Reading:

- 0.3, 0.5-.6

Problem 1.

Let J be an infinite set, and $(t_j)_{j\in J}$ nonnegative real numbers. We define $\sum_{j\in J} t_j = \sup_F \sum_{j\in F} t_j$ where the supremum is over all finite subsets of J, and is equal to ∞ if $\left\{\sum_{j\in F} t_j : F \subseteq J \text{ is finite}\right\}$ is not bounded above.

(i) Suppose that $\sum_{j\in J} t_j < \infty$. Prove that for every $\varepsilon > 0$, there is a finite $F \subseteq J$ so that $\sum_{j\in J\setminus F} t_j < \varepsilon$. (Hint: use Proposition 0.20).

(ii) Suppose that $(\alpha_j)_{j\in J}$ are complex numbers and $\sum_{j\in J} |\alpha_j| < \infty$. Suppose further that $J_0 = \{j \in J : \alpha_j \neq 0\}$ is infinite. Suppose that $\phi\colon \mathbb{N} \to J_0, \psi\colon \mathbb{N} \to J_0$ are two bijections. Prove that

$$\sum_{n=1}^{\infty} \alpha_{\phi(n)} = \sum_{n=1}^{\infty} \alpha_{\psi(n)}.$$

(Hint: reduce to the statement that the value of an absolutely convergent series does not change under rearrangement).

Problem 2.

It follows from Problem 1 that if $(\alpha_j)_{j\in J}$ are complex numbers and $\sum_{j\in J} |\alpha_j| < \infty$, we may define $\sum_{j\in J} \alpha_j$ as follows: let $J_0 = \{j : \alpha_j \neq 0\}$. If J_0 is finite, then $\sum_{j\in J} \alpha_j = \sum_{j\in J_0} \alpha_j$. If J_0 is infinite, choose a bijection $\phi\colon \mathbb{N} \to J_0$, and define

$$\sum_{j\in J} \alpha_j = \sum_{n=1}^{\infty} \alpha_{\phi(n)}.$$

Suppose that $(\alpha_j)_{j\in J}$ are complex numbers and $\sum_{j\in J} |\alpha_j| < \infty$. Show that $\sum_{j\in J} \alpha_j$ is the unique complex number s satisfying the following property. For every $\varepsilon > 0$, there is finite set $F \subseteq J$ so that if $F \subseteq E \subseteq J$ and E is finite, then

$$\left| s - \sum_{j\in E} \alpha_j \right| < \varepsilon.$$

Remark: There are two assertions to prove. One is that there is only one complex number s satisfying the above property. The second is that $\sum_{j\in J} \alpha_j$ satisfies the above property.

Problem 3.

Suppose that I, J are sets, and $(a_{ij})_{i\in I, j\in J}$ are nonnegative real numbers. Prove that

$$\sum_{j\in J}\left(\sum_{i\in I} a_{ij}\right) = \sum_{(i,j)\in I\times J} a_{ij} = \sum_{i\in I}\left(\sum_{j\in J} a_{ij}\right)$$

Additional problem to think about, do not turn in: Suppose $(a_{ij})_{i\in I, j\in J}$ are complex numbers with $\sum_{i,j} |a_{ij}| < \infty$. Show that

$$\sum_{j\in J}\left(\sum_{i\in I} a_{ij}\right) = \sum_{(i,j)\in I\times J} a_{ij} = \sum_{i\in I}\left(\sum_{j\in J} a_{ij}\right)$$