MATH 7820 Homework 4

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October 13, 2022

Problem 1

Show that the degree (mod 2) of a composition of two smooth maps f and g equals

$$\deg_2(f) \cdot \deg_2(g) \pmod 2.$$

Proof. Let $f : x \to Y$ and $g : Y \to Z$. We must assume X, Y are compact, Y, Z are connected, and $\dim(X) = \dim(Y) = \dim(Z) = n$ for all values above to be defined. Let $z \in Z$ be a regular value of $g \circ f$ and $\{x_1, \ldots, x_k\} = (g \circ f)^{-1}(z) = f^{-1}(g^{-1}(z))$. Fix $1 \leq i \leq k$. As $\dim T_{x_i}X = \dim T_{f(x_i)}Y = n$ and $d_{x_i}(g \circ f) = (d_{f(x_i)}g)(d_{x_i}f)$ is surjective, it follows that both $d_{f(x_i)}g$ and $d_{x_i}f$ are surjective. Thus, each $y_i = f(x_i)$ is a regular point of g as well as a regular value of f with corresponding regular point x_i.

By well definedness of degree mod 2, for each y_i, $|f^{-1}(y_i)| (\text{mod } 2) = \deg_2(f)$. Hence, we compute

$$\deg_2(g \circ f) \equiv |f^{-1}(g^{-1}(z))| = \sum_{i=1}^{k} |f^{-1}(y_i)| \equiv \sum_{i=1}^{k} \deg_2(f) \ (\text{mod } 2) = \deg_2(f) \cdot \deg_2(g).$$

$\square$

Problem 2

Give an example of manifolds M, N, and of a smooth function $F : M \to N$ with a regular value which is a limit point of critical values.

Example. Consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = e^{-x}\sin(x)$. Then $f'(x) = e^{-x}(\cos(x) - \sin(x))$.

$$0 = f'(x) = e^{-x}(\cos(x) - \sin(x)) \iff \cos(x) = \sin(x),$$

so the critical points of f are $x_k = \frac{\pi}{4} + \pi k$ for $k \in \mathbb{Z}$. These critical points correspond to the critical values

$$f(x_k) = e^{\frac{-\pi}{4} - \pi k}\left(\cos\left(\frac{-\pi}{4} - \pi k\right) + \sin\left(\frac{-\pi}{4} - \pi k\right)\right) \begin{cases} \sqrt{2}e^{-\frac{\pi}{4} - \pi k} & \text{when } k \text{ is even} \\ -\sqrt{2}e^{-\frac{\pi}{4} - \pi k} & \text{when } k \text{ is odd} \end{cases}.$$

Note $\pm\sqrt{2}e^{-\frac{\pi}{4} - \pi k} \xrightarrow{k \to \infty} 0$, thus 0 is a limit point of the critical values $\{f(x_k\}_{k \in \mathbb{Z}}$, but 0 is not a critical value of f as $x_k \neq 0$ for all $k \in \mathbb{Z}$, so 0 is a regular value of f which is a limit point of critical values of f. $\square$

Problem 3

If two smooth maps f, g from a manifold M to the unit sphere S^k satisfy

$$\|f(x) - g(x)\| < 2$$

for all $x \in M$, prove that f is smoothly homotopic g.

Proof. Fix $t \in [0,1]$ and $x \in M$. If $t = 0, 1$, then $\|(1-t)f(x) + tg(x)\| = 1 \neq 0$. If $t \in (0,1)$, then we compute

$$\|(1-t)f(x) + tg(x)\| = \|t(f(x) - g(x)) - f(x)\| \geq t\|f(x) - g(x)\| - \|f(x)\| \geq 2t - 1 > 0.$$

Thus, we may define the function $H : M \times [0,1] \to S^k$ by

$$H(x,t) = \frac{(1-t)f(x) + tg(x)}{\|(1-t)f(x) + tg(x)\|}.$$

This map is clearly continuous as, for each $t \in [0,1]$, the map $x \mapsto H(x,t)$ is smooth by smoothness of f and g and the fact that addition and scalar multiplication are smooth. Moreover, as f, g both map into the unit sphere already, $H(x,0) = f(x)$ and $H(x,1) = g(x)$. □

Problem 4

Consider a simple closed curve C in $\mathbb{R}^2 \setminus \{(0,0)\}$, and let $f : C \to \mathbb{R}$ be the distance to the origin: $f(x,y) = \sqrt{x^2 + y^2}$. Prove that the critical points of f are precisely the points where the curve C is tangent to some circle centered at the origin.

Hint: It may be helpfuls to consider the square of the function, and you may assume the curve C is parameterized, $x = x(t), y = y(t), a \leq t \leq b$.

Proof. Let $\gamma : [a,b] \to C$ parameterize C bijectively and let $\gamma(t) = (x(t), y(t))$. For any open subset $U \subseteq C$, (U, γ^{-1}) is a chart. Suppose $p \in C$ is a critical point. Write $p = \gamma(s)$. Then

$$0 = 2(f \circ \gamma)(s) \cdot (f \circ \gamma)'(s) = \frac{d}{dt}|_{t=s} f \circ \gamma = 2x(s)x'(s) + 2y(s)y'(s),$$

so $\langle \gamma(s), \gamma'(s) \rangle = 0$, whence γ is tangent to the circle of radius $\|\gamma(s)\|$ at the point $\gamma(s)$.

On the other hand, suppose that $p \in C$ is tangent to some circle centered at the origin. Then it must be true that the circle has radius $\|p\|$. Write $p = \gamma(s)$. Then, as before,

$$(f \circ \gamma)'(s) = \frac{\langle \gamma(s), \gamma'(s) \rangle}{(f \circ \gamma)(s)} = 0,$$

so p is a critical point of f. □