MATH 7310 Homework 7

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Problem 1

Let (X, Σ, μ) be a measure space.

(i): Prove that if $\mu(E_n) < +\infty$ for $n \in \mathbb{N}$ and $\mathbb{1}_{E_n} \to f$ in L^1, then f is (a.e. equal to) the characteristic function of a measurable set.

Proof. For $m \in \mathbb{N}$, let

$$F_m = \{x : \min\{|1 - f(x)|, |f(x)|\} > \frac{1}{m}\}.$$

Then $(F_m)_{m=1}^\infty$ is an increasing sequence of sets with $F = \{x : f(x) \notin \{0, 1\}\} = \bigcup_{n=1}^\infty A_n$.

Observe that, for fixed $m \in \mathbb{N}$,

$$\|\mathbb{1}_{E_n} - f\|_1 \geq \int_{F_m} |\mathbb{1}_{E_n} - f| \, d\mu \geq \int_{F_m} \frac{1}{m} \, d\mu = \frac{1}{m}\mu(F_m)$$

for all $n \in \mathbb{N}$, whence sending $n \to \infty$ it follows that $\mu(F_m) = 0$. Thus, it follows that $\mu(F) = 0$. Thus, $f = \mathbb{1}_{f^{-1}(\{1\})}$ almost everywhere. □

(ii): Let $\Sigma_f = \{E \in \Sigma : \mu(E) < +\infty\}$. Define an equivalence relation on Σ_f by $E \sim F$ if $\mu(E \Delta F) = 0$. Let $\Omega = \Sigma_f / \sim$, and define a metric ρ on Ω be $\rho([E], [F]) = \mu(E \Delta F)$. Show that the map $\iota : \Omega \to L^1(X, \mu)$ given by $\iota([E]) = \mathbb{1}_E$ is an isometry with closed image.

Proof. Observe that, if $E, F \in \Sigma_f$, then

$$\rho(\iota(E), \iota(F)) = \mu(E \Delta F) = \int \mathbb{1}_{E \Delta F} \, d\mu = \int |\mathbb{1}_E - \mathbb{1}_F| \, d\mu \,,$$

so ι is an isometry. Now suppose that $(f_n)_{n=1}^\infty$ is in $\iota(\Omega)$ and $f \in L^1(X, \mu)$ with $\|f_n - f\|_1 \xrightarrow{n \to \infty} 0$. Then for $n \in \mathbb{N}$, there are $E_n \in \Sigma_f$ such that $f_n = \mathbb{1}_{E_n}$, whence by part (i) there is some measurable $E \subseteq X$ such that $f = \mathbb{1}_E$. As $\mathbb{1}_E = f \in L^1(\mu)$, it follows that $\mu(E) < +\infty$ whence $[E] \in \Omega$ and thus $f = \iota([E]) \in \iota(\Omega)$. □

(iii): Show that (Ω, ρ) is a complete metric space.

Proof. Let $([E_n])_{n=1}^\infty$ be a Cauchy sequence in (Ω, ρ). Then as ι is an isometry, it follows that $(\mathbb{1}_{E_n})_{n=1}^\infty$ is a Cauchy sequence in $L^1(X, , \mu)$. By completeness of $L^1(X, \mu)$, there exists some $f \in L^1(X, \mu)$ such that $\|\mathbb{1}_{E_n} - f\|_1 \xrightarrow{n \to \infty} 0$. As the image of ι is closed, it follows that there is some $E \subseteq X$ with $\mu(E) < +\infty$ such that $f = \mathbb{1}_E = \iota([E])$ almost everywhere. Then, ι being an isometry implies that $\rho([E_n], [E]) \xrightarrow{n \to \infty} 0$. □

Problem 2

If X, Y are sets, and $f : X \to \mathbb{C}$, $g : Y \to \mathbb{C}$, we define $f \otimes g : X \times Y \to \mathbb{C}$ by $(f \otimes g)(x, y) = f(x)g(y)$. Fix $1 \le p < +\infty$.

(a): Let $(X, \Sigma, \mu), (Y, \mathcal{F}, \nu)$ be σ-finite measure spaces. Show that if $f \in L^p(X, \mu), g \in L^p(Y, \nu)$, then $\|f \otimes g\|_p = \|f\|_p \|g\|_p$.

Proof. By Tonelli's theorem,

$$\begin{aligned}\|f \otimes g\|_p^p &= \int_{X\times Y} |f \otimes g|^p \, d\mu \otimes \nu = \int_Y \int_X |f(x)|^p |g(y)|^p \, d\mu(x) \, d\nu(y) \\ &= \int_Y |g(y)|^p \int_X |f(x)|^p \, d\mu(x) \, d\nu(y) = \|f\|_p^p \int_Y |g(y)|^p \, d\nu(y) = \|f\|_p^p \|g\|_p^p.\end{aligned}$$

□

(b): Let $(Z, \mathcal{O}, \zeta)$ be a finite measure space. Suppose that $\mathcal{A} \subseteq \mathcal{O}$ is an algebra which generates the σ-algebra of $\mathcal{O}$. Use the monotone class lemma to show that $\{\mathbb{1}_A : A \in \mathcal{A}\}$ is dense in $\{\mathbb{1}_E : E \in \mathcal{O}\}$ in the L^p-norm for all $1 \le p < +\infty$.

Proof. By the monotone class lemma, $\mathcal{O} = \Sigma(\mathcal{A}) = M(\mathcal{A})$. Let $E \in \mathcal{O}$. Let

$$\mathscr{C} = \left\{ E \in \mathcal{O} : \mathbb{1}_E \in \overline{\{\mathbb{1}_A : A \in \mathcal{A}\}}^{\|\cdot\|_p} \right\}.$$

We will show that $\mathscr{C}$ is a monotone class as then $\mathcal{A} \subseteq \mathscr{C}$ would imply that $\mathcal{O} = M(\mathcal{A}) \subseteq \mathscr{C}$ and we would be done.

First, we compute that for any $A, B \in \mathscr{O}$, by problem 1(ii)

$$\|\mathbb{1}_A - \mathbb{1}_B\|_p^p = \int |\mathbb{1}_A - \mathbb{1}_B|^p \, d\zeta = \int \mathbb{1}_{A\Delta B} \, d\zeta = \|\mathbb{1}_A - \mathbb{1}_B\|_1 = \rho([A], [B]) = \zeta(A\Delta B).$$

Suppose that $E_1 \subseteq E_2 \subseteq \cdots$ is an increasing sequence of sets in $\mathscr{C}$ and let $E = \bigcup_{j\in\mathbb{N}} E_j$. Then, the finiteness of ζ and continuity from below,

$$\|\mathbb{1}_E - \mathbb{1}_{E_n}\|_p^p = \zeta(E\Delta E_n) = \zeta(E \setminus E_n) = \zeta(E) - \zeta(E_n) \xrightarrow{n\to\infty} 0,$$

whence $\mathbb{1}_E \in \mathscr{C}$.

On the other hand, suppose that $F_1 \supseteq F_2 \supseteq \cdots$ is a decreasing sequence of sets in $\mathscr{C}$ and let $F = \bigcup_{j\in\mathbb{N}} F_j$. Then by finiteness of ζ and continuity from below,

$$\|\mathbb{1}_F - \mathbb{1}_{F_n}\|_p^p = \zeta(F_n\Delta F) = \zeta(F_n \setminus F) = \zeta(F_n) - \zeta(F) \xrightarrow{n\to\infty} 0,$$

whence $\mathbb{1}_F \in \mathscr{C}$. □

(c): Let $(X, \Sigma, \mu), (Y, \mathcal{F}, \nu)$ be finite measure spaces. Use the previous part to show that $\{\mathbb{1}_E : E \in \Sigma\otimes\mathcal{F}\} \subseteq \overline{\mathrm{Span}}^{\|\cdot\|_p}\{\mathbb{1}_E \otimes \mathbb{1}_F : E \in \Sigma, F \in \mathcal{F}\}$. Use this to show that $\overline{\mathrm{Span}}^{\|\cdot\|_p}\{\mathbb{1}_E \otimes \mathbb{1}_F : E \in \Sigma, F \in \mathcal{F}\} = L^p(X \times Y, \mu \otimes \nu)$.

Proof. Let $\mathcal{A} \subseteq \Sigma \otimes \mathcal{F}$ be the algebra of finite disjoint unions of rectangles in $X \times Y$. Note that any $A \in \mathcal{A}$ has the form $A = \bigsqcup_{j=1}^n E_j \times F_j$ for some $E_j \in \Sigma$ and $F_j \in \mathcal{F}$, whence $\mathbb{1}_A = \sum_{j=1}^n \mathbb{1}_{E_j} \otimes \mathbb{1}_{F_j}$, so

$$\{\mathbb{1}_A : A \in \mathcal{A}\} \subseteq \mathrm{Span}\{\mathbb{1}_E \otimes \mathbb{1}_F : E \in \Sigma, F \in \mathcal{F}\}.$$

By definition of the product sigma algebra, $\Sigma(\mathcal{A}) = \Sigma \otimes \mathcal{F}$, so part (b) implies that

$$\{\mathbb{1}_E : E \in \Sigma \otimes \mathcal{F}\} = \overline{\{\mathbb{1}_A : A \in \mathcal{A}\}}^{\|\cdot\|_p} \subseteq \overline{\mathrm{Span}}^{\|\cdot\|_p}\{\mathbb{1}_E \otimes \mathbb{1}_F : E \in \Sigma, F \in \mathcal{F}\}.$$

As the left hand side is a linear subspace,

$$\mathrm{Span}\{\mathbb{1}_E : E \in \Sigma \otimes \mathcal{F}\} \subseteq \overline{\mathrm{Span}}^{\|\cdot\|_p}\{\mathbb{1}_E \otimes \mathbb{1}_F : E \in \Sigma, F \in \mathcal{F}\}$$

whence by density of simple functions in L^p and the finiteness of $\mu \otimes \nu$,

$$L^p(X \times Y, \mu \otimes \nu) \subseteq \overline{\mathrm{Span}}^{\|\cdot\|_p}\{\mathbb{1}_E : E \in \Sigma \otimes \mathcal{F}\} \subseteq \overline{\mathrm{Span}}^{\|\cdot\|_p}\{\mathbb{1}_E \otimes \mathbb{1}_F : E \in \Sigma, F \in \mathcal{F}\} \subseteq L^p(X \times Y, \mu \otimes \nu).$$

□

(d): Let $(X, \Sigma, \mu), (Y, \mathcal{F}, \nu)$ be σ-finite measure spaces. Suppose that $D_X \subseteq L^p(X, \mu)$, $D_Y \subseteq L^p(Y, \nu)$ and that

$$\overline{\mathrm{Span}}^{\|\cdot\|_p}(D_X) = L^p(X, \mu), \quad \overline{\mathrm{Span}}^{\|\cdot\|_p}(D_Y) = L^p(Y, \nu).$$

Show that $\overline{\mathrm{Span}}^{\|\cdot\|_p}(\{f \otimes g : f \in D_X, g \in D_Y\}) = L^p(X \times Y, \mu \otimes \nu)$.

Proof. First suppose that X, Y are finite measure spaces. By part (c), it suffices to show that

$$\{\mathbb{1}_E \otimes \mathbb{1}_F : E \in \Sigma, F \in \mathcal{F}\} \subseteq \overline{\mathrm{Span}}^{\|\cdot\|_p}\{f \otimes g : f \in D_X, g \in D_Y\} = L^p(X \times Y, \mu \otimes \nu).$$

To this end, take $E \in \Sigma$ and $F \in \mathcal{F}$. By finiteness, $\mathbb{1}_E \in L^p(X, \mu)$ and $\mathbb{1}_F \in L^p(Y, \nu)$. Then by assumption, there are sequences $(f_n)_{n=1}^\infty$ in $\mathrm{Span}(D_X)$ and $(g_n)_{n=1}^\infty$ in $\mathrm{Span}(D_Y)$ such that

$$\|f_n - \mathbb{1}_E\|_p, \|g_n - \mathbb{1}_F\|_p \xrightarrow{n \to \infty} 0.$$

Then, we compute using Tonelli's theorem that

$$\begin{aligned}
\|f_n \otimes g_n - \mathbb{1}_E \otimes \mathbb{1}_F\|_p^p &= \int |f_n \otimes g_n - \mathbb{1}_E \otimes \mathbb{1}_F|^p \, d(\mu \otimes \nu) \\
&= \int |f_n \otimes g_n - \mathbb{1}_E \otimes g_n + \mathbb{1}_E \otimes g_n - \mathbb{1}_E \otimes \mathbb{1}_F|^p \, d(\mu \otimes \nu) \\
&\leq \int |f_n \otimes g_n - \mathbb{1}_E \otimes g_n|^p + \mathbb{1}_E \otimes g_n - \mathbb{1}_E \otimes \mathbb{1}_F|^p \, d(\mu \otimes \nu) \\
&= \|f_n - \mathbb{1}_E\|_p^p \|g_n\|_p^p + \mu(E) \|g_n - \mathbb{1}_F\|_p^p \xrightarrow{n \to \infty} 0,
\end{aligned}$$

as desired. Thus, the claim holds for finite measure spaces. Now suppose X, Y are σ-finite and let $X_1 \subseteq X_2 \subseteq \cdots$ and $Y_1 \subseteq Y_2 \subseteq \cdots$ be measurable of finite measure such that $X = \bigcup X_i$ and $Y = \bigcup Y_j$. By part (c), for all $n \in \mathbb{N}$

$$\overline{\mathrm{Span}}^{\|\cdot\|_p}\{\mathbb{1}_{X_n \times Y_n}(f \otimes g) : f \in D_X, g \in D_Y\} = L^p(X_n \times Y_n, \mu \otimes \nu)$$

whence

$$L^p(X \times Y) = \bigcup_{n=1}^\infty \overline{\mathrm{Span}}^{\|\cdot\|_p}\{\mathbb{1}_{X_n \times Y_n}(f \otimes g) : f \in D_X, g \in D_Y\}.$$

As such, it suffices to show that

$$\{\mathbb{1}_{X_n \times Y_n}(f \otimes g) : f \in D_X, g \in D_Y\} \subseteq \overline{\operatorname{Span}}^{\|\cdot\|_p}\{f \otimes g : f \in D_X, g \in D_Y\}$$

for all $n \in \mathbb{N}$. Hence, let $n \in \mathbb{N}$, $f \in D_X$, and $g \in D_Y$. Then there are sequences $(f_k)_{k=1}^{\infty}$ in $\operatorname{Span}(D_X)$ and $(g_k)_{k=1}^{\infty}$ in $\operatorname{Span}(D_Y)$ such that

$$\|f_k - \mathbb{1}_{X_n} f\|_p, \|g_k - \mathbb{1}_{Y_n} g\|_p \xrightarrow{n\to\infty} 0.$$

Then we compute

$$\begin{aligned}
\|f_k \otimes g_k - \mathbb{1}_{X_n} f \otimes \mathbb{1}_{Y_n} g\|_p^p &= \int |f_k \otimes g_k - \mathbb{1}_{X_n} f \otimes \mathbb{1}_{Y_n} g|^p \, d(\mu \otimes \nu) \\
&= \int |f_k \otimes g_k - \mathbb{1}_{X_n} f \otimes g_k + \mathbb{1}_{X_n} f \otimes g_k - \mathbb{1}_{X_n} f \otimes \mathbb{1}_{Y_n} g|^p \, d(\mu \otimes \nu) \\
&\leq \int |f_k \otimes g_k - \mathbb{1}_{X_n} f \otimes g_k|^p + \mathbb{1}_{X_n} f \otimes g_k - \mathbb{1}_{X_n} f \otimes \mathbb{1}_{Y_n} g|^p \, d(\mu \otimes \nu) \\
&= \|f_k - \mathbb{1}_{X_n} f\|_p^p \|g_k\|_p^p + \|\mathbb{1}_{X_n} f\|_p^p \|g_k - \mathbb{1}_{Y_n} g\|_p^p \xrightarrow{k\to\infty} 0,
\end{aligned}$$

so $\mathbb{1}_{X_n \times Y_n}(f \otimes g) \in \overline{\operatorname{Span}}^{\|\cdot\|_p}\{f \otimes g : f \in D_X, g \in D_Y\}$. □

Problem 3

Suppose that $f \in L^p \cap L^\infty$ for some $p < +\infty$ so that $f \in L^q$ for all $q > p$. Prove that then $\|f\|_\infty = \lim_{q\to\infty} \|f\|_q$.

Proof. By Folland Proposition 6.10, we have that

$$\|f\|_q^{\frac{p}{q}} \leq \|f\|_p^{\frac{p}{q}} \|f\|_\infty^{\frac{p}{q}}$$

whence $\limsup_{q\to\infty} \|f\|_q \leq \|f\|_\infty$. On the other hand, for $n \in \mathbb{N}$ let $E_n = \{x : |f(x)| > \|f\|_\infty - \frac{1}{n}\}$. Then $(E_n)_{n=1}^{\infty}$ is a decreasing sequence of measurable sets with $E = \bigcap_{n=1}^{\infty} E_n = \{x : |f(x)| \geq \|f\|_\infty\}$ having $\mu(E) = 0$ by definition of the L^∞-norm. Observe that, for $n \in \mathbb{N}$ and $q > p$,

$$\|f\|_q \geq \left(\int_{E_n} |f|^q \, d\mu\right)^{\frac{1}{q}} > (\|f\|_\infty - \frac{1}{n})\mu(E_n)^{\frac{1}{q}}$$

whence

$$\liminf_{q\to\infty} \|f\|_q \geq \|f\|_\infty - \frac{1}{n}.$$

As this holds for all $n \in \mathbb{N}$, it follows that $\liminf_{q\to\infty} \|f\|_q \geq \|f\|_\infty$ as desired. □

Problem 4

If f is a measurable function on X, define the *essential range* R_f of f to be the set of all $z \in \mathbb{C}$ such that $\{x : |f(x) - z| < \varepsilon\}$ has positive measure for all $\varepsilon > 0$.

(a): Prove that R_f is closed.

Proof. Let $z \in \overline{R_f}$. Then there exists a sequence $(z_n)_{n=1}^{\infty}$ in R_f such that $z_n \to z$. Fix $\varepsilon > 0$. There is some $N \in \mathbb{N}$ such that $n \geq N \implies B_{\varepsilon/2}(z_n) \subseteq B_\varepsilon(z)$. Then $f^{-1}(B_{\varepsilon/2}(z_n)) \subseteq f^{-1}(B_\varepsilon(z))$, whence $0 < \mu(f^{-1}(B_{\varepsilon/2}(z_n))) \leq \mu(f^{-1}(B_\varepsilon(z)))$. Hence $z \in R_f$, so R_f is closed. □

(b): Prove that if $f \in L^\infty$, then R_f is compact and $\|f\|_\infty = \max\{|z| : z \in R_f\}$.

Proof. Fix $z \in R$ and let $M > 0$ be such that $\mu(f^{-1}(X \setminus \overline{B_M(0)})) = 0$. Suppose, for the sake of contradiction, that $|z| > M$. Then we may choose $\varepsilon > 0$ such that $B_\varepsilon(z) \subseteq X \setminus B_M(0)$. Then $f^{-1}(B_\varepsilon(z)) \subseteq f^{-1}(X \setminus B_M(0))$, whence $\mu(f^{-1}(B_\varepsilon(z))) = 0$ contradicting that $z \in R_f$. Thus $|z| \leq M$. As $M > 0$ was arbitrary for its condition, it follows that $|z| \leq \|f\|_\infty$. As $z \in R_f$ was arbitrary, it follows that $\sup_{z \in R_f} |z| \leq \|f\|_\infty < +\infty$. Hence R_f is compact by part (a) and Heine-Borel. Let $z_{max} \in R_f$ such that $|z_{\max}| = \max_{z \in R_f} |z| = \sup_{z \in R_f} |z|$.

We show that in fact $\mu(f^{-1}(\mathbb{C} \setminus B_{|z_{\max}|}(0))) = 0$, whence it would follow that $\|f\|_\infty \leq |z_{\max}|$ as desired. Let $N \in \mathbb{N}$ such that for all $n \geq N$ we have $|z_{\max}| < |z_{\max}| + \frac{1}{n} < \|f\|_\infty$. Then for $n \geq N$, set $U_n = (\mathbb{C} \setminus \overline{B_{|z_{\max}|+\frac{1}{n}}(0)}) \setminus (\mathbb{C} \setminus B_{\|f\|_\infty}(0))$. We may cover U_n by countably many balls $B_{r_j}(z_j)$ where $B_{r_j}(z_j) \subseteq U$, i.e. $U = \bigcup_{j=1}^\infty B_{r_j}(z_j)$. It follows then that $\mu(f^{-1}(B_{r_j}(z_j))) = 0$, whence $\mu(U_n) = 0$. As the U_n's are a decreasing sequence of measure zero sets with intersection $(\mathbb{C} \setminus B_{|z_{\max}|(0)}) \setminus (\mathbb{C} \setminus B_{\|f\|_\infty}(0)))$, it follows that $\mu(f^{-1}((\mathbb{C} \setminus B_{|z_{\max}|(0)}) \setminus (\mathbb{C} \setminus B_{\|f\|_\infty}(0))))) = 0$. Thus $\mu(f^{-1}(\mathbb{C} \setminus B_{|z_{\max}|}(0))) = 0$ as desired. □

Problem 5

Suppose that $1 \leq p < +\infty$ and $(f_n)_{n=1}^\infty$ in L^p. Prove that $(f_n)_{n=1}^\infty$ is Cauchy in the L^p-norm if and only if the following three conditions hold:

1. (f_n) is Cauchy in measure;
2. the sequence $(|f_n|^p)_{n=1}^\infty$ is uniformly integrable
3. for every $\varepsilon > 0$ there exists $E \subseteq X$ such that $\mu(E) < +\infty$ and $\int_{E^c} |f_n|^p \, d\mu < \varepsilon$ for all $n \in \mathbb{N}$.

Lemma 1. *Any finite subset $\{f_k\}_{k=1}^n \subseteq L^1(\mu)$ is uniformly integrable.*

Proof of Lemma 1. We show first that $f \in L^1(\mu)$ is uniformly integrable. Note that, if $f \in L^1(\mu)$, then $|f|\mathbb{1}_{\{|f|>m\}} \searrow 0$ pointwise a.e. as $\{|f| = +\infty\} = \bigcap_{M \in \mathbb{N}} \{|f| > M\}$ implies that $\lim_{M\to\infty} \mu(|f| > M) = \mu(\{|f| = +\infty\}) = 0$. Moreover, for all $M \in \mathbb{N}$, $|f \mathbb{1}_{|f|>M}| \leq |f| \in L^1(\mu)$, so by the dominated convergence theorem

$$\lim_{M\to\infty} \int_{\{|f|>M\}} |f| \, d\mu = 0. \tag{1}$$

For any $E \subseteq X$ measurable and $M \in \mathbb{N}$, we have that

$$\int_E |f| \, d\mu = \int_{E \cap \{|f| \leq M\}} |f| \, d\mu + \int_{E \cap \{|f|>M\}} |f| \, d\mu \leq M \cdot \mu(E) + \int_{\{|f|>M\}} |f| \, d\mu \,. \tag{2}$$

Fix $\varepsilon > 0$. By (1), there exists some $N \in \mathbb{N}$ such that $\int_{\{|f|>N\}} |f| \, d\mu < \frac{\varepsilon}{2}$. Choose $\delta = \frac{\varepsilon}{2N}$. Then, for any $E \subseteq X$ measurable such that $\mu(E) < \delta$, we have by (2) that

$$\left| \int_E f \, d\mu \right| \leq \int_E |f| \, d\mu < N \cdot \delta + \frac{\varepsilon}{2} = \varepsilon.$$

Now suppose that $\{f_k\}_{k=1}^n \subseteq L^1(\mu)$ is a finite subset of $L^1(\mu)$. Fix $\varepsilon > 0$. By uniform integrability of each of the singletons, for each $k \in \{1, \ldots, n\}$ there exists a $\delta_k > 0$ such that $\mu(E) < \delta_k \implies |\int_E f_k| < \varepsilon$. Choosing $\delta = \min\{\delta_1, \ldots, \delta_n\} > 0$, the claim follows. □

Lemma 2. *Suppose $(f_n)_{n=1}^\infty$ is a sequence in $L^1(\mu)$ and $f \in L^1(\mu)$ such that $\|f_n - f\|_1 \xrightarrow{n\to\infty} 0$. Then $\{f_n\}_{n=1}^\infty$ is uniformly integrable.*

Proof of Lemma 2. Observe that, for any measurable $E \subseteq X$ and $n \in \mathbb{N}$,

$$\int_E |f_n|\,d\mu \le \int_E |f|\,d\mu + \int_E |f_n - f|\,d\mu \le \int_E |f|\,d\mu + \|f_n - f\|_1.$$

Fix $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that for $n \ge N$ we have $\|f_n - f\|_1 < \frac{\varepsilon}{2}$. By Lemma 1, $\{f\}$ is uniformly integrable, so there is some $\delta' > 0$ such that $\mu(E) < \delta'$ implies that $\int_E |f|\,d\mu < \frac{\varepsilon}{2}$.

Again by Lemma 1, $\{f_k\}_{k=1}^{N-1}$ is uniformly integrable, so there is some $\delta'' > 0$ such that $\mu(E) < \delta''$ implies $\int_E |f_k|\,d\mu < \varepsilon$ for all $k \in \{1, \ldots, N-1\}$. Setting $\delta = \min\{\delta', \delta''\}$, the claim follows. $\square$

Lemma 3 (Lemma 3)**.** *Condition (3) holds for any finite subset $\{f_k\}_{k=1}^n \subseteq L^1(\mu)$.*

Proof of Lemma 3. We show first that the claim holds for just one function $f \in L^1(\mu)$. Suppose first that f is nonnegative and let $\varepsilon > 0$. Then by definition of the integral, there exists a simple function $0 \le g \le f$ such that

$$\int f\,d\mu - \int g\,d\mu < \varepsilon.$$

By monotonicity of the integral, $\int g \le \int f < +\infty$, whence it follows that the set $E = \{x : g(x) > 0\}$ has finite measure (as g takes finitely many values). Then

$$\int_{E^c} f\,d\mu = \int_{E^c} f - g\,d\mu \le \int f - g\,d\mu < \varepsilon.$$

Now suppose that f is real-valued and let $f^{\pm}$ be the positive and negative parts of f. Fix $\varepsilon > 0$. Then by the previous case there exist measurable $E^{\pm} \subseteq X$ with $\mu(E^{\pm}) < +\infty$ and $\int_{(E^{\pm})^c} f^{\pm}\,d\mu < \frac{\varepsilon}{2}$. Letting $E = E^+ \cup E^-$, it follows that $\mu(E) < +\infty$ and

$$\int_{E^c} |f|\,d\mu = \int_{(E^+)^c \cap (E^-)^c} f^+ + f^-\,d\mu \le \int_{(E^+)^c} f^+\,d\mu + \int_{(E^-)^c} f^-\,d\mu < \varepsilon.$$

Finally, suppose that f is complex-valued. Let $u = \mathrm{Re}(f)$ and $v = \mathrm{Im}(f)$. The claim then follows from applying the previous case to u, v and using the inequality $|f| \le |u| + |v|$.

Now suppose that we have a finite subset $\{f_k\}_{k=1}^n \subseteq L^1(\mu)$ and fix $\varepsilon > 0$. Then for $1 \le k \le n$ there exists $E_k \subseteq X$ with $\mu(E_k) < +\infty$ and $\int_{E_k^c} |f_k|\,d\mu < \varepsilon$. Let $E = E_1 \cup \cdots \cup E_n$. Then $\mu(E) < +\infty$ and for $1 \le k \le n$ we have

$$\int_{E^c} |f_k|\,d\mu = \int_{\bigcap_{j=1}^n E_j^c} |f_k|\,d\mu \le \int_{E_k^c} |f_k|\,d\mu < \varepsilon.$$

$\square$

Lemma 4 (Lemma 4)**.** *Suppose $(f_n)_{n=1}^\infty$ is a sequence in $L^1(\mu)$ and $f \in L^1(\mu)$ such that $\|f_n - f\|_1 \xrightarrow{n\to\infty} 0$. Then $\{f_n\}_{n=1}^\infty$ satisfies condition (3).*

Proof of Lemma 4. As in the proof of Lemma 2, we utilize that for any measurable $E \subseteq X$ and $n \in \mathbb{N}$,

$$\int_{E^c} |f_n|\,d\mu \le \int_{E^c} |f|\,d\mu + \|f_n - f\|_1.$$

Fix $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that for $n \ge N$ we have $\|f_n - f\|_1 < \frac{\varepsilon}{2}$. By Lemma 3, $\{f_k\}_{k=1}^{N-1}$ satisfies condition (3), so there is some $E_1 \subseteq X$ with $\mu(E_1) < +\infty$ such that $\int_{E_1^c} |f_k|\,d\mu < \varepsilon$ for all $k \in \{1, \ldots, N-1\}$. The singleton $\{f\}$ also satisfies condition 3, so there is some $E_2 \subseteq X$ with $\mu(E_2) < +\infty$ and $\int_{E_2^c} |f|\,d\mu < \frac{\varepsilon}{2}$.

Setting $E = E_1 \cup E_2$, it follows that $\mu(E) < +\infty$, $\int_{E^c} |f_k|\, d\mu \le \int_{E_1^c} |f_k|\, d\mu < \varepsilon$ for $k \in \{1, \ldots, N-1\}$, and for all $n \ge N$

$$\int_{E^c} |f_n|\, d\mu \le \int_{E^c} |f|\, d\mu + \|f_n - f\|_1 < \int_{E_2^c} |f|\, d\mu + \frac{\varepsilon}{2} < \varepsilon.$$

□

Proof of Theorem.

$\underline{\Longrightarrow}$: Suppose that $(f_n)_{n=1}^\infty$ is Cauchy in the L^p-norm. Then by completeness, there is some $f \in L^p(\mu)$ such $\|f - f_n\|_p \xrightarrow{n\to\infty} 0$. For $\varepsilon > 0$, noting that $\{|f_n - f| \ge \varepsilon\} = \{|f_n - f|^p/\varepsilon^p \ge 1\}$, we have that

$$\mu(\{|f_n - f| \ge \varepsilon\}) = \int_{\{|f_n - f| \ge \varepsilon\}} \frac{|f_n - f|^p}{\varepsilon^p}\, d\mu \le \frac{1}{\varepsilon^p} \|f_n - f\|_p^p \xrightarrow{n\to\infty} 0.$$

Thus $f_n \to f$ in measure, whence $(f_n)_{n=1}^\infty$ is Cauchy in measure.

By an inequality obtained from Gennady, $\||f_n|^p\|_1 \xrightarrow{n\to\infty} \||f|^p\|_1 \in L^1(\mu)$. Now by Lemma 2, $(|f_n|^p)_{n=1}^\infty$ is uniformly integrable. Also by Lemma 4, condition (3) holds for $(|f_n|^p)_{n=1}^\infty$.

$\underline{\Longleftarrow}$: Suppose that $(f_n)_{n=1}^\infty$ in $L^p(\mu)$ satisfies the three listed conditions. Fix $\varepsilon > 0$ and let $E \subseteq X$ be as in condition (3). Set $A_{mn} = \{x : |f_m(x) - f_n(x)| \ge \varepsilon\}$ and let $\delta > 0$ be as in condition (2).

By construction, observe that

$$\int_{E \setminus A_{mn}} |f_m - f_n|^p\, d\mu \le \int_{E \setminus A_{mn}} \varepsilon^p\, d\mu \le \mu(E)\varepsilon^p$$

As $(f_n)_{n=1}^\infty$ is Cauchy in measure, there exists some $N \in \mathbb{N}$ such that for $m, n \ge N$, we have $\mu(A_{mn}) < \delta$. It follows by condition (2) that for $m, n \ge N$,

$$\int_{A_{mn}} |f_m - f_n|^p\, d\mu \le \int_{A_{mn}} 2^{p-1}(|f_m|^p + |f_n|^p)\, d\mu < 2^p\varepsilon.$$

Lastly, by condition (3), for all $m, n \in \mathbb{N}$,

$$\int_{E^c} |f_m - f_n|^p\, d\mu \le \int_{A_{mn}} 2^{p-1}(|f_m|^p + |f_n|^p)\, d\mu < 2^p\varepsilon.$$

So, for $m, n \ge N$, we have that

$$\|f_m - f_n\|_p^p \le \mu(E)\varepsilon^p + 2^p\varepsilon + 2^p\varepsilon,$$

so $(f_n)_{n=1}^\infty$ is Cauchy in the L^p-norm.

□

Problem 6

Prove that if E is a subset of a Hilbert space $\mathcal{H}$, then $(E^\perp)^\perp$ is the smallest closed subspace of $\mathcal{H}$ containing E.

Claim. If M is a closed linear subspace of $\mathcal{H}$, then $(M^\perp)^\perp = M$.

Proof of Claim. Note that we have $\mathcal{H} = M \oplus M^\perp$. Let $y \in (M^\perp)^\perp$. Then there exist unique $x \in M$, $x^\perp \in M^\perp$ such that $y = x + x^\perp$. Noting that $M \subseteq (M^\perp)^\perp$, we have that $x^\perp = y - x \in M^\perp \cap (M^\perp)^\perp = \{0\}$, whence $x^\perp = 0$ and $y = x \in M$. Thus $M = (M^\perp)^\perp$. □

Proof. On one hand, note that $E \subseteq \overline{\mathrm{Span}(E)} \implies (E^\perp)^\perp \subseteq (\overline{\mathrm{Span}(E)}^\perp)^\perp \overset{\text{claim}}{=} \overline{\mathrm{Span}(E)}$. On the other hand, by the continuity and linearity of the inner product, $(E^\perp)^\perp$ is a closed linear subspace of $\mathcal{H}$. Thus, as $E \subseteq (E^\perp)^\perp$, it follows that $\overline{\mathrm{Span}(E)} \subseteq (E^\perp)^\perp$. $\square$