MATH 7752 - HOMEWORK 4
DUE FRIDAY 02/18/22

(1) Let V and W be finite dimensional vector spaces over a field F. Let $\{v_1, \ldots, v_n\}$, $\{w_1, \ldots, w_m\}$ be bases of V, W respectively. Consider the F-linear transformation $\varphi : V \otimes_F W \to M_{n\times m}(F)$ defined by $\varphi(v_i \otimes w_j) = e_{ij}$, where e_{ij} is the matrix with 1 at the (i, j)-entry and 0 elsewhere.

- (a) Verify that such a linear transformation exists and is in fact an isomorphism of F-vector spaces.
- (b) Prove that for every $A \in M_{n\times m}(F)$ the following statements are equivalent:
  - (i) There exists some $v \in V, w \in W$ such that $A = \varphi(v \otimes w)$ (v, w need not be basis elements).
  - (ii) $\mathrm{rk}(A) \leq 1$.

(2) Let $R = \oplus_{n=0}^{\infty} R_n$ be a graded ring. Recall that an element $r \in R$ is called *homogeneous* if $r \in R_n$, for some $n \geq 0$. Notice that every $r \in R$ can be written uniquely as $r = \displaystyle\sum_{n=0}^{\infty} r_n$, where $r_n \in R_n$ and all but finitely many r_n's are equal to zero. The elements $\{r_n\}$ are called *the homogeneous components* of r.

- (a) Let I be an ideal of R. Prove that the following statements are equivalent:
  - (i) I is a graded ideal, i.e. $I = \oplus_{n=0}^{\infty}(I \cap R_n)$.
  - (ii) For each $r \in I$, all homogeneous components of r lie also in I.
- (b) Let I be an ideal of R generated by homogeneous elements. Prove that I is graded.

(3)
- (a) Let R be a PID. Prove that R is Noetherian.
- (b) Let R be a commutative ring and M be an R-module. Recall that M is called *Noetherian* if every ascending chain $M_1 \subset M_2 \subset \cdots M_n \subset \cdots$ of submodules of M eventually stabilizes.
  - (i) Let N be a submodule of M. Prove that the following are equivalent:
    - (i) M is Noetherian.
    - (ii) N and M/N are both Noetherian.
  - (ii) Let R be a commutative Noetherian ring. Use (a) to prove that R^n is Noetherian, for every $n \geq 1$.
  - (iii) Prove that if R is Noetherian, then every submodule of a finitely generated R-module is finitely generated.

(4) Let A be a ring (with 1) and B be a subring of A. The ring B is called *a retract* of A if there exists a surjective ring homomorphism, $\varphi : A \to B$ such that $\varphi|_B = 1_B$.

Let M and N be R-modules. Prove that the tensor algebra $T(M)$ is (naturally isomorphic to) a subalgebra of $T(M \oplus N)$ and this subalgebra is a retract. Prove that the same is true for symmetric algebras.

(5) Finish the proof we started in class. Namely: Let V be a F-vector space of dimension n, where F is a field. Let $\varphi : V \to V$ be a F-linear transformation. Consider the linear transformation $\Phi_{ext,n} : \wedge^n(V) \to \wedge^n(V)$ induced by φ. Prove that $\Phi_{ext,n}$ is given by scalar multiplication by $\det(\varphi)$.