MATH 7752 - HOMEWORK 1
DUE FRIDAY 01/28/22 AT 1 P.M.

Convention: All rings considered below will have 1 (but not necessarily commutative, unless stated). Additionally, by an R-module we will always mean a left R-module.

(1) Let R be a ring and M an R-module.
   (a) Prove that for every $m \in M$, the map $r \mapsto rm$ from R to M is a homomorphism of R-modules.
   (b) Assume that R is commutative and M an R-module. Prove that there is an isomorphism $\mathrm{Hom}_R(R, M) \simeq M$ as R-modules.

(2) Give an explicit example of a map $f : A \to B$ with the following properties:
   - A, B are R-modules.
   - f is a group homomorphism.
   - f is not an R-module homomorphism.

(3) Let R be a ring and M an R-module.
   (a) Let N be a subset of M. The *annihilator* of N is defined to be the set

$$\mathrm{Ann}_R(N) := \{r \in R : rn = 0, \text{ for all } n \in N\}.$$

   Prove that $\mathrm{Ann}_R(N)$ is a left ideal of R.
   (b) Show that if N is an R-submodule of M, then $\mathrm{Ann}_R(N)$ is an ideal of R (i.e. it is two-sided ideal).
   (c) For a subset I of R the *annihilator* of I in M is defined to be the set,

$$\mathrm{Ann}_M(I) := \{m \in M : xm = 0, \text{ for all } x \in I\}.$$

   Find a natural condition on I that guarantees that $\mathrm{Ann}_M(I)$ is a submodule of M.
   (d) Let R be an integral domain. Prove that every finitely generated torsion R-module has a nonzero annihilator.

(4) In class we obtained a simple characterization of R-modules when $R = \mathbb{Z}$, and $R = F[x]$, with F a field. Imitate the method to find similar characterizations for R-modules in the following cases: (a) $R = \mathbb{Z}/n\mathbb{Z}$, for some $n \geq 2$; (b) $R = \mathbb{Z}[x]$; (c) $R = F[x, y]$.

(5) An R-module M is called *simple* (or *irreducible*) if its only submodules are $\{0\}$ and M. An R-module M is called *indecomposable* if M is not isomorphic to $N \oplus Q$ for some non-zero submodules N, Q. Show that every simple R-module is indecomposable, but the converse is not true.

(6) Let R be a ring. An R-module M is called *cyclic* if it is generated as an R-module by a single element.
   (a) Prove that every cyclic R-module is of the form R/I for some left ideal I of R.
   (b) Show that the simple R-modules are precisely the ones which are isomorphic to $R/\mathfrak{m}$ for some maximal left ideal $\mathfrak{m}$.

(c) Show that any non-zero homomorphism of simple R-modules is an isomorphism. Deduce that if M is simple, its endomorphism ring $\mathrm{End}_R(M) := \mathrm{Hom}_R(M, M)$ is a division ring. This result is known as *Schur's Lemma*.

(7) Show that $\mathbb{Q}$ is not a free $\mathbb{Z}$-module, that is $\mathbb{Q}$ is not isomorphic to a direct sum of the form $\bigoplus_I \mathbb{Z}$, for any index set I. More generally, let R be a PID which is not a field and $K = \mathrm{frac}(R)$ be its fraction field. Show that K is not a free R-module.

(8) Let R be a commutative ring. Recall that an ideal I of R is called *nilpotent* if there exists some $n \in \mathbb{N}$ such that $I^n = 0$.

(a) Let $i \in I$. Show that the element $r = 1 - i$ is invertible in R.

(b) Let M, N be R-modules and let $\phi : M \to N$ be an R-module homomorphism. Show that ϕ induces an R-module homomorphism, $\overline{\phi} : M/IM \to N/IN$.

(c) Prove that if $\overline{\phi}$ is sujective, then ϕ is itself surjective.

Extra Problem (optional)

(1) Let G be a finite group and k a field. Consider the group ring $k[G]$.

(a) Let M be a k-vector space with a G-action. Show that M becomes a $k[G]$-module. Conversely, if M is a $k[G]$-module, show that M is a G-set.

(b) Let M, N be two $k[G]$-modules. Show that $\mathrm{Hom}_k(M, N)$ becomes a $k[G]$-module with the following G-action: For $g \in G$ and $\phi : M \to N$ a $k[G]$-homomorphism define

$$(g \cdot \phi)(m) := g\phi(g^{-1}m), \text{ for } m \in M.$$