

MATH 7310 Homework 10

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Problem 1

If $F \in NBV$, let $G(x) = |\mu_F|((-\infty, x])$, Prove that $|\mu_F| = \mu_{T_F}$ by showing that $G = T_F$ via the following steps.

(a): From the definition of T_F , show that $T_F \leq G$.

Proof. Let $x \in \mathbb{R}$. Then for $x_0 < x_1 < \dots < x_n = x$, observe that

$$\sum_{j=1}^{\infty} |F(x_j) - F(x_{j-1})| = \sum_{j=1}^{\infty} |\mu_F((x_{j-1}, x_j])| \leq \sum_{j=1}^{\infty} |\mu_F|((x_{j-1}, x_j]) = |\mu_F|((x_0, x]),$$

whence $T_F(x) \leq \sup_{x_0 < x} |\mu_F|((x_0, x]) = |\mu_F|((-\infty, x]) = G(x)$. □

(b): $|\mu_F(E)| \leq \mu_{T_F}(E)$ when E is an interval, and hence when E is a Borel set.

Proof. Let $I = (a, b)$ be an interval. Then

$$|\mu_F(I)| = |F(b) - F(a)| \leq T_F(b) - T_F(a) = \mu_{T_F}(I).$$

Now suppose $U \subseteq \mathbb{R}$ is open. Then there exist countably many disjoint intervals $(a_j, b_j) \subseteq \mathbb{R}$ such that $U = \bigsqcup_{j=1}^{\infty} (a_j, b_j)$. Hence,

$$|\mu_F(U)| \leq \sum_{j=1}^{\infty} |\mu_F((a_j, b_j))| \leq \sum_{j=1}^{\infty} \mu_{T_F}((a_j, b_j)) = \mu_{T_F}(U).$$

The case where E is Borel follows by outer regularity. □

(c): $|\mu_F| \leq \mu_{T_F}$, and hence $G \leq T_F$ (Use exercise 21).

Proof. Observe that, by part (b),

$$\begin{aligned} |\mu_F|(E) &= \sup \left\{ \sum_1^{\infty} |\mu_F(E_j)| : E_1, E_2, \dots \text{ disjoint, } E = \bigsqcup_1^{\infty} E_j \right\} \\ &\leq \sup \left\{ \sum_1^{\infty} \mu_{T_F}(E_j) : E_1, E_2, \dots \text{ disjoint, } E = \bigsqcup_1^{\infty} E_j \right\} = \mu_{T_F}(E), \end{aligned}$$

so $T_F(x) = \mu_{T_F}((-\infty, x]) \geq |\mu_F|((-\infty, x]) = G(x)$. □

Problem 2

Let G be a continuous increasing function on $[a, b]$ and let $G(a) = c$, $G(b) = d$.

(a): If $E \subseteq [c, d]$ is a Borel set, then $m(E) = \mu_G(G^{-1}(E))$. (First consider the case where E is an interval.)

Proof. Suppose first that $(e, f] \subseteq [c, d]$. Then by elementary analysis there exist some $p < q \in [c, d]$ such that $G^{-1}((e, f]) = (p, q]$, $G(p) = e$, $G(q) = f$. So,

$$\mu_G(G^{-1}((e, f])) = G(q) - G(p) = e - f = m((e, f]).$$

Now suppose that $E \subseteq [c, d]$ is Borel. As G is real-valued and increasing, we may apply outer regularity of μ_G combined with the fact that $G^{-1}(E) \subseteq \bigcup I_j$ if and only if $E \subseteq \bigcup G(I_j)$ to conclude that $m(E) = \mu_G(G^{-1}(E))$. \square

(b): If f is a Borel measurable and integrable function on $[c, d]$, then $\int_c^d f(y) dy = \int_a^b f(G(x)) dG(x)$. In particular, $\int_c^d f(y) dy = \int_a^b f(G(x))G'(x) dx$ if G is absolutely continuous.

Proof. By linearity of the statement, it suffices to prove for real-valued nonnegative functions f , as then we would apply to positive and negative parts of real and imaginary parts. Thus, assume $f \geq 0$. Choose simple functions $0 \leq \phi_1 \leq \phi_2 \leq \dots \leq f$ such that $\phi_n \rightarrow f$ pointwise. Write $\phi_n = \sum a_j \mathbb{1}_{E_j}$ for some $a_j \geq 0$ and $E_j \subseteq [c, d]$ Borel. Then we compute by part (a) that

$$\int_{[c,d]} \phi_n dm = \sum a_j m(E_j) = \sum a_j \mu_G(G^{-1}(E_j)) = \int_{[a,b]} \phi_n \circ G d\mu_G,$$

whence by the monotone convergence theorem $\int_{[c,d]} f dm = \int_{[a,b]} f \circ G d\mu_G$. \square

(c): The validity of (b) may fail if G is merely right continuous rather than continuous.

Proof. Consider the function $G = \mathbb{1}_{[1/2, 1]}$ and $f(x) = x$. Then $\int_0^1 f(x) dx = 1$ but

$$\int_0^1 (f \circ G)(x) dG(x) = \int_0^1 \mathbb{1}_{[1/2, 1]}(x) dG(x) = \mu_G([1/2, 1]) = 0.$$

\square

Problem 3

Suppose $F : \mathbb{R} \rightarrow \mathbb{C}$. Prove that F is Lipschitz with constant M if and only if F is absolutely continuous and $|F'| \leq M$ a.e.

Proof.

\Rightarrow : Suppose that $M > 0$ is such that $|F(x) - F(y)| \leq M|x - y|$ for all $x, y \in \mathbb{R}$. Let $\varepsilon > 0$. Choose $\delta = \varepsilon/M$. Then, for any finite set of disjoint intervals $(a_1, b_1), \dots, (a_N, b_N)$ with $\sum_{j=1}^N (b_j - a_j) < \delta$, we have

$$\sum_{j=1}^N |F(b_j) - F(a_j)| \leq M \sum_{j=1}^N (b_j - a_j) < M \cdot \frac{\varepsilon}{M} = \varepsilon,$$

so F is absolutely continuous. Thus, F is differentiable almost everywhere. If $x \neq y$, then $|F(x) - F(y)|/|x - y| \leq M$, so for a.e. $x \in \mathbb{R}$ we have that $|F'| \leq M$.

\Leftarrow : Suppose that F is absolutely continuous and $|F'| \leq M$ a.e. Let $x, y \in \mathbb{R}$ and without loss of generality suppose that $x < y$. Then by the FTC for Lebesgue integrals,

$$|F(y) - F(x)| = \left| \int_x^y F' dt \right| \leq \int_x^y |F'| dt \leq M|y - x|.$$

□

Problem 4

Let $A \subseteq [0, 1]$ be a Borel set such that $0 < m(A \cap I) < m(I)$ for every subinterval I of $[0, 1]$.

(a): Let $F(x) = m([0, x] \cap A)$. Show that F is absolutely continuous and strictly increasing on $[0, 1]$, but $F' = 0$ on a set of positive measure.

Proof. If $y > x$, then $F(y) = m([0, y] \cap A) = m([0, x] \cap A) + m((x, y] \cap A) > F(x)$ by assumption, so F is strictly increasing. Now fix $\varepsilon > 0$ and set $\delta = \varepsilon$. Then, for any finite set of disjoint intervals $(a_1, b_1), \dots, (a_N, b_N)$ with $\sum_{j=1}^N (b_j - a_j) < \delta$,

$$\sum_{j=1}^N |F(b_j) - F(a_j)| \leq \sum_{j=1}^N m((a_j, b_j] \cap A) < \sum_{j=1}^N (b_j - a_j) < \varepsilon,$$

so F is absolutely continuous. Let μ_F be the unique Borel measure such that $F(x) = \mu_F([0, x])$. Then the absolute continuity of F implies that $\mu_F \ll m$ and $d\mu_F = F' dm$. On the other hand, for $a < b$ in $[0, 1]$,

$$\mu_F((a, b]) = F(b) - F(a) = m([0, b] \cap A) - m([0, a] \cap A) = m((a, b] \cap A) = \int_a^b \mathbb{1}_A dm,$$

whence $d\mu_F = \mathbb{1}_A dm$. By the uniqueness of Radon-Nikodym derivatives, $F' = \mathbb{1}_A$ almost everywhere, whence $F' = 0$ on $[0, 1] \setminus A$. Moreover $m([0, 1] \setminus A) = m([0, 1]) - m(A) = m([0, 1]) - m([0, 1] \cap A) > 0$ by assumption. □

(b): Let $G(x) = m([0, x] \cap A) - m([0, x] \setminus A)$. Show that G is absolutely continuous on $[0, 1]$, but G is not monotone on any subinterval of $[0, 1]$.

Proof. Let $\varepsilon > 0$. Again set $\delta = \varepsilon$. Then, for any finite set of disjoint intervals $(a_1, b_1), \dots, (a_N, b_N)$ with $\sum_{j=1}^N (b_j - a_j) < \delta$,

$$\sum_{j=1}^N |G(b_j) - G(a_j)| = \sum_{j=1}^N |m((a_j, b_j] \cap A) - m((a_j, b_j] \setminus A)| < \sum_{j=1}^N m((a_j, b_j]) < \varepsilon,$$

so G is absolutely continuous. Let μ be the measure given by $d\mu = G' dm$. Then for $a < b$,

$$\int_a^b G' dm = m((a, b] \cap A) - m((a, b] \setminus A) = \int_a^b (\mathbb{1}_A - \mathbb{1}_{[0, 1] \setminus A}) dm.$$

Then as intervals generate all borel subsets of $[0, 1]$, $G' dm = d\mu = (\mathbb{1}_A - \mathbb{1}_{[0, 1] \setminus A}) dm$, so by uniqueness of Radon-Nikodym derivatives $G' = \mathbb{1}_A - \mathbb{1}_{[0, 1] \setminus A}$ almost everywhere. Let $I \subseteq [0, 1]$ be an interval. Then $G' = 1$ on $I \cap A$ and $G' = -1$ on $I \setminus A$, but $m(I \cap A), m(I \setminus A) > 0$, so we must have that G is not monotonic on I . □

Problem 5

Problem 6

Let $a < b$ be real numbers and let $1 \leq p \leq +\infty$. Let X be the set of functions $f : [a, b] \rightarrow \mathbb{C}$ which are absolutely continuous and such that $f' \in L^p([a, b])$. Fix $x_0 \in [a, b]$. For $f \in X$, define

$$\|f\| = |f(x_0)| + \|f'\|_p.$$

Show that $\|\cdot\|$ is a norm which turns X into a Banach space.

Proof. That $\|\cdot\|$ is a norm follows from the fact that $|\cdot|$ and $\|\cdot\|_p$ are norms.

Suppose that $(f_n)_{n=1}^\infty$ is a Cauchy sequence in $(X, \|\cdot\|)$. Then

$$|f_n(x_0) - f_m(x_0)| + \|f'_n - f'_m\|_p = \|f_n - f_m\| \xrightarrow{n, m \rightarrow \infty} 0$$

whence $|f_n(x_0) - f_m(x_0)|, \|f'_n - f'_m\|_p \xrightarrow{n, m \rightarrow \infty} 0$. By completeness of \mathbb{C} and $L^p([a, b])$, there exists some $y_0 \in \mathbb{C}$ and $F \in L^p([a, b])$ such that $f_n(x_0) \rightarrow y_0$ and $f'_n \xrightarrow{L^p} F$. As $\|F\|_1 \leq m([a, b])^{1-\frac{1}{p}} \|F\|_p < +\infty$, $F \in L^1([a, b])$. Define $f : [a, b] \rightarrow \mathbb{C}$ by

$$f(x) = y_0 + \int_a^x F(t) dt.$$

Then $f(x) - f(x_0) = \int_a^x F(t) dt$, whence by Folland 3.35(a) f is absolutely continuous. Moreover, by Folland 3.35(c), f is differentiable almost everywhere, $f' \in L^1([a, b])$, and $f(x) - f(x_0) = \int_a^x f'(t) dt$. So

$$\int_a^x f'(t) dt = \int_a^x F(t) dt$$

for all $x \in [a, b]$, whence by uniqueness in premeasures give measures $f'(t) dt = F(t) dt$. However then by uniqueness of Radon-Nikodym derivatives, it follows that $f' = F$ almost everywhere, so $f' \in L^p([a, b])$. Thus $f \in X$.

As $f' = F$, we have that

$$\|f - f_n\| = |f(x_0) - f_n(x_0)| + \|f' - f'_n\|_p = |y_0 - f_n(x_0)| + \|F - f'_n\|_p \xrightarrow{n \rightarrow \infty} 0.$$

□