MATH 7752 Homework 3

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Problem 1

Let R be a commutative domain, and let M be a free R-module with basis $X = \{e_1, \ldots, e_k\}$, with $k \geq 2$. Prove that the element $e_1 \otimes e_2 + e_2 \otimes e_1$ cannot be written as simple tensor $m \otimes n$, for some $m, n \in M$.

Proof. Suppose, for the sake of contradiction, that there exist $m, n \in M$ such that $m \otimes n = e_1 \otimes e_2 + e_2 \otimes e_1$. Write $m = \sum_{i=1}^n r_i e_i$ and $n = \sum_{j=1}^n s_j e_j$ for some $r_i, s_j \in R$. Then

$$e_1 \otimes e_2 + e_2 \otimes e_1 = \left(\sum_{i=1}^n r_i e_i\right) \otimes \left(\sum_{j=1}^n s_j e_j\right) = \sum_{i,j} r_i s_j e_i \otimes e_j$$

Under this isomorphism $M \cong R^n$ induced by the basis X, we have that

$$M \otimes M \cong R^n \otimes R^n \cong (R^n \otimes R)^n \cong (R \otimes R)^{n^2} \cong R^{n^2}$$

as R-modules. By the previous homework, as R is commutative, it follows that $M \otimes M$ has well defined rank given by $rank(M) = n^2$. $\square$

Problem 2

Let R be a commutative ring (with 1) and $n, m \in \mathbb{N}$. Prove that there is an isomorphism of R-**algebras** $R^n \otimes R^m \simeq R^{nm}$. (Here by R^n we mean the direct sum $\underbrace{R \oplus \cdots \oplus R}_{n}$.)

Problem 3

(a) Let V be a finite-dimensional $\mathbb{C}$-vector space. Then V can be considered as a vector over $\mathbb{R}$ (by restriction of scalars), and it holds $\dim_{\mathbb{R}} V = 2 \dim_{\mathbb{C}} V$. Prove that $V \otimes_{\mathbb{C}} V$ is not isomorphic to $V \otimes_{\mathbb{R}} V$ as $\mathbb{R}$-vector spaces, and compute their dimensions over $\mathbb{R}$.

(b) Let R be an integral domain (commutative), and let K be its fraction field. Prove that there is an isomorphism of F-modules, $F \otimes_R F \simeq F \otimes_F F \simeq F$, where the F-module structure on $F \otimes_R F$ is given by **extension of scalars** (i.e. tensor product of Type I).

Problem 4

The purpose of this problem is to classify all 2-dimensional $\mathbb{R}$-algebras (where $\mathbb{R}$ are the real numbers). That means, to classify (up to algebra isomorphism) those $\mathbb{R}$-algebras that are 2-dimensional $\mathbb{R}$ vector spaces.
Let A be a 2-dimensional $\mathbb{R}$-algebra (with 1).

(a) Let $u \in A$ be any element that is $\mathbb{R}$-linearly independent from 1. Prove that

(i) u generates A as an $\mathbb{R}$-algebra. That is, the minimal $\mathbb{R}$-subalgebra of A containing u and 1 is A itself.

(ii) The element u satisfies a quadratic equation $au^2 + bu + c = 0$, for some $a, b, c \in \mathbb{R}$ with $a \neq 0$. Conclude that A is necessarily commutative.

Proof. Noting that the subalgebra generated by u contains $\mathrm{span}_{\mathbb{R}}(\{1, u\})$ which has dimension 2 as an $\mathbb{R}$-vector space, it follows that the subalgebra generated by u is in fact A.

Since the subalgebra generated by u is A, it follows that there exist $a, b \in \mathbb{R}$ such that $u^2 = au + b1$, whence $u^2 - au - b = 0$. This implies the algebra A is commutative as multiplication is hence defined by the relations $u \cdot 1 = u = 1 \cdot u$ and $1 = 1 \cdot 1$, which are all commutative. □

(b) Show that there exists some $v \in A$ which is $\mathbb{R}$-linearly independent from 1 and is such that $v^2 = -1$, or $v^2 = 1$, or $v^2 = 0$.

Proof. □

(c) Deduce from part (b) that A is isomorphic as an $\mathbb{R}$-algebra to one of the following: $\mathbb{R}[x]/(x^2+1)$, or $\mathbb{R}[x]/(x^2-1)$, or $\mathbb{R}[x]/(x^2)$.

(d) Prove that the algebras $\mathbb{R}[x]/(x^2+1)$, $\mathbb{R}[x]/(x^2-1)$, and $\mathbb{R}[x]/(x^2)$ are pairwise non-isomorphic. **Hint:** This can be shown with almost no computation.

Problem 5

The purpose of this problem is to prove the following theorem: Let D be a finite dimensional division algebra over $\mathbb{R}$. Then D is isomorphic to $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ (the quaternions). One way to proceed is to use the following steps:

(a) Let $\alpha \in D$ be an element $\mathbb{R}$-linearly independent from 1. Show that α satisfies a quadratic irreducible polynomial $p_\alpha(x) = x^2 + ax + b \in \mathbb{R}[x]$.

Proof. Since D is finite-dimensional over $\mathbb{R}$, there exists an $n \in \mathbb{N}$ such that the set $\{1, \alpha, \dots, \alpha^n\}$ is $\mathbb{R}$-linearly dependent. Hence α is algebraic over $\mathbb{R}$, so the set $I_\alpha = \{f(x) \in \mathbb{R}[x] : f(\alpha) = 0\}$.
As I_α is an ideal and $\mathbb{R}[x]$ is a PID, there exists a (without loss of generality) monic polynomial $p_\alpha(x) \in \mathbb{R}[x]$ such that $I_\alpha = (p_\alpha)$. As α is algebraic, $p_\alpha =\neq 0$. Moreover, p_α is nonconstant by $p_\alpha(\alpha) = 0$. Hence, p_α is not a unit in $\mathbb{R}[x]$. If $f \in I_\alpha = (p_\alpha)$ is irreducible, then in writing $f = gp_\alpha$ for some $g \in \mathbb{R}[x]$, irreducibility implies that $(f) = (p_\alpha) = I_\alpha$. Moreover, this implies that $\deg(f) = \deg(p_\alpha)$, so p_α being monic implies that p_α is the unique irreducible monic element of I_α.
As $\alpha \notin \mathbb{R} \cdot 1$, $\deg(p_\alpha) \geq 2$. By the Fundamental Theorem of Algebra, it follows then that p_α must be quadratic, so there exist $a, b \in \mathbb{R}[x]$ such that $p_\alpha(x) = x^2 + ax + b$.

□

(b) Let $V = \{\alpha \in D : \alpha^2 \in \mathbb{R}_{\leq 0}\}$. Show that V is an $\mathbb{R}$-linear subspace of D. **Hint:** Show there is an $\mathbb{R}$-linear map $f : D \to \mathbb{R}$ with kernel V.

Proof. For $\alpha \in D$, define an $\mathbb{R}$-endomorphism T_α of D via left multiplication by α. This furnished a linear map $D \to \mathrm{End}_{\mathbb{R}}(D)$. We claim that V is the kernel of the composition of the $\mathbb{R}$-linear maps

$$D \to \mathrm{End}_{\mathbb{R}}(D) \xrightarrow{\mathrm{Tr}} \mathbb{R}.$$

Fix $\alpha \in D$ such that $\alpha \notin \mathbb{R} \cdot 1$. Then by part (a) there exist $a, b \in \mathbb{R}$ such that α satisfies a quadratic irreducible polynomial $p_\alpha(x) = x^2 + ax + b$. Observe that, for $v \in D$,

$$p_\alpha(T_\alpha)(v) = T_\alpha^2(v) + aT_\alpha(v) + b(v) = \alpha^2 v + a\alpha v + bv = (\alpha^2 + a\alpha + b\alpha)(v) = 0$$

so $p_\alpha(T_\alpha) = 0 \in \mathrm{End}_{\mathbb{R}}(D)$. Irreducibility of p_α then implies that p_α is the minimal polynomial for the operator $T\alpha$. Let $\chi_\alpha(x)$ be the characteristic polynomial for T_α. Then $p_\alpha(x)|\chi_\alpha(x)$ and there exists a $k \in \mathbb{N}$ such that $\chi_\alpha(x)|(p_\alpha(x))^k$. As χ_α is monic and p_α is irreducible, there exists an $l \in \mathbb{N}$ such that $\chi_\alpha(x) = (p_\alpha(x))^l$. By multinomial expansion,

$$\chi_\alpha(x) = (p_\alpha(x))^l = \sum_{\substack{n_1+n_2+n_3=l \\ n_1,n_2,n_3 \geq 0}} \binom{l}{n_1,\ n_2,\ n_3} x^{2n_1+n_2} a^{n_2} b^{n_3}$$

This polynomial has x^{2l-1} coefficient

$$\binom{l}{l-1,\ 1,\ 0} a = l \cdot a$$

However, the x^{2l-1} coefficient of χ_α is also $\pm \mathrm{Tr}(T_\alpha)$, so $\pm \mathrm{Tr}(T_\alpha) = l \cdot a$. Moreover, as $p_\alpha(x)$ is irreducible, $a^2 - 4b < 0 \implies b > \frac{a^2}{4} \geq 0$. Hence, if α is such that $\mathrm{Tr}(\alpha) = 0$, then $a = 0$ whence $0 = p_\alpha(\alpha) = \alpha^2 + b \implies \alpha^2 = -b \leq 0$, i.e. $\alpha \in V$. Conversely, suppose that $\alpha \in D \setminus \{0\}$ is such that $\alpha^2 < 0$. Then α is linearly independent from 1, so there exist $a, b \in \mathbb{R}$. such that $\alpha^2 + a\alpha + b = 0$. Note that, as $\alpha^2 \in \mathbb{R}$, linear independence of α from 1 implies that $a = 0$ and $\alpha^2 + b = 0$. Then, $\mathrm{Tr}(T_\alpha) = 0$, as desired. $\square$

(c) Define $B : V \times V \to \mathbb{R}$, $B(\alpha, \beta) := -\dfrac{\alpha\beta + \beta\alpha}{2}$. Show that B defines an inner product on V (i.e. B is a symmetric, positive definite bilinear form on V).

(d) Let W be a linear subspace of V that generates D as an $\mathbb{R}$-algebra. Let $n = \dim_{\mathbb{R}} W$. Choose an orthonormal basis of W, i.e. a basis $\{e_i\}$ of W such that $B(e_i, e_i) = 1$ for all i and $B(e_i, e_j) = 0$ for all $i \neq j$ (such a basis always exists). Using this orthonormal basis show that if $n \geq 2$, then D has a subalgebra isomorphic to $\mathbb{H}$.

(e) Bonus: Suppose $n \geq 2$. Prove that $A = H$. **Hint:** One way to proceed is to show that if $n > 2$, then the multiplication in D cannot be associative.