MATH 7310 Homework 0

James Harbour

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Problem 1

Let J be an infinite set, and $(t_j)_{j \in J}$ nonnegative real numbers. We define $\sum_{j \in J} t_j = \sup_F \sum_{j \in F} t_j$ where the supremum is over all finite subsets of J, and is equal to ∞ if $\left\{ \sum_{j \in F} t_j : F \subseteq J \text{ is finite} \right\}$ is not bounded above.

(i) Suppose that $\sum_{j \in J} t_j < \infty$. Prove that for every $\varepsilon > 0$, there is a finite $F \subseteq J$ so that $\sum_{j \in J \setminus F} t_j < \varepsilon$.

Proof. Let $\varepsilon > 0$ an $J_0 = \{ j \in J : t_j > 0 \}$. By proposition 0.20, as $\sum_{j \in J} t_j < \infty$, J_0 is countably infinite. Moreover, letting $g : \mathbb{N} \to J_0$ be a bijection, proposition 0.20 gives that

$$\sum_{n=1}^{\infty} t_{g(n)} = \sum_{j \in J} t_j < \infty$$

As this sum is nonnegative and converges, there exists a $k \in \mathbb{N}$ such that

$$\sum_{n=k+1}^{\infty} t_{g(n)} < \varepsilon.$$

Letting $F = g(\{1, 2, \ldots, k\})$, it follows that

$$\sum_{j \in J \setminus F} t_j = \sum_{n=k+1}^{\infty} t_{g(n)} < \varepsilon.$$

$\square$

(ii) Suppose that $(\alpha_j)_{j \in J}$ are complex numbers and $\sum_{j \in J} |\alpha_j| < \infty$. Suppose further that $J_0 = \{ j \in J : \alpha_j \neq 0 \}$ is infinite. Suppose that $\phi : \mathbb{N} \to J_0$, $\psi : \mathbb{N} \to J_0$ are two bijections. Prove that

$$\sum_{n=1}^{\infty} \alpha_{\phi(n)} = \sum_{n=1}^{\infty} \alpha_{\psi(n)}.$$

Proof. By proposition 0.20, as $\sum_{j \in J} |\alpha_j| < \infty$, the set $\{ j \in J : \alpha_j \neq 0 \} = \{ j \in J : \alpha_j \neq 0 \} = J_0$ is countable. Moreover,

$$\sum_{n=1}^{\infty} |\alpha_{\phi(n)}| = \sum_{j \in J} |\alpha_j| < \infty$$

$$\sum_{n=1}^{\infty} |\alpha_{\psi(n)}| = \sum_{j \in J} |\alpha_j| < \infty,$$

so the sums $\sum_{n=1}^{\infty} \alpha_{\phi(n)}$ and $\sum_{n=1}^{\infty} \alpha_{\psi(n)}$ are absolutely convergent. As such, they are equal since the value of an absolutely convergent series is unchanged under rearrangement. $\square$

Problem 2

It follows from Problem 1 that if $(\alpha_j)_{j \in J}$ are complex numbers and $\sum_{j \in J} |\alpha_j| < \infty$, we may define $\sum_{j \in J}$ as follows: let $J_0 = \{j : \alpha_j \neq 0\}$. If J_0 is finite, then $\sum_{j \in J} = \sum_{j \in J_0}$. If J_0 is infinite, choose a bijection $\phi : \mathbb{N} \to J_0$, and define

$$\sum_{j \in J} \alpha_j = \sum_{n=1}^{\infty} \alpha_{\phi(n)}.$$

Suppose that $(\alpha_j)_{j \in J}$ are complex numbers and $\sum_{j \in J} |\alpha_j| < \infty$. Show that $\sum_{j \in J} |\alpha_j|$ is the unique complex number s satisfying the following property. For every $\varepsilon > 0$, there is a finite set $F \subseteq J$ so that if $F \subseteq E \subseteq J$ and E is finite, then

$$\left| s - \sum_{j \in E} \alpha_j \right| < \varepsilon.$$

Proof. Let $\varepsilon > 0$. Since $\sum_{j \in J} |\alpha_j| < \infty$, there exists a finite $F \subseteq J$ so that $\sum_{j \in J \setminus F} |\alpha_j| < \frac{\varepsilon}{2}$. If $F \subseteq E \subseteq J$ and E is finite, then $J \setminus E \subseteq J \setminus F$ implies that $\sum_{j \in J \setminus E} |\alpha_j| \leq \sum_{j \in J \setminus F} |\alpha_j| < \varepsilon$.
By the triangle inequality,

$$\left| s - \sum_{j \in E} \alpha_j \right| = \left| \sum_{j \in J \setminus E} \alpha_j \right| \leq \sum_{j \in J \setminus E} |\alpha_j| < \varepsilon.$$

Now suppose that $s, s' \in \mathbb{C}$ satisfy the desired property. Let $\varepsilon > 0$. Then there exists finite sets $F, F' \subseteq J$ satisyfing the above property for s and s' respectively. Then $F \cup F'$ is finite and $F, F' \subseteq F \cup F' \subseteq J$, so

$$|s - s'| = \left| s - \sum_{j \in F \cup F'} \alpha_j + \sum_{j \in F \cup F'} \alpha_j - s' \right| \leq \left| s - \sum_{j \in F \cup F'} \alpha_j \right| + \left| \sum_{j \in F \cup F'} \alpha_j - s' \right| < 2\varepsilon.$$

Hence $s = s'$. $\qquad\square$

Problem 3

Suppose that I, J are sets, and $(a_{ij})_{i \in I} j \in J$ are nonnegative real numbers. Prove that

$$\sum_{j \in J} \left(\sum_{i \in I} a_{ij} \right) = \sum_{(i,j) \in I \times J} a_{ij} = \sum_{i \in I} \left(\sum_{j \in J} a_{ij} \right)$$

Proof. On one hand, observe that, since $(a_{ij})_{i \in I} j \in J$ are nonnegative real numbers, a finite sum of suprema is equal to a suprema of finite sums. Hence,

$$\sup_{\substack{F \subseteq I \\ F \text{ finite}}} \sum_{i \in F} \left(\sup_{\substack{G \subseteq J \\ G \text{ finite}}} \sum_{j \in G} a_{ij} \right) = \sup_{\substack{F \subseteq I \\ F \text{ finite}}} \sup_{\substack{G \subseteq J \\ G \text{ finite}}} \left(\sum_{i \in F} \sum_{j \in G} a_{ij} \right) = \sup_{\substack{F \subseteq I \\ F \text{ finite}}} \sup_{\substack{G \subseteq J \\ G \text{ finite}}} \left(\sum_{(i,j) \in F \times G} a_{ij} \right)$$

$$= \sup_{\substack{X \subseteq I \times J \\ X \text{ finite}}} \left(\sum_{(i,j) \in X} a_{ij} \right) = \sum_{(i,j) \in I \times J} a_{ij}.$$

The other direction follows similarly. $\qquad\square$