MATH 7820 Homework 1

James Harbour

August 25, 2022

Problem 2

Let $\{(U_\alpha, \phi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ be atlases for smooth manifolds M and N of dimensions m and n respectively. Show that the collection $\{(U_\alpha \times V_\beta, \phi_\alpha \times \psi_\beta)\}$ of charts is an atlas on $M \times N$. Therefore, $M \times N$ is a smooth manifold of dimension $m + n$.

Proof. For $p \in M$ and $q \in N$, there exist α, β such that $p \in U_\alpha$ and $q \in V_\beta$ whence $(p, q) \in U_\alpha \times V_\beta$. Thus $M \times N = \bigcup_{\alpha,\beta} U_\alpha \times V_\beta$.

For each α, β, as ϕ_α and ψ_β are homeomorphisms onto $\phi_\alpha(U_\alpha)$ and $\psi_\beta(V_\beta)$ respectively, it follows that $\phi_\alpha \times \psi_\beta$ is a homeomorphism onto $\phi_\alpha \times \psi_\beta(U_\alpha \times V_\beta) = \phi_\alpha(U_\alpha) \times \psi(V_\beta)$.

Fix α, β. Then we compute the transition maps

$$(\phi_{\alpha'} \times \psi_{\beta'}) \circ (\phi_\alpha \times \psi_\beta)^{-1} = (\phi_{\alpha'} \times \psi_{\beta'}) \circ (\phi_\alpha^{-1} \times \psi_\beta^{-1}) = (\phi_{\alpha'} \circ \phi_\alpha^{-1}) \times (\psi_{\beta'} \circ \psi_\beta^{-1})$$
$$(\phi_\alpha \times \psi_\beta) \circ (\phi_{\alpha'} \times \psi_{\beta'})^{-1} = (\phi_\alpha \times \psi_\beta) \circ (\phi_{\alpha'}^{-1} \times \psi_{\beta'}^{-1}) = (\phi_\alpha \circ \phi_{\alpha'}^{-1}) \times (\psi_\beta \circ \psi_{\beta'}^{-1})$$

which are both smooth as $\phi_\alpha \circ \phi_{\alpha'}^{-1}$, $\phi_{\alpha'} \circ \phi_\alpha^{-1}$, $\psi_\beta \circ \psi_{\beta'}^{-1}$, $\psi_{\beta'} \circ \psi_\beta^{-1}$ are smooth. Thus $\{(U_\alpha \times V_\beta, \phi_\alpha \times \psi_\beta)\}$ is an atlas on $M \times N$. $\square$