MATH 7310 Homework 9

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Problem 1

Let ν be a complex measure on (X, Σ). If $\nu(X) = |\nu|(X)$, prove that $\nu = |\nu|$.

Proof. As $\nu \ll |\nu|$ and $|\nu|$ is a finite measure (so *a fortiori* a σ-finite measure), $d\nu = f\,d|\nu|$ where $f = \frac{d\nu}{d|\nu|}$. Moreover, by Proposition 3.13(b), $|f| = 1$ $|\nu|$-almost everywhere. As $|\operatorname{Re}(f)| \leq 1$ a.e., it follows that $1 - \operatorname{Re}(f) \geq$ a.e. Then, we observe by finiteness of $\nu(X)$ that

$$0 = |\nu|(X) - \nu(X) = \int 1 - f\,d|\nu| = \int 1 - \operatorname{Re}(f)\,d|\nu| - i\int \operatorname{Im}(f)\,d|\nu|\,.$$

Thus, the real and imaginary parts of the right side of the above equation must both be zero, so the almost everywhere positivity of $1 - \operatorname{Re}(f)$ implies that $\operatorname{Re}(f) = 1$ $|\nu|$-almost everywhere. After modifying out by the measure zero sets upon which $|f|$ and $\operatorname{Re}(f)$ are not both equal to one, we obtain that $\operatorname{Re}(f) = |f|$ $|\nu|$-almost everywhere, whence $f = \operatorname{Re}(f) = 1$ and $\operatorname{Im}(f) = 0$. Thus, for all $E \in \Sigma$,

$$\nu(E) = \int_E f\,d|\nu| = \int_E \operatorname{Re}(f)\,d|\nu| + i\int_E \operatorname{Im}(F)\,d|\nu| = \int_E 1\,d|\nu| = |\nu|(E).$$

$\square$

Problem 2

Let ν be a complex measure on (X, Σ). If $E \in \Sigma$, define

$$\mu_1(E) = \sup\left\{\sum_1^n |\nu(E_j)| : n \in \mathbb{N},\ E_1, \ldots, E_n \text{ disjoint},\ E = \bigsqcup_1^n E_j\right\},$$
$$\mu_2(E) = \sup\left\{\sum_1^\infty |\nu(E_j)| : E_1, E_2, \ldots \text{ disjoint},\ E = \bigsqcup_1^\infty E_j\right\}$$
$$\mu_3(E) = \sup\left\{\left|\int_E f\,d\nu\right| : |f| \leq 1\right\}.$$

Prove that $\mu_1 = \mu_2 = \mu_3$. (*Hint*: First show that $\mu_1 \leq \mu_2 \leq \mu_3$. To see that $\mu_3 = |\nu|$, let $f = \overline{d\nu / d|\nu|}$ and apply Proposition 3.13. To see that $\mu_3 \leq \mu_1$, approximate f by simple functions).

Proof. That $\mu_1 \leq \mu_2$ is clear as we may take the sequence $E_1, \ldots, E_n, \emptyset, \ldots$ in the set for μ_2 to recover the values in the set for μ_1.

Recall from page 46 of Folland that for any function $f : X \to \mathbb{C}$ we have its polar decomposition $f = \operatorname{sgn}(f)|f|$ where $\operatorname{sgn}(z) = z/|z|$ if $z \neq 0$ and $\operatorname{sgn}(0) = 0$. Moreover, if f is measurable with respect to some positive measure, then so are $\operatorname{sgn}(f)$ and $|f|$. From the polar decomposition of f, it follows that $\overline{\operatorname{sgn}(f)}f = |f|$. Using this idea, suppose that$E_1, E_2, \dots$ are disjoint with $E = \bigsqcup_{j=1}^{\infty} E_j$. Then we compute

$$\sum_{j=1}^{\infty} |\nu(E_j)| = \sum_{j=1}^{\infty} \overline{\operatorname{sgn}\nu(E_j)}\nu(E_j)$$

Hence, we are led to define $f = \sum_{j=1}^{\infty} \overline{\operatorname{sgn}(\nu(E_j))}\mathbb{1}_{E_j}$. This function is measurable as it is a pointwise limit of simple functions. Moreover, as the sets E_j are pairwise disjoint and $|\operatorname{sgn}(z)| \leq 1$ for all z, it follows that $|f| \leq 1$. Noting that $|\operatorname{Re}(f)|, |\operatorname{Im}(f)| \leq |f| \leq 1 \in L^1(|\nu|)$ (as $|\nu(X)| < +\infty$), we may apply the dominated convergence theorem to the positive and negative parts of the partial sums for both $\operatorname{Re}(f)$ and $\operatorname{Im}(f)$ with respect to the positive and negative parts of the signed measures $\operatorname{Re}(\nu)$ and $\operatorname{Im}(\nu)$ to obtain that

$$\left| \int_E f \, d\nu \right| = \left| \sum_{j=1}^{\infty} \int \overline{\operatorname{sgn}(\nu(E_j))}\mathbb{1}_{E_j} \, d\nu \right| = \left| \sum_{j=1}^{\infty} \overline{\operatorname{sgn}(\nu(E_j))}\nu(E_j) \right| = \sum_{j=1}^{\infty} |\nu(E_j)|,$$

so $\mu_2 \leq \mu_3$.

Now fix $E \in \Sigma$. One one hand, suppose that $|g| \leq 1$. Then by Proposition 3.13(c),

$$\left| \int_E g \, d\nu \right| \leq \int_E |g| \, d|\nu| \leq \int_E 1 \, d|\nu| = |\nu|(E),$$

whence $\mu_3(E) \leq |\nu|(E)$. On the other hand, let $f = \overline{d\nu \,/\, d|\nu|}$. By Proposition 3.13(b), $|\, d\nu \,/\, d|\nu|\,| = 1$ $|\nu|$-a.e. whence $d\nu \,/\, d|\nu| = 1/f$ $|\nu|$-a.e. Thus, we compute

$$\left| \int_E f \, d\nu \right| = \left| \int_E f \frac{d\nu}{d|\nu|} \, d|\nu| \right| = |\nu|(E),$$

whence $|\nu|(E) \leq \mu_3(E)$. So we have show that $\mu_3 = |\nu|$.

It remains to show that $\mu_3 \leq \mu_1$. Fix $E \in \Sigma$ and suppose f is measurable with $|f| \leq 1$. Chose simple functions $(\phi_k)_{k=1}^{\infty}$ such that $0 \leq |\phi_1| \leq |\phi_2| \leq \cdots \leq |f|$ and $\phi_k \to f$ pointwise. Write $\phi_k = \sum_{j=1}^{n_k} c_j^{(k)} \mathbb{1}_{E_j^{(k)}}$ where, for all $k \in \mathbb{N}$, $E_1^{(k)}, \dots, E_{n_k}^{(k)}$ are pairwise disjoint such that $X = \bigsqcup_{j=1}^{n_k} E_j^{(k)}$ and $c_j^{(k)} \in \mathbb{C}$ with $|c_j^{(k)}| \leq 1$.

As before, we may apply the dominated convergence theorem to the positive and negative parts for the sequences $\operatorname{Re}(\phi_k \mathbb{1}_E)$ and $\operatorname{Im}(\phi_k \mathbb{1}_E)$ with respect to the positive and negative parts of the signed measures $\operatorname{Re}(\nu)$ and $\operatorname{Im}(\nu)$ to obtain that

$$\begin{aligned}
\left| \int_E f \, d\nu \right| &= \left| \lim_{k\to\infty} \int \phi_k \mathbb{1}_E \, d\nu \right| = \left| \lim_{k\to\infty} \int \sum_{j=1}^{n_k} c_j^{(k)} \mathbb{1}_{E_j^{(k)} \cap E} \, d\nu \right| \\
&= \lim_{k\to\infty} \left| \sum_{j=1}^{n_k} c_j^{(k)} \nu(E_j^{(k)} \cap E) \right| \leq \lim_{k\to\infty} \sum_{j=1}^{n_k} |\nu(E_j^{(k)} \cap E)| \leq \mu_1(E).
\end{aligned}$$

As $|f| \leq 1$ was arbitrary, it follows that $\mu_3(E) \leq \mu_1(E)$ as desired. $\square$

Problem 3

(a): Let (X, Σ) be a measurable space. Let $M(\Sigma)$ be the vector space of complex measures on Σ with the total variation norm $\|\mu\| = |\mu|(X)$. Show that $M(\Sigma)$ is a Banach space.
Suggestion: it may be helpful to use that for $\mu \in M(\Sigma)$ we have

$$\sum_{n=1}^{\infty} |\mu(E_n)| \leq \|\mu\|$$

where $(E_n)_{n=1}^{\infty}$ is a sequence of pairwise disjoint elements of Σ (this is a consequence of a prior problem on this homework).

Proof. Let $(\mu_n)_{n=1}^{\infty}$ be a Cauchy sequence in $M(\Sigma)$. Define a new positive measure λ on Σ by

$$\lambda(E) = \sum_{n=1}^{\infty} 2^{-n} \frac{|\mu_n|(E)}{\|\mu_n\| + 1}.$$

Then by nonnegativity, we have that for all $n \in \mathbb{N}$, $\mu_n \ll \lambda$ whence there exists some measurable f_n such that $d\mu_n = f_n \, d\lambda$. Moreover, $f_n \in L^1(\lambda)$ as $\int |f_n| \, d\lambda = |\mu_n|(X) < +\infty$. Note that λ is necessarily a finite measure, so we utilize part (b) of this exercise. Let $J : L^1(X, \lambda) \to M(\Sigma)$ be as in part (b). Then $\mu_n = J(f_n)$ for all $n \in \mathbb{N}$. As J is an isometry, it follows that $(f_n)_{n=1}^{\infty}$ is Cauchy in $L^1(\lambda)$, whence by completeness there exists some $f \in L^1(\lambda)$ such that $\|f_n - f\|_{L^1(\lambda)} \xrightarrow{n\to\infty} 0$. Let $\mu = J(f)$. Then

$$\|\mu_n - \mu\| = \|f_n - f\|_{L^1(\lambda)} \xrightarrow{n\to\infty} 0,$$

so $\mu_n \to \mu$ in total variation norm. Thus $M(\Sigma)$ is Banach. □

(b): Fix a positive, σ-finite measure μ on Σ. Show that the map $J : L^1(X, \mu) \to M(\Sigma)$ given by $J(f) = f \, d\mu$ is a linear isometry with closed image.

Proof. Let We wish to show that for $f, g \in L^1(X, \mu)$ and $\alpha \in \mathbb{C}$, $J(\alpha f + g) = \alpha J(f) + J(g)$, after which showing that $\|J(f)\|_{M(\Sigma)} = \|f\|_{L^1(\mu)}$ would imply that J is a linear isometry.

Let $f \in L^1(X, \mu)$. We compute

$$\|J(f)\|_{M(\Sigma)} = |J(f)|(X) = J(f)(X) = \int_X f \, d\mu = \|f\|_{L^1(\mu)}.$$

Suppose that $(J(f_n))_{n=1}^{\infty}$ converges to ν in $M(\Sigma)$ where $(f_n)_{n=1}^{\infty}$ is in $L^1(X, \mu)$. So

$$\|J(f_n) - \nu\|_{M(\Sigma)} \xrightarrow{n\to\infty} 0$$

Suppose that $E \in \Sigma$ is null. Then as $J(f_n) \ll \mu$, $|J(f_n)|(E) = 0$ for all $n \in \mathbb{N}$. So by problem 2,

$$|\nu|(E) \leq |J(f_n)(E - \nu(E))| + |J(f_n)(E)| \leq \|J(f_n) - \nu\|_{M(\Sigma)} \xrightarrow{n\to\infty} 0$$

whence $|\nu|(E) = 0$ i.e. E is null for ν. Thus $\nu \ll \mu$. By the Lebesgue-Radon-Nikodym theorem, there exists some $f \in L^1(X, \mu)$ such that $\nu = f \, d\mu = J(f)$, so ν is in the image of J. □

(c): Suppose that $\mu, \nu \in M(\Sigma)$, and let $d\nu = f \, d\mu + d\lambda$ with $\lambda \perp \mu$ be the Lebesgue-Radon-Nikodym decomposition. Show that

$$\|\mu - \nu\| = \|1 - f\|_{L^1(\mu)} + \|\lambda\|.$$

Lemma 1 (Lemma 1)**.** *If $\nu_1, \nu_2 \in M(\Sigma)$ and $\nu_1 \perp \nu_2$, then $\|\nu_1 + \nu_2\| = \|\nu_1\| + \|\nu_2\|$.*

Proof of Lemma 1. Let $\mu = |\nu_1| + |\nu_2|$, so μ is a positive finite measure such that $\nu_j \ll \mu$. By Radon-Nikodym theorem, there is some f_j ($j = 1, 2$) such that $d\nu_j = f_j \, d\mu$. Then $d|\nu_j| = |f_j| \, d\mu$ and $d|\nu_1 + \nu_2| = |f_1 + f_2| \, d\mu$. Let $E, F \subseteq X$ such that $X = E \cup F$, $E \cap F = \emptyset$, E is null for ν_2, and F is null for ν_1. Then $0 = |\nu_1|(F) = \int_F |f_1| \, d\mu$ whence $f_1 = 0$ μ-almost everywhere on F and similarly we obtain that $f_2 = 0$ μ-almost everywhere on E. On the other hand, by the finiteness of $|\nu_1|, |\nu_2|$, it follows that $|\nu_1|(X) = |\nu_1|(E)$ and $|\nu_2|(X) = |\nu_2|(F)$. We compute

$$\begin{aligned}\|\nu_1 + \nu_2\| &= \int_X |f_1 + f_2| \, d\mu = \int_E |f_1 + f_2| \, d\mu + \int_F |f_1 + f_2| \, d\mu \\ &= \int_E |f_1| \, d\mu + \int_F |f_2| \, d\mu = |\nu_1|(E) + |\nu_2|(F) = \|\nu_1\| + \|\nu_2\|.\end{aligned}$$

□

Proof. For notational clarity, let $\nu = \delta_{ac} + \delta_s$ be the Lebesgue decomposition of ν with respect to μ, so $\delta_{ac} \ll \mu$ and $\delta_s \perp \mu$. Let $d\delta_{ac} = f \, d\mu$. As $\mu \ll \mu$ and $\delta_{ac} \ll \mu$, we have that $\mu - \delta_{ac} \ll \mu$, whence noting that $-\delta_s \perp \mu$ we conclude that $\mu - \delta_{ac} \perp -\delta_s$. Lastly, observe that for $E \in \Sigma$,

$$\mu(E) - \delta_{ac}(E) = \int_E 1 \, d\mu - \int_E f \, d\mu = \int_E 1 - f \, d\mu \,,$$

whence $\frac{d(\mu - \delta_{ac})}{d\mu} = 1 - f$. Hence, in the notation of part (b), $\mu - \delta_{ac} = J(1 - f)$. So, noting that J is an isometry and applying Lemma 1 to $\mu - \delta_{ac} \perp -\delta_s$, we find that

$$\|\mu - \nu\| = \|\mu - \delta - ac - \delta_s\| = \|\mu - \delta_{ac}\| + \|\delta_s\| = \|1 - f\|_{L^1(\mu)} + \|\delta_s\|.$$

□

Problem 4

If E is a Borel set in $\mathbb{R}^n$, the density $D_E(x)$ of E at x is defined as

$$D_E(x) = \lim_{r \to 0} \frac{m(E \cap B_r(x))}{m(B_r(x))},$$

whenever the limit exists.

(a): Show that $D_E(x) = 1$ for a.e. $x \in E$ and $D_E(x) = 0$ for a.e. $x \in E^c$.

Proof. Define a new Borel measure ν by $\nu(F) = \mu(E \cap F)$ for Borel F. Then $\nu \ll m$ and observe that, for Borel sets F,

$$\nu(F) = \int \mathbb{1}_{E \cap F} \, dm = \int \mathbb{1}_E \mathbb{1}_F \, dm = \int_F \mathbb{1}_E \, dm \,,$$

so $\frac{d\nu}{dm} = \mathbb{1}_E$ m-a.e. by uniqueness in Lebesgue-Radon-Nikodym theorem. Moreover, as $\nu(F) \leq m(F)$ for all Borel F, it follows that ν is finite on compacts and thus regular, so by the Lebesgue differentiation theorem, the following limit exists for m-a.e. x and is equal to

$$D_E(x) = \lim_{r \to 0} \frac{\nu(B_r(x))}{m(B_r(x))} = \mathbb{1}_E(x),$$

whence the claim follows. □

(b): Find examples of E and x such that $D_E(x)$ is a given number $\alpha \in (0,1)$, or such that $D_E(x)$ does not exist.

Solution. For an example where E and x are such that $D_E(x)$ is any given number $\alpha \in (0,1)$, consider $X = \mathbb{R}^2$, $x = (0,0)$, and E a sector of the unit disk centered at $(0,0)$ making an $\alpha \cdot 2\pi$ radians angle with $(0,0)$ and the x-axis. By definition of angles, $D_E((0,0))$ is clearly α.

$\square$

Problem 5

Let $\psi : \mathbb{R} \to \mathbb{R}$ be given as $\psi = \mathbb{1}_{[0,1/2)} - \mathbb{1}_{[1/2,1]}$. For $n, k \in \mathbb{Z}$ define $h_{n,k}(t) = 2^{n/2}\psi(2^n t - k)$. Show that $\mathcal{E} = \{1\} \cup \{h_{n,k} : n \in \mathbb{N} \cup \{0\},\, 0 \leq k < 2^n\}$ is an orthonormal basis for $L^2([0,1])$.

Proof. Note that

$$h_{n,k} = 2^{n/2}\left(\mathbb{1}_{[k2^{-n},(k+1/2)2^{-n})} - \mathbb{1}_{[(k+1/2)2^{-n},(k+1)2^{-n}]}\right)$$

Suppose that $f \in L^2([0,1])$ and $f \in \mathcal{E}^\perp$. For $n \in \mathbb{N} \cup \{0\}$, define a function $c_n : \{0, \ldots, 2^{n+1} - 1\}$ by $c_n(k) = \int_{[k2^{-(n+1)},(k+1)2^{-(n+1)}]} f(x)\,dx$. Then, for all $n \in \mathbb{N} \cup \{0\}$ and $0 \leq k < 2^n$,

$$0 = \langle f, h_{nk} \rangle \implies c_n(2k) = \int_{[k2^{-n},(k+1/2)2^{-n})} f(x)\,dx = \int_{[(k+1/2)2^{-n},(k+1)2^{-n}]} f(x)\,dx = c_n(2k+1).$$

We show that $c_n(\cdot)$ is constant for all $n \in \mathbb{N} \cup \{0\}$ by induction on n. For $n = 0$, we have already shown that $c_0(0) = c_0(1)$. Now suppose that $c_{n-1}(\cdot)$ is constant. Then, for $k \in \{0, \ldots, 2^{-(n+1)}\}$

$$c_{n-1}(0) = c_{n-1}(k) = c_n(2k) + c_n(2k+1) = 2c_n(2k) = 2c_n(2k+1),$$

so $c_n(\cdot)$ is constant.

However then Lebsegue differentiation theorem implies that f is constant as we may always find a nicely shrinking sequence of sets in our aforementioned family, however the expression we would obtain for $f(x)$ would not depend on x as c_n is constant. Now the fact that $\int_0^1 f(x)\,dx = 0$ implies that $f = 0$. $\square$

Problem 6

Fix $n \in \mathbb{N}$, and $1 \leq p < +\infty$. For $y \in \mathbb{R}^n$, define $\tau_y : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ by $\tau_y(f)(x) = f(x-y)$. Show that if $f \in L^p(\mathbb{R}^n)$, then

$$\|\tau_y f\|_p = \|f\|_p. \tag{1}$$

$$\lim_{y \to 0} \|\tau_y f - f\|_p = 0. \tag{2}$$

Hint: use (1) to show that the set of f's for which (2) is true is a closed, linear subspace of $L^p(\mathbb{R}^n)$. Then check (2) on a dense set of f's where (2) is easier to see.

Proof. Part (1) follows trivially from the change of variables $x - y \rightsquigarrow x$. Let $S \subseteq L^p(\mathbb{R}^n)$ be the set of f's in $L^p(\mathbb{R}^n)$ for which (2) is true. Suppose $(f_n)_{n=1}^\infty$ is a sequence in S and $f \in L^p(\mathbb{R}^n)$ such that $\|f_n - f\|_p \xrightarrow{n\to\infty} 0$. Observe that

$$\begin{aligned}\|\tau_y f - f\|_p \leq \|\tau_y f - f_n\|_p + \|f_n - f\|_p &\leq \|\tau_y(f - f_n)\|_p + \|\tau_y f_n - f_n\|_p + \|f_n - f\|_p \\ &= 2\|f_n - f\|_p + \|\tau_y f_n - f_n\|,\end{aligned}$$

which implies that $f \in S$.

Property (2) is clearly invariant under scaling by a real number, so to show S is a linear subspace of $L^p(\mathbb{R}^n)$ it suffices to show that it is closed under sums. Suppose that $f, g \in S$. Then

$$\|\tau_y(f+g)-(f+g)\|_p \leq \|\tau_y f - f\|_p + \|\tau_y g - g\|_p$$

whence $f+g \in S$.

We show that (2) holds for $C_c(\mathbb{R}^n)$. Fix $f \in C_c(\mathbb{R}^n)$ and let K be the closure of the support of f. By continuity of f, $\tau_y f \to f$ pointwise. Without loss of generality, we restrict our limit to over $y < 1$. Then, $|\tau_y f| \leq \sup_{x \in \mathbb{R}^n} |f(x)| \mathbb{1}_{K+B_1(0)} \in L^p(\mathbb{R}^n)$, whence by the dominated convergence theorem

$$\lim_{y \to 0} \int |f(x-y) - f(x)|^p \, dx = \int \lim_{y \to 0} |f(x-y) - f(x)|^p \, dx = 0.$$

Thus, (2) holds for a dense subspace of $L^p(\mathbb{R}^n)$, so it holds for all of $L^p(\mathbb{R}^n)$. $\square$