MATH 7310 Homework 2

James Harbour

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Problem 1

Let μ be a finitely additive measure.

(a) Prove that μ is a measure if and only if it is continuous from below as in Theorem 1.8c.

Proof. Theorem 1.8c shows the forward direction so it suffices to show the reverse direction. Suppose that μ is continuous from below. Let $(E_j)_{j=1}^{\infty}$ be a sequence of disjoint elements in the sigma algebra $\mathcal{M}$ corresponding to μ. Define a new sequence $(F_n)_{n=1}^{\infty}$ in $\mathcal{M}$ by $F_n = \bigsqcup_{j=1}^{n} E_j$. Then $\bigsqcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} F_n$. As $(F_n)_{n=1}^{\infty}$ is an increasing sequence in $\mathcal{M}$, we have that

$$\mu\left(\bigsqcup_{n=1}^{\infty} E_n\right) = \mu\left(\bigcup_{n=1}^{\infty} F_n\right) = \lim_{n\to\infty} \mu(F_n) \lim_{n\to\infty} \sum_{j=1}^{n} \mu(E_j) = \sum_{j=1}^{\infty} \mu(E_j),$$

so μ is a measure. $\square$

(b) If $\mu(X) < \infty$, prove that μ is a measure if and only if it is continuous from above as in Theorem 1.8d.

Proof. Theorem 1.8d shows the forward direction so it suffices to show the reverse direction. Suppose that μ is continuous from above. Let $(E_j)_{j=1}^{\infty}$ be a sequence of disjoint elements in $\mathcal{M}$. Define a new sequence $(F_n)_{n=1}^{\infty}$ in $\mathcal{M}$ by $F_n = \bigsqcup_{j=1}^{n} E_j$. Observe that $F_1^c \supset F_2^c \subset F_3^c \supset \cdots$ is a decreasing sequence in $\mathcal{M}$ with $\mu(F_1^c) = \mu(X) - \mu(F_1) < +\infty$. Hence, by continuity from above,

$$\begin{aligned}\mu\left(\bigsqcup_{j=1}^{\infty} E_j\right) &= \mu\left(\bigcup_{n=1}^{\infty} F_n\right) = \mu\left(X \setminus \bigcap_{n=1}^{\infty} F_n^c\right) = \mu(X) - \mu\left(\bigcap_{n=1}^{\infty} F_n^c\right) = \mu(X) - \lim_{n\to\infty}(F_n^c) \\ &= \mu(X) - \lim_{n\to\infty} \mu\left(X \setminus \bigsqcup_{j=1}^{n} E_j\right) = \mu(X) - \lim_{n\to\infty} \mu(X) - \mu\left(X \setminus \bigsqcup_{j=1}^{n} E_j\right) = \lim_{n\to\infty} \sum_{j=1}^{n} \mu(E_j) = \sum_{j=1}^{\infty} \mu(E_j),\end{aligned}$$

so μ is a measure. $\square$

Problem 2

Let $(X, \mathcal{M}, \mu)$ be a finite measure space.

(a) If $E, F \in \mathcal{M}$ and $\mu(E \Delta F) = 0$, then $\mu(E) = \mu(F)$.

Proof. Observe that

$$0 = \mu(E\Delta F) = \mu((E \setminus F) \sqcup (F \setminus E)) = \mu(E \setminus F) + \mu(F \setminus E).$$

As $\mu(E \setminus F), \mu(F \setminus E) \geq 0$, it follows that $\mu(E \setminus F), \mu(F \setminus E) = 0$. Then as $E = (E \setminus F) \sqcup (E \cap F)$ and $F = (F \setminus E) \sqcup (F \cap E)$, $\mu(E) = \mu(F)$. □

(b) Say that $E \sim F$ if $\mu(E\Delta F) = 0$; show that $\sim$ is an equivalence relation on $\mathcal{M}$.

Proof.
(Reflexivity): Note that $E\Delta E = E \setminus E = \emptyset \implies \mu(E\Delta E) = 0$, so $E \sim E$.
(Symmetry): Note that $E\Delta F = (E \setminus F) \sqcup (F \setminus E) = F\Delta E$, so $E \sim F \implies F \sim E$.
(Transitivity): Suppose that $E \sim F$ and $F \sim G$. Observe that

$$E \setminus G = ((E \setminus F) \sqcup (E \cap F)) \setminus G = ((E \setminus F) \setminus G) \cup ((E \cap F) \setminus G) \subseteq (E \setminus F) \cup (F \setminus G)$$
$$G \setminus E = ((G \setminus F) \sqcup (G \cap F)) \setminus E = ((G \setminus F) \setminus E) \cup ((G \cap F) \setminus E) \subseteq (G \setminus F) \cup (F \setminus E)$$

so by monotonicity and subadditivity,

$$\mu(E\Delta G) \leq \mu((E\setminus F)\cup(F\setminus G))+\mu((G\setminus F)\cup(F\setminus E)) \leq \mu(E\setminus F)+\mu(F\setminus E)+\mu(F\setminus G)+\mu(G\setminus F) = \mu(E\Delta F)+\mu(F\Delta G) =$$

hence $E \sim G$. □

(c) For $E, F \in \mathcal{M}$, define $\rho(E, F) = \mu(E\Delta F)$. Then $\rho(E, G) \leq \rho(E, F) + \rho(F, G)$, and hence ρ defines a metric on the space $\mathcal{M}/\sim$.

Proof. Note that the inequality used in the proof of transitivity above held regardless of the assumptions that the symmetric differences were zero, whence

$$\rho(E, G) = \mu(E\Delta G) \leq \mu(E\Delta F) + \mu(F\Delta G) = \rho(E, F) + \rho(F, G).$$

□

Problem 3

Let $\mathcal{A}$ be the collection of finite unions of sets of the form $(a, b] \cap \mathbb{Q})$ where $-\infty \leq a \leq b \leq +\infty$.

(i) Show that $\mathcal{A}$ is an algebra on $\mathbb{Q}$. (Use Proposition 1.7.)

Proof. Let $\mathcal{E}$ be the collection of sets of the form $(a, b] \cap \mathbb{Q}$ with $-\infty \leq a < b \leq +\infty$. By Proposition 1.7, it suffices to show that $\mathcal{E}$ is an elementary family.

Note that for an $a \in \mathbb{R}$, $(a, a] \cap \mathbb{Q} = \emptyset$, so $\emptyset \in \mathcal{E}$.
Suppose $E, F \in \mathcal{E}$.

□

(ii) Show that the σ-algebra generated by $\mathcal{A}$ is $\mathcal{P}(\mathbb{Q})$.

Proof. As $\mathcal{A} \subseteq \mathcal{P}(\mathbb{Q})$, by minimality $\Sigma(\dashv) \subseteq \mathcal{P}(\mathbb{Q})$. Now take $q \in Q$. Observe that $(q - \frac{1}{n}, q] \cap \mathbb{Q} \in \mathcal{A}$ for all $n \in \mathbb{N}$, whence $\{q\} = \bigcap_{n=1}^{\infty}(q - \frac{1}{n}, q] \cap \mathbb{Q} \in \Sigma(\mathcal{A})$. Hence, $\Sigma(\mathcal{A})$ contains all finite and countable subsets of $\mathbb{Q}$, so countability of $\mathbb{Q}$ implies that $\mathcal{P}(\mathbb{Q}) \subseteq \Sigma(\mathcal{A})$. □

(ii) Define μ_0 on $\mathcal{A}$ by $\mu_0(\emptyset) = 0$ and $\mu_0(A) = \infty$ for $A \neq \emptyset$. Prove that μ_0 is a premeasure on $\mathcal{A}$, and that there is more than one measure on $\mathcal{P}(\mathbb{Q})$ whose restriction to $\mathcal{A}$ is μ_0.

Proof. □

Problem 5

A *metric measure space* is a triple (X, d, μ) where (X, d) is a metric space and $\mu \mathcal{B}_{(X,d)} \to [0, +\infty]$ is a measure. We say that $E \subseteq X$ is a *continuity set*, if $\mu(\overline{E} \setminus \operatorname{Int}(E)) = 0$. For this problem, fix a metric measure space (X, d, μ).

(i) Show that the collection of continuity sets forms an algebra of sets.
(ii) Show that if $x \in X$, $r > 0$ and $\mu(B_r(x, d)) < +\infty$, then there is an $s \in (0, r)$ so that $B_s(x, d)$ is a continuity set.
(iii)

Problem 6

Let (X, d) be a metric space and μ, ν be finite Borel measures on X with $\mu(X) = \nu(X)$. Let $\mathcal{A} = \{E \in \mathcal{B}_{(X,d)} : \mu(E) = \nu(E)\}$.

(i) Show that if $F \subseteq E$ and $F, E \in \mathcal{A}$, then $E \setminus F \in \mathcal{A}$. Also show that if $(E_n)_{n=1}^{\infty}$ is an increasing sequence of elements of $\mathcal{A}$, then $\bigcup_{n=1}^{\infty} E_n \in \mathcal{A}$.

Proof. As $E, F \in \mathcal{A}$, $\mu(E) = \nu(E)$ and $\mu(F) = \nu(F)$. Then

$$\mu(E \setminus F) = \mu(E) - \mu(F) = \nu(E) - \nu(F) = \nu(E \setminus F)$$

so $E \setminus F \in \mathcal{A}$. Now suppose that $(E_n)_{n=1}^{\infty}$ is an increasing sequence of elements of $\mathcal{A}$. By continuity from above,

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n) = \lim_{n\to\infty} \nu(E_n) = \nu\left(\bigcup_{n=1}^{\infty} E_n\right),$$

so $\bigcup_{n=1}^{\infty} E_n \in \mathcal{A}$. $\square$

(ii) Given a nonempty $F \subseteq X$ closed and $x \in X$, define $d(x, F) = \inf_{y \in F} d(x, y)$. Show that $x \mapsto d(x, F)$ is continuous and $F = \{x \in X : d(x, F) = 0\}$.

Proof. Suppose $x, y \in X$. For $z \in F$,

$$d(x, F) \leq d(x, z) \leq d(x, y) + d(y, z) \implies d(x, F) - d(x, y) \leq d(y, z).$$

As this holds for arbitrary $z \in F$, it follows that $d(x, F) - d(x, y) \leq d(y, F)$, so $d(x, F) - d(y, F) \leq d(x, y)$. By symmetry, $d(y, F) - d(x, F) \leq d(x, y)$, so $|d(x, F) - d(y, F)| \leq d(x, y)$. Thus, the function $x \mapsto d(x, F)$ is $1 - \mathit{Lipschitz}$ whence it is continuous.

Clearly $F \subseteq \{x \in X : d(x, F) = 0\}$, so it suffices to show the reverse containment. Suppose that $x \in X$ such that $d(x, F) = 0$. For all $n \in \mathbb{N}$, there exists an $f_n \in F$ such that $0 \leq d(x, f_n) < \frac{1}{n}$. It follows that $d(x, f_n) \xrightarrow{n\to\infty} 0$, so $f_n \xrightarrow{n\to\infty} x$. Thus x is a limit point of F, so F being closed implies that $x \in F$. $\square$

(iii) Show that $\{U \subseteq X : U \text{ is open}\} \subseteq \mathcal{A}$ if an only if $\{F \subseteq X : F \text{ is closed}\} \subseteq \mathcal{A}$.

Proof.
$\underline{\implies}$: Suppose that $\{U \subseteq X : U \text{ is open}\} \subseteq \mathcal{A}$. Take $F \subseteq X$ such that F is closed.
$\underline{\impliedby}$:

$\square$