

MATH 7310 Homework 6

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April 21, 2022

Problem 1

Let $X = Y$ be an uncountable linearly ordered set such that for each $x \in X$, $\{y \in X : y < x\}$ is countable. Let $\mathcal{M} = \mathcal{N}$ be the σ -algebra of countable or co-countable sets, and let $\mu = \nu$ be defined on \mathcal{M} by $\mu(A) = 0$ if A is countable and $\mu(A) = 1$ if A is co-countable. Let $E = \{(x, y) \in X \times X : y < x\}$. Prove that E_x and E^y are measurable for all x, y , and that $\int \int \mathbb{1}_E d\mu d\nu$ and $\int \int \mathbb{1}_E d\nu d\mu$ exist but are not equal.

Proof. For $x \in X$, define the set $S(x) = \{y \in X : y < x\}$. Observe that, for $x \in X$, $E_x = \{y \in X : (x, y) \in E\} = \{y \in X : y < x\} = S(x)$ which is countable by assumption so E_x is measurable. On the other hand, for $y \in X$, since the ordering on X is total,

$$X \setminus E^y = \{x \in X : y \notin S(x)\} = \{x \in X : x = y \text{ or } x < y\} = \{y\} \cup S(y)$$

which is countable by assumption, so E^y is cocountable and thus measurable.

Thus, for $x, y \in X$, the x and y -sections of $\mathbb{1}_E$, i.e. $(\mathbb{1}_E)^y = \mathbb{1}_{E^y}$ and $(\mathbb{1}_E)_x = \mathbb{1}_{E_x}$, are measurable. Thus, the inner integrals in each of the iterated integrals exist. To see that both of the whole iterated integrals exist, we compute for fixed $y \in X$

$$\int \mathbb{1}_E(x, y) d\mu(x) = \int \mathbb{1}_{E^y}(x) d\mu(x) = \mu(E^y) = 1$$

and for fixed $x \in X$

$$\int \mathbb{1}_E(x, y) d\nu(y) = \int \mathbb{1}_{E_x}(y) d\nu(y) = \nu(E_x) = 0$$

which are both measurable functions as they are constant functions. Hence, both of the iterated integrals exist and we compute on one hand that

$$\int \int \mathbb{1}_E(x, y) d\mu(x) d\nu(y) = \int \mu(E^y) d\nu(y) = \int 1 d\nu(y) = \nu(X) = 1$$

and on the other hand that

$$\int \int \mathbb{1}_E(x, y) d\nu(y) d\mu(x) = \int \nu(E_x) d\mu(x) = \int 0 d\mu(x) = 0.$$

Thus $\int \int \mathbb{1}_E d\mu d\nu$ and $\int \int \mathbb{1}_E d\nu d\mu$ exist but are not equal. □

Problem 2

Prove Theorem 2.39 by using Theorem 2.37 and proposition 2.12 together with the following lemmas:

- (a) If $E \in \mathcal{M} \otimes \mathcal{N}$ and $\mu \otimes \nu(E) = 0$, then $\nu(E_x) = \mu(E^y) = 0$ for a.e. x and y .
- (b) If f is \mathcal{L} -measurable and $f = 0$ λ -a.e., then f_x and f^y are integrable for a.e. x and y , and $\int f_x d\nu = 0$ and $\int f^y d\mu = 0$ for a.e. x and y . (This uses completeness of μ and ν .)

Suppose that (X, \mathcal{M}, μ) and (Y, \mathcal{N}, ν) are complete, σ -finite measure spaces, and let $(X \times Y, \mathcal{L}, \lambda)$ be the completion of $(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \otimes \nu)$. If f is \mathcal{L} -measurable and either (a) $f \geq 0$ or (b) $f \in L^1(\lambda)$, then f_x is \mathcal{N} -measurable for a.e. x and f^y is \mathcal{M} -measurable for a.e. y , and in case (b) f_x and f^y are also integrable for a.e. x and y . Moreover, $x \mapsto \int f_x d\nu$ and $y \mapsto \int f^y d\mu$ are measurable, and in case (b) also integrable, and

$$\int f d\lambda = \int \int f(x, y) d\mu(x) d\nu(y) = \int \int f(x, y) d\nu(y) d\mu(x).$$

Proof of Lemma (a). By theorem 2.36, the functions $x \mapsto \nu(E_x)$ and $y \mapsto \mu(E^y)$ are measurable and

$$0 = \mu \otimes \nu(E) = \int \nu(E_x) d\mu(x) = \int \mu(E^y) d\nu(y),$$

whence by nonnegativity $\nu(E_x) = 0$ for a.e. x and $\mu(E^y) = 0$ for a.e. y . □

Proof of Lemma (b). Consider the set $E = \{(x, y) : f(x, y) \neq 0\}$. Then, by definition, there exists an $F \in \mathcal{M} \otimes \mathcal{N}$ such that $E \subseteq F$ and $\mu \otimes \nu(F) = 0$. Then, for all $x \in X$ and $y \in Y$, $F_x \in \mathcal{N}$ and $F^y \in \mathcal{M}$. As $E_x \subseteq F_x$ and $E^y \subseteq F^y$, the completeness of μ and ν gives that $E_x \in \mathcal{N}$ and $E^y \in \mathcal{M}$. Combining this with lemma (a), we find that $\nu(E_x) = 0$ and $\mu(E^y) = 0$. Now, for $x \in X$, $f_x = 0$ on $Y \setminus E_x$ whence $f_x = 0$ ν -a.e. and for $y \in Y$, $f^y = 0$ on $X \setminus E^y$ whence $f^y = 0$ μ -a.e. Thus f^y and f_x are integrable and both integrate to zero. □

Proof of Theorem 2.39. Suppose that f is \mathcal{L} -measurable. By proposition 2.12, there exists an $\mathcal{M} \otimes \mathcal{N}$ -measurable function g such that $f = g$ λ -a.e. As $\mathcal{L} \supseteq \mathcal{M} \otimes \mathcal{N}$, it follows that g is \mathcal{L} -measurable so $f - g$ is \mathcal{L} -measurable and equal to 0 λ -a.e.

By lemma (b), $f_x - g_x$ and $f^y - g^y$ are integrable for a.e. x and y and $\int f_x - g_x d\nu = 0$ and $\int f^y - g^y d\mu = 0$ for a.e. x and y . Now the claims for parts (a) and (b) of the theorem follow from their corresponding parts in Theorem 2.37. □

Problem 3

(a): Suppose (X, Σ, μ) is a σ -finite measure space and $f \in L^+(X)$. Let

$$G_f = \{(x, y) \in X \times [0, +\infty] : y \leq f(x)\}.$$

Show that G_f is $\Sigma \times \mathcal{B}_{\mathbb{R}}$ -measurable and $\mu \times m(G_f) = \int f d\mu$. Show also that the same is true if the inequality in the definition of G_f is made strict.

Proof. Let $\tilde{f} : X \times [0, +\infty] \rightarrow X \times [0, +\infty]$ be given by $(x, y) \mapsto (f(x), y)$ and $S : X \times [0, +\infty] \rightarrow [-\infty, +\infty]$ be given by $S(z, y) = z - y$ if z, y not both $\pm\infty$ and $S(z, y) = 0$ if $z = y = \infty$. Then S is measurable, and as $\pi_1 \circ \tilde{f}$ and $\pi_2 \circ \tilde{f}$ are measurable, so is \tilde{f} . Hence, as intermediate codomain and domain match for the corresponding measure spaces, $S \circ \tilde{f}$ is measurable. Noting that $G_f = (S \circ \tilde{f})^{-1}([0, +\infty])$, measurability of

$S \circ \tilde{f}$ implies that G_f is $\Sigma \times \mathcal{B}_{\mathbb{R}}$ -measurable.

Observe that, for $x \in X$, $m((G_f)_x) = m([0, f(x)]) = f(x)$. As G_f is measurable, by Theorem 2.36 in Folland, the function $x \mapsto m((G_f)_x)$ is measurable and

$$\mu \times m(G_f) = \int m((G_f)_x) d\mu(x) = \int f(x) d\mu(x).$$

□

(b): Let (X, μ) be a σ -finite measure space. Fix $p \in [1, +\infty)$. Show that if $f \in L^p(X, \mu)$, then

$$\|f\|_p^p = p \int_0^\infty t^{p-1} \mu(\{x : |f(x)| > t\}) dt.$$

Proof. Observe that, by part (a),

$$\|f\|_p^p = \int_X |f|^p d\mu = (\mu \times m)(G_{|f|^p}) = \int_{X \times [0, +\infty]} \mathbb{1}_{G_{|f|^p}}(x, t) d(\mu \times m)(x, t)$$

As $\|f\|_p^p < +\infty$, it follows that $\mathbb{1}_{G_{|f|^p}} \in L^1(X \times [0, +\infty], \mu \times m)$ whence by Fubini's theorem $(\mathbb{1}_{G_{|f|^p}})^t \in L^1(X, \mu)$ for almost every $t \in [0, +\infty]$, the a.e. defined function $\int (\mathbb{1}_{G_{|f|^p}})^t d\mu \in L^1([0, +\infty], m)$, and

$$\begin{aligned} \|f\|_p^p &= \int_{X \times [0, +\infty]} \mathbb{1}_{G_{|f|^p}} d(\mu \times m) = \int_0^\infty \left[\int_X (\mathbb{1}_{G_{|f|^p}})^t(x) d\mu(x) \right] dt \\ &= \int_0^\infty \left[\int_X (\mathbb{1}_{(G_{|f|^p})^t})(x) d\mu(x) \right] dt = \int_0^\infty \mu((G_{|f|^p})^t) dt \\ &= \int_0^\infty \mu(\{x : |f(x)|^p < t\}) dt \end{aligned}$$

Consider the functions $F : [0, +\infty] \rightarrow [0, \infty]$ and $\phi : [0, +\infty] \rightarrow [0, +\infty]$ given by $F(t) = \mu(\{x : |f(x)|^p > t\})$ and $\phi(t) = t^p$. For t nonnegative, observe that $\{x : |f(x)|^p > t^p\} = \{x : |f(x)| > t\}$, so $(F \circ \phi)(t) = \mu(\{x : |f(x)|^p > t\}) = \mu(\{x : |f(x)| > t\})$. Lastly, noting that F is measurable and ϕ is a C^1 -diffeomorphism, it follows that

$$\|f\|_p^p = \int_0^\infty F(t) dt = \int_0^\infty (F \circ \phi)(t) |\det D_t \phi| dt = p \int_0^\infty t^{p-1} \mu(\{x : |f(x)| > t\}) dt.$$

□

(c): Let (X, μ) be a σ -finite measure space. Show that if $f, g \in L^1(X, \mu)$ with $0 \leq f, g$ a.e., then

$$\|f - g\|_1 = \int_0^\infty \mu(\{x : f(x) > t\}) \Delta \mu(\{x : g(x) > t\}) dt.$$

Suggestion: it might be helpful to first show that for $a, b \in [0, +\infty)$ we have

$$|a - b| = \int_0^\infty |\mathbb{1}_{(t, \infty)}(a) - \mathbb{1}_{(t, \infty)}(b)| dt$$

Proof.

$$\begin{aligned}
\|f - g\|_1 &= \int_X |f - g| d\mu = \int_X \int_0^\infty |\mathbb{1}_{(t,+\infty)}(f(x)) - \mathbb{1}_{(t,+\infty)}(g(x))| dt d\mu(x) \\
&= \int_0^\infty \left[\int_X |\mathbb{1}_{f^{-1}((t,+\infty))}(x) - \mathbb{1}_{g^{-1}((t,+\infty))}(x)| \right] d\mu(x) dt \\
&= \int_0^\infty \left[\int_X \mathbb{1}_{f^{-1}((t,+\infty)) \Delta g^{-1}((t,+\infty))}(x) \right] d\mu(x) dt \\
&= \int_0^\infty \mu(\{x : f(x) > t\} \Delta \{x : g(x) > t\}) dt
\end{aligned}$$

□

Problem 4

If f is Lebesgue integrable on $(0, a)$ and $g(x) = \int_x^a t^{-1} f(t) dt$, then g is integrable on $(0, a)$ and $\int_0^a g(x) dx = \int_0^a f(x) dx$.

Proof. Define a set $E = \{(x, t) \in (0, a)^2 : x < t\}$. This set is measurable. Then, for fixed $x \in (0, a)$, $\mathbb{1}_{(x, a)}(t) = \mathbb{1}_{E_x}(t) = (\mathbb{1}_E)_x(t)$. Then we compute,

$$\begin{aligned}
\int_{(0, a)} |g(x)| dx &= \int_{(0, a)} \left| \int_{(0, a)} t^{-1} f(t) \mathbb{1}_E(x, t) dt \right| dx \leq \int_{(0, a)} \int_{(0, a)} |t^{-1} f(t)| \mathbb{1}_{E^t}(x) dx dt \\
&= \int_{(0, a)} t^{-1} |f(t)| m(E^t) dt = \int_{(0, a)} |f(t)| dt < +\infty
\end{aligned}$$

whence by Tonelli's theorem g is measurable. So, we may apply Fubini's theorem.

$$\begin{aligned}
\int_{(0, a)} g(x) dx &= \int_{(0, a)} \int_{(0, a)} t^{-1} f(t) \mathbb{1}_E(x, t) dt dx = \int_{(0, a)} \int_{(0, a)} t^{-1} f(t) \mathbb{1}_{E^t}(x) dx dt \\
&= \int_{(0, a)} t^{-1} f(t) m(E^t) dt = \int_{(0, a)} f(t) dt
\end{aligned}$$

as desired. □

Problem 5

Let \mathcal{E}_q be the set of products of the form $\prod_{j=1}^d I_j$ where each I_j is an h -interval with the property that all of its finite endpoints are rational.

(a): Show that \mathcal{E}_q is an elementary family which generates the Borel sets.

Proof. Intersections of rational intervals are rational intervals and intersections of products of sets are componentwise intersections, so \mathcal{E}_q is closed under intersections.

The complement of an element of \mathcal{E}_q is a disjoint union of finitely many boxes. Thus, \mathcal{E}_q is an elementary family.

Any h -box in \mathbb{R}^d can be written as a countable union of elements of \mathcal{E}_q , so \mathcal{E}_q generates the Borel sets in \mathbb{R}^d . □

(b): Suppose that μ is a Borel measure on \mathbb{R}^d with $0 < \mu((0, 1]^d) < +\infty$. If $\mu(E + x) = \mu(E)$ for every $x \in \mathbb{R}^d$, show that $\mu(E) = \mu((0, 1]^d)m(E)$ for every Borel $E \subseteq \mathbb{R}^d$.

Proof. As in the one dimensional case, we may disjointly cover $(0, 1]^d$ by n^d translates of the box $(0, \frac{1}{n}]^d$, whence

$$\mu((0, \frac{1}{n}]^d) = \mu((0, 1]^d) \frac{1}{n^d} = \mu((0, 1]^d)m((0, \frac{1}{n}]^d).$$

For $n, q \in \mathbb{N}$, we may disjointly cover $(0, \frac{q}{n}]^d$ by q^d many translates of the box $(0, \frac{1}{n}]^d$, whence

$$\mu((0, \frac{q}{n}]^d) = q^d \mu((0, \frac{1}{n}]^d) = \mu((0, 1]^d) \frac{q^d}{n^d} = \mu((0, 1]^d)m((0, \frac{q}{n}]^d).$$

Now, any rational box may be filled and computed as such from disjoint coverings of translates of boxes of the form $(0, \frac{q}{n}]^d$ and thus obtaining the corresponding formula.

Consider the algebra $\mathcal{A} = \{\text{finite disjoint unions of elements of } \mathcal{E}_q\}$. Note that, by part (a), $\Sigma(\mathcal{A}) = \mathcal{B}_{\mathbb{R}^d}$. Consider the measure $\nu : \mathcal{B}_{\mathbb{R}^d} \rightarrow [0, +\infty]$ given by $\nu(E) = \mu((0, 1]^d) \cdot m(E)$ for $E \in \mathcal{B}_{\mathbb{R}^d}$.

We have shown that $\mu|_{\mathcal{A}} = \nu|_{\mathcal{A}}$, whence $\mu = \nu$, as desired. □

Problem 6

Fix $d \in \mathbb{N}$.

(a): Let $s : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be the map $s(x, y) = x + y$. Let μ, ν be finite, Borel measures on \mathbb{R}^d . Define $\mu * \nu = s_*(\mu \otimes \nu)$. Show that for every Borel $E \subseteq \mathbb{R}^d$ we have

$$\mu * \nu(E) = \int \int \mathbb{1}_E(x + y) d\mu(x) d\nu(y)$$

and

$$\int \mu(E - y) d\nu(y) = \mu * \nu(E) = \int \nu(E - x) d\mu(x).$$

Show as a consequence that

$$\mu * \nu(X) = \mu(X)\nu(X).$$

Proof. On one hand, by finiteness of the measures and measurability of E , we may apply Fubini's theorem to see that

$$\int \int \mathbb{1}_E(s(x, y)) d\mu(x) d\nu(y) = \int \int \mathbb{1}_{s^{-1}(E)}(x, y) d\mu(x) d\nu(y) = \int \int \mathbb{1}_{s^{-1}(E)} d(\mu \otimes \nu) = \mu * \nu(E).$$

Moreover, noting that $(s^{-1}(E))^y = E - y$ and $(s^{-1}(E))_x = E - x$, theorem 2.36 gives that

$$\mu * \nu(E) = \mu \otimes \nu(s^{-1}(E)) = \int \nu(E - x) d\mu(x)$$

and

$$\mu * \nu(E) = \mu \otimes \nu(s^{-1}(E)) = \int \mu(E - y) d\nu(y)$$

It follows that

$$\mu * \nu(\mathbb{R}^d) = \int \mu(\mathbb{R}^d - y) d\nu(y) = \int \mu(\mathbb{R}^d) d\nu(y) = \mu(\mathbb{R}^d)\nu(\mathbb{R}^d).$$

□

(b): Show that for finite, Borel measures μ, ν, η on \mathbb{R}^d we have

$$(\mu * \nu) * \eta = \mu * (\nu * \eta).$$

Proof. Let $E \in \mathcal{B}_{\mathbb{R}^d}$. By the finiteness of the measures, we may apply Fubini freely, whence

$$\begin{aligned} ((\mu * \nu) * \eta)(E) &= \int (\mu * \nu)(E - z) d\eta(z) = \int \left[\int \nu(E - z - x) d\mu(x) \right] d\eta(z) \\ &= \int \left[\int \nu(E - x - z) d\eta(z) \right] d\mu(x) = \int (\nu * \eta)(E - x) d\mu(x) = (\mu * (\nu * \eta))(E). \end{aligned}$$

□

(c): For $f, g \in L^1(\mathbb{R}^d)$ show that

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(y)g(x - y)| dx dy = \|f\|_1 \|g\|_1.$$

Explain why this implies that $y \mapsto f(y)g(x - y)$ is in $L^1(\mathbb{R}^d)$ for almost every $x \in \mathbb{R}^d$ and why if we set $f * g(x) = \int_{\mathbb{R}^d} f(y)g(x - y) dy$ then we have that $f * g \in L^1(\mathbb{R}^d)$ and

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1.$$

Proof.

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(y)g(x - y)| dx dy = \int_{\mathbb{R}^d} |f(y)| \left[\int_{\mathbb{R}^d} |g(x - y)| dx \right] dy = \int_{\mathbb{R}^d} |f(y)| \left[\int_{\mathbb{R}^d} |g(x)| dx \right] dy = \|f\|_1 \|g\|_1$$

As $f, g \in L^1(\mathbb{R}^d)$, $\|f\|_1 \|g\|_1 < +\infty$ the above equality and Fubini's theorem give that the function $x \mapsto \int |f(y)g(x - y)| dy$ is $L^1(\mathbb{R}^d)$ for almost every $x \in \mathbb{R}^d$. Hence, $\{x : \int |f(y)g(x - y)| dy = +\infty\}$ is a null set, so $y \mapsto f(y)g(x - y)$ is in $L^1(\mathbb{R}^d)$ for almost every $x \in \mathbb{R}^d$. Thus,

$$\|f * g\|_1 = \int \left| \int f(y)g(x - y) dy \right| dx \leq \int \int |f(y)g(x - y)| dy dx = \|f\|_1 \|g\|_1.$$

□

(d): Adopt notation as in Problem 1 of HW5. Show that if $f, g \in L^1(\mathbb{R}^d)$ are nonnegative then $(f dm) * (g dm) = f * g dm$ with m being the Lebesgue measure.

Proof. Let $E \subseteq \mathbb{R}^d$ be measurable and let $d\mu = f dm$, $d\nu = g dm$, and $d\lambda = f * g dm$. On one hand, we compute that

$$\lambda(E) = \int_E (f * g)(y) dy = \int_{\mathbb{R}^d} \mathbb{1}_E(y) \left[\int_{\mathbb{R}^d} f(x)g(y - x) dx \right] dy.$$

On the other hand, by part (a), we have that

$$\begin{aligned} \mu * \nu(E) &= \int_{\mathbb{R}^d} \nu(E - x) d\mu(x) = \int_{\mathbb{R}^d} f(x) \left[\int_{E - x} g(y) dy \right] dx \\ &= \int_{\mathbb{R}^d} \left[\int_E f(x)g(y - x) dy \right] dx = \int_{\mathbb{R}^d} f(x) \int_{\mathbb{R}^d} \mathbb{1}_E(y) f(x)g(y - x) dy dx, \end{aligned}$$

which by nonnegativity and applying Tonelli's theorem, it follows that these two computed quantities are in fact equal. □

(e): Show that for $f, g, k \in L^1(\mathbb{R}^d)$ we have that

$$(f * g) * k = f * (g * k) \text{ almost everywhere.}$$

Proof. Applying associativity from part (b) and expanding using part (d), we observe that

$$\begin{aligned} (f * g) * k \, dm &= (f * g \, dm) * k \, dm = (f \, dm * g \, dm) * k \, dm = f \, dm * (g \, dm * k \, dm) \\ &= f \, dm * (g * k) \, dm = f * (g * k) \, dm. \end{aligned}$$

By uniqueness of Radon-Nikodym derivatives, it follows that $(f * g) * k = f * (g * k)$ almost everywhere. \square