FUNCTIONAL ANALYSIS HOMEWORK #1:

Problem 1. Let (X, μ) be a σ-finite measure space and let $L^p(X, \mu)$ be the space of all measurable functions $f\colon X \to \mathbb{C}$ such that $\int |f|^p \, d\mu < \infty$. Define, for $f \in L^p(X, \mu)$, $\|f\|_p = \int (|f|^p \, d\mu)^{1/p}$ (note: this is not a norm). As usual, we identify two measurable functions if they agree almost everywhere.

(a) Prove that if x, y are nonnegative real numbers and $0 < p < 1$, then $(x + y)^p \leq x^p + y^p$.

(b) Fix $0 < p < \infty$ and prove that $L^p(X, \mu)$ is a vector space under the natural operations of addition and scalar multiplication.

(c) Fix $0 < p < 1$ and define $d\colon L^p(X, \mu) \times L^p(X, \mu) \to [0, \infty)$ by $d(f, g) = \|f - g\|_p^p$. Prove that d is a metric and that the maps

$$L^p(X, \mu) \times L^p(X, \mu) \to L^p(X, \mu), \quad (f, g) \mapsto f + g,$$

$$\mathbb{C} \times L^p(X, \mu) \to \mathbb{C}, \ \ (\lambda, f) \mapsto \lambda f$$

are continuous with respect to d.

Problem 2. Let X be a Banach space.

(a) If Y, Z are Banach spaces, and $S \in B(X, Y), T \in B(Y, Z)$, prove that $\|TS\| \leq \|T\|\|S\|$.

(b) If $T \in B(X)$ and $\|T\| < 1$, prove that $(1 - T)$ is invertible.

(c) If $T \in B(X)$ Is invertible and $S \in B(X)$ has $\|S - T\| < \|T^{-1}\|^{-1}$, then S is invertible. Use this to show that the set of invertible elements in $B(X)$ is open.

Problem 3.

Exercise III.1.8

Problem 4.

Exercise III.4.5

Problem 5.

Exercise III.4.16

Challenge Problems. Do not turn in

Problem 6.

Fix $0 < p < 1$ and let $L^p([0, 1])$ be as in the previous exercise with the measure being Lebesgue measure. Let $\phi\colon L^p([0, 1]) \to \mathbb{C}$ be a continuous linear functional (with respect to the metric d given in problem 1). Follwing the following outline, prove that $\phi = 0$.

(a) Prove that there is a constant $C > 0$ so that $|\phi(f)| \leq C\|f\|_p$. Hint: it is enough to show that

$$\sup_{\|f\|_p=1} |\phi(f)| < \infty.$$

(Caution! $\|\cdot\|_p$ is not a norm).

(b) For $A \subseteq [0,1]$ Borel, let $\mu(A) = \phi(\chi_A)$. Prove that μ is a complex measure which is absolutely continuous with respect to Lebesgue measure.

(c) It follows from item (b) that there is a measurable function $f\colon [0,1] \to \mathbb{C}$ so that $\phi(\chi_A) = \int_A f(x)\,dx$ for all Borel $A \subseteq [0,1]$. Using item (a), prove that for every $x \in (0,1)$ we have

$$\lim_{\varepsilon\to 0} \frac{1}{2\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon} f(t)\,dt = 0.$$

Explain why this implies that $f(x) = 0$ almost everywhere.

(d) Use the above times to show that $\phi = 0$.