**Math 7820, Fall 2022**

**Homework 5, due on Thursday, October 27, electronically on Collab.**

**1.** In class we discussed the induced orientation of the tangent space $T_x(\partial M)$ of the boundary $\partial M$ of an oriented manifold $M$, at each $x \in \partial M$. Prove that this is in fact an orientation of $\partial M$, i.e. that it depends smoothly on the point $x \in \partial M$.

**2.** Show that the tangent bundle $TM$ of any (orientable or not) manifold $M$ is orientable.

[It's fine to assume $M$ is a submanifold of some Euclidean space $\mathbb{R}^k$, like in a recent homework problem which stated that $TM$ is then a submanifold of $\mathbb{R}^k \times \mathbb{R}^k$. It may be useful to consider charts $(U \times \mathbb{R}^k, \Phi)$ on $\mathbb{R}^k \times \mathbb{R}^k$ adapted to $TM$, where $(U, \phi)$ are charts on $\mathbb{R}^k$ adapted to $M$, and $\Phi(p, v) = (\phi(p), \mathrm{d}_p\phi(v))$.]

**3.** Given disjoint manifolds $M^m$, $N^n$ in $\mathbb{R}^{k+1}$, the linking map $\lambda\colon M \times N \longrightarrow S^k$ is defined by

$$\lambda(x, y) \;=\; \frac{x - y}{|x - y|}.$$

If $M, N$ are compact, oriented, and without boundary, and $m + n = k$, then the integer-valued degree of $\lambda$ is called the *linking number* $l(M, N)$. Prove that

$$l(N, M) \;=\; (-1)^{(m+1)(n+1)} l(M, N).$$

If $M$ bounds an oriented compact manifold $W$ disjoint from $N$, prove that $l(M, N) = 0$.

[Lemma 1 in §5 in Milnor's book maybe helpful for one of the questions in this problem. (The proof of this lemma is identical to the proof of the invariance of degree under homotopy, discussed in class.)]

**4.** Given any integer $n$, construct a smooth map $f\colon S^1 \times S^1 \longrightarrow S^1 \times S^1$ of degree $n$. (Here "degree" means the integer-valued degree, not its mod $2$ reduction.)

**5.** (1 point, extra credit) Consider the torus $S^1 \times S^1$ as the quotient of the unit square in the plane with the opposite sides identified. (Equivalently, it is the quotient of $\mathbb{R}^2$ under the equivalence relation $x \sim x + m$, $y \sim y + n$ for any integers $m, n$.) Then any $2 \times 2$ matrix $A$ with integer entries (viewed as a linear map $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ taking integer points to integer points) induces a smooth map from the torus to itself. What is the degree of the induced map on the torus in terms of the matrix $A$?