

MATH 7310 Homework 3

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February 13, 2022

Problem 2

Let (X, Σ, μ) be a measure space. We say that $E \subseteq X$ is an *atom* if

- $E \in \Sigma$,
- $\mu(E) > 0$,
- $\{\mu(F) : F \subseteq E, F \in \Sigma\} = \{0, \mu(E)\}$.

We say the μ is *diffuse* if it has no atoms.

(a) Let (X, d, μ) be a metric measure space. Assume that μ is outer regular, and that

$$\mu(E) = \sup\{\mu(K) : K \subseteq E \text{ compact}\} \text{ for all Borel } E \subseteq X.$$

If $\mu(\{p\}) = 0$ for all $p \in X$, show that μ is diffuse.

Proof. Suppose, for the sake of contradiction, that μ is not diffuse. Then there exists an atom $E \subseteq X$. \square

(b) Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be an increasing, right-continuous function. Show that for $p \in \mathbb{R}$ we have that $\{p\}$ is an atom of μ_F if and only if F is discontinuous at p . Show that μ_F is diffuse if and only if F is continuous.

Problem 4

Let (X, Σ, μ) be a diffuse σ -finite measure space. For $A \in \Sigma$, show that:

$$\{\mu(B) : B \subseteq A, B \in \Sigma\} = [0, \mu(A)].$$

Suggestions: Reduce to the finite case. It might be helpful to first show that for every $E \in \Sigma$ with $\mu(E) > 0$, we have $0 = \inf\{\mu(B) : B \subseteq E \text{ and } \mu(B) > 0\}$.

Proof.

(*reduction to finite case*): Write $X = \bigcup_{i=1}^{\infty} X_i$ where $X_i \in \Sigma$ and $\mu(X_i) < +\infty$.

Suppose that $E \in \Sigma$ with $\mu(E) > 0$. Since μ is diffuse, there exists a $B_1 \subseteq E$ such that $B_1 \in \Sigma$ and $0 < \mu(B_1) < \mu(E)$. Note that either $\mu(B_1)$ or $\mu(E \setminus B_1)$ is less than $2^{-1}\mu(E)$, so without loss of generality assume that $\mu(B_1) < 2^{-1}\mu(E)$. Now, again as μ is diffuse, there exists a $B_2 \subseteq B_1$ such that $B_2 \in \Sigma$ and $0 < \mu(B_2) < \mu(B_1) < \mu(E)$. Again, we may assume without loss of generality that $\mu(B_2) < 2^{-1}\mu(B_1) < 2^{-2}\mu(E)$. Continuing as such, we obtain a decreasing sequence of sets $E \supset B_1 \supset B_2 \supset \dots$ such that

$0 < \mu(B_n) < 2^{-n}\mu(E)$. It follows that $0 = \inf\{\mu(B) : B \subseteq E \text{ and } \mu(B) > 0\}$.

Zornification time. Consider the poset (\mathcal{S}, \preceq) where $\mathcal{S} = \{B \subseteq A : B \in \Sigma, \mu(B) \leq \beta\}$ and

$$B \preceq B' \iff \mu(B \setminus B') = 0$$

i.e. $B \subseteq B'$ μ -almost everywhere. Let \mathcal{C} be a chain in \mathcal{S} . Let $\alpha = \sup_{B \in \mathcal{C}} \mu(B)$. Then we may choose $B_1 \preceq B_2 \preceq B_3 \preceq \cdots$ in \mathcal{C} such that $\lim_{n \rightarrow \infty} \mu(B_n) = \alpha$ □