MATH 7752 - HOMEWORK 2
DUE WEDNESDAY 01/29/20

(1) Let D be a division ring (not necessarily commutative) and M be a D-module.
  (a) Let X be a generating set of M and Y a D-linearly independent subset of X. Prove that M has a D-basis B with $Y \subseteq B \subseteq X$.
  (b) Conclude that every non zero D-module M has a D-basis.

(2) Let R be a commutative domain. Let I be a non-principal ideal of R. Show that when I is considered as an R-module (by left multiplication), then I is indecomposable, but not cyclic. **Hint:** One way to proceed is to show that any two elements of I are R-linearly dependent.

(3) Let R be a commutative ring. An R-module M is called *torsion* if for any $m \in M$ there exists some nonzero $r \in R$ such that $rm = 0$. An R-module N is called *divisible* if for any nonzero $r \in R$ it holds $rN = N$. (Equivalently, N is divisible if for any $r \neq 0$ the R-module homomorphism $N \xrightarrow{\cdot r} N$ is surjective).
  (a) Suppose M is a torsion R-module and N is a divisible R-module. Prove that $M \otimes_R N = \{0\}$.
  (b) Consider the $\mathbb{Z}$-module $M = \mathbb{Q}/\mathbb{Z}$. Prove that $M \otimes_{\mathbb{Z}} M = \{0\}$.

(4) Let R be a PID and A be an R-module. Let K be the field of fractions of R, and consider the K-module $B = K \otimes_R A$. Prove that every $z \in B$ is a simple tensor. That is, z can be written in the form $z = q \otimes a$, for some $a \in A$ and $q \in K$.

(5) Let R be a commutative ring and M an R-module.
  (a) Let I be an ideal of R. Prove an isomorphism
$$R/I \otimes M \simeq M/IM.$$
  (b) Suppose that M is finitely generated free R-module. Show that the *rank* of M is well-defined, i.e. any two R-bases of M have the same number of elements. **Hint:** Imitate the proof we did in class for domains using the fact that R has at least one maximal ideal.

(6) Let $R \subseteq S$ be an inclusion of commutative rings. Consider the polynomial rings $R[x]$ and $S[x]$. Prove that there is an **isomorphism of S-modules**,
$$S \otimes_R R[x] \xrightarrow{\simeq} S[x].$$

(7) (Mostly for Practice) Let R be a commutative ring and $I_1, \dots, I_k$ be a finite collection of ideals of R. Let M be an R-module.
  (a) Prove that the map $f : M \to \dfrac{M}{I_1 M} \times \cdots \times \dfrac{M}{I_k M}$ defined by
$$m \mapsto (m + I_1 M, \dots, m + I_k M)$$

is an R-module homomorphism with kernel $I_1M \cap \ldots \cap I_kM$.

(b) Assume in addition that the ideals $I_1, \ldots, I_k$ are pairwise comaximal. That is, $I_i + I_j = R$, for $i \neq j$. Imitate the proof of the Chinese Remainder Theorem for rings to show that there is an isomorphism of R-modules,

$$\frac{M}{(I_1 \cdots I_k)M} \simeq \frac{M}{I_1M} \times \cdots \times \frac{M}{I_kM}.$$

Extra Problem (optional)

(1) Dummit & Foote: Problem 27, p. 351. (Example of a ring R for which $R \simeq R^2$ as R-modules).