MATH 7752 Homework 2

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Problem 1

Let D be a division ring (not necessarily commutative) and M be a D-module.

(a) Let X be a generating set of M and Y a D-linearly independent subset of X. Prove that M has a D-basis B with $Y \subseteq B \subseteq X$.

Proof. Consider the poset $\mathscr{S} = \{B \subseteq M : Y \subseteq B \subseteq X \text{ and } B \text{ is } D\text{-linearly independent}\}$ ordered by inclusion. Since Y is a D-linearly independent subset of X, we have that $Y \in \mathscr{S}$ so $\mathscr{S} \neq \emptyset$.

Suppose that $\mathscr{C} \subseteq \mathscr{S}$ is any linearly ordered chain in $\mathscr{S}$. Let $B = \bigcup \mathscr{C}$. Then $Y \subseteq B \subseteq X$. Suppose that $d_i \in D$ and $b_i \in B$ such that $\sum_{i=1}^n d_i \cdot b_i = 0$. Then for each $i \in \{1, \ldots, n\}$, there exists a $B_i \in \mathscr{C}$ such that $b_i \in B_i$. As $\mathscr{C}$ is a chain, there is some $l \in \{1, \ldots, n\}$ such that $B_i \subseteq B_l$ for all $1 \leq i \leq n$. It follows that $b_i \in B_l$ for all $1 \leq i \leq n$, whence B_l being D-linearly independent implies that $d_i = 0$ for all i. Thus B is D-linearly independent, so $B \in \mathscr{S}$.

Now by Zorn's lemma, there exists a maximal element $B \in \mathscr{S}$ of $\mathscr{S}$. We claim that B is in fact a D-basis for M. It suffices to show that B is a generating set for M. Let $N = \text{span}_D(B)$. Suppose, for the sake of contradiction, that $N \neq M$. As $B \subseteq X$ and X is a generating set for M, it follows that there exists an $x \in X \setminus \text{span}_D(B)$. Suppose $r, r_1, \ldots, r_n \in R$ are such that

$$0 = rx + r_1 b_1 + \cdots + r_n b_n.$$

If $r \neq 0$, then

$$x = (-r^{-1} r_1) \cdot b_1 + \cdots + (-r^{-1} r_1) \cdot r_n,$$

which would imply that $x \in \text{span}_D(B)$, contradicting the choice of x. Hence $r = 0$, so B being D-linearly independent implies that $r_i = 0$ for all i. Thus $B \cup \{x\}$ is D-linearly independent, contradicting the maximality of B. $\square$

(b) Conclude that every non zero D-module M has a D-basis.

Proof. Since $M \neq 0$, there exists an $y \in M \setminus \{0\}$. It follows that the singleton $\{y\}$ is a D-linearly independent subset of M. On the other hand, $M = 1 \cdot M$, so the set M is a generating set of M. Applying part (a) to $X = M$ and $Y = \{y\}$, it follows that M has a D-basis. $\square$

Problem 2

Let R be a commutative domain. Let I be a non-principal ideal of R. SHow that when I is considered as an R-module (by left multiplication), then I is indecomposable but not cyclic.

Proof. Since I is non-principal, by definition I is not cyclic as an R-module. Suppose, for the sake of contradiction, that $I = P \oplus Q$ for some nonzero proper R-submodules P, Q of I. Take $p \in P \setminus \{0\}$ and $q \in Q \setminus \{0\}$. Then $p \cdot q - q \cdot p = pq - qp = 0 \implies p \cdot q = q \cdot p$. As R is a domain $pq = qp \neq 0$, whence $pq = qp \in P \cap Q$ contradicts that the sum $P \oplus Q$ is direct. □

Problem 3

Let R be a commutative ring. An R-module M is called *torsion* if for any $m \in M$ there exists some nonzero $r \in R$ such that $rm = 0$. An R-module N is called *divisible* if for any nonzero $r \in R$ it holds that $rN = N$.

(a) Suppose M is a torsion R-module and N is a divisible R-module. Prove that $M \otimes_R N = \{0\}$.

Proof. Let $m \in M$ and $n \in N$. Since M is torsion, there exists a nonzero $r \in R$ such that $rm = 0$. Now, by divisibility of N, there exists an $n' \in N$ such that $rn' = n$. Hence

$$m \otimes n = m \otimes rn' = rm \otimes n' = 0 \otimes n' = 0.$$

Thus every simple tensor in $M \otimes_R N$ is 0, whence $M \otimes_R N = 0$. □

(b) Consider the $\mathbb{Z}$-module $M = \mathbb{Q}/\mathbb{Z}$. Prove that $M \otimes_{\mathbb{Z}} M = \{0\}$

Proof. We show that M is both torsion and divisible. Note that for any $\frac{p}{q} + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$, $q \neq 0$ and $q \cdot (\frac{p}{q} + \mathbb{Z}) = q \cdot \frac{p}{q} + \mathbb{Z} = p + \mathbb{Z} = \mathbb{Z}$, so $\mathbb{Q}/\mathbb{Z}$ is torsion.

On the other hand, suppose $n \in \mathbb{Z} \setminus \{0\}$. For $\frac{p}{q} + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$, observe that

$$n \cdot \left(\frac{p}{nq} + \mathbb{Z}\right) = \frac{np}{nq} + \mathbb{Z} = \frac{p}{q} + \mathbb{Z}$$

so $\mathbb{Q}/\mathbb{Z}$ is divisible. Appealing to part(a), it follows that $M \otimes_{\mathbb{Z}} M = 0$. □

Problem 4

Let R be a PID and A be an R-module. Let K be the field of fractions of R, and consider the K-module $B = K \otimes_R A$. Prove that every $z \in B$ is a simple tensor.

Proof. Let $z \in B$. Then there exists $\frac{x_1}{s_1}, \ldots, \frac{x_n}{s_n} \in K$, $a_1, \ldots, a_n \in A$ and $c_1, \ldots, c_n \in R$ such that

$$z = \sum_{i=1}^{n} c_i \cdot \left(\frac{x_i}{s_i} \otimes a_i\right) = \sum_{i=1}^{n} \frac{c_i x_i}{s_i} \otimes a_i = \sum_{i=1}^{n} \frac{c_i x_i \prod_{j \neq i} s_j}{s_1 \cdots s_n} \otimes a_i.$$

Since R is a PID, there exists an $s \in R$ such that $(s) = \left\langle c_i x_i \prod_{j \neq i} s_j : 1 \leq i \leq n \right\rangle$. Then for each $i \in \{1, \ldots, n\}$, there is an $r_i \in R$ such that $c_i x_i \prod_{j \neq i} s_j = r_i s$. Hence,

$$z = \sum_{i=1}^{n} \frac{c_i x_i \prod_{j \neq i} s_j}{s_1 \cdots s_n} \otimes a_i = \sum_{i=1}^{n} \frac{r_i s}{s_1 \cdots s_n} \otimes a_i = \sum_{i=1}^{n} \frac{s}{s_1 \cdots s_n} \otimes r_i a_i = \frac{s}{s_1 \cdots s_n} \otimes \left(\sum_{i=1}^{n} r_i a_i\right)$$

is a simple tensor. □

Problem 5

Let R be a commutative ring and M an R-module.
(a): Let I be an ideal of R. Prove an isomorphism

$$R/I \otimes M \simeq M/IM.$$

Proof. On one hand, consider the map $\widetilde{\Psi} : M \to R/I \otimes_R M$ given by $\widetilde{\Psi}(m) := (1 + I) \otimes m$ for $m \in M$. This map is clearly an R-module homomorphism by the second and third defining relations the tensor product. For $i \in I$ and $m \in M$, $\widetilde{\Psi}(im) = (1 + I) \otimes im = i \cdot (1 + I) \otimes m = 0$, so the generators of IM lie in $\ker(\widetilde{\Psi})$ whence $IM \subseteq \ker(\widetilde{\Psi})$. Hence, $\widetilde{\Psi}$ descends to an R-module homomorphism $\Psi : M/IM \to R/I \otimes_R M$ such that $\Psi(m + IM) = \widetilde{\Psi}(m)$. □

(b): Suppose that M is a finitely generated free R-module. Show that the *rank* of M is well-defined, i.e. any two R-bases of M have the same number of elements.

Proof. Let $\mathfrak{m} \subseteq R$ be a maximal ideal of R. Let $k = R/\mathfrak{m}$ be the corresponding residue field. Suppose that $n, l \in \mathbb{N}$ such that $R^l \cong M \cong R^n$. Then,

$$k^l \cong (R/\mathfrak{m} \otimes R)^l \cong R/\mathfrak{m} \otimes R^l \cong R/\mathfrak{m} \otimes R^n \cong (R/\mathfrak{m} \otimes R)^n \cong k^n$$

as R-modules. Let $\varphi : k^l \to k^n$ be the composition of the above R-module isomorphisms. Note that $\mathfrak{m} \subseteq \mathrm{Ann}_R(k^l), \mathrm{Ann}_R(k^n)$, so the k-module structures on k^l, k^m given by $(r + \mathfrak{m}) \cdot a := r \cdot a$ and $(r + \mathfrak{m}) \cdot b$ for $a \in k^l$ and $b \in k^n$ are well-defined. Moreover, for $r \in R$ and $a \in k^l$,

$$\varphi((r + \mathfrak{m}) \cdot a) = \varphi(r \cdot a) = r \cdot \varphi(a) = (r + \mathfrak{m}) \cdot \varphi(a),$$

so φ is also a k-module isomorphism. As k^l and k^n are isomorphic k-vector spaces, it follows that $l = n$.

□

Problem 6

Let $R \subseteq S$ be an inclusion of commutative rings. Consider the polynomial rings $R[x]$ and $S[x]$. Prove that there is an isomorphism of S-modules,

$$S \otimes_R R[x] \to S[x]$$