MATH 7310 Homework 11

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Problem 1

Problem 2

Let (X, Σ, μ) be a probability space. Fix $p \in [1, +\infty]$ and $f \in L^p(X, \Sigma, \mu)$, let $p' \in [1, +\infty]$ be the conjugate exponent. Let $\mathcal{F} \subseteq \Sigma$ be a sub-σ-algebra.

(a): Show that $f \in L^1(X, \Sigma, \mu)$.

Proof. By proposition 6.12, $\|f\|_1 \leq \mu(X)^{1-\frac{1}{p}} \|f\|_p < +\infty$, whence $f \in L^1(X, \Sigma, \mu)$. $\square$

(b): Let $\mathbb{E}_{\mathcal{F}}(f)$ be the conditional expectation onto $\mathcal{F}$. Show that

$$\|\mathbb{E}_{\mathcal{F}}(f) g\|_1 \leq \|f\|_p \|g\|_{p'}$$

for all $\mathcal{F}$-measurable simple functions g. Use this to show that $\mathbb{E}_{\mathcal{F}}(f) \in L^p(X, \mathcal{F}, \mu)$ and that

$$\|\mathbb{E}_{\mathcal{F}}(f)\|_p \leq \|f\|_p.$$

Proof. Let $\alpha : X \to \mathbb{C}$ be the $\mathcal{F}$-measurable function such that $\alpha \mathbb{E}_{\mathcal{F}}(f) = |\mathbb{E}_{\mathcal{F}}(f)|$ and $|\alpha| = 1$. Note that then by finiteness $\alpha \in L^\infty(X, \mathcal{F}, \mu|_{\mathcal{F}})$. Then, for $\mathcal{F}$-measurable simple functions g, by Homework 8 problems 5(a) and 4(b),

$$\begin{aligned} \|\mathbb{E}_{\mathcal{F}}(f) g\|_1 &= \int \mathbb{E}_{\mathcal{F}}(f) \alpha |g| \, d\mu|_{\mathcal{F}} = \int \mathbb{E}_{\mathcal{F}}(f\alpha) |g| \, d\mu|_{\mathcal{F}} \\ &= \int f \alpha |g| \, d\mu = \left| \int f \alpha |g| \, d\mu \right| \leq \int |f||g| \, d\mu \leq \|f\|_p \|g\|_{p'}. \end{aligned}$$

Now by L^p-$L^{p'}$ duality,

$$\begin{aligned} \|\mathbb{E}_{\mathcal{F}}(f)\|_p &= \sup \left\{ \left| \int \mathbb{E}_{\mathcal{F}}(f) g \, d\mu|_{\mathcal{F}} \right| : g \in L^{p'}(X, \mu|_{\mathcal{F}}) \text{ simple with } \|g\|_{p'} = 1 \right\} \\ &\leq \sup \left\{ \|\mathbb{E}_{\mathcal{F}}(f) g\|_1 : g \in L^{p'}(X, \mu|_{\mathcal{F}}) \text{ simple with } \|g\|_{p'} = 1 \right\} \\ &\leq \sup \left\{ \|f\|_p \|g\|_{p'} : g \in L^{p'}(X, \mu|_{\mathcal{F}}) \text{ simple with } \|g\|_{p'} = 1 \right\} = \|f\|_p \end{aligned}$$

$\square$

Problem 3

Suppose that (X, Σ, μ) and $(Y, \mathcal{F}, \nu)$ are σ-finite measure spaces and $K \in L^2(X \times Y, \mu \otimes \nu)$. If $f \in L^2(Y, \nu)$, the integral $Tf(x) = \int_Y K(x,y)f(y)\, d\nu(y)$ converges for a.e. $x \in X$, $Tf \in L^2(X, \mu)$, and $\|Tf\|_2 \leq \|K\|_2 \|f\|_2$.

Proof. By Holder's inequality with $p = 2$,

$$\int |K(x,y)||f(y)|\, d\nu(y) = \|K(x,\cdot)\|_{L^2(\nu)} \|f\|_{L^2(\nu)} < +\infty,$$

so the integral converges absolutely for a.e. x. Now, we compute

$$\|Tf\|_{L^2(\mu)} \leq \left\| \int |K(x,y)||f(y)|\, d\nu(y) \right\|_{L^2(\mu)} \leq \left\| \|K(\cdot,\cdot)\|_{L^2(\mu)} \right\|_{L^2(\nu)} \|f\|_{L^2(\nu)} = \|K\|_{L^2(\mu\otimes\nu)} \|f\|_{L^2(\nu)} < +\infty$$

so $Tf \in L^2(\mu)$. □

Problem 4

Let $\eta(t) = e^{-1/t}$ for $t > 0$ and $\eta(t) = 0$ for $t \leq 0$.

(a): For $k \in \mathbb{N}$, $t > 0$, prove that $\eta^{(k)}(t) = P_k(1/t)e^{-1/t}$ where P_k is a polynomial of degree $2k$.

Proof. We induct on $k \in \mathbb{N}$. We compute that $\eta'(t) = \frac{1}{t^2}e^{-1/t}$, so $P_1(x) = x^2$ is degree 2 and thus satisfies the hypothesis. Now suppose that $\eta^{(k)}(t) = P_k(\frac{1}{t})e^{-1/t}$ where P_k is a polynomial of degree $2k$. Then

$$\eta^{(k+1)}(t) = (P_k(\frac{1}{t})e^{-1/t})' = \frac{1}{t^2}P_k'(\frac{1}{t})e^{-1/t} + \frac{1}{t^2}P_k(\frac{1}{t})e^{-1/t} = (\frac{1}{t^2}P_k'(\frac{1}{t}) + \frac{1}{t^2}P_k(\frac{1}{t}))e^{-1/t}$$

so $P_{k+1}(x) = x^2 P_k'(x) + x^2 P_k(x)$ is a degree $2(k+1)$ polynomial we have satisfied the hypothesis. □

(b): Prove that $\eta^{(k)}(0)$ exists and is zero for all $k \in \mathbb{N}$.

Proof. We compute that $\lim_{t\to 0^+} \frac{\eta(t)}{t} = 0$, so $\eta'(0)$ exists and equals zero. Now suppose $\eta^{(k)}(0)$ exists and equals zero. Then

$$\lim_{t\to 0+} \frac{\eta^{(k)}(t) - \eta^{(k)}(0)}{t} = \lim_{t\to 0+} \frac{P_k(\frac{1}{t})e^{\frac{-1}{t}}}{t} = \lim_{t\to 0+} \frac{1}{tP_k(\frac{1}{t})e^{\frac{1}{t}}} = 0,$$

so $\eta^{(k+1)}(0)$ exists and equals zero. By induction, we are done. □

Problem 5

Let E be a measurable subset of $\mathbb{R}^n$ of positive measure. Show that $E - E$ contains an open set U with $0 \in U$.

Proof. Suppose first that $E \subseteq \mathbb{R}^n$ has $m(E) < +\infty$. Let $U = \{x : \mathbb{1}_E * \mathbb{1}_{-E}(x) > 0\}$. As $\mathbb{1}_E * \mathbb{1}_{-E}$ is continuous, $U = (\mathbb{1}_E * \mathbb{1}_{-E})^{-1}((0, +\infty))$ is open. Moreover,

$$\mathbb{1}_E * \mathbb{1}_{-E}(0) = \int \mathbb{1}_E(y)\mathbb{1}_{-E}(-y)\, dy = \int \mathbb{1}_E\, dy = m(E) > 0,$$

so $0 \in U$. Lastly, suppose $x \in U$. Then $0 < \mathbb{1}_E * \mathbb{1}_{-E}(x) = \int \mathbb{1}_E(y)\mathbb{1}_{-E}(x-y)\,dy$, whence by positivity of the integrand there exists some $y \in \mathbb{R}^n$ such that $\mathbb{1}_E(y)\mathbb{1}_{-E}(x-y) \neq 0$. But then $y \in E$ and $x - y \in -E$, so $x = y + (x-y) \in E - E$. Thus $U \subseteq E - E$.

Now suppose $m(E) = +\infty$. By σ-finiteness, there exists some $F \subseteq E$ measurable such that $0 < m(F) < +\infty$. By the previous case, we may find some open $U \subseteq F - F$ with $0 \in U$, whence $U \subseteq F - F \subseteq E - E$, as desired. $\square$