MATH 7820 Homework 5

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- A manifold M is *oriented* if for all $x \in M$ there is a choice of orientation on T_xM such that there is some chart (U, ϕ) around x such that $d_{\phi(x)}\phi^{-1}$ carries the standard orientation on $\mathbb{R}^n$ to the chosen orientation on T_xM.

Problem 1

In class we discussed the induced orientation of the tangent space $T_x(\partial M)$ of the boundary ∂M of an oriented manifold M, at each $x \in \partial M$. Prove that this is in fact an orientation of ∂M i.e. that it depends smoothly on the point $x \in \partial M$.

Proof. Choose an orientaton for M. We can induce an orientation on ∂M by for each $p \in \partial M$ choosing a positively oriented basis for T_pM $(v_1, \ldots, v_n)$ such that $\{v_2, \ldots, v_n\} \subseteq T_p(\partial M)$ and v_1 points inwards. There exists an open $U \subseteq M$ about p and a diffeomorphism $\phi : U \to \phi(U) \subseteq H^n$ such that $d_{\phi(x)}\phi^{-1}(e_i) = v_i$ where $\{e_1, \ldots, e_n\}$ is the standard oreintation on $\mathbb{R}^n$.
Since $\dim T_p(\partial M) = n - 1$ and $v_i \in T_p(\partial M)$ for $i \geq 2$, it follows that $(v_2, \ldots, v_n)$ is an ordered basis for $T_p(\partial M)$. Now let (U, ϕ) be a chart around p satisfying the property above. Let $\widetilde{\phi} := \phi|_{U\cap\partial M}$. Then $(U \cap \partial M, \widetilde{\phi})$ is a chart on ∂M around p. Observe that, for $i \geq 2$

$$(d_{\widetilde{\phi(p)}}\widetilde{\phi}^{-1})(e_i) = d_{\widetilde{\phi(p)}}\phi^{-1}|_{\phi(U\cap\partial M)}(e_i) = d_{\phi(p)}\phi^{-1} \circ d_{\phi(p)}\iota_{\phi(U\cap\partial M)\hookrightarrow\phi(U)}(e_i) = d_{\phi(p)}\phi^{-1}\big(I_n|0\big)e_i = d_{\phi(p)}\phi^{-1}(e_i) = v_i,$$

thus giving an orientation on ∂M.

$\square$

Problem 2

Show that the tangent bundle TM of any (orientable or not) manifold M is orientable.

Proof. Let $(U_\alpha, \phi_\alpha)_\alpha$ be an atlas on R^n adapted to M. Then $(V_\alpha, \Phi_\alpha)_\alpha$ given by $V_\alpha = U \times \mathbb{R}^n$ and $\Phi(p, v) = (\phi(p), d_p(v))$. Suppose that $U_\alpha \cap U_\beta \neq \emptyset$. Let $T = \Phi_\beta \circ \Phi_\alpha^{-1}$ and $t = \phi_\beta \circ \phi_\alpha^{-1}$. Fix $(x, y) = \Phi_\alpha(p, v) \in \Phi_\alpha(V_\alpha)$. Then we compute

$$T(x, y) = \Phi_\beta(p, v) = (\phi_\beta(p), d_p\phi_\beta(v)) = (t(x), d_p\phi_\beta d_x\phi_\alpha^{-1}(y)) = (t(x), d_xt(y)).$$

Note that with respect to the standard basis, $d_xt(y)$ does not depend on y and is a linear map, so by our previous homework $J(d_xt(y)) = d_xt(y) = J(t)(x)$. Thus $J(T)(x, y) = \begin{pmatrix} J(t)(x) & \cdot \\ \cdot & J(t)(x) \end{pmatrix}$, which has positive determinant as it is the square of the determinant of $J(t)(x)$. Thus this atlas is a positive atlas for TM, which implies that TM is orientable. $\square$

Problem 3

Given disjoint manifolds M^m, N^n in $\mathbb{R}^{k+1}$, the linking map $\lambda : M \times N \to S^k$ is defined by

$$\lambda(x, y) = \frac{x - y}{|x - y|}.$$

If M, N are compact, oriented, and without boundary, and $m + n = k$, then the integer valued degree of λ is called the *linking number* $l(M, N)$. Prove that

$$l(N, M) = (-1)^{(m+1)(n+1)} l(M, N).$$

If M bounds an oriented compact manifold W disjoint from N, prove that $l(M, N) = 0$.

Proof. Note that the orientations on M and N induce orientations on $M \times N$. Let $\tau : N \times M \to M \times N$ be the transposition map, $A : S^k \to S^k$ the antipodal map, and λ' the linking map on $N \times M$. Then the following diagram commutes:

$$\begin{tikzcd} N \times M \arrow[r, "\lambda'"] \arrow[d, "\tau"'] & S^k \\ M \times N \arrow[r, "\lambda"'] & S^k \arrow[u, "A"'] \end{tikzcd}$$

Note that, by previous homework, $\deg(\lambda') = \deg(A)\deg(\tau)\deg(\lambda)$. We compute the degree of τ. As τ is a bijection and every point is regular, for fixed $p = (y, x) \in N \times M$ we have that $\deg(\tau) = \operatorname{sgn}(d_p\tau)$. Given orientations for T_pN and T_pM, we patch them together to an orientation for $T_p(N \times M)$. Consider the corresponding basis for $T_p(M \times N)$. Under these bases, $d_p\tau$ is represented by the matrix $\begin{pmatrix} 0_{m\times n} & I_m \\ I_n & 0_{n\times m} \end{pmatrix}$. This matrix has determinant $(-1)^{m\cdot n}$, so $\deg(\tau) = (-1)^{m\cdot n}$ Thus,

$$l(N, M) = \deg(\lambda') = \deg(A)\deg(\tau) l(M, N) = (-1)^{k+1}(-1)^{m\cdot n} l(M, N) = (-1)^{(m+1)(n+1)} l(M, N).$$

$\square$

Problem 4

Given an integer n, construct a smooth map $f : S^1 \times S^1 \to S^1 \times S^1$ such that $\deg(f) = n$. (integer-valued degree).

Solution. Define $f(\theta, \phi) = (n \cdot \theta \bmod 2\pi, \phi)$. Note that $(0, 0))$ is a regular value for f, so

$$\deg(f) = \sum_{k=0}^{n} d_{(\frac{2\pi k}{n}, 0)} f = n$$

as each restriction to S^1 is orientation preserving.

$\square$