MATH 7752 - HOMEWORK 6
DUE FRIDAY 03/18/22 AT 1 P.M.

(1) (a) Prove that two 3×3 matrices over some field F are similar if and only if they have the same minimal and characteristic polynomials. Is the same true for 4×4 matrices?

(b) A matrix A is called idempotent if $A^2 = A$. Prove that two idempotent $n \times n$ matrices are similar if and only if they have the same rank. **Hint:** What is the minimal polynomial of an idempotent matrix? How does rank relate to eigenvalue 0?

(2) Let F be an algebraically closed field and V a finite dimensional F-vector space.

(a) Let $S, T \in \mathcal{L}(V)$ such that $ST = TS$. Let λ be an eigenvalue of S and $E_\lambda(S) \leq V$ be the corresponding eigenspace of S. Prove that $E_\lambda(S)$ is a T-invariant subspace.

(b) Assume that $T \in \mathcal{L}(V)$ is diagonalizable and let $W \leq V$ be a T-invariant subspace. Prove that $T|_W \in \mathcal{L}(W)$ is also diagonalizable.

(c) Assume again that $S, T \in \mathcal{L}(V)$ such that $ST = TS$. Prove that there exists a basis Ω of V such that $[T]_\Omega$, and $[S]_\Omega$ are both diagonal.

(d) Give an example of a vector space V with $\dim_F(V) \geq 3$ and two commuting linear transformations $S, T \in \mathcal{L}(V)$ such that NO basis Ω of V exists such that both $[T]_\Omega$, and $[S]_\Omega$ are in JCF.

(3) Find the number of distinct conjugacy classes in the group $GL_3(\mathbb{Z}/2\mathbb{Z})$, and specify one element in each conjugacy class.

(4) Let V be an n-dimensional vector space over an algebraically closed field and $T \in \mathcal{L}(V)$. Assume that T has just one eigenvalue λ and just one Jordan block. Let $S = T - \lambda I$.

(a) Prove that $\mathrm{rk}(S^k) = n - k$, for all $0 \leq k \leq n$. Deduce that $\mathrm{Im}(S^k) = \ker(S^{n-k})$, for all $0 \leq k \leq n$.

(b) Let $v \in V$ be any vector which lies outside of $\mathrm{Im}(S) = \ker(S^{n-1})$. Prove that $\{S^{n-1}v, \ldots, Sv, v\}$ is a Jordan basis for T.

(5) Assume again that V is an n-dimensional vector space over an algebraically closed field F and $T \in \mathcal{L}(V)$.

(a) Assume that T has unique eigenvalue 0 and two Jordan blocks: a 1×1 block and a 2×2 block (so $n = 3$ in this case). Justify the following algorithm for computing a Jordan basis for T: Take any $v \in V \setminus \ker(T)$ and choose $w \in \ker(T)$ such that $\{w, Tv\}$ is a basis for $\ker(T)$ (why is this possible?); then $\{w, Tv, v\}$ is a Jordan basis for T.

(b) Assume that T has unique eigenvalue 0 and two Jordan blocks, both of which are 2×2 (so $n = 4$). State an algorithm for finding a Jordan basis similar to the one in (a).

(c) Assume that for each $\lambda \in \operatorname{Spec}(T)$ there is only one Jordan λ-block in $JCF(T)$. Describe an algorithm for computing a Jordan basis of T. **Hint:** You just need a minor generalization of the algorithm in the previous problem.

(6) Compute the Jordan canonical form and a Jordan basis for each of the following matrices over $\mathbb{Q}$:

(a) $\begin{pmatrix} -1 & 3 & 0 \\ 0 & 2 & 0 \\ 2 & 1 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 0 \end{pmatrix}$.

(7) Let $F = \mathbb{F}_3$ be the field with 3 elements and let $A \in M_{12}(\mathbb{F}_3)$. Suppose that A satisfies all the following assumptions:

- $\operatorname{rk}(A) = 10, \quad \operatorname{rk}(A^2) = 9, \quad \operatorname{rk}(A^3) = 9.$
- $\operatorname{rk}(A - I) = 12.$
- $\operatorname{rk}(A - 2I) = 9, \quad \operatorname{rk}((A - 2I)^2) = 7, \quad \operatorname{rk}((A - 2I)^3) = 6.$

(a) Assume in addition that the characteristic polynomial $\chi_A(x)$ splits completely over F (i.e. it splits into linear factors in $F[x]$). Find the Jordan canonical form of A.

(b) Find all possible RCF's of matrices A satisfying all the bullet assumptions, but not necessarily the extra assumption in (a).

(8) **(Optional)** Let $V = C^\infty(\mathbb{R})$ be the space of all infinitely differentiable functions on the real line. (Observe that V is an infinite dimensional $\mathbb{R}$-vector space). Consider the linear transformation $T = \frac{d}{dx} : V \to V$ that sends a function f to its derivative. Find all eigenvalues of T and the corresponding **generalized eigenspaces**. **Hint:** Start by computing the generalized eigenspaces corresponding to the eigenvalue $\lambda = 0$.