Reading:

- For this homework: 2.5-2.6, 6.1
- For Wedneday, March 16: 6.1, 5.5
- For Monday, March 21: 5.5, 3.1

Problem 1.

Folland Chapter 2, Problem 47.

Problem 2.

Folland Chapter 2, Problem 49.

Problem 3.

(a) Folland Chapter 2, Problem 50

(b) Let (X, μ) be a σ-finite measure space. Fix $p \in [1, \infty)$. Show that if $f \in L^p(X, \mu)$, then

$$\|f\|_p^p = p \int_0^\infty t^{p-1} \mu(\{x : f(x) > t\}).$$

(It might be helpful to consider the measure in the previous part applied to an appropriate modification of f).

(c) Let (X, μ) be a σ-finite measure space. Show that if $f, g \in L^1(X, \mu)$ wih $0 \leq f, g$ a.e., then

$$\|f - g\|_1 = \int_0^\infty \mu(\{x : f(x) > t\} \Delta \{x : g(x) > t\})\, dt.$$

Suggestion: it might be helpful to first show that for $a, b \in [0, \infty)$ we have

$$|a - b| = \int_0^\infty 1_{(t,\infty)}(a) - 1_{(t,\infty)}(b)|\, dt.$$

Remark: the last two parts of this problem are sometimes called the *bathtub principle.*

Problem 4.

Folland Chapter 2, Problem 56.

Problem 5.

Recall that an h-rectangle in $\mathbb{R}^d$ is a product $\prod_{j=1}^d I_j$ of h-interval Let $\mathcal{E}_q$ be the set of products of the form $\prod_{j=1}^d I_j$ where each I_j is an h-interval with the property that all of its finite endpoints are rational.

(a) Show that $\mathcal{E}_q$ is an elementary family which generates the Borel sets.

(b) Suppose that μ is a Borel measure on $\mathbb{R}^d$ with $0 < \mu((0,1]^d) < +\infty$. If $\mu(E + x) = \mu(E)$ for every $x \in \mathbb{R}^d$, show that $\mu(E) = \mu((0,1]^d)m(E)$ for every Borel $E \subseteq \mathbb{R}^d$.

Problem 6.

Fix $d \in \mathbb{N}$.

(a) Let $s\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ be the map $s(x, y) = x + y$. Let μ, ν be finite, Borel measures on $\mathbb{R}^d$. Define $\mu * \nu = s_*(\mu \otimes \nu)$. Show that for every Borel $E \subseteq \mathbb{R}^d$ we have

$$\mu * \nu(E) = \int\int 1_E(x + y)\, d\mu(x)\, d\nu(y),$$

and

$$\int \mu(E - y)\, d\nu(y) = \mu * \nu(E) = \int \nu(E - x)\, d\mu(x).$$

Show as a consequence that

$$\mu * \nu(X) = \mu(X)\nu(X).$$

(b) Show that for finite, Borel measures μ, ν, η on $\mathbb{R}^d$ we have

$$(\mu * \nu) * \eta = \mu * (\nu * \eta).$$

(c) For $f, g\colon L^1(\mathbb{R}^d)$ show that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(y)g(x - y)|\, dx\, dy = \|f\|_1\|g\|_1.$$

Explain why this implies that $y \mapsto f(y)g(x - y)$ is in $L^1(\mathbb{R}^d)$ for almost every $x \in \mathbb{R}^d$ and why if we set $f * g(x) = \int_{\mathbb{R}^d} f(y)g(x - y)$ then we have that $f * g \in L^1(\mathbb{R}^d)$ and

$$\|f * g\|_1 \leq \|f\|_1\|g\|_1.$$

(d) Adopt notation as in Problem 1 of HW5. Show that if $f, g \in L^1(\mathbb{R}^d)$ are nonnegative then $(f\, dm) * (g\, dm) = f * g\, dm$ with m being Lebesgue measure.

(e) Show that for $f, g, k \in L^1(\mathbb{R}^d)$ we have that

$$(f * g) * k = f * (g * k) \text{ almost everywhere.}$$

Problems to think about. do not turn in

Problem 7.

Let $((X_n, \Sigma_n, \mu_n))_{n=1}^{\infty}$ be a sequence of probability spaces. Set $X = \prod_{n=1}^{\infty} X_n$ and $\Sigma = \bigotimes_{n=1}^{\infty} \Sigma_n$. For $k \in \mathbb{N}$, set $\mu^{(k)} = \bigotimes_{j=1}^{k} \mu_j$.

(a) For $r \in \mathbb{N}$, Let $\mathcal{A}_r$ be the collection of all sets of the form $E \times \prod_{j=k+1}^{\infty}$ for some $r \leq k$ and $E \in \bigotimes_{j=r}^{k} \Sigma_j$. Show that $\mathcal{A}_r$ is an algebra, and that there is a unique finitely-additive measure $\pi^{(r)}\colon \mathcal{A} \to [0, 1]$ so that

$$\pi^{(r)}\left(E \times \prod_{n=k}^{\infty} X_k\right) = \left(\bigotimes_{j=r}^{k} \mu_j\right)(E), \text{ for all } E \in \bigotimes_{j=r}^{k} \Sigma_j.$$

(b) If $E \in \mathcal{A}_r$, show that

$$\pi(E) = \int_{X_r} \pi^{(r+1)}(E_x)\, d\mu_r(x).$$

(As usual, $E_x = \left\{y \in \prod_{j=r+1}^{\infty} X_j : (x, y) \in E\right\}$). Show that if $\pi^{(r)}(E) \geq \varepsilon > 0$, then

$$\mu_r(\{x \in X_r : \pi^{(r+1)}(E_x) \geq \varepsilon/2\}) \geq \varepsilon/2.$$

(c) Suppose that $(E_n)_{n=1}^{\infty}$ is a decreasing sequence in $\mathcal{A}_1$ and that $\varepsilon = \inf_n \pi(E_n) > 0$.

$$F_n = \left\{x \in X_1 : \pi^{(2)}((E_n)_x) \geq \varepsilon/2\right\}.$$

Show that F_n is a decreasing sequence of sets with $\mu_1\left(\bigcap_{n=1}^{\infty} F_n\right) \geq \varepsilon/2$. Use this to show that there is an $x_1 \in X$ with $\pi^{(1)}((E_n)_{x_1}) \geq \varepsilon/2$ for every $n \in \mathbb{N}$.

(d) For $x \in \prod_{j=1}^{k} X_j$, and $E \subseteq \prod_{j=1}^{\infty} X_j$, we set

$$E_x = \left\{ y \in \prod_{j=k+1}^{\infty} X_j : (x, y) \in E \right\}$$

where we use the natural identification $\prod_{j=1}^{k} X_j \times \prod_{j=k+1}^{\infty} X_j = \prod_{j=1}^{\infty} X_j$. If $(E_n)_n$ are as in (c), show that there is a sequence $x = (x_k)_{k=1}^{\infty} \in X$ so that for every $n, k \in \mathbb{N}$ we have

$$\pi^{(k+1)}((E_n)_{(x_1, \cdots, x_k)}) \geq \varepsilon/2^k.$$

Show that $x \in \bigcap_{n=1}^{\infty} E_n$.

(e) Explain why Problem 4 (iii) on HW2 implies that there is a unique measure $\mu\colon \Sigma \to [0,1]$ so that $\mu\big|_{\mathcal{A}_1} = \pi^{(1)}$. (This is often called the countable product measure and is denoted $\bigotimes_{n=1}^{\infty} \mu_n$. It is of significant interest in probability theory).

Problem 8.

Folland Chapter 2, Problems 46-47, Folland Chapter 2, Problems 56-61.