MATH 7310 Homework 4

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Problem 1

(i): Let $(X, \Sigma), (Y, \mathcal{F})$ be two measurable spaces and let $\phi : X \to Y$ be measurable. Given a measure ν on Σ, define $\phi_*(\nu) : \mathcal{F} \to [0, +\infty]$ by $\phi_*(\nu)(E) = \nu(\phi^{-1}(E))$. Prove that $\phi_*(\nu)$ is a measure.

(ii): If $x \in [0,1]$, a *binary expansion* for x is a sequence $(a_n)_{n=1}^\infty \in \{0,1\}^{\mathbb{N}}$ so that $x = \sum_{n=1}^\infty a_n 2^{-n}$. Let N be the set of $x \in [0,1]$ whose binary expansion is not unique. Show that N is a Borel set of measure 0.

Proof. The set of all points in $[0,1]$ with nonunique binary expansion is precisely the set of all points of the form 2^{-n} for $n \in \mathbb{N} \cup \{0\}$. Thus, $N = \bigcup_{n=0}^\infty \{2^{-n}\}$ is Borel as singletons are Borel. As N is a countable set, it follows that $m(N) = 0$. $\square$

(iii): Let $C \subseteq [0,1]$ be the middle thirds Cantor set. For $k \in \mathbb{N}$, define

$$\phi_k, \phi : [0,1] \setminus N \to \mathbb{R}$$

by $\phi_k(\sum_{n=1}^\infty a_n 2^{-n}) = \sum_{n=1}^k 2a_n 3^{-n}$ and $\phi(\sum_{n=1}^\infty a_n 2^{-n}) = \sum_{n=1}^\infty 2a_n 3^{-n}$ for all $(a_n)_{n=1}^\infty \in \{0,1\}^{\mathbb{N}}$. Show that ϕ_k, ϕ are Borel and that $\phi_k([0,1] \setminus N)$ and $\phi([0,1] \setminus N)$ are subsets of C.

Proof. Noting that $\phi = \sup_k \phi_k$, it suffices to show that each ϕ_k is Borel. We claim that ϕ_k is in fact a finite linear combination of step functions. For $(a_1, \ldots, a_k) \in \{0,1\}^k$ consider the set

$$I_{(a_1,\ldots,a_k)} = \left\{ x \in [0,1] \setminus N : x = \sum_{n=1}^k a_k 2^{-k} + \sum_{j=k+1}^\infty b_j 2^{-j} \text{ where } b_j \in \{0,1\} \right\}$$

Note that ϕ_k is constant on each I_a for and is equal to $\sum_{n=1}^k a_n 2^{-n}$ for each $a \in \{0,1\}^k$, and each $x \in [0,1] \setminus N$ lies in some I_k, so

$$\phi_k = \sum_{(a_1,\ldots,a_k) \in \{0,1\}^k} \left(\sum_{n=1}^k a_n 2^{-n} \right) \mathbb{1}_{I_{(a_1,\ldots,a_k)}}.$$

Note that each element of the images of ϕ and ϕ_k have ternary decompositions with only 0s or 2s, so they clearly lie in the Cantor set. $\square$

(iv): Set $\mu = \phi_*(m)$, where m is the Lebesgue measure on $[0,1]$. Show that $\mu(C^c) = 0$ and that there is a unique, increasing continuous function $f : [0,1] \to [0,1]$ so that $f(0) = 0$ and $\mu([a,b]) = f(b) - f(a)$ for all $0 \le a < b \le 1$. (In particular, $f(1) = 1$).

Proof. Note that $\mu(C^c) = m(\phi^{-1}(C^c)) = m(\emptyset) = 0$. Moreover, μ is a Borel measure which is finite on compact sets, so there exists an increasing right continuous function $f : [0,1] \to [0,1]$ such that $\mu = \mu_f$, namely, $f(x) = \mu((0,x])$. Then $f(0) = 0$. As such f is determined up to a constant and $f(0)$ has been specified, f is unique. That f is continuous follows from the previous homework and the fact that μ is diffuse. □

(v): Show that $f(2\sum_{n=1}^{k} a_n 3^{-n}) = \sum_{n=1}^{k} a_n 2^{-n}$ for all $k \in \mathbb{N}$ and all $(a_n)_{n=1}^{k} \in \{0,1\}^k$. If (a,b) is an open interval disjoint from C, show that $f(b) = f(a)$.

Proof. $(a,b) \subseteq C^c$, so $\mu((a,b)) \leq \mu(C^c) = 0$. □

Problem 2

Let $f : [0,1] \to [0,1]$ be the Cantor function, and let $g(x) = f(x) + x$.

(a): Prove that g is a bijection from $[0,1]$ to $[0,2]$ and $h = g^{-1}$ is a continuous map from $[0,2]$ to $[0,1]$.

Proof. Note that g is strictly monotone increasing and continuous, so g is injective. As $g(1) = 2$, by the intermediate value theorem g is surjective. Morover, as g is a strictly monotone bijection between intervals, g^{-1} is continous. □

(b): If C is the Cantor set, $m(g(C)) = 1$.

Proof. Write $C^c = \bigcup_{j=1}^{\infty} I_j$ where $I_j = (a_j, b_j)$ for some $a_j < b_j$. Note that, as $(a_j, b_j) \subseteq C^c$, so f is constant on (a_j, b_j). Moreover,

$$g|_{I_j}(x) = f(x) + x = a_j + x \implies g(I_j) = I_j$$

$$m(g(C)) = 2 - m(g(C^c)) = 2 - \sum_{j=1}^{\infty} m(g(I_j)) = 2 - \sum_{j=1}^{\infty} m(I_j) = 2 - m(C^c) = 2 - 1 = 1$$

□

(c): By exercise 29 of chapter 1, $g(C)$ contains a Lebesgue nonmeasurable set A. Let $B = g^{-1}(A)$. Then B is Lebesgue measurable but not Borel.

Proof. To see that B is Lebesgue measurable, note that $B = g^{-1}(A) \subseteq g^{-1}(g(C)) = C$ and $m(C) = 0$, so by completeness, B is Lebesgue measurable. As g is a homeomorphism and $A = g(B)$, if B were Borel then so would A be, which is absurd as A is non-measurable. □

(d): There exist a Lebesgue measurable function F and a continuous function G on $\mathbb{R}$ such that $F \circ G$ is not Lebesgue measurable.

Proof. Take $F = \mathbb{1}_B$. As B is Lebesgue measurable, so is $\mathbb{1}_B$. Take $G = g$. Then $(F \circ G)^{-1}(\{1\}) = (G^{-1} \circ F^{-1})(\{1\}) = g^{-1}(B) = A$, which is not measurable, so $F \circ G$ is not Lebesgue measurable. □

Problem 3

Prove that the following hold if and only if the measure μ is complete:

(a): If f is measurable and $f = g$ μ-a.e., then g is measurable.

Proof.
$\underline{\Longrightarrow}$: Suppose that μ is complete. Let f be measurable and suppose that $f = g$ almost everywhere. Let $N = \{x : f(x) \neq g(x)\}$. If $E \subseteq \mathbb{C}$ is measurable, then

$$g^{-1}(E) = \{x : g(x) \in E \text{ and } f(x) = g(x)\} \cup \{x : g(x) \in E \text{ and } f(x) \neq g(x)\} = f^{-1}(E) \cup (g^{-1}(E) \cap N),$$

which is measurable by completeness of μ and measurability of f.

$\underline{\Longleftarrow}$: Let $N \in \Sigma$ with $\mu(N) = 0$ and suppose that $F \subseteq N$. Then $\mathbb{1}_N = \mathbb{1}_F$ on $X \setminus N$, so $\mathbb{1}_N = \mathbb{1}_F$ μ-a.e. whence by hypothesis $\mathbb{1}_F$ is measurable so $F \in \Sigma$. $\square$

(b): If f_n is measurable for $n \in \mathbb{N}$ and $f_n \to f$ μ-a.e., then f is measurable.

Proof.
$\underline{\Longrightarrow}$: Suppose μ is complete. Let f_n is measurable for $n \in \mathbb{N}$ and $f_n \to f$ μ-a.e. Let $N = \{x : f_n(x) \not\to f(x)\}$. Then $f_n \mathbb{1}_{X\setminus N} \to f \mathbb{1}_{X \setminus N}$ pointwise. Thus, $f \mathbb{1}_{X\setminus N}$ is measurable. But $f \mathbb{1}_{X\setminus N} = f$ μ-a.e., so by part (a), f is measurable.

$\underline{\Longleftarrow}$: Let $N \in \Sigma$ such that $\mu(N) = 0$. Suppose that $E \in \Sigma$. Then there exist simple functions $\{\phi_n\}_{n=1}^{\infty}$ with $0 \le |\phi_1| \le |\phi_2| \le \cdots \le |\mathbb{1}_N|$ and $\phi_n \to \mathbb{1}_N$ pointwise. On $X \setminus N$, $\phi_n \mathbb{1}_N = \phi_n \mathbb{1}_F$, so $\phi_n \mathbb{1}_N = \phi_n \mathbb{1}_F$ almost everywhere whence by part (a) $\phi_n \mathbb{1}_F$ is measurable. On $X \setminus N$, $\phi_n \mathbb{1}_F \to \mathbb{1}_N \mathbb{1}_F = \mathbb{1}_N$, so $\phi_n \to \mathbb{1}_F$ almost everywhere, whence by assumption $\mathbb{1}_F$ is measurable i.e. F is measurable.

$\square$

Problem 4

If $f \in L^+$ and $\int f \, d\mu < +\infty$, show that $\{x : f(x) = \infty\}$ is a null set and that $\{x : f(x) > 0\}$ is σ-finite.

Proof. Suppose, for the sake of contradiction, that $\mathcal{N} := \{x : f(x) = \infty\} = f^{-1}(\{\infty\}) \in \Sigma$ has positive measure. Let $\{\phi_n\}_{n\in\mathbb{N}}$ be a sequence of simple functions (valued in $[0, +\infty]$) with $0 \le \phi_1 \le \phi_2 \le \cdots \le f$ such that $\phi_n \to f$ pointwise. For $n \in \mathbb{N}$, define a new simple function ϕ_n' by

$$\phi_n' = \phi_n \mathbb{1}_{X\setminus\mathcal{N}} + n \cdot \mathbb{1}_{\mathcal{N}}.$$

Note that, as $\phi_n \equiv \phi_n'$ on $X \setminus \mathcal{N}$ and $\phi_n' \le f$ on $\mathcal{N}$, it follows that $0 \le \phi_1' \le \phi_2' \le \cdots \le f$ as well. Moreover, for $n \in \mathbb{N}$, as $\phi_n' \ge n \cdot \mathbb{1}_{\mathcal{N}}$, we have that

$$\int f \, d\mu \ge \int \phi_n' \, d\mu \ge \int n \cdot \mathbb{1}_{\mathcal{N}} \, d\mu = n \cdot \mu(\mathcal{N}) \to \infty \text{ as } n \to \infty.$$

Thus, $\int f \, d\mu = +\infty$, contradicting the assumption.

Let $X = \{x : f(x) > 0\}$ and consider the sets $\{A_n\}_{n=0}^{\infty}$ given by $A_0 = f^{-1}(\{\infty\})$, $A_n = f^{-1}([\frac{1}{n}, \frac{1}{n-1}))$ for $n \ge 1$. Then

$$X = \bigsqcup_{n=0}^{\infty} A_n$$

Suppose, for the sake of contradiction, that X is not σ-finite. Then, as $\mu(A_0) = 0$, some A_k for $k \geq 1$ must have infinite measure. As $f \geq f \cdot \mathbb{1}_{A_k} \geq \frac{1}{n}\mathbb{1}_{A_k}$, it follows that

$$\int f \, d\mu \geq \int f \cdot \mathbb{1}_{A_k} \, d\mu \geq \int \frac{1}{n}\mathbb{1}_{A_k} \, d\mu = \frac{1}{n}\mu(A_k) = \infty,$$

contradicting the assumption that $\int f \, d\mu < \infty$. $\square$

Problem 5

If $f \in L^+$, let $\lambda(E) = \int_E f \, d\mu$ for $E \in \Sigma$. Prove that λ is a measure on Σ, and that for any $g \in L^+$, $\int g \, d\lambda = \int fg \, d\mu$.

Proof. We first show that λ is a measure. Note that $\mathbb{1}_\emptyset$ is the zero function on X, so $\lambda(\emptyset) = \int_\emptyset f \, d\mu = \int f \mathbb{1}_\emptyset \, d\mu = 0$. If $E, F \in \Sigma$ are such that $E \subseteq F$, then $\mathbb{1}_E \leq \mathbb{1}_F \implies f\mathbb{1}_E \leq f\mathbb{1}_F$, so by monotonicity,

$$\lambda(E) = \int f\mathbb{1}_E \, d\mu \leq \int f\mathbb{1}_F \, d\mu = \lambda F.$$

Lastly, suppose that $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of disjoint elements of Σ. Set $A = \bigsqcup_{i=1}^\infty A_i$. Let $f_n = f \cdot \mathbb{1}_{\bigsqcup_{i=1}^n A_i}$. Then $0 \leq f_1 \leq f_2 \leq \cdots \leq f \cdot \mathbb{1}_A$ and $f_n \to f\mathbb{1}_A$ pointwise. By the monotone convergence theorem,

$$\lambda(A) = \int f\mathbb{1}_A \, d\mu = \lim_{n\to\infty} \int f\mathbb{1}_{\bigsqcup_{i=1}^n A_i} \, d\mu = \lim_{n\to\infty} \sum_{i=1}^n \int f\mathbb{1}_{A_i} \, d\mu = \sum_{i=1}^\infty \lambda(A_i).$$

Suppose that g is a simple function. Write $g = \sum_{i=1}^n c_i \mathbb{1}_{E_i}$ where $E_i \in \Sigma$ and $c_i \in [0, \infty)$. By definition,

$$\int g \, d\lambda = \sum_{i=1}^n c_i \lambda(E_i) = \sum_{i=1}^n c_i \int f\mathbb{1}_{E_i} \, d\mu = \int f \left(\sum_{i=1}^n c_i \mathbb{1}_{E_i} \right) d\mu = \int fg \, d\mu \, .$$

Now suppose that $g \in L^+$ is arbitrary. Then there exist a sequence of simple functions $0 \leq \phi_1 \leq \phi_2 \leq \cdots \leq g$ such that $\phi_n \to g$ pointwise. Then $0 \leq f\phi_1 \leq f\phi_2 \leq \cdots \leq fg$ and $f\phi_n \to fg$ pointwise. By applying the monotone convergence theorem twice, we see that

$$\int g \, d\lambda = \lim_{n\to\infty} \int \phi_n \, d\lambda = \lim_{n\to\infty} \int f\phi_n \, d\mu = \int fg \, d\mu \, .$$

$\square$

Problem 6

If $f \in L^+$ and $\int f \, d\mu < \infty$, show that for every $\varepsilon > 0$ there exists an $E \in \Sigma$ such that $\mu(E) < \infty$ and $\int_E f \, d\mu > (\int f \, d\mu) - \varepsilon$.

Proof. Let $\varepsilon > 0$. By definition, there exists a simple ϕ with $0 \leq \phi \leq f$ such that $\int \phi \, d\mu > (\int f \, d\mu) - \varepsilon$. Write ϕ as $\phi = \sum_{i=1}^n c_i \mathbb{1}_{E_i}$ for some $E_i \in \Sigma$ and $c_i \in [0, \infty)$. Note that, as $\sum_{i=1}^n c_i \mu(E_i) = \int \phi \, d\mu \leq \int f \, d\mu < \infty$, we have that $\mu(E_i) < \infty$ for all i. Set $E = \bigcup_{i=1}^n E_i$.

Noting that $\phi = \phi\mathbb{1}_E \leq f\mathbb{1}_E$, it follows that

$$\int_E f \, d\mu \geq \int f\mathbb{1}_E \, d\mu \geq \int \phi \, d\mu > (\int f \, d\mu) - \varepsilon$$

with $\mu(E) \leq \sum_{i=1}^n \mu(E_i) < \infty$ as desired. $\square$