MATH 7310 Homework 6

James Harbour

April 19, 2022

Problem 1

Let $X = Y$ be an uncountable linearly ordered set such that for each $x \in X$, $\{y \in X : y < x\}$ is countable. Let $\mathcal{M} = \mathcal{N}$ be the σ-algebra of countable or co-countable sets, and let $\mu = \nu$ be defined on $\mathcal{M}$ by $\mu(A) = 0$ if A is countable and $\mu(A) = 1$ if A is co-countable. Let $E = \{(x, y) \in X \times X : y < x\}$. Prove that E_x and E^y are measurable for all x, y, and that $\int\int \mathbb{1}_E \, d\mu \, d\nu$ and $\int\int \mathbb{1}_E \, d\nu \, d\mu$ exist but are not equal.

Proof. For $x \in X$, define the set $S(x) = \{y \in X : y < x\}$. Observe that, for $x \in X$, $E_x = \{y \in X : (x, y) \in E\} = \{y \in X : y < x\} = S(x)$ which is countable by assumption so E_x is measurable. On the other hand, for $y \in X$, since the ordering on X is total,

$$X \setminus E^y = \{x \in X : y \notin S(x)\} = \{x \in X : x = y \text{ or } x < y\} = \{y\} \cup S(y)$$

which is countable by assumption, so E^y is cocountable and thus measurable.

Thus, for $x, y \in X$, the x and y-sections of $\mathbb{1}_E$, i.e. $(\mathbb{1}_E)^y = \mathbb{1}_{E^y}$ and $(\mathbb{1})_x = \mathbb{1}_{E_x}$, are measurable. Thus, the inner integrals in each of the iterated integrals exist. To see that both of the whole iterated integrals exist, we compute for fixed $y \in X$

$$\int \mathbb{1}_E(x, y) \, d\mu(x) = \int \mathbb{1}_{E^y}(x) \, d\mu(x) = \mu(E^y) = 1$$

and for fixed $x \in X$

$$\int \mathbb{1}_E(x, y) \, d\nu(y) = \int \mathbb{1}_{E_x}(y) \, d\nu(y) = \nu(E_x) = 0$$

which are both measurable functions as they are constant functions. Hence, both of the interated integrals exist and we compute on one hand that

$$\int\int \mathbb{1}_E(x, y) \, d\mu(x) \, d\nu(y) = \int \mu(E^y) \, d\nu(y) = \int 1 \, d\nu(y) = \nu(X) = 1$$

and on the other hand that

$$\int\int \mathbb{1}_E(x, y) \, d\nu(y) \, d\mu(x) = \int \nu(E_x) \, d\mu(x) = \int 0 \, d\mu(x) = 0.$$

Thus $\int\int \mathbb{1}_E \, d\mu \, d\nu$ and $\int\int \mathbb{1}_E \, d\nu \, d\mu$ exist but are not equal. $\square$

Problem 2

Prove Theorem 2.39 by using Theorem 2.37 and proposition 2.12 together with the following lemmas:

(a) If $E \in \mathcal{M} \otimes \mathcal{N}$ and $\mu \otimes \nu(E) = 0$, then $\nu(E_x) = \mu(E^y) = 0$ for a.e. x and y.

(b) If f is $\mathscr{L}$-measurable and $f = 0$ λ-a.e., then f_x and f^y are integrable for a.e. x and y, and $\int f_x \, d\nu = 0$ and $\int f^y \, d\mu = 0$ for a.e. x and y. (This uses completeness of μ and ν.)

Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are complete, σ-finite measure spaces, and let $(X \times Y, \mathscr{L}, \lambda)$ be the completion of $(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \otimes \nu)$. If f is $\mathscr{L}$-measurable and either (a) $f \geq 0$ or (b) $f \in L^1(\lambda)$, then f_x is $\mathcal{N}$-measurable for a.e. x and f^y is $\mathcal{M}$-measurable for a.e. y, and in case (b) f_x and f^y are also integrable for a.e. x and y. Moreover, $x \mapsto \int f_x \, d\nu$ and $y \mapsto \int f^y \, d\mu$ are measurable, and in case (b) also integrable, and

$$\int f \, d\lambda = \int \int f(x,y) \, d\mu(x) \, d\nu(y) = \int \int f(x,y) \, d\nu(y) \, d\mu(x) \,.$$

Proof of Lemma (a). By theorem 2.36, the functions $x \mapsto \nu(E_x)$ and $y \mapsto \mu(E^y)$ are measurable and

$$0 = \mu \otimes \nu(E) = \int \nu(E_x) \, d\mu(x) = \int \mu(E^y) \, d\nu(y) \,,$$

whence by nonnegativity $\nu(E_x) = 0$ for a.e. x and $\mu(E^y) = 0$ for a.e. y. □

Proof of Lemma (b). Consider the set $E = \{(x,y) : f(x,y) \neq 0\}$. Then, by definition, there exists an $F \in \mathcal{M} \otimes \mathcal{N}$ such that $E \subseteq F$ and $\mu \otimes \nu(F) = 0$. Then, for all $x \in X$ and $y \in Y$, $F_x \in \mathcal{N}$ and $F^y \in \mathcal{M}$. As $E_x \subseteq F_x$ and $E^y \in F^y$, the completeness of μ and ν gives that $E_x \in \mathcal{N}$ and $E^y \in \mathcal{M}$. Combining this with lemma (a), we find that $\nu(E_x) = 0$ and $\mu(E^y) = 0$. Now, for $x \in X$, $f_x = 0$ on $Y \setminus E_x$ whence $f_x = 0$ ν-a.e. and for $y \in Y$, $f^y = 0$ on $X \setminus E^y$ whence $f^y = 0$ μ-a.e. Thus f^y and f_x are integrable and both integrate to zero. □

Proof of Theorem 2.39. Suppose that f is $\mathscr{L}$-measurable. By proposition 2.12, there exists an $\mathcal{M} \otimes \mathcal{N}$-measurable function g such that $f = g$ λ-a.e. As $\mathscr{L} \supseteq \mathcal{M} \otimes \mathcal{N}$, it follows that g is $\mathscr{L}$-measurable so $f - g$ is $\mathscr{L}$-measurable and equal to 0 λ-a.e.

By lemma (b), $f_x - g_x$ and $f^y - g^y$ are integrable for a.e. x and y and $\int f_x - g_x \, d\nu = 0$ and $\int f^y - g^y \, d\mu = 0$ for a.e. x and y. Now the claims for parts (a) and (b) of the theorem follow from their corresponding parts in Theorem 2.37. □

Problem 3

(a): Suppose (X, Σ, μ) is a σ-finite measure space and $f \in L^+(X)$. Let

$$G_f = \{(x,y) \in X \times [0, +\infty] : y \leq f(x)\}.$$

Show that G_f is $\Sigma \times \mathcal{B}_{\mathbb{R}}$-measurable and $\mu \times m(G_f) = \int f \, d\mu$. Show also that the same is true if the inequality in the definition of G_f is made strict.

Proof. Let $\tilde{f} : X \times [0, +\infty] \to X \times [0, +\infty]$ be given by $(x,y) \mapsto (f(x), y)$ and $S : X \times [0, +\infty] \to [-\infty, +\infty]$ be given by $S(z,y) = z - y$ if z, y not both $\pm\infty$ and $S(z,y) = 0$ if $z = y = \infty$. Then S is measurable, and as $\pi_1 \circ \tilde{f}$ and $\pi_2 \circ \tilde{f}$ are measurable, so is $\tilde{f}$. Hence, as intermediate codomain and domain match for the corresponding measure spaces, $S \circ \tilde{f}$ is measurable. Noting that $G_f = (S \circ \tilde{f})^{-1}([0, +\infty])$, measurability of

$S \circ \tilde{f}$ implies that G_f is $\Sigma \times \mathcal{B}_{\mathbb{R}}$-measurable.

Observe that, for $x \in X$, $m((G_f)_x) = m([0, f(x)]) = f(x)$. As G_f is measurable, by Theorem 2.36 in Folland, the function $x \mapsto m((G_f)_x)$ is measurable and

$$\mu \times m(G_f) = \int m((G_f)_x)\, d\mu(x) = \int f(x)\, d\mu(x)\,.$$

$\square$

(b): Let (X, μ) be a σ-finite measure space. Fix $p \in [1, +\infty)$. Show that if $f \in L^p(X, \mu)$, then

$$\|f\|_p^p = p \int_0^\infty t^{p-1} \mu(\{x : |f(x)| > t\})\, dt\,.$$

Proof. Observe that, by part (a),

$$\|f\|_p^p = \int_X |f|^p\, d\mu = (\mu \times m)(G_{|f|^p}) = \int_{X\times[0,+\infty]} \mathbb{1}_{G_{|f|^p}}(x,t)\, d(\mu \times m)\,(x,t)$$

As $\|f\|_p^p < +\infty$, it follows that $\mathbb{1}_{G_{|f|^p}} \in L^1(X\times[0,+\infty], \mu\times m)$ whence by Fubini's theorem $(\mathbb{1}_{G_{|f|^p}})^t \in L^1(X, \mu)$ for almost every $t \in [0, +\infty]$, the a.e. defined function $\int (\mathbb{1}_{G_{|f|^p}})^t\, d\mu \in L^1([0, +\infty], m)$, and

$$\begin{aligned}
\|f\|_p^p = \int_{X\times[0,+\infty]} \mathbb{1}_{G_{|f|^p}}\, d(\mu \times m) &= \int_0^\infty \left[\int_X (\mathbb{1}_{G_{|f|^p}})^t(x)\, d\mu(x)\right] dt \\
&= \int_0^\infty \left[\int_X (\mathbb{1}_{(G_{|f|^p})^t})(x)\, d\mu(x)\right] dt = \int_0^\infty \mu((G_{|f|^p})^t)\, dt \\
&= \int_0^\infty \mu(\{x : |f(x)|^p < t\})\, dt
\end{aligned}$$

Consider the functions $F : [0, +\infty] \to [0, \infty]$ and $\phi : [0, +\infty] \to [0, +\infty]$ given by $F(t) = \mu(\{x : |f(x)|^p > t\})$ and $\phi(t) = t^p$. For t nonnegative, observe that $\{x : |f(x)|^p > t^p\} = \{x : |f(x)| > t\}$, so $(F \circ \phi)(t) = \mu(\{x : |f(x)|^p > t\}) = \mu(\{x : |f(x)| > t\})$. Lastly, noting that F is measurable and ϕ is a C^1-diffeomorphism, it follows that

$$\|f\|_p^p = \int_0^\infty F(t)\, dt = \int_0^\infty (F \circ \phi)(t)|\det D_t\phi|\, dt = p \int_0^\infty t^{p-1} \mu(\{x : |f(x)| > t\})\, dt\,.$$

$\square$

(c): Let (X, μ) be a σ-finite measure space. Show that if $f, g \in L^1(X, \mu)$ with $0 \leq f, g$ a.e., then

$$\|f - g\|_1 = \int_0^\infty \mu(\{x : f(x) > t\}\Delta\{x : g(x) > t\})\, dt\,.$$

Suggestion: it might be helpful to first show that for $a, b \in [0, +\infty)$ we have

$$|a - b| = \int_0^\infty |\mathbb{1}_{(t,\infty)}(a) - \mathbb{1}_{(t,\infty)}(b)|\, dt$$

Proof.

$$\begin{aligned}\|f-g\|_1 = \int_X |f-g|\,d\mu &= \int_X \int_0^\infty |\mathbb{1}_{(t,+\infty)}(f(x)) - \mathbb{1}_{(t,+\infty)}(g(x))|\,dt\,d\mu(x) \\ &= \int_0^\infty \left[\int_X |\mathbb{1}_{f^{-1}((t,+\infty))}(x) - \mathbb{1}_{g^{-1}((t,+\infty))}(x)|\right] d\mu(x)\,dt \\ &= \int_0^\infty \left[\int_X \mathbb{1}_{f^{-1}((t,+\infty))\Delta g^{-1}((t,+\infty))}(x)\right] d\mu(x)\,dt \\ &= \int_0^\infty \mu(\{x : f(x) > t\}\Delta\{x : g(x) > t\})\,dt\end{aligned}$$

□

Problem 4

If f is Lebesgue integrable on $(0,a)$ and $g(x) = \int_x^a t^{-1} f(t)\,dt$, then g is integrable on $(0,a)$ and $\int_0^a g(x)\,dx = \int_0^a f(x)\,dx$.

Proof. Define a set $E = \{(x,t) \in (0,a)^2 : x < t\}$. This set is measurable. Then, for fixed $x \in (0,a)$, $\mathbb{1}_{(x,a)}(t) = \mathbb{1}_{E_x}(t) = (\mathbb{1}_E)_x(t)$. Then we compute,

$$\begin{aligned}\int_{(0,a)} |g(x)|\,dx = \int_{(0,a)} \left|\int_{(0,a)} t^{-1} f(t) \mathbb{1}_E(x,t)\,dt\right| dx &\leq \int_{(0,a)} \int_{(0,a)} |t^{-1} f(t)| \mathbb{1}_{E^t}(x)\,dx\,dt \\ = \int_{(0,a)} t^{-1} |f(t)| m(E^t)\,dt = \int_{(0,a)} |f(t)|\,dt < +\infty\end{aligned}$$

whence by Tonelli's theorem g is measurable. So, we may apply Fubini's theorem.

$$\begin{aligned}\int_{(0,a)} g(x)\,dx = \int_{(0,a)} \int_{(0,a)} t^{-1} f(t) \mathbb{1}_E(x,t)\,dt\,dx &= \int_{(0,a)} \int_{(0,a)} t^{-1} f(t) \mathbb{1}_{E^t}(x)\,dx\,dt \\ = \int_{(0,a)} t^{-1} f(t) m(E^t)\,dt = \int_{(0,a)} f(t)\,dt\end{aligned}$$

as desired. □

Problem 5

Let $\mathcal{E}_q$ be the set of products of the form $\Pi_{j=1}^d I_j$ where each I_j is an h-interval with the property that all of its finite endpoints are rational.

(a): Show that $\mathcal{E}_q$ is an elementary family which generates the Borel sets.

Proof. Intersections of rational intervals are rational intervals and intersections of products of sets are componentwise intersections, so $\mathcal{E}_q$ is closed under intersections.

The complement of an element of $\mathcal{E}_q$ is a a disjoint union of finitely many boxes. Thus, $\mathcal{E}_q$ is an elementary family.

Any h-box in $\mathbb{R}^d$ can be written as a countable union of elements of $\mathcal{E}_q$, so $\mathcal{E}_q$ generates the Borel sets in $\mathbb{R}^d$. □

(b): Suppose that μ is a Borel measure on $\mathbb{R}^d$ with $0 < \mu((0,1]^d) < +\infty$. If $\mu(E+x) = \mu(E)$ for every $x \in \mathbb{R}^d$, show that $\mu(E) = \mu((0,1])^d m(E)$ for every Borel $E \subseteq \mathbb{R}^d$.

Proof. □

Problem 6

Fix $d \in \mathbb{N}$.

(a): Let $s : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ be the map $s(x,y) = x + y$. Let μ, ν be finite, Borel measures on $\mathbb{R}^d$. Define $\mu * \nu = s_*(\mu \otimes \nu)$. Show that for every Borel $E \subseteq \mathbb{R}^d$ we have

$$\mu * \nu(E) = \int\int \mathbb{1}_E(x+y)\, d\mu(x)\, d\nu(y)$$

and

$$\int \mu(E-y)\, d\nu(y) = \mu * \nu(E) = \int \nu(E-x)\, d\mu(x).$$

Show as a consequence that

$$\mu * \nu(X) = \mu(X)\nu(X).$$

Proof. On one hand, by finiteness of the measures and measurability of E, we may apply Fubini's theorem to see that

$$\int\int \mathbb{1}_E(s(x,y))\, d\mu(x)\, d\nu(y) = \int\int \mathbb{1}_{s^{-1}(E)}(x,y)\, d\mu(x)\, d\nu(y) = \int\int \mathbb{1}_{s^{-1}(E)}\, d(\mu\otimes\nu) = \mu * \nu(E).$$

Moreover, noting that $(s^{-1}(E))^y = E - y$ and $(s^{-1}(E))_x = E - x$, theorem 2.36 gives that

$$\mu * \nu(E) = \mu \otimes \nu(s^{-1}(E)) = \int \nu(E-x)\, d\mu(x)$$

and

$$\mu * \nu(E) = \mu \otimes \nu(s^{-1}(E)) = \int \mu(E-y)\, d\nu(y)$$

It follows that

$$\mu * \nu(\mathbb{R}^d) = \int \mu(\mathbb{R}^d - y)\, d\nu(y) = \int \mu(R^d)\, d\nu(y) = \mu(R^d)\nu(R^d).$$

□

(b): Show that for finite, Borel measures μ, ν, η on $\mathbb{R}^d$ we have

$$(\mu * \nu) * \eta = \mu * (\nu * \eta).$$

Proof. Let $E \in \mathcal{B}_{\mathbb{R}^d}$. By the finiteness of the measures, we may apply Fubini freely, whence

$$\begin{aligned}((\mu*\nu)*\eta)(E) &= \int (\mu*\nu)(E-z)\, d\eta(z) = \int \left[\int \nu(E-z-x)\, d\mu(x)\right] d\eta(z) \\ &= \int \left[\int \nu(E-x-z)\, d\eta(z)\right] d\mu(x) = \int (\nu*\eta)(E-x)\, d\mu(x) = (\mu*(\nu*\eta))(E).\end{aligned}$$

□

(c): For $f, g \in L^1(\mathbb{R}^d)$ show that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(y)g(x-y)|\,dx\,dy = \|f\|_1\|g\|_1.$$

Explain why this implies that $y \mapsto f(y)g(x-y)$ is in $L^1(\mathbb{R}^d)$ for almost every $x \in \mathbb{R}^d$ and why if we set $f * g(x) = \int_{\mathbb{R}^d} f(y)g(x-y)\,dy$ then we have that $f * g \in L^1(\mathbb{R}^d)$ and

$$\|f * g\|_1 \leq \|f\|_1\|g\|_1.$$

Proof.

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(y)g(x-y)|\,dx\,dy = \int_{\mathbb{R}^d} |f(y)| \left[\int_{\mathbb{R}^d} |g(x-y)|\,dx\right] dy = \int_{\mathbb{R}^d} |f(y)| \left[\int_{\mathbb{R}^d} |g(x)|\,dx\right] dy = \|f\|_1\|g\|_1$$

As $f, g \in L^1(\mathbb{R}^d)$, $\|f\|_1\|g\|_1 < +\infty$ the above equality and Fubini's theorem give that the function $x \mapsto \int |f(y)g(x-y)|\,dy$ is $L^1(\mathbb{R}^d)$ for almost every $x \in \mathbb{R}^d$. Hence, $\{x : \int |f(y)g(x-y)|\,dy = +\infty\}$ is a null set, so $y \mapsto f(y)g(x-y)$ is in $L^1(\mathbb{R}^d)$ for almost every $x \in \mathbb{R}^d$. Thus,

$$\|f * g\|_1 = \int \left|\int f(y)g(x-y)\,dy\right| dx \leq \int\int |f(y)g(x-y)|\,dy\,dx = \|f\|_1\|g\|_1.$$

$\square$

(d): Adopt notation as in Problem 1 of HW5. Show that if $f, g \in L^1(R^d)$ are nonnegative than $(f\,dm) * (g\,dm) = f * g\,dm$ with m being the Lebesgue measure.

Proof. Let $E \subseteq \mathbb{R}^d$ be measurable and let $d\mu = f\,dm$, $d\nu = g\,dm$, and $d\lambda = f * g\,dm$. On one hand, we compute that

$$\lambda(E) = \int_E (f * g)(y)\,dy = \int_{\mathbb{R}^d} \mathbb{1}_E(y) \left[\int_{\mathbb{R}^d} f(x)g(y-x)\,dx\right] dy\,.$$

On the other hand, by part (a), we have that

$$\begin{aligned}
\mu * \nu(E) &= \int_{\mathbb{R}^d} \nu(E-x)\,d\mu(x) = \int_{\mathbb{R}^d} f(x) \left[\int_{E-x} g(y)\,dy\right] dx \\
&= \int_{\mathbb{R}^d} \left[\int_E f(x)g(y-x)\,dy\right] dx = \int_{\mathbb{R}^d} f(x) \int_{\mathbb{R}^d} \mathbb{1}_E(y) f(x)g(y-x)\,dy\,dx\,,
\end{aligned}$$

which by nonnegativity and applying Tonelli's theorem, it follows that these two computed quantities are in fact equal. $\square$

(e): Show that for $f, g, k \in L^1(\mathbb{R}^d)$ we have that

$$(f * g) * k = f * (g * k) \text{ almost everywhere.}$$

Proof. Applying associativity from part (b) and expanding using part (d), we observe that

$$\begin{aligned}
(f * g) * k\,dm &= (f * g\,dm) * k\,dm = (f\,dm * g\,dm) * k\,dm = f\,dm * (g\,dm * k\,dm) \\
&= f\,dm * (g * k)\,dm = f * (g * k)\,dm\,.
\end{aligned}$$

By uniqueness of Radon-Nikodym derivatives, it follows that $(f * g) * k = f * (g * k)$ almost everywhere. $\square$