Math 7800 **Algebraic Topology I** **Spring 2022**

Assignment 1 Due Wednesday, January 26

A remark about homeomorphisms: Given two spaces X and Y that one feels should be homeomorphic, it is sometimes easy to construct a continuous bijection in one direction, say $f : X \to Y$, but not so easy to directly show that the inverse function is also continuous (or, equivalently, check that f is an 'open map', i.e. takes an open set in X to an open set in Y). The following handy proposition, whose proof you hopefully have seen, is often used:

Proposition If X is compact, and Y is Hausdorff, then a continuous bijection $f : X \to Y$ is a homeomorphism.

You are encouraged to use this, e.g. in the problems below.

Some standard notation: The n disk is the space $D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$, and the $n-1$ sphere is the space $S^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\}$.

Some more standard notation, that sadly sometimes clashes: Given $A \subset X$, X/A usually denotes the quotient space $X/(\sim)$, where $a \sim a'$ for all $a, a' \in A$. Informally, we say that X/A is obtained form X by collapsing A to a point.)

If a group G acts on a space X (say on the right), X/G usually denotes the quotient space $X/(\sim)$, where $x \sim x'$ if there exists $g \in G$ such that $x' = xg$. X/G is thus the space of G–orbits in X.

These two notations occasionally clash, and when they do, the second generally wins out. For example, $\mathbb{R}/\mathbb{Z}$ means the quotient group from algebra, but viewed with a topology making it a topological group.

1. The circle, viewed in three different ways. Prove that the spaces $[0,1]/(0 \sim 1)$, $\mathbb{R}/\mathbb{Z}$, and S^1 are all homeomorphic. (One approach: construct continuous bijections $[0,1]/(0 \sim 1) \to \mathbb{R}/\mathbb{Z} \to S^1$, use the observation above to show that the composite is a homeomorphism, and then argue that each of the individual maps must also be.)

2. The homotopy relation. Two maps $f, g : X \to Y$ are *homotopic* if there exists a continuous $H : X \times [0,1] \to Y$ such that for all $x \in X$, $f(x) = H(x,0)$ and $g(x) = H(x,1)$. Two spaces X and Y are *homotopy equivalent* if there exist maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq 1_X : X \to X$ and $f \circ g \simeq 1_Y : Y \to Y$.
(a) Check that homotopy is an equivalence relation on the set of continuous functions from a space X to a space Y.
(b) Given maps $h : W \to X$, $g_0, g_1 : X \to Y$, and $f : Y \to Z$, check that $g_0 \simeq g_1$ implies that both $g_0 \circ h \simeq g_1 \circ h$ and $f \circ g_0 \simeq f \circ g_1$.
(c) Check that homotopy equivalence is an equivalence relation on the class of topological spaces. (Hint: this is pretty 'formal', given parts (a) and (b).)

3. Null homotopic maps from spheres.
(a) Show that the map $h : S^{n-1} \times [0,1] \longrightarrow D^n$, given by $h(\mathbf{x}, t) = t\mathbf{x}$, induces a homeomorphism $\bar{h} : (S^{n-1} \times [0,1])/(S^{n-1} \times \{0\}) \cong D^n$.
(b) We say that $f : X \longrightarrow Y$ is *null homotopic* if it is homotopic to a constant map. Show that $f : S^{n-1} \longrightarrow Y$ is null if and only if f extends to a continuous function $\bar{f} : D^n \longrightarrow Y$. (Hint: use part (a).)

A theme in algebraic topology is the search for invariants that behave well under 'piecing together', so that global information can be deduced from local information. The final problem illustrates this.

4. Axioms for Euler characteristics. For the moment, the following should be accepted on faith. For certain spaces X, there is defined an integer $\chi(X)$, called the *Euler characteristic of* X, and $\chi(X)$ satisfies the following properties:

(1) $\chi(\emptyset) = 0, \chi(\text{point}) = 1$.
(2) If $\chi(X)$ is defined, and Y is homotopy equivalent to X, then $\chi(Y)$ is defined, and $\chi(X) = \chi(Y)$.
(3) If $X = Int(U) \cup Int(V)$, and $\chi(U)$, $\chi(V)$, and $\chi(U \cap V)$ are all defined, then $\chi(X)$ is defined, and $\chi(X) = \chi(U) + \chi(V) - \chi(U \cap V)$.

Use these properties to make the following computations. (Your arguments can be a bit informal. This is supposed to be a fun problem!)
(a) Compute $\chi(S^n)$ (for $n \geq 0$). Hint: Decompose S^n into the union of the northern hemisphere (+ a little) and the southern hemisphere (+ a little).
(b) Compute $\chi(M_g)$, where M_g is a genus g surface: a compact oriented surface with g holes.
(c) Compute $\chi(K)$, where K is the Klein bottle: the surface obtained by gluing together the boundaries of two Möbius Bands.