Math 8851. Homework #1. To be completed by Thu, Feb 2

1. Prove the Schreier Subgroup Lemma (the statement is recalled below) without the extra assumption $1 \in T$. **Note:** You just need to slightly adjust the proof from class (where we assumed that $1 \in T$).

2. Let $F = F(X)$ for some set X and H a subgroup of F. Prove that H always has a Schreier transversal in F (with respect to X) in two different ways as follows:

   (a) Using Zorn's lemma
   (b) Using suitable total order on F.

Hint for (b): Choose an arbitrary total order on $X \sqcup X^{-1}$ and consider the corresponding lexicographical order on F: given two elements $f \neq f' \in F$, put $f < f'$ if one of the following holds:

   (i) $l(f) < l(f')$, where $l(\cdot)$ is the word length
   (ii) $l(f) = l(f')$, and if f and f' first differ in k^{th} position, then the k^{th} symbol in f is smaller than the k^{th} symbol in f'.

Then form a transversal by choosing the smallest element in each right coset of H.

3. Let $F = F(x, y)$ be the free group on two generators. Consider the following two subgroups of F:

   (a) $H = [F, F]$, the commutator subgroup of F
   (b) $H = \mathrm{Ker}\,\pi$ where π is the epimorphism from F onto S_3 (symmetric group on 3 letters) which sends x to (12) and y to (23).

For each of these subgroups do the following:

   (i) Find a Schreier transversal T for H (with respect to $X = \{x, y\}$).
   (ii) Draw the Schreier graph $Sch(H \setminus F, X)$ and the maximal tree $\mathcal{T}$ in $Sch(H \setminus F, X)$ corresponding to T (we will define the natural bijection between the Schreier transversals and maximal trees in class on Monday, Jan 29)
   (iii) Use the strong Nielsen-Schreier Theorem (the statement is recalled below) to find a free generating set for H.

4. Prove the *Schreier index formula*: If F is a free group of finite rank and H a subgroup of F of finite index, then

$$\mathrm{rk}(H) - 1 = (\mathrm{rk}(F) - 1) \cdot [F : H].$$

Hint: Count the number of vertices and edges in the Schreier graph $Sch(H \setminus F, X)$ and use the fact that $H \cong \pi_1(Sch(H \setminus F, X))$.

5. Use the Schreier Subgroup lemma to find a generating set with 2 elements for the alternating group A_n.

Lemma (Schreier Subgroup Lemma)**.** *Let G be a group, H a subgroup of G, S a generating set for G and T a right transversal for H in G. Then H is generated by the set*

$$U = U(S,T) = \{st \cdot \overline{st}^{-1} : s \in S, t \in T\}$$

where $\overline{g}$ is the unique element of T such that $H\overline{g} = Hg$.

Theorem (Strong Nielsen-Schreier Theorem)**.** *Let H a subgroup of $F(X)$, and let T be a (right) Schreier transversal for H (with respect to X). For every $x \in X$ and $t \in T$ let $h_{x,t} = xt \cdot \overline{xt}^{-1}$. Let*

$$I = \{(x,t) \in X \times T : h_{x,t} \neq 1\}.$$

Then the elements $\{h_{x,t} : (x,t) \in I\}$ are all distinct and form a free generating set for H.