MATH 7310 Homework 6

James Harbour

March 21, 2022

Problem 1

Let $X = Y$ be an uncountable linearly ordered set such that for each $x \in X$, $\{y \in X : y < x\}$ is countable. Let $\mathcal{M} = \mathcal{N}$ be the σ-algebra of countable or co-countable sets, and let $\mu = \nu$ be defined on $\mathcal{M}$ by $\mu(A) = 0$ if A is countable and $\mu(A) = 1$ if A is co-countable. Let $E = \{(x, y) \in X \times X : y < x\}$. Prove that E_x and E^y are measurable for all x, y, and that $\int\int \mathbb{1}_E \, d\mu \, d\nu$ and $\int\int \mathbb{1}_E \, d\nu \, d\mu$ exist but are not equal.

Proof. For $x \in X$, define the set $S(x) = \{y \in X : y < x\}$. Observe that, for $x \in X$, $E_x = \{y \in X : (x, y) \in E\} = \{y \in X : y < x\} = S(x)$ which is countable by assumption so E_x is measurable. On the other hand, for $y \in X$, since the ordering on X is total,

$$X \setminus E^y = \{x \in X : y \notin S(x)\} = \{x \in X : x = y \text{ or } x < y\} = \{y\} \cup S(y)$$

which is countable by assumption, so E^y is cocountable and thus measurable.

Thus, for $x, y \in X$, the x and y-sections of $\mathbb{1}_E$, i.e. $(\mathbb{1}_E)^y = \mathbb{1}_{E^y}$ and $(\mathbb{1})_x = \mathbb{1}_{E_x}$, are measurable. Thus, the inner integrals in each of the iterated integrals exist. To see that both of the whole iterated integrals exist, we compute for fixed $y \in X$

$$\int \mathbb{1}_E(x, y) \, d\mu(x) = \int \mathbb{1}_{E^y}(x) \, d\mu(x) = \mu(E^y) = 1$$

and for fixed $x \in X$

$$\int \mathbb{1}_E(x, y) \, d\nu(y) = \int \mathbb{1}_{E_x}(y) \, d\nu(y) = \nu(E_x) = 0$$

which are both measurable functions as they are constant functions. Hence, both of the interated integrals exist and we compute on one hand that

$$\int\int \mathbb{1}_E(x, y) \, d\mu(x) \, d\nu(y) = \int \mu(E^y) \, d\nu(y) = \int 1 \, d\nu(y) = \nu(X) = 1$$

and on the other hand that

$$\int\int \mathbb{1}_E(x, y) \, d\nu(y) \, d\mu(x) = \int \nu(E_x) \, d\mu(x) = \int 0 \, d\mu(x) = 0.$$

Thus $\int\int \mathbb{1}_E \, d\mu \, d\nu$ and $\int\int \mathbb{1}_E \, d\nu \, d\mu$ exist but are not equal. $\square$

Problem 3

(a): Suppose (X, Σ, μ) is a σ-finite measure space and $f \in L^+(X)$. Let

$$G_f = \{(x, y) \in X \times [0, +\infty] : y \leq f(x)\}.$$

Show that G_f is $\Sigma \times \mathcal{B}_{\mathbb{R}}$-measurable and $\mu \times m(G_f) = \int f \, d\mu$. Show also that the same is true if the inequality in the definition of G_f is made strict.

Proof. Let $\tilde{f} : X \times [0, +\infty] \to X \times [0, +\infty]$ be given by $(x, y) \mapsto (f(x), y)$ and $S : X \times [0, +\infty] \to [-\infty, +\infty]$ be given by $S(z, y) = z - y$ if z, y not both $\pm\infty$ and $S(z, y) = 0$ if $z = y = \infty$. Then S is measurable, and as $\pi_1 \circ \tilde{f}$ and $\pi_2 \circ \tilde{f}$ are measurable, so is $\tilde{f}$. Hence, as intermediate codomain and domain match for the corresponding measure spaces, $S \circ \tilde{f}$ is measurable. Noting that $G_f = (S \circ \tilde{f})^{-1}([0, +\infty])$, measurability of $S \circ \tilde{f}$ implies that G_f is $\Sigma \times \mathcal{B}_{\mathbb{R}}$-measurable.

Observe that, for $x \in X$, $m((G_f)_x) = m([0, f(x)]) = f(x)$. As G_f is measurable, by Theorem 2.36 in Folland, the function $x \mapsto m((G_f)_x)$ is measurable and

$$\mu \times m(G_f) = \int m((G_f)_x) \, d\mu(x) = \int f(x) \, d\mu(x) \, .$$

□

(b): Let (X, μ) be a σ-finite measure space. Fix $p \in [1, +\infty)$. Show that if $f \in L^p(X, \mu)$, then

$$\|f\|_p^p = p \int_0^\infty t^{p-1} \mu(\{x : |f(x)| > t\}) \, dt \, .$$

Proof. Observe that, by part (a),

$$\|f\|_p^p = \int_X |f|^p \, d\mu = (\mu \times m)(G_{|f|^p}) = \int_{X \times [0, +\infty]} \mathbb{1}_{G_{|f|^p}}(x, t) \, d(\mu \times m) \, (x, t)$$

As $\|f\|_p^p < +\infty$, it follows that $\mathbb{1}_{G_{|f|^p}} \in L^1(X \times [0, +\infty], \mu \times m)$ whence by Fubini's theorem $(\mathbb{1}_{G_{|f|^p}})^t \in L^1(X, \mu)$ for almost every $t \in [0, +\infty]$, the a.e. defined function $\int (\mathbb{1}_{G_{|f|^p}})^t \, d\mu \in L^1([0, +\infty], m)$, and

$$\begin{aligned}
\|f\|_p^p = \int_{X \times [0, +\infty]} \mathbb{1}_{G_{|f|^p}} \, d(\mu \times m) &= \int_0^\infty \left[\int_X (\mathbb{1}_{G_{|f|^p}})^t(x) \, d\mu(x) \right] dt \\
&= \int_0^\infty \left[\int_X (\mathbb{1}_{(G_{|f|^p})^t})(x) \, d\mu(x) \right] dt = \int_0^\infty \mu((G_{|f|^p})^t) \, dt \\
&= \int_0^\infty \mu(\{x : |f(x)|^p < t\}) \, dt
\end{aligned}$$

Consider the functions $F : [0, +\infty] \to [0, \infty]$ and $\phi : [0, +\infty] \to [0, +\infty]$ given by $F(t) = \mu(\{x : |f(x)|^p > t\})$ and $\phi(t) = t^p$. For t nonnegative, observe that $\{x : |f(x)|^p > t^p\} = \{x : |f(x)| > t\}$, so $(F \circ \phi)(t) = \mu(\{x : |f(x)|^p > t\}) = \mu(\{x : |f(x)| > t\})$. Lastly, noting that F is measurable and ϕ is a C^1-diffeomorphism, it follows that

$$\|f\|_p^p = \int_0^\infty F(t) \, dt = \int_0^\infty (F \circ \phi)(t) | \det D_t \phi| \, dt = p \int_0^\infty t^{p-1} \mu(\{x : |f(x)| > t\}) \, dt \, .$$

□

(c): Let (X, μ) be a σ-finite measure space. Show that if $f, g \in L^1(X, \mu)$ with $0 \le f, g$ a.e., then

$$\|f - g\|_1 = \int_0^\infty \mu(\{x : f(x) > t\} \Delta \{x : g(x) > t\})\, dt\,.$$

Suggestion: it might be helpful to first show that for $a, b \in [0, +\infty)$ we have

$$|a - b| = \int_0^\infty |\mathbb{1}_{(t,\infty)}(a) - \mathbb{1}_{(t,\infty)}(b)|\, dt$$

Proof.

$$\begin{aligned}
\|f - g\|_1 = \int_X |f - g|\, d\mu &= \int_X \int_0^\infty |\mathbb{1}_{(t,+\infty)}(f(x)) - \mathbb{1}_{(t,+\infty)}(g(x))|\, dt\, d\mu(x) \\
&= \int_0^\infty \left[\int_X |\mathbb{1}_{f^{-1}((t,+\infty))}(x) - \mathbb{1}_{g^{-1}((t,+\infty))}(x)|\right] d\mu(x)\, dt \\
&= \int_0^\infty \left[\int_X \mathbb{1}_{f^{-1}((t,+\infty))\Delta g^{-1}((t,+\infty))}(x)\right] d\mu(x)\, dt \\
&= \int_0^\infty \mu(\{x : f(x) > t\} \Delta \{x : g(x) > t\})\, dt
\end{aligned}$$

$\square$

Problem 4

If f is Lebesgue integrable on $(0, a)$ and $g(x) = \int_x^a t^{-1} f(t)\, dt$, then g is integrable on $(0, a)$ and $\int_0^a g(x)\, dx = \int_0^a f(x)\, dx$.

Problem 5

Let $\mathcal{E}_q$ be the set of products of the form $\Pi_{j=1}^d I_j$ where each I_j is an h-interval with the property that all of its finite endpoints are rational.

(a): Show that $\mathcal{E}_q$ is an elementary family which generates the Borel sets.

(b): Suppose that μ is a Borel measure on $\mathbb{R}^d$ with $0 < \mu((0,1]^d) < +\infty$. If $\mu(E + x) = \mu(E)$ for every $x \in \mathbb{R}^d$, show that $\mu(E) = \mu((0,1]^d) m(E)$ for every Borel $E \subseteq \mathbb{R}^d$.

Problem 6

Fix $d \in \mathbb{N}$.

(a): Let $s : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ be the map $s(x, y) = x + y$. Let μ, ν be finite, Borel measures on $\mathbb{R}^d$. Define $\mu * \nu = s_*(\mu \otimes \nu)$. Show that for every Borel $E \subseteq \mathbb{R}^d$ we have

$$\mu * \nu(E) = \int \int \mathbb{1}_E(x + y)\, d\mu(x)\, d\nu(y)$$

and

$$\int \mu(E - y)\, d\nu(y) = \mu * \nu(E) = \int \nu(E - x)\, d\mu(x)\,.$$

Show as a consequence that

$$\mu * \nu(X) = \mu(X)\nu(X).$$

Proof. On one hand, by finiteness of the measures and measurability of E, we may apply Fubini's theorem to see that

$$\int\int \mathbb{1}_E(s(x,y))\,d\mu(x)\,d\nu(y) = \int\int \mathbb{1}_{s^{-1}(E)}(x,y)\,d\mu(x)\,d\nu(y) = \int\int \mathbb{1}_{s^{-1}(E)}\,d(\mu\otimes\nu) = \mu * \nu(E).$$

Moreover, noting that $(s^{-1}(E))^y = E - y$ and $(s^{-1}(E))_x = E - x$, theorem 2.36 gives that

$$\mu * \nu(E) = \mu \otimes \nu(s^{-1}(E)) = \int \nu(E - x)\,d\mu(x)$$

and

$$\mu * \nu(E) = \mu \otimes \nu(s^{-1}(E)) = \int \mu(E - y)\,d\nu(y)$$

It follows that

$$\mu * \nu(\mathbb{R}^d) = \int \mu(\mathbb{R}^d - y)\,d\nu(y) = \int \mu(R^d)\,d\nu(y) = \mu(R^d)\nu(R^d).$$

□

(b): Show that for finite, Borel measures μ, ν, η on $\mathbb{R}^d$ we have

$$(\mu * \nu) * \eta = \mu * (\nu * \eta).$$

(c): For $f, g \in L^1(\mathbb{R}^d)$ show that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(y)g(x-y)|\,dx\,dy = \|f\|_1\|g\|_1.$$

Explain why this implies that $y \mapsto f(y)g(x-y)$ is in $L^1(\mathbb{R}^d)$ for almost every $x \in \mathbb{R}^d$ and why if we set $f * g(x) = \int_{\mathbb{R}^d} f(y)g(x-y)\,dy$ then we have that $f * g \in L^1(\mathbb{R}^d)$ and

$$\|f * g\|_1 \leq \|f\|_1\|g\|_1.$$

Proof.

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(y)g(x-y)|\,dx\,dy = \int_{\mathbb{R}^d} |f(y)| \left[\int_{\mathbb{R}^d} |g(x-y)|\,dx\right] dy = \int_{\mathbb{R}^d} |f(y)| \left[\int_{\mathbb{R}^d} |g(x)|\,dx\right] dy = \|f\|_1\|g\|_1$$

□

(d): Adopt notation as in Problem 1 of HW5. Show that if $f, g \in L^1(R^d)$ are nonnegative than $(f\,dm) * (g\,dm) = f * g\,dm$ with m being the Lebesgue measure.

(e): Show that for $f, g, k \in L^1(\mathbb{R}^d)$ we have that

$$(f * g) * k = f * (g * k) \text{ almost everywhere.}$$