MATH 7820 Homework 3

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Problem 1

Is the solution set to the system of equations

$$x^3 + y^3 + z^3 = 1, \quad z = xy$$

in $\mathbb{R}^3$ a smooth manifold? Prove your answer.

Proof. Let $S \subseteq \mathbb{R}^3$ be the solution set to the above system of equations. Define $F : \mathbb{R}^3 \to \mathbb{R}^2$ by $(u, v) = F(x, y, z) = (x^3 + y^3 + z^3, xy - z)$. Then $S = F^{-1}((1, 0))$. By the regular set theorem, it suffices to show that $F^{-1}((1, 0))$ is a regular set. Hence, we must show that d_pF is surjective for all $p \in S$, or equivalently, that $\operatorname{rank}(J(F)_p) = 2$ for all $p \in S$ where $J(F)$ denotes the Jacobian of F. We initially compute that

$$J(F) = \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \end{pmatrix} = \begin{pmatrix} 3x^2 & 3y^2 & 3z^2 \\ y & x & xy \end{pmatrix}$$

Now, suppose $p = (a, b, c) \in S$ is a critical point of F, i.e. $\operatorname{rank}(J(F)_p) < 2$. Then $a^3 + b^3 + c^3 = 1$ and $c = ab$, so

$$J(F)_p = \begin{pmatrix} 3a^2 & 3b^2 & 3c^2 \\ b & a & ab \end{pmatrix} = \begin{pmatrix} 3a^2 & 3b^2 & 3(ab)^2 \\ b & a & ab \end{pmatrix}.$$

If $c = 0$, then either a or b is 0, forcing the other to be 1, whence we obtain one of the following matrices,

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 3 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

which are both of full rank contradicting the the criticality of p. Thus, $c \neq 0$, so $a, b \neq 0$. Now by standard linear algebra, criticality of p is equivalent to the statement that all 2×2-minors of $J(F)_p$ vanish. Thus, we have that

$$\det \begin{pmatrix} 3a^2 & 3b^2 \\ b & a \end{pmatrix} = 0, \qquad \det \begin{pmatrix} 3a^2 & 3a^2b^2 \\ b & ab \end{pmatrix} = 0, \qquad \det \begin{pmatrix} 3b^2 & 3a^2b^2 \\ a & ab \end{pmatrix},$$

whence we obtain the relations,

$$\left.\begin{aligned} 0 &= 3a^3 - 3b^3 \\ 0 &= 3a^3b - 3a^2b^3 \\ 0 &= 3ab^3 - 3a^3b^2 \end{aligned}\right\} \overset{a,b\neq 0}{\Longrightarrow} \left.\begin{aligned} 0 &= a^3 - b^3 \\ 0 &= b - a^2 \\ 0 &= a - b^2 \end{aligned}\right\}.$$

Now, as $b = a^2$, it follows that $0 = a - b^2 = a - a^4$ whence $a \neq 0$ implies that $a^3 = 1$. As $a \in \mathbb{R}$, it follows that $a = 1$ and consequently $b = 1$,$c = 1$. However, this contradicts the fact that $p \in S$ as $3 = 1 + 1 + 1 = a^3 + b^3 + c^3 \neq 1$. Thus every point in S is regular, so $F^{-1}((1, 0))$ is a regular level set whence by the regular set theorem S is a smooth manifold. $\square$

Problem 2

A C^∞ map $f : N \to M$ is said to be *transversal* to a submanifold $S \subseteq M$ if for every $p \in f^{-1}(S)$,

$$f_*(T_pN) + T_{f(p)}S = T_{f(p)}M.$$

The goal of this exercise is to prove the *transversality theorem*: if a C^∞ map $f : N \to M$ is transversal to a regular submanifold S of codimension k in M, then $f^{-1}(S)$ is a regular submanifold of codimension k in N.

When S consists of a single point c, transversality of f to S simply means that $f^{-1}(c)$ is a regular level set. Thus the transversality theorem is a generalization of the regular level set theorem. It is especially useful in giving conditions under which the intersection of two submanifolds is a submanifold.

Let $p \in f^{-1}(S)$ and $(U, x^1, \ldots, x^n)$ be an adapted chart centered at $f(p)$ for M relative to S such that $U \cap S = Z(x^{m-k+1}, \ldots, x^m)$, the zero set of the functions $x^{m-k+1}, \ldots, x^m$. Define $g : U \to \mathbb{R}^k$ to be the map

$$g = (x^{m-k+1}, \ldots, x^m).$$

(a): Show that $f^{-1}(U) \cap f^{-1}(S) = (g \circ f)^{-1}(0)$.

Proof. Observe that

$$\begin{aligned} q \in (g \circ f)^{-1}(0) &\iff g(f(q)) = 0 \iff x^i(f(q)) = 0 \text{ for } i = m-k+1, \ldots, m \\ &\iff f(q) \in Z(x^{m-k+1}, \ldots, x^m) = U \cap S \iff q \in f^{-1}(U \cap S) = f^{-1}(U) \cap f^{-1}(S). \end{aligned}$$

□

(b): Show that $f^{-1}(U) \cap f^{-1}(S)$ is a regular level set of the function $g \circ f : f^{-1}(U) \to \mathbb{R}^k$.

Proof. Fix $p \in f^{-1}(S) \cap f^{-1}(U) = (g \circ f)^{-1}(0)$. We wish to show that p is a regular point for $g \circ f$. Suppose that $a \in T_0\mathbb{R}^k$. Then, noting that $dg_{f(p)}$ is surjective, there exists a $w \in T_{f(p)}M$ such that $dg_{f(p)}(w) = a$. Then, by transversality of f with respect to S, there exist $u \in T_pN$, $v \in T_{f(p)}S$ such that $w = df_p(u) + v = w$. Now, note that $g(U \cap S) = 0$, so it follows that $dg_{f(p)}(T_{f(p)}(S)) = 0$. Hence, we compute

$$a = dg_{f(p)}(w) = dg_{f(p)}(df_p(u) + v) = dg \circ f_p(u) + dg_{f(p)}(v) = dg \circ f_p(u),$$

so p is a regular point for $g \circ f$ as its differential is surjective. □

(c): Prove the transversality theorem.

Proof. By the regular level set theorem, $f^{-1}(U) \cap f^{-1}(S)$ is a codimension k submanifold of N. For $q \in S$, choose a chart (V_q, ϕ_q) adapted to q. Then $f^{-1}(V_q \cap S) \subseteq f^{-1}(S)$ and $f^{-1}(S) = \bigcup_{q \in S} f^{-1}(V_q \cap S)$, whence $f^{-1}(S)$ is also a codimension k submanifold of N, as desired. □

Problem 3

(a): Consider the "height map" $h : S^2 \to \mathbb{R}$. Here S^2 is the unit sphere in $\mathbb{R}^3$ and $h(x, y, z) = z$. Find the critical points and critical values for this map.

Proof. First suppose $p = (a, b, c) \in S^2$ with $c > 0$. Consider the chart (U, ϕ) on S^2 given by $U = \{(x, y, z) \in S^2 : z > 0\}$ and $\phi(x, y, z) = (x, y)$. Let $\widetilde{h} : \phi(U) \to \mathbb{R}$ be the coordinate representation of h with respect to this chart. Then $\widetilde{h}(x, y) = (h \circ \phi^{-1})(x, y) = h(x, y, \sqrt{1 - x^2 - y^2}) = \sqrt{1 - x^2 - y^2}$, whence

$$d\widetilde{h}_p = \begin{pmatrix} \frac{-x}{\sqrt{1-x^2-y^2}} & \frac{-y}{\sqrt{1-x^2-y^2}} \end{pmatrix},$$

which has rank 0 if and only if $x, y = 0$, whence $p = (0, 0, 1)$. Thus $p = (0, 0, 1)$ is the only critical point of h in U and has critical value 1.

If $p = (a, b, c) \in S_2$ with $c < 0$, then we have a similar situation except that $\widetilde{h}(x, y) = -\sqrt{1 - x^{2-y^2}}$ and our chart is (V, ψ) where $V = \{(x, y, z) \in S^2 : z < 0\}$ and $\psi^{-1}(x, y) = (x, y, -\sqrt{1 - x^2 - y^2})$. Thus, again $d\widetilde{h}_p$ has rank 0 if and only if $x, y = 0$. So, in this case $p = \psi^{-1}(0, 0) = (0, 0, -1)$ is the only critical point of h in V and has critical value -1.

Now must check points on the equator $E = \{(x, y, z) \in S^2 : z = 0\}$. Consider points p in the chart (W, ρ) with $W = \{(x, y, z) \in S^2 : y > 0\}$ and $\rho(x, y, z) = (x, z)$. We compute that the coordinate representation of h is then given by $\widetilde{h}(x, z) = z$, whence $dF_p = \begin{pmatrix} 0 & 1 \end{pmatrix}$ for all $p \in W$, so no points in W can be critical. Similarly, no points in $W' = \{(x, y, z) \in S^2 : y < 0\}$ can be critical either (identical calulation).

Thus, it remains to check the points $(1, 0, 0)$ and $(-1, 0, 0)$. Consider the chart (K, γ) given by $K = \{x > 0\}$ and $\gamma(x, y, z) = (y, z)$. Then again, $\widetilde{h}(y, z) = z$, whence no points in K can be critical. Similarly, no points in $K' = \{(x, y, z) \in S^2 : x > 0\}$ can be critical either (by the same calculation). □

(b): Show that any map $f : S^2 \to \mathbb{R}$ has at least two critical points. Generalize this proof from S^2 to any n-dimensional compact manifold.

Proof. Since f is continuous and S^2 is compact, it follows that there exist $p, q \in S^2$ such that $f(p) \leq f(x) \leq f(q)$ for all $x \in S^2$ and $p \neq q$. Thus p, q are global minima/maxima of the function f. Choose a chart (U, ϕ) on S^2 such that $p, q \in U$. Then the function $\widetilde{f} : \phi(U) \to \mathbb{R}$ given by $\widetilde{f} = f \circ \phi^{-1}$ is smooth as a function from $\mathbb{R}^2 \to R$ and has global minima/maxima p, q. Thus, by calculus 3,

$$\frac{\partial \widetilde{f}}{\partial x} = \frac{\partial \widetilde{f}}{\partial y} = \frac{\partial \widetilde{f}}{\partial x} = \frac{\partial \widetilde{f}}{\partial y} = 0,$$

whence df_p has rank 0 at both p and q, so p and q are critical points of f.

Now suppose that $F : M \to \mathbb{R}$ is a smooth map from an n-dimensional compact manifold M. Again, by continuity, there exist $p, q \in M$ such that $f(p) \leq f(x) \leq f(q)$ for all $x \in M$ and $p \neq q$. Again, choose a chart (U, ϕ) containing both p and q. Then $\widetilde{F} = F \circ \phi^{-1}$ is a smooth function from $\phi(U) \subseteq \mathbb{R}^n$ to $\mathbb{R}$ with global minima/maxima p, q. Again, for $i = 1, \ldots, n$, it follows that

$$\frac{\partial \widetilde{F}}{\partial x_i} = \frac{\partial \widetilde{F}}{\partial x_i} = 0,$$

whence the jacobian of $\widetilde{F}$ at p and at q has rank 0, making p, q critical points of F.

□

Problem 4

Consider a submanifold $M^n \subseteq \mathbb{R}^k$ and let $TM \subseteq \mathbb{R}^k \times \mathbb{R}^k$ be the set of all pairs (x, v) where x is a point in M and $v \in T_x M$. Show that TM is a smooth $2n$-dimensional submanifold of $\mathbb{R}^{2k}$.

Proof. Fix $(x, v) \in TM$ and let (U, ϕ) be a chart on $\mathbb{R}^k$ adapted to M about p. Set $V = U \cap M$, $\widetilde{U} = U \times \mathbb{R}^k$, and $\widetilde{V} = \bigsqcup_{y \in V} (\{y\} \times T_y M) \subseteq TM$. Let $\widetilde{\phi} : \widetilde{U} \to \mathbb{R}^k \times \mathbb{R}^k$ be given by $\widetilde{\phi}(y, w) = (\phi(y), d\phi_y(w))$. Note that, after identifying $T_y M \cong R^n \subseteq \mathbb{R}^k$ for each $y \in U$, we have that $\widetilde{V} = \widetilde{U} \cap TM$. . As (U, ϕ) is adapted to M, by definition $\phi(U) = \phi(U \cap M) \times \{0\} \subseteq \mathbb{R}^n \times \mathbb{R}^{k-n}$. Now compute that

$$\begin{aligned}
\widetilde{\phi}(\widetilde{U}) &= \{(\phi(p), d\phi_p(v)) : p \in V, v \in \mathbb{R}^n\} \\
&= \{(\phi|_V, 0, \ldots, 0, d\phi|_V(v), 0, \ldots, 0) \in \mathbb{R}^k \times \mathbb{R}^k : p \in V, v \in T_p M\} \\
&= \widetilde{\phi}(\widetilde{U} \cap TM) \times \mathbb{R}^{2(k-n)}.
\end{aligned}$$

Hence, it suffices to show that if $(U, \phi), (V, \psi)$ are two charts on $\mathbb{R}^k$ adapted to M, that the transition maps $\widetilde{\phi}| \circ \widetilde{\psi}^{-1}$ is smooth (the other direction would follow since the charts chosen are arbitrary). Let $p \in \psi(U \cap V), v \in \mathbb{R}^k$. Then

$$\widetilde{\phi}| \circ \widetilde{\psi}^{-1}(p, v) = (\phi \circ \psi^{-1}(p), d\phi_{\psi^{-1}(p)} \circ d\psi_p^{-1}(v)) = (\phi \circ \psi^{-1}(p), d(\phi \circ \psi^{-1})_p(v)),$$

which is smooth as the jacobian smoothly depends upon p, v. □