i\|_2^2 \geq 0$ by the trivial inequality, whence

$$p_S(x) = -\prod_{i=1}^m \|x - x_i\|_2^2 - (f(x) - 1)^2 \leq -\prod_{i=1}^m \|x - x_i\|_2^2 \tag{1}$$

$$p_S(x) = -\prod_{i=1}^m \|x - x_i\|_2^2 - (f(x) - 1)^2 \leq -(f(x) - 1)^2. \tag{2}$$

<u>($\Longleftarrow$)</u>: We proceed by contraposition. Suppose that $h_S(x) \neq 1$. We wish to show that $p_S(x) < 0$. Note that, as the image of h_S is $\{0, 1\}$, it follows that $h_S(x) = 0$. Now we have two cases.

If $x = x_j$ for some $j \in \{1, \ldots, m\}$, then $0 = h_S(x_j) = f(x_j)$, whence by (2),

$$p_S(x) \leq -(f(x) - 1)^2 = -1 < 0.$$

If $x \notin \{x_1, \ldots, x_m\}$, then $\|x - x_i\|_2^2 > 0$ for all i, whence by (1) we have

$$p_S(x) \leq -\prod_{i=1}^m \|x - x_i\|_2^2 < 0.$$

<u>($\Longrightarrow$)</u>: On the other hand, suppose that $h_S(x) = 1$. Then it follows that $x = x_j$ for some $j \in \{1, \ldots, m\}$ and $f(x_j) = h(x_j) = 1$. Hence,

$$p_S(x) = -\prod_{i=1}^m \|x - x_i\|_2^2 - (f(x) - 1)^2 = -\|x_j - x_j\|_2^2 \prod_{i=j} \|x - x_i\|_2^2 - (1 - 1)^2 = 0.$$

$\square$

2 Problem 2

Let $\mathcal{H}$ be a class of binary classifiers over a domain $\mathcal{X}$. Let $\mathcal{D}$ be an unknown distribution over $\mathcal{X}$, and let f be the target hypothesis in $\mathcal{H}$. Fix some $h \in \mathcal{H}$. Show that

$$\mathop{\mathbb{E}}_{S|_x \sim \mathcal{D}^m}[L_S(h)] = L_{(\mathcal{D},f)}(h).$$

Proof. This claim follows from the construction of the product measure, the definition of the pushforward measure, and some elementary manipulation. By definition,

$$L_{(\mathcal{D},f)}(h) = \mathcal{D}(\{x \in \mathcal{X} : f(x) \neq h(x)\}).$$

Let $\pi_i : \mathcal{X}^m \to \mathcal{X}$ denote the i^{th} projection map. Note that, for any measurable subset $A \subseteq \mathcal{X}$ and $(x_1, \ldots, x_m) \in \mathcal{X}^m$, we have

$$\mathbb{1}_A(x_i) = \mathbb{1}_A(\pi_i((x_1, \ldots, x_m))) = \mathbb{1}_{\pi_i^{-1}(A)}((x_1, \ldots, x_m)). \tag{3}$$

Recall, by construction of the product measure, that the product measure $\mathcal{D}^m$ pushes forward to the original measure $\mathcal{D}$ under each of the projection maps π_i, namely

$$\mathcal{D}^m(\pi_i^{-1}(A)) \overset{\text{definition}}{=} (\pi_i)_*\mathcal{D}^m(A) = \mathcal{D}(A) \text{ for all measurable } A \subseteq \mathcal{X}. \tag{4}$$

Finally, we expand the expectation and write,

$$\begin{aligned}
\mathop{\mathbb{E}}_{S|_x \sim \mathcal{D}^m}[L_S(h)] &= \int_{\mathcal{X}^m} L_S(h)\, d\mathcal{D}^m(x_1, \ldots, x_m) = \int_{\mathcal{X}^m} \frac{|\{\widetilde{x} \in S|_x : h(\widetilde{x}) \neq f(\widetilde{x})\}|}{m}\, d\mathcal{D}^m(x_1, \ldots, x_m) \\
&= \frac{1}{m} \int_{\mathcal{X}^m} \sum_{i=1}^{m} \mathbb{1}_{\{x \in \mathcal{X} : h(x) \neq f(x)\}}(x_i)\, d\mathcal{D}^m(x_1, \ldots, x_m) \\
&\overset{(3)}{=} \frac{1}{m} \sum_{i=1}^{m} \int_{\mathcal{X}^m} \mathbb{1}_{\pi_i^{-1}(\{x \in \mathcal{X} : h(x) \neq f(x)\})}((x_1, \ldots, x_n))\, d\mathcal{D}^m(x_1, \ldots, x_m) \\
&= \frac{1}{m} \sum_{i=1}^{m} \mathcal{D}^m(\pi_i^{-1}(\{x \in \mathcal{X} : h(x) \neq f(x)\})) \\
&\overset{(4)}{=} \frac{1}{m} \sum_{i=1}^{m} \mathcal{D}(\{x \in \mathcal{X} : h(x) \neq f(x)\}) = \mathcal{D}(\{x \in \mathcal{X} : h(x) \neq f(x)\}) = L_{(\mathcal{D},f)}(h)
\end{aligned}$$

□