Operator Algebras Reading Course

Meeting 1 Exercises

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Problem. Let X be a topological vector space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and $\phi : X \to \mathbb{F}$ a linear functional. Then ϕ is continuous if and only if $\ker(\phi)$ is closed.

Problem. Let V be a vector space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and $f, f_1, \ldots, f_n$ linear functionals on V. Then there exist $c_1, \ldots, c_n \in \mathbb{F}$ such that $f = \sum_j c_j f_j$ if and only if $\ker(f) \supseteq \bigcap_{j=1}^n \ker(f_j)$.

Proof. The forward direction is clear by definition. Now suppose that $\ker(f) \supseteq \bigcap_{j=1}^n \ker(f_j)$. If $f = 0$ then we are done so suppose that $f \neq 0$. Then f is surjective. Consider $T : V \to \mathbb{F}^n$ given by $T(x) = (f_1(x), \ldots, f_n(x))$. This map is linear and $\ker(T) = \bigcap_{j=1}^n \ker(f_j) \subseteq \ker(f)$, so f factors through $V/\ker(T)$. Moreover, $V/\ker(T)$ is isomorphic to $T(V)$ which is a subspace of $\mathbb{F}^n$, so there exists a map $g : \mathbb{F}^n \to \mathbb{F}$ such that $f = g \circ T$. Now, let $c_1, \ldots, c_n \in \mathbb{F}$ be such that $g(x_1, \cdots, x_n) = \sum_{j=1}^n c_j x_j$. Then $f(x) = h(f_1(x), \ldots, f_n(x)) = \sum_{j=1}^n c_j f_j(x)$ as desired. $\square$

Problem IV-1.23. Let X and Y be locally convex spaces and $T : X \to Y$ a linear transformation. Show that T is continuous if and only if for every continuous seminorm p on Y, $p \circ T$ is a continuous seminorm on X.

Proof. Note that if p is a seminorm on Y, then by linearity of T it is clear that $p \circ T$ is a seminorm on X. Hence, the forward direction follows immediately from transitivity of continuity.

Now suppose that for every continuous seminorm p on Y, $p \circ T$ is a continuous seminorm on X.
By linearity, it suffices to prove that T is continuous at 0. Moreover, by proposition 1.15, we have a basis for the neighborhood system at 0 given by the collection of all open, convex, balanced subsets of Y. As such, let $C \subseteq Y$ by open, convex, and balanced. It is enough to show that $T^{-1}(C)$ is open, as $0 \in T^{-1}(C)$.

By proposition 1.14, the Minkowski function p of C is the unique seminorm on Y such that

$$C = \{y \in Y : p(y) < 1\}.$$

As $\{y \in Y : p(y) < 1\}$ is open by assumption, proposition 1.3(b) implies that p is continuous whence $p \circ T$ is a continuous seminorm on X. Thus, proposition 1.3 implies that the set $V := \{x \in X : (p \circ T)(x) < 1\}$ is open. Noting that

$$x \in T^{-1}(C) \iff Tx \in C \iff (p \circ T)(x) < 1,$$

it follows that $T^{-1}(C) = V$ is open. $\square$

Problem IV-2.4. A subset B of a TVS X is *bounded* if for every open set U containing 0, there exists an $\varepsilon > 0$ such that $\varepsilon B \subseteq U$. Let X be a topological vector space. Prove the following.

(a): If B is a bounded subset of X, then so is $\overline{B}$.

Lemma 1. *If U is a neighborhood of 0, then there is a neighborhood V of 0 such that $\overline{V} \subseteq U$.*

Proof of Lemma 1. By an exercise we had in algebraic topology on topological groups, there exists a symmetric neighborhood V of 0 such that $V + V \subseteq U$. Take $x \in \overline{V}$. Then $x + V$ is an open neighborhood of x, so there is some $y \in V \cap (x + V)$. Writing $y = x + v$ for some $v \in V$, it follows that $x = y - v \in V - V = V + V \subseteq U$. □

Proof. □

(b): The finite union of bounded sets is bounded.

(c): Every compact set is bounded.

Lemma 2. *If U is a neighborhood of 0, then there is a balanced neighborhood W of 0 such that $W \subseteq U$.*

Proof of Lemma 2. The multiplication map $m : \mathbb{F} \times X \to X$ is continuous and U is a neighborhood of 0, so there exist $\delta > 0$ and a neighborhood V of 0 such that $m(B(0, \delta) \times V) \subseteq U$, i.e. $\alpha V \subseteq U$ for all $|\alpha| < \delta$. Let $W = \bigcup_{|\alpha|<\delta} \alpha V$.

Suppose $w \in W$ and $|\beta| < 1$. Writing $w = \alpha v$ for some $|\alpha| < \delta$ and $v \in V$, as $|\alpha\beta| < \delta$ it follows that $\beta w = \alpha\beta v \in W$. Thus W is a balanced open neighborhood of 0 constructed such that $W \subseteq U$. □

Proof. Let U be a neighborhood of 0. By the above lemma, there is a balanced neighborhood V of 0 such that $V \subseteq U$.
As V is absorbing, $X = \bigcup_{n=1}^{\infty} nV$. Suppose that K is compact. Then $K \subseteq \bigcup_{n=1}^{\infty} nV$ whence there are some $n_1 < \cdots < n_k$ such that

$$K \subseteq n_1 V \cup n_2 V \cdots n_k V.$$

As V is balanced, $n_1 V \cup n_2 V \cdots n_k V = n_k V$ □

(d): If $B \subseteq X$, then B is bounded if and only if for every sequence (x_n) contained in B and for every (α_n) in c_0, $\alpha_n x_n \to 0$ in X.

Proof.
$\underline{\Longrightarrow}$: Suppose (x_n) is a sequence in B and (α_n) in c_0. Let U be a neighborhood of 0. By the lemma, there is a balanced neighborhood V of 0 such that $V \subseteq U$. The boundedness of B implies that there is some $\varepsilon > 0$ such that $\varepsilon B \subseteq V$. Let $N \in \mathbb{N}$ be such that $|\alpha_n| < \varepsilon$ for all $n \geq N$. Then, as V is balanced, for $n \geq N$

$$\alpha_n x_n = \frac{\alpha}{\varepsilon} \varepsilon x_n \in \frac{\alpha}{\varepsilon} \varepsilon B \subseteq \frac{\alpha}{\varepsilon} V \subseteq V \subseteq U.$$

$\underline{\Longleftarrow}$: Suppose, for the sake of contradiction, that B is not bounded. Then there is some neighborhood U of 0 such that $\varepsilon B \not\subseteq U$ for all $\varepsilon > 0$. Thus, for $n \in \mathbb{N}$ there exists $x_n \in B \setminus nV$. Then $\frac{1}{n} x_n \notin V$ for all $n \in \mathbb{N}$, contradicting the assumption that $\frac{1}{n} x_n \to 0$ in X. □

(e): If Y is a TVS, $T : X \to Y$ continuous linear, and B is a bounded subset of X, then $T(B)$ is a bounded subset of Y.

Proof. Suppose that $V \subseteq Y$ is a neighborhood of 0. Then $T^{-1}(V)$ is a neighborhood of 0 in X, so there is some $\varepsilon > 0$ such that $\varepsilon B \subseteq T^{-1}(V)$. Applying T, it follows that $\varepsilon T(B) = T(\varepsilon B) \subseteq T(T^{-1}(V)) \subseteq V$. □

(f): If X is a LCS and $B \subseteq X$, then B is bounded if and only if for every continuous seminorm p, $\sup\{p(b) : b \in B\} < \infty$.

Proof.
$\underline{\implies}$: Suppose that p is a continuous seminorm. Then $V := \{x : p(x) < 1\}$ is an open neighborhood of 0, so there is some $\varepsilon > 0$ such that $\frac{1}{\varepsilon}B \subseteq V$, whence

$$B \subseteq \varepsilon V = \{\varepsilon x : p(x) < 1\} = \{x : p(x) < \varepsilon\}$$

so $\sup\{p(b) : b \in B\} \leq \varepsilon < +\infty$.

$\underline{\impliedby}$: Suppose that U is an open neighborhood of 0. As X is locally convex, there is an open balanced convex set C such that $C \subseteq U$. Let p be the Minkowski function corresponding to C, so $C = \{x : p(x) < 1\}$. Letting $\delta > \sup\{p(b) : b \in B\}$, it follows that

$$\delta C = \{\delta x : p(x) < 1\} = \{x : p(x) < \delta\} \implies B \subseteq \delta C \subseteq \delta U.$$

$\square$

(g): If X is a normed space and $B \subseteq X$, then B is bounded if and only if $\sup\{\|b\| : b \in B\} < \infty$.

Proof.
$\underline{\implies}$: The unit ball $B(0,1)$ is an open neighborhood of 0 so there is some $r > 0$ such that $B \subseteq rB(0,1) \subseteq B(0,r)$, so $\sup\{\|b\| : b \in B\} \leq r < \infty$.

$\underline{\impliedby}$: Suppose that U is an open neighborhood of 0. Then there exists some $\varepsilon > 0$ such that $B(0,\varepsilon) \subseteq U$. Letting $\delta = \sup\{\|b\| : b \in B\}$, it follows that

$$B \subseteq B(0,\delta) = \frac{\delta}{\varepsilon}B(0,\varepsilon) \subseteq \frac{\delta}{\varepsilon}U.$$

$\square$

(i): The translate of a bounded set is bounded.

Proof. Let $a \in X$. As $\{a\}$ is compact, the previous part implies that $\{a\}$ is bounded. Let U be a neighborhood of 0 and V a neighborhood of 0 such that $V + V \subseteq U$. Let $\varepsilon_1, \varepsilon_2 > 0$ such that $\varepsilon_1 B \subseteq V$ and $\varepsilon_2 a \in V$. Letting $\delta = \max\{\frac{1}{\varepsilon_1}, \frac{1}{\varepsilon_2}\}$, it follows that

$$B + a \subseteq \frac{1}{\varepsilon_1}V + \frac{1}{\varepsilon_2}V \subseteq \delta V + \delta V \subseteq \delta U.$$

$\square$

Problem. Do IV-3 problems 1, 2, 6, 8-13.