Reading:

- For this homework: 1.4.-1.5
- For Wednesday, February 9: 2.1-2.2
- For Monday, February 14: 2.2-2.3

Problem 1.

Folland Chapter 1, Problem 28

Problem 2.

Let (X, Σ, μ) be measure space. We say that $E \subseteq X$ is *an atom* if

- $E \in \Sigma$,
- $\mu(E) > 0$,
- $\{\mu(F) : F \subseteq E, F \in \Sigma\} = \{0, \mu(E)\}$.

We say that μ is *diffuse* if it has no atoms.

(a) Let (X, d, μ) be a metric measure space. Assume that μ is outer regular, and that

$$\mu(E) = \sup\{\mu(K) : K \subseteq E \text{ compact}\} \text{ for all Borel } E \subseteq X.$$

If $\mu(\{p\}) = 0$ for all $p \in X$, show that μ is diffuse.

(b) Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing, right-continuous function. Show that for $p \in \mathbb{R}$ we have that $\{p\}$ is an atom of μ_F if and only if F is discontinuous at p. Show that μ_F is diffuse if and only if F is continuous.

Problem 3.

Let (X, Σ, μ) be a σ-finite measure space.

(i) Suppose that $(E_j)_{j \in J}$ is a collection of sets with $E_j \in \Sigma$ for all $j \in J$ and with $\mu(E_j) > 0$ for all $j \in J$, and so that $\mu(E_j \cap E_k) = 0$ for $j \neq k$ in J. Show that J is countable.

(ii) Let (Ω, ρ) be the metric space defined in Problem 12 of Chapter 1 of Folland. For $E \in \Sigma$, let $[E]$ be its equivalence class in Ω. Show that

$$\{[E] : E \subseteq X \text{ is an atom}\},$$

is countable.

Problem 4.

Let (X, Σ, μ) be a diffuse σ-finite measure space. For $A \in \Sigma$, show that:

$$\{\mu(B) : B \subseteq A, B \in \Sigma\} = [0, \mu(A)].$$

Suggestions: Reduce to the finite case. It might be helpful to first show that for every $E \in \Sigma$ with $\mu(E) > 0$, we have $0 = \inf\{\mu(B) : B \subseteq E \text{ and } \mu(B) > 0\}$.

Problem 5.

Folland Chapter 1, Problem 29

Problem 6.

(a) Let $\mathcal{E}_q$ be the family of h-intervals in $\mathbb{R}$ with rational endpoints. Show that $\mathcal{E}_q$ is an elementary family and that the σ-algebra generated by this elementary family is all Borel subsets of $\mathbb{R}$.

(b) Suppose that $\mu : \mathcal{B}_{\mathbb{R}} \to [0, \infty]$ is a measure and that $\mu(E+x) = \mu(E)$ for all $E \in \mathcal{B}_{\mathbb{R}}, x \in \mathbb{R}$. Assume that $0 < \mu((0, 1]) < +\infty$. Show that $\mu(E) = \mu((0, 1])m(E)$ for all $E \in \mathcal{B}_{\mathbb{R}}$.