MATH 7310 Homework 7

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Problem 1

Let (X, Σ, μ) be a measure space.
(i): Prove that if $\mu(E_n) < +\infty$ for $n \in \mathbb{N}$ and $\mathbb{1}_{E_n} \to f$ in L^1, then f is (a.e. equal to) the characteristic function of a measurable set.

(ii): Let $\Sigma_f = \{E \in \Sigma : \mu(E) < +\infty\}$. Define an equivalence relation on Σ_f by $E \sim F$ if $\mu(E\Delta F) = 0$. Let $\Omega = \Sigma_f / \sim$, and define a metric ρ on Ω be $\rho([E],[F]) = \mu(E\Delta F)$. Show that the map $\iota : \Omega \to L^1(X,\mu)$ given by $\iota([E]) = \mathbb{1}_E$ is an isometry with closed image.

(iii): Show that (Ω, ρ) is a complete metric space.

Problem 2

If X, Y are sets, and $f : X \to \mathbb{C}$, $g : Y \to \mathbb{C}$, we define $f \otimes g : X \times Y \to \mathbb{C}$ by $(f \otimes g)(x, y) = f(x)g(y)$. Fix $1 \leq p < +\infty$.

(a): Let $(X, \Sigma, \mu), (Y, \mathcal{F}, \nu)$ be σ-finite measure spaces. Show that if $f \in L^p(X, \mu), g \in L^p(Y, \nu)$, then $\|f \otimes g\|_p = \|f\|_p \|g\|_p$.

(b): Let $(Z, \mathcal{O}, \zeta)$ be a finite measure space. Suppose that $\mathcal{A} \subseteq \mathcal{O}$ is an algebra which generates the σ-algebra of $\mathcal{O}$. Use the monotone class lemma to show that $\{\mathbb{1}_A : A \in \mathcal{A}\}$ is dense in $\{\mathbb{1}_E : E \in \mathcal{O}\}$ in the L^p-norm for all $1 \leq p < +\infty$.

(c): Let $(X, \Sigma, \mu), (Y, \mathcal{F}, \nu)$ be finite measure spaces. Use the previous part to show that $\{\mathbb{1}_E : E \in \Sigma \otimes \mathcal{F}\} \subseteq \overline{\mathrm{Span}}^{\|\cdot\|_p}\{\mathbb{1}_E \otimes \mathbb{1}_F : E \in \Sigma, F \in \mathcal{F}\}$. Use this to show that $\overline{\mathrm{Span}}^{\|\cdot\|_p}\{\mathbb{1}_E \otimes \mathbb{1}_F : E \in \Sigma, F \in \mathcal{F}\} = L^p(X \times Y, \mu \otimes \nu)$.

(d): Let $(X, \Sigma, \mu), (Y, \mathcal{F}, \nu)$ be σ-finite measure spaces. Suppose that $D_X \subseteq L^p(X, \mu)$, $D_Y \subseteq L^p(Y, \nu)$ and that

$$\overline{\mathrm{Span}}^{\|\cdot\|_p}(D_X) = L^1(X, \mu), \quad \overline{\mathrm{Span}}^{\|\cdot\|_p}(D_Y) = L^1(Y, \nu).$$

Show that $\overline{\mathrm{Span}}^{\|\cdot\|_p}(\{f \otimes g : f \in D_X, g \in D_Y\}) = L^p(X \times Y, \mu \otimes \nu)$.

Problem 3

Suppose that $f \in L^p \cap L^\infty$ for some $p < +\infty$ so that $f \in L^q$ for all $q > p$. Prove that then $\|f\|_\infty = \lim_{q\to\infty} \|f\|_q$.

Problem 4

If f is a measurable function on X, define the *essential range* R_f of f to be the set of all $z \in \mathbb{C}$ such that $\{x : |f(x) - z| < \varepsilon\}$ has positive measure for all $\varepsilon > 0$.

(a): Prove that R_f is closed.

Proof. Let $z \in \overline{R_f}$. Then there exists a sequence $(z_n)_{n=1}^\infty$ in R_f such that $z_n \to z$. Fix $\varepsilon > 0$. There is some $N \in \mathbb{N}$ such that $n \geq N \implies B_{\varepsilon/2}(z_n) \subseteq B_\varepsilon(z)$. Then $f^{-1}(B_{\varepsilon/2}(z_n)) \subseteq f^{-1}(B_\varepsilon(z))$, whence $0 < \mu(f^{-1}(B_{\varepsilon/2}(z_n))) \leq \mu(f^{-1}(B_\varepsilon(z)))$. Hence $z \in R_f$, so R_f is closed. $\square$

(b): Prove that if $f \in L^\infty$, then R_f is compact and $\|f\|_\infty = \max\{|z| : z \in R_f\}$.

Problem 5

Suppose that $1 \leq p < +\infty$ and $(f_n)_{n=1}^\infty$ in L^p. Prove that $(f_n)_{n=1}^\infty$ is Cauchy in the L^p-norm if and only if the following three conditions hold:

1. (f_n) is Cauchy in measure;
2. the sequence $(|f_n|^p)_{n=1}^\infty$ is uniformly integrable
3. for every $\varepsilon > 0$ there exists $E \subseteq X$ such that $\mu(E) < +\infty$ and $\int_{E^c} |f_n|^p \, d\mu < \varepsilon$ for all $n \in \mathbb{N}$.

Lemma 1. *Any finite subset $\{f_k\}_{k=1}^n \subseteq L^1(\mu)$ is uniformly integrable.*

Proof of Lemma 1. We show first that $f \in L^1(\mu)$ is uniformly integrable. Note that, if $f \in L^1(\mu)$, then $|f|\mathbb{1}_{\{|f|>m\}} \searrow 0$ pointwise a.e. as $\{|f| = +\infty\} = \bigcap_{M\in\mathbb{N}} \{|f| > M\}$ implies that $\lim_{M\to\infty} \mu(|f| > M) = \mu(\{|f| = +\infty\}) = 0$. Moreover, for all $M \in \mathbb{N}$, $|f\mathbb{1}_{|f|>M}| \leq |f| \in L^1(\mu)$, so by the dominated convergence theorem

$$\lim_{M\to\infty} \int_{\{|f|>M\}} |f| \, d\mu = 0. \tag{1}$$

For any $E \subseteq X$ measurable and $M \in \mathbb{N}$, we have that

$$\int_E |f| \, d\mu = \int_{E\cap\{|f|\leq M\}} |f| \, d\mu + \int_{E\cap\{|f|>M\}} |f| \, d\mu \leq M \cdot \mu(E) + \int_{\{|f|>M\}} |f| \, d\mu \,. \tag{2}$$

Fix $\varepsilon > 0$. By (1), there exists some $N \in \mathbb{N}$ such that $\int_{\{|f|>N\}} |f| \, d\mu < \frac{\varepsilon}{2}$. Choose $\delta = \frac{\varepsilon}{2N}$. Then, for any $E \subseteq X$ measurable such that $\mu(E) < \delta$, we have by (2) that

$$\left| \int_E f \, d\mu \right| \leq \int_E |f| \, d\mu < N \cdot \delta + \frac{\varepsilon}{2} = \varepsilon.$$

Now suppose that $\{f_k\}_{k=1}^n \subseteq L^1(\mu)$ is a finite subset of $L^1(\mu)$. Fix $\varepsilon > 0$. By uniform integrability of each of the singletons, for each $k \in \{1, \ldots, n\}$ there exists a $\delta_k > 0$ such that $\mu(E) < \delta_k \implies |\int_E f_k| < \varepsilon$. Choosing $\delta = \min\{\delta_1, \ldots, \delta_n\} > 0$, the claim follows. $\square$

Lemma 2. *Suppose $(f_n)_{n=1}^\infty$ is a sequence in $L^1(\mu)$ and $f \in L^1(\mu)$ such that $\|f_n - f\|_1 \xrightarrow{n\to\infty} 0$. Then $\{f_n\}_{n=1}^\infty$ is uniformly integrable.*

Proof of Lemma 2. Observe that, for any measurable $E \subseteq X$ and $n \in \mathbb{N}$,

$$\int_E |f_n|\,d\mu \leq \int_E |f|\,d\mu + \int_E |f_n - f|\,d\mu \leq \int_E |f|\,d\mu + \|f_n - f\|_1.$$

Fix $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that for $n \geq N$ we have $\|f_n - f\|_1 < \frac{\varepsilon}{2}$. By Lemma 1, $\{f\}$ is uniformly integrable, so there is some $\delta' > 0$ such that $\mu(E) < \delta'$ implies that $\int_E |f|\,d\mu < \frac{\varepsilon}{2}$.

Again by Lemma 1, $\{f_k\}_{k=1}^{N-1}$ is uniformly integrable, so there is some $\delta'' > 0$ such that $\mu(E) < \delta''$ implies $\int_E |f_k|\,d\mu < \varepsilon$ for all $k \in \{1, \ldots, N-1\}$. Setting $\delta = \min\{\delta', \delta''\}$, the claim follows. $\square$

Proof of Theorem.
$\underline{\implies}$: Suppose that $(f_n)_{n=1}^\infty$ is Cuachy in the L^p-norm. Then by completeness, there is some $f \in L^p(\mu)$ such $\|f - f_n\|_p \xrightarrow{n\to\infty} 0$. For $\varepsilon > 0$, noting that $\{|f_n - f| \geq \varepsilon\} = \{|f_n - f|^p/\varepsilon^p \geq 1\}$, we have that

$$\mu(\{|f_n - f| \geq \varepsilon\}) = \int_{\{|f_n - f| \geq \varepsilon\}} \frac{|f_n - f|^p}{\varepsilon^p}\,d\mu \leq \frac{1}{\varepsilon^p}\|f_n - f\|_p^p \xrightarrow{n\to\infty} 0.$$

Thus $f_n \to f$ in measure, whence $(f_n)_{n=1}^\infty$ is Cauchy in measure.

By the reverse triangle inequality,

$$\left|\|f_n\|_p - \|f\|_p\right| \leq \|f_n - f\|_p \xrightarrow{n\to\infty} 0,$$

so $\||f_n|^p\|_1 \xrightarrow{n\to\infty} \||f|^p\|_1 \in L^1(\mu)$. Now by Lemma 2, $(|f_n|^p)_{n=1}^\infty$ is uniformly integrable.

$\square$

Problem 6

Prove that if E is a subset of a Hilbert space $\mathcal{H}$, then $(E^\perp)^\perp$ is the smallest closed subspace of $\mathcal{H}$ containing E.

Claim. If M is a closed linear subspace of $\mathcal{H}$, then $(M^\perp)^\perp = M$.

Proof of Claim. Note that we have $\mathcal{H} = M \oplus M^\perp$. Let $y \in (M^\perp)^\perp$. Then there exist unique $x \in M$, $x^\perp \in M^\perp$ such that $y = x + x^\perp$. Noting that $M \subseteq (M^\perp)^\perp$, we have that $x^\perp = y - x \in M^\perp \cap (M^\perp)^\perp = \{0\}$, whence $x^\perp = 0$ and $y = x \in M$. Thus $M = (M^\perp)^\perp$. $\square$

Proof. On one hand, note that $E \subseteq \overline{\mathrm{Span}(E)} \implies (E^\perp)^\perp \subseteq (\overline{\mathrm{Span}(E)}^\perp)^\perp \overset{\text{claim}}{=} \overline{\mathrm{Span}(E)}$. On the other hand, as $(E^\perp)^\perp$ is a closed linear subspace of $\mathcal{H}$ and $E \subseteq (E^\perp)^\perp$, it follows that $\overline{\mathrm{Span}(E)} \subseteq (E^\perp)^\perp$. $\square$