MATH 7410 Homework 3 (In-Progress)

James Harbour

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Problem 1

(a): Let X be a separable Banach space. Show that $Ball(X^*) = \{\phi \in X^* : \|\phi\| \leq 1\}$ is wk^*-metrizable.

Proof. Note that it suffices to show that a countable subset of the seminorms defining the LCS topology on X^* in fact define the topology on $Ball(X^*)$.

Choose a norm dense sequence $(x_n)_{n=1}^{\infty}$ in X. We claim that the seminorms $\rho_{x_n} = |ev_{x_n}(\cdot)| : X^* \to [0, +\infty)$ define the restriction of the wk^*-topology to $Ball(X^*)$.

Suppose that $(\phi_\alpha)_{\alpha \in I}$ is a net in $Ball(X^*)$and $\phi \in Ball(X^*)$ is such that $\phi_\alpha(x_n) \to \phi(x)$ for all $n \in \mathbb{N}$. Now let $x \in X$ and fix $\varepsilon > 0$. Then by density there is some $n \in \mathbb{N}$ such that $\|x - x_n\| < \varepsilon/3$. Moreover, by assumption, there is some $\alpha_0 \in I$ such that for all $\alpha \geq \alpha_0$, $|\phi_\alpha(x_n) - \phi(x_n)| < \varepsilon/3$. Now, for all $\alpha \geq \alpha_0$,

$$\begin{aligned}|\phi(x) - \phi_\alpha(x)| \leq |\phi(x - x_n)| + |\phi(x_n) - \phi_\alpha(x)| &\leq \|x - x_n\| + |\phi_\alpha(x_n) - \phi(x_n)| + |\phi_\alpha(x_n) - \phi_\alpha(x)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \|\phi_\alpha\|\|x_n - x\| < \varepsilon\end{aligned}$$

so in fact, for nets in $Ball(X^*)$, pointwise convergence on $(x_n)_n^\infty$ implies pointwise convergence everywhere, and thus wk^*-convergence of the underlying nets. □

(b): If X is a Banach space, show that there is a compact space K such that X is isometrically isomorphic to a closed subspace of $C(K)$.

Proof. Let $\iota : X \hookrightarrow \mathbb{F}^{X^*}$ be the canonical evaluation map, and for brevity write $\hat{x} = \iota(x)$. Note that each $\hat{x}$ is weak-$*$ continuous, so $\iota(X) \subseteq C(X^*, wk^*)$. Let $K = (Ball(X^*), wk^*)$, so K is a compact space by Banach-Alaoglu. Define a map $\phi : X \to C(K)$ by $\phi(x) = \iota(x)|_{Ball(X^*)}$.

□

Problem 2

Let $Bil(X \times Y, Z)$ be the space of bounded, bilinear maps from $X \times Y \to Z$.

(a): Suppose that B_x, B^y are bounded for each $x \in X$,$y \in Y$. Prove that there is a constant $M > 0$ so that

$$\|B(x, y)\| \leq M\|x\|\|y\|$$

(use the Principle of Uniform Boundedness).

Proof. For $x \in X$, $y \in Ball(Y)$, note that $\|B_y(x)\| \leq \|B_x\|$. Then by the principle of uniform boundedness, $C := \sup_{y \in Ball(Y)} \|B_y\| < +\infty$. Now observe that

$$\sup_{x \in Ball(X),\, y \in Ball(Y)} \|B(x,y)\| = \sup_{x \in Ball(X),\, y \in Ball(Y)} \|B_y(x)\| \leq \sup_{y \in Ball(Y)} \|B_y\| = C,$$

whence the claim follows by scaling. $\square$

(b): Show that the map $\Phi : Bil(X \times Y, \mathbb{F}) \to B(X, Y^*)$ given by $[\Phi(B)(x)](y) = B(x,y)$ is a well-defined, isometric isomorphism.

Proof.
<u>well-defined</u>: For fixed $x \in X$, we have that

$$\|B_x\| = \sup_{\|y\| \leq 1} |B(x,y)| \leq \|B\| < +\infty$$

so $B_x \in Y^*$.

<u>isometric</u>: $B \in Bil(X \times Y, \mathbb{F})$, $x \in X$. On one hand, from

$$\|\Phi(B)\| = \sup_{\|x\| \leq 1} \|B(x, \cdot)\|_{Y^*} = \sup_{\|x\| \leq 1} \sup_{\|y\| \leq 1} |B(x,y)|$$

it is clear that $\|\Phi(B)\| \leq \|B\|$. Suppose that $x \in X$, $y \in Y$, so $|B(x,y)| \leq \|B\| \|x\| \|y\|$. Then

$$|B(x,y)| = |B_x(y)| \leq \|B_x\| \|y\| \leq \|\Phi(B)\| \|x\| \|y\|,$$

whence by minimality $\|B\| \leq \|\Phi(B)\|$. Thus Φ is isometric and $\Phi(B) \in B(X, Y^*)$.

<u>isomorphism</u>: It suffices to show surjectivity. Suppose that $\Gamma \in B(X, Y^*)$. Define a map $B : X \times Y \to \mathbb{F}$ by $B(x,y) = \Gamma(x)(y)$. For $x \in X$ and $y \in Y$, we have that

$$|B(x,y)| \leq \|\Gamma(x)\| \|y\| \leq \|\Gamma\| \|x\| \|y\|,$$

whence $B \in Bil(X \times Y, \mathbb{F})$. Moreover, by construction $\Phi(B) = \Gamma$. $\square$

(c): By switching names, it follows that the map $\widetilde{\Phi} : Bil(X \times Y, \mathbb{F}) \to B(Y, X^*)$ given by $[\widetilde{\Phi}(B)(y)](x) = B(x,y)$ is a well-defined, isometric isomorphism. So the map $\widetilde{\Phi} \circ \Phi^{-1}$ is an isometric isomorphism $B(X, Y^*) \cong B(Y, X^*)$. What is this isomorphism?

Problem 3

Let X, Y be Banach spaces. And let $(T_n)_{n=1}^{\infty}$ be a sequence in $B(X, Y)$.

Lemma 1. *If X is a Banach space, $(x_n)_{n=1}^{\infty}$ a sequence in X such that $\phi(x_n)$ is a bounded sequence for all $\phi \in X^*$, then $(x_n)_{n=1}^{\infty}$ is bounded in norm.*

Proof of Lemma. Let $\hat{x} \in X^{**}$ denote the canonical image of $x \in X$ inside X^{**}. For each $\phi \in X^*$, $\sup_{n \in \mathbb{N}} |\hat{x}_n(\phi)| = \sup_{n \in \mathbb{N}} |\phi(x_n)| < +\infty$, so by the principle of uniform boundedness, $\sup_{n \in \mathbb{N}} \|x_n\| = \sup_{n \in \mathbb{N}} \|\hat{x}_n\| < +\infty$. $\square$

(a): If T_n converges in the WOT to $T \in B(X, Y)$ show that $\sup_n \|T_n\| < +\infty$. (In particular, if T_n converges strongly, then it is norm).

Proof. Fix $x \in X$. Then for all $\phi \in Y^*$, $\phi(T_n x) \to \phi(Tx)$ so $\sup_{n\in\mathbb{N}} |\phi(T_n x)| < +\infty$. Now by the above lemma, $\sup_{n\in\mathbb{N}} \|T_n x\| < +\infty$. Hence, by the principle of uniform boundedness, $\sup_{n\in\mathbb{N}} \|T_n\| < +\infty$ as desired. $\square$

(b): If $\sup_n \|T_n\| < +\infty$ and there is a norm dense $D \subseteq X$ so that $T_n x$ converges for every $x \in D$, show that $T_n x$ converges for all $x \in X$, that $Tx = \lim_{n\to\infty} T_n x$ is a bounded operator, and that $\|Tx - T_n x\| \xrightarrow{n\to\infty} 0$ for every $x \in X$.

Proof. Fix $x \in X$, $\varepsilon > 0$ and choose $z \in D$ such that $\|x - z\| < \varepsilon$. Set $C = \sup_n \|T_n\|$. Then, we compute that

$$\begin{aligned}\|T_n x - T_m x\| &\le \|T_n x - T_n z\| + \|T_n z - T_m x\| < C\varepsilon + \|T_n z - T_m x\| \\ &\le C\varepsilon + \|T_n z - T_m z\| + \|T_m z - T_m x\| < 2C\varepsilon + \|T_n z - T_m z\| \xrightarrow{n,m\to\infty} 0,\end{aligned}$$

so by completeness, $T_n x$ converges for all $x \in X$.

Now define $Tx = \lim_{n\to\infty} T_n x$. This map is well-defined and linear, so it suffices to show that it is bounded. But

$$\|Tx\| = \lim_{n\to\infty} \|T_n x\| \le \liminf_{n\to\infty} \|T_n\|\|x\| \le C\|x\|,$$

whence $\|T\| \le C$ so T is a bounded operator. Lastly, suppose that $x \in X$. Then $\square$

Problem 4

Let X, Y be Banach spaces. And let $(T_n)_{n=1}^\infty$ be a sequence in $B(X, Y)$. Suppose that $\sup_n \|T_n\| < +\infty$ and that $D \subseteq X$, $G \subseteq Y^*$ are norm dense. Assume that $\lim_n \phi(T_n x)$ exists for all $\phi \in G$,$x \in D$.

(a): Show that $\lim_n \phi(T_n x)$ exists for all $\phi \in Y^*$, $x \in X$.

Proof. Fix $x \in X, \phi \in Y^*$, and $\varepsilon > 0$. Choose $y \in D, \psi \in G$ such that $\|x - y\|, \|\phi - \psi\| < \varepsilon$. Set $C = \sup_{n\in\mathbb{N}} \|T_n\|$ Then, for $n, m \in \mathbb{N}$, we compute that

$$\begin{aligned}\|\phi(T_n x) - \phi(T_m x)\| &\le \|\phi(T_n x) - \psi(T_n x)\| + \|\psi(T_n x) - \phi(T_m x)\| \\ &\le C\varepsilon\|x\| + \|\psi(T_n x) - \psi(T_n y)\| + \|\psi(T_n y) - \phi(T_m x)\| \\ &\le C\varepsilon(\|x\| + \|\psi\|) + \|\psi(T_n y) - \psi(T_m x)\| + \|\psi(T_m x) - \phi(T_m x)\| \\ &\le C\varepsilon(2\|x\| + \|\psi\|) + \|\psi(T_n y) - \psi(T_m y)\| + \|\psi(T_m y) - \psi(T_m x)\| \\ &\le C\varepsilon(2\|x\| + 2\|\psi\|) + \|\psi(T_n y) - \psi(T_m y)\| \xrightarrow{n,m\to\infty,\, \varepsilon\to 0} 0.\end{aligned}$$

Hence, by completeness of $\mathbb{F}$, the desired limit exists. $\square$

(b): Show that for every $x \in X$, there is a well-defined bounded operator $S : X \to Y^{**}$ given by $S(x)(\phi) = \lim_{n\to\infty} \phi(T_n x)$.

Proof. This limit exists for every $x \in X$ and $\phi \in Y^*$ by part (a), and it is clearly linear in both x and ϕ. Now suppose that $\phi \in Y^{**}$. Then

$$|\lim_{n\to\infty} \phi(T_n x)| = \lim_{n\to\infty} |\phi(T_n x)| \le \liminf_{n\to\infty} \|\phi\|\|T_n x\| \le \liminf_{n\to\infty} \|\phi\|\|T_n\|\|x\| \le \|\phi\|\|x\| \sup_{n\in\mathbb{N}} \|T_n\|,$$

so $\|S(x)\| \le \|x\| \sup_{n\in\mathbb{N}} \|T_n\| < +\infty$, whence $S(x) \in Y^{**}$. Thus S is a well-defined operator from X to Y^{**}. Moreover, the above inequality implies that $\|S\| \le \sup_{n\in\mathbb{N}} \|T_n\| < +\infty$, so $S \in B(X, Y^{**})$. $\square$

(c): If $T_n x$ converges weakly to an element of Y for every $x \in D$, show that $S(X) \subseteq Y$, and that $T_n \to S$ WOT.

Proof. Since Y is norm-closed inside Y^{**}, S is norm-continuous, and $D \subseteq S^{-1}(Y)$, it follows that $X \subseteq S^{-1}(Y)$ i.e. $S(X) \subseteq Y$. Now, suppose that $x \in X$ and $\phi \in Y^{***}$. Then there is some $y \in Y$ and $\psi \in Y^*$ such that $S(x) = \widehat{y}$ and $\phi|_Y = \psi$.
By construction, $S(x)(\psi) = \lim_{n\to\infty} \psi(T_n x)$, so we compute that

$$\phi(\widehat{T_n x}) = \widehat{\psi T_n x} = \widehat{T_n x}(\psi) = \psi(T_n x) \to S(x)(\psi) = \phi(S(x)) = \phi(\widehat{y}).$$

Hence, $T_n \to S$ WOT.

□

Problem 5

Let G be a countable, discrete, group and $\lambda : G \to B(l^2(G))$ be given by $(\lambda(g)\xi)(h) = \xi(g^{-1}h)$.

(a): Let $(g_n)_{n=1}^{\infty}$ be a sequence in G so that for every finite $F \subseteq G$ we have $\{n : g_n \in F\}$ is finite. Show that $\lim_{n\to\infty} \lambda(g_n) = 0$ in WOT. (Hint: consider first acting on pairs of vectors which are finitely supported and applying the preceding problem to reduce to this case).

Proof. Suppose first that $\xi, \eta \in l^2(G)$ both have finite support, and let $\operatorname{supp}(\xi) = \{x_1, \dots, x_k\}$, $\operatorname{supp}(\eta) = \{y_1, \dots, y_l\}$, $\alpha_i = \xi(x_i)$, $\beta_j = \eta(y_j)$. Then, using finite supportedness to justify interchanges of summations, we compute that

$$\langle \lambda(g_n)\xi, \eta\rangle = \sum_{x\in G} \xi(g_n^{-1}x)\bar{\xi(x)} = \sum_{x\in G}\sum_{i,j=1}^{k,l} \alpha_i\bar{\beta}_j\delta_{x_i}(g_n^{-1}x)\delta_{y_j}(x) = \sum_{i,j=1}^{k,l}\sum_{x\in G} \alpha_i\bar{\beta}_j\delta_{x_i}(g_n^{-1}x)\delta_{y_j}(x) = \sum_{i,j=1}^{k,l} \alpha_i\bar{\beta}_j\delta_{g_n}(y_j x_i^{-1}).$$

If $g_n \notin \bigcup_{j=1}^{l}\bigcup_{i=1}^{k}\{y_j x_i^{-1}\}$, then the above expression is zero. As this set is finite, the assumption on the given sequence implies that $\langle\lambda(g_n)\xi, \eta\rangle$ is eventually equal to zero past some fixed index, whence it converges to zero.

□

(b): Suppose G is infinite. If $\mathcal{K} \subseteq l^2(G)$ is closed and $\lambda(g)\mathcal{K} = \mathcal{K}$ for every $g \in G$, and $\mathcal{K} \neq 0$, show that $\mathcal{K}$ is not finite-dimensional. (Hint: construct a sequence satisfies the hypothesis of the preceding problem. If $\mathcal{K}$ is finite-dimensional, then λ applied to the sequence restricted to $\mathcal{K}$ converges to 0 in WOT, and hence in any other LCS topology on $B(\mathcal{K})$. Consider using this for one of the other operator topologies to get a contradiction).

Proof. Suppose, for the sake of contradiction, that $\mathcal{K}$ is finite dimensional. Let $(g_n)_{n=1}^{\infty}$ be a sequence of pairwise distinct elements of G. This sequence satisfies the hypothesis of part (a), whence $\lambda(g_n) \xrightarrow{\text{WOT}} 0$. As $\mathcal{K}$ is $\lambda(G)$-invariant, we have that $\lambda(g_n)|_{\mathcal{K}} \in B(\mathcal{K})$ whence $\lambda(g_n)|_{\mathcal{K}} \xrightarrow{\text{WOT}} 0$ in $B(\mathcal{K})$.

As $\mathcal{K}$ is finite dimensional, $B(\mathcal{K})$ is also a finite dimensional LCS. Thus, every locally convex topology on $B(K)$ is equal, whence $\lambda(g_n)|_{\mathcal{K}} \xrightarrow{\text{SOT}} 0$. Let $\xi \in B(\mathcal{K})$ with $\xi \neq 0$. Then

$$\|\xi\| = \|\lambda(g_n)\xi\| = \|\lambda(g_n)|_{\mathcal{K}}\xi\| \xrightarrow{n\to\infty} 0$$

which implies that $\xi = 0$, contradicting the choice of ξ.

□