Homework Assignment 1

1. (a) Let A be the ring $C[0,1]$ of continuous real-valued functions on the interval $I = [0,1]$. Show that for any proper ideal $\mathfrak{a} \subset A$ there exists $p_0 \in I$ such that $f(p_0) = 0$ for all $f \in \mathfrak{a}$. (*Hint.* Suppose that for each $p \in I$ there exists $f_p \in \mathfrak{a}$ such that $f_p(p) \neq 0$. Then there exists an open neighborhood $U_p \ni p$ such that $f_p(q) \neq 0$ for any $q \in U_p$. Pick a finite cover $U_{p_1}, \ldots, U_{p_m}$ of I and consider $f = f_{p_1}^2 + \cdots + f_{p_m}^2$. Show that $f \in \mathfrak{a}$ and $f(x) \neq 0$ for all $x \in I$.)

(b) Let $B = C(-\infty, \infty)$. Show that the set $\mathfrak{b} \subset B$ of functions with compact support is a proper ideal of B but there is no point $p \in (-\infty, \infty)$ such that $f(p) = 0$ for all $f \in \mathfrak{b}$.

2. Let again $A = C[0,1]$ and $I = [0,1]$.

(a) Show that for any $p \in I$, the set $\mathfrak{m}_p = \{f \in A \mid f(p) = 0\}$ is a maximal ideal of A.

(b) Show that the correspondence $p \mapsto \mathfrak{m}_p$ defines a bijection $\theta \colon I \to \operatorname{Specm} A$. (*Hint.* The surjectivity follows from Problem 1(a); for the injectivity observe that $\mathfrak{m}_{p_1} + \mathfrak{m}_{p_2} = A$ if $p_1 \neq p_2$ - cf. Problem 3(b).)

(c) Show that if I is given the natural topology and $\operatorname{Specm} A$ the topology induced from $\operatorname{Spec} A$ then θ becomes a homeomorphism. (*Hint.* Let $\mathfrak{a} \subset A$ be an ideal, and let $X = V(\mathfrak{a}) \cap \operatorname{Specm} A = \{\mathfrak{m} \mid \mathfrak{m} \supset \mathfrak{a}\}$. Then $\theta^{-1}(X) = \{p \in I \mid f(p) = 0 \text{ for all } f \in \mathfrak{a}\}$, hence closed in I. Conversely, let $Y \subset I$ be a closed subset. Set $\mathfrak{a} = \{f \in A \mid f(p) = 0 \text{ for all } p \in Y\}$. Show that $\theta(Y) = V(\mathfrak{a}) \cap \operatorname{Specm} A$. The inclusion $\subset$ is obvious. Now, suppose $p \in I \setminus Y$. Use Urysohn's lemma to find $f \in \mathfrak{a}$ such that $f(p) \neq 0$, hence $\mathfrak{m}_p \notin V(\mathfrak{a}) \cap \operatorname{Specm} A$.)

3. Let again $A = C[0,1]$.

(a) Let $\mathfrak{p} \in \operatorname{Spec} A$. It follows from Problem 2(b) that there exists $p \in I$ such that $\mathfrak{p} \subset \mathfrak{m}_p$. Show that $\mathfrak{p}$ contains

$$\mathfrak{l}_p := \{f \in A \mid f = 0 \text{ on some neighborhood of } p\}.$$

(*Hint.* Show that for any $f \in \mathfrak{l}_p$ there exists $g \in A$ such that $g(p) \neq 0$ and $fg = 0$, hence $fg \in \mathfrak{p}$.)

(b) Show that for $p_1, p_2 \in I$, $p_1 \neq p_2$, we have $\mathfrak{l}_{p_1} + \mathfrak{l}_{p_2} = A$. Deduce that every $\mathfrak{p} \in \operatorname{Spec} A$ is contained in a *unique* $\mathfrak{m} \in \operatorname{Specm} A$. It follows that every closed irreducible subset of $\operatorname{Spec} A$ contains a unique closed point.

Question. Is A Noetherian? You can answer this question based on Problem 3(b), but it is also easy to find the answer directly.

4. (a) For $A = C[0,1]$, show that every $\mathfrak{m} \in \operatorname{Specm} A$ properly contains some $\mathfrak{p} \in \operatorname{Spec} A$, and in particular, $\operatorname{Spec} A \neq \operatorname{Specm} A$. (*Hint.* Let $\mathfrak{m} = \mathfrak{m}_p$ with $p \in I$. Pick $f \in \mathfrak{m}$ so that $f(q) \neq 0$ for $q \neq p$, and consider the multiplicative sets $S_1 = A \setminus \mathfrak{m}$ and $S_2 = \{1, f, f^2, \ldots\}$. Then $S = S_1 S_2$ is also a multiplicative set. Using Zorn's lemma, find an ideal $\mathfrak{p} \subset A$ maximal with respect to the property $\mathfrak{p} \cap S = \emptyset$. Argue that $\mathfrak{p}$ is prime and $\mathfrak{p} \subsetneq \mathfrak{m}$.)

(b) Show that $\operatorname{Specm} A$ is dense in $\operatorname{Spec} A$. (*Hint.* Use principal open sets $D(f)$.)

(c) It follows from Problem 2(c) that $\operatorname{Specm} A$ is compact in the topology induced from $\operatorname{Spec} A$. At the same time, it is dense in $\operatorname{Spec} A$, which is strictly bigger. So, there appears to be a contradiction with the basics facts from topology. Resolve this contradiction.