Problem 1.

Conway V.1.7

Problem 2.

Conway V.1.8

Problem 3.

Conway VI.1.11

Problem 4.

Let X, Y be a Banach spaces over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. For $C \subseteq B(X, Y)$ convex and $F \subseteq X$ finite, set $C_F = \{(Tx)_{x \in F} : T \in C\} \subseteq Y^{\oplus F}$. Equip $Y^{\oplus F}$ with the norm

$$\|(y_x)_{x \in F}\| = \sum_{x \in F} \|y_x\|.$$

(a) Let $C \subseteq B(X, Y)$ be convex. Show that $T \in \overline{C}^{SOT}$ if and only if for every $F \subseteq X$ finite, we have that $(Tx)_{x \in F} \in \overline{C_F}^{\|\cdot\|}$. Show that $T \in \overline{C}^{WOT}$ if and only if for every $F \subseteq X$ finite we have that $(Tx)_{x \in F} \in \overline{C_F}^{weak}$.

(b) Suppose that $C \subseteq B(X, Y)$ is convex. Show that $\overline{C}^{WOT} = \overline{C}^{SOT}$.

(c) If $\phi \colon B(X, Y) \to \mathbb{F}$ is linear, show that ϕ is WOT-continuous if and only if it is SOT-continuous.

Remark: One can show that ϕ as in the third part is WOT-continuous if and only if there exists $x_1, \cdots, x_n \in X, \psi_1, \cdots, \psi_n \in Y^*$ so that

$$\phi(T) = \sum_{j=1}^{n} \psi_j(Tx_j).$$

This follows as in our characterizations of $(X, wk)^*$, $(X^*, wk^*)^*$.

Problem 5.

Let I be a set and $\mathcal{M}$ be the set of $m \in \ell^\infty(I)^*$ such that:

- $m(f) \geq 0$ for all $f \geq 0$,
- $m(1) = 1$

Identify $\mathrm{Prob}(I)$ with $\{f \in \ell^1(I) : f \geq 0, \|f\|_1 = 1\}$ and view $\ell^1(I) \subseteq \ell^\infty(I)^*$ by $f \mapsto \phi_f$ where $\phi_f(g) = \sum_{i \in I} f(i)g(i)$. Show that $\mathrm{Prob}(I)$ is weak*-dense in $\mathcal{M}$.

Remark: it can be shown the map $m \mapsto (E \mapsto m(1_E))$ is a bijection between $\mathcal{M}$ and all finitely-additive probabiltiy measure $\mathcal{P}(I) \to [0, 1]$. Finitely-additive probability measures which are not coutnably additive are somewhat pathological objects which exist only via the axiom of choice. This is an explicit case where we can approximate pathological objects by classical ones.

Problems to think about. do not turn in

Problem 6.

Conway V.1.9

Problem 7.

Conway V.1.10

Problem 8.

Let X be a compact, Hausdorff space and let μ be a Borel probability measure on X.

(a) Show that $C(X)$ is weak*-dense in $L^\infty(X, \mu)$.

Hint: it might be helpful to use that the Riesz reperesentation theorem implies that if a Borel measure integrates to 0 on every continuous function, then that measure is 0.

(b) Show that $\{f \in C(X) : 0 \leq f \leq 1\}$ is weak*-dense in $\{f \in L^\infty(X, \mu) : 0 \leq f \leq 1 \text{ almost everywhere}\}$.