

MATH 7310 Homework 2

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Problem 1

Let μ be a finitely additive measure.

(a) Prove that μ is a measure if and only if it is continuous from below as in Theorem 1.8c.

Proof. Theorem 1.8c shows the forward direction so it suffices to show the reverse direction. Suppose that μ is continuous from below. Let $(E_j)_{j=1}^\infty$ be a sequence of disjoint elements in the sigma algebra \mathcal{M} corresponding to μ . Define a new sequence $(F_n)_{n=1}^\infty$ in \mathcal{M} by $F_n = \bigsqcup_{j=1}^n E_j$. Then $\bigsqcup_{n=1}^\infty E_n = \bigcup_{n=1}^\infty F_n$. As $(F_n)_{n=1}^\infty$ is an increasing sequence in \mathcal{M} , we have that

$$\mu\left(\bigsqcup_{n=1}^\infty E_n\right) = \mu\left(\bigcup_{n=1}^\infty F_n\right) = \lim_{n \rightarrow \infty} \mu(F_n) = \lim_{n \rightarrow \infty} \sum_{j=1}^n \mu(E_j) = \sum_{j=1}^\infty \mu(E_j),$$

so μ is a measure. □

(b) If $\mu(X) < \infty$, prove that μ is a measure if and only if it is continuous from above as in Theorem 1.8d.

Proof. Theorem 1.8d shows the forward direction so it suffices to show the reverse direction. Suppose that μ is continuous from above. Let $(E_j)_{j=1}^\infty$ be a sequence of disjoint elements in \mathcal{M} . Define a new sequence $(F_n)_{n=1}^\infty$ in \mathcal{M} by $F_n = \bigsqcup_{j=1}^n E_j$. Observe that $F_1^c \supset F_2^c \supset F_3^c \supset \dots$ is a decreasing sequence in \mathcal{M} with $\mu(F_1^c) = \mu(X) - \mu(F_1) < +\infty$. Hence, by continuity from above,

$$\begin{aligned} \mu\left(\bigsqcup_{j=1}^\infty E_j\right) &= \mu\left(\bigcup_{n=1}^\infty F_n\right) = \mu\left(X \setminus \bigcap_{n=1}^\infty F_n^c\right) = \mu(X) - \mu\left(\bigcap_{n=1}^\infty F_n^c\right) = \mu(X) - \lim_{n \rightarrow \infty} \mu(F_n^c) \\ &= \mu(X) - \lim_{n \rightarrow \infty} \mu\left(X \setminus \bigsqcup_{j=1}^n E_j\right) = \mu(X) - \lim_{n \rightarrow \infty} \left(\mu(X) - \mu\left(\bigsqcup_{j=1}^n E_j\right)\right) = \lim_{n \rightarrow \infty} \sum_{j=1}^n \mu(E_j) = \sum_{j=1}^\infty \mu(E_j), \end{aligned}$$

so μ is a measure. □

Problem 2

Let (X, \mathcal{M}, μ) be a finite measure space.

(a) If $E, F \in \mathcal{M}$ and $\mu(E \Delta F) = 0$, then $\mu(E) = \mu(F)$.

Proof. Observe that

$$0 = \mu(E \Delta F) = \mu((E \setminus F) \sqcup (F \setminus E)) = \mu(E \setminus F) + \mu(F \setminus E).$$

As $\mu(E \setminus F), \mu(F \setminus E) \geq 0$, it follows that $\mu(E \setminus F), \mu(F \setminus E) = 0$. Then as $E = (E \setminus F) \sqcup (E \cap F)$ and $F = (F \setminus E) \sqcup (F \cap E)$, $\mu(E) = \mu(F)$. \square

(b) Say that $E \sim F$ if $\mu(E \Delta F) = 0$; show that \sim is an equivalence relation on \mathcal{M} .

Proof.

(Reflexivity): Note that $E \Delta E = E \setminus E = \emptyset \implies \mu(E \Delta E) = 0$, so $E \sim E$.

(Symmetry): Note that $E \Delta F = (E \setminus F) \sqcup (F \setminus E) = F \Delta E$, so $E \sim F \implies F \sim E$.

(Transitivity): Suppose that $E \sim F$ and $F \sim G$. Observe that

$$\begin{aligned} E \setminus G &= ((E \setminus F) \sqcup (E \cap F)) \setminus G = ((E \setminus F) \setminus G) \cup ((E \cap F) \setminus G) \subseteq (E \setminus F) \cup (F \setminus G) \\ G \setminus E &= ((G \setminus F) \sqcup (G \cap F)) \setminus E = ((G \setminus F) \setminus E) \cup ((G \cap F) \setminus E) \subseteq (G \setminus F) \cup (F \setminus E) \end{aligned}$$

so by monotonicity and subadditivity,

$$\mu(E \Delta G) \leq \mu((E \setminus F) \cup (F \setminus G)) + \mu((G \setminus F) \cup (F \setminus E)) \leq \mu(E \setminus F) + \mu(F \setminus E) + \mu(F \setminus G) + \mu(G \setminus F) = \mu(E \Delta F) + \mu(F \Delta G) = 0$$

hence $E \sim G$. \square

(c) For $E, F \in \mathcal{M}$, define $\rho(E, F) = \mu(E \Delta F)$. Then $\rho(E, G) \leq \rho(E, F) + \rho(F, G)$, and hence ρ defines a metric on the space \mathcal{M}/\sim .

Proof. Note that the inequality used in the proof of transitivity above held regardless of the assumptions that the symmetric differences were zero, whence

$$\rho(E, G) = \mu(E \Delta G) \leq \mu(E \Delta F) + \mu(F \Delta G) = \rho(E, F) + \rho(F, G).$$

\square

Problem 3

Let \mathcal{A} be the collection of finite unions of sets of the form $(a, b] \cap \mathbb{Q}$ where $-\infty \leq a \leq b \leq +\infty$.

(i) Show that \mathcal{A} is an algebra on \mathbb{Q} . (Use Proposition 1.7.)

Proof. Let \mathcal{E} be the collection of sets of the form $(a, b] \cap \mathbb{Q}$ with $-\infty \leq a < b \leq +\infty$. By Proposition 1.7, it suffices to show that \mathcal{E} is an elementary family.

Note that for an $a \in \mathbb{R}$, $(a, a] \cap \mathbb{Q} = \emptyset$, so $\emptyset \in \mathcal{E}$.

Suppose $E, F \in \mathcal{E}$. \square

(ii) Show that the σ -algebra generated by \mathcal{A} is $\mathcal{P}(\mathbb{Q})$.

Proof. As $\mathcal{A} \subseteq \mathcal{P}(\mathbb{Q})$, by minimality $\Sigma(\mathcal{A}) \subseteq \mathcal{P}(\mathbb{Q})$. Now take $q \in \mathbb{Q}$. Observe that $(q - \frac{1}{n}, q] \cap \mathbb{Q} \in \mathcal{A}$ for all $n \in \mathbb{N}$, whence $\{q\} = \bigcap_{n=1}^{\infty} (q - \frac{1}{n}, q] \cap \mathbb{Q} \in \Sigma(\mathcal{A})$. Hence, $\Sigma(\mathcal{A})$ contains all finite and countable subsets of \mathbb{Q} , so countability of \mathbb{Q} implies that $\mathcal{P}(\mathbb{Q}) \subseteq \Sigma(\mathcal{A})$. \square

(ii) Define μ_0 on \mathcal{A} by $\mu_0(\emptyset) = 0$ and $\mu_0(A) = \infty$ for $A \neq \emptyset$. Prove that μ_0 is a premeasure on \mathcal{A} , and that there is more than one measure on $\mathcal{P}(\mathbb{Q})$ whose restriction to \mathcal{A} is μ_0 .

Proof. \square

Problem 4

Let \mathcal{A} be an algebra, and let $\mu : \mathcal{A} \rightarrow [0, +\infty]$ be a finitely additive measure.

(i) Suppose $(A_j)_{j=1}^\infty$ are pairwise disjoint elements of \mathcal{A} , and that $A = \bigcup_{j=1}^\infty A_j \in \mathcal{A}$. Show that

$$\mu(A) \geq \sum_{j=1}^\infty \mu(A_j).$$

Proof. Since μ is finitely additive, it is also finitely subadditive. Then by monotonicity, for any $n \in \mathbb{N}$,

$$\mu(A) \geq \mu\left(\bigcup_{j=1}^n A_j\right) = \sum_{j=1}^n \mu(A_j).$$

Hence, it follows that $\mu(A) \geq \sum_{j=1}^\infty \mu(A_j)$. □

(ii) Show that the following are equivalent:

1. μ is a premeasure,
2. $\mu\left(\bigcup_{j=1}^\infty A_j\right) \leq \sum_{j=1}^\infty \mu(A_j)$ for any sequence $(A_j)_{j=1}^\infty$ with $\bigcup_{j=1}^\infty A_j \in \mathcal{A}$,
3. for any increasing sequence $(E_j)_{j=1}^\infty$ in \mathcal{A} with $\bigcup_{j=1}^\infty E_j \in \mathcal{A}$, we have

$$\mu\left(\bigcup_j E_j\right) = \lim_{n \rightarrow \infty} \mu(E_n).$$

Proof.

(1 \implies 2): Let $(A_j)_{j=1}^\infty$ be a sequence in \mathcal{A} such that $\bigcup_{j=1}^\infty A_j \in \mathcal{A}$. Define a new sequence $(A'_n)_{n=1}^\infty$ in \mathcal{A} by $A'_1 = A_1$ and $A'_n = A_n \setminus \bigcup_{j=1}^{n-1} A_j$ for $n \geq 2$. Then by disjoint countable additivity and monotonicity,

$$\mu\left(\bigcup_{j=1}^\infty A_j\right) = \mu\left(\bigsqcup_{j=1}^\infty A'_j\right) = \sum_{j=1}^\infty \mu(A'_j) \leq \sum_{j=1}^\infty \mu(A_j).$$

(1 \implies 3):

(3 \implies 1): Follows identically to the proof in problem 1 part (b) by assuming the sequence is such that the countable union stays in \mathcal{A} .

(3 \implies 2): Suppose that $(A_j)_{j=1}^\infty$ is a sequence in \mathcal{A} with $\bigcup_{j=1}^\infty A_j \in \mathcal{A}$. Then, by finite subadditivity (which follows from finite additivity),

$$\mu\left(\bigcup_{j=1}^\infty A_j\right) = \mu\left(\bigcup_{n=1}^\infty \bigcup_{j=1}^n A_j\right) = \lim_{n \rightarrow \infty} \mu\left(\bigcup_{j=1}^n A_j\right) \leq \lim_{n \rightarrow \infty} \sum_{j=1}^n \mu(A_j) = \sum_{j=1}^\infty \mu(A_j).$$

□

(iii) If $\mu(X) < +\infty$, show that μ is a premeasure if and only if for every decreasing sequence $(E_n)_{n=1}^\infty$ of sets in \mathcal{A} with $\bigcap_{n=1}^\infty E_n = \emptyset$, we have

$$\lim_{n \rightarrow \infty} \mu(E_n) = 0.$$

Proof.

\implies : Let $(E_n)_{n=1}^\infty$ be a decreasing sequence of sets in \mathcal{A} with $\bigcap_{n=1}^\infty E_n = \emptyset$

□

Problem 5

A *metric measure space* is a triple (X, d, μ) where (X, d) is a metric space and $\mu : \mathcal{B}_{(X,d)} \rightarrow [0, +\infty]$ is a measure. We say that $E \subseteq X$ is a *continuity set* if $\mu(\overline{E} \setminus \text{Int}(E)) = 0$. For this problem, fix a metric measure space (X, d, μ) .

(i) Show that the collection of continuity sets forms an algebra of sets.

Proof. Suppose that $E_1, \dots, E_n \subseteq X$ are continuity sets. Then $\mu(\overline{E_j} \setminus \text{Int}(E_j)) = 0$ for $1 \leq j \leq n$. As there are finitely many sets, the union of closures is equal to the closure of the union. Hence

$$\overline{\bigcup_{j=1}^n E_j} \setminus \text{Int}\left(\bigcup_{j=1}^n E_j\right) = \bigcup_{j=1}^n \overline{E_j} \setminus \text{Int}\left(\bigcup_{j=1}^n E_j\right) \subseteq \bigcup_{j=1}^n \overline{E_j} \setminus \bigcup_{j=1}^n \text{Int}(E_j) = \bigcup_{j=1}^n \overline{E_j} \setminus \text{Int}(E_j),$$

so by subadditivity,

$$\mu\left(\overline{\bigcup_{j=1}^n E_j} \setminus \text{Int}\left(\bigcup_{j=1}^n E_j\right)\right) = \mu\left(\bigcup_{j=1}^n \overline{E_j} \setminus \text{Int}(E_j)\right) \leq \sum_{j=1}^n \mu(\overline{E_j} \setminus \text{Int}(E_j)) = 0$$

whence $E_1 \cup \dots \cup E_n$ is a continuity set. Now suppose that $E \subseteq X$ is a continuity set.

$$(\overline{X \setminus E}) \setminus \text{Int}(X \setminus E) = (X \setminus \text{Int}(E)) \setminus \text{Int}(X \setminus E) = (X \setminus \text{Int}(E)) \setminus (X \setminus \overline{E}) = \overline{E} \setminus \text{Int}(E)$$

so $\mu((\overline{X \setminus E}) \setminus \text{Int}(X \setminus E)) = \mu(\overline{E} \setminus \text{Int}(E)) = 0$, whence $X \setminus E$ is also a continuity set. \square

(ii) Show that if $x \in X$, $r > 0$ and $\mu(B_r(x, d)) < +\infty$, then there is an $s \in (0, r)$ so that $B_s(x, d)$ is a continuity set.

Proof. Suppose, for the sake of contradiction, that $\mu(\overline{B_s(x)} \setminus \text{Int}(B_s(x))) \neq 0$ for all $s \in (0, r)$. For $n \in \mathbb{N}$, define a set

$$A_n = \left\{s \in (0, r) : \frac{1}{n} \leq \mu(\overline{B_s(x)} \setminus \text{Int}(B_s(x))) < \frac{1}{n-1}\right\}$$

where $1/0 := \infty$ by convention. Then $(0, r) = \bigcup_{n=1}^{\infty} A_n$, so there exists an $n \in \mathbb{N}$ such that A_n is infinite. Take a countably infinite subset $\{s_1, s_2, \dots\} \subseteq A_n$. Note that, for any fixed $t \in (0, +\infty)$, $\overline{B_t(x)} \setminus \text{Int}(B_t(x)) \subseteq \{y \in X : d(x, y) = t\}$, whence

$$\mu\left(\bigcup_{j=1}^{\infty} \overline{B_{s_j}(x)} \setminus \text{Int}(B_{s_j}(x))\right) = \sum_{j=1}^{\infty} \mu(\overline{B_{s_j}(x)} \setminus \text{Int}(B_{s_j}(x))) = \infty$$

contradicting that $\mu(B_r(x)) < \infty$. \square

(iii) Suppose that (X, d) is separable and that for every $x \in X$, there is an $r > 0$ so that $\mu(B_r(x, d)) < +\infty$. Show that there is a countable basis consisting of open continuity sets. (Hint: given a countable dense $D \subseteq X$ and $x \in D$, use the preceding part to choose a countable set $J_x \subseteq (0, +\infty)$ with the property that $\inf_{t \in J_x} t = 0$ and so that $B_t(x, d)$ is a continuity set for all $t \in J_x$).

Proof. Let $D \subseteq X$ be a countable dense subset of X . Fix $x \in D$. For $n \in \mathbb{N}$, appeal to part (i) to find a $t_n \in (0, r)$ such that $B_{t_n}(x, d)$ is a continuity set. Letting $J_x = \{t_n : n \in \mathbb{N}\}$, it follows that J_x has the property that $\inf_{t \in J_x} t = 0$ and so that $B_t(x, d)$ is a continuity set for all $t \in J_x$. Let

$$\mathcal{J} = \{(x, t) : x \in D, t \in J_x\}$$

Note that \mathcal{J} is countable. Let $\mathcal{B} = \{B_t(x, d) : (x, t) \in \mathcal{J}\}$. We claim that \mathcal{B} is a basis for the metric topology on (X, d) . As \mathcal{B} covers D and D is dense in X , it is clear that \mathcal{B} covers X .

Suppose that $x \in X$ and $B_t(y, d), B_{t'}(z, d) \in \mathcal{B}$ such that $x \in B_t(y, d) \cap B_{t'}(z, d)$. As $\inf_{t \in J_x} t = 0$, there exists a $t'' \in J_x$ such that $t \leq t', t''$, whence $x \in B_{t''}(x, d) \subseteq B_t(x, d) \cap B_{t'}(x, d)$. \square

Problem 6

Let (X, d) be a metric space and μ, ν be finite Borel measures on X with $\mu(X) = \nu(X)$. Let $\mathcal{A} = \{E \in \mathcal{B}_{(X, d)} : \mu(E) = \nu(E)\}$.

(i) Show that if $F \subseteq E$ and $F, E \in \mathcal{A}$, then $E \setminus F \in \mathcal{A}$. Also show that if $(E_n)_{n=1}^\infty$ is an increasing sequence of elements of \mathcal{A} , then $\bigcup_{n=1}^\infty E_n \in \mathcal{A}$.

Proof. As $E, F \in \mathcal{A}$, $\mu(E) = \nu(E)$ and $\mu(F) = \nu(F)$. Then

$$\mu(E \setminus F) = \mu(E) - \mu(F) = \nu(E) - \nu(F) = \nu(E \setminus F)$$

so $E \setminus F \in \mathcal{A}$. Now suppose that $(E_n)_{n=1}^\infty$ is an increasing sequence of elements of \mathcal{A} . By continuity from above,

$$\mu\left(\bigcup_{n=1}^\infty E_n\right) = \lim_{n \rightarrow \infty} \mu(E_n) = \lim_{n \rightarrow \infty} \nu(E_n) = \nu\left(\bigcup_{n=1}^\infty E_n\right),$$

so $\bigcup_{n=1}^\infty E_n \in \mathcal{A}$. □

(ii) Given a nonempty $F \subseteq X$ closed and $x \in X$, define $d(x, F) = \inf_{y \in F} d(x, y)$. Show that $x \mapsto d(x, F)$ is continuous and $F = \{x \in X : d(x, F) = 0\}$.

Proof. Suppose $x, y \in X$. For $z \in F$,

$$d(x, F) \leq d(x, z) \leq d(x, y) + d(y, z) \implies d(x, F) - d(x, y) \leq d(y, z).$$

As this holds for arbitrary $z \in F$, it follows that $d(x, F) - d(x, y) \leq d(y, F)$, so $d(x, F) - d(y, F) \leq d(x, y)$. By symmetry, $d(y, F) - d(x, F) \leq d(x, y)$, so $|d(x, F) - d(y, F)| \leq d(x, y)$. Thus, the function $x \mapsto d(x, F)$ is 1-Lipschitz whence it is continuous.

Clearly $F \subseteq \{x \in X : d(x, F) = 0\}$, so it suffices to show the reverse containment. Suppose that $x \in X$ such that $d(x, F) = 0$. For all $n \in \mathbb{N}$, there exists an $f_n \in F$ such that $0 \leq d(x, f_n) < \frac{1}{n}$. It follows that $d(x, f_n) \xrightarrow{n \rightarrow \infty} 0$, so $f_n \xrightarrow{n \rightarrow \infty} x$. Thus x is a limit point of F , so F being closed implies that $x \in F$. □

(iii) Show that $\{U \subseteq X : U \text{ is open}\} \subseteq \mathcal{A}$ if and only if $\{F \subseteq X : F \text{ is closed}\} \subseteq \mathcal{A}$.

Proof.

\implies : Suppose that $\{U \subseteq X : U \text{ is open}\} \subseteq \mathcal{A}$. Take $F \subseteq X$ such that F is closed.

\impliedby :

□