MATH 7410 Homework 4 (In-Progress)

James Harbour

October 31, 2022

Problem 1

Let X be a normed space and x_n a sequence in X such that $x_n \to x$ weakly. Show that there is a sequence y_n such that $y_n \in co\{x_1, \dots, x_n\}$ and $\|y_n - x\| \to 0$.

Proof. Let $C_n = co\{x_1, \dots, x_n\}$ and $C = \bigcup_{n=1}^{\infty} C_n$. Note that $C = co\{x_i : i \in \mathbb{N}\}$ is convex. Thus $x \in \overline{C}^{wk} = \overline{C}^{\|\cdot\|}$, whence there is some sequence $(z_n)_{n=1}^{\infty}$ in C such that $\|z_n - x\| \to 0$. Let $k_n \in \mathbb{N}$ be such that $z_n \in C_{k_n}$.

Note that a sequence a_n in X converges to 0 in norm if and only if for every $\varepsilon > 0$, the set $\{n \in \mathbb{N} : \|a_n\| \geq \varepsilon\}$ is finite. This condition is invariant under rearrangements, so without loss of generality we may take the sequence $(k_n)_{n\in\mathbb{N}}$ to be nondecreasing. Construct a new sequence $(y_m)_{m\in\mathbb{N}}$ as follows. For $m < k_1$, set $y_m = 0$. For $n \in \mathbb{N}$ and $k_n \leq m < k_{n+1}$, set $y_m = z_n$.

The sequence $y_m - x$ still has the condition that for all $\varepsilon > 0$ the set of indices whose corresponding elements have norm at least ε is finite, as we have only added a finite number of elements to this set. Thus $y_n \in C_n$ for all $n \in \mathbb{N}$ and $\|y_n - x\| \to 0$. $\square$

Problem 2

If $\mathcal{H}$ is a Hilbert space and h_n is a sequence in $\mathcal{H}$ such that $h_n \to h$ weakly and $\|h_n\| \to \|h\|$, show that $\|h_n - h\| \to 0$.

Proof. $\square$

Problem 3

If X, Y are Banach spaces and $B \in B(Y^*, X^*)$, then $B = A^*$ for some $A \in B(X, Y)$ if and only if B is wk^*-continuous.

Proof.
$(\Longrightarrow)$: Suppose that $B = A^*$ for some $A \in B(X, Y)$ and let $(\psi_\alpha)_{\alpha\in I}$ be a net in Y^* such that $\psi_\alpha \to \psi \in Y^*$ weak*. Fix $x \in X$. Then

$$B(\psi_\alpha)(x) = A^*(\psi_\alpha)(x) = \psi_\alpha(Ax) \xrightarrow{\alpha\in I} \psi(Ax) = B(\psi)(x),$$

so $B(\psi_\alpha) \to B(\psi)$ weak*, i.e. B is weak*-continuous.

$\underline{(\Longleftarrow)}$: Suppose that B is wk^*-continuous. Let ι_X, ι_Y be the canonical injections into the corresponding double-duals. For shorthand, we may write $\hat{x} := \iota_X(x)$ and similarly for Y. Noting that $B^* \in B(X^{**}, Y^{**})$ we investigate what occurs when B^* is restricted to the image of X inside its double dual.

Fix $x \in X$. We claim that $B^*(\hat{x}) \in (Y^*, wk^*)^*$. To this end, let $(\phi_\alpha)_{\alpha \in I}$ be net in Y^* such that $\phi_\alpha \to \phi \in Y^*$ weak*. Then,

$$B^*(\hat{x})(\phi_\alpha) = \hat{x}(B(\phi_\alpha)) = B(\phi_\alpha)(x) \to B(\phi)(x) = B^*(\hat{x})(\phi),$$

so $B^*(\hat{x})$ is weak*-continuous, whence there exists some $y \in Y$ such that $B^*(\hat{x}) = \hat{y}$.

□

Problem 4

Let X, Y be Banach spaces over $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$. For $C \subseteq B(X,Y)$ convex and $F \subseteq X$ finite, set $C_F = \{(Tx)_{x \in F} : T \in C\} \subseteq Y^{\oplus F}$. Equip $Y^{\oplus F}$ with the norm

$$\|(y_x)_{x \in F}\| = \sum_{x \in F} \|y_x\|.$$

(a): Let $C \subseteq B(X,Y)$ be convex. Show that $T \in \overline{C}^{SOT}$ if and only if for every $F \subseteq X$ finite, we have that $(Tx)_{x \in F} \in \overline{C_F}^{\|\cdot\|}$. Show that $T \in \overline{C}^{WOT}$ if and only if for every $F \subseteq X$ finite, we have that $(Tx)_{x \in F} \in \overline{C_F}^{weak}$.

Proof.
$\underline{\Longrightarrow}$: Suppose $T \in \overline{C}^{SOT}$, so there is some net $(T_\alpha)_{\alpha \in I}$ in C such that $T_\alpha \to T$ SOT. Let $F \subseteq X$ finite. Then $\|T_\alpha x - Tx\| \to 0$ for every $x \in F$. Since F is finite,

$$\|(T_\alpha x)_{x \in F} - (Tx)_{x \in F}\| = \sum_{x \in F} \|T_\alpha x - Tx\| \xrightarrow{\alpha \in I} 0$$

so $T \in \overline{C}^{SOT}$.

$\underline{\Longleftarrow}$:

□

(b): Suppose that $C \subseteq B(X,Y)$ is convex. Show that $\overline{C}^{WOT} = \overline{C}^{SOT}$.

(c): IF $\phi : B(X,Y) \to \mathbb{F}$ is linear, show that ϕ is WOT-continuous if and only if it is SOT-continuous.

Problem 5

Let I be a set and $\mathcal{M}$ be the set of all $m \in l^\infty(I)^*$ such that :

- $m(f) \geq 0$ for all $f \geq 0$,
- $m(1) = 1$.

Identify Prob(I) with $\{f \in l^1(I) : f \geq 0, \|f\|_1 = 1\}$ and view $l^1(I) \subseteq l^\infty(I)^*$ by $f \mapsto \phi_f$ where $\phi_f(g) = \sum_{i \in I} f(i)g(i)$. Show that Prob$(I)$ is weak*-dense in $\mathcal{M}$.

Proof. Suppose, for the sake of contradiction, that there is some $\psi \in \mathcal{M} \setminus \overline{\mathrm{Prob}(I)}^{wk^*}$. Then there exists some wk^*-continuous linear functional $T : l^\infty(I)^* \to \mathbb{F}$ such that $\ker(T) \supseteq \overline{\mathrm{Prob}(I)}^{wk^*}$ and $T(\psi) = 1$. Note that $T \in (l^\infty(I)^*, wk^*)^* = l^\infty(I)$, so there is some $f \in l^\infty(I)$ such that $T = \hat{f}$, i.e. $T(\phi) = \phi(f)$ for all $\phi \in l^\infty(I)^*$. Then, for $i \in I$

$$0 = T(\delta_i) = \delta_i(f) = f(i),$$

whence $f = 0$ and consequently $T = 0$, contradicting that $T(\psi) = 1$.

$\square$