

# MATH 7410 Homework 1

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## Problem 1

Let  $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear map. For any  $p \in \mathbb{R}^n$ , there is a canonical identification  $T_p(\mathbb{R}^n) \rightarrow \mathbb{R}^n$  given by

$$\sum a^i \frac{\partial}{\partial x^i} \Big|_p \mapsto a = (a^1, \dots, a^n)$$

Show that the differential  $L_{*,p} : T_p(\mathbb{R}^n) \rightarrow T_{L(p)}(\mathbb{R}^m)$  is the map  $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$  itself, with the identification of the tangent spaces as above.

## Problem 2

If  $M$  and  $N$  are manifolds, let  $\pi_1 : M \times N \rightarrow M$  and  $\pi_2 : M \times N \rightarrow N$  be the two projections. Prove that for  $(p, q) \in M \times N$ ,

$$(d(\pi_1)_{(p,q)}, d(\pi_2)_{(p,q)}) : T_{(p,q)}(M \times N) \rightarrow T_p M \times T_q N$$

is an isomorphism.

*Proof.* Let  $\iota_1 : M \hookrightarrow M \times N$  and  $\iota_2 : N \hookrightarrow M \times N$  be given by  $\iota_1(m) = (m, q)$  and  $\iota_2(n) = (p, n)$ , the natural inclusions. Then  $\pi_1 \circ \iota_1 = id_M$ ,  $\pi_2 \circ \iota_2 = id_N$ ,  $\pi_2 \circ \iota_1 = q$ ,  $\pi_1 \circ \iota_2 = p$ . Taking differentials with respect to  $p$  and  $q$  and applying the chain rule, we compute that

$$\begin{aligned} d(\pi_1)_{(p,q)} \circ d(\iota_1)_p &= id_{T_p M} & d(\pi_2)_{(p,q)} \circ d(\iota_2)_q &= id_{T_q N} \\ d(\pi_2)_{(p,q)} \circ d(\iota_1)_p &= 0 & d(\pi_1)_{(p,q)} \circ d(\iota_2)_q &= 0. \end{aligned}$$

Define a map  $F : T_p M \times T_q N \rightarrow T_{(p,q)}(M \times N)$  by  $F(X_p, Y_q) = d(\iota_1)_p(X_p) + d(\iota_2)_q(Y_q)$ . Then we compute

$$\begin{aligned} (d(\pi_1)_{(p,q)}, d(\pi_2)_{(p,q)}) \circ F &= (d(\pi_1)_{(p,q)} d(\iota_1)_p + d(\pi_1)_{(p,q)} d(\iota_2)_q, d(\pi_2)_{(p,q)} d(\iota_1)_p + d(\pi_2)_{(p,q)} d(\iota_2)_q) \\ &= (id_{T_p M}, id_{T_q N}) = id_{T_p M \times T_q N}. \end{aligned}$$

Thus the desired map is right invertible and hence surjective. As the map is linear and the dimensions of the domain and target space are finite and equal, this map is in fact an isomorphism. □

## Problem 3

Let  $G$  be a Lie group with multiplication map  $\mu : G \times G \rightarrow G$ , inverse map  $\iota : G \rightarrow G$ , and identity element  $e$ .

(a): Show that the differential at the identity of the multiplication map  $\mu$  is addition:

$$\begin{aligned}\mu_{*,(e,e)} : T_e G \times T_e G &\rightarrow T_e G, \\ \mu_{*,(e,e)}(X_e, Y_e) &= X_e + Y_e.\end{aligned}$$

(Hint: First, compute  $\mu_{*,(e,e)}(X_e, 0)$  and  $\mu_{*,(e,e)}(0, Y_e)$  using Proposition 8.18).

*Proof.* Let  $X_e, Y_e \in T_e G$  and choose curves  $\alpha_1, \alpha_2 : (-\epsilon, \epsilon) \rightarrow G$  such that  $\alpha_1(0) = \alpha_2(0) = e$  and  $\frac{d\alpha_1}{dt}|_0 = X_e$ ,  $\frac{d\alpha_2}{dt}|_0 = Y_e$ . Consider  $\beta_1, \beta_2$  given by  $\beta_1(t) = (\alpha_1(t), e)$  and  $\beta_2(t) = (e, \alpha_2(t))$ . Then  $\frac{d\beta_1}{dt}|_0 = (X_e, e)$ ,  $\frac{d\beta_2}{dt}|_0 = (e, Y_e)$ .

Now we compute, by Proposition 8.18, that

$$\mu_{*,(e,e)}(X_e, 0) = \left. \frac{d(\mu \circ \beta_1)}{dt} \right|_0 = \left. \frac{d\alpha_1}{dt} \right|_0 = X_e.$$

Likewise  $\mu_{*,(e,e)}(0, Y_e) = Y_e$ . So by linearity of the differential, the claim follows.  $\square$

(b): Show that the differential at the identity of  $\iota$  is the negative:

$$\begin{aligned}\iota_{*,e} : T_e G &\rightarrow T_e G \\ \iota_{*,e}(X_e) &= -X_e.\end{aligned}$$

(Hint: Take the differential of  $\mu(c(t), (t \circ c)(t)) = e$ .)

*Proof.* Let  $X_e \in T_e G$  and choose smooth  $\alpha : (-\epsilon, \epsilon) \rightarrow G$  such that  $\alpha(0) = e$  and  $\frac{d\alpha}{dt}|_0 = X_e$ . Consider the map  $(-\epsilon, \epsilon) \rightarrow G$  given by  $t \mapsto \mu(\alpha(t), (\iota \circ \alpha)(t)) = e$ . This is a constant smooth map and

$$0 = \left. \frac{d\mu(\alpha(t), (\iota \circ \alpha)(t))}{dt} \right|_0 = X_e + \left. \frac{d(\iota \circ \alpha)}{dt} \right|_0 = X_e + \iota_{*,e}(X_e),$$

whence the claim follows.  $\square$

## Problem 4

Show that  $T_p^1 M \subseteq T_p^2 M$ .