MATH 7752 - HOMEWORK 5
DUE FRIDAY 02/25/22

(1) Let $F = \mathbb{Z}^3$ be the free $\mathbb{Z}$-module of rank 3. Let N be the submodule of F generated by $v_1 = (1, 2, 3), (5, 4, 6)$, and $(7, 8, 9)$.
   (a) Find compatible bases for F and N, that is, bases satisfying the submodule theorem 1.
   (b) Describe the quotient F/N in the IF form.
   (c) Describe in IF form the abelian group given by the presentation
$$\langle a, b, c \mid a + 2b + 3c = 0, 5a + 4b + 6c = 0, 7a + 8b + 9c = 0\rangle.$$

(2) Let R be a PID. For an R-module M define $\mathrm{rk}(M)$ to be the minimal size of a generating set of M.
   (a) Let M be a finitely generated R-module and $R/a_1R \oplus \cdots \oplus R/a_mR \oplus R^s$ be its invariant factor decomposition. That is, $s \geq 0$ and the elements $a_1, \ldots, a_m$ are non-zero, non-units such that $a_1|a_2| \cdots |a_m$. Prove that $\mathrm{rk}(M) = m + s$. **Warning:** It is not true in general that $\mathrm{rk}(P \oplus Q) = \mathrm{rk}(P) \oplus \mathrm{rk}(Q)$.
   (b) Let F be a free R-module of rank n with basis $e_1, \ldots e_n$. Let N be the submodule of F generated by some elements $v_1, \ldots, v_n \in F$. Let $A \in Mat_n(F)$ be the matrix such that
$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = A \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}.$$
   Find a simple condition on the entries of A which holds if and only if $\mathrm{rk}(F/N) = n$.

(3) In this problem R will be a commutative domain. An R-module P is called *projective* if it is a direct summand of a free R-module. That is, if there exist a free R-module F and a submodule Q of F such that $F = P \oplus Q$.
   (a) Let P, M, N be R-modules and suppose $f : M \to N$ is a surjective R-module homomorphism. The map f induces a homomorphism of R-modules,
$$\begin{aligned} f_\star : \quad & \mathrm{Hom}_R(P, M) \to \mathrm{Hom}_R(P, N) \\ & [\phi : P \to M] \mapsto [f \circ \phi : P \to N]. \end{aligned}$$
   Prove that if P is finitely generated and projective, then $f_\star$ is surjective.
   Hint: The universal property of free R-modules will be useful.
   (b) Show that if R is a PID and P is finitely generated, then P is projective if and only if P is free.

(4) Determine the number of possible RCF's of 8×8 matrices A over $\mathbb{Q}$ with $\chi_A(x) = x^8 - x^4$. Explain your argument in detail.

(5) **(Optional: extra practice for the midterm)** Solve Problem 9 from p. 489 of Dummit-Foote.