Reading:

- For this homework: 1.3-1.4
- For Wednesday, February 2: 1.5 up to Theorem 1.11
- For Monday, February 7 : 2.1

Problem 1.

Folland Chapter 1, Problem 11

Problem 2.

Folland Chapter 1, Problem 12 (we will later see that this metric space is complete).

Problem 3.

Folland Chapter 1, Problem 23

Problem 4.

Let $\mathcal{A}$ be an algebra, and let $\mu\colon \mathcal{A} \to [0, +\infty]$ be a finitely additve measure.

(i) Suppose $(A_j)_{j=1}^{\infty}$ are pairwise disjoint subsets of $\mathcal{A}$, and that $A = \bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$. Show that

$$\mu(A) \geq \sum_{j=1}^{\infty} \mu(A_j).$$

(ii) Show that the following are equivalent:
- μ is a premeasure,
- $\mu\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} \mu(A_j)$ for any sequence $(A_j)_{j=1}^{\infty}$ with $\bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$,
- for any increasing sequence $(E_j)_{j=1}^{\infty}$ in $\mathcal{A}$ with $\bigcup_{j=1}^{\infty} E_j \in \mathcal{A}$, we have

$$\mu\left(\bigcup_j E_j\right) = \lim_{n\to\infty} \mu(E_n).$$

(iii) If $\mu(X) < +\infty$, show that μ is a premeasure if and only if for every decreasing sequence $(E_n)_{n=1}^{\infty}$ of sets with $\bigcap_{n=1}^{\infty} E_n = \varnothing$, we have

$$\lim_{n\to\infty} \mu(E_n) = 0.$$

Problem 5.

A *metric measure space* is a triple (X, d, μ) where (X, d) is a metric space and $\mu\colon \mathcal{B}_{(X,d)} \to [0, +\infty]$ is a measure. We say that $E \subseteq X$ is a *continuity set*, if $\mu(\overline{E} \setminus \mathrm{Int}(E)) = 0$. Here $\mathrm{Int}(E)$ is the interior of E. For this problem, fix a metric measure space (X, d, μ).

(i) Show that the collection of continuity sets forms an algebra of sets.

(ii) Show that if $x \in X$, $r > 0$ and $\mu(B_r(x, d)) < +\infty$, then there is an $s \in (0, r)$ so that $B_s(x, d)$ is a continuity set.

(iii) Suppose that (X, d) is separable and that for every $x \in X$, there is an $r > 0$ so that $\mu(B_r(x, d)) < +\infty$. Show that there is a countable basis consisting of open continuity sets. (Hint: given a countable, dense $D \subseteq X$ and $x \in D$, use the preceding part to choose a countable set $J_x \subseteq (0, +\infty)$ with the property that $\inf_{t\in J_x} t = 0$ and so that $B_t(x, d)$ is a continuity set for all $t \in J_x$).

Problem 6.

Let (X, d) be a metric space an μ, ν be finite, Borel measures on X with $\mu(X) = \nu(X)$. Let $\mathcal{A} = \{E \in \mathcal{B}_{(X,d)} : \mu(E) = \nu(E)\}$.

(i) Show that $\mathcal{A}$ is an algebra.

(ii) Given a nonempty $F \subseteq X$ closed, and $x \in X$, define $d(x, F) = \inf_{y \in F} d(x, y)$. Show that $x \mapsto d(x, F)$ is continuous and $F = \{x \in X : d(x, F) = 0\}$.

(iii) Suppose that $\{U \subseteq X : U \text{ is open}\} \subseteq \mathcal{A}$ if and only if $\{F \subseteq X : F \text{ is closed}\} \subseteq \mathcal{A}$. (Hint: use the metric to write a closed subset of X as a countable intersection of open sets. Similarly, use the metric to write an open set as a countable union of closed sets).

We'll see later that either of the conditions in (iii) imply that $\mathcal{A} = \mathcal{B}_X$.

Problems to think about, do not turn in:

Problem 7.

Folland Chapter 1, Problem 3. (Problem 4 on the last homework, and Folland Chapter 1, Problem 5 might be helfpul).

Problem 8.

Suppose that X is a set, that $\mathcal{A} \subseteq \mathcal{P}(X)$ is an algebra and that $\mu \colon \mathcal{A} \to [0, +\infty]$ is a finitely additive measure with $\mu(X) = 1$. Suppose that $T_j \colon X \to X, j = 1, \cdots, k$ are bijections so that $\mathcal{A} = \{T_j(A) : A \in \mathcal{A}\}$ for all $j = 1, \cdots, k$. Suppose that there exists $B \subseteq X$ with $0 < \mu(B) < +\infty$, an integer$1 \leq s \leq k$, and sets $A_1, \cdots, A_k \in \mathcal{A}$ so that

$$B \supseteq \bigsqcup_{i=1}^{k} A_i,$$

and so that

$$B \sqcup T_s(B) \subseteq \bigcup_{i=1}^{k} T_i(A_i).$$

Show that there is an integer $1 \leq l \leq k$, and an $A \in \mathcal{A}$ with $\mu(T_i(A)) \neq \mu(A)$.

(Note: the existence of such a set where B is a unit ball in $\mathbb{R}^3$, μ is a finitely additive measure defined on $\mathcal{P}(\mathbb{R}^3)$ extending Lebesque measure, and $T_1, \cdots, T_k$ are isometries is the Banach-Tarski paradox).