Homework Assignment 5

1. Let F be a presheaf on a topological space X, and let $f\colon X \to Y$ be a continuous map. The *direct image* f_*F is defined by the following data:

$$(f_*F)(V) = F(f^{-1}(V)) \text{ for open } V \subset Y, \text{ and } \rho(f_*F)^{V_1}_{V_2} = \rho(F)^{f^{-1}(V_1)}_{f^{-1}(V_2)} \text{ for open } V_2 \subset V_1.$$

Show that f_*F is a presheaf. Furthermore, show that if F is a sheaf then f_*F is also a sheaf.

2. Let X be a closed subspace of a topological space Y, and let $\iota\colon X \to Y$ be the identity embedding. Let $Sh(X)$ and $Sh(Y)$ be the categories of sheaves of abelian groups on X and Y respectively. Given $F \in Sh(X)$, identify the stalks of the sheaf ι_*F. Use this to show that the functor $\iota_*\colon Sh(X) \to Sh(Y)$ is exact.

3. Let $X = Y$ be the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$, and let $f\colon X \to Y$ be given by $f(z) = z^2$. Fix an abelian group S and let F be the constant sheaf of abelian groups on X with value group S (recall that for an open subset $U \subset X$, the group $F(U)$ consists of all locally constant functions $U \to S$). Show that for any $y \in Y$, the stalk $(f_*F)_y$ is isomorphic to $S \times S$. On the other hand, let G be the constant sheaf on Y with value group $S \times S$. While G has the same stalks as f_*F, show that $G \not\simeq f_*F$. (*Hint.* Compute global sections.)

4. Let X be a topological space, and let $\mathcal{B}$ be a basis of the topology. Given two sheaves F and G on X, assume that for every $U \in \mathcal{B}$ there is a morphism $f_U\colon F(U) \to G(U)$, and that for $U, V \in \mathcal{B}$ such that $V \subset U$ the diagram

$$\begin{array}{ccc} F(U) & \xrightarrow{f_U} & G(U) \\ \rho(F)^U_V \downarrow & & \downarrow \rho(G)^U_V \\ F(V) & \xrightarrow{f_V} & G(V) \end{array}$$

commutes. Show that there exists a unique morphism of sheaves $f\colon F \to G$ for which f_U for $U \in \mathcal{B}$ coincide with the given ones. Furthermore, show that if f_U is surjective (resp., injective) for every $U \in \mathcal{B}$, then f is surjective (resp., injective) as a *morphism of sheaves*.

5. *(Gluing morphisms)* Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be ringed spaces. Suppose we are given an open covering $\{U_i\}_{i\in I}$ of X, and for each $i \in I$ a morphism $(f_i, f_i^\#)\colon (U_i, \mathcal{O}_X|U_i) \to (Y, \mathcal{O}_Y)$. If $(f_i, f_i^\#)$ and $(f_j, f_j^\#)$ coincide on the overlap $U_i \cap U_j$ for all $i, j \in I$ then there exists a morphism $(f, f^\#)$ that restricts to $(f_i, f_i^\#)$ on $(U_i, \mathcal{O}_X|U_i)$.

6. Let A be a commutative ring, and let $\{M_i, \tau_i^j\}$ be a direct system of A-modules over a directed set I. For any A-module N, one can consider the direct system $\{M_i \otimes_A N, \tau_i^j \otimes \mathrm{id}_N\}$. Show that there is a natural isomorphism of A-modules $(\varinjlim M_i) \otimes_A N \simeq \varinjlim (M_i \otimes_A N)$.

7. A commutative ring A is called *reduced* if its nilradical is zero. Show that A is reduced if and only if all localizations $A_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Spec} A$ are reduced. This can be generalized to schemes: a scheme X is defined to be reduced if the ring $\mathcal{O}_X(U)$ is reduced for all open $U \subset X$. Show that a scheme X is reduced if and only if all stalks $\mathcal{O}_{X,x}$ are reduced.

8. As in class, consider $A = K[x, y]$ and $X = \operatorname{Spec} A$. Let M denote the A-module A/I where $I = xA$. Calculate the stalks $\widetilde{M}_{\mathfrak{p}}$ of the corresponding sheaf $\widetilde{M}$ for all $\mathfrak{p} \in X$. Is $\tilde{M}$ a skyscraper sheaf?