

MATH 7310 Homework 8

James Harbour

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Problem 1

Let \mathcal{H} be a Hilbert space.

(a): Prove that, for any $x, y \in \mathcal{H}$,

$$\langle x, y \rangle = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2)$$

Proof.

$$\begin{aligned}\|x \pm y\|^2 &= \|x\|^2 + \|y\|^2 \pm 2\operatorname{Re}(\langle x, y \rangle) \\ \|x \pm iy\|^2 &= \|x\|^2 + \|y\|^2 \pm 2\operatorname{Re}(\langle x, iy \rangle) = \|x\|^2 + \|y\|^2 \pm 2\operatorname{Im}(\langle x, y \rangle).\end{aligned}$$

We compute that

$$\begin{aligned}\frac{1}{4}(\|x + y\|^2 - \|x - y\|^2) &= \operatorname{Re}(\langle x, y \rangle) \\ \frac{1}{4}(\|x + iy\|^2 - \|x - iy\|^2) &= \operatorname{Im}(\langle x, y \rangle)\end{aligned}$$

whence the identity follows by noting $\langle x, y \rangle = \operatorname{Re}\{\langle x, y \rangle\} + i\operatorname{Im}(\langle x, y \rangle)$. □

(b): If \mathcal{H}' is another Hilbert space, prove that a linear map from \mathcal{H} to \mathcal{H}' is unitary if and only if it is isometric and surjective.

Proof.

\implies : Suppose that $T : \mathcal{H} \rightarrow \mathcal{H}'$ is unitary. Then T is surjective by definition. Moreover, for $x \in \mathcal{H}$, $\|x\| = \langle x, x \rangle = \langle Tx, Tx \rangle = \|Tx\|$, whence T is an isometry by linearity.

\impliedby : Suppose that $T : \mathcal{H} \rightarrow \mathcal{H}'$ is isometric and surjective. Then

$$\begin{aligned}\langle Tx, Ty \rangle &= \frac{1}{4}(\|T(x) + T(y)\|^2 - \|T(x) - T(y)\|^2 + i\|T(x) + T(iy)\|^2 - i\|T(x) - T(iy)\|^2) \\ &= \frac{1}{4}(\|T(x + y)\|^2 - \|T(x - y)\|^2 + i\|T(x + iy)\|^2 - i\|T(x - iy)\|^2) \\ &= \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2) = \langle x, y \rangle,\end{aligned}$$

so T is an isometry. Now suppose that $T(x) = 0$. Then $0 = \langle Tx, Tx \rangle = \langle x, x \rangle$ whence $x = 0$, so T is also injective and thus unitary. □

Problem 2

For $n \in \mathbb{Z}$, define $e_n : [0, 1] \rightarrow \mathbb{C}$ by $e_n(t) = e^{2\pi i n t}$.

(a): Show that $\{e_n\}_{n \in \mathbb{Z}}$ is an orthonormal set in $L^2([0, 1])$.

Proof. Observe that, for $n, m \in \mathbb{Z}$ with $n \neq m$

$$\langle e_n, e_m \rangle = \int_0^1 e^{2\pi i n t} \overline{e^{2\pi i m t}} dt = \int_0^1 e^{2\pi i (n-m)t} dt = \left[\frac{1}{2\pi i (n-m)} e^{2\pi i (n-m)t} \right]_{t=0}^{t=1} = \frac{1}{2\pi i (n-m)} (e^{2\pi i (n-m)} - 1) = 0.$$

On the other hand, for $n \in \mathbb{Z}$,

$$\langle e_n, e_n \rangle = \int_0^1 e^{2\pi i n t} \overline{e^{2\pi i n t}} dt = \int_0^1 1 dt = 1$$

so $\{e_n\}_{n \in \mathbb{Z}}$ is an orthonormal set in $L^2([0, 1])$. □

(b): Show that $\{f \in C([0, 1]) : f(1) = f(0)\} = \{g \circ e_1 : g \in C(S^1)\}$, where $S^1 = \{z \in \mathbb{C} : |z| = 1\}$.

Proof. Noting that e_1 is just the composition of the canonical projection map $[0, 1] \rightarrow [0, 1]/(0 \sim 1)$ with the homeomorphism $[0, 1]/(0 \sim 1) \cong S^1$ given by the same formula (where well-definedness follows from the quotiented set), the claim follows from the universal property of the quotient topology. □

(c): The Stone-Weierstrass theorem says that if (X, d) is a compact metric space and $A \subseteq C(X)$ is a linear subspace so that:

- $1 \in A$,
- $f \in A$ implies $\bar{f} \in A$,
- $f, g \in A$ implies that $fg \in A$,
- If $x \in X$, then there are $f, g \in A$ with $f(x) \neq g(x)$,

then A is dense in $C(X)$ for the uniform norm $\|f\|_u = \sup_{x \in X} |f(x)|$. Use the Stone-Weierstrass theorem to show that $\overline{\text{Span}^{\|\cdot\|_u} \{e_n : n \in \mathbb{Z}\}} = \{f \in C([0, 1]) : f(1) = f(0)\}$.

Proof. For $n \in \mathbb{Z}$, define $p_n : S^1 \rightarrow \mathbb{C}$ by $p_n(x) = x^n$ and set $A = \text{Span}\{p_n : n \in \mathbb{Z}\} \subseteq C(S^1)$. Note that $\overline{p_n} = p_{-n}$, $p_n p_m = p_{nm}$, $1 = p_0$, whence by linearity the first three properties above hold for A .

To see that the last property holds for A , take arbitrary $x \in S^1$. If $x^n \neq 1$ for all $n \in \mathbb{Z} \setminus \{0\}$, then $p_1(x) = x \neq x^2 = p_2(x)$. Suppose that $x^n = 1$ for some $n \in \mathbb{Z} \setminus \{0\}$. Then $x^{-n} = 1$, so we may assume without loss of generality that $n \in \mathbb{N}$. If $x = 1$, then $p_1(x) = 1 \neq 0 = p_0(x)$ and $p_1, p_0 \in A$. Otherwise, let $m \geq 2$ be the smallest such $m \in \mathbb{N}$ such that $x^m = 1$. Then $p_m(x) = x^m \neq x^{m-1} = p_{m-1}(x)$, as desired.

Thus, by the Stone-Weierstrass theorem, $\overline{A}^{\|\cdot\|_u} = C(S^1)$. Now take $f \in \{f \in C([0, 1]) : f(1) = f(0)\}$. Then by part (b) there exists a $g \in C(S^1)$ such that $f = g \circ e_1$. Now take a sequence $(a_n)_{n=1}^\infty$ in A such that $a_n \rightarrow g$ with respect to $\|\cdot\|_u$.

Then $a_n \circ e_1 \rightarrow g \circ e_1$ with respect to $\|\cdot\|_u$. Lastly, noting that $a_n \circ e_n \in \text{Span}\{e_n : n \in \mathbb{Z}\}$, it follows that $f \in \overline{\text{Span}^{\|\cdot\|_u} \{e_n : n \in \mathbb{Z}\}}$. □

(d): Show that $\text{Span}\{e_n : n \in \mathbb{Z}\}$ is dense in $L^2([0, 1])$ and use this to show that $\{e_n\}_{n \in \mathbb{Z}}$ is an orthonormal basis for $L^2([0, 1])$.

Proof. Observe that, for any measurable $f : [0, 1] \rightarrow \mathbb{C}$, by Hölder's inequality applied twice we have

$$\|f\|_2 \leq \|f\|_1 \leq \|f\|_\infty \|1\|_1 = \|f\|_\infty m([0, 1]) = \|f\|_\infty \leq \|f\|_u.$$

Now suppose that $f \in C([0, 1])$ such that $f(0) = f(1)$. Then there exists $f_n \in \text{Span}\{e_n : n \in \mathbb{Z}\}$ such that $\|f_n - f\|_u \rightarrow 0$. Then $\|f_n - f\|_2 \leq \|f_n - f\|_u \rightarrow 0$, so $\overline{\text{Span}}^{\|\cdot\|_2}\{e_n : n \in \mathbb{Z}\} = \{f \in C([0, 1]) : f(1) = f(0)\}$ which equals $C([0, 1])$ modulo almost-everywhere equality. Moreover, the closure of $C([0, 1])$ in the L^2 -norm contains equivalence classes of the indicator functions of intervals via the hill approximation, so it is in fact all of L^2 . Thus by transitivity of topological density, $\text{Span}\{e_n : n \in \mathbb{Z}\}$ is dense in $L^2([0, 1])$.

As $\{e_n : n \in \mathbb{Z}\}$ is orthonormal and the L^2 -norm-closure of its span is in fact all of $L^2([0, 1])$, it follows that $\{e_n\}_{n \in \mathbb{Z}}$ is an orthonormal basis for $L^2([0, 1])$. \square

Problem 3

(a): Let (X, Σ, μ) , (Y, \mathcal{F}, ν) be σ -finite measure spaces such that $L^2(\mu)$ and $L^2(\nu)$ are separable. If $\{f_m\}$ and $\{g_n\}$ are orthonormal bases for $L^2(\mu)$ and $L^2(\nu)$ and $h_{mn}(x, y) = f_m(x)g_n(y)$, then $\{h_{mn}\}$ is an orthonormal basis for $L^2(\mu \otimes \nu)$.

Proof.

$$\overline{\text{Span}}^{\|\cdot\|_2}\{f_m : m \in \mathbb{Z}\} = L^2(\mu) \quad \overline{\text{Span}}^{\|\cdot\|_2}\{g_m : m \in \mathbb{Z}\} = L^2(\nu)$$

By Fubini-Tonelli,

$$\begin{aligned} \langle f_m \otimes g_n, f_k \otimes g_l \rangle &= \int_{X \times Y} f_m \otimes g_n \overline{f_k \otimes g_l} d(\mu \otimes \nu) = \int_{X \times Y} h_{mn} \overline{h_{kl}} d(\mu \otimes \nu) \\ &= \int_X f_m(x) \overline{f_k(x)} \int_Y g_n(y) \overline{g_l(y)} d\nu(y) d\mu(x) = \int_X f_m(x) \overline{f_k(x)} \delta_{nl} d\mu(x) = \delta_{mk} \delta_{nl} = \delta_{(m,n),(k,l)} \end{aligned}$$

so $\{h_{mn}\}$ is an orthonormal set in $L^2(\mu \otimes \nu)$. By homework 7 problem 1 part (d), as $\overline{\text{Span}}^{\|\cdot\|_2}\{f_m : m \in \mathbb{Z}\} = L^2(\mu)$ and $\overline{\text{Span}}^{\|\cdot\|_2}\{g_m : m \in \mathbb{Z}\} = L^2(\nu)$, it follows that

$$\overline{\text{Span}}^{\|\cdot\|_2}\{h_{mn} : m, n \in \mathbb{Z}\} = \overline{\text{Span}}^{\|\cdot\|_2}\{f_m \otimes g_n : m, n \in \mathbb{Z}\} = L^2(\mu \otimes \nu)$$

so $\{h_{mn}\}$ is in fact an orthonormal basis for $L^2(\mu \otimes \nu)$. \square

(b): For $k \in \mathbb{N}$, and $n = (n_1, \dots, n_k) \in \mathbb{Z}^k$, define $e_n \in L^2([0, 1]^k)$ by

$$e_n(x) = \prod_{j=1}^k e^{2\pi i n_j x}.$$

Show that $\{e_n\}_{n \in \mathbb{Z}^k}$ is an orthonormal basis of $L^2([0, 1]^k)$.

Problem 4

(a): Let (X, Σ, μ) be a σ -finite measure space, \mathcal{F} a sub- σ -algebra of Σ , and $\nu = \mu|_{\mathcal{F}}$. If $f \in L^1(\mu)$, prove that there exists $g \in L^1(\nu)$ (thus g is \mathcal{F} -measurable) such that $\int_E f d\mu = \int_E g d\nu$ for all $E \in \mathcal{F}$; also prove that if g' is another such function then $g = g'$ ν -a.e.

Proof. Define a new measure λ by $\lambda(E) = \int_E f d\mu$ for all $E \in \Sigma$. If $E \in \mathcal{F}$ with $\mu|_{\mathcal{F}}(E) = \nu(E) = 0$, then $\lambda(E) = \int_E f d\mu = 0$, so $\lambda|_{\mathcal{F}} \ll \nu$. Let $g = \frac{d\lambda|_{\mathcal{F}}}{d\nu} \in L^1(\nu)$, so $d\lambda|_{\mathcal{F}} = g d\nu$. Then, for all $E \in \mathcal{F}$,

$$\int_E g d\nu = \lambda(E) = \int_E f d\mu.$$

Now suppose that g' is another such function. □

(b): Show that $\int gh d\nu = \int fh d\mu$ for all $h \in L^1(\nu)$.

Proof. We have the following relation

$$f d\mu = d\lambda_f = g d\nu,$$

so by homework 5 problem 1 part (a),

$$\int fh d\mu = \int h d\lambda_f = \int gh d\nu.$$

□

Problem 5

Let (X, Σ, μ) be a probability space. For a sub- σ -algebra $\mathcal{F} \subseteq \Sigma$, and $f \in L^1(X, \Sigma, \mu)$, let $\mathbb{E}_{\mathcal{F}}(f)$ be the conditional expectation of f onto \mathcal{F} .

(a): Show that $\mathbb{E}_{\mathcal{F}}(fg) = \mathbb{E}_{\mathcal{F}}(f)g$ for all $g \in L^\infty(X, \mathcal{F}, \mu)$.

Proof. Noting that, for $E \in \mathcal{F}$, as $\mu(X) < +\infty$ and $g \in L^\infty(X, \mathcal{F}, \mu|_{\mathcal{F}})$, we have that $g \cdot \mathbb{1}_E \in L^1(X, \mathcal{F}, \mu|_{\mathcal{F}})$ whence by problem 4 part (b)

$$\int_E \mathbb{E}_{\mathcal{F}}(f)g d\mu|_{\mathcal{F}} = \int \mathbb{E}_{\mathcal{F}}(f)(g\mathbb{1}_E) d\mu|_{\mathcal{F}} = \int fg\mathbb{1}_E d\mu = \int_E fg d\mu.$$

Thus, by the uniqueness of conditional expectation in problem 4, $\mathbb{E}_{\mathcal{F}}(fg) = \mathbb{E}_{\mathcal{F}}(f)g$. □

(b): If $f \in L^2(X, \Sigma, \mu)$, show that $\mathbb{E}_{\mathcal{F}}(f)$ is the orthogonal projection of f onto $L^2(X, \mathcal{F}, \mu)$ in the decomposition

$$L^2(X, \Sigma, \mu) = L^2(X, \mathcal{F}, \mu) + L^2(X, \mathcal{F}, \mu)^\perp.$$

Note: one difficulty you'll a priori face is that we do not yet know that $f \in L^2$ implies that $\mathbb{E}_{\mathcal{F}}(f) \in L^2$. However, one can note that you can characterize the orthogonal projection g of f onto $L^2(X, \mathcal{F}, \mu)$ by $\langle f, h \rangle = \langle g, h \rangle$ for all $h \in L^2(X, \mathcal{F}, \mu)$ (you should prove this if you use it), and this can be used to show that this projection is the conditional expectation.

Problem 6

Show that if ν is a signed measure, then E is ν -null if and only if $|\nu|(E) = 0$. Also, prove that if μ and ν are signed measures, then $\nu \perp \mu$ if and only if $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.

Proof.

\implies : Suppose that E is ν -null. Let (P, N) be a *Hahn* decomposition for ν and consider positive measures ν^+ , ν^- such that $\nu = \nu^+ - \nu^-$. By uniqueness of these positive measures, $\nu^+(E) = \nu(E \cap P)$ and $\nu^-(E) = -\nu(E \cap N)$. Nullity of E for ν then implies that $\nu^+(E) = \nu(E \cap P) = 0$ and $\nu^-(E) = -\nu(E \cap N) = 0$. Thus $|\nu|(E) = \nu^+(E) + \nu^-(E) = 0$.

\Leftarrow : Suppose that $|\nu|(E) = 0$. Let $F \subseteq E$ such that $F \in \Sigma$. As $|\nu|$ is a positive measure on Σ , $|\nu|(F) = 0$, whence $\nu^+(F) = 0 = \nu^-(F)$. It follows that $\nu(F) = \nu^+(F) - \nu^-(F) = 0$, so E is ν -null.

\implies : Suppose that $\nu \perp \mu$. So there exist $E, F \in \Sigma$ such that E is μ -null and F is ν -null, $E \cap F = \emptyset$, and $E \cup F = X$. Then by the previously proven equivalence, $|\nu|(F) = 0$ whence $\nu^+(F) = 0 = \nu^-(F)$. As ν^+ , ν^- are positive measures, this implies that F is ν^+ -null and ν^- -null, so the initial decomposition of X giving singularity of ν and μ also gives $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.

\Leftarrow : Suppose that $\nu^+ \perp \mu$ and $\nu^- \perp \mu$. Then there exist $E^+, F^+, E^-, F^- \in \Sigma$ such that $E^\pm \cap F^\pm = \emptyset$, $E^\pm \cup F^\pm = X$, E^\pm is μ -null, and F^\pm is ν^\pm -null. Consider the sets $A = E^+ \cup E^-$ and $B = F^+ \cap F^-$. Note that A is a union of μ -null sets and is thus μ -null, whilst $\nu^+(B) = 0 = \nu^-(B)$ implies that $|\nu|(B) = 0$, so B is ν -null. A and B are clearly disjoint and

$$X \setminus A = X \setminus (E^+ \cup E^-) = (X \setminus E^+) \cap (X \setminus E^-) = F^+ \cap F^- = B \implies A \cup B = X.$$

□