Problem 1

(a) Let (X, μ) be a measure space. For $f : X \to [0, +\infty]$ measurable, we define a measure ν by $\nu(E) = \int_E f \, d\mu$ where $E \subseteq X$ is measurable. If $g : X \to \mathbb{C}$ is measurable, show that $g \in L^1(X, \nu)$ if and only if $gf \in L^1(X, \mu)$ and that $\int g \, d\nu = \int fg \, d\mu$ for all $g \in L^1(X, \nu)$.

Proof.
$\underline{\Longrightarrow}$: Suppose $g \in L^1(X, \nu)$, so $\int |g| \, d\nu < +\infty$. Thus $|g| \in L^+(X, \nu)$, so by problem 5 on homework 4, $\int |g| \, d\nu = \int |g| f \, d\mu = \int |gf| \, d\mu$, so $gf \in L^1(X, \mu)$.

$\underline{\Longrightarrow}$: Suppose $gf \in L^1(X, \mu)$. So $\int |g| \, d\nu = \int |g| f \, d\mu = \int |gf| \, d\mu < +\infty$, whence $g \in L^1(X, \nu)$.

Now let $g \in L^1(X, \nu)$ and write $g = u + iv$ where $u = \mathrm{Re}(g)$ and $v = \mathrm{Im}(g)$. Let u^+, u^-, v^+, v^- be the positive and negative parts of u and v respectively. As $g \in L^1(X, \nu)$, each of these functions are in $L^+(X, \nu)$. Then, using nonnegativity of these functions and problem 5 of homework 4,

$$\begin{aligned} \int g \, d\nu = \int u \, d\nu + i \int v \, d\nu &= \int u^+ \, d\nu - \int u^- \, d\nu + i \int v \, d\nu - i \int v \, d\nu \\ &= \int u^+ f \, d\mu - \int u^- f \, d\mu + i \int v^+ f \, d\mu - i \int v^- f \, d\mu = \int gf \, d\mu \, . \end{aligned}$$

□

(b) Let $(X, \Sigma), (Y, \mathcal{F})$ be measurable spaces and let $\mu : \Sigma \to [0, +\infty]$ be a measure. Let $\phi : X \to Y$ be measurable. If $f : Y \to \mathbb{C}$ is measurable, show that $f \in L^1(Y, \phi_*(\mu))$ if and only if $f \circ \phi \in L^1(X, \mu)$ and that $\int f \, d(\phi_*(\mu)) = \int f \circ \phi \, d\mu$ for all $f \in L^1(Y, \phi_*(\mu))$.

Proof.
$\underline{\Longrightarrow}$: Suppose that $f \in L^1(Y, \phi_*(\mu))$. Then

$$\int |f| \, d(\phi_*(\mu)) < +\infty$$

□

Problem 2

Let $f(x) = x^{-1/2}$ if $0 < x < 1$, $f(x) = 0$ otherwise. Let $(r_n)_{n=1}^{\infty}$ be an enumeration of the rationals, and set $g(x) = \sum_{n=1}^{\infty} 2^{-n} f(x - r_n)$.

(a) Show that $g \in L^1(m)$, and in particular that $g < \infty$ a.e.

Proof. □

(b) Prove that g is discontinuous at every point and unbounded on every interval, and it remains so after any modification on a Lebesgue null set.

(c) Prove that $g^2 < \infty$ almost everywhere, but g^2 is not integrable on any interval.

Problem 3

Compute the following limits and justify the calculations:

(a) $\lim_{n\to\infty}\int_0^\infty (1+(x/n))^{-n}\sin(x/n)\,dx$.

Proof. Let $f_n(x) = (1+(x/n))^{-n}\sin(x/n)$ for $x \in [0,+\infty)$. Then for all $x \in [0,+\infty)$, $f(x) := \lim_{n\to\infty} f_n(x) = 0/e^x = 0$, so $\int f(x)\,dx = 0$. On the other hand, we estimate via cherrypicking terms in the binomial expansion that for $n \in \mathbb{N}\setminus\{1\}$,

$$|f_n| = \frac{|\sin(\frac{x}{n})|}{(1+\frac{x}{n})^n} \le \frac{1}{(1+\frac{x}{n})^n} \le \frac{1}{1+\binom{n}{2}x^2} \le \frac{1}{1+x^2}$$

which is in L^1. Hence, by the dominated convergence theorem, $\lim_{n\to\infty}\int_0^\infty f_n(x)\,dx = \int_0^\infty f(x)\,dx = 0$. □

(b) $\lim_{n\to\infty}\int_0^1 (1+nx^2)(1+x^2)^{-n}\,dx$.

Proof. Let $f_n(x) = (1+nx^2)(1+x^2)^{-n}$ on $[0,1]$. Let $f(x) = \lim_{n\to\infty}(1+nx^2)(1+x^2)^{-n} =$ By Bernoulli's inequality, for $n \in \mathbb{N}$

$$|f_n| \le (1+x^2)^n(1+x^2)^{-n} = 1$$

which is in $L^1([0,1],m)$. Thus, by the dominated convergence theorem

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx = \int_0^1 \lim_{n\to\infty}(1+nx^2)(1+x^2)^{-n} = \int_0^1 \lim_{n\to\infty}\frac{x^2}{(1+x^2)^n\ln(1+x^2)2x}\,dx = 0.$$

□

(c) $\lim_{n\to\infty}\int_0^\infty n\sin(x/n)[x(1+x^2)]^{-1}\,dx$

Proof. Let $f_n(x) = n\sin(x/n)[x(1+x^2)]^{-1}$ and $f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\frac{\cos(x/n)}{1+x^2} = \frac{1}{1+x^2}$. For $n \in \mathbb{N}$, note that

$$|f_n| \le \frac{1}{1+x^2}$$

which is in L^+, so by the dominated convergence theorem $\lim_{n\to\infty}\int_0^\infty n\sin(x/n)[x(1+x^2)]^{-1}\,dx = \int_0^\infty \frac{1}{1+x^2}\,dx = \pi/2$. □

(d) $\lim_{n\to\infty}\int_a^\infty n(1+n^2x^2)^{-1}\,dx$.

Proof. We compute

$$\int_a^\infty n(1+n^2x^2)^{-1}\,dx = \int_{na}^\infty \frac{1}{1+x^2}\,dx = \frac{\pi}{2} - \arctan(na).$$

If $a>0$, $\lim_{n\to\infty}\frac{\pi}{2} - \arctan(na) = 0$.
If $a=0$, $\lim_{n\to\infty}\frac{\pi}{2} = \frac{\pi}{2}$.
If $a<0$, $\lim_{n\to\infty}\frac{\pi}{2} - \arctan(na) = \frac{\pi}{2} - (-\frac{\pi}{2}) = \pi$.

□

Problem 4

(a) Suppose $\mu(X) < \infty$. If f and g are complex-valued measurable functions on X, define

$$\rho(f, g) = \int \frac{|f - g|}{1 + |f - g|} \, d\mu \, .$$

Then ρ is a metric on the space of measurable functions if we identify functions that are equal a.e., and $f_n \to f$ with respect to this metric if and only if $f_n \to f$ in measure.

(b) Suppose (X, μ) is a finite measure space. Let ρ be the metric in (a). Show that a sequence of measurable functions $f_n : X \to \mathbb{C}$ is Cauchy in measure if and only if it is Cauchy with respect to ρ.

Problem 5

Suppose that $|f_n| \leq g \in L^1$ and $f_n \to f$ in measure.

(a) Prove that $\int f \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu$.

(b) Prove that $f_n \to f$ in L^1.

Problem 6

If $f : [a, b] \to \mathbb{C}$ is Lebesgue measurable and $\varepsilon > 0$, there is a compact set $E \subseteq [a, b]$ such that $\mu(E^c) < \varepsilon$ and $f|_E$ is continuous. (*Hint*: Use Egoroff's theorem and Theorem 2.26)