

# MATH 7310 Homework 2

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## Problem 1

Let  $\mu$  be a finitely additive measure.

(a) Prove that  $\mu$  is a measure if and only if it is continuous from below as in Theorem 1.8c.

*Proof.* Theorem 1.8c shows the forward direction so it suffices to show the reverse direction. Suppose that  $\mu$  is continuous from below. Let  $(E_j)_{j=1}^\infty$  be a sequence of disjoint elements in the sigma algebra  $\mathcal{M}$  corresponding to  $\mu$ . Define a new sequence  $(F_n)_{n=1}^\infty$  in  $\mathcal{M}$  by  $F_n = \bigsqcup_{j=1}^n E_j$ . Then  $\bigsqcup_{n=1}^\infty E_n = \bigcup_{n=1}^\infty F_n$ . As  $(F_n)_{n=1}^\infty$  is an increasing sequence in  $\mathcal{M}$ , we have that

$$\mu\left(\bigsqcup_{n=1}^\infty E_n\right) = \mu\left(\bigcup_{n=1}^\infty F_n\right) = \lim_{n \rightarrow \infty} \mu(F_n) = \lim_{n \rightarrow \infty} \sum_{j=1}^n \mu(E_j) = \sum_{j=1}^\infty \mu(E_j),$$

so  $\mu$  is a measure. □

(b) If  $\mu(X) < \infty$ , prove that  $\mu$  is a measure if and only if it is continuous from above as in Theorem 1.8d.

*Proof.* Theorem 1.8d shows the forward direction so it suffices to show the reverse direction. Suppose that  $\mu$  is continuous from above. Let  $(E_j)_{j=1}^\infty$  be a sequence of disjoint elements in  $\mathcal{M}$ . Define a new sequence  $(F_n)_{n=1}^\infty$  in  $\mathcal{M}$  by  $F_n = \bigsqcup_{j=1}^n E_j$ . Observe that  $F_1^c \supset F_2^c \supset F_3^c \supset \dots$  is a decreasing sequence in  $\mathcal{M}$  with  $\mu(F_1^c) = \mu(X) - \mu(F_1) < +\infty$ . Hence, by continuity from above,

$$\begin{aligned} \mu\left(\bigsqcup_{j=1}^\infty E_j\right) &= \mu\left(\bigcup_{n=1}^\infty F_n\right) = \mu\left(X \setminus \bigcap_{n=1}^\infty F_n^c\right) = \mu(X) - \mu\left(\bigcap_{n=1}^\infty F_n^c\right) = \mu(X) - \lim_{n \rightarrow \infty} \mu(F_n^c) \\ &= \mu(X) - \lim_{n \rightarrow \infty} \mu\left(X \setminus \bigsqcup_{j=1}^n E_j\right) = \mu(X) - \lim_{n \rightarrow \infty} \left(\mu(X) - \mu\left(\bigsqcup_{j=1}^n E_j\right)\right) = \lim_{n \rightarrow \infty} \sum_{j=1}^n \mu(E_j) = \sum_{j=1}^\infty \mu(E_j), \end{aligned}$$

so  $\mu$  is a measure. □

## Problem 2

Let  $(X, \mathcal{M}, \mu)$  be a finite measure space.

(a) If  $E, F \in \mathcal{M}$  and  $\mu(E \Delta F) = 0$ , then  $\mu(E) = \mu(F)$ .

*Proof.* Observe that

$$0 = \mu(E \Delta F) = \mu((E \setminus F) \sqcup (F \setminus E)) = \mu(E \setminus F) + \mu(F \setminus E).$$

As  $\mu(E \setminus F), \mu(F \setminus E) \geq 0$ , it follows that  $\mu(E \setminus F), \mu(F \setminus E) = 0$ . Then as  $E = (E \setminus F) \sqcup (E \cap F)$  and  $F = (F \setminus E) \sqcup (F \cap E)$ ,  $\mu(E) = \mu(F)$ .  $\square$

**(b)** Say that  $E \sim F$  if  $\mu(E \Delta F) = 0$ ; show that  $\sim$  is an equivalence relation on  $\mathcal{M}$ .

*Proof.*

(Reflexivity): Note that  $E \Delta E = E \setminus E = \emptyset \implies \mu(E \Delta E) = 0$ , so  $E \sim E$ .

(Symmetry): Note that  $E \Delta F = (E \setminus F) \sqcup (F \setminus E) = F \Delta E$ , so  $E \sim F \implies F \sim E$ .

(Transitivity): Suppose that  $E \sim F$  and  $F \sim G$ . Observe that

$$\begin{aligned} E \setminus G &= ((E \setminus F) \sqcup (E \cap F)) \setminus G = ((E \setminus F) \setminus G) \cup ((E \cap F) \setminus G) \subseteq (E \setminus F) \cup (F \setminus G) \\ G \setminus E &= ((G \setminus F) \sqcup (G \cap F)) \setminus E = ((G \setminus F) \setminus E) \cup ((G \cap F) \setminus E) \subseteq (G \setminus F) \cup (F \setminus E) \end{aligned}$$

so by monotonicity and subadditivity,

$$\mu(E \Delta G) \leq \mu((E \setminus F) \cup (F \setminus G)) + \mu((G \setminus F) \cup (F \setminus E)) \leq \mu(E \setminus F) + \mu(F \setminus E) + \mu(F \setminus G) + \mu(G \setminus F) = \mu(E \Delta F) + \mu(F \Delta G) = 0$$

hence  $E \sim G$ .  $\square$

**(c)** For  $E, F \in \mathcal{M}$ , define  $\rho(E, F) = \mu(E \Delta F)$ . Then  $\rho(E, G) \leq \rho(E, F) + \rho(F, G)$ , and hence  $\rho$  defines a metric on the space  $\mathcal{M}/\sim$ .

*Proof.* Note that the inequality used in the proof of transitivity above held regardless of the assumptions that the symmetric differences were zero, whence

$$\rho(E, G) = \mu(E \Delta G) \leq \mu(E \Delta F) + \mu(F \Delta G) = \rho(E, F) + \rho(F, G).$$

$\square$

## Problem 3

Let  $\mathcal{A}$  be the collection of finite unions of sets of the form  $(a, b] \cap \mathbb{Q}$  where  $-\infty \leq a < b \leq +\infty$ .

**(i)** Show that  $\mathcal{A}$  is an algebra on  $\mathbb{Q}$ . (Use Proposition 1.7.)

*Proof.* Let  $\mathcal{E} = \{(a, b] \cap \mathbb{Q} : -\infty \leq a < b \leq +\infty\} \cup \{\emptyset\}$ . By Proposition 1.7, it suffices to show that  $\mathcal{E}$  is an elementary family.

Suppose  $(a, b] \cap \mathbb{Q}, (c, d] \cap \mathbb{Q} \in \mathcal{E}$ , with  $a < b$  and  $c < d$ . If  $b \leq c$ , then  $((a, b] \cap \mathbb{Q}) \cap ((c, d] \cap \mathbb{Q}) = \emptyset \in \mathcal{E}$ . If  $b > c$ , then  $((a, b] \cap \mathbb{Q}) \cap ((c, d] \cap \mathbb{Q}) = (c, b] \cap \mathbb{Q} \in \mathcal{E}$ .

Lastly, suppose that  $(a, b] \cap \mathbb{Q}$  with  $a < b$ . If  $a = -\infty$  and  $b = +\infty$ , then  $\mathbb{Q} \setminus ((-\infty, +\infty] \cap \mathbb{Q}) = \emptyset$ . If  $a = -\infty$  and  $b \neq +\infty$ , then  $\mathbb{Q} \setminus ((-\infty, b] \cap \mathbb{Q}) = (b, +\infty] \cap \mathbb{Q}$ . If  $a \neq -\infty$  and  $b = +\infty$ , then  $\mathbb{Q} \setminus ((a, +\infty] \cap \mathbb{Q}) = (-\infty, a] \cap \mathbb{Q}$ . Finally, if  $a \neq -\infty$  and  $b \neq +\infty$ , then  $\mathbb{Q} \setminus ((a, b] \cap \mathbb{Q}) = ((-\infty, a] \cap \mathbb{Q}) \sqcup ((b, +\infty] \cap \mathbb{Q})$ . So  $\mathcal{E}$  is an elementary family.  $\square$

**(ii)** Show that the  $\sigma$ -algebra generated by  $\mathcal{A}$  is  $\mathcal{P}(\mathbb{Q})$ .

*Proof.* As  $\mathcal{A} \subseteq \mathcal{P}(\mathbb{Q})$ , by minimality  $\Sigma(\mathcal{A}) \subseteq \mathcal{P}(\mathbb{Q})$ . Now take  $q \in \mathbb{Q}$ . Observe that  $(q - \frac{1}{n}, q] \cap \mathbb{Q} \in \mathcal{A}$  for all  $n \in \mathbb{N}$ , whence  $\{q\} = \bigcap_{n=1}^{\infty} (q - \frac{1}{n}, q] \cap \mathbb{Q} \in \Sigma(\mathcal{A})$ . Hence,  $\Sigma(\mathcal{A})$  contains all finite and countable subsets of  $\mathbb{Q}$ , so countability of  $\mathbb{Q}$  implies that  $\mathcal{P}(\mathbb{Q}) \subseteq \Sigma(\mathcal{A})$ .  $\square$

(ii) Define  $\mu_0$  on  $\mathcal{A}$  by  $\mu_0(\emptyset) = 0$  and  $\mu_0(A) = \infty$  for  $A \neq \emptyset$ . Prove that  $\mu_0$  is a premeasure on  $\mathcal{A}$ , and that there is more than one measure on  $\mathcal{P}(\mathbb{Q})$  whose restriction to  $\mathcal{A}$  is  $\mu_0$ .

*Proof.* To see that  $\mu_0$  is a premeasure, suppose that  $(A_j)_{j=1}^{\infty}$  is a sequence of pairwise disjoint elements of  $\mathcal{A}$  such that  $\bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$ . If  $A_j = \emptyset$  for all  $j \in \mathbb{N}$ , then  $\bigcup_{j=1}^{\infty} A_j = \emptyset$  whence  $\mu_0(\bigcup_{j=1}^{\infty} A_j) = 0 = \sum_{j=1}^{\infty} 0 = \sum_{j=1}^{\infty} \mu_0(A_j)$ . If there exists a  $k \in \mathbb{N}$  such that  $A_k \neq \emptyset$ , then  $A_k \subseteq \bigcup_{j=1}^{\infty} A_j \neq \emptyset$ , so  $\mu_0(\bigcup_{j=1}^{\infty} A_j) = +\infty = \sum_{j=1}^{\infty} \mu_0(A_j)$ .

On one hand, we have an outer measure

$$\mu_0^*(E) = \inf \left\{ \sum_{j=1}^{\infty} \mu_0(A_j) : A_j \in \mathcal{A}, E \subseteq \bigcup_{j=1}^{\infty} A_j \right\}$$

for  $E \in \mathcal{P}(\mathbb{Q})$ . Note that,  $\mu_0^*(E) = 0$  if  $E = \emptyset$  and  $\mu_0^*(E) = +\infty$  if  $E \neq \emptyset$ . Moreover, this outer measure is in fact a measure on  $E \in \mathcal{P}(\mathbb{Q})$  extending  $\mu_0$  by the same reasoning showing  $\mu_0$  is a premeasure, so let  $\mu = \mu_0^*$ .

On the other hand, consider the counting measure  $\nu : \mathcal{P}(\mathbb{Q}) \rightarrow [0, +\infty]$ . Note that, if  $A \in \mathcal{A}$  and  $A \neq \emptyset$ , then  $A$  must contain infinitely many elements, whence  $\nu(A) = \infty$ . Hence  $\nu$  agrees with  $\mu_0$  on  $\mathcal{A}$ . However,  $\nu$  has finite, nonzero value on finite, nonempty subsets of  $\mathbb{Q}$ , so  $\nu \neq \mu$ .  $\square$

## Problem 4

Let  $\mathcal{A}$  be an algebra, and let  $\mu : \mathcal{A} \rightarrow [0, +\infty]$  be a finitely additive measure.

(i) Suppose  $(A_j)_{j=1}^{\infty}$  are pairwise disjoint elements of  $\mathcal{A}$ , and that  $A = \bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$ . Show that

$$\mu(A) \geq \sum_{j=1}^{\infty} \mu(A_j).$$

*Proof.* Since  $\mu$  is finitely additive, it is also finitely subadditive. Then by monotonicity, for any  $n \in \mathbb{N}$ ,

$$\mu(A) \geq \mu \left( \bigcup_{j=1}^n A_j \right) = \sum_{j=1}^n \mu(A_j).$$

Hence, it follows that  $\mu(A) \geq \sum_{j=1}^{\infty} \mu(A_j)$ .  $\square$

(ii) Show that the following are equivalent:

1.  $\mu$  is a premeasure,
2.  $\mu \left( \bigcup_{j=1}^{\infty} A_j \right) \leq \sum_{j=1}^{\infty} \mu(A_j)$  for any sequence  $(A_j)_{j=1}^{\infty}$  with  $\bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$ ,
3. for any increasing sequence  $(E_j)_{j=1}^{\infty}$  in  $\mathcal{A}$  with  $\bigcup_{j=1}^{\infty} E_j \in \mathcal{A}$ , we have

$$\mu \left( \bigcup_j E_j \right) = \lim_{n \rightarrow \infty} \mu(E_n).$$

*Proof.*

(2  $\implies$  1): Suppose that  $(A_j)_{j=1}^\infty$  are pairwise disjoint elements of  $\mathcal{A}$  with  $A = \bigcup_{j=1}^\infty A_j \in \mathcal{A}$ . by part (i),  $\mu(\bigcup_{j=1}^\infty A_j) \geq \sum_{j=1}^\infty \mu(A_j)$ . On the other hand, by assumption  $\mu(\bigcup_{j=1}^\infty A_j) \leq \sum_{j=1}^\infty \mu(A_j)$ , so

$$\mu\left(\bigcup_{j=1}^\infty A_j\right) = \sum_{j=1}^\infty \mu(A_j).$$

Hence,  $\mu$  is a premeasure.

(1  $\implies$  3): Let  $(E_j)_{j=1}^\infty$  be an increasing sequence in  $\mathcal{A}$  with  $\bigcup_{j=1}^\infty E_j \in \mathcal{A}$ . Define a new sequence in  $\mathcal{A}$  by  $E'_1 = E_1$  and  $E'_j = E_j \setminus E_{j-1}$  for  $j \geq 2$ . Then,

$$\mu\left(\bigcup_{j=1}^\infty E_j\right) = \mu\left(\bigcup_{j=1}^\infty E'_j\right) = \sum_{j=1}^\infty \mu(E'_j) = \lim_{n \rightarrow \infty} \sum_{j=1}^n \mu(E'_j) = \lim_{n \rightarrow \infty} \mu(E_n).$$

(3  $\implies$  2): Suppose that  $(A_j)_{j=1}^\infty$  is a sequence in  $\mathcal{A}$  with  $\bigcup_{j=1}^\infty A_j \in \mathcal{A}$ . Then, by finite subadditivity (which follows from finite additivity),

$$\mu\left(\bigcup_{j=1}^\infty A_j\right) = \mu\left(\bigcup_{n=1}^\infty \bigcup_{j=1}^n A_j\right) = \lim_{n \rightarrow \infty} \mu\left(\bigcup_{j=1}^n A_j\right) \leq \lim_{n \rightarrow \infty} \sum_{j=1}^n \mu(A_j) = \sum_{j=1}^\infty \mu(A_j).$$

□

(iii) If  $\mu(X) < +\infty$ , show that  $\mu$  is a premeasure if and only if for every decreasing sequence  $(E_n)_{n=1}^\infty$  of sets in  $\mathcal{A}$  with  $\bigcap_{n=1}^\infty E_n = \emptyset$ , we have

$$\lim_{n \rightarrow \infty} \mu(E_n) = 0.$$

*Proof.*

$\implies$ : Let  $(E_n)_{n=1}^\infty$  be a decreasing sequence of sets in  $\mathcal{A}$  with  $\bigcap_{n=1}^\infty E_n = \emptyset$ . Note that then the sequence of sets  $(X \setminus E_n)_{n=1}^\infty$  is increasing, so by number 3 in part (ii) and utilizing finiteness of  $\mu(X)$ ,

$$\mu\left(\bigcup_{n=1}^\infty X \setminus E_n\right) = \lim_{n \rightarrow \infty} \mu(X \setminus E_n) = \lim_{n \rightarrow \infty} \mu(X) - \mu(E_n).$$

Hence,

$$0 = \mu\left(\bigcap_{n=1}^\infty E_n\right) = \mu\left(X \setminus \bigcup_{n=1}^\infty (X \setminus E_n)\right) = \mu(X) - \mu\left(\bigcup_{n=1}^\infty X \setminus E_n\right) = \mu(X) - \lim_{n \rightarrow \infty} (\mu(X) - \mu(E_n)) = \lim_{n \rightarrow \infty} \mu(E_n)$$

$\Leftarrow$ : Let  $(A_j)_{j=1}^\infty$  be a sequence of pairwise disjoint elements of  $\mathcal{A}$  such that  $A = \bigcup_{j=1}^\infty A_j \in \mathcal{A}$ . Define a new sequence in  $\mathcal{A}$  by  $E_n = A \setminus \bigcup_{j=1}^n A_j$ . Then  $(E_n)_{n=1}^\infty$  is a decreasing sequence in  $\mathcal{A}$  such that  $\bigcap_{n=1}^\infty E_n = \emptyset$ . Hence, by assumption,

$$0 = \lim_{n \rightarrow \infty} \mu(E_n) = \lim_{n \rightarrow \infty} \mu(A) - \mu\left(\bigcup_{j=1}^n A_j\right) = \mu(A) - \lim_{n \rightarrow \infty} \sum_{j=1}^n \mu(A_j) \implies \mu\left(\bigcup_{j=1}^\infty A_j\right) = \sum_{j=1}^\infty \mu(A_j),$$

so  $\mu$  is a premeasure.

□

## Problem 5

A *metric measure space* is a triple  $(X, d, \mu)$  where  $(X, d)$  is a metric space and  $\mu : \mathcal{B}_{(X,d)} \rightarrow [0, +\infty]$  is a measure. We say that  $E \subseteq X$  is a *continuity set* if  $\mu(\overline{E} \setminus \text{Int}(E)) = 0$ . For this problem, fix a metric measure space  $(X, d, \mu)$ .

(i) Show that the collection of continuity sets forms an algebra of sets.

*Proof.* Suppose that  $E_1, \dots, E_n \subseteq X$  are continuity sets. Then  $\mu(\overline{E_j} \setminus \text{Int}(E_j)) = 0$  for  $1 \leq j \leq n$ . As there are finitely many sets, the union of closures is equal to the closure of the union. Hence

$$\overline{\bigcup_{j=1}^n E_j} \setminus \text{Int}\left(\bigcup_{j=1}^n E_j\right) = \bigcup_{j=1}^n \overline{E_j} \setminus \text{Int}\left(\bigcup_{j=1}^n E_j\right) \subseteq \bigcup_{j=1}^n \overline{E_j} \setminus \bigcup_{j=1}^n \text{Int}(E_j) = \bigcup_{j=1}^n \overline{E_j} \setminus \text{Int}(E_j),$$

so by subadditivity,

$$\mu\left(\overline{\bigcup_{j=1}^n E_j} \setminus \text{Int}\left(\bigcup_{j=1}^n E_j\right)\right) = \mu\left(\bigcup_{j=1}^n \overline{E_j} \setminus \text{Int}(E_j)\right) \leq \sum_{j=1}^n \mu(\overline{E_j} \setminus \text{Int}(E_j)) = 0$$

whence  $E_1 \cup \dots \cup E_n$  is a continuity set. Now suppose that  $E \subseteq X$  is a continuity set.

$$(\overline{X \setminus E}) \setminus \text{Int}(X \setminus E) = (X \setminus \text{Int}(E)) \setminus \text{Int}(X \setminus E) = (X \setminus \text{Int}(E)) \setminus (X \setminus \overline{E}) = \overline{E} \setminus \text{Int}(E)$$

so  $\mu((\overline{X \setminus E}) \setminus \text{Int}(X \setminus E)) = \mu(\overline{E} \setminus \text{Int}(E)) = 0$ , whence  $X \setminus E$  is also a continuity set.  $\square$

(ii) Show that if  $x \in X$ ,  $r > 0$  and  $\mu(B_r(x, d)) < +\infty$ , then there is an  $s \in (0, r)$  so that  $B_s(x, d)$  is a continuity set.

*Proof.* We show the stronger statement that there are only countably many  $s \in (0, r)$  such that  $B_s(x, d)$  is not a continuity set.

Suppose, for the sake of contradiction, that there exists uncountably many counterexamples  $s \in (0, r)$  such that  $\mu(\overline{B_s(x)} \setminus \text{Int}(B_s(x))) \neq 0$ . For  $n \in \mathbb{N}$ , define a set

$$A_n = \left\{s \in (0, r) : \frac{1}{n} \leq \mu(\overline{B_s(x)} \setminus \text{Int}(B_s(x))) < \frac{1}{n-1}\right\}$$

where  $1/0 := \infty$  by convention. Then, by assumption,  $\bigcup_{n=1}^{\infty} A_n$  is uncountable, so there exists an  $n \in \mathbb{N}$  such that  $A_n$  is infinite. Take a countably infinite subset  $\{s_1, s_2, \dots\} \subseteq A_n$ . Note that, for any fixed  $t \in (0, +\infty)$ ,  $\overline{B_t(x)} \setminus \text{Int}(B_t(x)) \subseteq \{y \in X : d(x, y) = t\}$ , whence the following union is disjoint:

$$\mu\left(\bigsqcup_{j=1}^{\infty} \overline{B_{s_j}(x)} \setminus \text{Int}(B_{s_j}(x))\right) = \sum_{j=1}^{\infty} \mu(\overline{B_{s_j}(x)} \setminus \text{Int}(B_{s_j}(x))) = \infty.$$

However, this contradicts that  $\mu(B_r(x)) < \infty$ .  $\square$

(iii) Suppose that  $(X, d)$  is separable and that for every  $x \in X$ , there is an  $r > 0$  so that  $\mu(B_r(x, d)) < +\infty$ . Show that there is a countable basis consisting of open continuity sets. (Hint: given a countable dense  $D \subseteq X$  and  $x \in D$ , use the preceding part to choose a countable set  $J_x \subseteq (0, +\infty)$  with the property that  $\inf_{t \in J_x} t = 0$  and so that  $B_t(x, d)$  is a continuity set for all  $t \in J_x$ ).

*Proof.* Let  $D \subseteq X$  be a countable dense subset of  $X$ . Fix  $x \in D$ . Then the set  $\{r \in (0, +\infty) : \mu(B_{r_x}(x, d)) < +\infty\}$  is nonempty, so define  $R_x = \sup\{r \in (0, +\infty) : \mu(B_{r_x}(x, d)) < +\infty\} \in [0, +\infty]$ . By part (ii), we may take  $J_x \subseteq (0, R_x)$  to be a countable, dense subset of  $(0, R_x)$  such that  $B_t(x, d)$  is a continuity set for all  $t \in J_x$ , whence  $\inf_{t \in J_x} t = 0$

$$\mathcal{J} = \bigsqcup_{x \in D} \{x\} \times J_x.$$

By the axiom of choice,  $\mathcal{J}$  is countable. Let  $\mathcal{B} = \{B_t(x, d) : (x, t) \in \mathcal{J}\}$ . We claim that  $\mathcal{B}$  is a basis for the metric topology on  $(X, d)$ . Let  $U \subseteq X$  be open and  $p \in U$ . Then there exist  $r, r' > 0$  such that  $B_r(p, d) \subseteq U$  and  $\mu(B_{r'}(p, d)) < +\infty$ . Take  $\varepsilon = \min\{r, r'\}$ , so  $B_\varepsilon(p, d) \subseteq U$  and  $\mu(B_\varepsilon(p, d)) < +\infty$ . By density, there exists an  $x \in D$  such that  $d(x, p) < \frac{\varepsilon}{4}$ . Note that  $B_{\frac{3\varepsilon}{4}}(x, d) \subseteq B_\varepsilon(p, d)$ , whence  $\mu(B_{\frac{3\varepsilon}{4}}(x, d)) < +\infty$  so  $\frac{2\varepsilon}{4} < \frac{3\varepsilon}{4} \leq R_x$ . Then, by density of  $J_x$  in  $(0, R_x)$ , there exists a  $t \in J_x$  such that  $|t - \frac{2\varepsilon}{4}| < \frac{\varepsilon}{100}$ . Then  $p \in B_t(x, d)$  and  $B_t(x, d)$  is a continuity set in  $U$ . □

## Problem 6

Let  $(X, d)$  be a metric space and  $\mu, \nu$  be finite Borel measures on  $X$  with  $\mu(X) = \nu(X)$ . Let  $\mathcal{A} = \{E \in \mathcal{B}_{(X, d)} : \mu(E) = \nu(E)\}$ .

(i) Show that if  $F \subseteq E$  and  $F, E \in \mathcal{A}$ , then  $E \setminus F \in \mathcal{A}$ . Also show that if  $(E_n)_{n=1}^\infty$  is an increasing sequence of elements of  $\mathcal{A}$ , then  $\bigcup_{n=1}^\infty E_n \in \mathcal{A}$ .

*Proof.* As  $E, F \in \mathcal{A}$ ,  $\mu(E) = \nu(E)$  and  $\mu(F) = \nu(F)$ . Then

$$\mu(E \setminus F) = \mu(E) - \mu(F) = \nu(E) - \nu(F) = \nu(E \setminus F)$$

so  $E \setminus F \in \mathcal{A}$ . Now suppose that  $(E_n)_{n=1}^\infty$  is an increasing sequence of elements of  $\mathcal{A}$ . By continuity from above,

$$\mu\left(\bigcup_{n=1}^\infty E_n\right) = \lim_{n \rightarrow \infty} \mu(E_n) = \lim_{n \rightarrow \infty} \nu(E_n) = \nu\left(\bigcup_{n=1}^\infty E_n\right),$$

so  $\bigcup_{n=1}^\infty E_n \in \mathcal{A}$ . □

(ii) Given a nonempty  $F \subseteq X$  closed and  $x \in X$ , define  $d(x, F) = \inf_{y \in F} d(x, y)$ . Show that  $x \mapsto d(x, F)$  is continuous and  $F = \{x \in X : d(x, F) = 0\}$ .

*Proof.* Suppose  $x, y \in X$ . For  $z \in F$ ,

$$d(x, F) \leq d(x, z) \leq d(x, y) + d(y, z) \implies d(x, F) - d(x, y) \leq d(y, z).$$

As this holds for arbitrary  $z \in F$ , it follows that  $d(x, F) - d(x, y) \leq d(y, F)$ , so  $d(x, F) - d(y, F) \leq d(x, y)$ . By symmetry,  $d(y, F) - d(x, F) \leq d(x, y)$ , so  $|d(x, F) - d(y, F)| \leq d(x, y)$ . Thus, the function  $x \mapsto d(x, F)$  is 1-Lipschitz whence it is continuous.

Clearly  $F \subseteq \{x \in X : d(x, F) = 0\}$ , so it suffices to show the reverse containment. Suppose that  $x \in X$  such that  $d(x, F) = 0$ . For all  $n \in \mathbb{N}$ , there exists an  $f_n \in F$  such that  $0 \leq d(x, f_n) < \frac{1}{n}$ . It follows that  $d(x, f_n) \xrightarrow{n \rightarrow \infty} 0$ , so  $f_n \xrightarrow{n \rightarrow \infty} x$ . Thus  $x$  is a limit point of  $F$ , so  $F$  being closed implies that  $x \in F$ . □

(iii) Show that  $\{U \subseteq X : U \text{ is open}\} \subseteq \mathcal{A}$  if and only if  $\{F \subseteq X : F \text{ is closed}\} \subseteq \mathcal{A}$ .

*Proof.*

$\implies$ : Suppose that  $\{U \subseteq X : U \text{ is open}\} \subseteq \mathcal{A}$ . Take  $F \subseteq X$  such that  $F$  is closed. Then  $X \setminus F$  is open, whence by finiteness of  $\mu$  and  $\nu$ ,

$$\mu(X) - \mu(F) = \mu(X \setminus F) = \nu(X \setminus F) = \nu(X) - \nu(F) \implies \mu(F) = \nu(F)$$

so  $F \in \mathcal{A}$ .

$\impliedby$ : Likewise, suppose that  $\{U \subseteq X : U \text{ is closed}\} \subseteq \mathcal{A}$ . Take  $U \subseteq X$  such that  $U$  is open. Then  $X \setminus U$  is closed, whence by finiteness of  $\mu$  and  $\nu$ ,

$$\mu(X) - \mu(U) = \mu(X \setminus U) = \nu(X \setminus U) = \nu(X) - \nu(U) \implies \mu(U) = \nu(U)$$

so  $U \in \mathcal{A}$ .

□