MATH 7752 - HOMEWORK 9
DUE FRIDAY 04/08/22

Reminder: A field F of characteristc $p > 0$ is perfect if the Frobenius map $\varphi : x \mapsto x^p$ is surjective.

(1) Let $K/L/F$ be a tower of algebraic extensions. Show that K/F is separable if and only if K/L and L/F are separable.

(2) Let K/F be a finite separable extension. Show that there is a finite number of fields L such that $F \subseteq L \subseteq K$.

(3) Let F be a field of $\mathrm{char}(F) = p > 0$. Show that F admits a finite inseparable extension K/F if and only if F is not perfect.

(4) Let F be a field of characteristic $p > 0$ and let K/F be an extension.
  (a) Let $E = \{\alpha \in K : \alpha^{p^n} \in F, \text{ for some } n \geq 1\}$. Prove that E is a subfield of K.
  (b) Show that every F-automorphism of K is automatically an E-automorphism.

(5) Let F be a field of characteristic $p > 0$ and let K/F be a finite extension.
  (i) Let $\alpha \in K$. Show that either $\alpha^{p^n} \in F$ for some $n \geq 1$, or there exists some $m \geq 1$ such that $\alpha^{p^m} \notin F$ and the element α^{p^m} is separable over F.
  (ii) Suppose that no element of $K \setminus F$ is separable over F. (Such extensions are called *purely inseparable*). Deduce that for every $\alpha \in K$ there exists some $n \geq 1$ (depending on α) such that $\alpha^{p^n} \in F$.

(6) The purpose of this problem is to show that the primitive element theorem is not true for inseparable extensions. Let p be a prime number. Let t be a transcendental element over $\mathbb{F}_p$ and let $F = \mathbb{F}_p(t)$. Let s be a transcendental element over F and let $K = F(s)$. Consider the polynomial $f(x) = (x^p - t)(x^p - s) \in K[x]$ and let L be its splitting field.
  (a) Prove that $[L : K] = p^2$.
  (b) Show that for every $\gamma \in L$, it follows that $\gamma^p \in K$.
  (c) Show that the extension L/K is not simple.