MATH 7752 Homework 2

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Problem 1

Let D be a division ring (not necessarily commutative) and M be a D-module.

(a) Let X be a generating set of M and Y a D-linearly independent subset of X. Prove that M has a D-basis B with $Y \subseteq B \subseteq X$.

Proof. Consider the poset $\mathscr{S} = \{B \subseteq M : Y \subseteq B \subseteq X \text{ and } B \text{ is } D\text{-linearly independent}\}$ ordered by inclusion. Since Y is a D-linearly independent subset of X, we have that $Y \in \mathscr{S}$ so $\mathscr{S} \neq \emptyset$.

Suppose that $\mathscr{C} \subseteq \mathscr{S}$ is any linearly ordered chain in $\mathscr{S}$. Let $B = \bigcup \mathscr{C}$. Then $Y \subseteq B \subseteq X$. Suppose that $d_i \in D$ and $b_i \in B$ such that $\sum_{i=1}^n d_i \cdot b_i = 0$. Then for each $i \in \{1, \dots, n\}$, there exists a $B_i \in \mathscr{C}$ such that $b_i \in B_i$. As $\mathscr{C}$ is a chain, there is some $l \in \{1, \dots, n\}$ such that $B_i \subseteq B_l$ for all $1 \leq i \leq n$. It follows that $b_i \in B_l$ for all $1 \leq i \leq n$, whence B_l being D-linearly independent implies that $d_i = 0$ for all i. Thus B is D-linearly independent, so $B \in \mathscr{S}$.

Now by Zorn's lemma, there exists a maximal element $B \in \mathscr{S}$ of $\mathscr{S}$. We claim that B is in fact a D-basis for M. It suffices to show that B is a generating set for M. Let $N = \operatorname{span}_D(B)$. Suppose, for the sake of contradiction, that $N \neq M$. As $B \subseteq X$ and X is a generating set for M, it follows that there exists an $x \in X \setminus \operatorname{span}_D(B)$. Suppose $r, r_1, \dots, r_n \in R$ are such that

$$0 = rx + r_1 b_1 + \cdots + r_n b_n.$$

If $r \neq 0$, then

$$x = (-r^{-1} r_1) \cdot b_1 + \cdots + (-r^{-1} r_1) \cdot r_n,$$

which would imply that $x \in \operatorname{span}_D(B)$, contradicting the choice of x. Hence $r = 0$, so B being D-linearly independent implies that $r_i = 0$ for all i. Thus $B \cup \{x\}$ is D-linearly independent, contradicting the maximality of B. $\square$

(b) Conclude that every non zero D-module M has a D-basis.

Proof. Since $M \neq 0$, there exists an $y \in M \setminus \{0\}$. It follows that the singleton $\{y\}$ is a D-linearly independent subset of M. On the other hand, $M = 1 \cdot M$, so the set M is a generating set of M. Applying part (a) to $X = M$ and $Y = \{y\}$, it follows that M has a D-basis. $\square$

Problem 2

Let R be a commutative domain. Let I be a non-principal ideal of R. SHow that when I is considered as an R-module (by left multiplication), then I is indecomposable but not cyclic.

Proof. Since I is non-principal, by definition I is not cyclic as an R-module. Suppose, for the sake of contradiction, that $I = P \oplus Q$ for some nonzero proper R-submodules P, Q of I. Take $p \in P \setminus \{0\}$ and $q \in Q \setminus \{0\}$. Then $p \cdot q - q \cdot p = pq - qp = 0 \implies p \cdot q = q \cdot p$. As R is a domain $pq = qp \neq 0$, whence $pq = qp \in P \cap Q$ contradicts that the sum $P \oplus Q$ is direct. □

Problem 3

Let R be a commutative ring. An R-module M is called *torsion* if for any $m \in M$ there exists some nonzero $r \in R$ such that $rm = 0$. An R-module N is called *divisible* if for any nonzero $r \in R$ it holds that $rN = N$.

(a) Suppose M is a torsion R-module and N is a divisible R-module. Prove that $M \otimes_R N = \{0\}$.

Proof. Let $m \in M$ and $n \in N$. Since M is torsion, there exists a nonzero $r \in R$ such that $rm = 0$. Now, by divisibility of N, there exists an $n' \in N$ such that $rn' = n$. Hence

$$m \otimes n = m \otimes rn' = rm \otimes n' = 0 \otimes n' = 0.$$

Thus every simple tensor in $M \otimes_R N$ is 0, whence $M \otimes_R N = 0$. □

(b) Consider the $\mathbb{Z}$-module $M = \mathbb{Q}/\mathbb{Z}$. Prove that $M \otimes_{\mathbb{Z}} M = \{0\}$

Proof. We show that M is both torsion and divisible. Note that for any $\frac{p}{q} + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$, $q \neq 0$ and $q \cdot (\frac{p}{q} + \mathbb{Z}) = q \cdot \frac{p}{q} + \mathbb{Z} = p + \mathbb{Z} = \mathbb{Z}$, so $\mathbb{Q}/\mathbb{Z}$ is torsion.

On the other hand, suppose $n \in \mathbb{Z} \setminus \{0\}$. For $\frac{p}{q} + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$, observe that

$$n \cdot \left(\frac{p}{nq} + \mathbb{Z} \right) = \frac{np}{nq} + \mathbb{Z} = \frac{p}{q} + \mathbb{Z}$$

so $\mathbb{Q}/\mathbb{Z}$ is divisible. Appealing to part(a), it follows that $M \otimes_{\mathbb{Z}} M = 0$. □

Problem 4

Let R be a PID and A be an R-module. Let K be the field of fractions of R, and consider the K-module $B = K \otimes_R A$. Prove that every $z \in B$ is a simple tensor.

Proof. Let $z \in B$. Then there exists $\frac{x_1}{s_1}, \ldots, \frac{x_n}{s_n} \in K$, $a_1, \ldots, a_n \in A$ and $c_1, \ldots, c_n \in R$ such that

$$z = \sum_{i=1}^{n} c_i \cdot \left(\frac{x_i}{s_i} \otimes a_i \right) = \sum_{i=1}^{n} \frac{c_i x_i}{s_i} \otimes a_i = \sum_{i=1}^{n} \frac{c_i x_i \prod_{j \neq i} s_j}{s_1 \cdots s_n} \otimes a_i.$$

Since R is a PID, there exists an $s \in R$ such that $(s) = \left\langle c_i x_i \prod_{j \neq i} s_j : 1 \leq i \leq n \right\rangle$. Then for each $i \in \{1, \ldots, n\}$, there is an $r_i \in R$ such that $c_i x_i \prod_{j \neq i} s_j = r_i s$. Hence,

$$z = \sum_{i=1}^{n} \frac{c_i x_i \prod_{j \neq i} s_j}{s_1 \cdots s_n} \otimes a_i = \sum_{i=1}^{n} \frac{r_i s}{s_1 \cdots s_n} \otimes a_i = \sum_{i=1}^{n} \frac{s}{s_1 \cdots s_n} \otimes r_i a_i = \frac{s}{s_1 \cdots s_n} \otimes \left(\sum_{i=1}^{n} r_i a_i \right)$$

is a simple tensor. □

Problem 5

Let R be a commutative ring and M an R-module.

(a): Let I be an ideal of R. Prove an isomorphism

$$R/I \otimes M \simeq M/IM.$$

Proof. On one hand, consider the map $\widetilde{\Psi} : M \to R/I \otimes_R M$ given by $\widetilde{\Psi}(m) := (1 + I) \otimes m$ for $m \in M$. This map is clearly an R-module homomorphism by the second and third defining relations the tensor product. For $i \in I$ and $m \in M$, $\widetilde{\Psi}(im) = (1 + I) \otimes im = i \cdot (1 + I) \otimes m = 0$, so the generators of IM lie in $\ker(\widetilde{\Psi})$ whence $IM \subseteq \ker(\widetilde{\Psi})$. Hence, $\widetilde{\Psi}$ descends to an R-module homomorphism $\Psi : M/IM \to R/I \otimes_R M$ such that $\Psi(m+IM) = \widetilde{\Psi}(m)$. Observe that, for $r \in R$ and $m \in M$, $\Psi(rm+IM) = (1+I) \otimes r \cdot m = (r+I) \otimes m$, so $\Psi(M/IM)$ contains all simple tensors, whence by linearity Ψ is surjective.

On the hand, consider the map $\widetilde{\Phi} : R/I \times M \to M/IM$ given by $\widetilde{\Phi}(r + I, m) = rm + IM$. To see that this is well-defined, if $r + I = r' + I \in R/I$, then $r - r' \in I$ whence $(r - r') \cdot m + IM = IM$. Suppose now that $m, n \in M$, $r + I, s + I \in R/I$, and $x \in R$. Then

$$\begin{aligned}
\widetilde{\Phi}((r+I)+(s+I), m) &= \widetilde{\Phi}((r+s)+I, m) = (r+s) \cdot m + IM \\
&= (r \cdot m + IM) + (s \cdot m + IM) = \widetilde{\Phi}(r+I, m) + \widetilde{\Phi}(s+I, m) \\
\widetilde{\Phi}(r+I, m+n) &= r \cdot (m+n) + IM = r \cdot m + r \cdot n + IM = (r \cdot m + IM) + (r \cdot n + IM) \\
&= \widetilde{\Phi}(r+I, m) + \widetilde{\Phi}(r+I, n) \\
\widetilde{\Phi}(x \cdot (r+I), m) &= \widetilde{\Phi}(xr+I, m) = (xr) \cdot m + IM = x \cdot (r \cdot m + IM) = x \cdot \widetilde{\Phi}(r+I, m) \\
\widetilde{\Phi}(r+I, x \cdot m) &= r \cdot (x \cdot m) + IM = x \cdot (r \cdot m + IM) = x \cdot \widetilde{\Phi}(r+I, m),
\end{aligned}$$

so $\widetilde{\Phi}$ is an R-bilinear map. By the universal property of tensor products, there exists a unique R-module homomorphism $\Phi : R/I \otimes M \to M/IM$ such that $\Phi((r + I) \otimes m) = \widetilde{\Phi}(r + I, m)$ for all $r \in R$ and $m \in M$.

Now we show that Φ and Ψ are mutual inverses. On one hand, for $m + IM \in M/IM$,

$$\Phi(\Psi(m + IM)) = \Phi((1 + I) \otimes m) = 1 \cdot m + IM = m + IM,$$

so $\Phi \circ \Psi = id_{M/IM}$. For the other direction, it suffices by linearity to prove that $\Psi \circ \Phi$ is the identity on just the simple tensors. For $(r + I) \otimes m \in R/I \otimes M$,

$$\Psi(\Phi((r + I) \otimes m)) = \Psi(rm + IM) = (1 + I) \otimes rm = (r + I) \otimes m$$

so $\Psi \circ \Phi$ is the identity on simple tensors, whence $\Psi \circ \Phi = id_{R/I \otimes M}$.

□

(b): Suppose that M is a finitely generated free R-module. Show that the *rank* of M is well-defined, i.e. any two R-bases of M have the same number of elements.

Proof. Let $\mathfrak{m} \subseteq R$ be a maximal ideal of R. Let $k = R/\mathfrak{m}$ be the corresponding residue field. Suppose that $n, l \in \mathbb{N}$ such that $R^l \cong M \cong R^n$. Then,

$$k^l \cong (R/\mathfrak{m} \otimes R)^l \cong R/\mathfrak{m} \otimes R^l \cong R/\mathfrak{m} \otimes R^n \cong (R/\mathfrak{m} \otimes R)^n \cong k^n$$

as R-modules. Let $\varphi : k^l \to k^n$ be the composition of the above R-module isomorphisms. Note that $\mathfrak{m} \subseteq \mathrm{Ann}_R(k^l), \mathrm{Ann}_R(k^n)$, so the k-module structures on k^l, k^m given by $(r+\mathfrak{m}) \cdot a := r \cdot a$ and $(r+\mathfrak{m}) \cdot b$ for $a \in k^l$ and $b \in k^n$ are well-defined. Moreover, for $r \in R$ and $a \in k^l$,

$$\varphi((r+\mathfrak{m}) \cdot a) = \varphi(r \cdot a) = r \cdot \varphi(a) = (r+\mathfrak{m}) \cdot \varphi(a),$$

so φ is also a k-module isomorphism. As k^l and k^n are isomorphic k-vector spaces, it follows that $l = n$.

□

Problem 6

Let $R \subseteq S$ be an inclusion of commutative rings. Consider the polynomial rings $R[x]$ and $S[x]$. Prove that there is an isomorphism of S-modules,

$$S \otimes_R R[x] \to S[x].$$

Proof. On one hand, there is an obvious map $\Psi : S[x] \to S \otimes_R R[x]$ defined by

$$\Psi\left(\sum_{k=0}^{n} s_k x^k\right) := \sum_{k=0}^{n} s_k \otimes x^k.$$

Then, for monomials $s_k x^k, t_k x^k \in S[x]$ and $s \in S$,

$$\Psi(s \cdot (s_k x^k) + t_k x^k) = \Psi((ss_k + t_k)x^k) = (ss_k + t_k) \otimes x^k = s \cdot (s_k \otimes x^k) + t_k \otimes x^k = s \cdot \Psi(s_k x^k) + \Psi(t_k x^k),$$

whence via extending linearly and applying this relation, it follows that Ψ is an S-module homomorphism.

On the other hand, consider the map $\widetilde{\Phi} : S \times R[x] \to S[x]$ given by $(s, f(x)) \mapsto sf(x)$. This map is clearly R-bilinear as it is a multiplication map, so there exists a unique R-module homomorphism $\Phi : S \otimes R[x] \to S[x]$ such $\Phi(s \otimes f(x)) = \widetilde{\Phi}(s, f(x)) = sf(x)$. Moreover, we will show that this map is in fact an S-module homomorphism. It suffices to show this relation on simple tensors, whence linearity would imply that it holds on all of $S \otimes_R R[x]$ considered as an S-module. Let $s, s' \in S$ and $f(x) \in R[x]$. Then,

$$\Phi(s \cdot (s' \otimes f(x))) = \Phi((ss') \otimes f(x)) = ss' f(x) = s \cdot \Phi(s' \otimes f(x)).$$

Now we show that the S-module homomorphisms Ψ and Φ are mutual inverses. On one hand, suppose that $f(x) = \sum_{k=0}^{n} s_k x^k \in S[x]$. Then

$$\Phi(\Psi(f(x))) = \Phi\left(\sum_{k=0}^{n} s_k \otimes x^k\right) = \sum_{k=0}^{n} \Phi(s_k \otimes x^k) = \sum_{k=0}^{n} s_k x^k = f(x),$$

so $\Phi \circ \Psi = id_{S[x]}$. On the other hand, it suffices to show that $\Psi \circ \Phi$ agrees with the identity on simple tensors, whence by linearity it would agree with the identity on all of $S \otimes_R R[x]$. Suppose $f(x) = \sum_{k=0}^{n} r_k x^k \in R[x]$ and $s \in S$. Then,

$$\Psi(\Phi(s \otimes f(x))) = \Psi(sf(x)) = \sum_{k=0}^{n} \Psi(sr_k x^k) = \sum_{k=0}^{n} (sr_k) \otimes x^k = \sum_{k=0}^{n} s \otimes r_k x^k = s \otimes \sum_{k=0}^{n} r_k x^k = s \otimes f(x),$$

so $\Psi \circ \Phi = id_{S \otimes R[x]}$.

□

Problem 7

Let R be a commutative ring and $I_1, \ldots, I_k$ be a finite collection of ideals of R. Let M be an R-module.

(a) Prove that the map $f : M \to \frac{M}{I_1 M} \times \cdots \times \frac{M}{I_k M}$ defined by

$$m \mapsto (m + I_1 M, \ldots, m + I_k M)$$

is an R-module homomorphism with kernel $I_1 M \cap \cdots \cap I_k M$.

Proof. Let $m, n \in M$ and $r \in R$. Then

$$f(r\cdot m+n) = (r\cdot m+n+I_1M, \ldots, r\cdot m+n+I_kM) = r\cdot(m+I_1M, \ldots, m+I_kM)+(n+I_1M, \ldots, n+I_kM) = r\cdot f(m)+f(n)$$

so f is an R-module homomorphism. It is clear that $I_1 M \cap \cdots \cap I_k M \subseteq \ker(f)$, so it remains to show the reverse inclusion. Suppose that $m \in \ker(f)$. Then $(I_1 M, \ldots, I_k M) = f(m) = (m + I_1 M, \ldots, m + I_k M)$, whence $m \in I_j M$ for $1 \leq j \leq k$, so $m \in I_1 M \cap \cdots \cap I_k M$. □

(b) Assume in addition that the ideals $I_1, \ldots, I_k$ are pairwise comaximal. Show that there is an isomorphism of R-modules,

$$\frac{M}{(I_1 \cdots I_k)M} \cong \frac{M}{I_1 M} \times \cdots \times \frac{M}{I_k M}.$$

Proof. We proceed by induction on the integer $k \geq 2$. Suppose first that $k = 2$. Then by comaximality, there exists an $r \in I_1$ and $s \in I_2$ such that $r + s = 1$. Then $1 - s = r \in I_1$, so for $m \in M$,

$$\begin{aligned} f(rm) &= (rm + I_1 M, rm + I_2 M) = (I_1 M, (1 - s) \cdot m + I_2 M) = (I_1 M, m + I_2 M) \\ f(sm) &= (sm + I_1 M, sm + I_2 M) = ((1 - r) \cdot m + I_1 M, I_2 M) = (m + I_1 M, I_2 M). \end{aligned}$$

Hence, for a fixed $(m + I_1 M, n + I_2 M) \in M/I_1 M \times M/I_2 M$,

$$f(rn + sm) = f(rn) + f(sm) = (m + I_1 M, n + I_2 M),$$

so f is surjective. Now we show that $I_1 \cap I_2 = I_1 I_2$. The reverse inclusion is true *a priori*, regardless of comaximality of the ideals. For the forward inclusion, suppose that $x \in I_1 \cap I_2$. Then $x = x(r+s) = xr + xs \in I_1 I_2$. Hence $\ker(f) = I_1 I_2$ by part (a), so the claim follows via the first isomorphism theorem.

Now fix $k > 2$ suppose that the claim holds for all integers less than k. Consider the ideals $I = I_1$ and $J = I_2 \cdots I_k$. We claim that these ideals are comaximal. For $j \in \{2, \ldots, k\}$, choose $r_j \in I$ and $s_j \in I_i$ such that $r_j + s_j = 1$. Then, every term in the expansion of the product $1 = (x_2 + y_2) \cdots (x_k + y_k)$ is in I except for the term $y_2 \cdots y_k \in J$, so $1 \in I + J$ i.e. I, J are comaximal. By the induction hypothesis,

$$\frac{M}{(I_1 \cdots I_k)M} = \frac{M}{(IJ)M} \cong \frac{M}{I_1 M} \times \frac{M}{(I_2 \cdots I_k)M} \cong \frac{M}{I_1 M} \times \cdots \times \frac{M}{I_k M}.$$

□