

MATH 7752 Homework 2

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Problem 1

Let D be a division ring (not necessarily commutative) and M be a D -module.

(a) Let X be a generating set of M and Y a D -linearly independent subset of X . Prove that M has a D -basis B with $Y \subseteq B \subseteq X$.

Proof. Consider the poset $\mathcal{S} = \{B \subseteq M : Y \subseteq B \subseteq X \text{ and } B \text{ is } D\text{-linearly independent}\}$ ordered by inclusion. Since Y is a D -linearly independent subset of X , we have that $Y \in \mathcal{S}$ so $\mathcal{S} \neq \emptyset$.

Suppose that $\mathcal{C} \subseteq \mathcal{S}$ is any linearly ordered chain in \mathcal{S} . Let $B = \bigcup \mathcal{C}$. Then $Y \subseteq B \subseteq X$. Suppose that $d_i \in D$ and $b_i \in B$ such that $\sum_{i=1}^n d_i \cdot b_i = 0$. Then for each $i \in \{1, \dots, n\}$, there exists a $B_i \in \mathcal{C}$ such that $b_i \in B_i$. As \mathcal{C} is a chain, there is some $l \in \{1, \dots, n\}$ such that $B_i \subseteq B_l$ for all $1 \leq i \leq n$. It follows that $b_i \in B_l$ for all $1 \leq i \leq n$, whence B_l being D -linearly independent implies that $d_i = 0$ for all i . Thus B is D -linearly independent, so $B \in \mathcal{S}$.

Now by Zorn's lemma, there exists a maximal element $B \in \mathcal{S}$ of \mathcal{S} . We claim that B is in fact a D -basis for M . It suffices to show that B is a generating set for M . Let $N = \text{span}_D(B)$. Suppose, for the sake of contradiction, that $N \neq M$. As $B \subseteq X$ and X is a generating set for M , it follows that there exists an $x \in X \setminus \text{span}_D(B)$. Suppose $r, r_1, \dots, r_n \in R$ are such that

$$0 = rx + r_1b_1 + \dots + r_nb_n.$$

If $r \neq 0$, then

$$x = (-r^{-1}r_1) \cdot b_1 + \dots + (-r^{-1}r_n) \cdot b_n,$$

which would imply that $x \in \text{span}_D(B)$, contradicting the choice of x . Hence $r = 0$, so B being D -linearly independent implies that $r_i = 0$ for all i . Thus $B \cup \{x\}$ is D -linearly independent, contradicting the maximality of B . \square

(b) Conclude that every non zero D -module M has a D -basis.

Proof. Since $M \neq 0$, there exists an $y \in M \setminus \{0\}$. It follows that the singleton $\{y\}$ is a D -linearly independent subset of M . On the other hand, $M = 1 \cdot M$, so the set M is a generating set of M . Applying part (a) to $X = M$ and $Y = \{y\}$, it follows that M has a D -basis. \square

Problem 2

Let R be a commutative domain. Let I be a non-principal ideal of R . Show that when I is considered as an R -module (by left multiplication), then I is indecomposable but not cyclic.

Proof. Since I is non-principal, by definition I is not cyclic as an R -module. Suppose, for the sake of contradiction, that $I = P \oplus Q$ for some nonzero proper R -submodules P, Q of I . Take $p \in P \setminus \{0\}$ and $q \in Q \setminus \{0\}$. Then $p \cdot q - q \cdot p = pq - qp = 0 \implies p \cdot q = q \cdot p$. As R is a domain $pq = qp \neq 0$, whence $pq = qp \in P \cap Q$ contradicts that the sum $P \oplus Q$ is direct. \square

Problem 3

Let R be a commutative ring. An R -module M is called *torsion* if for any $m \in M$ there exists some nonzero $r \in R$ such that $rm = 0$. An R -module N is called *divisible* if for any nonzero $r \in R$ it holds that $rN = N$.

(a) Suppose M is a torsion R -module and N is a divisible R -module. Prove that $M \otimes_R N = \{0\}$.

Proof. Let $m \in M$ and $n \in N$. Since M is torsion, there exists a nonzero $r \in R$ such that $rm = 0$. Now, by divisibility of N , there exists an $n' \in N$ such that $rn' = n$. Hence

$$m \otimes n = m \otimes rn' = rm \otimes n' = 0 \otimes n' = 0.$$

Thus every simple tensor in $M \otimes_R N$ is 0, whence $M \otimes_R N = 0$. \square

(b) Consider the \mathbb{Z} -module $M = \mathbb{Q}/\mathbb{Z}$. Prove that $M \otimes_{\mathbb{Z}} M = \{0\}$

Proof. We show that M is both torsion and divisible. Note that for any $\frac{p}{q} + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$, $q \neq 0$ and $q \cdot (\frac{p}{q} + \mathbb{Z}) = q \cdot \frac{p}{q} + \mathbb{Z} = p + \mathbb{Z} = \mathbb{Z}$, so \mathbb{Q}/\mathbb{Z} is torsion.

On the other hand, suppose $n \in \mathbb{Z} \setminus \{0\}$. For $\frac{p}{q} + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$, observe that

$$n \cdot \left(\frac{p}{nq} + \mathbb{Z} \right) = \frac{np}{nq} + \mathbb{Z} = \frac{p}{q} + \mathbb{Z}$$

so \mathbb{Q}/\mathbb{Z} is divisible. Appealing to part(a), it follows that $M \otimes_{\mathbb{Z}} M = 0$. \square

Problem 4

Let R be a PID and A be an R -module. Let K be the field of fractions of R , and consider the K -module $B = K \otimes_R A$. Prove that every $z \in B$ is a simple tensor.

Proof. Let $z \in B$. Then there exists $\frac{x_1}{s_1}, \dots, \frac{x_n}{s_n} \in K$, $a_1, \dots, a_n \in A$ and $c_1, \dots, c_n \in R$ such that

$$z = \sum_{i=1}^n c_i \cdot \left(\frac{x_i}{s_i} \otimes a_i \right) = \sum_{i=1}^n \frac{c_i x_i}{s_i} \otimes a_i = \sum_{i=1}^n \frac{c_i x_i \prod_{j \neq i} s_j}{s_1 \cdots s_n} \otimes a_i.$$

Since R is a PID, there exists an $s \in R$ such that $(s) = \langle c_i x_i \prod_{j \neq i} s_j : 1 \leq i \leq n \rangle$. Then for each $i \in \{1, \dots, n\}$, there is an $r_i \in R$ such that $c_i x_i \prod_{j \neq i} s_j = r_i s$. Hence,

$$z = \sum_{i=1}^n \frac{c_i x_i \prod_{j \neq i} s_j}{s_1 \cdots s_n} \otimes a_i = \sum_{i=1}^n \frac{r_i s}{s_1 \cdots s_n} \otimes a_i = \sum_{i=1}^n \frac{s}{s_1 \cdots s_n} \otimes r_i a_i = \frac{s}{s_1 \cdots s_n} \otimes \left(\sum_{i=1}^n r_i a_i \right)$$

is a simple tensor. \square

Problem 5

Let R be a commutative ring and M an R -module.

(a): Let I be an ideal of R . Prove an isomorphism

$$R/I \otimes M \simeq M/IM.$$

Proof. On one hand, consider the map $\tilde{\Psi} : M \rightarrow R/I \otimes_R M$ given by $\tilde{\Psi}(m) := (1 + I) \otimes m$ for $m \in M$. This map is clearly an R -module homomorphism by the second and third defining relations of the tensor product. For $i \in I$ and $m \in M$, $\tilde{\Psi}(im) = (1 + I) \otimes im = i \cdot (1 + I) \otimes m = 0$, so the generators of IM lie in $\ker(\tilde{\Psi})$ whence $IM \subseteq \ker(\tilde{\Psi})$. Hence, $\tilde{\Psi}$ descends to an R -module homomorphism $\Psi : M/IM \rightarrow R/I \otimes_R M$ such that $\Psi(m + IM) = \tilde{\Psi}(m)$. \square

(b): Suppose that M is a finitely generated free R -module. Show that the *rank* of M is well-defined, i.e. any two R -bases of M have the same number of elements.

Proof. Let $\mathfrak{m} \subseteq R$ be a maximal ideal of R . Let $k = R/\mathfrak{m}$ be the corresponding residue field. Suppose that \square

Problem 6

Let $R \subseteq S$ be an inclusion of commutative rings. Consider the polynomial rings $R[x]$ and $S[x]$. Prove that there is an isomorphism of S -modules,

$$S \otimes_R R[x] \rightarrow S[x]$$