Math 7820, Fall 2022

Homework 4, due on Thursday, October 13, electronically on Collab.

1. Show that the degree (mod 2) of a composition of two smooth maps f and g equals

$$deg_2(f) \cdot deg_2(g) \pmod 2.$$

2. Give an example of manifolds M, N, and of a smooth function $f\colon M \longrightarrow N$ with a regular value which is a limit point of critical values. Here M, N are smooth manifolds without boundary; no other topological assumptions are made about the manifolds.

3. If two smooth maps f, g from a manifold M to the unit sphere S^k satisfy

$$\|f(x) - g(x)\| < 2$$

for all $x \in M$, prove that f is smoothly homotopic to g.

4. Consider a simple closed curve C in $\mathbb{R}^2 \setminus \{(0,0)\}$, and let $f\colon C \longrightarrow \mathbb{R}$ be the distance to the origin: $f(x,y) = \sqrt{x^2+y^2}$. Prove that the critical points of f are precisely the points where the curve C is tangent to some circle centered at the origin.

Hint: It may be helpful to consider the square of the function, $f^2(x,y)$, and you may assume the curve C is parametrized, $x = x(t), y = y(t), a \le t \le b$.