MATH 7752 Homework 2

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Problem 3

Let R be a commutative ring. An R-module M is called *torsion* if for any $m \in M$ there exists some nonzero $r \in R$ such that $rm = 0$. An R-module N is called *divisible* if for any nonzero $r \in R$ it holds that $rN = N$.

(a) Suppose M is a torsion R-module and N is a divisible R-module. Prove that $M \otimes_R N = \{0\}$.

Proof. Let $m \in M$ and $n \in N$. Since M is torsion, there exists a nonzero $rinR$ such that $rm = 0$. Now, by divisibility of N, there exists an $n' \in N$ such that $rn' = n$. Hence

$$m \otimes n = m \otimes rn' = rm \otimes n' = 0 \otimes n' = 0.$$

Thus every simple tensor in $M \otimes_R N$ is 0, whence $M \otimes_R N = 0$. □

(b) Consider the $\mathbb{Z}$-module $M = \mathbb{Q}/\mathbb{Z}$. Prove that $M \otimes_{\mathbb{Z}} M = \{0\}$

Proof. We show that M is both torsion and divisible. Note that □