MATH 7410 Homework 1

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Problem 1

Let (X, μ) be a σ-finite measure space.

(a): Prove that if $x, y \in \mathbb{R}_+$ and $0 < p < 1$, then $(x+y)^p \leq x^p + y^p$.

Proof. Fix $y \in \mathbb{R}_+$ and consider the function $f : [0, +\infty) \to \mathbb{R}$ given by $f(x) = (x+y)^p - (x^p + y^p)$. Note that $f(0) = 0$. As $0 < p < 1$,

$$x^{p-1} \geq (x+y)^{p-1} \implies 0 \geq p\left((x+y)^{p-1} - x^{p-1}\right) = f'(x)$$

so f is nonincreasing and $f(0) = 0$ whence $f(x) \leq 0$ for all $x \in \mathbb{R}_+$ as desired. □

(b): Fix $0 < p < \infty$ and prove that $L^p(X, \mu)$ is a vector space under the natural operations of addition and scalar multiplication.

Proof. Fix $f, g \in L^p(X, \mu)$. Then as $|f+g| \leq 2(|f| + |g|)$ and $x \mapsto x^p$ is increasing,

$$\|f+g\|_p^p = \int |f+g|^p \, d\mu \leq \int 2^p(|f| + |g|)^p \, d\mu = 2^p(\|f\|_p^p + \|g\|_p^p) < +\infty,$$

so $f + g \in L^p(X, \mu)$. For $\lambda \in \mathbb{C}$,

$$\|\lambda f\|_p = \left(\int |\lambda f|^p \, d\mu\right)^{1/p} = |\lambda| \|f\|_p < +\infty,$$

so $\lambda f \in L^p(X, \mu)$. Thus $L^p(X, \mu)$ is a vector space. □

(c): Fix $0 < p < 1$ and define $d : L^p(X, \mu) \times L^p(X, \mu) \to [0, \infty)$ by $d(f, g) = \|f - g\|_p^p$. Prove that d is a metric and that addition and multiplication are continuous with respect to d.

Proof. Let $f, g \in L^p(X, \mu)$. Applying part (a) and that $x \mapsto x^p$ is increasing,

$$\|f+g\|_p^p = \int |f+g|^p \, d\mu \leq \int (|f| + |g|)^p \, d\mu \overset{(a)}{\leq} \int |f|^p + |g|^p \, d\mu = \|f\|_p^p + \|g\|_p^p.$$

Then for $f, g, h \in L^p(X, \mu)$,

$$d(f, h) = \|f - h\|_p^p = \|f - g + g - h\|_p^p \leq \|f - g\|_p^p + \|g - h\|_p^p = d(f, g) + d(g, h).$$

Clearly, $d(f, f) = 0$ and $d(f, g) = d(g, f)$. Suppose that $d(f, g) = 0$. Then $\int |f - g|^p \, d\mu = 0$, whence $|f - g| = 0$ a.e. so $f = g$ a.e. Thus d is a metric.

Now suppose that $f_n \to f$ and $g_n \to g$ in $L^p(X, \mu)$ with respect to d. Then applying the above inequality,

$$d(f_n + g_n, f + g) = \|f_n - f + g_n - g\|_p^p \le \|f_n - f\|_p^p + \|g_n - g\|_p^p = d(f_n, f) + d(g_n, g) \xrightarrow{n\to\infty} 0,$$

so addition is continuous with respect to d.

Lastly, assume that $(\lambda_n, f_n) \to (\lambda, f)$ in the product topology, i.e. $\lambda_n \to \lambda$ in $\mathbb{C}$ and $f_n \to f$ with respect to d. Note that

$$\left| \|f_n\|_p^p - \|f\|_p^p \right| \le \|f_n - f\|_p^p \xrightarrow{n\to\infty} 0,$$

so there exists some $C > 0$ such that $\|f_n\|_p^p \le C$ for all $n \in \mathbb{N}$. We compute,

$$d(\lambda_n f_n, \lambda f) = \|\lambda_n f_n - \lambda f\|_p^p \le \|\lambda_n f_n - \lambda f_n\|_p^p + \|\lambda f_n - \lambda f\|_p^p \le C|\lambda_n - \lambda|^p + |\lambda|^p \|f_n - f\|_p^p \xrightarrow{n\to\infty} 0,$$

so scalar multiplication is continuous with respect to d. □

Problem 2

Let X be a Banach space.

(a): If Y, Z are Banach spaces, and $S \in B(X, Y), T \in B(Y, Z)$, prove that $\|TS\| \le \|T\|\|S\|$.

Proof. For all $x \in X$, observe that

$$\|TSx\| \le \|T\|\|Sx\| \le \|T\|\|S\|\|x\|,$$

so by definition $\|TS\| \le \|T\|\|S\|$. □

(b): If $T \in B(X)$ and $\|T\| < 1$, prove that $1 - T$ is invertible.

Proof. Define $S_n = \sum_{k=0}^n T^k$. Note that

$$\|S_n\| \le \sum_{k=0}^{n} \|T\|^k \le \sum_{k=0}^{\infty} \|T\|^k = (1 - \|T\|)^{-1}.$$

As X is Banach and S_n is Cauchy, there exists an $S \in B(X)$ such that $\|S_n - S\| \xrightarrow{n\to\infty} 0$.

Fix $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that for $n \ge N$, $\|T\|^n < \varepsilon/2$ and $\|S - S_n\| < \frac{\varepsilon}{2\|1-T\|}$. Then for $n \ge N$,

$$\|S(1 - T) - 1\| \le \|(S - S_n)(1 - T)\| + \|S_n(1 - T) - 1\| \le \|S - S_n\|\|1 - T\| + \|S_n(1 - T) - 1\|$$
$$< \frac{\varepsilon}{2} + \left\| \sum_{k=0}^{n} T^k(1 - T) - 1 \right\| = \frac{\varepsilon}{2} \|T^{k+1}\| < \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, $S(1 - T) = 1$. □

(c): IF $T \in B(X)$ is invertible and $S \in B(X)$ has $\|S - T\| < \|T^{-1}\|^{-1}$, then S is invertible. Use this to show that the set of invertible elements $Inv(B(X))$ is open.

Proof. Observe that

$$\|T^{-1}S - 1\| \le \|T^{-1}\|\|S - T\| < 1$$

by assumption, so part (b) implies that $-T^{-1}S = 1 - (T^{-1}S - 1)$ is invertible, whence S is invertible as the invertible elements of $B(X)$ form a group with multiplication.

Hence if $T \in Inv(B(X))$ then $B_{\|T^{-1}\|^{-1}}(T, \|\cdot\|) \subseteq Inv(B(X))$, so $Inv(B(X))$ is open as it is the union of all such open balls. □

Problem 3

Show that l^∞ is not separable.

Proof. For each $I \in \mathscr{P}(\mathbb{N})$, define an element $a_I \in l^\infty(\mathbb{N})$ by $a_I(n) = \mathbb{1}_I(n)$. Then for $I \neq J \in \mathscr{P}(\mathbb{N})$, $B_{1/2}(a_I) \cap B_{1/2}(a_J) = \emptyset$. Thus, we have an uncountable family of pairwise disjoint balls $\{B_{1/2}(a_I)\}_{I \in \mathscr{P}(\mathbb{N})}$. Any dense subset of $l^\infty(\mathbb{N})$ must intersect each of these balls, whence by disjointness this set must be uncountable. By contraposition $l^\infty(\mathbb{N})$ is not separable. □

Problem 4

Prove that if X is a normed space, $M \leq X$, and both M and X/M are complete, then X is complete.

Proof. Let $Q : X \to X/M$ be the natural map and $(x_n)_{n=1}^\infty$ a Cauchy sequence in X. Then by completeness, there exists a $z \in X$ such that $\|Q(x_n - z)\| \xrightarrow{n \to \infty} 0$. Choose a sequence $(m_n)_{n=1}^\infty$ in M such that for $n \in \mathbb{N}$,

$$\|Q(x_n - x)\| + \frac{1}{n} \geq \|x_n - x - m_n\|.$$

Then $\|x_n - x - m_n\| \xrightarrow{n \to \infty} 0$. We claim that $(m_n)_{n=1}^\infty$ is Cauchy. To see this, observe that

$$\begin{aligned} \|m_n - m_k\| &\leq \|x_k - x - m_k\| + \|x_k - x - m_n\| \\ &\leq \|x_k - x - m_k\| + \|x_n - x - m_n\| + \|x_k - x_n\| \xrightarrow{k,n \to \infty} 0. \end{aligned}$$

By completeness, there is some $m \in M$ such that $m_n \to m$. Lastly, note that

$$\|x_n - x - m\| \leq \|x_n - x - m_n\| + \|m_n - m\| \xrightarrow{n \to \infty} 0,$$

so $x_n \to x + m$, whence X is complete. □

Problem 5

Let $\mathcal{H}$ be a Hilbert space and suppose $\mathcal{M} \leq \mathcal{H}$. Show that if $Q : \mathcal{H} \to \mathcal{H}/\mathcal{M}$ is the natural map, then $Q : \mathcal{M}^\perp \to \mathcal{H}/\mathcal{M}$ is an isometric isomorphism.

Proof. Let $f \in \mathcal{H}$. There are unique $f^\parallel \in \mathcal{M}$ and $f^\perp \in \mathcal{M}^\perp$ such that $f = f^\parallel + f^\perp$, whence $Q(f^\perp) = f^\perp + \mathcal{M} = f^\perp + f^\parallel + \mathcal{M} = f + \mathcal{M}$. Thus $Q|_{\mathcal{M}^\perp}$ is surjective.

Moreover, note that $f^\parallel$ is such that

$$\left\|f^\perp\right\| = \left\|f - f^\parallel\right\| = dist(f, \mathcal{M}) = \inf\{\|f + m\| : m \in M\} = \left\|Q(f^\perp)\right\|$$

so $Q|_{\mathcal{M}^\perp}$ is isometric and thus injective. As Q is bounded and $\mathcal{M}^\perp$ is a closed linear subspace of $\mathcal{H}$, $Q|_{\mathcal{M}^\perp}$ is continuous. Thus $Q|_{\mathcal{M}^\perp}$ is a continuous bijection of Banach spaces, so by the Inverse mapping theorem $(Q|_{\mathcal{M}^\perp})^{-1}$ is continuous. □