MATH 7752 Homework 6

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Problem 1

(a) Prove that two 3×3 matrices over some field F are similar if and only if they have the same minimal and characteristic polynomials. Is the same true for 4×4 matrices?

Proof. The forward direction is clear and true for $n \times n$ matrices, so suppose that $A, B \in M_3(F)$ such that $\mu_A = \mu_B = \mu$ and $\chi_A = \chi_B = \chi$. Let $\alpha_1 | \cdots | \alpha_m$ and $\beta_1 | \cdots | \beta_n$ be the invariant factors for A and B respectively. As $\sum \deg(\alpha_i) = 3$ and $\sum \deg(\beta_i) = 3$, it follows that $m, n \leq 3$. To show that A and B are similar, it suffices to show that their invariant factors are the same as then their RCFs would be the same.

Suppose, without loss of generality, that $m \geq n$. If $m = n = 1$, then $\alpha_1 = \chi = \beta_1$. If $n = 1$ and $m > 1$, then

$$\alpha_1 \cdots \alpha_{m-1} \mu = \chi = \beta_1 = \mu \implies \alpha_1 \cdots \alpha_{m-1} = 1$$

contradicting that the invariant factors are nonunits.

If $m = n = 2$, then $\alpha_1 \mu = \beta_1 \mu \implies \alpha_1 = \beta_1$.

If $m = 3$ and $n = 2$, then all the α_i's are degree one monic whence successive divisibility forces them to be equal, say $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$ with α monic of degree 1. But then $\alpha^2 = \alpha_1 \alpha_2 = \beta_1$, whence β_1 is degree 2 and β_2 is degree at least 2, contradicting that their degrees sum to 3.

If $m = n = 3$, then each of the α_i's are equal to some linear α and each of the β_i's are equal to some linear β. Moreover, $\alpha = \alpha_3 = \mu = \beta_3 = \beta$, so all of the invariant factors are the same.

The same is not true for 4×4 matrices. Consider the matrices

$$A = \begin{pmatrix} 1 & 1 & & \\ & 1 & & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}$$

Then $\chi_A = (x-1)^4 = \chi_B$ and $\mu_A = (x-2)^2 = \mu_B$; however, A and B are already in RCF and not equal, so they are not similar. $\square$

(b) A matrix A is called idempotent if $A^2 = A$. Prove that two idempotent $n \times n$ matrices are similar if and only if they have the same rank. **Hint:** What is the minimal polynomial of an idempotent matrix? How does rank relate to eigenvalue 0?

Proof. Note that a matrix M being idempotent implies that $x^2 - x \in \mathrm{Ann}(M)$, so $\mu_M | x^2 - x$ whence $\mu_M \in \{x, x-1, x^2 - x\}$. Let A, B be two idempotent $n \times n$ matrices.

The forward direction is true in general, so it suffices to show the reverse direction. Suppose that A and B have the same rank $0 \le k \le n$. If $k = 0$, then $A = B = 0$ whence A, B are equal and thus similar, so suppose $k \neq 0$. By the rank-nullity theorem, $\dim(E_0(A)) = n - \mathrm{rk}(A) = n - k = n - \mathrm{rk}(B) = \dim(E_0(B))$. By idempotence, $\mu_A, \mu_B \in \{x, x-1, x^2 - x\}$. If either μ_A or μ_B is x, then the corresponding matrix is 0 whence $k = 0$ contradicting that $k \neq 0$. Thus $\mu_A, \mu_B \in \{x-1, x^2-x\}$. If either μ_A or μ_B is $x-1$ then the corresponding matrix is the identity whence it does not have 0 as an eigenvalue and thus $0 = \dim(E_0(A)) = \dim(E_0(B))$. Note that the other minimal polynomial cannot be $x^2 - x$ as otherwise the eigenspace corresponding to zero would have positive dimension. Thus in this case both matrices are the identity and thus similar. Lastly, suppose that $\mu_A = x^2 - x = x(x-1) = \mu_B$. Then all Jordan blocks in the JCF of A,B have size 1. As A, B have the same dimensions of their 0-eigenspaces, it follows that they have the same number of 0 blocks. But then they have the same number of 1 blocks as this is the only other eigenvalue and all blocks have size 1. Thus, they have the same JCF and are similar. □

Problem 2

Let F be an algebraically closed field and V a finite dimensional F-vector space.

(a) Let $S, T \in \mathcal{L}(V)$ such that $ST = TS$. Let λ be an eigenvalue of S and $E_\lambda(S) \le V$ be the corresponding eigenspace of S. Prove that $E_\lambda(S)$ is a T-invariant subspace.

Proof. Let $v \in E_\lambda(S)$, so $Sv = \lambda v$. Then

$$S(Tv) = (ST)(v) = (TS)(v) = T(Sv) = T(\lambda v) = \lambda \cdot (Tv)$$

so $Tv \in E_\lambda(S)$. Thus $T(E_\lambda(S)) \subseteq E_\lambda(S)$. □

(b) Assume that $T \in \mathcal{L}(V)$ is diagonalizable and let $W \le V$ be a T-invariant subspace. Prove that $T|_W \in \mathcal{L}(W)$ is also diagonalizable.

Proof. Since T is diagonalizable, $E_\lambda(T) = V_\lambda(T)$ for all $\lambda \in \mathrm{Spec}(T)$. Let $\lambda \in \mathrm{Spec}(T|_W) \subseteq \mathrm{Spec}(T)$ and $w \in W_\lambda(T|_W)$. Then, for some $k \in \mathbb{N}$, $(T - \lambda I)^k(w) = (T|_W - \lambda I|_W)^k(w) = 0$. Hence $w \in V_\lambda(T) = E_\lambda(T)$. But $w \in W$ so then $w \in E_\lambda(T|_W)$ whence $E_\lambda(T|_W) = W_\lambda(T|_W)$. So $T|_W$ is diagonalizable. □

(c) Assume again that $S, T \in \mathcal{L}(V)$ such that $ST = TS$. Prove that there exists a basis Ω of V such that $[T]_\Omega$, and $[S]_\Omega$ are both diagonal.

Proof. I am quite sure that this claim is false as stated, so I will add the assumption that S, T are both diagonalizable.

Then as S is diagonalizable,

$$V = \bigoplus_{\lambda \in \mathrm{Spec}(S)} V_\lambda(S) = \bigoplus_{\lambda \in \mathrm{Spec}(S)} E_\lambda(S).$$

Fix $\lambda \in \mathrm{Spec}(S)$. By part (a), $E_\lambda(S)$ is T-invariant whence part (b) implies that $T|_{E_\lambda(S)}$ is diagonalizable. So

$$E_\lambda(S) = \bigoplus_{\delta \in \mathrm{Spec}(T|_{E_\lambda(S)})} E_\delta(T|_{E_\lambda(S)}).$$

Now we write

$$V = \bigoplus_{\lambda \in \mathrm{Spec}(S)} \bigoplus_{\delta \in \mathrm{Spec}(T|_{E_\lambda(S)})} E_\delta(T|_{E_\lambda(S)}).$$

Since this sum is direct, we may form a basis for V from bases for $E_\delta(T|_{E\lambda(S)})$ over this double direct sum, whence this basis is both an eigenbasis for T and S. □

(d) Give an example of a vector space V with $\dim_F(V) \geq 3$ and two commuting linear transformations $S, T \in \mathcal{L}(V)$ such that NO basis Ω of V exists such that both $[T]_\Omega$, and $[S]_\Omega$ are in JCF.

Proof. Consider the matrices

$$S = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Note that ST and TS are both the zero matrix so these matrices commute. Moreover, S is already in JCF and the JCF of T is precisely S. Thus, if we pick a basis bringing T into JCF, we would end up knocking S out of JCF and vice versa. □

Problem 3

Find the number of distinct conjugacy classes in the group $GL_3(\mathbb{Z}/2\mathbb{Z})$, and specify one element in each conjugacy class.

Proof. Fix $A \in \mathrm{GL}_3(\mathbb{Z}/2\mathbb{Z})$. Let $\alpha_1 | \cdots | \alpha_m = \mu_A$ be the invariant factors for A. Then $\deg(\alpha_1) + \cdots + \deg(\alpha_m) = 3$ and $\alpha_1 \cdots \alpha_m = \chi_A$. As $\det(A) \neq 0$, it follows that $\det(A) = 1$ so $\chi_A(x) = x^3 + ax^2 + bx + 1$.

$\underline{a = 0,\ b = 0}$: $\chi_A(x) = x^3 + 1 = (x+1)(x^2+x+1)$. Both $x+1$ and x^2+x+1 are irreducible over $\mathbb{Z}/2\mathbb{Z}$, so each $\alpha_i \in \{x+1, x^2+x+1, x^3+1\}$. Note that $\alpha_m = x+1, x^2+x+1$ are both impossible as they are both not equal to the characteristic polynomial and thus irreducibility would force lower factors to equal α_m and thus would miss the other respective factor. So $\alpha_m = x^3+1$, whence $m = 1$ and A is similar to

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$\underline{a = 1,\ b = 0}$: $\chi_A(x) = x^3 + x^2 + 1$. This polynomial is irreducible over $\mathbb{Z}_2$ and already degree 3, so $m = 1$ and A is similar to

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$\underline{a = 0,\ b = 1}$: $\chi_A(x) = x^3 + x + 1$. This polynomial is irreducible over $\mathbb{Z}_2$ and already degree 3, so $m = 1$ and A is similar to

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$\underline{a = 1,\ b = 1}$: $\chi_A(x) = x^3 + x^2 + x + 1 = (x+1)^3$. In this case, via partitions of 3 we have either $m = 1$ so $\alpha_1 = \chi_A$, $m = 2$ and $\alpha_1 = x+1$ and $\alpha_2 = (x+1)^2$, or $m = 3$ and $\alpha_1 = \alpha_2 = \alpha_3 = x+1$. Hence A is similar to one of the following:

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

□

Problem 4

Let V be an n-dimensional vector space over an algebraically closed field and $T \in \mathcal{L}(V)$. Assume that T has just one eigenvalue λ and just one Jordan block. Let $S = T - \lambda I$.

(a) Prove that $\mathrm{rk}(S^k) = n - k$, for all $0 \leq k \leq n$. Deduce that $\mathrm{Im}(S^k) = \ker(S^{n-k})$, for all $0 \leq k \leq n$.

Proof. Note that $n_T(k, \lambda) = 1$ for $0 \leq k \leq n$ by assumption.

We induct on $0 \leq k \leq n$. For $k = 0$, $S^0 = I$ so $\mathrm{rk}(S^0) = n = n - 0$.

Now suppose $0 < k \leq n$ and that the claim holds for $k - 1$. On one hand, by the induction hypothesis $\mathrm{rk}(S^{k-1}) = n - (k-1)$. On the other hand

$$1 = n_T(k, \lambda) = \mathrm{rk}((T-\lambda I)^{k-1}) - \mathrm{rk}((T-\lambda I)^k) = \mathrm{rk}(S^{k-1}) - \mathrm{rk}(S^k) = n - k + 1 - \mathrm{rk}(S^k) \implies \mathrm{rk}(S^k) = n - k.$$

To see that $\mathrm{Im}(S^k) = \ker(S^{n-k})$, note that by the rank nullity theorem we have

$$\dim\ker(S^{n-k}) = n - \mathrm{rk}(S^{n-k}) = n - (n - (n-k)) = n - k = \mathrm{rk}(S^k) = \dim \mathrm{Im}(S^k),$$

so it suffices to show that $\mathrm{Im}(S^k) \subseteq \ker(S^{n-k})$.

Take $w \in \mathrm{Im}(S^k)$. Then $w = S^k v$ for some $v \in V$. Noting that $\mathrm{rk}(S^n) = 0 \implies S^n = \mathrm{O}$, we have that $S^{n-k}w = S^{n-k}(S^k v) = S^n v = \mathrm{Ov} = 0$, so $w \in \ker(S^{n-k})$. □

(b) Let $v \in V$ be any vector which lies outside of $\mathrm{Im}(S) = \ker(S^{n-1})$. Prove that $\{S^{n-1}v, \dots, Sv, v\}$ is a Jordan basis for T.

Proof. As $v \notin \ker(S^{n-1})$, we have that $S^{n-1}v \neq 0$ whilst

$$(T - \lambda I)S^{n-1}v = S^n v = \mathrm{Ov} = 0 \implies \mathrm{T(S^{n-1}v) = \lambda \cdot (S^{n-1}v)}.$$

If $0 \leq k < n - 1$,

$$(T - \lambda I)S^k v = S(S^k v) = S^{k+1}v \implies T(S^k v) = \lambda S^k v + S^{k+1} v$$

Hence, $\Omega = \{S^{n-1}v, \dots, Sv, v\}$ is a Jordan chain. For ease of notation, let $v_k = S^k v$ for $0 \leq k \leq n-1$. Then for each $0 \leq k \leq n-1$, $(T - \lambda I)^{n-k} v_k = 0$ and $(T - \lambda I)^{n-k-1}$ so $v_k \in \ker(T - \lambda I)^{n-k} \setminus \ker(T - \lambda I)^{n-(k+1)}$. Moreover, we have the series

$$0 = \ker(T - \lambda I)^{n-n} \subseteq \ker(T - \lambda I)^{n-(n-1)} \subseteq \cdots \subseteq \ker(T - \lambda I)^{n-1} \subseteq \ker(T - \lambda I)^{n-0} = V_\lambda(T) = V,$$

so at each step we add in a vector linearly independent from the previously added vectors, whence induction implies that Ω is a linearly independent set and thus a basis by dimensionality. As Ω is a Jordan chain, it follows that $[T]_\Omega = J(n, \lambda)$. □

Problem 6

Compute the Jordan canonical form and a Jordan basis for each of the following matrices over $\mathbb{Q}$:

(a) $\begin{pmatrix} -1 & 3 & 0 \\ 0 & 2 & 0 \\ 2 & 1 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 0 \end{pmatrix}$.

(a):

Proof. First we compute the characteristic polynomial of A:

$$\chi_A(x) = \det(xI - A) = (x-2)(x+1)^2$$

so $\operatorname{Spec}(A) = \{-1, 2\}$. Now we find the number of Jordan blocks corresponding to each eigenvalue via computing the corresponding eigenspaces and consequently their dimensions.

Note that $E_{-1}(A) = \ker(A+I) = \operatorname{span}\{\left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right)\}$ and $E_2(A) = \ker(A - 2I) = \left(\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right)\}$. As $\dim(E_{-1}(A)) = 1$ and the power of $x+1$ in χ_A is 2 (the sum of the sizes of the blocks), it follows that $JCF(A)$ has one -1-block of size 2 and thus we are forced to have

$$JCF(A) = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

To find a Jordan basis, note that $\dim V_{-1}(A) = 2$ as it is the sum of the sizes of all the -1-blocks in $JCF(A)$. To obtain a Jordan cycle, we solve for a w such that $\left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right) = (A+I)w$. We compute that $w = \left(\begin{smallmatrix} 1/2 \\ 0 \\ 0 \end{smallmatrix}\right)$ works as a solution, whence we obtain an actual basis for $V_{-1}(A)$ by dimensionality. Hence, our Jordan basis is

$$\left\{\left(\begin{smallmatrix} 1/2 \\ 0 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right)\right\}.$$

$\square$

(b):

Proof. First we compute the characteristic polynomial of B:

$$\chi_B(x) = \det\begin{pmatrix} x-1 & 1 & -1 \\ -1 & x+1 & -1 \\ -1 & 1 & x \end{pmatrix} = x^3$$

so $\operatorname{Spec}(B) = \{0\}$. We now compute eigenspaces and generalized eigenspaces. We compute that $E_0(B) = \ker(B) = \operatorname{span}\{\left(\begin{smallmatrix} 1 \\ 1 \\ 0 \end{smallmatrix}\right)\}$, so $\dim(E_0(B)) = 1$ whence there is one 0-block and thus

$$JCF(A) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Let $v_2 = \left(\begin{smallmatrix} 1 \\ 1 \\ 0 \end{smallmatrix}\right)$. We seek nonzero $v_1, v_0 \in V_0(B)$ such that $v_2 = Av_1$ and $v_1 = Av_0$. We compute that the choices $v_1 = \left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right)$ and $v_0 = \left(\begin{smallmatrix} 1 \\ 0 \\ -1 \end{smallmatrix}\right)$ possess the desired properties. Thus our Jordan basis is given by

$$\left\{\left(\begin{smallmatrix} 1 \\ 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 0 \\ -1 \end{smallmatrix}\right)\right\}.$$

$\square$

Problem 7

Let $F = \mathbb{F}_3$ be the field with 3 elements and let $A \in M_{12}(\mathbb{F}_3)$. Suppose that A satisfies all the following assumptions:

- $\text{rk}(A) = 10, \qquad \text{rk}(A^2) = 9, \qquad \text{rk}(A^3) = 9.$
- $\text{rk}(A - I) = 12.$
- $\text{rk}(A - 2I) = 9, \qquad \text{rk}((A - 2I)^2) = 7, \qquad \text{rk}((A - 2I)^3) = 6.$

(a) Assume in addition that the characteristic polynomial $\chi_A(x)$ splits completely over F (i.e. it splits into linear factors in $F[x]$). Find the Jordan canonical form of A.

Proof. For brevity, we write $n(k, \lambda) = n_A(k, \lambda)$. Recall that $n_A(k, \lambda) = \text{rk}((A - \lambda I)^{k-1}) - \text{rk}((A - \lambda I)^k)$ gives the number of Jordan blocks corresponding to λ of size at least k. We compute,

$$\begin{aligned} n(1, 0) &= \text{rk}(I) - \text{rk}(A) = 12 - 10 = 2 \\ n(2, 0) &= \text{rk}(A) - \text{rk}(A^2) = 10 - 9 = 1 \\ n(3, 0) &= \text{rk}(A^2) - \text{rk}(A^3) = 9 - 9 = 0 \end{aligned}$$

whence we have one 0-block of size 1, and one 0-block of size 2,

$$n(1, 1) = \text{rk}(I) - \text{rk}(A - I) = 12 - 12 = 0$$

giving that 1 is not an eigenvalue of A, and

$$\begin{aligned} n(1, 2) &= \text{rk}(I) - \text{rk}(A - 2I) = 12 - 9 = 3 \\ n(2, 2) &= \text{rk}(A - 2I) - \text{rk}((A - 2I)^2) = 9 - 7 = 2 \\ n(3, 2) &= \text{rk}((A - 2I)^2) - \text{rk}((A - 2I)^3) = 7 - 6 = 1 \end{aligned}$$

implying that we have one 2-block of size 1, one 2-block of size 2, and the remaining space given by a single 2-block of size $12 - (2 + 1 + 2 + 1) = 6$.

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & & & & & & \\ 0 & 2 & 1 & 0 & 0 & 0 & & & & & & \\ 0 & 0 & 2 & 1 & 0 & 0 & & & & & & \\ 0 & 0 & 0 & 2 & 1 & 0 & & & & & & \\ 0 & 0 & 0 & 0 & 2 & 1 & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 2 & & & & & & \\ & & & & & & 2 & 1 & & & & \\ & & & & & & 0 & 2 & & & & \\ & & & & & & & & 2 & & & \\ & & & & & & & & & 0 & 1 & \\ & & & & & & & & & & 0 & \\ & & & & & & & & & & & 0 \end{pmatrix} \tag{1}$$

□

(b) Find all possible RCF's of matrices A satisfying all the bullet assumptions, but not necessarily the extra assumption in (a).

Proof. Write $V = \mathbb{F}_3^{12}$. Then, writing V_A in invariant factor form:

$$V_A \cong \frac{F[x]}{(\alpha_1(x))} \oplus \frac{F[x]}{(\alpha_2(x))} \oplus \cdots \oplus \frac{F[x]}{(\alpha_m(x))}$$

In elementary divisor form, this becomes

$$V_A \cong \frac{F[x]}{(x)} \oplus \frac{F[x]}{(x^2)} \oplus \frac{F[x]}{((x-2))} \oplus \frac{F[x]}{((x-2)^2)} \oplus \frac{F[x]}{((x-2)^k)} \oplus \frac{F[x]}{(p(x))}$$

where $p(x) \in \mathbb{F}_3[x]$, $k \geq 3$, and there is no factor corresponding to $x-1$ as $\operatorname{rk}(A-I) = 12$ implies that 1 is not an eigenvalue of A. Note that $\mu_A = x^2(x-2)^k p(x)$. As $12 = \sum \deg(\alpha_i) = 1+2+1+2+k+\deg(p) \implies k = 6 - \deg(p) \leq 6$.

$\underline{k=3}$: Then $\deg(p) = 6-3 = 3$, whence p must be irreducible as otherwise it would have a linear fact and thus contribute an eigenspace which we have already filled by rank restrictions. Hence, the RCFs appear in corresponding partitions of the factors $x^2(x-2)^3$ into lower α_i's with successive division and degrees summing to 12.

$\underline{k=4}$: The $\deg(p) = 6-4 = 2$, whence p must be irreducible by the above logic. Hence, the RCFs appear in corresponding partitions of the factors $x^2(x-2)^4$ into lower α_i's with successive division and degrees summing to 12.

$\underline{k=5}$: Then $\deg(p) = 1$, which is impossible as then we would get another addition to the already filled eigenspace.

$\underline{k=6}$: Then $\deg(p) = 0$, so $p = 1$. The RCFs appear then in corresponding partitions of the factors $x^2(x-2)^6$ in the lower α_i's to obtain total degree sum of 12. $\square$