MATH 7310 Homework 3

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Problem 1

Let F be increasing and right continuous, and let μ_F be the associated measure. Then $\mu_F(\{a\}) = F(a) - F(a-)$, $\mu_F([a,b)) = F(b-) - F(a-)$, $\mu_F([a,b]) = F(b) - F(a-)$, and $\mu_F((a,b)) = F(b-) - F(a)$.

Proof. Let $a, b \in \mathbb{R}$ with $a < b$. Then, take $x < a$. Then by definition, as $(x,a]$ is an h-interval,

$$\mu_F(\{a\}) \leq F(a) - F(x) \implies F(x) \leq F(a) - \mu_F(\{a\}).$$

Hence, taking the infimum over all such $x < a$, we have that

$$F(a-) \leq F(a) - \mu_F(\{a\}) \implies \mu_F(\{a\}) \leq F(a) - F(a-).$$

On the other hand, note that

$$\mu_F(\{a\}) = \inf\left(\sum_{i=1}^{\infty} \mu_F((a_i, b_i]) : a \in \bigcup_{i=1}^{\infty}(a_i, b_i]\right) \leq \inf\left(\sum_{i=1}^{\infty} \mu_F((a_i, a]) : a \in \bigcup_{i=1}^{\infty}(a_i, a]\right) = \inf\{\mu_F((a_i, a]) : a_i < a\} =$$

Also, observe that

$$\begin{aligned} F(b) - F(a) = \mu_F((a,b]) &= \mu_F(([a,b) \setminus \{a\}) \sqcup \{b\}) = \mu_F([a,b)) - \mu_F(\{a\}) + \mu_F(\{b\}) \\ &= \mu_F([a,b)) - F(a) + F(a-) + F(b) - F(b-) \implies \mu_F([a,b)) = F(b-) - F(a-) \end{aligned}$$

$$F(b) - F(a) = \mu_F((a,b]) = \mu_F(([a,b] \setminus \{a\}) = \mu_F([a,b]) - F(a) + F(a-) \implies \mu_F([a,b]) = F(b) - F(a-)$$

$$\mu_F((a,b)) = \mu((a,b] \setminus \{b\}) = F(b) - F(a) - F(b) + F(b-) = F(b-) - F(a).$$

$\square$

Problem 2

Let (X, Σ, μ) be a measure space. We say that $E \subseteq X$ is *an atom* if

- $E \in \Sigma$,
- $\mu(E) > 0$,

- $\{\mu(F) : F \subseteq E, F \in \Sigma\} = \{0, \mu(E)\}$.

We say the μ is *diffuse* if it has no atoms.

(a) Let (X, d, μ) be a metric measure space. Assume that μ is outer regular, and that

$$\mu(E) = \sup\{\mu(K) : K \subseteq E \text{ compact}\} \text{ for all Borel } E \subseteq X.$$

If $\mu(\{p\}) = 0$ for all $p \in X$, show that μ is diffuse.

Proof. Note that, for all $p \in X$, by outer regularity we have that

$$0 = \mu(\{p\}) = \inf\{\mu(U) : U \supset \{p\} \text{ open}\}.$$

Suppose, for the sake of contradiction, that μ is not diffuse. Then there exists an atom $E \subseteq X$. As $0 < \mu(E) = \sup\{\mu(K) : K \subseteq E \text{ compact}\}$ and $\{\mu(K) : K \subseteq E \text{ compact}\} = \{0, \mu(E)\}$, there exists a $K \subseteq E$ compact such that $\mu(K) = \mu(E) > 0$.

For each $p \in K$, as $0 = \inf\{\mu(U) : U \supset \{p\} \text{ open}\}$, there exists an open U_p such that $\mu(U_p) < \mu(K)$. Then $\{U_p : p \in K\}$ is an open cover for K, so there exist $p_1, \ldots, p_n \in K$ such that $\{U_{p_1}, \ldots, U_{p_n}\}$ covers K. Then, as E is an atom, $\mu(U_p \cap K) \leq \mu(U_p) < \mu(K) \implies \mu(U_p \cap K) = 0$. But then, as $K = \bigcup_{i=1}^n U_{p_i} \cap K$,

$$\mu(K) \leq \mu\left(\bigcup_{i=1}^n U_{p_i} \cap K\right) \leq \sum_{i=1}^n \mu(U_{p_i} \cap K) = 0,$$

contradicting that $\mu(K) > 0$. $\square$

(b) Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing, right-continuous function. Show that for $p \in \mathbb{R}$ we have that $\{p\}$ is an atom of μ_F if and only if F is discontinuous at p. Show that μ_F is diffuse if and only if F is continuous.

Proof. Suppose that $\{p\}$ is an atom of μ_F. Then $\mu(p) > 0$. By problem (1), $0 < \mu(\{p\}) = F(p) - F(p-) \implies F(p) > F(p-)$, whence $F(p) \neq F(p-)$ so F is discontinuous at p.

Coversely, suppose that F is discontinuous at p. As F is already right continous, it follows that $F(p) \neq F(p-)$, as otherwise F would be continous at p. Then $\mu(p) = F(p) - F(p-) > 0$ as *a priori* $F(p) \geq F(p-)$, so $\{p\}$ is an atom.

Now suppose that μ_F is diffuse. Then, for $p \in \mathbb{R}$, $\{p\}$ is not an atom whence $\mu(\{p\}) = 0$ so part (a) implies that $0 = \mu(\{p\}) = F(p) - F(p-)$. Thus $F(p+) = F(p) = F(p-)$, so F is continuous.

Conversely, suppose that F is continuous. Then for all $p \in \mathbb{R}$, $\mu(\{p\}) = F(p) - F(p-) = F(p) - F(p) = 0$. As μ_F is outer regular and inner regular with respect to compacts, by part (a) μ_F we have that is diffuse. $\square$

Problem 3

Let (X, Σ, μ) be a σ-finite measure space.

(i) Suppose that $(E_j)_{j \in J}$ is a collection of sets with $E_j \in \Sigma$ for all $j \in J$ and with $\mu(E_j) > 0$ for all $j \in J$, and so that $\mu(E_j \cap E_k) = 0$ for all $j \neq k$ in J. Show that J is countable.

Proof. Without loss of generality, assume that $X = \bigsqcup_{n=1}^{\infty} X_n$ where $X_n \in \Sigma$, $\mu(X_n > 0)$ for all $n \in \mathbb{N}$, and $X_i \cap X_j = \emptyset$ for $i \neq j$.

Suppose, for the sake of contradiction, that J is uncountable. For $j \in J$, note that

$$0 < \mu(E_j) = \sum_{n=1}^{\infty} \mu(E_j \cap X_n),$$

whence there exists an $n_j \in \mathbb{N}$ such that $\mu(E_j \cap X_n) > 0$. As there can only be countably many such n_j's and there are uncountably many E_j's, there exists a $k \in \mathbb{N}$ and $J_0 \subseteq J$ uncountable such that $\mu(E_j \cap X_k) > 0$ for all $j \in J_0$. By a pigeonhole argument, there exists a $b > 0$ and an infinite $J_0' \subseteq J_0$ such that $\mu(E_j \cap X_k) > b$ for all $j \in J_0'$.

Choose a countable sequence $(j_l)_{l=1}^{\infty}$ in J_0 such that $j_l \neq j_s$ for $l \neq s$. For $n \in \mathbb{N}$, set $F_n = E_j \cap X_k$. Note that, for $l \neq s$, we have that $\mu(F_l \cap F_s) = 0$.

We claim that $\mu(\sum_{l=1}^{n} F_l) = \sum_{l=1}^{n} \mu(F_l)$ for all $n \in \mathbb{N}$ by induction. Observe that

$$\mu(F_1 \cup F_2) = \mu(F_1) + \mu(F_2) - \mu(F_1 \cap F_2) = \mu(F_1) + \mu(F_2).$$

Now fix $n > 2$ and suppose that $\mu(\sum_{l=1}^{n-1} F_l) = \sum_{l=1}^{n-1} \mu(F_l)$. Observe that

$$\mu\left(F_n \cap \bigcup_{j=1}^{n-1} F_j\right) = \mu\left(\bigcup_{j=1}^{n-1} F_n \cap F_j\right) \leq \sum_{j=1}^{n-1} \mu(F_n \cap F_j) = 0,$$

so

$$\mu\left(\bigcup_{j=1}^{n} F_j\right) = \mu\left(F_n \cup \bigcup_{j=1}^{n-1} F_j\right) = \mu(F_n) + \sum_{j=1}^{n-1} \mu(F_j) - \mu\left(F_n \cap \bigcup_{j=1}^{n-1} F_j\right) = \sum_{j=1}^{n} \mu(F_j).$$

By induction, the claim holds for all $n \in \mathbb{N}$.

Observe that, for all $n \in \mathbb{N}$,

$$\mu\left(\bigcup_{j=1}^{\infty} F_j\right) \geq \mu\left(\bigcup_{j=1}^{n} F_j\right) = \sum_{j=1}^{n} \mu(F_j) \geq n \cdot b.$$

Hence $\mu\left(\bigcup_{j=1}^{\infty} F_j\right) = +\infty$, contradicting the fact that $\mu(X_k) < +\infty$.

□

(ii) Let (Ω, ρ) be the metric space defined in Problem 12 of Chapter 1 of Folland. For $E \in \Sigma$, let $[E]$ be its equivalence class in Ω. Show that

$$\{[E] : E \subseteq X \text{ is an atom}\},$$

is countable.

Proof. Let $\sim$ be the equivalence relation $E \sim F \iff \mu(E\Delta F) = 0$. Let $\mathcal{E} = \{E : E \subseteq X \text{ is an atom}\}$. Let $\pi : \mathcal{E} \to \mathcal{E}/\sim$ be the canonical surjection. By the axiom of choice, there exists a section $s : \mathcal{E}/\sim \to \mathcal{E}$ such that $\pi \circ s = id_{\mathcal{E}/\sim}$. Take $E \neq F \in s(\mathcal{E}/\sim)$. Then, as s is injective, $[E] \neq [F]$, so $\mu(E_i \Delta E_j) > 0$.

Suppose, without loss of generality, that $\mu(E_i \setminus E_j) > 0$. Then, as E_i is an atom, $\mu(E_i \setminus E_j) = \mu(E_i)$. Hence

$$\mu(E_i) = \mu(E_i \setminus E_j) + \mu(E_i \cap E_j) = \mu(E_i) + \mu(E_i \cap E_j) \implies \mu(E_i \cap E_j) = 0$$

Hence, $s(\mathcal{E}/\sim)$ has the properties of the collection in part (i), so $s(\mathcal{E}/\sim)$ is countable whence injectivity implies that $\mathcal{E}/\sim$ is countable. □

Problem 4

Let (X, Σ, μ) be a diffuse σ-finite measure space. For $A \in \Sigma$, show that:

$$\{\mu(B) : B \subseteq A, B \in \Sigma\} = [0, \mu(A)].$$

Suggestions: Reduce to the finite case. It might be helpful to first show that for every $E \in \Sigma$ with $\mu(E) > 0$, we have $0 = \inf\{\mu(B) : B \subseteq E \text{ and } \mu(B) > 0\}$.

Proof.
(*reduction to finite case*): Suppose we have shown the claim for finite measure spaces. Write $X = \bigcup_{i=1}^{\infty} X_i$ where $X_i \in \Sigma$ and $\mu(X_i) < +\infty$, and without loss of generality the X_i's are pairwise disjoint.

If $\mu(E) < +\infty$, then we are done as we assumed that we have already shown the finite case. So, suppose that $\mu(E) = +\infty$. Then $E \subseteq E$ is a witness for $\mu(E) = +\infty$, so take $b \in (0, +\infty)$.
As

$$+\infty = \mu(E) = \sum_{i=1}^{\infty} \mu(E \cap X_i),$$

there exist $k, l \in \mathbb{N}$ such that $\sum_{i=k}^{l} \mu(E \cap X_i) > b$. Noting that $\mu(\bigsqcup_{i=k}^{l} E \cap X_i) = \sum_{i=k}^{l} \mu(E \cap X_i) < +\infty$, by the finite case there exists a $B \subseteq \bigsqcup_{i=k}^{l} E \cap X_i$ with $B \in \Sigma$ such that $\mu(B) = b$.

(*finite case*): Suppose that $E \in \Sigma$ with $\mu(E) > 0$. Since μ is diffuse, there exists a $B_1 \subseteq E$ such that $B_1 \in \Sigma$ and $0 < \mu(B_1) < \mu(E)$. Note that either $\mu(B_1)$ or $\mu(E \setminus B_1)$ is less than $2^{-1}\mu(E)$, so without loss of generality assume that $\mu(B_1) < 2^{-1}\mu(E)$. Now, again as μ is diffuse, there exists a $B_2 \subseteq B_1$ such that $B_2 \in \Sigma$ and $0 < \mu(B_2) < \mu(B_1) < \mu(E)$. Again, we may assume without loss of generality that $\mu(B_2) < 2^{-1}\mu(B_1) < 2^{-2}\mu(E)$. Continuing as such, we obtain a decreasing sequence of sets $E \supset B_1 \supset B_2 \supset \cdots$ such that $0 < \mu(B_n) < 2^{-n}\mu(E)$. It follows that

$$0 = \inf\{\mu(B) : B \subseteq E \text{ and } \mu(B) > 0\}. \tag{1}$$

Suppose, for the sake of contradiction, that the claim is false. Then there exists an $A \in \Sigma \setminus \{\emptyset\}$ and $b \in (0, \mu(A))$ such that $\mu(B) \neq b$ for all $B \subseteq A$ with $B \in \Sigma$.
We proceed via transfinite recursion.

First, note that we may choose B_0 such that $0 < \mu(B_0) \leq b$. If $\mu(B_0) = b$ then stop; otherwise, we have that $0 < \mu(B_0) < b$. Suppose now that α is an ordinal and we have constructed $(B_\eta)_{\eta<\alpha}$ pairwise disjoint elements of Σ which are subsets of A such that $\mu(B_\eta) > 0$ for all $\eta \in \alpha$, and $b - \sum_{\eta\in\alpha} \mu(B_\eta) > 0$. By (1), there exists a $B_\alpha \in \Sigma$ with $B_\alpha \subseteq A \setminus \bigsqcup_{\eta\in\alpha} B_\eta$ such that

$$0 < \mu(B_\alpha) \leq b - \sum_{\eta\in\alpha} \mu(B_\eta).$$

If $\mu(B_\alpha) = b - \sum_{\eta\in\alpha} \mu(B_\eta)$, stop; otherwise, we have $0 < \sum_{\eta\leq\alpha} \mu(B_\eta) < b$.

We claim that this recursion halts at some countable ordinal. Suppose, for the sake of contradiction, that this recursion does not halt at some countable ordinal. Then we reach ω_1, so we have pairwise disjoint subsets $(B_\eta)_{\eta<\omega_1}$ of A in Σ such that $\mu(B_\eta) > 0$ for all $\eta < \omega_1$. Noting that each $\mu(B_\eta)$ is finite, we have that

$$\{\eta < \omega_1 : \mu(B_\eta) > 0\} = \bigcup_{n=1}^{\infty} \{\eta < \omega_1 : \frac{1}{n} \leq \mu(B_\eta) < \frac{1}{n-1}\}$$

where $1/0 := +\infty$. By uncountability of ω_1, there exists an $n \in \mathbb{N}$ such that $\{\eta < \omega_1 : \frac{1}{n} \leq \mu(B_\eta) < \frac{1}{n-1}\}$ is infinite. Take a countable sequence $\eta_1, \eta_2, \ldots$ in $\{\eta < \omega_1 : \frac{1}{n} \leq \mu(B_\eta) < \frac{1}{n-1}\}$ with $\eta_i \neq \eta_j$ for $i \neq j$. Then $\bigsqcup_{i=1}^{\infty} B_{\eta_i} \in \Sigma$, whence for all $N \in \mathbb{N}$,

$$\mu\left(\bigsqcup_{i=1}^{\infty} B_{\eta_i}\right) = \sum_{i=1}^{\infty} \mu(B_{\eta_i}) \geq \sum_{i=1}^{N} \mu(B_{\eta_i}) \geq \frac{N}{n}$$

so $\mu(\bigsqcup_{i=1}^{\infty} B_{\eta_i}) = +\infty$, contradicting that $\mu(A) < +\infty$.

Hence, the recursion halts at some countable ordinal α. Then $\sum_{\eta\in\alpha} \mu(B_\eta) = \sum_{\eta<\alpha} \mu(B_\eta) = b$. Let $\varphi : \mathbb{N} \to \alpha$ be a bijection. By nonnegativity, $b = \sum_{\eta\in\alpha} \mu_{B_\eta} = \sum_{i=1}^{\infty} \mu(B_{\varphi(i)})$. Moreover, $\bigsqcup_{i=1}^{\infty} B_{\varphi(i)} \in \Sigma$, so

$$b = \sum_{i=1}^{\infty} \mu(B_{\varphi(i)}) = \mu\left(\bigsqcup_{i=1}^{\infty} B_{\varphi(i)}\right)$$

as desired. □

Problem 5

Let E be a Lebesgue measurable set.

(a) Let $E \subseteq N$ where N is the nonmeasurable set described in section 1.1. Prove that $m(E) = 0$.

Proof. As in class, we have that $\bigsqcup_{q\in[-1,1]\cap\mathbb{Q}} E + q \subseteq \bigsqcup_{q\in[-1,1]\cap\mathbb{Q}} N + q \subseteq [-2, 3]$, so

$$\sum_{q\in[-1,1]\cap\mathbb{Q}} m(E) = \sum_{q\in[-1,1]\cap\mathbb{Q}} m(E + q) = m\left(\bigsqcup_{q\in[-1,1]\cap\mathbb{Q}} E + q\right) \leq 5$$

so $\sum_{q\in[-1,1]\cap\mathbb{Q}} m(E)$ is whence we must have that $m(E) = 0$. □

(b) Prove that if $m(E) > 0$, then E contains a nonmeasurable set.

Proof. As $m(E) > 0$, $E \neq \emptyset$ so there exists a $k \in \mathbb{Z}$ such $(E + k) \cap [0, 1] \neq \emptyset$.

Performing the same construction of the Vitali set on $(E + k) \cap [0, 1]$, we obtain a set $V \subseteq (E + k) \cap [0, 1]$ which is non measurable. Hence, by translation invariance, $V - k$ is non-measurable. □

Problem 6

(a) Let $\mathcal{E}_q$ be the family of h-intervals in $\mathbb{R}$ with rational endpoints. Show that $\mathcal{E}_q$ is an elementary family and that the σ-algebra generated by this elementary family is all Borel subsets of $\mathbb{R}$.

Proof. We have shown in previous homework that $\mathcal{E}_q$ is indeed an elementary family.

Take $(a, b] \in \mathcal{E}_q$. As $(a, b] = \bigcap_{n=1}^{\infty}(a, b + \frac{1}{n})$, $(a, b] \in \mathcal{B}_\mathbb{R}$. Thus $\Sigma(\mathcal{E}_q) \subseteq \mathcal{B}_\mathbb{R}$.

On the other hand, any $(\alpha, \beta) \in \mathcal{B}_\mathbb{R}$ can be written as a countable union of open rational intervals, so $(\alpha, \beta) \in \Sigma(\mathcal{E}_q)$. Hence, as open intervals generate $\mathcal{B}_\mathbb{R}$, it follows that $\mathcal{B}_\mathbb{R} \subseteq \Sigma(\mathcal{E}_q)$. □

(b) Suppose that $\mu : \mathcal{B}_{\mathbb{R}} \to [0, \infty]$ is a measure such that $\mu(E + x) = \mu(E)$ for all $E \in \mathcal{B}_{\mathbb{R}}, x \in \mathbb{R}$. Assume that $0 < \mu((0, 1]) < +\infty$. Show that $\mu(E) = \mu((0, 1])m(E)$ for all $E \in \mathcal{B}_{\mathbb{R}}$.

Proof. First, suppose that $n \in \mathbb{N}$. Observe that

$$\mu((0,1]) = \mu\left(\bigsqcup_{j=0}^{n-1}(\frac{j}{n}, \frac{j+1}{n}]\right) = \sum_{j=0}^{n-1}\mu((\frac{j}{n}, \frac{j+1}{n}]) = n\cdot\mu((0, \frac{1}{n}]) \implies \mu((0, \frac{1}{n}]) = \frac{1}{n}\mu((0,1]) = \mu((0,1])m((0, \frac{1}{n}])$$

Now let $q = \frac{k}{n} \in \mathbb{Q}$. Then

$$\mu((0,q]) = \sum_{j=0}^{k}\mu\left((\frac{k}{n}, \frac{k+1}{n}]\right) = \sum_{j=0}^{k}\mu\left((0, \frac{1}{n}]\right) = k\cdot\mu\left((0, \frac{1}{n}]\right) = k\cdot\mu((0,1])\frac{1}{n} = \mu((0,1])m((0,q]).$$

Consider the algebra $\mathcal{A} = \{$finite disjoint unions of elements of $E_q\}$. Note that, by part (a), $\Sigma(\mathcal{A}) = \mathcal{B}_{\mathbb{R}}$. Consider the measure $\nu : \mathcal{B}_{\mathbb{R}} \to [0, +\infty]$ given by $\nu(E) = \mu((0, 1]) \cdot m(E)$ for $E \in \mathcal{B}_{\mathbb{R}}$.

We have shown that $\mu|_{\mathcal{A}} = \nu|_{\mathcal{A}}$, whence $\mu = \nu$, as desired. $\square$