MATH 7752 - HOMEWORK 10
DUE WEDNESDAY 04/15/22 AT 1 P.M.

(1) Let F be a field, $f(x) \in F[x]$ be an irreducible separable polynomial over F of degree n and let K be a splitting field of $f(x)$.
   (a) Prove that $|\operatorname{Gal}(K/F)|$ is a multiple of n and divides $n!$.
   (b) Let $n = 3$. Prove that $\operatorname{Gal}(K/F)$ is isomorphic to either $\mathbb{Z}/3\mathbb{Z}$ or S_3.
   (c) Let $n = 4$ and assume that $|\operatorname{Gal}(K/F)| = 8$. Determine the isomorphism class of $\operatorname{Gal}(K/F)$.

(2) Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial of degree n, and let K be a splitting field of $f(x)$ contained in $\mathbb{C}$. Label the roots of $f(x)$ by $\alpha_1, \ldots, \alpha_n$ (in some order), and let $\rho : \operatorname{Gal}(K/\mathbb{Q}) \hookrightarrow S_n$ be the associated embedding.
   (a) Assume that $f(x)$ has at least one non-real root. Prove that the complex conjugation gives an element τ of $\operatorname{Gal}(K/\mathbb{Q})$ of order 2. What can you say about τ if $f(x)$ has precisely two non-real roots?
   (b) Suppose that the degree n of $f(x)$ is a prime number, and that $f(x)$ has precisely two non-real roots. Prove that $\operatorname{Gal}(K/\mathbb{Q})$ is isomorphic to S_n. **Hint:** You might need to recall some facts from Algebra I about generators of S_n.

(3) Let K be the splitting field of $f(x) = x^4 - 2 \in \mathbb{Q}[x]$.
   (a) Choose an order on the set of roots of $f(x)$ and describe the associated embedding $\operatorname{Gal}(K/\mathbb{Q}) \hookrightarrow S_4$. (You can use the information you obtained in Homework 8).
   (b) Describe all subgroups of $\operatorname{Gal}(K/\mathbb{Q})$ and the corresponding subfields of K.

(4) Let K/F and L/F be field extensions.
   (a) Assume that L/F is finite Galois. Show that KL/K is also Galois.
   (b) Suppose that both K/F and L/F are Galois extensions.
      (i) Prove that the extension KL/F is also Galois and there is a natural embedding $\iota : \operatorname{Gal}(KL/F) \to \operatorname{Gal}(K/F) \times \operatorname{Gal}(L/F)$.
      (ii) Assume now that K/F and L/F are both finite. Prove that the map ι in part (i) is an isomorphism if and only if $K \cap L = F$.