Problem 1. Let X, Y, Z be normed linear spaces with $Y \leq X$ and $T\colon X \to Z$ a bouned linear opeartor such that $T\big|_Y = 0$. Then there is a unique $\overline{T}\colon X/Y \to Z$ so that $\overline{T} \circ Q = T$ where $Q\colon X \to X/Y$ is the quotient map (this is proved in linear algebra, you may take this for granted). Show that $\|\overline{T}\| = \|T\|$.

Problem 2. (a) Given $Y \leq X$ normed linear spaces, with $Y \neq X$, prove that for all $\varepsilon > 0$ there is an $x \in X$ with $\|x\| = 1$ and so that the distance from x to Y is at least $1 - \varepsilon$.

Hint: it might be useful to use that the previous problem implies that the quotient map has norm one.

(b) Prove that if X is a normed linear space so that $\mathrm{Ball}(X) = \{x \in X : \|x\| \leq 1\}$ is compact, then X is finite-dimensional.

Problem 3.

(a) Conway III.12.5

(b) Let X, Y, A as in the previous part. Let V be the ℓ^∞-direct sum of X, so $V = \{(x_n)_{n=1}^\infty \in X^{\mathbb{N}} : \sup_n \|x_n\| < +\infty\}$. Define

$$\mathrm{approxker}(A) = \frac{\{(x_n)_n \in V : \|Ax_n\| \to 0\}}{\{(x_n)_n \in V : \|x_n\| \to 0\}}.$$

Show that A is injective with closed image if and only if $\mathrm{approxker}(A) = \{0\}$.

Hint for one of the implications: if the condition in previous item fails, then for every $\varepsilon > 0$ there is an $x \in X$ with $\|x\| = 1$ and $\|Ax\| < \varepsilon$.

Remark: being an injection with closed image implies that the operator is a homeomorphism onto its image. In some sense this is the appropriate "topological" generalization of an injective linear transformation on a finite-dimensional vector space. What this problem says is that one has to make the assumption of having trivial kernel quantitative in order to have this "topological" generalization. This is a running theme in funcitonal analysis.

Problem 4. Conway III.12.7

Problem 5. Conway III.12.8

Challenge Problems. Do not turn in

Problem 6. Let E_n be the set of all $f \in C([0,1])$ for which there is some $x_0 \in [0,1]$ (x_0 depending upon f with $|f(x) - f(x_0)| \leq n|x - x_0|$ for all $x \in [0,1]$.

(1) Prove that E_n^c is an open, dense set. (Hint: To prove that E_n^c is dense, you may use that piecewise linear functions are dense in $C([0,1])$. Given a piecewise linear function f, We may then add a very small, but very wiggly piece-wise linear function g to f so that the slopes of $f + g$ all have absolute value at least $2n$, then $f + g \notin E_n$, and $f + g$ can be made arbitrarily close to f.)
(2) Prove that the set of nowhere differentiable functions is dense in $C([0,1])$.

Problem 7 (Grothendieck, I believe). Let $\mathcal{H}$ be a closed subspace of $L^2([0,1])$. Suppose that in fact $\mathcal{H} \subseteq C([0,1])$.

(a) Show that there is a constant $C > 0$ so that $\|f\|_\infty \leq C\|f\|_2$ for all $f \in \mathcal{H}$.

(b) Prove that for every $x \in [0,1]$ there is a $g_x \in \mathcal{H}$ so that $\langle f, g_x\rangle = f(x)$ for all $f \in \mathcal{H}$ and $\|g_x\|_2 \leq C$.

(c) Prove that $\mathcal{H}$ is finite-dimensional, and in fact the dimension of $\mathcal{H}$ is at most C^2 (Hint: if $\{f_j\}_{j=1}^k$ are orthonormal for some k, then part (b) implies $\sum_{j=1}^k |f_j(x)| \leq C^2$.)