Problem 1.

Conway V.6.6.

Problem 2.

Conway V.7.6.

Problem 3.

Conway V.7.2.

(Recall that if (X, μ) is a measure space, then an *atom* of (X, μ) is a measuarable $E \subseteq X$ with $\mu(E) > 0$ and so that if $F \subseteq E$ is measurable with $\mu(F) > 0$, then $\mu(F) \in \{0, \mu(E)\}$.)

problems to think about. do not turn in

Problem 4.

Let V be a separable Banach space. For a σ-finite measure space (X, μ) and $1 \leq p \leq \infty$, we let $L^p(X, \mu; V)$ be the space of all weakly measurable functions $f \colon X \to V$ so that $\|f\|_p = \left(\int \|f(x)\|^p \, d\mu(x)\right)^{1/p} < +\infty$.

(a) Prove that $L^p(X, \mu; V)$ is a Banach space. Hint: if $(f_n)_n \in L^p(X, \mu; V)$ and $\sum_n \|f_n\|_p < +\infty$ argue that

$$\int \left(\sum_{n=1}^{\infty} \|f_n(x)\|\right)^p d\mu(x) < +\infty.$$

Use this to show that $f(x) = \sum_{n=1}^{\infty} f_n(x)$ converges a.e x and that $\|f - \sum_{n=1}^{N} f_n\|_p \to_{N\to\infty} 0$.

(b) Prove that $I \colon L^1(X, \mu; V) \to V$ given by $I(f) = \int f(x) \, d\mu(x)$ is a bounded linear contraction.

Problem 5.

Prove that if (X, μ) is σ-finite and $L^1(X, \mu)$ is the dual of a Banach space, then there is a countable collection of atoms $(E_j)_{j\in J}$ so that $\mu\left(X \setminus \bigcup_{j\in J} E_j\right) = 0$.