Reading:

- For this homework: 1.1-1.3
- For Wednesday, January 26: 1.3, and 1.4 up to Theorem 1.11
- For Monday, January 31: 1.4, and 1.5 up to Theorem 1.16

Problem 1.

Folland Chapter 1, Problem 1

Problem 2.

Folland Chapter 1, Problem 4

Problem 3.

Let X, Y be sets and $f\colon X \to Y$.

(i) If $\Sigma \subseteq \mathcal{P}(X)$ is a σ-algebra, show that

$$\{E \subseteq Y : f^{-1}(E) \in \Sigma\}$$

is a σ-algebra.

(ii) If $\Sigma \subseteq \mathcal{P}(Y)$ is a σ-algebra, show that

$$\{E \subseteq X : E = f^{-1}(F) \text{ for some } F \in \Sigma\}$$

is a σ-algebra.

(iii) If Y is countable, show that

$$\{E \subseteq X : E = f^{-1}(F) \text{ for some } F \subseteq Y\}$$

is the σ-algebra generated by $\{f^{-1}(\{y\}) : y \in Y\}$.

Problem 4.

(i) Suppose that X is a set. Let J be a countable set and suppose that $X = \bigsqcup_{j \in J} A_j$ with $A_j \neq \varnothing$ for all $j \in J$. Show that the σ-algebra generated by $(A_j)_{j \in J}$ is $\{\bigcup_{j \in J_0} A_j : J_0 \subseteq J\}$.

(ii) Show that the σ-algebra generated by a finite collection of sets is finite.

Problem 5.

Let (X, d_X) be a metric space. A collection $\mathcal{B}$ of open sets in X is said to be *a basis* if for every open $U \subseteq X$ and every $x \in U$, there is a $B \in \mathcal{B}$ so that $x \in B \subseteq U$. If $\mathcal{B}$ is a *countable* basis of open sets in X, show that the σ-algebra generated by $\mathcal{B}$ coincides with the Borel sets in X.

Remark: Existence of a countable basis for a metric space is equivalent to separability (not needed for the problem, but good to think about if you haven't seen this before). This problem generalizes what was mentioned in class that intervals with rational endpoints, or rectangles with rational vertices generate the family of Borel sets.

Problem to think about, do not have to turn in: Folland Chapter 1, Problem 5.

(Problem 4 may be relevant).