Problem 1.

(a) Let X be a separable Banach space. Show that $\mathrm{Ball}(X^*) = \{\phi \in X^* : \|\phi\| \leq 1\}$ is wk*-metrizable. (remark: in class, we outlined that X^* is not wk*-metrizable).

(b) Conway V.3.3

Problem 2. Let X, Y, Z be Banach spaces, a bilinear map is a map $B\colon X \times Y \to Z$ so that for each fixed $x \in X, y \in Y$ the maps: $B_x\colon Y \to Z, B^y\colon X \to Z$ defined by $B_x(y) = B(x,y), B^y(x) = B(x,y)$ are linear. We say that B is bounded if there is a constant $C > 0$ so that $\|B(x,y)\| \leq C\|x\|\|y\|$ i.e.

$$\|B\| = \sup_{\|x\|\leq 1, \|y\| \leq 1} \|B(x,y)\| < \infty.$$

Let $\mathrm{Bil}(X \times Y, Z)$ be the space of bounded, bilinear maps from $X \times Y \to Z$.

(a) Suppose that B_x, B^y are bounded for each $x \in X, y \in Y$. Prove that there is a constant $M > 0$ so that

$$\|B(x,y)\| \leq M\|x\|\|y\|$$

(use the Priniciple of Uniform Boundedness).

(b) Show that the map $\Phi\colon \mathrm{Bil}(X \times Y, \mathbb{F}) \to B(X, Y^*)$ given by $[\Phi(B)(x)](y) = B(x,y)$ is a well-defined, isometric isomorphism.

(c) By switching names it follows that the map $\widetilde{\Phi}\colon \mathrm{Bil}(X \times Y, \mathbb{F}) \to B(Y, X^*)$ given by $[\widetilde{\Phi}(B)(y)](x) = B(x,y)$ is a well-defined, isometric isomorphism. So the map $\widetilde{\Phi} \circ \Phi^{-1}$ is an isometric isomorphism $B(X, Y^*) \cong B(Y, X^*)$. What is this isomorphism?

Problem 3.

Let X, Y be Banach spaces. And let $(T_n)_{n=1}^{\infty}$ be a sequence in $B(X,Y)$.

(a) If T_n converges in the WOT to $T \in B(X,Y)$ show that $\sup_n \|T_n\| < +\infty$. (In particular, if T_n converges strongly, then it is norm).

(b) If $\sup_n \|T_n\| < +\infty$ and there is a norm dense $D \subseteq X$ so that $T_n x$ converges for every $x \in D$, show that $T_n x$ converges for all $x \in X$, that $Tx = \lim_{n\to\infty} T_n x$ is a bounded operator, and that $\|Tx - T_n x\| \to_{n\to\infty} 0$ for every $x \in X$.

Problem 4.

Let X, Y be Banach spaces. And let $(T_n)_{n=1}^{\infty}$ be a sequence in $B(X,Y)$. Suppose that $\sup_n \|T_n\| < +\infty$ and that $D \subseteq X, G \subseteq Y^*$ are norm dense. Assume that $\lim_n \phi(T_n x)$ exists for all $\phi \in G, x \in D$.

(a) Show that $\lim_n \phi(T_n x)$ exists for all $\phi \in Y^*, x \in X$.

(b) Show that for every $x \in X$, there is a well-defined bounded operator $S\colon X \to Y^{**}$ given by $S(x)(\phi) = \lim_{n\to\infty} \phi(T_n x)$.

(c) If $T_n x$ converges weakly to an element of Y for every $x \in D$, show that $S(X) \subseteq Y$, and that $T_n \to S$ WOT.

Problem 5.

Let G be a countable, discrete, group and $\lambda\colon G \to B(\ell^2(G))$ be given by $(\lambda(g)\xi)(h) = \xi(g^{-1}h)$.

(a) Let $(g_n)_{n=1}^{\infty}$ be a sequence in G so that for every finite $F \subseteq G$ we have $\{n : g_n \in F\}$ is finite. Show that $\lim_{n\to\infty} \lambda(g_n) = 0$ in WOT. (Hint: consider acting first on pairs of vectors which are finitely supported and applying the preceding problem to reduce to this case).

(b) Suppose G is infinite. If $\mathcal{K} \subseteq \ell^2(G)$ is closed and $\lambda(g)\mathcal{K} = \mathcal{K}$ for every $g \in G$, and $\mathcal{K} \neq \{0\}$, show that $\mathcal{K}$ is finite-dimensional.

(Hint: construct a sequence g_n satisfying the hypotheses of the preceding problem. If $\mathcal{K}$ is finite-dimensional, then $\lambda(g_n)\big|_{\mathcal{K}}$ converges to 0 in WOT, hence in any other LCS topology on $B(\mathcal{K})$. Consider using this for one of the other operator topologies to get a contradiction).

Problems to think about. Do not turn in

Problem 6.

These are applications of some of the above problems.

(a) If (X,μ) is a probability space, $1 \leq p < +\infty$, and $(f_n)_{n=1}^{\infty}, f$ are in $L^\infty(X,\mu)$ define $T_n, T \in B(L^p(X,\mu))$ by $T_n(g) = f_n g$ show that $T(g) = fg$, show that $T_n \to T$ in SOT if and only if $\sup_n \|f_n\| < +\infty$ and $\|f_n - f\|_p \to_{n\to\infty} 0$.

(b) For $n \in \mathbb{N}$, define $U_n \in B(L^2([0,1]))$ by $U_n(f)(x) = e^{2\pi i n x} f(x)$. Show that $U_n \to_{n\to\infty} 0$ WOT.

Problem 7.

Let X be a Banach space. Show that if X^* is separable, then X is separable.

Hint: consider a dense sequence $(\phi_n)_{n=1}^{\infty}$. For each n, choose $x_n \in X$ with $\|x_n\| = 1$ and $|\phi_n(x_n)| \geq (1 - 2^{-n})\|\phi_n\|$. Show that $\operatorname{span}\{x_n : n \in \mathbb{N}\}$ is dense.

Problem 8.

(a) Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces and T a bounded operator and suppose that $\operatorname{Im}(T)$ is closed. Show that

$$\begin{tikzcd} \mathcal{H} \arrow[r, "T"] \arrow[d, "P_{\ker(T)^\perp}"] & \mathcal{K} \\ \ker(T)^\perp \arrow[r, "S"] & \operatorname{Im}(T) \arrow[u, "\iota"'] \end{tikzcd}$$

commutes, where $\iota\colon \operatorname{Im}(T) \to \mathcal{K}$ is the inclusion map, $P_{\ker(T)^\perp}$ is the orthogonal projection onto $\ker(T)^\perp$ and S is the restriction of T to $\ker(T)^\perp$. Show additionally that S is a bounded, linear, bijection.

(b) Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces and T a bounded operator and suppose that $\operatorname{Im}(T)$ is closed. Show that $\operatorname{Im}(T^*)$ is closed. Hint: use the previous part and the fact that $\iota_{\mathcal{K}}^* = P_{\mathcal{K}}$.

(c) Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces and $T \in B(\mathcal{H}, \mathcal{K})$. Show that T is invertible if and only if there is a constant $C > 0$ so that

$$\|T\xi\| \geq C\|\xi\|, \text{ for all } \xi \in \mathcal{H}$$

and

$$\|T^*\xi\| \geq C\|\xi\|, \text{ for all } \xi \in \mathcal{K}.$$

(Remark: this remains true in Banach spaces, though the argument is slight more invovled. See Theorem 1.10 of Conway.