MATH 7310 Homework 1

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Problem 1

A family of sets $\mathcal{R} \subseteq \mathcal{P}(X)$ is called a *ring* if it is closed under finite unions and differences. A ring that is closed under countable unions is called a *σ-ring*.
(a) Rings (resp. σ-rings) are closed under finite (resp. countable) intersections.

Proof. Let $\mathcal{R}$ be ring (resp. σ-ring). Let $(E_i)_{i\in I}$ be a finite (resp. countable) family of elements of $\mathcal{R}$ with finite (resp. countable) indexing set I. Let $j \in I$ be arbitrary. For any set $F \subseteq X$, note that

$$E_j \setminus (E_j \setminus F) = (E_j \cap F) \cup (E_j \setminus E_j) = E_j \cap F$$

Hence, utilizing De Morgan's laws, we write

$$\bigcap_{i\in I} E_i = E_j \cap \bigcap_{i\in I\setminus\{j\}} E_i = E_j \setminus \left(E_j \setminus \left(\bigcap_{i\in I\setminus\{j\}} E_i\right)\right) = E_j \setminus \left(\bigcup_{i\in I\setminus\{j\}} (E_j \setminus E_i)\right) \in \mathcal{R}$$

as desired. $\square$

(b) If $\mathcal{R}$ is a ring (resp. σ-ring), then $\mathcal{R}$ is an algebra (resp. σ-algebra) if and only if $X \in \mathcal{R}$.

Proof.
$\underline{\Longrightarrow}$: Let $E \in \mathcal{R}$. Then $\emptyset = E \setminus E \in \mathcal{R}$. As $\mathcal{R}$ is an algebra (resp. σ-algebra), it follows that $X = \emptyset^c \in \mathcal{R}$.
$\underline{\Longleftarrow}$: Suppose that $X \in \mathcal{R}$. As $\mathcal{R}$ is already a ring (resp. σ-algebra), to show $\mathcal{R}$ is an algebra (resp.σ-algebra) it suffices to show that $\mathcal{R}$ is closed under complements. Suppose $E \in \mathcal{R}$. As $X \in \mathcal{R}$ and $\mathcal{R}$ is closed under differences, it follows that $E^c = X \setminus E \in \mathcal{R}$. $\square$

(c) If $\mathcal{R}$ is a σ-ring, then $\{E \subseteq X : E \in \mathcal{R} \text{ or } E^c \in \mathcal{R}\}$ is a σ-algebra.

Proof. Let $\Sigma = \{E \subseteq X : E \in \mathcal{R} \text{ or } E^c \in \mathcal{R}\}$. By construction, Σ is closed under complements. Suppose that $(E_i)_{i\in I}$ is a countable family of elements of Σ, where I is an indexing set. Let $I_0 = \{i \in I : E_i \in \mathcal{R}\}$. Note that $\bigcup_{i\in I_0} E_i \in \mathcal{R} \subseteq \Sigma$. Moreover

$$\left(\bigcup_{j\in I\setminus I_0} E_j\right)^c = \bigcap_{j\in I\setminus I_0} E_j^c \in \mathcal{R},$$

so $\bigcap_{j\in I\setminus I_0} E_j \in \Sigma$. Hence

$$\bigcup_{i\in I} E_i = \bigcup_{i\in I_0} E_i \cup \bigcup_{j\in I\setminus I_0} E_j.$$

Thus, it suffices to show that, if $E, F \in \Sigma$ such that $E \in \mathcal{R}$ and $F^c \in \mathcal{R}$, then $E \cup F \in \Sigma$. Observe that

$$E \cup F = E \cup E \cup F = E \cup (E \setminus F^c) \in \mathcal{R},$$

so Σ is a σ-algebra. $\square$

(d) If $\mathcal{R}$ is a σ-ring, then $\{E \subseteq X : E \cap F \in \mathcal{R} \text{ for all } F \in \mathcal{R}\}$ is a σ-algebra.

Proof. Let $\Sigma = \{E \subseteq X : E \cap F \in \mathcal{R} \text{ for all } F \in \mathcal{R}\}$. Suppose $(E_i)_{i \in I}$ is a countable family of elements of Σ. For any $F \in \mathcal{R}$,

$$F \cap \bigcup_{i \in I} E_i = \bigcup_{i \in I} F \cap E_i \in \mathcal{R}$$

by definition, so $\bigcup_{i \in I} E_i \in \Sigma$. Now suppose $E \in \Sigma$. Let $F \in \mathcal{R}$. Observe that, as $E \cap F \in \mathcal{R}$ and $\mathcal{R}$ is closed under differences,

$$E^c \cap F = (X \setminus E) \cap F = X \cap (F \setminus E) = F \setminus E = (F \setminus E) \cup (F \setminus F) = F \setminus (E \cap F)$$

Thus, $E^c \in \Sigma$, so Σ is a σ-algebra. $\square$

Problem 2

An algebra $\mathcal{A}$ is a σ-algebra if and only if $\mathcal{A}$ is closed under countable increasing unions.

Proof. The forward direction follows *a fortiori* by the definition of a σ-algebra. For the reverse direction, suppose that $\mathcal{A}$ is closed under countable increasing unions. As $\mathcal{A}$ is already an algebra, it suffices to show that $\mathcal{A}$ is closed under arbitrary countable unions. Let $(E_i)_{i=1}^{\infty}$ be a countable family of elements of $\mathcal{A}$. Define a new sequence $(E'_j)_{j=1}^{\infty}$ by $E'_j := E_1 \cup \cdots \cup E_j$. Note that, as $\mathcal{A}$ is closed under finite unions, $E'_j \in \mathcal{A}$ for all $j \in \mathbb{N}$. Moreover, $(E'_j)_{j=1}^{\infty}$ is an increasing sequence of sets by construction, so

$$\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{n} E_i = \bigcup_{n=1}^{\infty} E'_n \in \mathcal{A}.$$

$\square$

Problem 3

Let X, Y be sets and $f : X \to Y$.
(i) If $\Sigma \subseteq \mathcal{P}(X)$ is a σ-algebra, show that

$$\{E \subseteq Y : f^{-1}(E) \in \Sigma\}$$

is a σ-algebra.

Proof. Let $\mathfrak{A} = \{E \subseteq Y : f^{-1}(E) \in \Sigma\}$. Suppose $(E_i)_{i \in I}$ is a countable collection of elements of $\mathfrak{A}$ where I is an indexing set. Then

$$f^{-1}\left(\bigcup_{i \in I} E_i\right) = \bigcup_{i \in I} f^{-1}(E_i) \in \Sigma$$

by definition, so $\bigcup_{i \in I} E_i \in \mathfrak{A}$. Now let $E \in \mathfrak{A}$.

$$f^{-1}(Y \setminus E) = f^{-1}(Y) \setminus f^{-1}(E) = X \setminus f^{-1}(E) \in \Sigma,$$

so $E^c = Y \setminus E \in \mathfrak{A}$. $\square$

(ii) If $\Sigma \subseteq \mathcal{P}(X)$ is a σ-algebra, show that

$$\{E \subseteq X : E = f^{-1}(F) \text{ for some } F \in \Sigma\}$$

is a σ-algebra.

Proof. Let $\mathfrak{A} = \{E \subseteq X : E = f^{-1}(F) \text{ for some } F \in \Sigma\}$. Suppose that $(E_i)_{i\in I}$ is a countable collection of elements of $\mathfrak{A}$. Then there exists a countable collection $(F_i)_{i\in I}$ of elements of Σ such that $E_i = f^{-1}(F_i)$ for all $i \in I$. Observe that

$$\bigcup_{i\in I} E_i = \bigcup_{i\in I} f^{-1}(F_i) = f^{-1}\left(\bigcup_{i\in I} F_i\right),$$

so $\bigcup_{i\in I} E_i \in \mathfrak{A}$. Now let $E \in \mathfrak{A}$. There exists an $F \in \Sigma$ such that $E = f^{-1}(F)$. Then

$$X \setminus E = X \setminus f^{-1}(F) = f^{-1}(Y) \setminus f^{-1}(F) = f^{-1}(Y \setminus F),$$

so $E^c = X \setminus E \in \mathfrak{A}$. □

(iii) If Y is a countable, show that

$$\{E \subseteq X : E = f^{-1}(F) \text{ for some } F \subseteq Y\}$$

is the σ-algebra generated by $\{f^{-1}(\{y\}) : y \in Y\}$.

Proof. Let $\mathfrak{A} = \{E \subseteq X : E = f^{-1}(F) \text{ for some } F \subseteq Y\}$ and $\mathcal{S} = \{f^{-1}(\{y\}) : y \in Y\}$. Note that, by part (ii) with $\Sigma = \mathcal{P}(X)$, $\mathfrak{A}$ is a σ-algebra. Clearly $\mathcal{S} \subseteq \mathfrak{A}$, so by minimality $\Sigma(\mathcal{S}) \subseteq \mathfrak{A}$.

On the other hand, suppose $E \in \mathfrak{A}$. Then there exists some $F \subseteq Y$ such that $E = f^{-1}(F)$. Observe that then,

$$E = f^{-1}(F) = f^{-1}\left(\bigcup_{y\in F} \{y\}\right) = \bigcup_{y\in F} f^{-1}(\{y\}) \in \Sigma(\mathcal{S})$$

since $F \subseteq Y$ is countable. Hence $\mathfrak{A} \subseteq \Sigma(\mathcal{S})$. □

Problem 4

(i) Suppose that X is a set. Let J be a countable set and suppose that $X = \bigsqcup_{j\in J} A_j$ with $A_j \neq \emptyset$ for all $j \in J$. Show that the σ-algebra generated by $(A_j)_{j\in J}$ is $\{\bigcup_{j\in J_0} A_j : J_0 \subseteq J\}$.

Proof. Let $\mathfrak{A} = \{\bigcup_{j\in J_0} A_j : J_0 \subseteq J\}$ and $\mathcal{A} = \{A_j : j \in J\}$. We show first that $\mathfrak{A}$ is indeed a σ-algebra. Suppose that $(J_n)_{n\in\mathbb{N}}$ is a sequence of subsets of J and let $I = \bigcup_{n=1}^{\infty} J_n \subseteq J$. Then

$$\bigcup_{n=1}^{\infty} \bigcup_{j\in J_n} A_j = \bigcup_{j\in I} A_j \in \mathfrak{A}.$$

Now let $J_0 \subseteq J$. Then

$$\left(\bigsqcup_{j\in J_0} A_j\right) = X \setminus \bigsqcup_{j\in J_0} A_j = \bigsqcup_{j\in J\setminus J_0} A_j \in \mathfrak{A},$$

so $\mathfrak{A}$ is a σ-algebra. Clearly $\mathcal{A} \subseteq \mathfrak{A}$, so by minimality $\Sigma(\mathcal{A}) \subseteq \mathfrak{A}$.

On the other hand, as J is countable, each element of $\mathfrak{A}$ is an at most countable union of elements of $\mathcal{A}$, so $\mathfrak{A} \subseteq \Sigma(\mathcal{A})$. □

(ii) Show that the σ-algebra generated by a finite collection of sets is finite.

Proof. Let $\mathcal{A} = \{A_1, \ldots, A_n\} \subseteq \mathcal{P}(X)$ be a finite collection of sets. Define a new collection $\mathcal{A}'$ as follows:

For all $(x_1, \ldots, x_n) \in (\mathbb{Z}/2\mathbb{Z})^n$, define a set $A'_{(x_1,\ldots,x_n)} \in \mathcal{A}'$ by

$$A'_{(x_1,\ldots,x_n)} = \left(\bigcap_{\substack{1 \le i \le n \\ \text{s.t. } x_i = 1}} A_i \right) \setminus \left(\bigcup_{\substack{1 \le j \le n \\ \text{s.t. } x_j = 0}} A_j \right).$$

Note that

$$A_i = \bigsqcup_{\substack{(x_1,\ldots,x_n) \in (\mathbb{Z}/2\mathbb{Z})^n \\ x_i = 1}} A'_{(x_1,\ldots,x_n)},$$

so $\mathcal{A} \subseteq \Sigma(\mathcal{A}') \implies \Sigma(\mathcal{A}) \subseteq \Sigma(\mathcal{A}')$. As each $A' \in \mathcal{A}'$ is a finite intersection, complement, and union of elements of $\mathcal{A}$, $\mathcal{A}' \subseteq \Sigma(\mathcal{A}) \implies \Sigma(\mathcal{A}') \subseteq \Sigma(\mathcal{A})$. Hence, $\Sigma(\mathcal{A}) = \Sigma(\mathcal{A}')$.

The elements of $\mathcal{A}'$ are pairwise disjoint by constuction, and there are at most 2^n of them. Label the elements of $\mathcal{A}'$ as $A'_1, \ldots, A'_l$ where $l \le 2^n$. Hence, by part (i), $\Sigma(\mathcal{A}') = \{\bigcup_{j \in J_0} A_j : J_0 \subseteq \{1, \ldots, k\}\}$ is finite of order at most $2^k \le 2^{2^n}$.

□

Problem 5

Let (X, d_X) be a metric space. A collection $\mathcal{B}$ of open sets in X is said to be a *basis* if for every open $U \subseteq X$ and every $x \in U$, there is a $B \in \mathcal{B}$ so that $x \in B \subseteq U$. If $\mathcal{B}$ is a *countable* basis of open sets in X, show that the σ-algebra generated by $\mathcal{B}$ coincides with the Borel sets in X.

Proof. Let $\mathfrak{B}$ be the collection of Borel sets in X. This collection is a σ-algebra by definition. As $\mathcal{B}$ is a collection of open sets in X, $\mathcal{B} \subseteq \mathfrak{B}$. Hence, by minimality, $\Sigma(\mathcal{B}) \subseteq \mathfrak{B}$.

Suppose that $U \subseteq X$ is open. For all $x \in U$, there exists a $B_x \in \mathcal{B}$ such such that $x \in B_x \subseteq U$. As $\mathcal{B}$ is countable, the union $U = \bigcup_{x \in U} B_x$ is in fact countable, so $U = \bigcup_{x \in U} B_x \in \Sigma(\mathcal{B})$. Thus, $\Sigma(\mathcal{B})$ contains all open sets in X, whence by minimality of $\mathfrak{B}$, $\Sigma(\mathcal{B}) \subseteq \mathfrak{B}$. □