MATH 7410 Homework 1

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Problem 1

Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. For any $p \in \mathbb{R}^n$, there is a canonical identification $T_p(\mathbb{R}^n) \to \mathbb{R}^n$ given by

$$\sum a^i \frac{\partial}{\partial x^i}|_p \mapsto a = (a^1, \dots, a^n)$$

Show that the differential $L_{*,p} : T_p(R^n) \to T_{L(p)}(R^m)$ is the map $L : \mathbb{R}^n \to \mathbb{R}^m$ itself, with the identification of the tangent spaces as above.

Problem 2

If M and N are manifolds, let $\pi_1 : M \times N \to M$ and $\pi_2 : M \times N \to N$ be the two projections. Prove that for $(p, q) \in M \times N$,

$$(\pi_{1*}, \pi_{2*} : T_{(p,q)}(M \times N) \to T_pM \times T_qN$$

is an isomorphism.

Problem 3

Let G be a Lie group with multiplication map $\mu : G \times G \to G$, inverse map $\iota : G \to G$, and identity element e.

(a): Show that the differential at the identity of the multiplication map μ is addition:

$$\begin{aligned} \mu_{*,(e,e)} &: T_eG \times T_eG \to T_eG, \\ \mu_{*,(e,e)}(X_e, Y_e) &= X_e + Y_e. \end{aligned}$$

(*Hint*: First, compute $\mu_{*,(e,e)}(X_e, 0)$ and $\mu_{*,(e,e)}(0, Y_e)$ using Proposition 8.18).

(b): Show that the differential at the identity of ι is the negative:

$$\begin{aligned} \iota_{*,e} &: T_eG \to T_eG \\ \iota_{*,e}(X_e) &= -X_e. \end{aligned}$$

(*Hint*: Take the differential of $\mu(c(t), (t \circ c)(t) = e$.)

Problem 4

Show that $T_p^1 M \subseteq T_p^2 M$.