MATH 7410 Homework 1

James Harbour

September 13, 2022

Problem 1

Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. For any $p \in \mathbb{R}^n$, there is a canonical identification $T_p(\mathbb{R}^n) \to \mathbb{R}^n$ given by

$$\sum a^i \frac{\partial}{\partial x^i}|_p \mapsto a = (a^1, \dots, a^n)$$

Show that the differential $L_{*,p} : T_p(R^n) \to T_{L(p)}(R^m)$ is the map $L : \mathbb{R}^n \to \mathbb{R}^m$ itself, with the identification of the tangent spaces as above.

Problem 2

If M and N are manifolds, let $\pi_1 : M \times N \to M$ and $\pi_2 : M \times N \to N$ be the two projections. Prove that for $(p, q) \in M \times N$,

$$(\pi_{1*}, \pi_{2*} : T_{(p,q)}(M \times N) \to T_pM \times T_qN$$

is an isomorphism.

Problem 3

Let G be a Lie group with multiplication map $\mu : G \times G \to G$, inverse map $\iota : G \to G$, and identity element e.

(a): Show that the differential at the identity of the multiplication map μ is addition:

$$\mu_{*,(e,e)} : T_eG \times T_eG \to T_eG,$$
$$\mu_{*,(e,e)}(X_e, Y_e) = X_e + Y_e.$$

(*Hint*: First, compute $\mu_{*,(e,e)}(X_e, 0)$ and $\mu_{*,(e,e)}(0, Y_e)$ using Proposition 8.18).

Proof. Let $X_e, Y_e \in T_eG$ and choose curves $\alpha_1, \alpha_2 : (-\epsilon, \epsilon) \to G$ such that $\alpha_1(0) = \alpha_2(0) = e$ and $\frac{d\alpha_1}{dt}|_0 = X_e$, $\frac{d\alpha_2}{dt}|_0 = Y_e$. Consider β_1, β_2 given by $\beta_1(t) = (\alpha_1(t), e)$ and $\beta_2(t) = (e, \alpha_2(t))$. Then □

(b): Show that the differential at the identity of ι is the negative:

$$\iota_{*,e} : T_eG \to T_eG$$
$$\iota_{*,e}(X_e) = -X_e.$$

(*Hint*: Take the differential of $\mu(c(t), (t \circ c)(t)) = e$.)

Problem 4

Show that $T_p^1 M \subseteq T_p^2 M$.