Problem 1.

Let G be a finitely generated group with finite generation set S. Suppose that $S = S^{-1}$ and that $e \in S$. We let

$$B_S(n) = \{s_1 \cdots s_n : s_i \in S, i = 1, \cdots, n\}.$$

Suppose that G has *subexponential growth*, namely $\limsup_{n\to\infty} |B_S(n)|^{1/n} = 1$ (note: this implies that $\lim_{n\to\infty} |B_S(n)|^{1/n} = 1$). Show that there is a sequence $n_1 < n_2 < \cdots$ of natural numbers so that $(B_S(n_k))_{k=1}^{\infty}$ is a Følner sequence.

Hint: it might be helpful to use/prove that for every sequence $(a_n)_{n=1}^{\infty}$ of positive real numbers we have

$$\liminf_{n\to\infty} \frac{a_n}{a_{n-k}} \leq \liminf_{n\to\infty} a_n^{k/n}$$

for every $k > 0$.

Problem 2.

Let G be a countable, discrete group. For $p \in [1, +\infty)$ we say that $(f_n)_{n=1}^{\infty}$ in $\ell^p(G)$ are almost invariant vectors if $\|f_n\|_p = 1$ and if

$$\|\lambda_g f_n - f_n\|_p \to_{n\to\infty} 0 \text{ for all } g \in G.$$

(i) For $p \in [1, +\infty)$ and $f \in \ell^p(G)$ prove that $\|\lambda_g|f| - |f|\|_p \leq \|\lambda_g f - f\|_p$ for all $g \in G$.

(ii) For $a, b \in [0, +\infty)$ and $p \in [1, +\infty)$ prove that $|a^{1/p} - b^{1/p}| \leq |a - b|^{1/p}$ and

$$|a^p - b^p| \leq p|a - b| \max(a^{p-1}, b^{p-1}) \leq p|a - b|(a^{p-1} + b^{p-1}).$$

(iii) Suppose $p \in [1, +\infty)$. Prove that there are almost invariant vectors in $\ell^p(G)$ if and only if G is amenable.

Problems to think about, do not turn in.

Problem 3.

Let G be a countable discrete group. Let (X, μ) be a σ-finite measure space and suppose that $G \curvearrowright (X, \mu)$ by measure-preserving transformations. We define an action on measurable functions by $\alpha_g(f)(x) = f(g^{-1}x)$. Prove that the following are equivalent:

(i) There is a $\Phi \in L^\infty(X, \mu)^*$ so that $\Phi(\alpha_g f) = \Phi(f)$ for every $f \in L^\infty(X, \mu), g \in G$ and so that $\Phi(f) \geq 0$ for all $f \geq 0$ and with $\Phi(1) = 1$.

(ii) There is a sequence $(f_n)_{n=1}^{\infty}$ in $L^1(X, \mu)$ with $f_n \geq 0$ and $\|f_n\|_1 = 1$ and so that $\|\alpha_g(f_n) - f_n\|_1 \to_{n\to\infty} 0$ for every $g \in G$

(iii) There is a sequence $(f_n)_{n=1}^{\infty}$ in $L^2(X, \mu)$ with $\|f_n\|_2 = 1$ and $\|\alpha_g(f_n) - f_n\|_2 \to_{n\to\infty} 0$ for every $g \in G$.

(iv) There is a sequence $(E_n)_{n=1}^{\infty}$ of finite measure, measurable subsets of X with $0 < \mu(E_n)$ and so that $\frac{\mu(gE_n \Delta E_n)}{\mu(E_n)} \to_{n\to\infty} 0$ for every $g \in G$.

(v) For every $\mu \in \mathrm{Prob}(G)$ we have $\left\|\sum_{g\in G} \mu(g)\alpha_g\right\|_{B(L^2(X,\mu))} = 1$.