# MATH 7310 Homework 2

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February 7, 2022

## Problem 1

Let $\mu$ be a finitely additive measure.

**(a)** Prove that $\mu$ is a measure if and only if it is continuous from below as in Theorem 1.8c.

*Proof.* Theorem 1.8c shows the forward direction so it suffices to show the reverse direction. Suppose that $\mu$ is continuous from below. Let $(E_j)_{j=1}^{\infty}$ be a sequence of disjoint elements in the sigma algebra $\mathcal{M}$ corresponding to $\mu$. Define a new sequence $(F_n)_{n=1}^{\infty}$ in $\mathcal{M}$ by $F_n = \bigsqcup_{j=1}^{n} E_j$. Then $\bigsqcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} F_n$. As $(F_n)_{n=1}^{\infty}$ is an increasing sequence in $\mathcal{M}$, we have that

$$\mu\left(\bigsqcup_{n=1}^{\infty} E_n\right) = \mu\left(\bigcup_{n=1}^{\infty} F_n\right) = \lim_{n\to\infty} \mu(F_n) \lim_{n\to\infty} \sum_{j=1}^{n} \mu(E_j) = \sum_{j=1}^{\infty} \mu(E_j),$$

so $\mu$ is a measure. $\square$

**(b)** If $\mu(X) < \infty$, prove that $\mu$ is a measure if and only if it is continuous from above as in Theorem 1.8d.

*Proof.* Theorem 1.8d shows the forward direction so it suffices to show the reverse direction. Suppose that $\mu$ is continuous from above. Let $(E_j)_{j=1}^{\infty}$ be a sequence of disjoint elements in $\mathcal{M}$. Define a new sequence $(F_n)_{n=1}^{\infty}$ in $\mathcal{M}$ by $F_n = \bigsqcup_{j=1}^{n} E_j$. Observe that $F_1^c \supset F_2^c \subset F_3^c \supset \cdots$ is a decreasing sequence in $\mathcal{M}$ with $\mu(F_1^c) = \mu(X) - \mu(F_1) < +\infty$. Hence, by continuity from above,

$$\begin{aligned}\mu\left(\bigsqcup_{j=1}^{\infty} E_j\right) &= \mu\left(\bigcup_{n=1}^{\infty} F_n\right) = \mu\left(X \setminus \bigcap_{n=1}^{\infty} F_n^c\right) = \mu(X) - \mu\left(\bigcap_{n=1}^{\infty} F_n^c\right) = \mu(X) - \lim_{n\to\infty}(F_n^c) \\ &= \mu(X) - \lim_{n\to\infty} \mu\left(X \setminus \bigsqcup_{j=1}^{n} E_j\right) = \mu(X) - \lim_{n\to\infty} \mu(X) - \mu\left(X \setminus \bigsqcup_{j=1}^{n} E_j\right) = \lim_{n\to\infty} \sum_{j=1}^{n} \mu(E_j) = \sum_{j=1}^{\infty} \mu(E_j),\end{aligned}$$

so $\mu$ is a measure. $\square$

## Problem 2

Let $(X, \mathcal{M}, \mu)$ be a finite measure space.

**(a)** If $E, F \in \mathcal{M}$ and $\mu(E \Delta F) = 0$, then $\mu(E) = \mu(F)$.

*Proof.* Observe that

$$0 = \mu(E\Delta F) = \mu((E \setminus F) \sqcup (F \setminus E)) = \mu(E \setminus F) + \mu(F \setminus E).$$

As $\mu(E \setminus F), \mu(F \setminus E) \geq 0$, it follows that $\mu(E \setminus F), \mu(F \setminus E) = 0$. Then as $E = (E \setminus F) \sqcup (E \cap F)$ and $F = (F \setminus E) \sqcup (F \cap E)$, $\mu(E) = \mu(F)$. □

**(b)** Say that $E \sim F$ if $\mu(E\Delta F) = 0$; show that $\sim$ is an equivalence relation on $\mathcal{M}$.

*Proof.*
*(Reflexivity):* Note that $E\Delta E = E \setminus E = \emptyset \implies \mu(E\Delta E) = 0$, so $E \sim E$.
*(Symmetry):* Note that $E\Delta F = (E \setminus F) \sqcup (F \setminus E) = F\Delta E$, so $E \sim F \implies F \sim E$.
*(Transitivity):* Suppose that $E \sim F$ and $F \sim G$. Observe that

$$E \setminus G = ((E \setminus F) \sqcup (E \cap F)) \setminus G = ((E \setminus F) \setminus G) \cup ((E \cap F) \setminus G) \subseteq (E \setminus F) \cup (F \setminus G)$$
$$G \setminus E = ((G \setminus F) \sqcup (G \cap F)) \setminus E = ((G \setminus F) \setminus E) \cup ((G \cap F) \setminus E) \subseteq (G \setminus F) \cup (F \setminus E)$$

so by monotonicity and subadditivity,

$$\mu(E\Delta G) \leq \mu((E\setminus F)\cup(F\setminus G))+\mu((G\setminus F)\cup(F\setminus E)) \leq \mu(E\setminus F)+\mu(F\setminus E)+\mu(F\setminus G)+\mu(G\setminus F) = \mu(E\Delta F)+\mu(F\Delta G) =$$

hence $E \sim G$. □

**(c)** For $E, F \in \mathcal{M}$, define $\rho(E, F) = \mu(E\Delta F)$. Then $\rho(E, G) \leq \rho(E, F) + \rho(F, G)$, and hence $\rho$ defines a metric on the space $\mathcal{M}/\sim$.

*Proof.* Note that the inequality used in the proof of transitivity above held regardless of the assumptions that the symmetric differences were zero, whence

$$\rho(E, G) = \mu(E\Delta G) \leq \mu(E\Delta F) + \mu(F\Delta G) = \rho(E, F) + \rho(F, G).$$

□

# Problem 3

Let $\mathcal{A}$ be the collection of finite unions of sets of the form $(a, b] \cap \mathbb{Q}$ where $-\infty \leq a < b \leq +\infty$.

**(i)** Show that $\mathcal{A}$ is an algebra on $\mathbb{Q}$. (Use Proposition 1.7.)

*Proof.* Let $\mathcal{E} = \{(a, b] \cap \mathbb{Q} : -\infty \leq a < b \leq +\infty\} \cup \{\emptyset\}$. By Proposition 1.7, it suffices to show that $\mathcal{E}$ is an elementary family.

Suppose $(a, b] \cap \mathbb{Q}, (c, d] \cap \mathbb{Q} \in \mathcal{E}$, with $a < b$ and $c < d$. If $b \leq c$, then $((a, b] \cap \mathbb{Q}) \cap ((c, d] \cap \mathbb{Q}) = \emptyset \in \mathcal{E}$. If $b > c$, then $((a, b] \cap \mathbb{Q}) \cap ((c, d] \cap \mathbb{Q}) = (c, b] \cap \mathbb{Q} \in \mathcal{E}$.

Lastly, suppose that $(a, b] \cap \mathbb{Q}$ with $a < b$. If $a = -\infty$ and $b = +\infty$, then $\mathbb{Q} \setminus ((-\infty, +\infty] \cap \mathbb{Q}) = \emptyset$. If $a = -\infty$ and $b \neq +\infty$, then $\mathbb{Q} \setminus ((-\infty, b] \cap \mathbb{Q}) = (b, +\infty] \cap \mathbb{Q}$. If $a \neq -\infty$ and $b = +\infty$, then $\mathbb{Q} \setminus ((a, +\infty] \cap \mathbb{Q}) = (-\infty, a] \cap \mathbb{Q}$. Finally, if $a \neq -\infty$ and $b \neq +\infty$, then $\mathbb{Q} \setminus ((a, b] \cap \mathbb{Q}) = ((-\infty, a] \cap \mathbb{Q}) \sqcup ((b, +\infty] \cap \mathbb{Q})$. So $\mathcal{E}$ is an elementary family. □

**(ii)** Show that the $\sigma$-algebra generated by $\mathcal{A}$ is $\mathcal{P}(\mathbb{Q})$.

*Proof.* As $\mathcal{A} \subseteq \mathcal{P}(\mathbb{Q})$, by minimality $\Sigma(\mathcal{A}) \subseteq \mathcal{P}(\mathbb{Q})$. Now take $q \in Q$. Observe that $(q - \frac{1}{n}, q] \cap \mathbb{Q} \in \mathcal{A}$ for all $n \in \mathbb{N}$, whence $\{q\} = \bigcap_{n=1}^{\infty}(q - \frac{1}{n}, q] \cap \mathbb{Q} \in \Sigma(\mathcal{A})$. Hence, $\Sigma(\mathcal{A})$ contains all finite and countable subsets of $\mathbb{Q}$, so countability of $\mathbb{Q}$ implies that $\mathcal{P}(\mathbb{Q}) \subseteq \Sigma(\mathcal{A})$. $\square$

**(ii)** Define $\mu_0$ on $\mathcal{A}$ by $\mu_0(\emptyset) = 0$ and $\mu_0(A) = \infty$ for $A \neq \emptyset$. Prove that $\mu_0$ is a premeasure on $\mathcal{A}$, and that there is more than one measure on $\mathcal{P}(\mathbb{Q})$ whose restriction to $\mathcal{A}$ is $\mu_0$.

*Proof.* To see that $\mu_0$ is a premeasure, suppose that $(A_j)_{j=1}^{\infty}$ is a sequence of pairwise disjoint elements of $\mathcal{A}$ such that $\bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$. If $A_j = \emptyset$ for all $j \in \mathbb{N}$, then $\bigcup_{j=1}^{\infty} A_j = \emptyset$ whence $\mu_0(\bigcup_{j=1}^{\infty} A_j) = 0 = \sum_{j=1}^{\infty} 0 = \sum_{j=1}^{\infty} \mu_0(A_j)$. If there exists a $k \in \mathbb{N}$ such that $A_k \neq \emptyset$, then $A_k \subseteq \bigcup_{j=1}^{\infty} A_j \neq \emptyset$, so $\mu_0(\bigcup_{j=1}^{\infty} A_j) = +\infty = \sum_{j=1}^{\infty} \mu_0(A_j)$.

On one hand, we have an outer measure

$$\mu_0^*(E) = \inf \left\{ \sum_{j=1}^{\infty} \mu_0(A_j) : A_j \in \mathcal{A},\ E \subseteq \bigcup_{j=1}^{\infty} A_j \right\}$$

for $E \in \mathcal{P}(\mathbb{Q})$. Note that, $\mu_0^*(E) = 0$ if $E = \emptyset$ and $\mu_0^*(E) = +\infty$ if $E \neq \emptyset$. Moreover, this outer measure is in fact a measure on $E \in \mathcal{P}(\mathbb{Q})$ extending $\mu_0$ by the same reasoning showing $\mu_0$ is a premeasure, so let $\mu = \mu_0^*$.

On the other hand, consider the counting measure $\nu : \mathcal{P}(\mathbb{Q}) \to [0, +\infty]$. Note that, if $A \in \mathcal{A}$ and $A \neq \emptyset$, then $A$ must contain infinitely many elements, whence $\nu(A) = \infty$. Hence $\nu$ agrees with $\mu_0$ on $\mathcal{A}$. However, $\nu$ has finite, nonzero value on finite, nonempty subsets of $\mathbb{Q}$, so $\nu \neq \mu$. $\square$

# Problem 4

Let $\mathcal{A}$ be an alegbra, and let $\mu : \mathcal{A} \to [0, +\infty]$ be a finitely additive measure.

**(i)** Suppose $(A_j)_{j=1}^{\infty}$ are pairwise disjoint elements of $\mathcal{A}$, and that $A = \bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$. Show that

$$\mu(A) \geq \sum_{j=1}^{\infty} \mu(A_j).$$

*Proof.* Since $\mu$ is finitely additive, it is also finitely subadditive. Then by monotonicity, for any $n \in \mathbb{N}$,

$$\mu(A) \geq \mu\left(\bigsqcup_{j=1}^{n} A_j\right) = \sum_{j=1}^{n} \mu(A_j).$$

Hence, it follows that $\mu(A) \geq \sum_{j=1}^{\infty} \mu(A_j)$. $\square$

**(ii)** Show that the following are equivalent:

1. $\mu$ is a premeasure,
2. $\mu\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} \mu(A_j)$ for any sequence $(A_j)_{j=1}^{\infty}$ with $\bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$,
3. for any increasing sequence $(E_j)_{j=1}^{\infty}$ in $\mathcal{A}$ with $\bigcup_{j=1}^{\infty} E_j \in \mathcal{A}$, we have

$$\mu\left(\bigcup_j E_j\right) = \lim_{n\to\infty} \mu(E_n).$$

*Proof.*
*(2 $\implies$ 1)*: Suppose that $(A_j)_{j=1}^{\infty}$ are pairwise disjoint elements of $\mathcal{A}$ with $A = \bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$. by part (i), $\mu(\bigsqcup_{j=1}^{\infty} A_j) \geq \sum_{j=1}^{\infty} \mu(A_j)$. On the other hand, by assumption $\mu(\bigsqcup_{j=1}^{\infty} A_j) \leq \sum_{j=1}^{\infty} \mu(A_j)$, so

$$\mu\left(\bigsqcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j).$$

Hence, $\mu$ is a premeasure.
*(1 $\implies$ 3)*: Let $(E_j)_{j=1}^{\infty}$ be an increasing sequence in $\mathcal{A}$ with $\bigcup_{j=1}^{\infty} E_j \in \mathcal{A}$. Define a new sequence in $\mathcal{A}$ by $E_1' = E_1$ and $E_j' = E_j \setminus E_{j-1}$ for $j \geq 2$. Then,

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \mu\left(\bigsqcup_{j=1}^{\infty} E_j'\right) = \sum_{j=1}^{\infty} \mu(E_j') = \lim_{n\to\infty} \sum_{j=1}^{n} \mu(E_j') = \lim_{n\to\infty} \mu(E_n).$$

*(3 $\implies$ 2)*: Suppose that $(A_j)_{j=1}^{\infty}$ is a sequence in $\mathcal{A}$ with $\bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$. Then, by finite subadditivity (which follows from finite additivity),

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \mu\left(\bigcup_{n=1}^{\infty}\bigcup_{j=1}^{n} A_j\right) = \lim_{n\to\infty} \mu\left(\bigcup_{j=1}^{n} A_j\right) \leq \lim_{n\to\infty} \sum_{j=1}^{n} \mu(A_j) = \sum_{j=1}^{\infty} \mu(A_j).$$

$\square$

**(iii)** If $\mu(X) < +\infty$, show that $\mu$ is a premeasure if and only if for every decreasing sequence $(E_n)_{n=1}^{\infty}$ of sets in $\mathcal{A}$ with $\bigcap_{n=1}^{\infty} E_n = \emptyset$, we have

$$\lim_{n\to\infty} \mu(E_n) = 0.$$

*Proof.*
$\underline{\implies}$: Let $(E_n)_{n=1}^{\infty}$ be a decreasing sequence of sets in $\mathcal{A}$ with $\bigcap_{n=1}^{\infty} E_n = \emptyset$. Note that then the sequence of sets $(X \setminus E_n)_{n=1}^{\infty}$ is increasing, so by number 3 in part (ii) and utilizing finiteness of $\mu(X)$,

$$\mu\left(\bigcup_{n=1}^{\infty} X \setminus E_n\right) = \lim_{n\to\infty} \mu(X \setminus E_n) = \lim_{n\to\infty} \mu(X) - \mu(E_n).$$

Hence,

$$0 = \mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \mu\left(X \setminus \bigcap_{n=1}^{\infty} (X \setminus E_n)\right) = \mu(X) - \mu\left(\bigcup_{n=1}^{\infty} X \setminus E_n\right) = \mu(X) - \lim_{n\to\infty} (\mu(X) - \mu(E_n)) = \lim_{n\to\infty} \mu(E_n)$$

$\underline{\impliedby}$: Let $(A_j)_{j=1}^{\infty}$ be a sequence of pairwise disjoint elements of $\mathcal{A}$ such that $A = \bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$. Define a new sequence in $\mathcal{A}$ by $E_n = A \setminus \bigsqcup_{j=1}^{n} A_j$. Then $(E_n)_{n=1}^{\infty}$ is a decreasing sequence in $\mathcal{A}$ such that $\bigcap_{n=1}^{\infty} = 0$. Hence, by assumption,

$$0 = \lim_{n\to\infty} \mu(E_n) = \lim_{n\to\infty} \mu(A) - \mu\left(\bigsqcup_{j=1}^{n} A_j\right) = \mu(A) - \lim_{n\to\infty} \sum_{j=1}^{n} \mu(A_j) \implies \mu\left(\bigsqcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j),$$

so $\mu$ is a premeasure. $\square$

# Problem 5

A *metric measure space* is a triple $(X, d, \mu)$ where $(X, d)$ is a metric space and $\mu : \mathcal{B}_{(X,d)} \to [0, +\infty]$ is a measure. We say that $E \subseteq X$ is a *continuity set* if $\mu(\overline{E} \setminus \operatorname{Int}(E)) = 0$. For this problem, fix a metric measure space $(X, d, \mu)$.

**(i)** Show that the collection of continuity sets forms an algebra of sets.

*Proof.* Suppose that $E_1, \ldots, E_n \subseteq X$ are continuity sets. Then $\mu(\overline{E_j} \setminus \operatorname{Int}(E_j)) = 0$ for $1 \leq j \leq n$. As there are finitely many sets, the union of closures is equal to the closure of the union. Hence

$$\overline{\bigcup_{j=1}^{n} E_j} \setminus \operatorname{Int}\left(\bigcup_{j=1}^{n} E_j\right) = \bigcup_{j=1}^{n} \overline{E_j} \setminus \operatorname{Int}\left(\bigcup_{j=1}^{n} E_j\right) \subseteq \bigcup_{j=1}^{n} \overline{E_j} \setminus \bigcup_{j=1}^{n} \operatorname{Int}(E_j) = \bigcup_{j=1}^{n} \overline{E_j} \setminus \operatorname{Int}(E_j),$$

so by subadditivity,

$$\mu\left(\overline{\bigcup_{j=1}^{n} E_j} \setminus \operatorname{Int}\left(\bigcup_{j=1}^{n} E_j\right)\right) = \mu\left(\bigcup_{j=1}^{n} \overline{E_j} \setminus \operatorname{Int}(E_j)\right) \leq \sum_{j=1}^{n} \mu(\overline{E_j} \setminus \operatorname{Int}(E_j)) = 0$$

whence $E_1 \cup \cdots \cup E_n$ is a continuity set. Now suppose that $E \subseteq X$ is a continuity set.

$$(\overline{X \setminus E}) \setminus \operatorname{Int}(X \setminus E) = (X \setminus \operatorname{Int}(E)) \setminus \operatorname{Int}(X \setminus E) = (X \setminus \operatorname{Int}(E)) \setminus (X \setminus \overline{E}) = \overline{E} \setminus \operatorname{Int}(E)$$

so $\mu((\overline{X \setminus E}) \setminus \operatorname{Int}(X \setminus E)) = \mu(\overline{E} \setminus \operatorname{Int}(E)) = 0$, whence $X \setminus E$ is also a continuity set. $\square$

**(ii)** Show that if $x \in X$, $r > 0$ and $\mu(B_r(x, d)) < +\infty$, then there is an $s \in (0, r)$ so that $B_s(x, d)$ is a continuity set.

*Proof.* We show the stronger statement that there are only countably many $s \in (0, r)$ such that $B_s(x, d)$ is not a continuity set.

Suppose, for the sake of contradiction, that there exists uncountably many counterexamples $s \in (0, r)$ such that $\mu(\overline{B_s(x)} \setminus \operatorname{Int}(B_s(x))) \neq 0$. For $n \in \mathbb{N}$, define a set

$$A_n = \{s \in (0, r) : \frac{1}{n} \leq \mu(\overline{B_s(x)} \setminus \operatorname{Int}(B_s(x))) < \frac{1}{n-1}\}$$

where $1/0 := \infty$ by convention. Then, by assumption, $\bigcup_{n=1}^{\infty} A_n$ is uncountable, so there exists an $n \in \mathbb{N}$ such that $A_n$ is infinite. Take a countably infinite subset $\{s_1, s_2, \ldots\} \subseteq A_n$. Note that, for any fixed $t \in (0, +\infty)$, $\overline{B_t(x)} \setminus \operatorname{Int}(B_t(x)) \subseteq \{y \in X : d(x, y) = t\}$, whence the following union is disjoint:

$$\mu\left(\bigsqcup_{j=1}^{\infty} \overline{B_{s_j}(x)} \setminus \operatorname{Int}(B_{s_j}(x))\right) = \sum_{j=1}^{\infty} \mu(\overline{B_{s_j}(x)} \setminus \operatorname{Int}(B_{s_j}(x))) = \infty.$$

However, this contradicts that $\mu(B_r(x)) < \infty$. $\square$

**(iii)** Suppose that $(X, d)$ is separable and that for every $x \in X$, there is an $r > 0$ so that $\mu(B_r(x, d)) < +\infty$. Show that there is a countable basis consisting of open continuity sets. (Hint: given a countable dense $D \subseteq X$ and $x \in D$, use the preceding part to choose a countable set $J_x \subseteq (0, +\infty)$ with the property that $\inf_{t \in J_x} t = 0$ and so that $B_t(x, d)$ is a continuity set for all $t \in J_x$).

*Proof.* Let $D \subseteq X$ be a countable dense subset of $X$. Fix $x \in D$. Then the set $\{r \in (0, +\infty) : \mu(B_{r_x}(x, d)) < +\infty\}$ is nonempty, so define $R_x = \sup\{r \in (0, +\infty) : \mu(B_{r_x}(x, d)) < +\infty\} \in [0, +\infty]$. By part (ii), we may take $J_x \subseteq (0, R_x)$ to be a countable, dense subset of $(0, R_x)$ such that $B_t(x, d)$ is a continuity set for all $t \in J_x$, whence $\inf_{t \in J_x} t = 0$

$$\mathscr{J} = \bigsqcup_{x \in D} \{x\} \times J_x.$$

By the axiom of choice, $\mathscr{J}$ is countable. Let $\mathscr{B} = \{B_t(x, d) : (x, t) \in \mathscr{J}\}$. We claim that $\mathscr{B}$ is a basis for the metric topology on $(X, d)$. Let $U \subseteq X$ be open and $p \in U$. Then there exist $r, r' > 0$ such that $B_r(p, d) \subseteq U$ and $\mu(B_{r'}(p, d)) < +\infty$. Take $\varepsilon = \min\{r, r'\}$, so $B_\varepsilon(p, d) \subseteq U$ and $\mu(B_\varepsilon(p, d)) < +\infty$. By density, there exists an $x \in D$ such that $d(x, p) < \frac{\varepsilon}{4}$. Note that $B_{\frac{3\varepsilon}{4}}(x, d) \subseteq B_\varepsilon(p, d)$, whence $\mu(B_{\frac{3\varepsilon}{4}}(x, d)) < +\infty$ so $\frac{2\varepsilon}{4} < \frac{3\varepsilon}{4} \leq R_x$. Then, by density of $J_x$ in $(0, R_x)$, there exists a $t \in J_x$ such that $|t - \frac{2\varepsilon}{4}| < \frac{\varepsilon}{100}$. Then $p \in B_t(x, d)$ and $B_t(x, d)$ is a continuity set in $U$.

□

# Problem 6

Let $(X, d)$ be a metric space and $\mu, \nu$ be finite Borel measures on $X$ with $\mu(X) = \nu(X)$. Let $\mathcal{A} = \{E \in \mathcal{B}_{(X,d)} : \mu(E) = \nu(E)\}$.

**(i)** Show that if $F \subseteq E$ and $F, E \in \mathcal{A}$, then $E \setminus F \in \mathcal{A}$. Also show that if $(E_n)_{n=1}^\infty$ is an increasing sequence of elements of $\mathcal{A}$, then $\bigcup_{n=1}^\infty E_n \in \mathcal{A}$.

*Proof.* As $E, F \in \mathcal{A}$, $\mu(E) = \nu(E)$ and $\mu(F) = \nu(F)$. Then

$$\mu(E \setminus F) = \mu(E) - \mu(F) = \nu(E) - \nu(F) = \nu(E \setminus F)$$

so $E \setminus F \in \mathcal{A}$. Now suppose that $(E_n)_{n=1}^\infty$ is an increasing sequence of elements of $\mathcal{A}$. By continuity from above,

$$\mu\left(\bigcup_{n=1}^\infty E_n\right) = \lim_{n\to\infty} \mu(E_n) = \lim_{n\to\infty} \nu(E_n) = \nu\left(\bigcup_{n=1}^\infty E_n\right),$$

so $\bigcup_{n=1}^\infty E_n \in \mathcal{A}$. □

**(ii)** Given a nonempty $F \subseteq X$ closed and $x \in X$, define $d(x, F) = \inf_{y \in F} d(x, y)$. Show that $x \mapsto d(x, F)$ is continuous and $F = \{x \in X : d(x, F) = 0\}$.

*Proof.* Suppose $x, y \in X$. For $z \in F$,

$$d(x, F) \leq d(x, z) \leq d(x, y) + d(y, z) \implies d(x, F) - d(x, y) \leq d(y, z).$$

As this holds for arbitrary $z \in F$, it follows that $d(x, F) - d(x, y) \leq d(y, F)$, so $d(x, F) - d(y, F) \leq d(x, y)$. By symmetry, $d(y, F) - d(x, F) \leq d(x, y)$, so $|d(x, F) - d(y, F)| \leq d(x, y)$. Thus, the function $x \mapsto d(x, F)$ is $1 - $ *Lipschitz* whence it is continuous.

Clearly $F \subseteq \{x \in X : d(x, F) = 0\}$, so it suffices to show the reverse containment. Suppose that $x \in X$ such that $d(x, F) = 0$. For all $n \in \mathbb{N}$, there exists an $f_n \in F$ such that $0 \leq d(x, f_n) < \frac{1}{n}$. It follows that $d(x, f_n) \xrightarrow{n\to\infty} 0$, so $f_n \xrightarrow{n\to\infty} x$. Thus $x$ is a limit point of $F$, so $F$ being closed implies that $x \in F$. □

**(iii)** Show that $\{U \subseteq X : U \text{ is open}\} \subseteq \mathcal{A}$ if an only if $\{F \subseteq X : F \text{ is closed}\} \subseteq \mathcal{A}$.

*Proof.*
$\underline{\Longrightarrow}$: Suppose that $\{U \subseteq X : U \text{ is open}\} \subseteq \mathcal{A}$. Take $F \subseteq X$ such that $F$ is closed. Then $X \setminus F$ is open, whence by finiteness of $\mu$ and $\nu$,

$$\mu(X) - \mu(F) = \mu(X \setminus F) = \nu(X \setminus F) = \nu(X) - \nu(F) \implies \mu(F) = \nu(F)$$

so $F \in \mathcal{A}$.
$\underline{\Longleftarrow}$: Likewise, suppose that $\{U \subseteq X : U \text{ is closed}\} \subseteq \mathcal{A}$. Take $U \subseteq X$ such that $U$ is open. Then $X \setminus F$ is closed, whence by finiteness of $\mu$ and $\nu$,

$$\mu(X) - \mu(U) = \mu(X \setminus U) = \nu(X \setminus U) = \nu(X) - \nu(U) \implies \mu(U) = \nu(U)$$

so $U \in \mathcal{A}$.

$\square$