Math 8750, Spring 2022

Homework 1, due on Thursday, September 1, in class.

Exercises 5.3, 5.5, 6.3, 6.4 in Tu's textbook:

1. Let S^2 be the unit sphere
$$x^2 + y^2 + z^2 = 1$$
in $\mathbb{R}^3$. Define in S^2 the six charts corresponding to the six hemispheres:
$$U_1 = \{(x, y, z) \in S^2 | x > 0,\ \phi_1(x, y, z) = (y, z)\},$$
and the analogous
$$U_2 = \{x < 0\},\ U_3 = \{y > 0\},\ U_4 = \{y < 0\},\ U_5 = \{z > 0\},\ U_6 = \{z < 0\}.$$
Describe the domain $\phi_4(U_1 \cap U_4)$ of $\phi_1 \circ \phi_4^{-1}$ and show that $\phi_1 \circ \phi_4^{-1}$ is smooth on its domain.

2. Let $\{(U_\alpha, \phi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ be atlases for smooth manifolds M and N of dimensions m and n, respectively. Show that the collection $\{(U_\alpha \times V_\beta, \phi_\alpha \times \psi_\beta)\}$ of charts is an atlas on $M \times N$. Therefore, $M \times N$ is a smooth manifold of dimension $m + n$.

3. Let V be a finite-dimensional vector space over $\mathbb{R}$, and $\mathrm{GL}(V)$ the group of all linear automorphisms of V. Relative to an ordered basis $e = (e_1, ..., e_n)$ for V, a linear automorphism $L \in \mathrm{GL}(V)$ is represented by a matrix $[a^i_j]$ defined by
$$L(e_j) = \sum_i a^i_j\, e_i.$$
The map
$$\phi_e \colon \mathrm{GL}(V) \longrightarrow \mathrm{GL}(n, \mathbb{R}),\quad L \mapsto [a^i_j],$$
is a bijection with an open subset of $\mathbb{R}^{n\times n}$ that makes $\mathrm{GL}(V)$ into a smooth manifold, which we denote temporarily by $\mathrm{GL}(V)_e$. If $\mathrm{GL}(V)_u$ is the manifold structure induced from another ordered basis $u = (u_1, ..., u_n)$ for V, show that $\mathrm{GL}(V)_e$ is the same as $\mathrm{GL}(V)_u$.

4. Find all points in $\mathbb{R}^3$ in a neighborhood of which the functions $x, x^2 + y^2 + z^2 - 1, z$ can serve as a local coordinate system.