Reading:

- For this homework: 1.5-2.2
- For Wednesday, February 16: 2.2-2.3
- For Monday, February 18: 2.3-2.4

Problem 1.

(i) Let (X, Σ),$(Y, \mathcal{F})$ be two measurable spaces and let $\phi\colon X \to Y$ be measurable. Given a measure ν on Σ, define $\phi_*(\nu)\colon \mathcal{F} \to [0, +\infty]$ by $\phi_*(\nu)(E) = \nu(\phi^{-1}(E))$. Prove that $\phi_*(\nu)$ is a measure.

(ii) If $x \in [0, 1]$ a *binary expansion* for x is a sequence $(a_n)_{n=1}^{\infty} \in \{0, 1\}^{\mathbb{N}}$ so that $x = \sum_{n=1}^{\infty} a_n 2^{-n}$. Let N be the set of $x \in [0, 1]$ whose binary expansion is not unique. Show that N is a Borel set of measure 0 (there's a simple characterization of the points whose binary expansion is not unique, you are allowed to use this).

(iii) Let $C \subseteq [0, 1]$ be the middle thirds Cantor set. For $k \in \mathbb{N}$, define

$$\phi_k, \phi\colon [0, 1] \setminus N \to \mathbb{R}$$

by $\phi_k\left(\sum_{n=1}^{\infty} a_n 2^{-n}\right) = \sum_{n=1}^{k} 2a_n 3^{-n}$ and $\phi\left(\sum_{n=1}^{\infty} a_n 2^{-n}\right) = \sum_{n=1}^{\infty} 2a_n 3^{-n}$ for all $(a_n)_{n=1}^{\infty} \in \{0, 1\}^{\mathbb{N}}$. Show that ϕ_k, ϕ are Borel and that $\phi_k(\mathbb{R})$ and $\phi(\mathbb{R})$ are subsets of C (you may use the ternary expansion characterization of points of the Cantor set).

(iv) Set $\mu = \phi_*(m)$, where m is Lebesgue measure on $[0, 1]$. Show that $\mu(C^c) = 0$ and that there is a unique, increasing continuous function $f\colon [0, 1] \to [0, 1]$ so that $f(0) = 0$ and $\mu([a, b]) = f(b) - f(a)$ for all $0 \leq a < b \leq 1$. (In particular, $f(1) = 1$).

(v) Show that $f\left(2\sum_{n=1}^{k} a_n 3^{-n}\right) = \sum_{n=1}^{k} a_n 2^{-n}$ for all $k \in \mathbb{N}$ and all $(a_n)_{n=1}^{k} \in \{0, 1\}^k$. If (a, b) is a open interval disjoint from C, show that $f(b) = f(a)$.

(This problems shows that f is increasing with derivative 0 almost everywhere, but that f is not constant). The function f is called the Cantor-Lebesgue function (sometimes just called the Cantor function).

Problem 2.

Folland, Chapter 2, Problem 9.

Problem 3.

Folland, Chapter 2, Problem 10.

Problem 4.

Folland, Chapter 2, Problem 12.

Problem 5.

Folland, Chatper 2, Problem 14.

Problem 6.

Folland, Chatper 2, Problem 16.