MATH 7410 Homework 5

James Harbour

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Problem 1

Let $\mathbb{N}$ have the discrete topology. Let $\{r_n : n \in \mathbb{N}\}$ be an enumeration of the rational numbers in $[0, 1]$. Let $S = \mathbb{Q} \cap [0, 1]$ and for each $s \in S$ let $\{r_n : n \in N_s\}$ be a subsequence such that $s = \lim\{r_n : n \in N_s\}$.

(a): Show that, if $s, t \in S$ and $s \neq t$, then $N_s \cap N_t$ is finite.

Proof. Choose $\varepsilon > 0$ small enough such that $B_\varepsilon(s) \cap B_\varepsilon(t) = \emptyset$. Let $A = \{n \in N_s : r_n \notin B_\varepsilon(s)\}$ and $B = \{n \in N_t : r_n \notin B_\varepsilon(t)\}$. By assumption, A, B are both finite. Suppose that $n \in N_s \cap N_t$ and $n \notin A$. Then $r_n \in B_\varepsilon(s)$, whence $r_n \notin B_\varepsilon(t)$ and thus $n \in B$. Hence we have shown that $N_s \cap N_t \subseteq A \cup B$, which is finite. $\square$

(b): If for each $s \in S$, $\overline{N_s}$ in $\beta\mathbb{N}$ and $A_s = \overline{N_s} \setminus \mathbb{N}$, show that $\{A_s : s \in S\}$ are pairwise disjoint subsets of $\beta\mathbb{N} \setminus \mathbb{N}$ that are both open and closed.

Proof. Throughout this proof we identify $\beta\mathbb{N} = \Sigma(l^\infty(\mathbb{N}))$. Suppose first that $h \in A_s \cap A_t$. Then there exist nets $(n_\alpha)_\alpha$ in N_s and $(m_\beta)_\beta$ in N_t such that $\widehat{n_\alpha} \xrightarrow{wk^*} h$ and $\widehat{m_\beta} \xrightarrow{wk^*} h$. Considering r as a function $r : \mathbb{N} \to S$, it follows that $r \in l^\infty(\mathbb{N})$ so

$$s = \lim_\alpha r_{n_\alpha} = \lim_\alpha \widehat{n_\alpha}(r) = h(r) = \lim_\beta \widehat{m_\beta}(r) = \lim_\beta r_{m_\beta} = t,$$

and thus $s = t$. By contraposition, it follows that all of the sets A_s are pairwise disjoint.

To see that A_s is closed in $\beta\mathbb{N} \setminus \mathbb{N}$, note that $A_s = \overline{N_s} \setminus \mathbb{N} = \overline{N_s} \cap (\beta\mathbb{N} \setminus \mathbb{N})$ and appeal to the definition of the subspace topology. On the other hand, consider the function $f : \mathbb{N} \to \{0, 1\}$ givem by $f = \mathbb{1}_{N_s}$. This function is continuous and $\{0, 1\}$ is compact, so by universality there is some continuous $\widetilde{f} : \beta\mathbb{N} \to \{0, 1\}$ extending f. Now by continuity, as $\widetilde{f}^{-1}(1) \supseteq N_s$ it follows that $\widetilde{f}^{-1}(1) \supseteq \overline{N_s}$. Hence, by disjointness of the two inverse images of the separate points, $\widetilde{f}^{-1}(1) = \overline{N_s}$ and $\widetilde{f}^{-1}(0) = \beta\mathbb{N} \setminus \overline{N_s}$. Thus, $\widetilde{f} = \mathbb{1}_{\overline{N_s}}$. Since $\widetilde{f}$ is continuous and $\{1\}$ is open, $\overline{N_s} = \widetilde{f}^{-1}(1)$ is also open.

$\square$

Problem 2

Show that $Ball(l^1)$ is the norm closure of the convex hull of its extreme points.

Proof. By problem 3, $ext(K) = \{\alpha\delta_n : n \in \mathbb{N}, \alpha \in \mathbb{D}\}$. Let $K = Ball(l^1)$. Now suppose $f \in Ball(l^1)$. Set $f_n = \sum_{i=1}^n f(i)\delta_i$, so $f_n \to f$ in norm. Let $S_n = \sum_{i=1}^n |f(i)|$. Then we compute that

$$f_n = \sum_{i=1}^n |f(i)| \left(\frac{f(i)}{|f(i)|}\delta_i\right) = \sum_{i=1}^n |f(i)| \left(\frac{f(i)}{|f(i)|}\delta_i\right) + \frac{1-S_n}{2}\delta_{n+1} + \frac{1-S_n}{2}(-\delta_{n+1})$$

and $\sum_{i=1}^n |f(i)| + \frac{1-S_n}{2} + \frac{1-S_n}{2} = 1$, so $f_n \in co(ext(K))$ for all $n \in \mathbb{N}$. □

Problem 3

If (X, Σ, μ) is a σ-finite measure space, the set of extreme points of $Ball(L^1(\mu))$ is $\{\alpha\mathbb{1}_E : E \text{ is an atom of } \mu, \alpha \in \mathbb{F}, \text{ and } |\alpha| = \mu(E)^{-1}\}$.

Lemma 1. *Measurable functions are constant a.e. on atoms.*

Proof of lemma. Let $E \subseteq X$ be an atom and $f : E \to \mathbb{C}$ measurable. Without loss of generality assume that $\mu(supp(f)) > 0$, as otherwise we would be done. Fix $N \in \mathbb{N}$, and choose a sequence $(y_{N,n})_{n=1}^\infty$ in $\mathbb{C}$ such that $\mathbb{C} = \bigcup_{n=1}^\infty B_{\frac{1}{N}}(y_{N,n})$. Then

$$supp(f) = \bigcup_{n=1}^\infty \{x \in E : f(x) \in B_{\frac{1}{N}}(y_{N,n})\} = \bigcup_{n=1}^\infty f^{-1}(B_{\frac{1}{N}}(y_{N,n})),$$

so by assumption it follows that there is some $n = n(N) \in \mathbb{N}$ such that $\mu(f^{-1}(B_{\frac{1}{N}}(y_{N,n})) > 0$. Write $x_N = y_{N,n}$ for brevity. We claim that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy. Fix $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\varepsilon}{2}$. Fix $e \in E$. Then for all $n, m \geq N$,

$$|x_n - x_m| \leq |x_n - f(e)| + |f(e) - x_m| \leq \frac{1}{n} + \frac{1}{m} < \varepsilon.$$

By completeness there is some $x \in \mathbb{C}$ such that $x_n \to x$. We claim that $f = x$ a.e.

Fix $e \in E$. Then $|f(e) - x| = \lim |f(e) - x_n| = 0$, so $f(e) = x$. □

Proof. On one hand, suppose that $E \subseteq X$ is an atom of μ, $\alpha \in \mathbb{F}$, $|\alpha| = \mu(E)^{-1}$, and $f = \alpha\mathbb{1}_E$. Suppose that $g, h \in Ball(L^1(\mu))$ and $t \in [0, 1]$ are such that $f = (1-t) \cdot g + t \cdot h$. If $g = 0$ or $h = 0$, then by the fact that $\|f\| = 1$, it would follow that $t = 0, 1$ and we would be done. Hence, assume $g, h \neq 0$. Since $\mathrm{supp}(f) \subseteq E$,

$$f = f\mathbb{1}_E = (1-t)g\mathbb{1}_E + th\mathbb{1}_E.$$

now we compute that

$$1 = \|f\| \leq (1-t)\|g\mathbb{1}_E\| + t\|h\mathbb{1}_E\| \implies \|g\mathbb{1}_E\|, \|h\mathbb{1}_E\| = 1$$

as we have assumed $g, h \in Ball(L^1(\mu))$. Thus $\|g\| = \|g\mathbb{1}_E\|$ and $\|h\| = \|h\mathbb{1}_E\|$, so $g, h = 0$ a.e. on $X \setminus E$. Setting $N(f) = \{x : f(x) \neq 0\}$, it follows that $N(g), N(h) \subseteq E$. Now note that $E = N(f) \subseteq N(g) \cup N(h)$, whence it follows that

$$E = (E \cap N(g)) \cup (E \cap N(h)) = N(g) \cup N(h).$$

Since E is an atom, it follows that $\mu(N(g)), \mu(N(h)) \in \{0, \mu(E)\}$. As we have assumed $g, h \neq 0$, $\mu(N(g)) = \mu(E) = \mu(N(h))$.

Let $f \in K$, and suppose for the sake of contradiction that $N(f)$ is an atom but $f \notin ext(K)$. Then $f = (1-t)g + th$ with $t \in (0,1)$, $g, h \neq f$. Since $N(f)$ is atomic and $g, h \neq f$, it follows that $\mu(N(f) \backslash (N(g) \cup N(h))) = 0$.

$$0 = f\mathbb{1}_{X\backslash N(f)} = ((1-t)g + th)\mathbb{1}_{X\backslash N(f)}$$

hence restricting to $N(f)$ we compute that

$$f = (1-t)g + th = ((1-t)g + th)\mathbb{1}_{N(f)}$$

Note that, as $N(f)$ is an atom, all measurable function are constant a.e. on $N(f)$, whence there are some $\alpha, \beta, \gamma \in \mathbb{C}$ such that

$$f\mathbb{1}_{N(f)} = \alpha, \quad g\mathbb{1}_{N(f)} = \beta \quad , h\mathbb{1}_{N(f)} = \gamma.$$

By definition, $\alpha \neq 0$, hence $\alpha = (1-t)\beta + t\gamma$ implies that at least one of β, γ must be nonzero. Let $a = \frac{\beta}{\alpha}$ and $b = \frac{\gamma}{\alpha}$. Then $g\mathbb{1}_{N(f)} = af\mathbb{1}_{N(f)}$ and $h\mathbb{1}_{N(f)} = bf\mathbb{1}_{N(f)}$. Note that this implies $|a|, |b| \leq 1$ as $f, g, h \in K$. Now

$$f\mathbb{1}_{N(f)} = (1-t)g\mathbb{1}_{N(f)} + th\mathbb{1}_{N(f)} = (1-t)\alpha f\mathbb{1}_{N(f)} + t\beta f\mathbb{1}_{N(f)} \implies 1 = (1-t)a + tb,$$

which has no solution for $a, b \in \overline{\mathbb{D}}$ as 1 is an extreme point for the closed unit disk. Now we have shown that if $f \in K$ and $N(f)$ is an atom, then $f \in ext(K)$.

Suppose that E is an atom, $\alpha \in \mathbb{F}$, $|\alpha| = \mu(E)^{-1}$, and $f = \alpha\mathbb{1}_E$. Then $f \in K$ and $N(f) = E$ is an atom, so $f \in ext(K)$.

On the other hand, suppose that $f \in K$ is an extreme point, and suppose for the sake of contradiction that $E = N(f)$ is non-atomic. Then there exist measurable $A, B \subseteq E$ such that $A \cap B = \emptyset$, $0 < \mu(A), \mu(B) < \mu(E)$, and $\|f\mathbb{1}_A\|, \|f\mathbb{1}_B\| > 0$. Let $g = \frac{1}{\|f\mathbb{1}_A\|} f\mathbb{1}_A$ and $h = \frac{1}{\|f\mathbb{1}_B\|} f\mathbb{1}_B$.Then $g, h \in K$, $g, h \neq 0$. Note by extremality of f that $1 = \|f\| = \|f\mathbb{1}_E\|$. Then, observe setting $t = \|f\mathbb{1}_A\| \in (0,1)$, it follows that

$$f = f\mathbb{1}_E = f\mathbb{1}_A + f\mathbb{1}_B = \|f\mathbb{1}_A\|g + \|f\mathbb{1}_B\|h = tg + (1-t)h,$$

contradicting that f is an extreme point.
Now we have that if $f \in ext(K)$ then $N(f)$ is an atom. However then by Lemma 1, f is constant a.e. on $N(f)$, so there is some $\alpha \in \mathbb{C}$ such that $f = \alpha$ on $N(f)$. Outside of $N(f)$, $f = 0$ so $f = \alpha\mathbb{1}_{N(f)}$. Moreover, as $f \in ext(K)$, we have that $\|f\| = 1$ whence $|\alpha| = \mu(N(f))^{-1}$. $\square$