MATH 7310 Homework 0

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January 23, 2022

Problem 1

Let J be an infinite set, and $(t_j)_{j\in J}$ nonnegative real numbers. We define $\sum_{j\in J} t_j = \sup_F \sum_{j\in F} t_j$ where the supremum is over all finite subsets of J, and is equal to ∞ if $\left\{\sum_{j\in F} t_j : F \subseteq J \text{ is finite}\right\}$ is not bounded above.

(i) Suppose that $\sum_{j\in J} t_j < \infty$. Prove that for every $\varepsilon > 0$, there is a finite $F \subseteq J$ so that $\sum_{j\in J\setminus F} t_j < \varepsilon$.

Proof. Let $\varepsilon > 0$ an $J_0 = \{j \in J : t_j > 0\}$. By proposition 0.20, as $\sum_{j\in J} t_j < \infty$, J_0 is countably infinite. Moreover, letting $g : \mathbb{N} \to J_0$ be a bijection, proposition 0.20 gives that

$$\sum_{n=1}^{\infty} t_{g(n)} = \sum_{j\in J} t_j < \infty$$

As this sum is nonnegative and converges, there exists a $k \in \mathbb{N}$ such that

$$\sum_{n=k+1}^{\infty} t_{g(n)} < \varepsilon.$$

Letting $F = g(\{1, 2, \dots, k\})$, it follows that

$$\sum_{j\in J\setminus F} t_j = \sum_{n=k+1}^{\infty} t_{g(n)} < \varepsilon.$$

$\square$

(ii) Suppose that $(\alpha_j)_{j\in J}$ are complex numbers and $\sum_{j\in J} |\alpha_j| < \infty$. Suppose further that $J_0 = \{j \in J : \alpha_j \neq 0\}$ is infinite. Suppose that $\phi : \mathbb{N} \to J_0,\ \psi : \mathbb{N} \to J_0$ are two bijections. Prove that

$$\sum_{n=1}^{\infty} \alpha_{\phi(n)} = \sum_{n=1}^{\infty} \alpha_{\psi(n)}.$$

Proof. By proposition 0.20, as $\sum_{j\in J} |\alpha_j| < \infty$, the set $\{j \in J : \alpha_j \neq 0\} = \{j \in J : \alpha_j \neq 0\} = J_0$ is countable. Moreover,

$$\sum_{n=1}^{\infty} |\alpha_{\phi(n)}| = \sum_{j\in J} |\alpha_j| < \infty$$

$$\sum_{n=1}^{\infty} |\alpha_{\psi(n)}| = \sum_{j\in J} |\alpha_j| < \infty,$$

so the sums $\sum_{n=1}^{\infty} \alpha_{\phi(n)}$ and $\sum_{n=1}^{\infty} \alpha_{\psi(n)}$ are absolutely convergent. As such, they are equal since the value of an absolutely convergent series is unchanged under rearrangement. $\square$

Problem 2

It follows from Problem 1 that if $(\alpha_j)_{j\in J}$ are complex numbers and $\sum_{j\in J}|\alpha_j| < \infty$, we may define $\sum_{j\in J}$ as follows: let $J_0 = \{j : \alpha_j \neq 0\}$. If J_0 is finite, then $\sum_{j\in J} = \sum_{j\in J_0}$. If J_0 is infinite, choose a bijection $\phi : \mathbb{N} \to J_0$, and define

$$\sum_{j\in J}\alpha_j = \sum_{n=1}^{\infty}\alpha_{\phi(n)}.$$

Suppose that $(\alpha_j)_{j\in J}$ are complex numbers and $\sum_{j\in J}|\alpha_j| < \infty$. Show that $\sum_{j\in J}|\alpha_j|$ is the unique complex number s satisfying the following property. For every $\varepsilon > 0$, there is a finite set $F \subseteq J$ so that if $F \subseteq E \subseteq J$ and E is finite, then

$$\left|s - \sum_{j\in E}\alpha_j\right| < \varepsilon.$$

Proof. Let $\varepsilon > 0$. Since $\sum_{j\in J}|\alpha_j| < \infty$, there exists a finite $F \subseteq J$ so that $\sum_{j\in J\setminus F}|\alpha_j| < \frac{\varepsilon}{2}$. If $F \subseteq E \subseteq J$ and E is finite, then $J \setminus E \subseteq J \setminus F$ implies that $\sum_{j\in J\setminus E}|\alpha_j| \leq \sum_{j\in J\setminus F}|\alpha_j| < \varepsilon$.
By the triangle inequality,

$$\left|s - \sum_{j\in E}\alpha_j\right| = \left|\sum_{j\in J\setminus E}\alpha_j\right| \leq \sum_{j\in J\setminus E}|\alpha_j| < \varepsilon.$$

Now suppose that $s, s' \in \mathbb{C}$ satisfy the desired property. Let $\varepsilon > 0$. Then there exists finite sets $F, F' \subseteq J$ satisyfing the above property for s and s' respectively. Then $F \cup F'$ is finite and $F, F' \subseteq F \cup F' \subseteq J$, so

$$|s - s'| = \left|s - \sum_{j\in F\cup F'}\alpha_j + \sum_{j\in F\cup F'}\alpha_j - s'\right| \leq \left|s - \sum_{j\in F\cup F'}\alpha_j\right| + \left|\sum_{j\in F\cup F'}\alpha_j - s'\right| < 2\varepsilon.$$

Hence $s = s'$. $\square$

Problem 3

Suppose that I, J are sets, and $(a_{ij})_{i\in I}j \in J$ are nonnegative real numbers. Prove that

$$\sum_{j\in J}\left(\sum_{i\in I}a_{ij}\right) = \sum_{(i,j)\in I\times J}a_{ij} = \sum_{i\in I}\left(\sum_{j\in J}a_{ij}\right)$$

Proof. On one hand, observe that, since $(a_{ij})_{i\in I}j \in J$ are nonnegative real numbers, a finite sum of suprema is equal to a suprema of finite sums. Hence,

$$\begin{aligned}\sup_{\substack{F\subseteq I\\ F\text{ finite}}}\sum_{i\in F}\left(\sup_{\substack{G\subseteq J\\ G\text{ finite}}}\sum_{j\in G}a_{ij}\right) &= \sup_{\substack{F\subseteq I\\ F\text{ finite}}}\sup_{\substack{G\subseteq J\\ G\text{ finite}}}\left(\sum_{i\in F}\sum_{j\in G}a_{ij}\right) = \sup_{\substack{F\subseteq I\\ F\text{ finite}}}\sup_{\substack{G\subseteq J\\ G\text{ finite}}}\left(\sum_{(i,j)\in F\times G}a_{ij}\right)\\ &= \sup_{\substack{X\subseteq I\times J\\ X\text{ finite}}}\left(\sum_{(i,j)\in X}a_{ij}\right) = \sum_{(i,j)\in I\times J}a_{ij}.\end{aligned}$$

The other direction follows similarly. $\square$