MATH 7752 Homework 1

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Problem 1

Let R be a ring and M an R-module.

(a) Prove that for every $m \in M$, the map $r \mapsto rm$ from R to M is a homomorphism of R-modules.

Proof. Fix $m \in M$ and let φ denote the map $r \mapsto rm$. Fix $x, y \in R$ and $r \in R$. Observe that

$$\varphi(x+y) = (x+y)m = xm + ym = \varphi(x) + \varphi(y)$$

and

$$\varphi(rx) = (rx)m = r(xm) = r\varphi(x),$$

so φ is an R-module homomorphism. $\square$

(b) Assume that R is commutative and M an R-module. Prove that there is an isomorphism $\mathrm{Hom}_R(R, M) \simeq M$ as R-modules.

Proof. For $m \in M$, let φ_m denote the R-module homomorphism in part (a).
Consider the map $\psi : M \to \mathrm{Hom}_R(R, M)$ given by $\psi(m) = \varphi_m$. For $m, n \in M$ and $r, x \in R$,

$$\psi(m+n)(x) = \varphi_{m+n}(x) = x(m+n) = xm + xn = \varphi_m(x) + \varphi_n(x) = (\psi(m) + \psi(n))(x)$$

so $\psi(m+n) = \psi(m) + \psi(n)$, and

$$\psi(rm)(x) = \varphi_{rm}(x) = x(rm) = r(xm) = r\varphi_m(x) = (r\psi(m))(x)$$

so $\psi(rm) = r\psi(m)$.

Suppose $\psi(m) = \psi(n)$. Then $m = \varphi_m(1) = \psi(m)(1) = \psi(n)(1) = \varphi_n(1) = n$, so ψ is injective.

Suppose $\varphi \in \mathrm{Hom}_R(R, M)$. For $r \in R$,

$$\psi_{\varphi(1)}(r) = r\varphi(1) = \varphi(r),$$

so $\psi_{\varphi(1)} = \varphi$, i.e. ψ is surjective. $\square$

Problem 2

Give an explicit example of a map $f : A \to B$ with the following properties:

- A, B are R-modules.
- f is a group homomorphism.
- f is not an R-module homomorphism.

Problem 3

Let R be a ring and M an R-module.

(a) Let N be a subset of M. The *annihilator* of N is defined to be the set

$$\mathrm{Ann}_R(N) := \{r \in R : rn = 0, \text{ for all } n \in N\}.$$

Prove that $\mathrm{Ann}_R(N)$ is a left ideal of R.

Proof. Let $x, y \in I$ and $r \in R$. Fix $n \in N$. Noting that $xn = 0 = yn$, it follows that

$$(x + ry)n = xn + (ry)n = xn + r(yn) = 0.$$

Thus $x + ry \in \mathrm{Ann}_R(N)$. Since all elements chosen were arbitrary, $\mathrm{Ann}_R(N)$ is a left ideal of R. □

(b) Show that if N is an R-submodule of M, then $\mathrm{Ann}_R(N)$ is an ideal of R (i.e. it is two-sided ideal).

Proof. By part (a), it suffices to show that $\mathrm{Ann}_R(N)$ is a right ideal of R. Moreover, part (a) shows *a fortiori* that $\mathrm{Ann}_R(N)$ is already an abelian group, so we need only address its multiplicative structure. Let $y \in \mathrm{Ann}_R(N)$ and $r \in R$. Fix $n \in N$. As N is an R-submodule of M, $yn \in N$, whence $(yr)n = y(rn) = 0$ by definition. Hence $\mathrm{Ann}_R(N)$ is a two-sided ideal of R. □

(c) For a subset I of R the *annihilator* of I in M is defined to be the set,

$$\mathrm{Ann}_M(I) := \{m \in M : xm = 0, \text{ for all } x \in I\}.$$

Find a natural condition on I that guarantees that $\mathrm{Ann}_M(I)$ is a submodule of M.

Claim. *$\mathrm{Ann}_M(I)$ is an R-submodule of M if I is a right ideal of R.*

Proof. Suppose I is a right ideal of R. As $x \cdot 0 = 0$ for all $x \in I$, $\mathrm{Ann}_M(I) \neq \emptyset$. Suppose $m, n \in \mathrm{Ann}_M(I)$ and $r \in R$. Fix $x \in I$. By definition $x \cdot m = 0$. As I is a right ideal, $xr \in I$, so $x \cdot (m + r \cdot n) = x \cdot m + (xr) \cdot n = 0$. Thus $\mathrm{Ann}_M(I)$ is an R-submodule of M. □

(d) Let R be an integral domain. Prove that every finitely generated torsion R-module has a nonzero annihilator.

Proof. Let M be a finitely generated torsion R-module. Taking a generating set $m_1, \ldots, m_n \in M$ of M, for each $k \in \{1, \ldots, n\}$ there exists an $x_k \in R^\times = R \setminus \{0\}$ such that $x_k m_k = 0$. As $R^\times$ is closed under multiplication, $r := x_1 \cdots x_n \in R^\times$ whence $r \neq 0$.
Now suppose that $m \in M$. Then there exist $r_1, \ldots, r_n \in R$ such that $m = r_1 m_1 + \cdots + r_n m_n$. Observe that, by the commutativity of R,

$$rm = (x_1 \cdots x_n)(r_1 m_1 + \cdots + r_n m_n) = \sum_{k=1}^{n} \left(\prod_{i \neq k} x_i \right) (x_k m_k) = 0.$$

Thus $0 \neq r \in \mathrm{Ann}_R(M)$, so M has nonzero annihilator. □

Problem 4

In class we obtained a simple characterization of R-modules when $R = \mathbb{Z}$, and $R = F[x]$, with F a field. Imitate the method to find similar characterizations for R-modules in the following cases:

(a) $R = \mathbb{Z}/n\mathbb{Z}$, for some $n \geq 2$.

(b) $R = \mathbb{Z}[x]$.

(c) $R = F[x, y]$.

Problem 5

An R-module M is called *simple* (or *irreducible*) if its only submodules are $\{0\}$ and M. An R-module M is called *indecomposable* if M is not isomorphic to $N \oplus Q$ for some non-zero submodules N, Q. Show that every simple R-module is indecomposable, but the converse is not true.

Problem 6

Let R be a ring. An R-module M is called *cyclic* if it is generated as an R-module by a single element.

(a) Prove that every cyclic R-module is of the form R/I for some left ideal I of R.

Proof. Let M be a cyclic R-module. Then there exists an $m \in M$ such that $M = Rm$. Consider the map $\varphi : R \to M$ given by $\varphi(r) = rm$ for $r \in R$. By problem 1 part (a), φ is an R-module homomorphism; moreover, φ is surjective since m generates M. Let $I = \ker(\varphi)$, a left ideal of R (actually two-sided, but we are identifying R with its left regular module over itself so a priori I is just a left R-submodule). Then, by the first isomorphism theorem, $M = \varphi(R) \cong R/\ker(\varphi) = R/I$. □

(b) Show that the simple R-modules are precisely the ones which are isomorphic to $R/\mathfrak{m}$ for some maximal left ideal $\mathfrak{m}$.

Proof. On one hand, $\mathfrak{m}$ be a maximal left ideal of R. By the correspondence theorem applied to the natural projection, the only R-submodules of $R/\mathfrak{m}$ are $\{0\}$ and $R/\mathfrak{m}$, so $R/\mathfrak{m}$ is simple (and so is every R-module isomorphic to it).

On the other hand, suppose M is a nonzero simple R-module. Take $m \in M \setminus \{0\}$. Then by the simplicity of M, $Rm = M$ i.e. M is a cyclic module generated by m. Part (a) implies that there is some left ideal $\mathfrak{m}$ of R such that $M \cong R/\mathfrak{m}$. Suppose that I is a proper left ideal of R such that $\mathfrak{m} \subseteq I \subsetneq R$. Applying the natural projection, we see that $0 \subseteq I/\mathfrak{m} \subsetneq R/\mathfrak{m}$, whence simplicity of $R/\mathfrak{m}$ implies that $I/\mathfrak{m}$ is trivial i.e. $I = \mathfrak{m}$. Thus by definition $\mathfrak{m}$ is a maximal left ideal. $\square$

(c) Show that any non-zero homomorphism of simple R-modules is an isomorphism. Deduce that if M is simple, its endomorphism ring $\mathrm{End}_R(M) := \mathrm{Hom}_R(M, M)$ is a division ring. This result is known as *Schur's Lemma*.

Problem 7

Show that $\mathbb{Q}$ is not a free $\mathbb{Z}$-module, that is $\mathbb{Q}$ is not isomorphic to a direct sum of the form $\bigoplus_I \mathbb{Z}$, for any index set I. More generally, let R be a PID which is not a field and $K = \mathrm{frac}(R)$ be its fraction field. Show that K is not a free R-module.

Problem 8

Let R be a commutative ring. Recall that an ideal I of R is called *nilpotent* if there exists some $n \in \mathbb{N}$ such that $I^n = 0$.

(a) Let $i \in I$. Show that the element $r = 1 - i$ is invertible in R.

Proof. As I is a nilpotent ideal, there exists an $n \in \mathbb{N}$ such that $I^n = 0$. Then $i^n = 0$, so

$$1 = 1 - i^n = (1-i)(1 + i + \cdots + i^{n-1}),$$

whence $1 - i \in R^\times$. $\square$

(b) Let M, N be R-modules and let $\varphi : M \to N$ be an R-module homomorphism. Show that φ induces an R-module homomorphism, $\overline{\varphi} : M/IM \to N/IN$.

Proof. Let $\pi_M : M \to M/IM$ and $\pi_N : N \to N/IN$ be the natural projections. Define a map $\overline{\varphi} : M/IM \to N/IN$ by $\overline{\varphi}(m + IM) := \varphi(m) + IN = (\pi_N \circ \varphi)(m)$. To see that this map is well defined, suppose that $m + IM = m' + IM$. Then there exist $i_1, \ldots, i_s \in I$ and $m_1, \ldots, m_s \in M$ such that $m - m' = i_1 m_1 + \cdots + i_s m_s$. So

$$\varphi(m - m') = \varphi(i_1 m_1 + \cdots + i_s m_s) = i_1 \varphi(m_1) + \cdots + i_s \varphi(m_s) \in IN,$$

whence $\pi_N(\varphi(m)) - \pi_N(\varphi(m')) = \pi_N(\varphi(m - m')) = 0$, so $\pi_N(\varphi(m)) = \pi_N(\varphi(m'))$.
To see that this map is an R-module homomorphism, note that $\overline{\varphi} = \pi_N \circ \varphi$. $\square$

(c) Prove that if $\overline{\varphi}$ is sujective, then φ is itself surjective.

Proof. $\square$

Problem 9

Let G be a finite group and k a field. Consider the group ring $k[G]$.

(a) Let M be a k-vector space with a G-action. Show that M becomes a $k[G]$-module. Conversely, if M is a $k[G]$-module, show that M is a G-set.

(b) Let M, N be two $k[G]$-modules. Show that $\mathrm{Hom}_k(M, N)$ becomes a $k[G]$-module with the following G-action: For $g \in G$ and $\varphi : M \to N$ a $k[G]$-homomorphism define

$$(g \cdot \varphi)(m) := g\varphi(g^{-1}m), \text{ for } m \in M.$$