Reading:

- For this homework: 6.2-6.3/8.1-8.2
- For Wedneday, April 20: 8.1-8.2
- For Monday, April 25: 8.8.2-8.3 Rest of the semester: 8.3-8.5

Problem 1.

Let (X, μ) be a σ-finite measure space. Let $p \in [1, +\infty)$, and let $p' \in [1, +\infty)$ with $\frac{1}{p} + \frac{1}{p'} = 1$. Modify the proofs in class to obtain the following more general results.

(a) Suppose that $f \in L^+(X, \mu)$. Let $D_+ \subseteq \{f \in L^{p'}(X, \mu) : f \geq 0\}$. Suppose that D_+ is closed under scaling by nonnegative elements of $[0, +\infty)$ and under sums (such sets are called cones), and that $\overline{D_+}^{\|\cdot\|_{p'}} = \{f \in L^{p'}(X, \mu) : f \geq 0\}$. Set

$$s = \sup_{g \in D_+, \|g\|_{p'} = 1} \int fg \, d\mu.$$

If $s < +\infty$, show that $f \in L^p(X, \mu)$ and $\|f\|_p = s$.

Suggestion: as in class, one inequality is easier. For the reverse, let $X = \bigcup_n E_n$ where E_n are measurable and $\mu(E_n) < +\infty$, and set $F_n = \{x : |f(x)| \leq n\}$. Approximate $1_{E_N \cap F_N} f^{p-1}$ by elements of D_+ in the p'-norm. It may also be useful to use Fatou's lemma.

(b) Suppose that $X = \bigcup_{n=1}^{\infty} E_n$ with $\mu(E_n) < +\infty$. Let $f \colon X \to \mathbb{C}$ with $f\big|_{E_n} \in L^1(E_n, \mu)$ for all n. Suppose $D \subseteq L^p(X, \mu)$ is a linear subspace satisfying the following. For every simple function ϕ with $\phi 1_{E_n} = \phi$ for some n, there is a $C \geq 0$, an integer $k \geq n$, and a sequence of functions $\phi_m \in D$ with $|\phi_m| \leq C$, $\phi_m 1_{E_k} = \phi_m$ a.e. and $\|\phi_m - \phi\|_{p'} \to_{m \to \infty} 0$.

Assume that

$$s = \sup_{g \in D, \|g\|_{p'} = 1} \left| \int f(x) g(x) \, d\mu(x) \right|.$$

Show that $f \in L^p(X, \mu)$ with $\|f\|_p = s$.

Suggestion: first, show that for every $n \in \mathbb{N}$ we have

$$\left| \int f(x) g(x) \, d\mu(x) \right| \leq s \|g\|_{p'},$$

for every simple function g with $g = g 1_{E_n}$ a.e. Then follow the argument in class to show that

$$\|f 1_{E_n \cap F_n}\|_p \leq s$$

where F_n is as in (a).

Note: Let $C_c^{\infty}(\mathbb{R}^n)$ be all smooth, compactly supported functions. We'll see later that the positive elements of $C_c^{\infty}(\mathbb{R}^n)$ satisfy (a) and $C_c^{\infty}(\mathbb{R}^n)$ satisfies (b).

Problem 2.

Let (X, Σ, μ) be a probability space. Fix $p \in [1, +\infty]$ and $f \in L^p(X, \Sigma, \mu)$, let $p' \in [1, +\infty]$ be the conjugate exponent (i.e. $\frac{1}{p} + \frac{1}{p'} = 1$). Let $\mathcal{F} \subseteq \Sigma$ be a sub-σ-algebra.

(a) Show that $f \in L^1(X, \Sigma, \mu)$. (Hint: find a theorem from class/the book which implies this quickly).

(b) Let $\mathbb{E}_{\mathcal{F}}(f)$ be the conditional expectation onto $\mathcal{F}$ as defined in a past homework problem. Show that

$$\|\mathbb{E}_{\mathcal{F}}(f)g\|_1 \leq \|f\|_p \|g\|_{p'}$$

for all $\mathcal{F}$-measurable simple functions g. Use this to show that $\mathbb{E}_{\mathcal{F}}(f) \in L^p(X, \mathcal{F}, \mu)$ and that

$$\|\mathbb{E}_{\mathcal{F}}(f)\|_p \leq \|f\|_p.$$

Problem 3.

Folland, Chapter 6, Problem 32.

Problem 4.

Folland, Chapter 8, Problem 3.

Problem 5.

Let E be a measurable subset of $\mathbb{R}^n$ of positive measure. Show that $E - E$ contains an open set U with $0 \in U$.

Hint: reduce to the case E has finite measure. Show that $U = \{x \in \mathbb{R}^n : 1_E * 1_{-E}(x) > 0\}$ works.

Remark: this can be used to show that every positive measure set has cardinality of the continuum. It also allows one to show that measurable homomorphisms on $\mathbb{R}$ with values in "reasonable" topological groups are automatically contiuous.

Problems to think about

Problem 6.

Folland, Chapter 8, Problem 8.

Note: I use $\tau_y f$ for what Folland calls f^y.