MATH 7410 Homework 2

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Problem 1

(a): Let X be a separable Banach space. Show that $Ball(X^*) = \{\phi \in X^* : \|\phi\| \leq 1\}$ is wk^*-metrizable.

Proof. Note that it suffices to show that a countable subset of the seminorms defining the LCS topology on X^* in fact define the topology on $Ball(X^*)$.

Choose a norm dense sequence $(x_n)_{n=1}^\infty$ in X. We claim that the seminorms $\rho_{x_n} = |ev_{x_n}(\cdot)| : X^* \to [0, +\infty)$ define the restriction of the wk^*-topology to $Ball(X^*)$.

Suppose that $(\phi_\alpha)_{\alpha \in I}$ is a net in $Ball(X^*)$and $\phi \in Ball(X^*)$ is such that $\phi_\alpha(x_n) \to \phi(x)$ for all $n \in \mathbb{N}$. Now let $x \in X$ and fix $\varepsilon > 0$. Then by density there is some $n \in \mathbb{N}$ such that $\|x - x_n\| < \varepsilon/3$. Moreover, by assumption, there is some $\alpha_0 \in I$ such that for all $\alpha \geq \alpha_0$, $|\phi_\alpha(x_n) - \phi(x_n)| < \varepsilon/3$. Now, for all $\alpha \geq \alpha_0$,

$$
\begin{aligned}
|\phi(x) - \phi_\alpha(x)| \leq |\phi(x - x_n)| + |\phi(x_n) - \phi_\alpha(x)| &\leq \|x - x_n\| + |\phi_\alpha(x_n) - \phi(x_n)| + |\phi_\alpha(x_n) - \phi_\alpha(x)| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \|\phi_\alpha\|\|x_n - x\| < \varepsilon
\end{aligned}
$$

so in fact, for nets in $Ball(X^*)$, pointwise convergence on $(x_n)_n^\infty$ implies pointwise convergence everywhere, and thus wk^*-convergence of the underlying nets. ☐

(b): If X is a Banach space, show that there is a compact space K such that X is isometrically isomorphic to a closed subspace of $C(K)$.

Problem 2

Let $Bil(X \times Y, Z)$ be the space of bounded, bilinear maps from $X \times Y \to Z$.

(a): Suppose that B_x, B^y are bounded for each $x \in X$,$y \in Y$. Prove that there is a constant $M > 0$ so that

$$\|B(x, y)\| \leq M\|x\|\|y\|$$

(use the Principle of Uniform Boundedness).

Proof. For $x \in X$, $y \in Ball(Y)$, note that $\|B_y(x)\| \leq \|B_x\|$. Then by the principle of uniform boundedness, $C := \sup_{y \in Ball(Y)} \|B_y\| < +\infty$. Now observe that

$$\sup_{x \in Ball(X), y \in Ball(Y)} \|B(x, y)\| = \sup_{x \in Ball(X), y \in Ball(Y)} \|B_y(x)\| \leq \sup_{y \in Ball(Y)} \|B_y\| = C,$$

whence the claim follows by scaling. ☐

(b): Show that the map $\Phi : Bil(X \times Y, \mathbb{F}) \to B(X, Y^*)$ given by $[\widetilde{\Phi}(B)(x)](y) = B(x, y)$ is a well-defined, isometric isomorphism.

(c): By switching names, it follows that the map $\widetilde{\Phi} : Bil(X \times Y, \mathbb{F}) \to B(Y, X^*)$ given by $[\widetilde{\Phi}(B)(y)](x) = B(x, y)$ is a well-defined, isometric isomorphism. So the map $\widetilde{\Phi} \circ \Phi^{-1}$ is an isometric isomorphism $B(X, Y^*) \cong B(Y, X^*)$. What is this isomorphism?

Problem 3

Let X, Y be Banach spaces. And let $(T_n)_{n=1}^{\infty}$ be a sequence in $B(X, Y)$.

(a): If T_n converges in the WOT to $T \in B(X, Y)$ show that $\sup_n \|T_n\| < +\infty$. (In particular, if T_n converges strongly, then it is norm).

Proof. Fix $x \in X$. Then for all $\phi \in Y^*$, $\phi(T_n x) \to \phi(Tx)$ so $\sup_{n \in \mathbb{N}} |\phi(T_n x)| < +\infty$. □

(b): If $\sup_n \|T_n\| < +\infty$ and there is a norm dense $D \subseteq X$ so that $T_n x$ converges for every $x \in D$, show that $T_n x$ converges for all $x \in X$, that $Tx = \lim_{n \to \infty} T_n x$ is a bounded operator, and that $\|Tx - T_n x\| \xrightarrow{n \to \infty} 0$ for every $x \in X$.

Problem 4

Let X, Y be Banach spaces. And let $(T_n)_{n=1}^{\infty}$ be a sequence in $B(X, Y)$. Suppose that $\sup_n \|T_n\| < +\infty$ and that $D \subseteq X$, $G \subseteq Y^*$ are norm dense. Assume that $\lim_n \phi(T_n x)$ exists for all $\phi \in G$,$x \in D$.

Proof. Fix $x \in X, \phi \in Y^*$, and $\varepsilon > 0$. Choose $y \in D, \psi \in G$ such that $\|x - y\|, \|\phi - \psi\| < \varepsilon$. Then, for $n, m \in \mathbb{N}$, we compute that □

(a): Show that $\lim_n \phi(T_n x)$ exists for all $\phi \in Y^*$, $x \in X$.

(b): Show that for every $x \in X$, there is a well-defined bounded operator $S : X \to Y^{**}$ given by $S(x)(\phi) = \lim_{n \to \infty} \phi(T_n x)$.

(c): If $T_n x$ converges weakly to an element of Y for every $x \in D$, show that $S(X) \subseteq Y$, and that $T_n \to S$ WOT.

Problem 5

Let G be a countable, discrete, group and $\lambda : G \to B(l^2(G))$ be given by $(\lambda(g)\xi)(h) = \xi(g^{-1}h)$.

(a): Let $(g_n)_{n=1}^{\infty}$ be a sequence in G so that for every finite $F \subseteq G$ we have $\{n : g_n \in F\}$ is finite. Show that $\lim_{n \to \infty} \lambda(g_n) = 0$ in WOT. (Hint: consider first acting on pairs of vectors which are finitely supported and applying the preceding problem to reduce to this case).

(b): Suppose G is infinite. If $\mathcal{K} \subseteq l^2(G)$ is closed and $\lambda(g)\mathcal{K} = \mathcal{K}$ for every $g \in G$, and $\mathcal{K} \neq 0$, show that $\mathcal{K}$ is finite-dimensional. (Hint: construct a sequence satisfies the hypothesis of the preceding problem. If $\mathcal{K}$ is finite-dimensional, then λ applied to the sequence restricted to $\mathcal{K}$ converges to 0 in WOT, and hence in any other LCS topology on $B(\mathcal{K})$. Consider using this for one of the other operator topologies to get a contradiction).