MATH 7752 - HOMEWORK 7
DUE FRIDAY 03/25/22

(1) (a) Consider the field $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Prove that $[K : \mathbb{Q}] = 4$.
   (b) Let $L = \mathbb{Q}(\sqrt{2} + \sqrt{3})$. Show that $L = K$.

(2) Let $S = \{n_1, \dots, n_r\}$ be a finite set of positive integers with $n_i \geq 2$. For each $j \in \{1, \dots, r\}$ let $\mathbb{Q}_j = \mathbb{Q}(\sqrt{n_1}, \dots, \sqrt{n_j})$. Moreover, set $\mathbb{Q}_0 = \mathbb{Q}$.
   (a) Prove that $[\mathbb{Q}_r : \mathbb{Q}] = 2^m$ for some integer $0 \leq m \leq r$. Moreover, show that the following set spans $\mathbb{Q}_r$ over $\mathbb{Q}$,
   $$P(S) = \{1\} \cup \{\sqrt{n} : n \text{ is a product of distinct elements from } S\}.$$
   (b) Prove that $[\mathbb{Q}_r : \mathbb{Q}] < 2^r$ if and only if n_1 is a complete square, or there exists $2 \leq j \leq r$ such that $\sqrt{n_j} = \alpha + \beta\sqrt{n_{j-1}}$, for some $\alpha, \beta \in \mathbb{Q}_{j-2}$.
   (c) Suppose that the integers $n_1, \dots, n_r$ are square-free and pairwise relatively prime. Prove that $[\mathbb{Q}_r : \mathbb{Q}] = 2^r$. Conclude that the extension $L = \mathbb{Q}(T)$, where $T = \{\sqrt{n} : n \in \mathbb{N}, n \text{ square free}\}$ is an infinite algebraic extension of $\mathbb{Q}$.

(3) Let F be a field and α an algebraic element of odd degree over F (i.e. the degree $[F(\alpha) : F]$ is odd). Show that $F(\alpha^2) = F(\alpha)$.

(4) Let K/F be an algebraic extension.
   (a) Let $F \subset R \subset K$ where R is a subring of K. Prove that R must be a subfield.
   (b) Show that (a) would be false if we dropped the assumption that K/F is algebraic.

(5) Let K/F be a finite field extension, $n = [K : F]$, and fix some basis $\Omega = \{\alpha_1, \dots, \alpha_n\}$ of K over F. For any $\alpha \in K$ define $T_\alpha : K \to K$ by $\beta \mapsto \alpha\beta$. Note that $T_\alpha \in \mathrm{End}_F(K)$. Let $A_\alpha = [T_\alpha]_\Omega \in M_n(F)$ be the matrix of T_α with respect to Ω.
   (a) Prove that the map $K \xrightarrow{\rho} M_n(F)$ given by $\alpha \mapsto A_\alpha$ is an injective ring homomorphism.
   (b) Prove that the minimal polynomial of α over F and the minimal polynomial of A_α coincide.

(6) Let K/F be an extension of fields and let $F \subseteq K_1 \subseteq K$ and $F \subseteq K_2 \subseteq K$ be two subextensions of K/F. The *compositum* of K_1 and K_2 is the smallest subfield of K that contains both K_1 and K_2. **Notation:** We denote the compositum by K_1K_2.
   (a) Consider the F-algebra $K_1 \otimes_F K_2$. Show that there exists a unique F-algebra homomorphism $\Phi : K_1 \otimes_F K_2 \to K_1K_2$ such that $\Phi(a \otimes b) = ab$. Conclude that $[K_1K_2 : F] \leq [K_1 : F][K_2 : F]$.
   (b) Show that $K_1 \otimes_F K_2$ is a field if and only if the above $\leq$ becomes an equality.
   (c) Suppose that $K_1 \cap K_2 \neq F$. Prove that $K_1 \otimes_F K_2$ is not a field.