Reading:

- For this homework: 2.2-2.4
- For Wedneday, March 2: 2.5-2.6
- For Monday, March 14: 2.6-5.1/6.1

Problem 1.

(a) Let (X, μ) be a measure space. For $f\colon X \to [0, +\infty]$ measurable, we define a measure ν by $\nu(E) = \int_E f\,d\mu$ where $E \subseteq X$ is measurable (you don't have to prove that this is a measure, this follows by a previous homework problem). If $g\colon X \to \mathbb{C}$ is measurable, show that $g \in L^1(X, \nu)$ if and only if $gf \in L^1(X, \mu)$ and that $\int g\,d\nu = \int fg\,d\mu$ for all $g \in L^1(X, \nu)$. (We often denote ν by $f\,d\mu$).

(b) Let $(X, \Sigma), (Y, \mathcal{F})$ be measurable spaces and let $\mu\colon \Sigma \to [0, +\infty]$ be a measure. Let $\phi\colon X \to Y$ be measurable. If $f\colon Y \to \mathbb{C}$ is measurable, show that $f \in L^1(Y, \phi_*(\mu))$ if and only if $f \circ \phi \in L^1(X, \mu)$ and that $\int f\,d(\phi_*)(\mu) = \int f \circ \phi\,d\mu$ for all $f \in L^1(Y, \phi_*(\mu))$.

Problem 2.

Folland Chapter 2, Problem 25

Problem 3.

Folland Chapter 2, Problem 28

Problem 4.

(a) Folland Chapter 2, Problem 32

(b) Suppose (X, μ) is a finite measure space. Let ρ be the metric in (a). Show that a sequence of measurable functions $f_n\colon X \to \mathbb{C}$ is Cauchy in measure if and only if it is Cauchy with respect to ρ.

Problem 5.

Folland Chapter 2, Problem 34

Suggestion: it might be helpful to use that if x_n is a sequence in a metric space then x_n converges to x if and only if given any subsequence x_{n_k} there is a subsubsequence $x_{n_{k_j}}$ so that $x_{n_{k_j}} \to x$. You should prove this fact if you use it.

Problem 6.

Folland, Chapter 2, Problem 44

Problems to think about. do not turn in

Problem 7.

Folland Chapter 2, Problems 20-21, 29-31

Problem 8.

Folland Chapter 2, Problem 36-38

Problem 9.

For a set X, a *unital algebra* of functions is a subsets of $\mathbb{C}^X = \{f\colon X \to \mathbb{C}\}$ which contains the constant function 1, and is closed under poinwise addition, pointwise sum, and scaling by complex numbers.

Let A be the smallest unital algebra of functions in $\mathbb{C}^{\mathbb{R}^d}$ which contains all continuous functions and which satisfies the following property: for every sequence $(f_n)_n$ in A for which there is a $C > 0$ with $|f_n| \leq C$ and so that if $f_n \to f$ pointwise, then $f \in A$. Show that A consists of all bounded Borel functions.

Suggestion: it might be helpful to first show that

$$\{E \in \mathcal{B}_{\mathbb{R}^d} : 1_E \in A\}$$

is a σ-algebra.

Problem 10.

Let $f \colon \mathbb{R}^d \to \mathbb{C}$ be measurable.

(a) Show that for every compact $K \subseteq \mathbb{R}^d$ and every $\varepsilon > 0$ there is a continuous function $g \colon \mathbb{R}^d \to \mathbb{C}$ so that

$$\mu(\{x \in K : f(x) \neq g(x)\}) < \varepsilon.$$

(b) Show that there is a sequence $(f_n)_n$ of continuous functions so that $f_n \to f$ almost everywhere. If there is a $C > 0$ so that $|f| \leq C$ a.e., show that we can take $|f_n| \leq C$.