Sobolev Spaces Notes

James Harbour

July 20, 2023

1 Notation

Throughout this document, Ω denotes an open set in $\mathbb{R}^n$.

2 Integer Sobolev Spaces

Definition 1. For $f \in L^1_{loc}(\Omega)$, we say that a function $g \in L^1_{loc}(\Omega)$ is an i-th weak derivative of f if

$$\int_\Omega g\varphi \, dx = -\int_\Omega f \frac{\partial \varphi}{\partial x_i} \, dx \text{ for all } \varphi \in C^1_c(\Omega).$$

Weak derivatives, if they exist, are well defined up to sets of measure zero. Let $W^{k,p}(\Omega)$ be the set of all locally integrable functions with weak derivatives in L^p up to order k.

These spaces are Banach spaces with norm

$$\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}$$

3 Fractional Sobolev Spaces

Definition 2. Fix $1 \leq p < +\infty$ and let $s \in (0,1)$ be a fractional exponent. For $u \in L^p(\Omega)$, define the *Gagliardo (semi)norm* of u to be the quantity

$$[u]_{W^{s,p}(\Omega)} := \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \, dx \, dy \right)^{\frac{1}{p}}.$$

and define the *fractional Sobolev space* $W^{s,p}(\Omega) := \{u \in L^p(\Omega) : [u]_{W^{s,p}(\Omega)} < +\infty\}$. This space is in fact Banach with the natural norm

$$\|u\|_{W^{s,p}(\Omega)} := \left(\|u\|^p_{L^p(\Omega)} + [u]^p_{W^{s,p}(\Omega)} \right)^{\frac{1}{p}}$$

In a somewhat precise sense, the fraction sobolev spaces $W^{s,p}(\Omega)$ are intermediary spaces between $L^p(\Omega)$ and the classical Sobolev space $W^{1,p}(\Omega)$.

Definition 3. Fix $p \in (1, +\infty)$ and $s \in \mathbb{R}$. Define the *Bessel potential* to be the operator

$$\mathscr{J}_s f(\xi) := \mathscr{F}^{-1} \left[(1 + |\xi|^2)^{\frac{s}{2}} \mathscr{F} f \right].$$

We define the *Bessel Potential Spaces* $H^{s,p}(\mathbb{R}^n)$ as follows:

$$H^{s,p}(\mathbb{R}^n) := \{f \in \mathscr{S}' : \mathscr{J}_s f \in L^p(\mathbb{R}^n)\}$$

$$\|u\|_{H^{s,p}(\mathbb{R}^n)} := \int_{\mathbb{R}^n} (1+|\xi|^2)^s |\widehat{u}(\xi)|^2 \, d\xi$$

Proposition 1. *Let $p \in [1, +\infty)$ and $0 < s \leq s' < 1$. Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $u : \Omega \to \mathbb{R}$ a measurable function. Then there exists a constant $C \geq 1$ depending only on n,s, p, such that*

$$\|u\|_{W^{s,p}(\Omega)} \leq C \|u\|_{W^{s',p}(\Omega)}$$

Hence, there is a continuous inclusion $W^{s',p}(\Omega) \subseteq W^{s,p}(\Omega)$.