

MATH 7310 Homework 9

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Problem 3

(a): Let (X, Σ) be a measurable space. Let $M(\Sigma)$ be the vector space of complex measures on Σ with the total variation norm $\|\mu\| = |\mu|(X)$. Show that $M(\Sigma)$ is a Banach space.

Suggestion: it may be helpful to use that for $\mu \in M(\Sigma)$ we have

$$\sum_{n=1}^{\infty} |\mu(E_n)| \leq \|\mu\|$$

where $(E_n)_{n=1}^{\infty}$ is a sequence of pairwise disjoint elements of Σ (this is a consequence of a prior problem on this homework).

(b): Fix a positive, σ -finite measure μ on Σ . Show that the map $J : L^1(X, \mu) \rightarrow M(\Sigma)$ given by $J(f) = f d\mu$ is a linear isometry with closed image.

Proof. Let We wish to show that for $f, g \in L^1(X, \mu)$ and $\alpha \in \mathbb{C}$, $J(\alpha f + g) = \alpha J(f) + J(g)$, after which showing that $\|J(f)\|_{M(\Sigma)} = \|f\|_{L^1(\mu)}$ would imply that J is a linear isometry.

Let $f \in L^1(X, \mu)$. We compute

$$\|J(f)\|_{M(\Sigma)} = |J(f)|(X) = J(f)(X) = \int_X f d\mu = \|f\|_{L^1(\mu)}.$$

Suppose that $(J(f_n))_{n=1}^{\infty}$ converges to ν in $M(\Sigma)$ where $(f_n)_{n=1}^{\infty}$ is in $L^1(X, \mu)$. So

$$\|J(f_n) - \nu\|_{M(\Sigma)} \xrightarrow{n \rightarrow \infty} 0$$

□

(c): Suppose that $\mu, \nu \in M(\Sigma)$, and let $d\nu = f d\mu + d\lambda$ with $\lambda \perp \mu$ be the Lebesgue-Radon-Nikodym decomposition. Show that

$$\|\mu - \nu\| = \|1 - f\|_{L^1(\mu)} + \|\lambda\|.$$

Problem 5

Let $\psi : \mathbb{R} \rightarrow \mathbb{R}$ be given as $\psi = \mathbb{1}_{[0,1/2)} - \mathbb{1}_{[1/2,1]}$. For $n, k \in \mathbb{Z}$ define $h_{n,k}(t) = 2^{n/2}\psi(2^n t - k)$. Show that $\mathcal{E} = \{1\} \cup \{h_{n,k} : n \in \mathbb{N} \cup \{0\}, 0 \leq k < 2^n\}$ is an orthonormal basis for $L^2([0, 1])$.