MATH 7310 Homework 7

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Problem 1

Let (X, Σ, μ) be a measure space.

(i): Prove that if $\mu(E_n) < +\infty$ for $n \in \mathbb{N}$ and $\mathbb{1}_{E_n} \to f$ in L^1, then f is (a.e. equal to) the characteristic function of a measurable set.

Proof. For $m \in \mathbb{N}$, let

$$F_m = \{x : \min\{|1 - f(x)|, |f(x)|\} > \frac{1}{m}\}.$$

Then $(F_m)_{m=1}^{\infty}$ is an increasing sequence of sets with $F = \{x : f(x) \notin \{0, 1\}\} = \bigcup_{n=1}^{\infty} A_n$.

Observe that, for fixed $m \in \mathbb{N}$,

$$\|\mathbb{1}_{E_n} - f\|_1 \geq \int_{F_m} |\mathbb{1}_{E_n} - f| \, d\mu \geq \int_{F_m} \frac{1}{m} \, d\mu = \frac{1}{m}\mu(F_m)$$

for all $n \in \mathbb{N}$, whence sending $n \to \infty$ it follows that $\mu(F_m) = 0$. Thus, it follows that $\mu(F) = 0$. Thus, $f = \mathbb{1}_{f^{-1}(\{1\})}$ almost everywhere. $\square$

(ii): Let $\Sigma_f = \{E \in \Sigma : \mu(E) < +\infty\}$. Define an equivalence relation on Σ_f by $E \sim F$ if $\mu(E \Delta F) = 0$. Let $\Omega = \Sigma_f / \sim$, and define a metric ρ on Ω be $\rho([E], [F]) = \mu(E \Delta F)$. Show that the map $\iota : \Omega \to L^1(X, \mu)$ given by $\iota([E]) = \mathbb{1}_E$ is an isometry with closed image.

Proof. Observe that, if $E, F \in \Sigma_f$, then

$$\rho(\iota(E), \iota(F)) = \mu(E \Delta F) = \int \mathbb{1}_{E \Delta F} \, d\mu = \int |\mathbb{1}_E - \mathbb{1}_F| \, d\mu \,,$$

so ι is an isometry. Now suppose that $(f_n)_{n=1}^{\infty}$ is in $\iota(\Omega)$ and $f \in L^1(X, \mu)$ with $\|f_n - f\|_1 \xrightarrow{n \to \infty} 0$. Then for $n \in \mathbb{N}$, there are $E_n \in \Sigma_f$ such that $f_n = \mathbb{1}_{E_n}$, whence by part (i) there is some measurable $E \subseteq X$ such that $f = \mathbb{1}_E$. As $\mathbb{1}_E = f \in L^1(\mu)$, it follows that $\mu(E) < +\infty$ whence $[E] \in \Omega$ and thus $f = \iota([E]) \in \iota(\Omega)$. $\square$

(iii): Show that (Ω, ρ) is a complete metric space.

Proof. Let $([E_n])_{n=1}^{\infty}$ be a Cauchy sequence in (Ω, ρ). Then as ι is an isometry, it follows that $(\mathbb{1}_{E_n})_{n=1}^{\infty}$ is a Cauchy sequence in $L^1(X, , \mu)$. By completeness of $L^1(X, \mu)$, there exists some $f \in L^1(X, \mu)$ such that $\|\mathbb{1}_{E_n} - f\|_1 \xrightarrow{n \to \infty} 0$. As the image of ι is closed, it follows that there is some $E \subseteq X$ with $\mu(E) < +\infty$ such that $f = \mathbb{1}_E = \iota([E])$ almost everywhere. Then, ι being an isometry implies that $\rho([E_n], [E]) \xrightarrow{n \to \infty} 0$. $\square$

Problem 2

If X, Y are sets, and $f : X \to \mathbb{C}$, $g : Y \to \mathbb{C}$, we define $f \otimes g : X \times Y \to \mathbb{C}$ by $(f \otimes g)(x, y) = f(x)g(y)$. Fix $1 \leq p < +\infty$.

(a): Let $(X, \Sigma, \mu), (Y, \mathcal{F}, \nu)$ be σ-finite measure spaces. Show that if $f \in L^p(X, \mu), g \in L^p(Y, \nu)$, then $\|f \otimes g\|_p = \|f\|_p \|g\|_p$.

Proof. By Tonelli's theorem,

$$\begin{aligned}\|f \otimes g\|_p^p &= \int_{X \times Y} |f \otimes g|^p \, d\mu \otimes \nu = \int_Y \int_X |f(x)|^p |g(y)|^p \, d\mu(x) \, d\nu(y) \\ &= \int_Y |g(y)|^p \int_X |f(x)|^p \, d\mu(x) \, d\nu(y) = \|f\|_p^p \int_Y |g(y)|^p \, d\nu(y) = \|f\|_p^p \|g\|_p^p.\end{aligned}$$

□

(b): Let $(Z, \mathcal{O}, \zeta)$ be a finite measure space. Suppose that $\mathcal{A} \subseteq \mathcal{O}$ is an algebra which generates the σ-algebra of $\mathcal{O}$. Use the monotone class lemma to show that $\{\mathbb{1}_A : A \in \mathcal{A}\}$ is dense in $\{\mathbb{1}_E : E \in \mathcal{O}\}$ in the L^p-norm for all $1 \leq p < +\infty$.

Proof. By the monotone class lemma, $\mathcal{O} = \Sigma(\mathcal{A}) = M(\mathcal{A})$. Let $E \in \mathcal{O}$. Then □

(c): Let $(X, \Sigma, \mu), (Y, \mathcal{F}, \nu)$ be finite measure spaces. Use the previous part to show that $\{\mathbb{1}_E : E \in \Sigma \otimes \mathcal{F}\} \subseteq \overline{\text{Span}}^{\|\cdot\|_p}\{\mathbb{1}_E \otimes \mathbb{1}_F : E \in \Sigma, F \in \mathcal{F}\}$. Use this to show that $\overline{\text{Span}}^{\|\cdot\|_p}\{\mathbb{1}_E \otimes \mathbb{1}_F : E \in \Sigma, F \in \mathcal{F}\} = L^p(X \times Y, \mu \otimes \nu)$.

(d): Let $(X, \Sigma, \mu), (Y, \mathcal{F}, \nu)$ be σ-finite measure spaces. Suppose that $D_X \subseteq L^p(X, \mu)$, $D_Y \subseteq L^p(Y, \nu)$ and that

$$\overline{\text{Span}}^{\|\cdot\|_p}(D_X) = L^1(X, \mu), \quad \overline{\text{Span}}^{\|\cdot\|_p}(D_Y) = L^1(Y, \nu).$$

Show that $\overline{\text{Span}}^{\|\cdot\|_p}(\{f \otimes g : f \in D_X, g \in D_Y\}) = L^p(X \times Y, \mu \otimes \nu)$.

Problem 3

Suppose that $f \in L^p \cap L^\infty$ for some $p < +\infty$ so that $f \in L^q$ for all $q > p$. Prove that then $\|f\|_\infty = \lim_{q \to \infty} \|f\|_q$.

Proof. By Folland Proposition 6.10, we have that

$$\|f\|_q^{\frac{p}{q}} \leq \|f\|_p^{\frac{p}{q}} \|f\|_\infty^{\frac{p}{q}}$$

whence $\limsup_{q \to \infty} \|f\|_q \leq \|f\|_\infty$. On the other hand, for $n \in \mathbb{N}$ let $E_n = \{x : |f(x)| > \|f\|_\infty - \frac{1}{n}\}$. Then $(E_n)_{n=1}^\infty$ is a decreasing sequence of measurable sets with $E = \bigcap_{n=1}^\infty E_n = \{x : |f(x)| \geq \|f\|_\infty\}$ having $\mu(E) = 0$ by definition of the L^∞-norm. Observe that, for $n \in \mathbb{N}$ and $q > p$,

$$\|f\|_q \geq \left(\int_{E_n} |f|^q \, d\mu \right)^{\frac{1}{q}} > (\|f\|_\infty - \frac{1}{n}) \mu(E_n)^{\frac{1}{q}}$$

whence

$$\liminf_{q \to \infty} \|f\|_q \geq \|f\|_\infty - \frac{1}{n}.$$

As this holds for all $n \in \mathbb{N}$, it follows that $\liminf_{q \to \infty} \|f\|_q \geq \|f\|_\infty$ as desired. □

Problem 4

If f is a measurable function on X, define the *essential range* R_f of f to be the set of all $z \in \mathbb{C}$ such that $\{x : |f(x) - z| < \varepsilon\}$ has positive measure for all $\varepsilon > 0$.

(a): Prove that R_f is closed.

Proof. Let $z \in \overline{R_f}$. Then there exists a sequence $(z_n)_{n=1}^{\infty}$ in R_f such that $z_n \to z$. Fix $\varepsilon > 0$. There is some $N \in \mathbb{N}$ such that $n \geq N \implies B_{\varepsilon/2}(z_n) \subseteq B_\varepsilon(z)$. Then $f^{-1}(B_{\varepsilon/2}(z_n)) \subseteq f^{-1}(B_\varepsilon(z))$, whence $0 < \mu(f^{-1}(B_{\varepsilon/2}(z_n))) \leq \mu(f^{-1}(B_\varepsilon(z)))$. Hence $z \in R_f$, so R_f is closed. $\square$

(b): Prove that if $f \in L^\infty$, then R_f is compact and $\|f\|_\infty = \max\{|z| : z \in R_f\}$.

Proof. Fix $z \in R$ and let $M > 0$ be such that $\mu(f^{-1}(X \setminus \overline{B_M(0)})) = 0$. Suppose, for the sake of contradiction, that $|z| > M$. Then we may choose $\varepsilon > 0$ such that $B_\varepsilon(z) \subseteq X \setminus B_M(0)$. Then $f^{-1}(B_\varepsilon(z)) \subseteq f^{-1}(X \setminus B_M(0))$, whence $\mu(f^{-1}(B_\varepsilon(z))) = 0$ contradicting that $z \in R_f$. Thus $|z| \leq M$. As $M > 0$ was arbitrary for its condition, it follows that $|z| \leq \|f\|_\infty$. As $z \in R_f$ was arbitrary, it follows that $\sup_{z \in R_f} |z| \leq \|f\|_\infty < +\infty$. Hence R_f is compact by part (a) and Heine-Borel. Let $z_{max} \in R_f$ such that $|z_{\max}| = \max_{z \in R_f} |z| = \sup_{z \in R_f} |z|$.

We show that in fact $\mu(f^{-1}(\mathbb{C} \setminus B_{|z_{\max}|}(0))) = 0$, whence it would follow that $\|f\|_\infty \leq |z_{\max}|$ as desired. Let $N \in \mathbb{N}$ such that for all $n \geq N$ we have $|z_{\max}| < |z_{\max}| + \frac{1}{n} < \|f\|_\infty$. Then for $n \geq N$, set $U_n = (\mathbb{C} \setminus \overline{B_{|z_{\max}|+\frac{1}{n}}(0)}) \setminus (\mathbb{C} \setminus B_{\|f\|_\infty}(0))$. We may cover U_n by countably many balls $B_{r_j}(z_j)$ where $B_{r_j}(z_j) \subseteq U$, i.e. $U = \bigcup_{j=1}^{\infty} B_{r_j}(z_j)$. It follows then that $\mu(f^{-1}(B_{r_j}(z_j))) = 0$, whence $\mu(U_n) = 0$. As the U_n's are a decreasing sequence of measure zero sets with intersection $(\mathbb{C} \setminus B_{|z_{\max}|(0)}) \setminus (\mathbb{C} \setminus B_{\|f\|_\infty}(0)))$, it follows that $\mu(f^{-1}((\mathbb{C} \setminus B_{|z_{\max}|(0)}) \setminus (\mathbb{C} \setminus B_{\|f\|_\infty}(0))))) = 0$. Thus $\mu(f^{-1}(\mathbb{C} \setminus B_{|z_{\max}|}(0))) = 0$ as desired. $\square$

Problem 5

Suppose that $1 \leq p < +\infty$ and $(f_n)_{n=1}^{\infty}$ in L^p. Prove that $(f_n)_{n=1}^{\infty}$ is Cauchy in the L^p-norm if and only if the following three conditions hold:

1. (f_n) is Cauchy in measure;
2. the sequence $(|f_n|^p)_{n=1}^{\infty}$ is uniformly integrable
3. for every $\varepsilon > 0$ there exists $E \subseteq X$ such that $\mu(E) < +\infty$ and $\int_{E^c} |f_n|^p \, d\mu < \varepsilon$ for all $n \in \mathbb{N}$.

Lemma 1. *Any finite subset $\{f_k\}_{k=1}^{n} \subseteq L^1(\mu)$ is uniformly integrable.*

Proof of Lemma 1. We show first that $f \in L^1(\mu)$ is uniformly integrable. Note that, if $f \in L^1(\mu)$, then $|f|\mathbb{1}_{\{|f|>m\}} \searrow 0$ pointwise a.e. as $\{|f| = +\infty\} = \bigcap_{M \in \mathbb{N}} \{|f| > M\}$ implies that $\lim_{M \to \infty} \mu(|f| > M) = \mu(\{|f| = +\infty\}) = 0$. Moreover, for all $M \in \mathbb{N}$, $|f\mathbb{1}_{|f|>M}| \leq |f| \in L^1(\mu)$, so by the dominated convergence theorem

$$\lim_{M \to \infty} \int_{\{|f|>M\}} |f| \, d\mu = 0. \tag{1}$$

For any $E \subseteq X$ measurable and $M \in \mathbb{N}$, we have that

$$\int_E |f| \, d\mu = \int_{E \cap \{|f| \leq M\}} |f| \, d\mu + \int_{E \cap \{|f|>M\}} |f| \, d\mu \leq M \cdot \mu(E) + \int_{\{|f|>M\}} |f| \, d\mu \,. \tag{2}$$

Fix $\varepsilon > 0$. By (1), there exists some $N \in \mathbb{N}$ such that $\int_{\{|f|>N\}} |f| \, d\mu < \frac{\varepsilon}{2}$. Choose $\delta = \frac{\varepsilon}{2N}$. Then, for any $E \subseteq X$ measurable such that $\mu(E) < \delta$, we have by (2) that

$$\left| \int_E f \, d\mu \right| \leq \int_E |f| \, d\mu < N \cdot \delta + \frac{\varepsilon}{2} = \varepsilon.$$

Now suppose that $\{f_k\}_{k=1}^n \subseteq L^1(\mu)$ is a finite subset of $L^1(\mu)$. Fix $\varepsilon > 0$. By uniform integrability of each of the singletons, for each $k \in \{1, \dots, n\}$ there exists a $\delta_k > 0$ such that $\mu(E) < \delta_k \implies |\int_E f_k| < \varepsilon$. Choosing $\delta = \min\{\delta_1, \dots, \delta_n\} > 0$, the claim follows. $\square$

Lemma 2. *Suppose $(f_n)_{n=1}^\infty$ is a sequence in $L^1(\mu)$ and $f \in L^1(\mu)$ such that $\|f_n - f\|_1 \xrightarrow{n \to \infty} 0$. Then $\{f_n\}_{n=1}^\infty$ is uniformly integrable.*

Proof of Lemma 2. Observe that, for any measurable $E \subseteq X$ and $n \in \mathbb{N}$,

$$\int_E |f_n| \, d\mu \leq \int_E |f| \, d\mu + \int_E |f_n - f| \, d\mu \leq \int_E |f| \, d\mu + \|f_n - f\|_1.$$

Fix $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that for $n \geq N$ we have $\|f_n - f\|_1 < \frac{\varepsilon}{2}$. By Lemma 1, $\{f\}$ is uniformly integrable, so there is some $\delta' > 0$ such that $\mu(E) < \delta'$ implies that $\int_E |f| \, d\mu < \frac{\varepsilon}{2}$.

Again by Lemma 1, $\{f_k\}_{k=1}^{N-1}$ is uniformly integrable, so there is some $\delta'' > 0$ such that $\mu(E) < \delta''$ implies $\int_E |f_k| \, d\mu < \varepsilon$ for all $k \in \{1, \dots, N-1\}$. Setting $\delta = \min\{\delta', \delta''\}$, the claim follows. $\square$

Lemma 3 (Lemma 3)**.** *Condition (3) holds for any finite subset $\{f_k\}_{k=1}^n \subseteq L^1(\mu)$.*

Proof of Lemma 3. We show first that the claim holds for just one function $f \in L^1(\mu)$. Suppose first that f is nonnegative and let $\varepsilon > 0$. Then by definition of the integral, there exists a simple function $0 \leq g \leq f$ such that

$$\int f \, d\mu - \int g \, d\mu < \varepsilon.$$

By monotonicity of the integral, $\int g \leq \int f < +\infty$, whence it follows that the set $E = \{x : g(x) > 0\}$ has finite measure (as g takes finitely many values). Then

$$\int_{E^c} f \, d\mu = \int_{E^c} f - g \, d\mu \leq \int f - g \, d\mu < \varepsilon.$$

Now suppose that f is real-valued and let $f^\pm$ be the positive and negative parts of f. Fix $\varepsilon > 0$. Then by the previous case there exist measurable $E^\pm \subseteq X$ with $\mu(E^\pm) < +\infty$ and $\int_{(E^\pm)^c} f^\pm \, d\mu < \frac{\varepsilon}{2}$. Letting $E = E^+ \cup E^-$, it follows that $\mu(E) < +\infty$ and

$$\int_{E^c} |f| \, d\mu = \int_{(E^+)^c \cap (E^-)^c} f^+ + f^- \, d\mu \leq \int_{(E^+)^c} f^+ \, d\mu + \int_{(E^-)^c} f^- \, d\mu < \varepsilon.$$

Finally, suppose that f is complex-valued. Let $u = \mathrm{Re}(f)$ and $v = \mathrm{Im}(f)$. The claim then follows from applying the previous case to u, v and using the inequality $|f| \leq |u| + |v|$.

Now suppose that we have a finite subset $\{f_k\}_{k=1}^n \subseteq L^1(\mu)$ and fix $\varepsilon > 0$. Then for $1 \leq k \leq n$ there exists $E_k \subseteq X$ with $\mu(E_k) < +\infty$ and $\int_{E_k^c} |f_k| \, d\mu < \varepsilon$. Let $E = E_1 \cup \dots \cup E_n$. Then $\mu(E) < +\infty$ and for $1 \leq k \leq n$ we have

$$\int_{E^c} |f_k| \, d\mu = \int_{\bigcap_{j=1}^n E_j^c} |f_k| \, d\mu \leq \int_{E_k^c} |f_k| \, d\mu < \varepsilon.$$

$\square$

Lemma 4 (Lemma 4)**.** *Suppose $(f_n)_{n=1}^\infty$ is a sequence in $L^1(\mu)$ and $f \in L^1(\mu)$ such that $\|f_n - f\|_1 \xrightarrow{n\to\infty} 0$. Then $\{f_n\}_{n=1}^\infty$ satisfies condition (3).*

Proof of Lemma 4. As in the proof of Lemma 2, we utilize that for any measurable $E \subseteq X$ and $n \in \mathbb{N}$,

$$\int_{E^c} |f_n|\, d\mu \le \int_{E^c} |f|\, d\mu + \|f_n - f\|_1.$$

Fix $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that for $n \ge N$ we have $\|f_n - f\|_1 < \frac{\varepsilon}{2}$. By Lemma 3, $\{f_k\}_{k=1}^{N-1}$ satisfies condition (3), so there is some $E_1 \subseteq X$ with $\mu(E_1) < +\infty$ such that $\int_{E_1^c} |f_k|\, d\mu < \varepsilon$ for all $k \in \{1, \ldots, N-1\}$. The singleton $\{f\}$ also satisfies condition 3, so there is some $E_2 \subseteq X$ with $\mu(E_2) < +\infty$ and $\int_{E_2^c} |f|\, d\mu < \frac{\varepsilon}{2}$. Setting $E = E_1 \cup E_2$, it follows that $\mu(E) < +\infty$, $\int_{E^c} |f_k|\, d\mu \le \int_{E_1^c} |f_k|\, d\mu < \varepsilon$ for $k \in \{1, \ldots, N-1\}$, and for all $n \ge N$

$$\int_{E^c} |f_n|\, d\mu \le \int_{E^c} |f|\, d\mu + \|f_n - f\|_1 < \int_{E_2^c} |f|\, d\mu + \frac{\varepsilon}{2} < \varepsilon.$$

□

Proof of Theorem.
$\underline{\Longrightarrow}$: Suppose that $(f_n)_{n=1}^\infty$ is Cauchy in the L^p-norm. Then by completeness, there is some $f \in L^p(\mu)$ such $\|f - f_n\|_p \xrightarrow{n\to\infty} 0$. For $\varepsilon > 0$, noting that $\{|f_n - f| \ge \varepsilon\} = \{|f_n - f|^p/\varepsilon^p \ge 1\}$, we have that

$$\mu(\{|f_n - f| \ge \varepsilon\}) = \int_{\{|f_n - f| \ge \varepsilon\}} \frac{|f_n - f|^p}{\varepsilon^p}\, d\mu \le \frac{1}{\varepsilon^p}\|f_n - f\|_p^p \xrightarrow{n\to\infty} 0.$$

Thus $f_n \to f$ in measure, whence $(f_n)_{n=1}^\infty$ is Cauchy in measure.

By the reverse triangle inequality,

$$\Big|\|f_n\|_p - \|f\|_p\Big| \le \|f_n - f\|_p \xrightarrow{n\to\infty} 0,$$

so $\||f_n|^p\|_1 \xrightarrow{n\to\infty} \||f|^p\|_1 \in L^1(\mu)$. Now by Lemma 2, $(|f_n|^p)_{n=1}^\infty$ is uniformly integrable. Also by Lemma 4, condition (3) holds for $(|f_n|^p)_{n=1}^\infty$.

$\underline{\Longleftarrow}$: Suppose that $(f_n)_{n=1}^\infty$ in $L^p(\mu)$ satisfies the three listed conditions. Fix $\varepsilon > 0$ and let $E \subseteq X$ be as in condition (3). Set $A_{mn} = \{x : |f_m(x) - f_n(x)| \ge \varepsilon\}$ and let $\delta > 0$ be as in condition (2).

By construction, observe that

$$\int_{E\setminus A_{mn}} |f_m - f_n|^p\, d\mu \le \int_{E\setminus A_{mn}} \varepsilon^p\, d\mu \le \mu(E)\varepsilon^p$$

As $(f_n)_{n=1}^\infty$ is Cauchy in measure, there exists some $N \in \mathbb{N}$ such that for $m, n \ge N$, we have $\mu(A_{mn}) < \delta$. It follows by condition (2) that for $m, n \ge N$,

$$\int_{A_{mn}} |f_m - f_n|^p\, d\mu \le \int_{A_{mn}} 2^{p-1}(|f_m|^p + |f_n|^p)\, d\mu < 2^p\varepsilon.$$

Lastly, by condition (3), for all $m, n \in \mathbb{N}$,

$$\int_{E^c} |f_m - f_n|^p\, d\mu \le \int_{A_{mn}} 2^{p-1}(|f_m|^p + |f_n|^p)\, d\mu < 2^p\varepsilon.$$

So, for $m, n \ge N$, we have that

$$\|f_m - f_n\|_p^p \le \mu(E)\varepsilon^p + 2^p\varepsilon + 2^p\varepsilon,$$

so $(f_n)_{n=1}^\infty$ is Cauchy in the L^p-norm.

□

Problem 6

Prove that if E is a subset of a Hilbert space $\mathcal{H}$, then $(E^{\perp})^{\perp}$ is the smallest closed subspace of $\mathcal{H}$ containing E.

Claim. If M is a closed linear subspace of $\mathcal{H}$, then $(M^{\perp})^{\perp} = M$.

Proof of Claim. Note that we have $\mathcal{H} = M \oplus M^{\perp}$. Let $y \in (M^{\perp})^{\perp}$. Then there exist unique $x \in M$, $x^{\perp} \in M^{\perp}$ such that $y = x + x^{\perp}$. Noting that $M \subseteq (M^{\perp})^{\perp}$, we have that $x^{\perp} = y - x \in M^{\perp} \cap (M^{\perp})^{\perp} = \{0\}$, whence $x^{\perp} = 0$ and $y = x \in M$. Thus $M = (M^{\perp})^{\perp}$. $\square$

Proof. On one hand, note that $E \subseteq \overline{\mathrm{Span}(E)} \implies (E^{\perp})^{\perp} \subseteq (\overline{\mathrm{Span}(E)}^{\perp})^{\perp} \overset{\text{claim}}{=} \overline{\mathrm{Span}(E)}$. On the other hand, as $(E^{\perp})^{\perp}$ is a closed linear subspace of $\mathcal{H}$ and $E \subseteq (E^{\perp})^{\perp}$, it follows that $\overline{\mathrm{Span}(E)} \subseteq (E^{\perp})^{\perp}$. $\square$