MATH 7310 Homework 9

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April 12, 2022

Problem 3

(a): Let (X, Σ) be a measurable space. Let $M(\Sigma)$ be the vector space of complex measures on Σ with the total variation norm $\|\mu\| = |\mu|(X)$. Show that $M(\Sigma)$ is a Banach space.
Suggestion: it may be helpful to use that for $\mu \in M(\Sigma)$ we ahve

$$\sum_{n=1}^{\infty} |\mu(E_n)| \leq \|\mu\|$$

where $(E_n)_{n=1}^{\infty}$ is a sequence of pairwise disjoint elements of Σ (this is a consequence of a prior problem on this homework).

(b): Fix a positive, σ-finite measure μ on Σ. Show that the map $J : L^1(X, \mu) \to M(\Sigma)$ given by $J(f) = f\, d\mu$ is a linear isometry with closed image.

Proof. Let We wish to show that for $f, g \in L^1(X, \mu)$ and $\alpha \in \mathbb{C}$, $J(\alpha f + g) = \alpha J(f) + J(g)$, after which showing that $\|J(f)\|_{M(\Sigma)} = \|f\|_{L^1(\mu)}$ would imply that J is a linear isometry.

Let $f \in L^1(X, \mu)$. We compute

$$\|J(f)\|_{M(\Sigma)} = |J(f)|(X) = J(f)(X) = \int_X f\, d\mu = \|f\|_{L^1(\mu)}.$$

Suppose that $(J(f_n))_{n=1}^{\infty}$ converges to ν in $M(\Sigma)$ where $(f_n)_{n=1}^{\infty}$ is in $L^1(X, \mu)$. So

$$\|J(f_n) - \nu\|_{M(\Sigma)} \xrightarrow{n\to\infty} 0$$

Suppose that $E \in \Sigma$ is null. Then as $J(f_n) \ll \mu$, $|J(f_n)|(E) = 0$ for all $n \in \mathbb{N}$. So by problem 2,

$$|\nu|(E) \leq |J(f_n)(E - \nu(E))| + |J(f_n)(E)| \leq \|J(f_n) - \nu\|_{M(\Sigma)} \xrightarrow{n\to\infty} 0$$

whence $|\nu|(E) = 0$ i.e. E is null for ν. Thus $\nu \ll \mu$. By the Lebesgue-Radon-Nikodym theorem, there exists some $f \in L^1(X, \mu)$ such that $\nu = f\, d\mu = J(f)$, so ν is in the image of J. □

(c): Suppose that $\mu, \nu \in M(\Sigma)$, and let $d\nu = f\, d\mu + d\lambda$ with $\lambda \perp \mu$ be the Lebesgue-Radon-Nikodym decomposition. Show that

$$\|\mu - \nu\| = \|1 - f\|_{L^1(\mu)} + \|\lambda\|.$$

Proof. Observe that □

Problem 4

If E is a Borel set in $\mathbb{R}^n$, the density $D_E(x)$ of E at x is defined as

$$D_E(x) = \lim_{r\to 0} \frac{m(E \cap B_r(x))}{m(B_r(x))},$$

whenever the limit exists.

(a): Show that $D_E(x) = 1$ for a.e. $x \in E$ and $D_E(x) = 0$ for a.e. $x \in E^c$.

Proof. Define a new Borel measure ν by $\nu(F) = \mu(E \cap F)$ for Borel F. Then $\nu \ll m$ and observe that, for Borel sets F,

$$\nu(F) = \int \mathbb{1}_{E\cap F}\, dm = \int \mathbb{1}_E \mathbb{1}_F\, dm = \int_F \mathbb{1}_E\, dm\,,$$

so $\frac{d\nu}{dm} = \mathbb{1}_E$ m-a.e. by uniqueness in Lebesgue-Radon-Nikodym theorem. Moreover, as $\nu(F) \le m(F)$ for all Borel F, it follows that ν is finite on compacts and thus regular, so by the Lebesgue differentiation theorem, the following limit exists for m-a.e. x and is equal to

$$D_E(x) = \lim_{r\to 0} \frac{\nu(B_r(x))}{m(B_r(x))} = \mathbb{1}_E(x),$$

whence the claim follows. □

(b): Find examples of E and x such that $D_E(x)$ is a given number $\alpha \in (0, 1)$, or such that $D_E(x)$ does not exist.

Problem 5

Let $\psi : \mathbb{R} \to \mathbb{R}$ be given as $\psi = \mathbb{1}_{[0,1/2)} - \mathbb{1}_{[1/2,1]}$. For $n, k \in \mathbb{Z}$ define $h_{n,k}(t) = 2^{n/2}\psi(2^n t - k)$. Show that $\mathcal{E} = \{1\} \cup \{h_{n,k} : n \in \mathbb{N} \cup \{0\{,\ 0 \le k < 2^n\}$ is an orthonormal basis for $L^2([0, 1])$.