# MATH 7752 Homework 4

James Harbour

February 21, 2022

## Problem 1

Let $V$ and $W$ be finite dimensional vector spaces over a field $F$. Let $\{v_1, \ldots, v_n\}$, $\{w_1, \ldots, w_m\}$ be bases of $V, W$ respectively. Consider the $F$-linear transformation $\varphi : V \otimes_F W \to M_{n\times m}(F)$ defined by $\varphi(v_i \otimes w_j) = e_{ij}$, where $e_{ij}$ is the matrix with 1 at the $(i, j)$-entry and 0 elsewhere.

**(a)** Verify that such a linear transformation exists and is in fact an isomorphism of $F$-vector spaces.

*Proof.* Define a map $\Phi : V \times W \to M_{n\times m}(F)$ by $\Phi(v_i, w_j) = e_{ij}$ and extending bilinearly. Then, by construction, $\Phi$ is an F-bilinear map, so universality of tensor product implies that there exists an $F$-linear $\varphi : V \otimes_F W \to M_{n\times m}(F)$ such that $\varphi(v_i \otimes w_j) = \Phi(v_i, w_j) = e_{ij}$. As $\{v_i \otimes w_j\}_{i,j}$ is a basis for $V \otimes_F W$ which $\varphi$ maps linearly to $\{e_{ij}\}_{i,j}$, a basis for $M_{n\times m}(F)$, it follows that $\varphi$ is an isomorphism. □

**(b)** Prove that for every $A \in M_{n\times m}(F)$ the following statements are equivalent:

(i) There exists some $v \in V, w \in W$ such that $A = \varphi(v \otimes w)$ ($v, w$ need not be basis elements).

(ii) $\mathrm{rk}(A) \leq 1$.

*Proof.*
<u>$(i \implies ii)$</u>: Suppose that there exist some $v \in V$ and $w \in W$ such that $A = \varphi(v \otimes w)$. If $v$ or $w$ is zero, then $A$ would be zero and thus have rank zero, so suppose $v, w \neq 0$. Write $v = \sum_{i=1}^n a_i v_i$ and $w = \sum_{j=1}^m b_j w_j$. Then

$$A = \varphi(v \otimes w) = \varphi\left(\sum_{i,j} a_i b_j v_i \otimes w_j\right) = \sum_{i,j} a_i b_j e_{ij}$$

so each row has the form $\rho_r = (a_r b_1, a_r b_2, \ldots, a_r b_m) = a_r \cdot (b_1, \ldots, b_m)$. If all but one row is zero, then $\mathrm{rk}(A) = 1$. If $\rho_r, \rho_s$ are two nonzero rows with $r \neq s$, then $a_r, a_s \neq 0$ and $a_s \cdot \rho_r - a_r \cdot \rho_s = 0$, so the cardinality of a maximal linearly independent subset of the rows of $A$ is at most 1, whence $\mathrm{rk}(A) \leq 1$.

<u>$(ii \implies i)$</u>: Suppose that $\mathrm{rk}(A) = 1$. Then there exist bases $\{v_1', \ldots, v_n'\}$, $\{w_1', \ldots, w_m'\}$ of $V, W$ respectively such that the matrix of $A$ is equal to $e_{11}$. Take $v = v_1'$ and $w = w_1'$. Then $\varphi(v \otimes w) = A$. □

## Problem 2

Let $R = \oplus_{n=0}^{\infty} R_n$ be a graded ring. Recall that an element $r \in R$ is called *homogeneous* if $r \in R_n$, for some $n \geq 0$. Notice that every $r \in R$ can be written uniquely as $r = \sum_{n=0}^{\infty} r_n$, where $r_n \in R_n$ and all but finitely many $r_n$'s are equal to zero. The elements $\{r_n\}$ are called *the homogeneous components* of $r$

**(a)** Let $I$ be an ideal of $R$. Prove that the following statements are equivalent:

(i) $I$ is a graded ideal, i.e. $I = \oplus_{n=0}^{\infty}(I \cap R_n)$.

(ii) For each $r \in I$, all homogeneous components of $r$ lie also in $I$.

*Proof.*
<u>$(i \implies ii)$</u>: Suppose $I$ is a graded ideal and let $r \in I$. Then, as $I$ is graded, there exist $r_k \in I \cap R_k$ for $k \geq 0$ such that $r = \sum_{k \geq 0} r_k$ with all but finitely many $r_k$ nonzero. On the other hand, since $R_k \cap I \subseteq R_k$ for all $k \geq 0$, the directness of the sum decomposition of $R$ into a grading implies by uniqueness that the $r_k$'s are precisely the homogenous components of $r$.

<u>$(ii \implies i)$</u>: Suppose that, for each $r \in I$, all homogenous components of $r$ lie also in $I$. Take $i \in I$. Using the decomposition of $R$ to write $i$ as $i = \sum_{k \geq 0} r_k$ for some $r_k \in R_k$. By assumption, $r_k \in I$ for $k \in \mathbb{N}$, so $i \in \sum_{k \geq 0} I \cap R_k$. Hence $I \subseteq \sum_{k \geq 0} I \cap R_k$. On the other hand, $\sum_{k \geq 0} I \cap R_k \subseteq I$ as each summand is in $I$. Moreover, the sum $\sum_{k \geq 0} I \cap R_k$ is in fact direct as each $I \cap R_k$ is inside $R_k$ and the sum decomposition for $R$ is direct. Thus $I = \oplus_{k \geq 0} I \cap R_k$. ☐

**(b)** Let $I$ be an ideal of $R$ generated by homogeneous elements. Prove that $I$ is graded.

*Proof.* Let $\{X \subseteq I\}$ be a set of homogenous elements which generate $I$. For $r \in I$, there exist $x_1, \ldots, x_n \in X$ such that $r = \sum_{i=1}^{n} x_i$ and each $x_i \in R_{k_i}$ for some $k_i \geq 0$. By uniqueness of direct sum decomposition, these $x_i$ are the homogenous components of $r$, so $x_i \in \{X\} \subseteq I$ implies by part (a) that $I$ is a graded ideal. ☐

# Problem 3

**(a)** Let $R$ be a PID. Prove that $R$ is Noetherian.

*Proof.* Suppose $I_1 \subseteq I_2 \subseteq \cdots$ is an ascending chain of ideals in $R$. Then $I = \bigcup_{i=1}^{\infty} I_i$ is an ideal of $R$, so by PID there exists an $r \in R$ such that $I = (r)$. But then, as $r \in I$, there exists a $k \in \mathbb{N}$ such that $r \in I_k$. Hence $(r) \subseteq I_k \subseteq I_{k+1} \subseteq \cdots \subseteq I = (a)$, so $I_k = I_{k+1} = \cdots$ as desired. ☐

**(b)** Let $R$ be a commutative ring and $M$ be an $R$-module. Recall that $M$ is called *Noetherian* if every ascending chain $M_1 \subset M_2 \subset \cdots M_n \subset \cdots$ of submodules of $M$ eventually stabilizes.

**(i)** Let $N$ be a submodule of $M$. Prove that the following are equivalent:

1. $M$ is Noetherian.

2. $N$ and $M/N$ are both Noetherian.

*Proof.* The forward direction is immediate as ascending chains in $N$ are ascending chains in $M$ and ascending chains in $M/N$ may be pulled back to chains in $M$ by the correspondence theorem. Hence, it suffices to prove the reverse direction.

Suppose that $M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$ is an ascending chain of submodules in $M$. Then $\frac{M_1+N}{N} \subseteq \frac{M_2+N}{N} \subseteq \frac{M_3+N}{N} \subseteq \cdots$ and $M_1 \cap N \subseteq M_2 \cap N \subseteq \cdots$ stabilize at some $k, s \in \mathbb{N}$. Take $l = \max\{k, s\}$. Then $M_l \cap N = M_{l+1} \cap N$ and $M_l + N = M_{l+1} + N$. Let $x \in M_{l+1}$. Then $x \in M_{l+1} + N = M_l + N$, so there exists some $m \in M_l$ and $n \in N$ such that $x = m + n$. Then $x - m \in M_{l+1} \cap N = M_l \cap N \subseteq M_l$, so $x = (x - m) + m \in M_l$. Thus $M_l = M_{l+1} = \cdots$. ☐

**(ii)** Let $R$ be a commutative Noetherian ring. Use (a) to prove that $R^n$ is Noetherian, for every $n \geq 1$.

*Proof.* We induct on $n \in \mathbb{N}$. The base case follows by assumption. Now fix $n \geq 2$ and suppose that $R^{n-1}$ is noetherian. Then $R^n/R \cong R^{n-1}$ is noetherian and $R$ is noetherian, so part (i) implies that $R^n$ is noetherian. $\square$

**(iii)** Prove that if $R$ is Noetherian, then every submodule of a finitely generated $R$-module is finitely generated.

*Proof.* Since $M$ is finitely generated, there exists a surjective $R$-module homomorphism $R^n \rightarrowtail M$ for some $n \in \mathbb{N}$. Thus, $M \cong R^n/N$ where $N$ is the kernel of the aforementioned map. So, by parts (i) and (ii), $M$ is noetherian. As every submodule of a noetherian module is noetherian, it suffices to show that every noetherian module is finitely generated. So, let $M$ be any noetherian module. If $M = 0$, then $M$ is generated by 0 so we would be done, so suppose $M \neq 0$. Take $m_1 \in M \setminus \{0\}$. If $\langle m_1 \rangle = M$, then done, otherwise, take $m_2 \in M \setminus \langle m_1 \rangle$. Continuing as such, we build an ascending chain

$$\langle m_1 \rangle \subseteq \langle m_1, m_2 \rangle \subseteq \langle m_1, m_2, m_3 \rangle \subseteq \cdots,$$

so by the noetherian condition, there exists a $k \in \mathbb{N}$ such that this chain terminates at step $k$. Thus $M = \langle m_1, \ldots, m_k \rangle$, so $M$ is finitely generated. $\square$

# Problem 4

Let $A$ be a ring (with 1) and $B$ be a subring of $A$. The ring $B$ is called *a retract* of $A$ if there exists a surjective ring homomorphism, $\phi : A \to B$ such that $\varphi|_B = 1_B$.
Let $M$ and $N$ be $R$-modules. Prove that the tensor algebra $T(M)$ is (naturally isomorphic to) a subalgebra of $T(M \oplus N)$ and this subalgebra is a retract. Prove that the same is true for symmetric algebras.

*Proof.* Let $i : M \to M \oplus N$ be the the natural injection and $j : M \oplus N \to T(M \oplus N)$ be the natural inclusion. By universality of $T(M)$, there exists a unique $R$-algebra homomorphism $\Phi : T(M) \to T(M \oplus N)$ such that $\Phi|_M = j \circ i$. On the other hand, let$\pi : M \oplus N \to M$ be the natural projection and $i' : M \to T(M)$ the natural inclusion. By universality of $T(M \oplus N)$, there exists a unique $R$-algebra homorphism $\Psi : T(M \oplus N) \to T(M)$ such that $\Psi|_{M \oplus N} = i' \circ \pi$.

We claim that $\Psi|_{\mathrm{im}(\Phi)}$ and $\Phi$ are mutual inverses. It suffices to check this on the $R$-algebra generators of $T(M)$, i.e. $T^1(M) = M$. On one hand, suppose that $m \in M$. Then

$$\Psi(\Phi(m)) = \Psi((m, 0)) = m$$

So $\Phi$ is injective, whence it is isomorphic to $\operatorname{im} \Phi$. Moreover, $\Psi$ gives that $\operatorname{im} \Phi$ this is a retract of $T(M \oplus N)$. $\square$

# Problem 5

Finish the proof we started in class. Namely: Let $V$ be a $F$-vector space of dimension $n$, where $F$ is a field. Let $\varphi : V \to V$ be a $F$-linear transformation. Consider the linear transformation $\Phi_{ext,n} : \wedge^n(V) \to \wedge^n(V)$ induced by $\phi$. Prove that $\Phi_{ext,n}$ is given by scalar multiplication by $\det(\varphi)$.

*Proof.* Let $\varphi(e_i) = \sum_{j=1}^n a_{ij} e_j$ where $a_{ij} \in F$. Then, via the properties of wedge products,

$$\begin{aligned}
\varphi(e_1 \wedge \cdots \wedge e_n) = \varphi(e_1) \wedge \cdots \wedge \varphi(e_n) &= \left( \sum_{j=1}^n a_{1j} e_j \right) \wedge \cdots \wedge \left( \sum_{j=1}^n a_{nj} e_j \right) \\
&= \left( \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) a_{1\sigma(1)} \cdots a_{n\sigma(n)} \right) e_1 \wedge \cdots \wedge e_n = \det(\varphi) e_1 \wedge \cdots \wedge e_n
\end{aligned}$$

$\square$