MATH 7310 Homework 8

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Problem 1

Let $\mathcal{H}$ be a Hilbert space.

(a): Prove that, for any $x, y \in \mathcal{H}$,

$$\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2)$$

Proof.

$$\begin{aligned}
\|x \pm y\|^2 &= \|x\|^2 + \|y\|^2 \pm 2\operatorname{Re}(\langle x, y\rangle) \\
\|x \pm iy\|^2 &= \|x\|^2 + \|y\|^2 \pm 2\operatorname{Re}(\langle x, iy\rangle) = \|x\|^2 + \|y\|^2 \pm 2\operatorname{Im}(\langle x, y\rangle).
\end{aligned}$$

We compute that

$$\begin{aligned}
\frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) &= \operatorname{Re}(\langle x, y\rangle) \\
\frac{1}{4}(\|x+iy\|^2 - \|x-iy\|^2) &= \operatorname{Im}(\langle x, y\rangle)
\end{aligned}$$

whence the identity follows by noting $\langle x, y\rangle = \operatorname{Re}\{\langle x, y\rangle\} + i\operatorname{Im}(\langle x, y\rangle)$. □

(b): If $\mathcal{H}'$ is another Hilbert space, prove that a linear map from $\mathcal{H}$ to $\mathcal{H}'$ is unitary if and only if it is isometric and surjective.

Proof.
$\underline{\implies}$: Suppose that $T : \mathcal{H} \to \mathcal{H}'$ is unitary. Then T is surjective by definition. Moreover, for $x \in \mathcal{H}$, $\|x\| = \langle x, x\rangle = \langle Tx, Tx\rangle = \|Tx\|$, whence T is an isometry by linearity.

$\underline{\impliedby}$: Suppose that $T : \mathcal{H} \to \mathcal{H}'$ is isometric and surjective. Then

$$\begin{aligned}
\langle Tx, Ty\rangle &= \frac{1}{4}(\|T(x)+T(y)\|^2 - \|T(x)-T(y)\|^2 + i\|T(x)+T(iy)\|^2 - i\|T(x)-T(iy)\|^2) \\
&= \frac{1}{4}(\|T(x+y)\|^2 - \|T(x-y)\|^2 + i\|T(x+iy)\|^2 - i\|T(x-iy)\|^2) \\
&= \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2) = \langle x, y\rangle,
\end{aligned}$$

so T is an isometry. Now suppose that $T(x) = 0$. Then $0 = \langle Tx, Tx\rangle = \langle x, x\rangle$ whence $x = 0$, so T is also injective and thus unitary. □

Problem 2

For $n \in \mathbb{Z}$, define $e_n : [0,1] \to \mathbb{C}$ by $e_n(t) = e^{2\pi int}$.

(a): Show that $\{e_n\}_{n\in\mathbb{Z}}$ is an orthonormal set in $L^2([0,1])$.

Proof. Observe that, for $n, m \in \mathbb{Z}$ with $n \neq m$

$$\langle e_n, e_m\rangle = \int_0^1 e^{2\pi int}\overline{e^{2\pi imt}}\,dt = \int_0^1 e^{2\pi i(n-m)t}\,dt = \left[\frac{1}{2\pi i(n-m)}e^{2\pi i(n-m)t}\right]_{t=0}^{t=1} = \frac{1}{2\pi i(n-m)}(e^{2\pi i(n-m)-1}) = 0.$$

On the other hand, for $\in \mathbb{Z}$,

$$\langle e_n, e_n\rangle = \int_0^1 e^{2\pi int}\overline{e^{2\pi int}}\,dt = \int_0^1 1\,dt = 1$$

so $\{e_n\}_{n\in\mathbb{Z}}$ is an orthonormal set in $L^2([0,1])$. $\square$

(b): Show that $\{f \in C([0,1]) : f(1) = f(0)\} = \{g \circ e_1 : g \in C(S^1)\}$, where $S^1 = \{z \in \mathbb{C} : |z| = 1\}$.

Proof. Noting that e_1 is just the composition of the canonical projection map $[0,1] \to [0,1]/(0 \sim 1)$ with the homeomorphism $[0,1]/(0\sim 1) \cong S^1$ given by the same formula (where well-definedness follows from the quotiented set), the claim follows from the universal property of the quotient topology. $\square$

(c): The Stone-Weierstrass theorem says that if (X,d) is a compact metric space and $A \subseteq C(X)$ is a linear subspace so that:

- $1 \in A$,
- $f \in A$ implies $\overline{f} \in A$,
- $f, g \in A$ implies that $fg \in A$,
- If $x \in X$, then there are $f, g \in A$ with $f(x) \neq g(x)$,

then A is dense in $C(X)$ for the uniform norm $\|f\|_u = \sup_{x\in X}|f(x)|$. Use the Stone-Weierstrass theorem to show that $\overline{\text{Span}}^{\|\cdot\|_u}\{e_n : n \in \mathbb{Z}\} = \{f \in C([0,1]) : f(1) = f(0)\}$.

Proof. For $n \in \mathbb{Z}$, define $p_n : S^1 \to \mathbb{C}$ by $p_n(x) = x^n$ and set $A = \text{Span}\{p_n : n \in \mathbb{Z}\} \subseteq C(S^1)$. Note that $\overline{p_n} = p_{-n}$, $p_n p_m = p_{nm}$, $1 = p_0$, whence by linearity the first three properties above hold for A.

To see that the last property holds for A, take arbitrary $x \in S^1$. If $x^n \neq 1$ for all $n \in \mathbb{Z}\setminus\{0\}$, then $p_1(x) = x \neq x^2 = p_2(x)$. Suppose that $x^n = 1$ for some $n \in \mathbb{Z}\setminus\{0\}$. Then $x^{-n} = 1$, so we may assume without loss of generality that $n \in \mathbb{N}$. If $x = 1$, then $p_1(x) = 1 \neq 0 = 0(x)$ and $p_1, 0 \in A$. Otherwise, let $m \geq 2$ be the smallest such $m \in \mathbb{N}$ such that $x^m = 1$. Then $p_m(x) = x^m \neq x^{m-1} = p_{m-1}(x)$, as desired.

Thus, by the Stone-Weierstrass theorem, $\overline{A}^{\|\cdot\|_u} = C(S^1)$. Now take $f \in \{f \in C([0,1]) : f(1) = f(0)\}$. Then by part (b) there exists a $g \in C(S^1)$ such that $f = g \circ e_1$. Now take a sequence $(a_n)_{n=1}^{\infty}$ in A such that $a_n \to g$ with respect to $\|\cdot\|_u$.

Then $a_n \circ e_1 \to g \circ e_1$ with respect to $\|\cdot\|_u$. Lastly, noting that $a_n \circ e_n \in \text{Span}\{e_n : n \in \mathbb{Z}\}$, it follows that $f \in \overline{\text{Span}}^{\|\cdot\|_u}\{e_n : n \in \mathbb{Z}\}$. $\square$

(d): Show that $\text{Span}\{e_n : n \in \mathbb{Z}\}$ is dense in $L^2([0,1])$ and use this to show that $\{e_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2([0,1])$.

Proof. Observe that, for any measurable $f : [0,1] \to \mathbb{C}$, by Hölder's inequality applied twice we have

$$\|f\|_2 \le \|f\|_1 \le \|f\|_\infty \|1\|_1 = \|f\|_\infty m([0,1]) = \|f\|_\infty \le \|f\|_u.$$

Now suppose that $f \in C([0,1])$ such that $f(0) = f(1)$. Then there exists $f_n \in \operatorname{Span}\{e_n : n \in \mathbb{Z}\}$ such that $\|f_n - f\|_u \to 0$. Then $\|f_n - f\|_2 \le \|f_n - f\|_u \to 0$, so $\overline{\operatorname{Span}}^{\|\cdot\|_2}\{e_n : n \in \mathbb{Z}\} = \{f \in C([0,1]) : f(1) = f(0)\}$ which equals $C([0,1])$ modulo almost-everywhere equality. Moreover, the closure of $C([0,1])$ in the L^2-norm contains equivalence classes of the indicator functions of intervals via the hill approximation, so it in fact is all of L^2. Thus by transitivity of topological density, $\operatorname{Span}\{e_n : n \in \mathbb{Z}\}$ is dense in $L^2([0,1])$.

As $\{e_n : n \in \mathbb{Z}\}$ is orthonormal and the L^2-norm-closure of its span is in fact all of $L^2([0,1])$, it follows that $\{e_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2([0,1])$. □

Problem 3

(a): Let (X, Σ, μ), $(Y, \mathcal{F}, \nu)$ be σ-finite measure spaces such that $L^2(\mu)$ and $L^2(\nu)$ are separable. If $\{f_m\}$ and $\{g_n\}$ are orthonormal bases for $L^2(\mu)$ and $L^2(\nu)$ and $h_{mn}(x,y) = f_m(x)g_n(y)$, then $\{h_{mn}\}$ is an orthonormal basis for $L^2(\mu \otimes \nu)$.

Proof.

$$\overline{\operatorname{Span}}^{\|\cdot\|_2}\{f_m : m \in \mathbb{Z}\} = L^2(\mu) \qquad \overline{\operatorname{Span}}^{\|\cdot\|_2}\{g_m : m \in \mathbb{Z}\} = L^2(\nu)$$

By Fubini-Tonelli,

$$\begin{aligned}\langle f_m \otimes g_n, f_k \otimes g_l\rangle &= \int_{X\times Y} f_m \otimes g_n \, \overline{f_k \otimes g_l} \, d(\mu\otimes\nu) = \int_{X\times Y} h_{mn} \, \overline{h_{nl}} \, d(\mu\otimes\nu) \\ &= \int_X f_m(x)\overline{f_k(x)} \int_Y g_n(y)\, \overline{g_l(y)} \, d\nu(y)\, d\mu(x) = \int_X f_m(x)\overline{f_k(y)}\delta_{nl}\, d\mu(x) = \delta_{mk}\delta_{nl} = \delta_{(m,n),(k,l)}\end{aligned}$$

so $\{h_{mn}\}$ is an orthonormal set in $L^2(\mu\otimes\nu)$. By homework 7 problem 1 part (d), as $\overline{\operatorname{Span}}^{\|\cdot\|_2}\{f_m : m \in \mathbb{Z}\} = L^2(\mu)$ and $\overline{\operatorname{Span}}^{\|\cdot\|_2}\{g_m : m \in \mathbb{Z}\} = L^2(\nu)$, it follows that

$$\overline{\operatorname{Span}}^{\|\cdot\|_2}\{h_{mn} : m,n \in \mathbb{Z}\} = \overline{\operatorname{Span}}^{\|\cdot\|_2}\{f_m \otimes g_n : m,n \in \mathbb{Z}\} = L^2(\mu\otimes\nu)$$

so $\{h_{mn}\}$ is in fact an orthonormal basis for $L^2(\mu\otimes\nu)$.

□

(b): For $k \in \mathbb{N}$, and $n = (n_1, \ldots, n_k) \in \mathbb{Z}^k$, define $e_n \in L^2([0,1]^k)$ by

$$e_n(x) = \prod_{j=1}^{k} e^{2\pi i n_j x}.$$

Show that $\{e_n\}_{n\in\mathbb{Z}^k}$ is an orthonormal basis of $L^2([0,1]^k)$.

Proof. We induct on $k \in \mathbb{N}$. Note that we have already shown the base case $k = 1$ in problem 2. Now suppose $k > 1$ and that the claim is true for $k-1$. Then $\{e_n\}_{n\in\mathbb{Z}^{k-1}}$ is an orthonormal basis of $L^2([0,1]^{k-1})$. By part (a), $\{e_n e_m\}_{n\in\mathbb{Z}^{k-1}, m\in\mathbb{Z}} = \{e_n\}_{n\in\mathbb{Z}^k}$ is an orthonormal basis for $L^2([0,1]^{k-1}\times[0,1]) = L^2([0,1]^k)$, and thus we are done. □

Problem 4

(a): Let (X, Σ, μ) be a σ-finite measure space, $\mathcal{F}$ a sub-σ-algebra of Σ, and $\nu = \mu|_{\mathcal{F}}$. If $f \in L^1(\mu)$, prove that there exists $g \in L^1(\nu)$ (thus g is $\mathcal{F}$-measurable) such that $\int_E f\,d\mu = \int_E g\,d\nu$ for all $E \in \mathcal{F}$; also prove that if g' is another such function then $g = g'$ ν-a.e.

Proof. Define a new measure λ by $\lambda(E) = \int_E f\,d\mu$ for all $E \in \Sigma$. If $E \in \mathcal{F}$ with $\mu|_{\mathcal{F}}(E) = \nu(E) = 0$, then $\lambda(E) = \int_E f\,d\mu = 0$, so $\lambda|_{\mathcal{F}} \ll \nu$. Let $g = \frac{d\lambda|_{\mathcal{F}}}{d\nu} \in L^1(\nu)$, so $d\lambda|_{\mathcal{F}} = g\,d\nu$. Then, for all $E \in \mathcal{F}$,

$$\int_E g\,d\nu = \lambda(E) = \int_E f\,d\mu\,.$$

Now suppose that g' is another such function. Then $g' = \frac{d\lambda|_{\mathcal{F}}}{d\nu}$ ν-a.e. by the uniqueness portion of the Radon-Nikodym theorem, and thus $g = g'$ ν-a.e.

□

(b): Show that $\int gh\,d\nu = \int fh\,d\mu$ for all $h \in L^1(\nu)$.

Proof. We have the following relation

$$f\,d\mu = d\lambda_f = g\,d\nu\,,$$

so by homework 5 problem 1 part (a),

$$\int fh\,d\mu = \int h\,d\lambda_f = \int gh\,d\nu\,.$$

□

Problem 5

Let (X, Σ, μ) be a probability space. For a sub-σ-algebra $\mathcal{F} \subseteq \Sigma$, and $f \in L^1(X, \Sigma, \mu)$, let $\mathbb{E}_{\mathcal{F}}(f)$ be the conditional expectation of f onto $\mathcal{F}$.

(a): Show that $\mathbb{E}_{\mathcal{F}}(fg) = \mathbb{E}_{\mathcal{F}}(f)g$ for all $g \in L^\infty(X, \mathcal{F}, \mu)$.

Proof. Noting that, for $E \in \mathcal{F}$, as $\mu(X) < +\infty$ and $g \in L^\infty(X, \mathcal{F}, \mu|_{\mathcal{F}})$, we have that $g \cdot \mathbb{1}_E \in L^1(X, \mathcal{F}, \mu|_{\mathcal{F}})$ whence by problem 4 part (b)

$$\int_E \mathbb{E}_{\mathcal{F}}(f) g\,d\mu|_{\mathcal{F}} = \int \mathbb{E}_{\mathcal{F}}(f)(g\mathbb{1}_E)\,d\mu|_{\mathcal{F}} = \int fg\mathbb{1}_E\,d\mu = \int_E fg\,d\mu\,.$$

Thus, by the uniqueness of conditional expectation in problem 4, $\mathbb{E}_{\mathcal{F}}(fg) = \mathbb{E}_{\mathcal{F}}(f)g$. □

(b): If $f \in L^2(X, \Sigma, \mu)$, show that $\mathbb{E}_{\mathcal{F}}(f)$ is the orthogonal projection of f onto $L^2(X, \mathcal{F}, \mu)$ in the decomposition

$$L^2(X, \Sigma, \mu) = L^2(X, \mathcal{F}, \mu) + L^2(X, \mathcal{F}, \mu)^\perp.$$

Note: one difficulty you'll a priori face is that we do not yet know that $f \in L^2$ implies that $\mathbb{E}_{\mathcal{F}}(f) \in L^2$. However, one can note that you can characterize the orthogonal projection g of f onto $L^2(X, \mathcal{F}, \mu)$ by $\langle f, h\rangle = \langle g, h\rangle$ for all $h \in L^2(X, \mathcal{F}, \mu)$ (you should prove this if you use it), and this can be used to show that this projection is the conditional expectation.

Proof. For $h \in L^2(X, \mathcal{F}, \mu)$, observe that

$$\langle f - \mathbb{E}_{\mathcal{F}}(f), h\rangle = \int (f - \mathbb{E}_{\mathcal{F}}(f))\overline{h}\, d\mu = \int f\overline{h}\, d\mu - \int \mathbb{E}_{\mathcal{F}}(f)\overline{h}\, d(\mu|_{\mathcal{F}}) = \int f\overline{h}\, d\mu - \int f\overline{h}\, d(\mu|_{\mathcal{F}}) = 0$$

so $\langle f, h\rangle = \langle g, h\rangle$ for all $h \in L^2(X, \mathcal{F}, \mu)$. Noting that $M = L^2(X, \mathcal{F}, \mu)$ is a closed linear subspace of $\mathcal{H} = L^2(X, \Sigma, \mu)$, consider the decomposition $\mathcal{H} = M \oplus M^\perp$. For $x \in \mathcal{H}$ with unique decomposition $x = x^\parallel + x^\perp$ with $x^\parallel \in M$ and $x^\perp \in M^\perp$, we claim that if $g \in M$ is such that $\langle x, h\rangle = \langle g, h\rangle$ for all $h \in M$, then $g = x^\parallel$. This is in fact clear as then $x - g \in M^\perp$ and $x = g + (x - g)$, so by uniqueness $g = x^\perp$. Thus, projection is the conditional expectation. □

Problem 6

Show that if ν is a signed measure, then E is ν-null if and only if $|\nu|(E) = 0$. Also, prove that if μ and ν are signed measures, then $\nu \perp \mu$ if and only if $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.

Proof.
$\underline{\Longrightarrow}$: Suppose that E is ν-null. Let (P, N) be a *Hahn* decomposition for ν and consider positive measures ν^+, ν^- such that $\nu = \nu^+ - \nu^-$. By uniqueness of these positive measures, $\nu^+(E) = \nu(E \cap P)$ and $\nu^-(E) = -\nu(E \cap N)$. Nullity of E for ν then implies that $\nu^+(E) = \nu(E \cap P) = 0$ and $\nu^-(E) = -\nu(E \cap N) = 0$. Thus $|\nu|(E) = \nu^+(E) + \nu^-(E) = 0$.

$\underline{\Longleftarrow}$: Suppose that $|\nu|(E) = 0$. Let $F \subseteq E$ such that $F \in \Sigma$. As $|\nu|$ is a positive measure on Σ, $|\nu|(F) = 0$, whence $\nu^+(F) = 0 = \nu^-(F)$. It follows that $\nu(F) = \nu^+(F) - \nu^-(F) = 0$, so E is ν-null.

$\underline{\Longrightarrow}$: Suppose that $\nu \perp \mu$. So there exist $E, F \in \Sigma$ such that E is μ-null and F is ν-null, $E \cap F = \emptyset$, and $E \cup F = X$. Then by the previously proven equivalence, $|\nu|(F) = 0$ whence $\nu^+(F) = 0 = \nu^-(F)$. As ν^+, ν^- are positive measures, this implies that F is ν^+-null and ν^--null, so the initial decomposition of X giving singularity of ν and μ also gives $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.

$\underline{\Longleftarrow}$: Suppose that $\nu^+ \perp \mu$ and $\nu^- \perp \mu$. Then there exist $E^+, F^+, E^-, F^- \in \Sigma$ such that $E^\pm \cap F^\pm = \emptyset$, $E^\pm \cup F^\pm = X$, $E^\pm$ is μ-null, and $F^\pm$ is $\nu^\pm$-null. Consider the sets $A = E^+ \cup E^-$ and $B = F^+ \cap F^-$. Note that A is a union of μ-null sets and is thus μ-null, whilst $\nu^+(B) = 0 = \nu^-(B)$ implies that $|\nu|(B) = 0$, so B is ν-null. A and B are clearly disjoint and

$$X \setminus A = X \setminus (E^+ \cup E^-) = (X \setminus E^+) \cap (X \setminus E^-) = F^+ \cap F^- = B \implies A \cup B = X.$$

□