MATH 7410 Homework 2

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Problem 1

Let X, Y, Z be normed linear spaces with $Y \leq X$ and $T : X \to Z$ a bounded linear operator such that $T|_Y = 0$. Then there is a unique $\bar{T} : X/Y \to Z$ so that $\bar{T} \circ Q = T$ where $Q : X \to X/Y$ is the quotient map. Show that $\|\bar{T}\| = \|T\|$.

Proof. On one hand, we know that $\|T\| = \|\bar{T} \circ Q\| \leq \|\bar{T}\|\|Q\| \leq \|\bar{T}\|$. Note that $\mathrm{ran}(\bar{T}) \subseteq \mathrm{ran}(T)$. We claim that Q

Note that, for $y \in Y$, $\|Tx\| = \|T(x+y)\| \leq \|T\|\|x+y\|$. Thus, by taking the infimum over $y \in Y$, we obtain that $\|\bar{T}(x+Y)\|\|Tx\| \leq \|T\|\|x+Y\|$. Thus, $\|\bar{T}\| \leq \|T\|$. □

Problem 2

(a): Given $Y \leq X$ normed linear spaces with $Y \neq X$, prove that for all $\varepsilon > 0$ there is an $x \in X$ with $\|x\| = 1$ so that the distance from x to Y is at least $1 - \varepsilon$.

Proof. Suppose that the statement fails. Then there is some $\varepsilon \in (0,1)$ at which it fails. Then for $x \in X$ such that $\|x\| = 1$, $dist(x, Y) < 1 - \varepsilon$. Let x_0 satisfy the previous statement and be such that $x_0 \notin Y$. By Hahn Banach, there exists a $\phi \in X^*$ such that $\|\phi\| = \frac{1}{dist(x_0,Y)}$, $\phi(x_0) = 1$, and $\phi|_Y = 0$. So, $\|\phi\| > \frac{1}{1-\varepsilon}$. □

(b): Prove that if X is a normed linear space so that $Ball(X) = \{x \in X : \|x\| \leq 1\}$ is compact, then X is finite dimensional.

Proof. By compactness, there exist $x_1, \ldots, x_n \in Ball(X)$ such that $Ball(X) \subseteq \bigcup_{1 \leq i \leq n} B_{1/2}(x_i) = \bigcup_{1 \leq i \leq n} x_i + \frac{1}{2} \cdot Ball(X)$. Let $Y = \mathrm{Span}\{x_1, \ldots, x_n\}$, so

$$Ball(X) \subseteq Y + \frac{1}{2} \cdot Ball(X) \subseteq Y + \frac{1}{2}\left(Y + \frac{1}{2} Ball(X)\right) \subseteq \cdots \subseteq Y + \frac{1}{2^n} Ball(X).$$

Now let $x \in Ball(X)$. Then there exist $y \in Y$ and $z_n \in Ball(X)$ such that for all $n \in \mathbb{N}$, $x = y + \frac{1}{2^n}$. Then

$$\|x - y\| = \frac{1}{2^n}\|z^n\| \leq \frac{1}{2^n} \xrightarrow{n \to \infty} 0,$$

so $Ball(X) \subseteq Y$, whence by linearity of Y, $X = Y$. □

Problem 3

(a): Let X, Y be Banach spaces and $A \in B(X,Y)$. Show that there is a $c > 0$ such that $\|Ax\| \geq c\|x\|$ for all $x \in X$ if and only if $\ker(A) = 0$ and $\operatorname{ran}(A)$ is closed.

Proof.
$\underline{\Longrightarrow}$: If $x \in \ker(A)$, then $\|x\| \leq \frac{1}{c}\|Ax\| = 0$, so $\ker(A) = 0$. Suppose that $y_n = Ax_n$ is a sequence in $\operatorname{ran}(A)$ such that $y_n \to y \in Y$. Then $\|x_n - x_m\| \leq \frac{1}{c}\|y_n - y_m\|$, so $(x_n)_n$ is Cauchy, whence by completeness there is some $x \in X$ such that $x_n \to x$. Since A is bounded, it follows that

$$\|y_n - Ax\| = \|Ax_n - Ax\| \leq \|A\| \cdot \|x_n - x\| \xrightarrow{n\to\infty} 0,$$

so since X is Hausdorff $y = Ax$ is in $\operatorname{ran}(A)$.

$\underline{\Longleftarrow}$: Since $\operatorname{ran}(A)$ is a closed subspace of Y, $\operatorname{ran}(A)$ is also Banach. Thus, by the inverse mapping theorem, $A^{-1} \in B(\operatorname{ran}(A), X)$. Then, for $x \in X$ and $c = (\|A^{-1}\| + 1)^{-1} > 0$,

$$\|x\| = \left\|A^{-1}Ax\right\| \leq \left\|A^{-1}\right\|\|Ax\| \leq \frac{1}{c}\|Ax\| \implies \|Ax\| \geq c\|x\|.$$

□

(b): Let X, Y, A be as in the previous part. Let V be the l^∞-direct sum of X so $V = \{(x_n)_{n=1}^\infty \in X^{\mathbb{N}} : \sup_n \|x_n\| < +\infty\}$. Define

$$\mathit{approxker}(A) = \frac{\{(x_n)_n \in V : \|Ax_n\| \to 0\}}{\{(x_n)_n \in V : \|x_n\| \to 0\}}$$

Show that A is injective with closed image if and only if $\mathit{approxker}(A) = \{0\}$. (*Hint*: For one of the implications, if the previous item fails, then for every $\varepsilon > 0$ there is an $x \in X$ with $\|x\| = 1$ and $\|Ax\| < \varepsilon$.)

Proof.
$\underline{\Longrightarrow}$: Suppose that A is injective with closed image, and let $(x_n)_n \in V$ such that $\|Ax_n\| \to 0$. Then by part (a),

$$\|x_n\| \leq \frac{1}{c}\|Ax_n\| \xrightarrow{n\to\infty} 0,$$

so $\mathit{approxker}(A) = 0$.
$\underline{\Longleftarrow}$: We proceed by contraposition. Suppose that A fails to be injective with closed image. Then by part (a), for all $n \in \mathbb{N}$ there is some $x_n \in X$ such that $\|x_n\| = 1$ and $\|Ax_n\| < \frac{1}{n}$. So, $(x_n)_n \in V$ and $\|Ax_n\| \to 0$, but $\|x_n\| \not\to 0$, so $\mathit{approxker}(A) \neq \{0\}$. □

Problem 4

Let $1 \leq p \leq \infty$ and suppose (α_{ij}) is a matrix such that $(Af)(i) = \sum_{j=1}^\infty \alpha_{ij} f(j)$ defines an element Af of l^p for every f in l^p. Show that $A \in B(l^p)$.

Proof. We first claim that for each fixed $i \in \mathbb{N}$, $(\alpha_{ij})_j \in l^q$. So fix $i \in \mathbb{N}$
Suppose that $(f_n)_n$ is a sequence in $l^p(\mu)$ such that $f_n \xrightarrow{l^p} 0$ and $g \in l^p(\mu)$ is such that $Af_n \xrightarrow{l^p} g$. We show that $g = 0$. Since the measure is counting measure, it suffices to show that $(Af_n)(i) \xrightarrow{n\to\infty} 0$ for all $k \in \mathbb{N}$.

For $k \in \mathbb{N}$, define $T_k \in (l^p)^*$ by $T_k(f) = \sum_{j=1}^k \alpha_{ij} f(j)$. Note that each T_k is bounded. Now, for fixed $f \in l^p$ and all $k \in \mathbb{N}$,

$$|T_k(f)| \leq \sum_{j=k}^{k} |\alpha_{ij} f(j)| \leq \sum_{j=1}^{\infty} |\alpha_{ij} f(j)| < +\infty,$$

so by the uniform boundedness principle $M := \sup_{k\in\mathbb{N}} \|T_k\| < +\infty$. Thus, for $f \in l^p$, we have that $|\sum_{j=1}^\infty \alpha_{ij} f(j)| \leq \liminf_{k\to\infty} |T_k f| \leq M\|f\|_p$, so by the Riesz representation theorem $(\alpha_{ij})_j \in l^q$.
Now, by Holder's inequality,

$$|(Af_n)(i)| = \left|\sum_{j=1}^{\infty} \alpha_{ij} f_n(j)\right| \leq \|(\alpha_{ij})_j\|_q \|f_n\|_p \xrightarrow{n\to\infty} 0.$$

So, by the closed graph theorem, A is bounded. □

Problem 5

Let (X, Σ, μ) be a σ-finite measure space, $1 \leq p < \infty$, and suppose that $k : X \times X \to \mathbb{F}$ is a $\Sigma \times \Sigma$ measurable function such that for $f \in L^p(\mu)$ and a.e. x, $k(x, \cdot)f(\cdot) \in L^1(\mu)$ and $(Kf)(x) = \int k(x, y)f(y)\, d\mu(y)$ defines an element Kf of $L^p(\mu)$. Show that $K : L^p(\mu) \to L^p(\mu)$ is a bounded operator.

Proof. For $x \in X$ such that $k(x, \cdot)f(\cdot) \in L^1(\mu)$, consider the map $K_x : L^p(\mu) \to L^1(\mu)$ given by $K_x f = k(x, \cdot)f(\cdot)$. This map is well defined by assumption. Suppose that $(f_n)_n$ is a sequence in $L^p(\mu)$ such that $f_n \xrightarrow{L^p} 0$ and $g \in L^p(\mu)$ is such that $Kf_n \xrightarrow{L^p} g$. We show that $g = 0$. By passing to a subsequence, it suffices to assume $f_n \to 0$ pointwise a.e., and passing to a further subsequence we can assume that $Kf_n \to g$ pointwise a.e. as well. We shall now justify an application of DCT.

Since $(f_n)_n$ converges in L^p-norm, there is a subsequence $(f_{n_k})_k$ such that $\|f_m - f_{n_k}\|_p < \frac{1}{2^k}$ for all $m \geq n_k$. Let $F' = \sum_{k=1}^\infty |f_{n_{k+1}} - f_{n_k}|$. Each partial sum for F' has L^p-norm less than 1 by the above estimate and Minkowski's inequality, whence Fatou's lemma implies that $\|F'\|_p \leq 1$. Letting $F = f_{n_1} + \sum_{k=1}^\infty f_{n_{k+1}} - f_{n_k}$. By the previous estimate, $F \in L^p$. Thus, $|F| + F' \in L^p(\mu)$, and for all $k \in \mathbb{N}$, we have that $|f_{n_k}| \leq |F| + F'$ pointwise a.e.

Now, without loss of generality, assume that $n_k = k$ for all $k \in \mathbb{N}$. Let $h = |F| + F'$. Then by assumption, for a.e. $x \in X$ we have $k(x, \cdot)h(\cdot) \in L^1(\mu)$. Now, for a.e. $x \in X$, $|K_x(f_n)| \leq |K_x h|$ pointwise almost everywhere with $K_x h \in L^1(\mu)$. So, by the dominated convergence theorem,

$$\lim_{n\to\infty} \|K_x f_n\|_1 = 0.$$

Thus $|Kf_n(x)| \leq \|K_x f_n\|_1 \to 0$. Now, by an identical argument to above, we can find an $\widetilde{h} \in L^p(\mu)$ such that $|Kf_n| \leq \widetilde{h}$ pointwise a.e. So, by the dominated convergence theorem, $\|Kf_n\|_p \to 0$, whence $g = 0$ a.e. So the closed graph theorem implies that K is bounded. □