

MATH 8620 Homework 1

James Harbour

August 24, 2022

Problem 1

(a): Let A be the ring $C[0, 1]$. Show that for any proper ideal $\mathfrak{a} \subsetneq A$ there exists $p_0 \in I$ such that $f(p_0) = 0$ for all $f \in \mathfrak{a}$.

Proof. Let $V(\mathfrak{a}) = \{p \in I : f(p) = 0 \text{ for all } f \in \mathfrak{a}\}$. We proceed by contraposition. Suppose that $V(\mathfrak{a}) = \emptyset$. Then for each $x \in I$, there exists an $f_x \in \mathfrak{a}$ such that $f_x(x) \neq 0$. By continuity, for each $x \in I$ there is some open neighborhood $U_x \subseteq I$ of x such that $f_x \neq 0$ on U_x . By compactness, there exist $x_1, \dots, x_n \in I$ such that $\{U_{x_j}\}_{j=1}^n$ covers I .

Then $f = \sum_{j=1}^n f_{x_j}^2$ does not vanish on I , so $f \in A^\times$. As $f \in \mathfrak{a}$, it follows that $\mathfrak{a} = A$ is not proper. By contraposition, the result follows. \square

(b): Let $B = C(-\infty, \infty)$. Show that the set $\mathfrak{b} \subseteq B$ of functions with compact support is a proper ideal of B but there is no point $p \in (-\infty, \infty)$ such that $f(p) = 0$ for all $f \in \mathfrak{b}$.

Proof. Note that the constant function $1 \in B$ has support $(-\infty, \infty)$ which is not compact, so $\mathfrak{b} \subsetneq B$. Suppose that $f \in \mathfrak{b}$ and $\varphi \in B$. Then $\text{supp}(\varphi f) \subseteq \text{supp}(f)$, whence as a closed subset of a compact set, $\text{supp}(\varphi f)$ is itself compact. Also for $f, g \in \mathfrak{b}$, $\text{supp}(f + g) \subseteq \text{supp}(f) \cup \text{supp}(g)$, whence it is similarly compact. So \mathfrak{b} is a proper ideal of B .

Suppose, for the sake of contradiction, that there exists a $p \in I$ such that $f(p) = 0$ for all $f \in \mathfrak{b}$. For $f \in B$, as \mathfrak{b} is dense in B with respect to the sup-norm, there exists a sequence $\{f_n\}_{n=1}^\infty$ in \mathfrak{b} such that $\|f - f_n\|_{\text{sup}} \xrightarrow{n \rightarrow \infty} 0$. Then $f_n \rightarrow f$ pointwise, so $f(p) = \lim_{n \rightarrow \infty} f_n(p) = 0$ by assumption. So every function $f \in B$ vanishes at p , which is absurd and contradicts Uryshon's lemma. \square

Problem 2

Let $A = C[0, 1]$ and $I = [0, 1]$.

(a): Show that for any $p \in I$, the set $\mathfrak{m}_p = \{f \in A : f(p) = 0\}$ is a maximal ideal of A .

Proof. Consider the evaluation map $\varepsilon_p : A \rightarrow \mathbb{R}$ given by $\varepsilon_p(f) = f(p)$. By definition, $\ker(\varepsilon_p) = \mathfrak{m}_p$. Clearly ε_p is surjective as A contains the constant functions. Moreover, ε_p is a ring homomorphism, so by the first isomorphism theorem $A/\mathfrak{m}_p = A/\ker(\varepsilon_p) \cong \mathbb{R}$. Since \mathbb{R} is a field, it follows that \mathfrak{m}_p is a maximal ideal. \square

(b): Show that the correspondence $p \mapsto \mathfrak{m}_p$ defines a bijection $\theta : I \rightarrow \text{Specm}(A)$.

Proof. On one hand, suppose $\mathfrak{m} \in \text{Specm}(A)$. Then by Problem 1(a) there exists a $p \in I$ such that $f(p) = 0$ for all $f \in \mathfrak{m}$. Thus $\mathfrak{m}_p \subseteq \mathfrak{m}$, whence by maximality $\mathfrak{m} = \mathfrak{m}_p = \theta(p)$, so θ is surjective.

On the other hand. Suppose that $p_1 \neq p_2$. Choose open $U \subseteq I$ such that $p_1 \in U$ and $p_2 \in I \setminus U$. By complete regularity of I , there exists an $f \in A$ such that $f(p_1) = 0$ and $f \equiv 1$ on $I \setminus U$, whence $f(p_2) = 1$ so $f \in \mathfrak{m}_{p_1} \setminus \mathfrak{m}_{p_2}$ whence $\theta(p_1) = \mathfrak{m}_{p_1} \neq \mathfrak{m}_{p_2} = \theta(p_2)$. \square

(c): Show that if I is given the natural topology and $\text{Specm}(A)$ the topology induced from $\text{Spec}(A)$ then θ becomes a homeomorphism.

Proof. On one hand, suppose that $X \subseteq \text{Specm}(A)$ is closed. Then by definition there is some ideal $\mathfrak{a} \subseteq A$ such that $X = \text{Specm}(A) \cap V(\mathfrak{a}) = \{\mathfrak{m} \in \text{Specm}(A) : \mathfrak{m} \supseteq \mathfrak{a}\}$. Then,

$$\begin{aligned} \theta^{-1}(X) &= \{p \in I : \mathfrak{m}_p \in X\} = \{p \in I : \mathfrak{m}_p \supseteq \mathfrak{a}\} \\ &= \{p \in I : f(p) = 0 \text{ for all } f \in \mathfrak{a}\} = \bigcap_{f \in \mathfrak{a}} f^{-1}(\{0\}) \end{aligned}$$

which is closed in I by continuity of each $f \in \mathfrak{a}$.

On the other hand, suppose that $Y \subseteq I$ is closed and set $\mathfrak{a} = \{f \in A : f(p) = 0 \text{ for every } p \in Y\}$. We claim that $\theta(Y) = \text{Specm}(A) \cap V(\mathfrak{a})$. On one hand, $\mathfrak{a} = \bigcap_{p \in Y} \mathfrak{m}_p$ so $p \in Y$ implies that $\mathfrak{m}_p \supseteq \mathfrak{a}$ whence $\mathfrak{m}_p \in \text{Specm}(A) \cap V(\mathfrak{a})$. On the other hand, suppose $p \in I \setminus Y$, so $\mathfrak{m}_p \notin \theta(Y)$. By Uryshon's lemma (actually just complete regularity), there exists an $f \in A$ such that $f|_Y = 0$ and $f(p) = 1$. Then $f \in \mathfrak{a}$ and $f \notin \mathfrak{m}_p$, whence $\mathfrak{a} \not\subseteq \mathfrak{m}_p$ so $\mathfrak{m}_p \notin \text{Specm}(A) \cap V(\mathfrak{a})$. \square

Problem 3

Let again $A = C[0, 1]$.

(a): Let $\mathfrak{p} \in \text{Spec}(A)$. It follows from Problem 2(b) that there exists $p \in I$ such that $\mathfrak{p} \subseteq \mathfrak{m}_p$. Show that \mathfrak{p} contains

$$\mathfrak{l}_p := \{f \in A : f = 0 \text{ on some neighborhood of } p\}.$$

Proof. Let $f \in \mathfrak{l}_p$ and $U \subseteq I$ an open neighborhood of p such that $f|_U = 0$. By Uryshon's lemma, there exists a $g \in A$ such that $g|_{I \setminus U} = 0$ and $g(p) = 1$. Then $fg = 0 \in \mathfrak{p}$, whence by primality $f \in \mathfrak{p}$ as $g \notin \mathfrak{m}_p \supseteq \mathfrak{p}$. \square

(b): Show that for $p_1, p_2 \in I, p_1 \neq p_2$, we have $\mathfrak{l}_{p_1} + \mathfrak{l}_{p_2} = A$. Deduce that every $\mathfrak{p} \in \text{Spec}(A)$ is contained in a *unique* $\mathfrak{m} \in \text{Specm}(A)$. It follows that every closed irreducible subset of $\text{Spec}(A)$ contains a unique closed point.

Question. Is A Noetherian?

Problem 4

(a): For $A = C[0, 1]$, show that every $\mathfrak{m} \in \text{Specm}(A)$ properly contains some \mathfrak{p} and in particular, $\text{Spec}(A) \neq \text{Specm}(A)$.

(b): Show that $\text{Specm}(A)$ is dense in $\text{Spec}(A)$.

(c): It follows from Problem 2(c) that $\text{Specm}(A)$ is compact in the topology induced from $\text{Spec}(A)$. At the same time, it is dense in $\text{Spec}(A)$, which is strictly bigger. So, there appears to be a contradiction with the basic facts from topology. Resolve this contradiction.