MATH 8620 Homework 1

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Problem 1

(a): Let A be the ring $C[0,1]$. Show that for any proper ideal $\mathfrak{a} \subsetneq A$ there exists $p_0 \in I$ such that $f(p_0) = 0$ for all $f \in \mathfrak{a}$.

Proof. Let $V(\mathfrak{a}) = \{p \in I : f(p) = 0 \text{ for all } f \in \mathfrak{a}\}$. We proceed by contraposition. Suppose that $V(\mathfrak{a}) = \emptyset$. Then for each $x \in I$, there exists an $f_x \in \mathfrak{a}$ such that $f_x(x) \neq 0$. By continuity, for each $x \in I$ there is some open neighborhood $U_x \subseteq I$ of x such that $f_x \not\equiv 0$ on U_x. By compactness, there exist $x_1, \ldots, x_n \in I$ such that $\{U_{x_j}\}_{j=1}^n$ covers I.

Then $f = \sum_{j=1}^n f_{x_j}^2$ does not vanish on I, so $f \in A^\times$. As $f \in \mathfrak{a}$, it follows that $\mathfrak{a} = A$ is not proper. By contraposition, the result follows. $\square$

(b):Let $B = C(-\infty, \infty)$. Show that the set $\mathfrak{b} \subseteq B$ of functions with compact support is a proper ideal of B but there is no point $p \in (-\infty, \infty)$ such that $f(p) = 0$ for all $f \in \mathfrak{b}$.

Proof. Note that the constant function $1 \in B$ has support $(-\infty, \infty)$ which is not compact, so $\mathfrak{b} \subsetneq B$. Suppose that $f \in \mathfrak{b}$ and $\varphi \in B$. Then $\operatorname{supp}(\varphi f) \subseteq \operatorname{supp}(f)$, whence as a closed subset of a compact set, $\operatorname{supp}(\varphi f)$ is itself compact. Also for $f, g \in \mathfrak{b}$, $\operatorname{supp}(f + g) \subseteq \operatorname{supp}(f) \cup \operatorname{supp}(g)$, whence it is similarly compact. So $\mathfrak{b}$ is a proper ideal of B.

Suppose, for the sake of contradiction, that there exists a $p \in I$ such that $f(p) = 0$ for all $f \in \mathfrak{b}$. For $f \in B$, as $\mathfrak{b}$ is dense in B with respect to the sup-norm, there exists a sequence $\{f_n\}_{n=1}^\infty$ in $\mathfrak{b}$ such that $||f - f_n||_{\sup} \xrightarrow{n\to\infty} 0$. Then $f_n \to f$ pointwise, so $f(p) = \lim_{n\to\infty} f_n(p) = 0$ by assumption. So every function $f \in B$ vanishes at p, which is absurd and contradicts Uryshon's lemma. $\square$

Problem 2

Let $A = C[0,1]$ and $I = [0,1]$.

(a): Show that for any $p \in I$, the set $\mathfrak{m}_p = \{f \in A : f(p) = 0\}$ is a maximal ideal of A.

Proof. Consider the evaluation map $\varepsilon_p : A \to \mathbb{R}$ given my $\varepsilon_p(f) = f(p)$. By definition, $\ker(\varepsilon_p) = \mathfrak{m}_p$. Clearly ε_p is surjective as A contains the constant functions. Moreover, ε_p is a ring homomorphism, so by the first isomorphism theorem $A/\mathfrak{m}_p = A/\ker(\varepsilon_p) \cong \mathbb{R}$. Since $\mathbb{R}$ is a field, it follows that $\mathfrak{m}_p$ is a maximal ideal. $\square$

(b): Show that the correspondence $p \mapsto \mathfrak{m}_p$ defines a bijection $\theta : I \to \operatorname{Specm}(A)$.

Proof. On one hand, suppose $\mathfrak{m} \in \mathrm{Specm}(A)$. Then by Problem 1(a) there exists a $p \in I$ such that $f(p) = 0$ for all $f \in \mathfrak{m}$. Thus $\mathfrak{m}_p \subseteq \mathfrak{m}$, whence by maximality $\mathfrak{m} = \mathfrak{m}_p = \theta(p)$, so θ is surjective.

On the other hand. Suppose that $p_1 \neq p_2$. Choose open $U \subseteq I$ such that $p_1 \in U$ and $p_2 \in I \setminus U$. By complete regularity of I, there exists an $f \in A$ such that $f(p_1) = 0$ and $f \equiv 1$ on $I \setminus U$, whence $f(p_2) = 1$ so $f \in \mathfrak{m}_{p_1} \setminus \mathfrak{m}_{p_2}$ whence $\theta(p_1) = \mathfrak{m}_{p_1} \neq \mathfrak{m}_{p_2} = \theta(p_2)$. □

(c): Show that if I is given the natural topology and $\mathrm{Specm}(A)$ the topology induced from $\mathrm{Spec}(A)$ then θ becomes a homeomorphism.

Proof. On one hand, suppose that $X \subseteq \mathrm{Specm}(A)$ is closed. Then by definition there is some ideal $\mathfrak{a} \subseteq A$ such that $X = \mathrm{Specm}(A) \cap V(\mathfrak{a}) = \{\mathfrak{m} \in \mathrm{Specm}(A) : \mathfrak{m} \supseteq \mathfrak{a}\}$. Then,

$$\begin{aligned} \theta^{-1}(X) &= \{p \in I : \mathfrak{m}_p \in X\} = \{p \in I : \mathfrak{m}_p \supseteq \mathfrak{a}\} \\ &= \{p \in I : f(p) = 0 \text{ for all } f \in \mathfrak{a}\} = \bigcap_{f \in \mathfrak{a}} f^{-1}(\{0\}) \end{aligned}$$

which is closed in I by continuity of each $f \in \mathfrak{a}$.

On the other hand, suppose that $Y \subseteq I$ is closed and set $\mathfrak{a} = \{f \in A : f(p) = 0 \text{ for every } p \in Y\}$. We claim that $\theta(Y) = \mathrm{Specm}(A) \cap V(\mathfrak{a})$. On one hand, $\mathfrak{a} = \bigcap_{p \in Y} \mathfrak{m}_p$ so $p \in Y$ implies that $\mathfrak{m}_p \supseteq \mathfrak{a}$ whence $\mathfrak{m}_p \in \mathrm{Specm}(A) \cap V(\mathfrak{a})$. On the other hand, suppose $p \in I \setminus Y$, so $\mathfrak{m}_p \notin \theta(Y)$. By Uryshon's lemma (actually just complete regularity), there exists an $f \in A$ such that $f|_Y = 0$ and $f(p) = 1$. Then $f \in \mathfrak{a}$ and $f \notin \mathfrak{m}_p$, whence $\mathfrak{a} \not\subseteq \mathfrak{m}_p$ so $\mathfrak{m}_p \notin \mathrm{Specm}(A) \cap V(\mathfrak{a})$. □

Problem 3

Let again $A = C[0, 1]$.

(a): Let $\mathfrak{p} \in \mathrm{Spec}(A)$. It follows from Problem 2(b) that there exists $p \in I$ such that $\mathfrak{p} \subseteq \mathfrak{m}_p$. Show that $\mathfrak{p}$ contains

$$\mathfrak{l}_p := \{f \in A : f = 0 \text{ on some neighborhood of } p\}.$$

Proof. Let $f \in \mathfrak{l}_p$ and $U \subseteq I$ an open neighborhood of p such that $f|_U = 0$. By Uryshon's lemma, there exists a $g \in A$ such that $g|_{I \setminus U} = 0$ and $g(p) = 1$. Then $fg = 0 \in \mathfrak{p}$, whence by primality $f \in \mathfrak{p}$ as $g \notin \mathfrak{m}_p \supseteq \mathfrak{p}$. □

(b): Show that for $p_1, p_2 \in I, p_1 \neq p_2$, we have $\mathfrak{l}_{p_1} + \mathfrak{l}_{p_2} = A$. Deduce that every $\mathfrak{p} \in \mathrm{Spec}(A)$ is contained in a *unique* $\mathfrak{m} \in \mathrm{Specm}(A)$. It follows that every closed irreducible subset of $\mathrm{Spec}(A)$ contains a unique closed point.

Proof. Suppose, for the sake of contradiction, that $\mathfrak{p} \subseteq \mathfrak{m}_{p_1}, \mathfrak{m}_{p_2}$ and $p_1 \neq p_2$. Then by part (a), $\mathfrak{l}_{\mathfrak{p}_1}, \mathfrak{l}_{\mathfrak{p}_2} \subseteq \mathfrak{p}$, whence $A = \mathfrak{l}_{\mathfrak{p}_1} + \mathfrak{l}_{\mathfrak{p}_2} \subseteq \mathfrak{p}$ contradicting that prime ideals are proper.

Suppose that $X \subseteq \mathrm{Spec}(A)$ is irreducible and suppose $\mathfrak{p}_1, \mathfrak{m}_2 \in X$ are closed points. □

Question. Is A Noetherian?

Problem 4

(a): For $A = C[0,1]$, show that every $\mathfrak{m} \in \text{Specm}(A)$ properly contains some $\mathfrak{p}$ and in particular, $\text{Spec}(A) \neq \text{Specm}(A)$.

Proof. Let $p \in I$ such that $\mathfrak{m}_p = \mathfrak{m}$. Choose some $f \in \mathfrak{m}$ so that $f(q) \neq 0$ for $q \neq p$. Consider the sets $S_1 = A \setminus \mathfrak{m}$ and $S_2 = \{f^k : k \geq 0\}$. Note that S_1 is multiplicative as $\mathfrak{m}$ is maximal hence prime and S_2 is clearly multiplicative. Thus $S = S_1 S_2$ is multiplicative. Consider the poset

$$\mathscr{S} = \{\text{ideals } \mathfrak{a} \subseteq A : \mathfrak{a} \cap S = \emptyset\}$$

ordered by inclusion. Suppose that $(\mathfrak{a}_i)_{i \in J}$ is a chain in $\mathscr{S}$. By total ordering, $\mathfrak{a} := \bigcup_{i \in J} \mathfrak{a}_\mathfrak{i}$ is an ideal and $\mathfrak{a} \cap S = \bigcup_{i \in J} \mathfrak{a}_\mathfrak{i} \cap S = \emptyset$, so $\mathfrak{a} \in \mathscr{S}$. Thus $\mathfrak{a}$ is an upper bound in $\mathscr{S}$ for the chain.

Now by Zorn's lemma, there exists an ideal $\mathfrak{p} \subseteq A$ maximal with respect to the property that $\mathfrak{p} \cap S =$. We claim that $\mathfrak{p}$ is prime. Suppose, for the sake of contradiction, that $\mathfrak{p}$ is not prime. Then there exist $a, b \in A \setminus \mathfrak{p}$ such that $ab \in \mathfrak{p}$. Let $I_a = (a) + \mathfrak{p}$, $I_b = (b) + \mathfrak{p}$. Then $I_a, I_b \supsetneq \mathfrak{p}$, so by maximality of $\mathfrak{p}$ in the poset $\mathscr{S}$ there exist $s_1 \in I_a \cap S$ and $s_2 \in I_b \cap S$. We compute that

$$s_1 s_2 \in I_a I_b = ((a) + \mathfrak{p})((b) + \mathfrak{p}) = (ab) + \mathfrak{p} = \mathfrak{p},$$

however as S is multiplicative $s_1 s_2 \in S$, contradicting that $\mathfrak{p} \cap S = \emptyset$. Thus $\mathfrak{p} \in \text{Spec}(A)$

Noting that $S_1 \subseteq S$, $\mathfrak{p} \cap S = \emptyset$ implies $\mathfrak{p} \subseteq A \setminus S \subseteq A \setminus S_1 = \mathfrak{m}$. Also, $S_2 \subseteq S$ implies that $\mathfrak{p} \cap S_2 = \emptyset$ whence $f \in \mathfrak{m} \setminus \mathfrak{p}$, so $\mathfrak{p} \subsetneq \mathfrak{m}$. □

(b): Show that $\text{Specm}(A)$ is dense in $\text{Spec}(A)$.

Proof. First let $X \subseteq \text{Spec}(A)$ be an arbitrary subset (we shall later set $X = \text{Spec}(A)$). Then we compute

$$\text{Spec}(A) \setminus \overline{X} = \text{Spec}(A) \setminus \bigcap_{\substack{X \subseteq V(\mathfrak{a}) \\ \mathfrak{a} \subseteq A}} V(\mathfrak{a}) = \bigcup_{\substack{X \subseteq V(\mathfrak{a}) \\ \mathfrak{a} \subseteq A}} \text{Spec}(A) \setminus V(\mathfrak{a})$$

Suppose, for the sake of contradiction, that there is some $\mathfrak{p} \in \text{Spec}(A) \setminus \overline{\text{Specm}(A)}$. Then by the above computation, there exists some ideal $\mathfrak{a} \subseteq A$ such that $\text{Specm}(A) \subseteq V(\mathfrak{a})$ and $\mathfrak{p} \in \text{Spec}(A) \setminus V(\mathfrak{a})$. However, then $\mathfrak{a} \subseteq \mathfrak{m}_p$ for all $p \in I$, whence $\mathfrak{a} = 0$. Thus $\mathfrak{p} \in \text{Spec}(A) \setminus V(\mathfrak{a}) = \text{Spec}(A) \setminus V(0) = \emptyset$, which is absurd. □

(c): It follows from Problem 2(c) that $\text{Specm}(A)$ is compact in the topology induced from $\text{Spec}(A)$. At the same time, it is dense in $\text{Spec}(A)$, which is strictly bigger. So, there appears to be a contradiction with the basic facts from topology. Resolve this contradiction.

Solution. As the space $\text{Spec}(A)$ with the Zariski topology is non-Hausdorff, compactness does not necessarily imply closedness. Hence, $\text{Specm}(A)$ is compact but not closed in $\text{Spec}(A)$, resolving this problem. □