MATH 7310 Homework 6

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Problem 1

Let $X = Y$ be an uncountable linearly ordered set such that for each $x \in X$, $\{y \in X : y < x\}$ is countable. Let $\mathcal{M} = \mathcal{N}$ be the σ-algebra of countable or co-countable sets, and let $\mu = \nu$ be defined on $\mathcal{M}$ by $\mu(A) = 0$ if A is countable and $\mu(A) = 1$ if A is co-countable. Let $E = \{(x, y) \in X \times X : y < x\}$. Prove that E_x and E^y are measurable for all x, y, and that $\int \int \mathbb{1}_E \, d\mu \, d\nu$ and $\int \int \mathbb{1}_E \, d\nu \, d\mu$ exist but are not equal.

Problem 3

(a): Suppose (X, Σ, μ) is a σ-finite measure space and $f \in L^+(X)$. Let

$$G_f = \{(x, y) \in X \times [0, +\infty] : y \leq f(x)\}.$$

Show that G_f is $\Sigma \times \mathcal{B}_{\mathbb{R}}$-measurable and $\mu \times m(G_f) = \int f \, d\mu$. Show also that the same is true if the inequality in the definition of G_f is made strict.

(b): Let (X, μ) be a σ-finite measure space. Fix $p \in [1, +\infty)$. Show that if $f \in L^p(X, \mu)$, then

$$\|f\|_p^p = p \int_0^\infty t^{p-1} \mu(\{x : f(x) > t\}) \, dt \, .$$

(c): Let (X, μ) be a σ-finite measure space. Show that if $f, g \in L^1(X, \mu)$ with $0 \leq f, g$ a.e., then

$$\|f - g\|_1 = \int_0^\infty \mu(\{x : f(x) > t\} \Delta \{x : g(x) > t\}) \, dt \, .$$

Suggestion: it might be helpful to first show that for $a, b \in [0, +\infty)$ we have

$$|a - b| = \int_0^\infty |\mathbb{1}_{(t,\infty)}(a) - \mathbb{1}_{(t,\infty)}(b)| \, dt$$

Problem 4

If f is Lebesgue integrable on $(0, a)$ and $g(x) = \int_x^a t^{-1} f(t) \, dt$, then g is integrable on $(0, a)$ and $\int_0^a g(x) \, dx = \int_0^a f(x) \, dx$

Problem 5

Let $\mathcal{E}_q$ be the set of products of the form $\Pi_{j=1}^d I_j$ where each I_j is an h-interval with the property that all of its finite endpoints are rational.

(a): Show that $\mathcal{E}_q$ is an elementary family which generates the Borel sets.

(b): Suppose that μ is a Borel measure on $\mathbb{R}^d$ with $0 < \mu((0,1]^d) < +\infty$. If $\mu(E+x) = \mu(E)$ for every $x \in \mathbb{R}^d$, show that $\mu(E) = \mu((0,1]^d)m(E)$ for every Borel $E \subseteq \mathbb{R}^d$.

Problem 6

Fix $d \in \mathbb{N}$.

(a): Let $s : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ be the map $s(x,y) = x + y$. Let μ, ν be finite, Borel measures on $\mathbb{R}^d$. Define $\mu * \nu = s_*(\mu \otimes \nu)$. Show that for every Borel $E \subseteq \mathbb{R}^d$ we have

$$\mu * \nu(E) = \int\int \mathbb{1}_E(x+y)\, d\mu(x)\, d\nu(y)$$

and

$$\int \mu(E-y)\, d\nu(y) = \mu * \nu(E) = \int \nu(E-x)\, d\mu(x)\,.$$

Show as a consequence that

$$\mu * \nu(X) = \mu(X)\nu(X).$$

(b): Show that for finite, Borel measures μ, ν, η on $\mathbb{R}^d$ we have

$$(\mu * \nu) * \eta = \mu * (\nu * \eta).$$

(c): For $f, g \in L^1(\mathbb{R}^d)$ show that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(y)g(x-y)|\, dx\, dy = \|f\|_1 \|g\|_1.$$

Explain why this implies that $y \mapsto f(y)g(x-y)$ is in $L^1(\mathbb{R}^d)$ for almost every $x \in \mathbb{R}^d$ and why if we set $f * g(x) = \int_{\mathbb{R}^d} f(y)g(x-y)\, dy$ then we have that $f * g \in L^1(\mathbb{R}^d)$ and

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1.$$

(d): Adopt notation as in Problem 1 of HW5. Show that if $f, g \in L^1(R^d)$ are nonnegative than $(f\, dm) * (g\, dm) = f * g\, dm$ with m being the Lebesgue measure.

(e): Show that for $f, g, k \in L^1(\mathbb{R}^d)$ we have that

$$(f * g) * k = f * (g * k) \text{ almost everywhere.}$$