

# MATH 7752 Homework 2

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## Problem 1

Let  $D$  be a division ring (not necessarily commutative) and  $M$  be a  $D$ -module.

(a) Let  $X$  be a generating set of  $M$  and  $Y$  a  $D$ -linearly independent subset of  $X$ . Prove that  $M$  has a  $D$ -basis  $B$  with  $Y \subseteq B \subseteq X$ .

*Proof.* Consider the poset  $\mathcal{S} = \{B \subseteq M : Y \subseteq B \subseteq X \text{ and } B \text{ is } D\text{-linearly independent}\}$  ordered by inclusion. Since  $Y$  is a  $D$ -linearly independent subset of  $X$ , we have that  $Y \in \mathcal{S}$  so  $\mathcal{S} \neq \emptyset$ .

Suppose that  $\mathcal{C} \subseteq \mathcal{S}$  is any linearly ordered chain in  $\mathcal{S}$ . Let  $B = \bigcup \mathcal{C}$ . Then  $Y \subseteq B \subseteq X$ . Suppose that  $d_i \in D$  and  $b_i \in B$  such that  $\sum_{i=1}^n d_i \cdot b_i = 0$ . Then for each  $i \in \{1, \dots, n\}$ , there exists a  $B_i \in \mathcal{C}$  such that  $b_i \in B_i$ . As  $\mathcal{C}$  is a chain, there is some  $l \in \{1, \dots, n\}$  such that  $B_i \subseteq B_l$  for all  $1 \leq i \leq n$ . It follows that  $b_i \in B_l$  for all  $1 \leq i \leq n$ , whence  $B_l$  being  $D$ -linearly independent implies that  $d_i = 0$  for all  $i$ . Thus  $B$  is  $D$ -linearly independent, so  $B \in \mathcal{S}$ .

Now by Zorn's lemma, there exists a maximal element  $B \in \mathcal{S}$  of  $\mathcal{S}$ . We claim that  $B$  is in fact a  $D$ -basis for  $M$ . It suffices to show that  $B$  is a generating set for  $M$ . Let  $N = \text{span}_D(B)$ . Suppose, for the sake of contradiction, that  $N \neq M$ . As  $B \subseteq X$  and  $X$  is a generating set for  $M$ , it follows that there exists an  $x \in X \setminus \text{span}_D(B)$ . Suppose  $r, r_1, \dots, r_n \in R$  are such that

$$0 = rx + r_1 b_1 + \dots + r_n b_n.$$

If  $r \neq 0$ , then

$$x = (-r^{-1}r_1) \cdot b_1 + \dots + (-r^{-1}r_n) \cdot b_n,$$

which would imply that  $x \in \text{span}_D(B)$ , contradicting the choice of  $x$ . Hence  $r = 0$ , so  $B$  being  $D$ -linearly independent implies that  $r_i = 0$  for all  $i$ . Thus  $B \cup \{x\}$  is  $D$ -linearly independent, contradicting the maximality of  $B$ .  $\square$

(b) Conclude that every non zero  $D$ -module  $M$  has a  $D$ -basis.

*Proof.* Since  $M \neq 0$ , there exists an  $y \in M \setminus \{0\}$ . It follows that the singleton  $\{y\}$  is a  $D$ -linearly independent subset of  $M$ . On the other hand,  $M = 1 \cdot M$ , so the set  $M$  is a generating set of  $M$ . Applying part (a) to  $X = M$  and  $Y = \{y\}$ , it follows that  $M$  has a  $D$ -basis.  $\square$

## Problem 2

Let  $R$  be a commutative domain. Let  $I$  be a non-principal ideal of  $R$ . Show that when  $I$  is considered as an  $R$ -module (by left multiplication), then  $I$  is indecomposable but not cyclic.

*Proof.* Since  $I$  is non-principal, by definition  $I$  is not cyclic as an  $R$ -module. Suppose, for the sake of contradiction, that  $I = P \oplus Q$  for some nonzero proper  $R$ -submodules  $P, Q$  of  $I$ . Take  $p \in P \setminus \{0\}$  and  $q \in Q \setminus \{0\}$ . Then  $p \cdot q - q \cdot p = pq - qp = 0 \implies p \cdot q = q \cdot p$ . As  $R$  is a domain  $pq = qp \neq 0$ , whence  $pq = qp \in P \cap Q$  contradicts that the sum  $P \oplus Q$  is direct.  $\square$

## Problem 3

Let  $R$  be a commutative ring. An  $R$ -module  $M$  is called *torsion* if for any  $m \in M$  there exists some nonzero  $r \in R$  such that  $rm = 0$ . An  $R$ -module  $N$  is called *divisible* if for any nonzero  $r \in R$  it holds that  $rN = N$ .

(a) Suppose  $M$  is a torsion  $R$ -module and  $N$  is a divisible  $R$ -module. Prove that  $M \otimes_R N = \{0\}$ .

*Proof.* Let  $m \in M$  and  $n \in N$ . Since  $M$  is torsion, there exists a nonzero  $r \in R$  such that  $rm = 0$ . Now, by divisibility of  $N$ , there exists an  $n' \in N$  such that  $rn' = n$ . Hence

$$m \otimes n = m \otimes rn' = rm \otimes n' = 0 \otimes n' = 0.$$

Thus every simple tensor in  $M \otimes_R N$  is 0, whence  $M \otimes_R N = 0$ .  $\square$

(b) Consider the  $\mathbb{Z}$ -module  $M = \mathbb{Q}/\mathbb{Z}$ . Prove that  $M \otimes_{\mathbb{Z}} M = \{0\}$

*Proof.* We show that  $M$  is both torsion and divisible. Note that for any  $\frac{p}{q} + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$ ,  $q \neq 0$  and  $q \cdot (\frac{p}{q} + \mathbb{Z}) = q \cdot \frac{p}{q} + \mathbb{Z} = p + \mathbb{Z} = \mathbb{Z}$ , so  $\mathbb{Q}/\mathbb{Z}$  is torsion.

On the other hand, suppose  $n \in \mathbb{Z} \setminus \{0\}$ . For  $\frac{p}{q} + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$ , observe that

$$n \cdot \left( \frac{p}{nq} + \mathbb{Z} \right) = \frac{np}{nq} + \mathbb{Z} = \frac{p}{q} + \mathbb{Z}$$

so  $\mathbb{Q}/\mathbb{Z}$  is divisible. Appealing to part(a), it follows that  $M \otimes_{\mathbb{Z}} M = 0$ .  $\square$

## Problem 4

Let  $R$  be a PID and  $A$  be an  $R$ -module. Let  $K$  be the field of fractions of  $R$ , and consider the  $K$ -module  $B = K \otimes_R A$ . Prove that every  $z \in B$  is a simple tensor.

*Proof.* Let  $z \in B$ . Then there exists  $\frac{x_1}{s_1}, \dots, \frac{x_n}{s_n} \in K$ ,  $a_1, \dots, a_n \in A$  and  $c_1, \dots, c_n \in R$  such that

$$z = \sum_{i=1}^n c_i \cdot \left( \frac{x_i}{s_i} \otimes a_i \right) = \sum_{i=1}^n \frac{c_i x_i}{s_i} \otimes a_i = \sum_{i=1}^n \frac{c_i x_i \prod_{j \neq i} s_j}{s_1 \cdots s_n} \otimes a_i.$$

Since  $R$  is a PID, there exists an  $s \in R$  such that  $(s) = \langle c_i x_i \prod_{j \neq i} s_j : 1 \leq i \leq n \rangle$ . Then for each  $i \in \{1, \dots, n\}$ , there is an  $r_i \in R$  such that  $c_i x_i \prod_{j \neq i} s_j = r_i s$ . Hence,

$$z = \sum_{i=1}^n \frac{c_i x_i \prod_{j \neq i} s_j}{s_1 \cdots s_n} \otimes a_i = \sum_{i=1}^n \frac{r_i s}{s_1 \cdots s_n} \otimes a_i = \sum_{i=1}^n \frac{s}{s_1 \cdots s_n} \otimes r_i a_i = \frac{s}{s_1 \cdots s_n} \otimes \left( \sum_{i=1}^n r_i a_i \right)$$

is a simple tensor.  $\square$

## Problem 5

Let  $R$  be a commutative ring and  $M$  an  $R$ -module.

(a): Let  $I$  be an ideal of  $R$ . Prove an isomorphism

$$R/I \otimes M \simeq M/IM.$$

*Proof.* Consider the exact sequence

$$0 \rightarrow I \rightarrow R \rightarrow R/I \rightarrow 0.$$

Upon applying the right exact functor  $- \otimes_R M$  to the above exact sequence, we obtain the exact sequence

$$I \otimes_R M \rightarrow R \otimes_R M \cong M \rightarrow R/I \otimes_R M \rightarrow 0$$

The image of  $I \otimes_R M$  under the identification  $R \otimes_R M \cong M$  via  $r \otimes m \mapsto r \cdot m$  is  $IM$ , hence by the first isomorphism theorem  $M/IM \cong R/I \otimes_R M$ .  $\square$

(b): Suppose that  $M$  is a finitely generated free  $R$ -module. Show that the *rank* of  $M$  is well-defined, i.e. any two  $R$ -bases of  $M$  have the same number of elements.

*Proof.* Let  $\mathfrak{m} \subseteq R$  be a maximal ideal of  $R$ . Let  $k = R/\mathfrak{m}$  be the corresponding residue field. Suppose that  $\mathcal{B}$  is a basis of  $M$  over  $R$ . Then  $\mathcal{B}$  is also a basis of  $M/\mathfrak{m}M$  over  $k$ .  $\square$

## Problem 6

Let  $R \subseteq S$  be an inclusion of commutative rings. Consider the polynomial rings  $R[x]$  and  $S[x]$ . Prove that there is an isomorphism of  $S$ -modules,

$$S \otimes_R R[x] \rightarrow S[x]$$