Reading:

- For this homework: 5.1/6.1, 5.5
- For Wedneday, March 23:
- For Monday, March 28:

Problem 1.

Let (X, Σ, μ) be a measure space.

(i) Folland Chapter 2, Problem 36.

(ii) Let $\Sigma_f = \{E \in \Sigma : \mu(E) < +\infty\}$. Define an equivalence relation on Σ_f by $E \sim F$ if $\mu(E \Delta F) = 0$. Let Ω be the space of equivalence classes of elements of Σ_f, and define a metric ρ on Ω by $\rho([E], [F]) = \mu(E\Delta F)$ (this is a minor modification of Problem 12 from Chapter 1 of Folland, you do not have to prove this is a metric). Show that the map $\iota \colon \Omega \to L^1(X, \mu)$ given by $\iota([E]) = 1_E$ is an isometry with closed image.

(iii) Show that (Ω, ρ) is a complete metric space.

Note: applying the Baire Category Theorem together with Problem 4 of the practice midterm gives a (less messy, but less constructive) solution to Problem 30 of Chapter 1 of Folland.

Problem 2.

If X, Y are sets, and $f \colon X \to \mathbb{C}, g \colon Y \to \mathbb{C}$, we define $f \otimes g \colon X \times Y \to \mathbb{C}$ by $(f \otimes g)(x, y) = f(x)g(y)$. Fix $1 \leq p < +\infty$.

(a) Let $(X, \Sigma, \mu), (Y, \mathcal{F}, \nu)$ be σ-finite measure spaces. Show that if $f \in L^p(X, \mu), g \in L^p(Y, \nu)$, then $\|f \otimes g\|_p = \|f\|_p \|g\|_p$.

(b) Let $(Z, \mathcal{O}, \zeta)$ be a finite measure space. Suppose that $\mathcal{A} \subseteq \mathcal{O}$ is an algebra which generates the σ-algebra of $\mathcal{O}$. Use the monotone class lemma to show that $\{1_A : A \in \mathcal{A}\}$ is dense in $\{1_E : E \in \mathcal{O}\}$ in the L^p-norm for all $1 \leq p < +\infty$. Hint: this is similar to a proof from the Wednesday, March 2nd class.

(c) Let $(X, \Sigma, \mu), (Y, \mathcal{F}, \nu)$ be finite measure spaces. Use the previous part to show Show that $\{1_E : E \in \Sigma \otimes \mathcal{F}\} \subseteq \overline{\mathrm{Span}}^{\|\cdot\|_p}\{1_E \otimes 1_F : E \in \Sigma, F \in \mathcal{F}\}$. Use this to show that $\overline{\mathrm{Span}}^{\|\cdot\|_p}\{1_E \otimes 1_F : E \in \Sigma, F \in \mathcal{F}\} = L^p(X \times Y, \mu \otimes \nu)$.

Suggestion: show that if $\mathcal{A}$ is the algebra generated by $\{E \times F : E \in \Sigma, F \in \mathcal{F}\}$, then $\{1_A : A \in \mathcal{A}\} \subseteq \mathrm{Span}\{1_E \otimes 1_F : E \in \Sigma, F \in \mathcal{F}\}$.

(d) Let $(X, \Sigma, \mu), (Y, \mathcal{F}, \nu)$ be σ-finite measure spaces. Suppose that $D_X \subseteq L^p(X, \mu), D_Y \subseteq L^p(Y, \nu)$ and that

$$\overline{\mathrm{Span}}^{\|\cdot\|_p}(D_X) = L^1(X, \mu), \overline{\mathrm{Span}}^{\|\cdot\|_p}(D_Y) = L^1(X, \mu).$$

Show that $\overline{\mathrm{Span}}^{\|\cdot\|_p}(\{f \otimes g : f \in D_X, g \in D_Y\}) = L^p(X \times Y, \mu \otimes \nu)$.

Problem 3.

Folland, Chapter 6, Problem 7.

Problem 4.

Folland, Chapter 6, Problem 11

Problem 5.

Folland, Chapter 6, Problem 15

Problem 6.

Folland, Chapter 5, Problem 56.

Problems to think about, do not turn in

Problem 7.

Fix $n \in \mathbb{N}$. For $y \in \mathbb{R}^n$, define $\tau_y \colon L^1(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ by $\tau_y(f)(x) = f(x-y)$.

(a) Show that if $f \in L^1(\mathbb{R}^n)$, then

$$\lim_{y \to 0} \|\tau_y f - f\|_1 = 0.$$

Hint: first show that the set of f's for which this is true is a closed, linear subspace of $L^1(\mathbb{R}^n)$. Then reduce to a class of f's for which this limiting formula is easier to see.

(b) If $f \in L^1(\mathbb{R}^n), g \in L^\infty(\mathbb{R}^n)$. Show that

$$\|f * g\|_{\sup} \le \|f\|_1 \|g\|_\infty \text{ where } \|h\|_{\sup} = \sup_{x \in \mathbb{R}^n} |h(x)|.$$

(c) Show that

$$\|\tau_y(f * g) - f * g\|_{\sup} \le \|\tau_y f - f\|_1 \|g\|_\infty.$$

Use this to show that $f * g$ is a uniformly continuous function.

(d) Suppose that $E \subseteq \mathbb{R}^n$ is Lebesgue measurable with $m(E) > 0$. Show that there is an open $U = \{x : 1_E * 1_{-E}(x) > 0\}$ is a open set in $\mathbb{R}^n$ with $0 \in U$ and $U \subseteq E - E$.

Problem 8.

Folland, Chapter 5, Problem 55