Reading:

- For this homework: 5.5/3.1-3.2
- For Wedneday, March 30: 3.2-3.4
- For Monday, April 4: 3.4-3.5

Problem 1.

Folland, Chapter 5, Problem 55

Problem 2.

For $n \in \mathbb{Z}$, define $e_n : [0,1] \to \mathbb{C}$ by $e_n(t) = e^{2\pi i n t}$.

(a) Show that $\{e_n\}_{n\in\mathbb{Z}}$ is an orthonormal set in $L^2([0,1])$.
(b) Show that $\{f \in C([0,1]) : f(1) = f(0)\} = \{g \circ e_1 : g \in C(S^1)\}$, where $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. (Here $C(X)$ is the space of continuous complex valued functions on X when (X,d) is a metric space).
(c) The Stone-Weierestrass theorem says that if (X,d) is a compact metric space, and $A \subseteq C(X)$ is a linear subspace so that:
  - $1 \in A$,
  - $f \in A$ implies $\overline{f} \in A$,
  - $f, g \in A$ implies that $fg \in A$ (the mutliplication here is pointwise multiplication),
  - If $x \in X$, then there are $f, g \in A$ with $f(x) \neq g(x)$.

  then A is dense in $C(X)$ for the uniform norm $\|f\|_u = \sup_{x\in X} |f(x)|$. Use the Stone-Weierestrass theorem to show that $\overline{\mathrm{Span}\{e_n : n \in \mathbb{Z}\}}^{\|\cdot\|_u} = \{f \in C([0,1]) : f(1) = f(0))\}$.
(d) Show that $\mathrm{Span}\{e_n : n \in \mathbb{Z}\}$ is dense in $L^2([0,1])$ and use this to show that $\{e_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2([0,1])$. (Remark: one of our equivalent conditions in class for an orthonormal set to be a basis will be easier to apply).

Problem 3.

(a) Folland Chapter 5, Problem 60.
(b) For $k \in \mathbb{N}$, and $n = (n_1, \cdots, n_k) \in \mathbb{Z}^k$, define $e_n \in L^2([0,1]^d)$ by

$$e_n(x) = \prod_{j=1}^{k} e^{2\pi i n_j x}.$$

Show that $\{e_n\}_{n\in\mathbb{Z}^d}$ is an orthonormal basis of $L^2([0,1]^d)$.

Problem 4.

(a) Folland Chapter 3, Problem 17.
(b) Show that $\int gh\, d\nu = \int fh\, d\mu$ for all $h \in L^1(\nu)$.

Note: depending upon how you solve the first part, the second might be short.

Problem 5.

Let (X, Σ, μ) be a probability space. For a sub-σ-algebra $\mathcal{F} \subseteq \Sigma$, and $f \in L^1(X, \Sigma, \mu)$, let $\mathbb{E}_{\mathcal{F}}(f)$ be the conditional expectaion of f onto $\mathcal{F}$.

(a) Show that $\mathbb{E}_{\mathcal{F}}(fg) = \mathbb{E}_{\mathcal{F}}(f)g$ for all $g \in L^\infty(X, \mathcal{F}, \mu)$.

(b) If $f \in L^2(X, \Sigma, \mu)$, show that $\mathbb{E}_{\mathcal{F}}(f)$ is the orthogonal projection of f onto $L^2(X, \mathcal{F}, \mu)$ in the decomoposition

$$L^2(X, \Sigma, \mu) = L^2(X, \mathcal{F}, \mu) + L^2(X, \mathcal{F}, \mu)^{\perp}.$$

Note: one difficulty you'll a priori face is that we do not yet know that $f \in L^2$ implies that $\mathbb{E}_{\mathcal{F}}(f) \in L^2$. However, one can note that you can characterize the orthogonal projection g of f onto $L^2(X, \mathcal{F}, \mu)$ by $\langle f, h\rangle = \langle g, h\rangle$ for all $h \in L^2(X, \mathcal{F}, \mu)$ (you should prove this if you use it), and this can be used to show that this projection is the conditional expectation.

Problem 6.

Folland, Chapter 3, Problem 2
Problems to think about, do not turn in

Problem 7.

Folland, Chapter 3, Problems 3-7, 8, 11-14.
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