MATH 7752 Homework 3

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February 10, 2022

Problem 1

Let R be a commutative domain, and let M be a free R-module with basis $X = \{e_1, \ldots, e_k\}$, with $k \geq 2$. Prove that the element $e_1 \otimes e_2 + e_2 \otimes e_1$ cannot be written as simple tensor $m \otimes n$, for some $m, n \in M$.

Proof. Suppose, for the sake of contradiction, that there exist $m, n \in M$ such that $m \otimes n = e_1 \otimes e_2 + e_2 \otimes e_1$. Write $m = \sum_{i=1}^n r_i e_i$ and $n = \sum_{j=1}^n s_j e_j$ for some $r_i, s_j \in R$. Then

$$e_1 \otimes e_2 + e_2 \otimes e_1 = \left(\sum_{i=1}^n r_i e_i\right) \otimes \left(\sum_{j=1}^n s_j e_j\right) = \sum_{i,j} r_i s_j e_i \otimes e_j$$

As $M \otimes M$ is free with basis $\{e_i \otimes e_j\}_{i,j=1}^n$, it follows that $r_1 s_2 = 1$, $r_2 s_1 = 1$, and $r_i s_j = 0$ for all $(i, j) \neq (1, 2), (2, 1)$. However, then $r_1 s_1 = 0$ whence $r_1 = 0$ or $s_1 = 0$, contradicting $r_1, s_1 \in R^\times$.

□

Problem 2

Let R be a commutative ring (with 1) and $n, m \in \mathbb{N}$. Prove that there is an isomorphism of R-**algebras** $R^n \otimes R^m \simeq R^{nm}$. (Here by R^n we mean the direct sum $\underbrace{R \oplus \cdots \oplus R}_{n}$.)

Proof. Define a map $\Phi : R^n \times R^m \to R^{nm}$ by $\Phi((r_1, \ldots, r_n), (r'_1, \ldots, r'_m)) =$ □

Problem 3

(a) Let V be a finite-dimensional $\mathbb{C}$-vector space. Then V can be considered as a vector over $\mathbb{R}$ (by restriction of scalars), and it holds $\dim_{\mathbb{R}} V = 2 \dim_{\mathbb{C}} V$. Prove that $V \otimes_{\mathbb{C}} V$ is not isomorphic to $V \otimes_{\mathbb{R}} V$ as $\mathbb{R}$-vector spaces, and compute their dimensions over $\mathbb{R}$.

Proof. Let $k = \dim_{\mathbb{C}} V$. Then $\dim_{\mathbb{R}} V = 2k$. As a $\mathbb{C}$-vector space, $\dim_{\mathbb{C}} V \otimes_{\mathbb{C}} V = (\dim_{\mathbb{C}} V)^2 = k^2$, whence by restricton of scalars $\dim_{\mathbb{R}} V \otimes_{\mathbb{C}} V = 2 \dim_{\mathbb{C}} V \otimes_{\mathbb{C}} V = 2k^2$.

On the other hand $\dim_{\mathbb{R}} V \otimes_{\mathbb{R}} V = (\dim_{\mathbb{R}} V)^2 = (2k)^2 = 4k^2$. Hence

$$\dim_{\mathbb{R}} V \otimes_{\mathbb{C}} V = 2k^2 \neq 4k^2 = \dim_{\mathbb{R}} V \otimes_{\mathbb{R}} V$$

so $V \otimes_{\mathbb{C}} V$ and $V \otimes_{\mathbb{R}} V$ are not isomorphic as $\mathbb{R}$-vector spaces.

□

(b) Let R be an integral domain (commutative), and let K be its fraction field. Prove that there is an isomorphism of F-modules, $F \otimes_R F \simeq F \otimes_F F \simeq F$, where the F-module structure on $F \otimes_R F$ is given by **extension of scalars** (i.e. tensor product of Type I).

Problem 4

The purpose of this problem is to classify all 2-dimensional $\mathbb{R}$-algebras (where $\mathbb{R}$ are the real numbers). That means, to classify (up to algebra isomorphism) those $\mathbb{R}$-algebras that are 2-dimensional $\mathbb{R}$ vector spaces.
Let A be a 2-dimensional $\mathbb{R}$-algebra (with 1).

(a) Let $u \in A$ be any element that is $\mathbb{R}$-linearly independent from 1. Prove that

(i) u generates A as an $\mathbb{R}$-algebra. That is, the minimal $\mathbb{R}$-subalgebra of A containing u and 1 is A itself.

(ii) The element u satisfies a quadratic equation $au^2 + bu + c = 0$, for some $a, b, c \in \mathbb{R}$ with $a \neq 0$. Conclude that A is necessarily commutative.

Proof. Noting that the subalgebra generated by u contains $\mathrm{span}_{\mathbb{R}}(\{1, u\})$ which has dimension 2 as an $\mathbb{R}$-vector space, it follows that the subalgebra generated by u is in fact A.

Since the subalgebra generated by u is A, it follows that there exist $a, b \in \mathbb{R}$ such that $u^2 = au + b1$, whence $u^2 - au - b = 0$. This implies the algebra A is commutative as multiplication is hence defined by the relations $u \cdot 1 = u = 1 \cdot u$ and $1 = 1 \cdot 1$, which are all commutative. □

(b) Show that there exists some $v \in A$ which is $\mathbb{R}$-linearly independent from 1 and is such that $v^2 = -1$, or $v^2 = 1$, or $v^2 = 0$.

Proof. □

(c) Deduce from part (b) that A is isomorphic as an $\mathbb{R}$-algebra to one of the following: $\mathbb{R}[x]/(x^2 + 1)$, or $\mathbb{R}[x]/(x^2 - 1)$, or $\mathbb{R}[x]/(x^2)$.

(d) Prove that the algebras $\mathbb{R}[x]/(x^2 + 1)$, $\mathbb{R}[x]/(x^2 - 1)$, and $\mathbb{R}[x]/(x^2)$ are pairwise non-isomorphic. **Hint:** This can be shown with almost no computation.

Problem 5

The purpose of this problem is to prove the following theorem: Let D be a finite dimensional division algebra over $\mathbb{R}$. Then D is isomorphic to $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ (the quaternions). One way to proceed is to use the following steps:

(a) Let $\alpha \in D$ be an element $\mathbb{R}$-linearly independent from 1. Show that α satisfies a quadratic irreducible polynomial $p_\alpha(x) = x^2 + ax + b \in \mathbb{R}[x]$.

Proof. Since D is finite-dimensional over $\mathbb{R}$, there exists an $n \in \mathbb{N}$ such that the set $\{1, \alpha, \ldots, \alpha^n\}$ is $\mathbb{R}$-linearly dependent. Hence α is algebraic over $\mathbb{R}$, so the set $I_\alpha = \{f(x) \in \mathbb{R}[x] : f(\alpha) = 0\}$.
As I_α is an ideal and $\mathbb{R}[x]$ is a PID, there exists a (without loss of generality) monic polynomial $p_\alpha(x) \in \mathbb{R}[x]$ such that $I_\alpha = (p_\alpha)$. As α is algebraic, $p_\alpha =\neq 0$. Moreover, p_α is nonconstant by $p_\alpha(\alpha) = 0$. Hence, p_α is not a unit in $\mathbb{R}[x]$. If $f \in I_\alpha = (p_\alpha)$ is irreducible, then in writing $f = gp_\alpha$ for some $g \in \mathbb{R}[x]$, irreducibility implies that $(f) = (p_\alpha) = I_\alpha$. Moreover, this implies that $\deg(f) = \deg(p_\alpha)$, so p_α being monic implies that p_α is the unique irreducible monic element of I_α.

As $\alpha \notin \mathbb{R} \cdot 1$, $\deg(p_\alpha) \geq 2$. By the Fundamental Theorem of Algebra, it follows then that p_α must be quadratic, so there exist $a, b \in \mathbb{R}[x]$ such that $p_\alpha(x) = x^2 + ax + b$.

□

(b) Let $V = \{\alpha \in D : \alpha^2 \in \mathbb{R}_{\leq 0}\}$. Show that V is an $\mathbb{R}$-linear subspace of D. **Hint:** Show there is an $\mathbb{R}$-linear map $f : D \to \mathbb{R}$ with kernel V.

Proof. For $\alpha \in D$, define an $\mathbb{R}$-endomorphism T_α of D via left multiplication by α. This furnished a linear map $D \to \mathrm{End}_{\mathbb{R}}(D)$. We claim that V is the kernel of the composition of the $\mathbb{R}$-linear maps

$$D \to \mathrm{End}_{\mathbb{R}}(D) \xrightarrow{\mathrm{Tr}} \mathbb{R}.$$

Fix $\alpha \in D$ such that $\alpha \notin \mathbb{R} \cdot 1$. Then by part (a) there exist $a, b \in \mathbb{R}$ such that α satisfies a quadratic irreducible polynomial $p_\alpha(x) = x^2 + ax + b$. Observe that, for $v \in D$,

$$p_\alpha(T_\alpha)(v) = T_\alpha^2(v) + aT_\alpha(v) + b(v) = \alpha^2 v + a\alpha v + bv = (\alpha^2 + a\alpha + b\alpha)(v) = 0$$

so $p_\alpha(T_\alpha) = 0 \in \mathrm{End}_{\mathbb{R}}(D)$. Irreducibility of p_α then implies that p_α is the minimal polynomial for the operator $T\alpha$. Let $\chi_\alpha(x)$ be the characteristic polynomial for T_α. Then $p_\alpha(x) | \chi_\alpha(x)$ and there exists a $k \in \mathbb{N}$ such that $\chi_\alpha(x) | (p_\alpha(x))^k$. As χ_α is monic and p_α is irreducible, there exists an $l \in \mathbb{N}$ such that $\chi_\alpha(x) = (p_\alpha(x))^l$. By multinomial expansion,

$$\chi_\alpha(x) = (p_\alpha(x))^l = \sum_{\substack{n_1+n_2+n_3=l \\ n_1,n_2,n_3 \geq 0}} \binom{l}{n_1,\ n_2,\ n_3} x^{2n_1+n_2} a^{n_2} b^{n_3}$$

This polynomial has x^{2l-1} coefficient

$$\binom{l}{l-1,\ 1,\ 0} a = l \cdot a$$

However, the x^{2l-1} coefficient of χ_α is also $\pm \mathrm{Tr}(T_\alpha)$, so $\pm \mathrm{Tr}(T_\alpha) = l \cdot a$. Moreover, as $p_\alpha(x)$ is irreducible, $a^2 - 4b < 0 \implies b > \frac{a^2}{4} \geq 0$. Hence, if α is such that $\mathrm{Tr}(\alpha) = 0$, then $a = 0$ whence $0 = p_\alpha(\alpha) = \alpha^2 + b \implies \alpha^2 = -b \leq 0$, i.e. $\alpha \in V$. Conversely, suppose that $\alpha \in D \setminus \{0\}$ is such that $\alpha^2 < 0$. Then α is linearly independent from 1, so there exist $a, b \in \mathbb{R}$. such that $\alpha^2 + a\alpha + b = 0$. Note that, as $\alpha^2 \in \mathbb{R}$, linear independence of α from 1 implies that $a = 0$ and $\alpha^2 + b = 0$. Then, $\mathrm{Tr}(T_\alpha) = 0$, as desired. □

(c) Define $B : V \times V \to \mathbb{R}$, $B(\alpha, \beta) := -\dfrac{\alpha\beta + \beta\alpha}{2}$. Show that B defines an inner product on V (i.e. B is a symmetric, positive definite bilinear form on V).

Proof. Observe that, for $\alpha, \beta \in V$, $\alpha^2, \beta^2, (\alpha + \beta)^2 \in \mathbb{R}$, whence $\alpha\beta + \beta\alpha = (\alpha + \beta)^2 - \alpha^2 - \beta^2 \in \mathbb{R}$, so B is in fact real-valued.

Fix $\alpha, \alpha', \beta, \beta' \in V$ and $\lambda \in \mathbb{R}$. Then

$$\begin{aligned}
B(\alpha, \beta) &= -\frac{\alpha\beta + \beta\alpha}{2} = -\frac{\beta\alpha + \alpha\beta}{2} = B(\beta, \alpha) \\
B(\alpha + \lambda\alpha', \beta) &= -\frac{(\alpha + \lambda\alpha')\beta + \beta(\alpha + \lambda\alpha')}{2} = -\frac{\alpha\beta + \beta\alpha}{2} - \lambda\frac{\alpha'\beta + \beta\alpha'}{2} = B(\alpha, \beta) + \lambda B(\alpha', \beta). \\
B(\alpha, \beta + \lambda\beta') &= -\frac{\alpha(\beta + \lambda\beta') + (\beta + \lambda\beta')\alpha}{2} = -\frac{\alpha\beta + \beta\alpha}{2} - \lambda\frac{\alpha\beta' + \beta'\alpha}{2} = B(\alpha, \beta) + \lambda B(\alpha, \beta'),
\end{aligned}$$

so B is a symmetric bilinear form. Moreover, as $\alpha \in V \setminus \{0\}$ implies that $\alpha^2 \in \mathbb{R}_{<0}$, we have then that

$$B(\alpha, \alpha) = -\frac{\alpha\alpha + \alpha\alpha}{2} = -\alpha^2 > 0$$

so B is also positive-definite. $\square$

(d) Let W be a linear subspace of V that generates D as an $\mathbb{R}$-algebra. Let $n = \dim_{\mathbb{R}} W$. Choose an orthonormal basis of W, i.e. a basis $\{e_i\}$ of W such that $B(e_i, e_i) = 1$ for all i and $B(e_i, e_j) = 0$ for all $i \neq j$ (such a basis always exists). Using this orthonormal basis show that if $n \geq 2$, then D has a subalgebra isomorphic to $\mathbb{H}$.

Proof. We must first show that such a subspace W of V with the prescribed property actually exists. Let $\psi : D \to \mathbb{R}$ denote the linear map constructed in part (b), i.e. $\psi(\alpha) = \operatorname{Tr}(T_\alpha)$ where T_α is the left multiplication by α operator. We claim that $\operatorname{im} \psi \neq 0$, whence $\operatorname{im} \psi = \mathbb{R}$. To show this, consider $\lambda \in R \setminus \{0\}$. With respect to any $\mathbb{R}$-basis of D, the matrix of T_λ is diagonal with nonzero entries λ along the diagonal, so $\operatorname{Tr}(T_\lambda) = \lambda \dim_{\mathbb{R}} D \in \mathbb{R} \setminus \{0\}$, as desired.

By rank-nullity theorem,

$$\dim D = \dim \ker \psi + \dim \operatorname{im} \psi = \dim V + \dim \mathbb{R} = \dim V + 1$$

so $\dim V = \dim D - 1$. The remaining direct summand of D is spanned by 1 and is thus $\mathbb{R}$, so as the subalgebra generated by V contains 1 and V, it must be all of D.

Now suppose that W is a linear subspace of V that generates D as an $\mathbb{R}$-algebra. Let $n = \dim_{\mathbb{R}} W$. Choose an orthonormal basis $\{e_i\}_{i=1}^n$ of W with respect to B. Then, for $i \in \{1, \dots, n\}$,

$$1 = B(e_i, e_i) = -\frac{e_i^2 + e_i^2}{2} = -e_i^2 \implies e_i^2 = -1.$$

Also, for $i, j \in \{1, \dots, n\}$,

$$0 = B(e_i, e_j) = -\frac{e_i e_j + e_j e_i}{2} \implies e_i e_j = -e_j e_i.$$

Let A be the subalgebra of D generated by $\{1, e_1, e_2\}$. We will show that $A \cong \mathbb{H}$ as $\mathbb{R}$-algebras.

As $(e_1 e_2)^2 = (-e_2 e_1)(e_1 e_2) = -e_1^2 e_2^2 \in R_{<0}$, it follows that $e_1 e_2 \in V$. Observe that

$$B(e_1e_2, e_1) = -\frac{e_1e_2e_1 + e_1e_1e_2}{2} = -\frac{e_1e_2e_1 - e_1e_2e_1}{2} = 0$$
$$B(e_1e_2, e_2) = -\frac{e_1e_2e_2 + e_2e_1e_2}{2} = -\frac{-e_2e_1e_2 + e_2e_1e_2}{2} = 0$$

Since B is an inner product on V and $e_1 e_2$ is orthogonal to $\{e_1, e_2\}$ with respect to B, it follows that $e_1 e_2$ is linearly independent from $\{e_1, e_2\}$. As $1 \notin V$, 1 is linearly independent from $\{e_1, e_2, e_1 e_2\}$.

As $\{1, i, j\} \subseteq \mathbb{H}$ satisfy the relations for $\{1, e_1, e_2\} \subseteq A$, there exists an $\mathbb{R}$-algebra homomorphism $\varphi : A \to \mathbb{H}$ such that $\varphi(1) = 1$, $\varphi(e_1) = i$, $\varphi(e_2) = j$. Observe that then $\varphi(e_1 e_2) = \varphi(e_1)\varphi(e_2) = ij = k$, so φ is surjective by linearity. On the other hand, there exists an $\mathbb{R}$-algebra homomorphism $\psi : \mathbb{H} \to A$ such that $\psi(1) = 1$, $\psi(i) = e_1, \psi(j) = e_2$. The maps ψ, φ are clearly mutual inverses, so $A \cong \mathbb{H}$ as $\mathbb{R}$-algebras. $\square$

(e) Bonus: Suppose $n \geq 2$. Prove that $D = H$. **Hint:** One way to proceed is to show that if $n > 2$, then the multiplication in D cannot be associative.

Proof. Suppose, for the sake of contradiction, that $n > 2$. $\square$