

MATH 8620 Homework 5

James Harbour

December 11, 2022

Problem 1

Let F be a presheaf on a topological space X , and let $f : X \rightarrow Y$ be a continuous map. The *direct image* f_*F is defined by the following data:

$$(f_*F)(V) = F(f^{-1}(V)) \text{ for open } V \subseteq Y, \text{ and } \rho(f_*F)_{V_2}^{V_1} = \rho(F)_{f^{-1}(V_2)}^{f^{-1}(V_1)} \text{ for open } V_2 \subseteq V_1.$$

Show that f_*F is a presheaf. Furthermore, show that if F is a sheaf then f_*F is also a sheaf.

Problem 2

Let X be a closed subspace of a topological space Y , and let $\iota : X \rightarrow Y$ be the identity embedding. Let $Sh(X)$ and $Sh(Y)$ be the categories of sheaves of abelian groups on X and Y respectively. Given $F \in Sh(X)$, identify the stalks of the sheaf ι_*F . Use this to show that the functor $\iota_* : Sh(X) \rightarrow Sh(Y)$ is exact.

Problem 3

Let $X = Y$ be the unit circle $\{z \in \mathbb{C} : |z| = 1\}$, and let $f : X \rightarrow Y$ be given by $f(z) = z^2$. Fix an abelian group S and let F be the constant sheaf of abelian groups on X with value group S (recall that for an open subset $U \subseteq X$, the group $F(U)$ consists of all locally constant functions $U \rightarrow S$). Show that for any $y \in Y$, the stalk $(f_*F)_y$ is isomorphic to $S \times S$. On the other hand, let G be the constant sheaf on Y with value group $S \times S$. While G has the same stalks as f_*F , show that $G \not\cong f_*F$. (*Hint.* Compute global sections.)

Problem 4

Let X be a topological space, and let \mathcal{B} be a basis of the topology. Given two sheaves F and G on X , assume that for every $U \in \mathcal{B}$ there is a morphism $f_U : F(U) \rightarrow G(U)$, and that for $U, V \in \mathcal{B}$ such that $V \subseteq U$, the diagram

$$\begin{array}{ccc} F(U) & \xrightarrow{f_U} & G(U) \\ \rho(F)_V^U \downarrow & & \downarrow \rho(G)_V^U \\ F(V) & \xrightarrow{f_V} & G(V) \end{array}$$

commutes. Show that there exists a unique morphism of sheaves $f : F \rightarrow G$ for which f_U for $U \in \mathcal{B}$ coincide with the given ones. Furthermore, show that if f_U is surjective (resp., injective) for every $U \in \mathcal{B}$, then f is surjective (resp., injective) as a *morphism of sheaves*.

Problem 5

Let (X, \mathcal{O}_X) and (Y, \mathcal{O}_Y) be ringed spaces. Suppose we are given an open covering $\{U_i\}_{i \in I}$ of X , and for each $i \in I$ a morphism $(f_i, f_i^\#) : (U_i, \mathcal{O}_X|_{U_i}) \rightarrow (Y, \mathcal{O}_Y)$. If $(f_i, f_i^\#)$ and $(f_j, f_j^\#)$ coincide on the overlap $U_i \cap U_j$ for all $i, j \in I$ then there exists a morphism $(f, f^\#)$ that restricts to $(f_i, f_i^\#)$ on $(U_i, \mathcal{O}_X|_{U_i})$.

Problem 6

Let A be a commutative ring, and let $\{M_i, \tau_i^j\}$ be a direct system of A -modules over a directed set I . For any A -module N , one can consider the direct system $\{M_i \otimes_A N, \tau_i^j \otimes id_N\}$. Show that there is a natural isomorphism of A -modules $(\lim_{\rightarrow} M_i) \otimes_A N \cong \lim_{\rightarrow} (M_i \otimes_A N)$.

Problem 7

A commutative ring A is called *reduced* if its nilradical is zero. Show that A is reduced if and only if all localizations $A_{\mathfrak{p}}$ for all $\mathfrak{p} \in \text{Spec}(A)$ are reduced. This can be generalized to schemes: a scheme X is defined to be reduced if the ring $\mathcal{O}_X(U)$ is reduced for all open $U \subseteq X$. Show that a scheme X is reduced if and only if all stalks $\mathcal{O}_{X,x}$ are reduced.

Problem 8

As in class, consider $A = K[x, y]$ and $X = \text{Spec}(A)$. Let M denote the A -module A/I where $I = xA$. Calculate the stalks $\widetilde{M}_{\mathfrak{p}}$ of the corresponding sheaf \widetilde{M} for all $\mathfrak{p} \in X$. Is \widetilde{M} a skyscraper sheaf?