MATH 7310 Homework 10

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May 1, 2022

Problem 1

If $F \in NBV$, let $G(x) = |\mu_F|((-\infty, x])$, Prove that $|\mu_F| = \mu_{T_F}$ by showing that $G = T_F$ via the following steps.

(a): From the definition of T_F, show that $T_F \leq G$.

Proof. Let $x \in \mathbb{R}$. Then for $x_0 < x_1 < \cdots < x_n = x$, observe that

$$\sum_{j=1}^{\infty} |F(x_j) - F(x_{j-1})| = \sum_{j=1}^{\infty} |\mu_F((x_{j-1}, x_j])| \leq \sum_{j=1}^{\infty} |\mu_F|((x_{j-1}, x_j]) = |\mu_F|((x_0, x]),$$

whence $T_F(x) \leq \sup_{x_0 < x} |\mu_F|((x_0, x]) = |\mu_F|((-\infty, x]) = G(x)$. $\square$

(b): $|\mu_F(E)| \leq \mu_{T_F}(E)$ when E is an interval, and hence when E is a Borel set.

Proof. Let $I = (a, b]$ be an interval. Then

$$|\mu_F(I)| = |F(b) - F(a)| \leq T_F(b) - T_F(a) = \mu_{T_F}(I).$$

Now suppose that E is Borel and let $(a_j, b_j]$ be a countable sequence of h-intervals such that $E \subseteq \bigcup_{j=1}^{\infty} (a_j, b_j]$. Then

$$\left| \sum_{j=1}^{\infty} \mu_F((a_j, b_j]) \right| \leq \sum_{j=1}^{\infty} |\mu_F((a_j, b_j])| \leq \sum_{j=1}^{\infty} \mu_{T_F}((a_j, b_j]),$$

whence $|\mu_F(E)| \leq \sum_{j=1}^{\infty} \mu_{T_F}((a_j, b_j])$. As $E \subseteq \bigcup_{j=1}^{\infty} (a_j, b_j]$ was arbitrary, after taking an infimum it follows by outer regularity of μ_{T_F} that $|\mu_F(E)| \leq \mu_{T_F}(E)$. $\square$

(c): $|\mu_F| \leq \mu_{T_F}$, and hence $G \leq T_F$ (Use exercise 21).

Proof. Observe that, by part (b),

$$\begin{aligned} |\mu_F|(E) &= \sup \left\{ \sum_{1}^{\infty} |\mu_F(E_j)| : E_1, E_2, \ldots \text{ disjoint, } E = \bigsqcup_{1}^{\infty} E_j \right\} \\ &\leq \sup \left\{ \sum_{1}^{\infty} \mu_{T_F}(E_j) : E_1, E_2, \ldots \text{ disjoint, } E = \bigsqcup_{1}^{\infty} E_j \right\} = \mu_{T_F}(E), \end{aligned}$$

so $T_F(x) = \mu_{T_F}((-\infty, x]) \geq |\mu_F|((-\infty, x]) = G(x)$.

$\square$

Problem 2

Let G be a continuous increasing function on $[a,b]$ and let $G(a) = c$, $G(b) = d$.

(a): If $E \subseteq [c,d]$ is a Borel set, then $m(E) = \mu_G(G^{-1}(E))$. (First consider the case where E is an interval.)

Proof. □

(b): If f is a Borel measurable and integrable function on $[c,d]$, then $\int_c^d f(y)\,dy = \int_a^b f(G(x))\,dG(x)$. In particular, $\int_c^d f(y)\,dy = \int_a^b f(G(x))G'(x)\,dx$ if G is absolutely continuous.

(c): The validity of (b) may fail if G is merely right continuous rather than continuous.

Problem 3

Suppose $F : \mathbb{R} \to \mathbb{C}$. Prove that F is Lipschitz with constant M if and only if F is absolutely continuous and $|F'| \leq M$ a.e.

Proof.
$\Longrightarrow$: Suppose that $M > 0$ is such that $|F(x) - F(y)| \leq M|x-y|$ for all $x, y \in \mathbb{R}$. Let $\varepsilon > 0$. Choose $\delta = \varepsilon/M$. Then, for any finite set of disjoint intervals $(a_1, b_1), \ldots, (a_N, b_N)$ with $\sum_{j=1}^N (b_j - a_j) < \delta$, we have

$$\sum_{j=1}^N |F(b_j) - F(a_j)| \leq M \sum_{j=1}^N (b_j - a_j) < M \cdot \frac{\varepsilon}{M} = \varepsilon,$$

so F is absolutely continuous. Thus, F is differentiable almost everywhere. If $x \neq y$, then $|F(x) - F(y)|/|x - y| \leq M$, so for a.e. $x \in \mathbb{R}$ we have that $|F'| \leq M$.

$\Longleftarrow$: Suppose that F is absolutely continuous and $|F'| \leq M$ a.e. Let $x, y \in \mathbb{R}$ and without loss of generality suppose that $x < y$. Then by the FTC for Lebesgue integrals,

$$|F(y) - F(x)| = \left| \int_x^y F'\,dt \right| \leq \int_x^y |F'|\,dt \leq M|y - x|.$$

□

Problem 4

Let $A \subseteq [0,1]$ be a Borel set such that $0 < m(A \cap I) < m(I)$ for every subinterval I of $[0,1]$.

(a): Let $F(x) = m([0,x] \cap A)$. Show that F is absolutely continuous and strictly increasing on $[0,1]$, but $F' = 0$ on a set of positive measure.

Proof. If $y > x$, then $F(y) = m([0,y] \cap A) = m([0,x] \cap A) + m((x,y] \cap A) > F(x)$ by assumption, so F is strictly increasing. Now fix $\varepsilon > 0$ and set $\delta = \varepsilon$. Then, for any finite set of disjoint intervals $(a_1, b_1), \ldots, (a_N, b_N)$ with $\sum_{j=1}^N (b_j - a_j) < \delta$,

$$\sum_{j=1}^N |F(b_j) - F(a_j)| \leq \sum_{j=1}^N m((a_j, b_j] \cap A) < \sum_{j=1}^N (b_j - a_j) < \varepsilon,$$

so F is absolutely continuous. Let μ_F be the unique Borel measure such that $F(x) = \mu_F([0,x])$. Then the absolute continuity of F implies that $\mu_F \ll m$ and $d\mu_F = F'\,dm$. On the other hand, for $a < b$ in $[0,1]$,

$$\mu_F((a,b]) = F(b) - F(a) = m([0,b] \cap A) - m([0,a] \cap A) = m([a,b] \cap A) = \int_a^b \mathbb{1}_A \, dm\,,$$

whence $d\mu_F = \mathbb{1}_A \, dm$. By the uniqueness of Radon-Nikodym derivatives, $F' = \mathbb{1}_A$ almost everywhere, whence $F' = 0$ on $[0,1] \setminus A$. Moreover $m([0,1] \setminus A) = m([0,1]) - m(A) = m([0,1]) - m([0,1] \cap A) > 0$ by assumption. $\square$

(b): Let $G(x) = m([0,x] \cap A) - m([0,x] \setminus A)$. Show that G is absolutely continuous on $[0,1]$, but G is not monotone on any subinterval of $[0,1]$.

Proof. Let $\varepsilon > 0$. Again set $\delta = \varepsilon$. Then, for any finite set of disjoint intervals $(a_1, b_1), \ldots, (a_N, b_N)$ with $\sum_{j=1}^N (b_j - a_j) < \delta$,

$$\sum_{j=1}^N |G(b_j) - G(a_j)| = \sum_{j=1}^N m((a_j, b_j] \cap A) - m((a_j, b_j] \setminus A) < \sum_{j=1}^N m((a_j, b_j]) < \varepsilon,$$

so G is absolutely continuous. Let μ be the measure given by $d\mu = G'\,dm$. Then for $a < b$,

$$\int_a^b G' \, dm = m((a,b] \cap A) - m((a,b] \setminus A) = \int_a^b \mathbb{1}_A - \mathbb{1}_{[0,1]\setminus A} \, dm\,.$$

Then as intervals generate all borel subsets of $[0,1]$, $G'\,dm = d\mu = (\mathbb{1}_A - \mathbb{1}_{[0,1]\setminus A})\,dm$, so by uniqueness of Radon-Nikodym derivatives $G' = \mathbb{1}_A - \mathbb{1}_{[0,1]\setminus A}$ almost everywhere. Let $I \subseteq [0,1]$ be an interval. Then $G' = 1$ on $I \cap A$ and $G' = -1$ on $I \setminus A$, but $m(I \cap A), m(I \setminus A) > 0$, so we must have that G is not monotonic on I. $\square$

Problem 5

Problem 6

Let $a < b$ be real numbers and let $1 \leq p \leq +\infty$. Let X be the set of functions $f : [a,b] \to \mathbb{C}$ which are absolutely continuous and such that $f' \in L^p([a,b])$. Fix $x_0 \in [a,b]$. For $f \in X$, define

$$\|f\| = |f(x_0)| + \|f'\|_p.$$

Show that $\|\cdot\|$ is a norm which turns X into a Banach space.