MATH 7820 Homework 6

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Problem 1

Denote the standard coordinates on $\mathbb{R}^2$ by x, y, and let

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \quad \text{and} \quad Y = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

be vector fields on $\mathbb{R}^2$. Find a 1-form ω on $\mathbb{R}^2 \setminus \{(0,0)\}$ such that $\omega(X) = 1$ and $\omega(Y) = 0$.

Solution. Let dx and dy be dual to $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ respectively. Let $f, g : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ be given by

$$f(x,y) = \frac{-y}{x^2+y^2} \quad \text{and} \quad g(x,y) = \frac{x}{x^2+y^2}.$$

Now define a 1-form ω by $\omega = f(x,y)\,dx + g(x,y)\,dy$. Then we compute that, for $p = (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$,

$$\omega_p(X_p) = (f(x,y)\,dx + g(x,y)\,dy)\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) = \frac{x^2}{x^2+y^2} + \frac{y_2}{x^2+y^2} = 1,$$

$$\omega_p(Y_p) = (f(x,y)\,dx + g(x,y)\,dy)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) = \frac{-xy}{x^2+y^2} + \frac{xy}{x^2+y^2} = 0.$$

□

Problem 2

Suppose $(U, x^1, \ldots, x^n)$ and $(V, y^1, \ldots, y^n)$ are two charts on M with $U \cap V \neq \emptyset$. Then a C^∞ 1-form ω on $U \cap V$ has two different local expressions:

$$\omega = \sum_j a_j\,dx^j = \sum_i b_i\,dy^i\,.$$

Find a formula for a_j in terms of b_i.

Proof. Fix $p \in U \cap V$. For $1 \leq k \leq n$, there exist $c_1^k, \ldots, c_n^k \in \mathbb{R}$ such that

$$\left.\frac{\partial}{\partial x^k}\right|_p = \sum_{l=1}^n c_l^k \left.\frac{\partial}{\partial y^l}\right|_p.$$

Now applying both sides of this expression to the coordinate function y^m, we see for $1 \leq m \leq n$ that

$$c_m^k = \sum_{l=1}^n c_l^k \left.\frac{\partial y^m}{\partial y^l}\right|_p = \left.\frac{\partial y^m}{\partial x^k}\right|_p.$$

Now, we compute

$$a_k(p) = \omega_p\left(\frac{\partial}{\partial x^k}|_p\right) = \sum_i b_i(p)(dy^i)_p\left(\frac{\partial}{\partial x^k}\bigg|_p\right)$$
$$= \sum_i b_i(p)(dy^i)_p\left(\sum_l \frac{\partial y^l}{\partial x^k}\bigg|_p \frac{\partial}{\partial y^l}\bigg|_p\right) = \sum_i b_i(p)\frac{\partial y^i}{\partial x^k}\bigg|_p.$$

□

Problem 3

Prove that a vector bundle whose fiber is an n-dimensional vector space is trivial (i.e. is isomorphic to a product bundle) if and only if it admits n sections $s_1, \ldots, s_n$ such that $s_1(x), \ldots, s_n(x)$ are linearly independent for each point x in the base.

Proof.
$\underline{\implies}$: Suppose we are given homeomorphisms f, g as below such that f is a linear isomorphism on fibers and the following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\pi"] & B \times \mathbb{R}^n \arrow[d, "p"] \\ B \arrow[r, "g"] & B \end{tikzcd}$$

Let $e_1, \ldots, e_n \in \mathbb{R}^n$ be the standard basis for $\mathbb{R}^n$, and for $1 \leq i \leq n$ define $s_i : B \to E$ by $s_i(x) = f^{-1}(g(x), e_i)$. Now suppose that $x \in B$ and $\lambda_i \in \mathbb{R}$ are such that in $\pi^{-1}(\{x\})$ we have $0 = \sum_i \lambda_i s_i(x)$. Then we use linearity of f^{-1} on fibers to compute that

$$0 = \sum_i \lambda_i s_i(x) = \sum_i \lambda_i f^{-1}(g(x), e_i) = f^{-1}\left(g(x), \sum_i \lambda_i e_i\right),$$

whence by linearity $\sum_i \lambda_i e_i = 0$ in $p^{-1}(g(x))$, so by independence $\lambda_i = 0$ for all i. Hence $s_1(x), \ldots, s_n(x)$ are linearly independent.

$\underline{\impliedby}$: For $x \in B$, let $\beta(x)$ denote the basis $\beta(x) = \{s_i(x) : 1 \leq i \leq n\}$. Define a map $\phi : E \to B \times \mathbb{R}^n$ as follows: for $p \in E$ there exists a unique $x \in B$ such that $p \in \pi^{-1}(\{x\})$, so set $\phi(p) = (x, [p]_{\beta(x)})$. Note that, on each fiber, the map ϕ is given by $p \mapsto [p]_{\beta(x)}$ and is thus a linear isomorphism by assumption. Moreover, the map ϕ satisfies the following commutative diagram:

$$\begin{tikzcd} E \arrow[r, "\phi"] \arrow[d, "\pi"] & B \times \mathbb{R}^n \arrow[d, "p"] \\ B \arrow[r, equal, "id_B"] & B \end{tikzcd}$$

Now define a map $\psi : B \times \mathbb{R}^n \to E$ by $\psi(x, v_1, \ldots, v_n) = \sum_i v_i s_i(x)$. Note that this map is again clearly a linear isomorphism on fibers by assumption. Moreover, it is clear that ψ is the inverse of ϕ, thus giving an isomorphism of vector bundles.

□

Problem 4

Suppose $E_1 \to B$, $E_2 \to B$ are two vector bundles over the same base. It may be assumed without loss of generality that they are both trivialized over the same collection of charts $\{U^i\}$ covering B. Denote their transition maps $\phi_{ij} : U_i \cap U_j \to \mathrm{GL}_m(\mathbb{R})$, $\psi_{ij} : U_i \cap U_j \to \mathrm{GL}_n(\mathbb{R})$. The tensor product of these two bundles is a vector bundle over the same base B, defined by taking the tensor product of the transition maps at each point, with values in $\mathrm{GL}_{mn}(\mathbb{R})$.

Carry out this construction in the case of two copies of the non-trivial line bundle over the circle discussed in class, and identify their tensor product.

Proof. Under the construction in class, we have two sets U, V covering S^1 with intersection two intervals. Moreover, the transition map $\phi : U \cap V \to \mathrm{GL}_1(\mathbb{R})$ is given sending one interval to $+1$ and the other interval to -1. This line bundle then becomes the Mobius band. Taking the tensor product of this map with itself and then applying the standard isomorphism $\mathbb{R} \otimes \mathbb{R} \to \mathbb{R}$ given by multiplication, it follows that the new transition map is $\phi \otimes \phi = \phi \cdot \phi = 1$, whence the new line bundle simply becomes the trivial bundle. $\square$