MATH 7310 Homework 4

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Problem 1

(i): Let $(X, \Sigma), (Y, \mathcal{F})$ be two measurable spaces and let $\phi : X \to Y$ be measurable. Given a measure ν on Σ, define $\phi_*(\nu) : \mathcal{F} \to [0, +\infty]$ by $\phi_*(\nu)(E) = \nu(\phi^{-1}(E))$. Prove that $\phi_*(\nu)$ is a measure.

(ii): If $x \in [0, 1]$, a *binary expansion* for x is a sequence $(a_n)_{n=1}^{\infty} \in \{0, 1\}^{\mathbb{N}}$ so that $x = \sum_{n=1}^{\infty} a_n 2^{-n}$. Let N be the set of $x \in [0, 1]$ whose binary expansion is not unique. Show that N is a Borel set of measure 0.

Proof. The set of all points in $[0, 1]$ with nonunique binary expansion is precisely the set of all points of the form 2^{-n} for $n \in \mathbb{N} \cup \{0\}$. Thus, $N = \bigcup_{n=0}^{\infty} \{2^{-n}\}$ is Borel as singletons are Borel. As N is a countable set, it follows that $m(N) = 0$. $\square$

(iii): Let $C \subseteq [0, 1]$ be the middle thirds Cantor set. For $k \in \mathbb{N}$, define

$$\phi_k, \phi : [0, 1] \setminus N \to \mathbb{R}$$

by $\phi_k(\sum_{n=1}^{\infty} a_n 2^{-n}) = \sum_{n=1}^{k} 2a_n 3^{-n}$ and $\phi(\sum_{n=1}^{\infty} a_n 2^{-n}) == \sum_{n=1}^{\infty} 2a_n 3^{-n}$ for all $(a_n)_{n=1}^{\infty} \in \{0, 1\}^{\mathbb{N}}$. Show that ϕ_k, ϕ are Borel and that $\phi_k([0, 1] \setminus N)$ and $\phi([0, 1] \setminus N)$ are subsets of C.

(iv): Set $\mu = \phi_*(m)$, where m is the Lebesgue measure on $[0, 1]$. Show that $\mu(C^c) = 0$ and that there is a unique, increasing continuous function $f : [0, 1] \to [0, 1]$ so that $f(0) = 0$ and $\mu([a, b]) = f(b) - f(a)$ for all $0 \leq a < b \leq 1$. (In particular, $f(1) = 1$).

(v): Show that $f(2 \sum_{n=1}^{k} a_n 3^{-n}) = \sum_{n=1}^{k} a_n 2^{-n}$ for all $k \in \mathbb{N}$ and all $(a_n)_{n=1}^{k} \in \{0, 1\}^k$. If (a, b) is an open interval disjoint from C, show that $f(b) = f(a)$.

Problem 2

Let $f : [0, 1] \to [0, 1]$ be the Cantor function, and let $g(x) = f(x) + x$.

(a): Prove that g is a bijection from $[0, 1]$ to $[0, 2]$ and $h = g^{-1}$ is a continuous map from $[0, 2]$ to $[0, 1]$.

(b): If C is any Cantor set, $m(g(C)) = 1$.

(c): By exercise 29 of chapter 1, $g(C)$ contains a Lebesgue nonmeasurable set A. Let $B = g^{-1}(A)$. Then B is Lebesgue measurable but not Borel.

(d): There exist a Lebesgue measurable function F and a continuous function G on $\mathbb{R}$ such that $F \circ G$ is not Lebesge measurable.

Problem 3

Prove that the following hold if and only if the measure μ is complete:

(a): If f is measurable and $f = g$ μ-a.e., then g is measurable.

Proof.
$\underline{\Longrightarrow}$: Suppose that μ is complete. Let f be measurable and suppose that $f = g$ almost everywhere.
$\underline{\Longleftarrow}$: Suppose that μ is not complete. Then there exists a null set $N \in \Sigma$ and a subset $E \subseteq N$ such that $E \notin \Sigma$. $\square$

(b): If f_n is measurable for $n \in \mathbb{N}$ and $f_n \to f$ μ-a.e., then f is measurable.

Problem 4

If $f \in L^+$ and $\int f \, d\mu < +\infty$, show that $\{x : f(x) = \infty\}$ is a null set and that $\{x : f(x) > 0\}$ is σ-finite.

Proof. Suppose, for the sake of contradiction, that $\mathcal{N} := \{x : f(x) = \infty\} = f^{-1}(\{\infty\}) \in \Sigma$ has positive measure. Let $\{\phi_n\}_{n\in\mathbb{N}}$ be a sequence of simple functions (valued in $[0, +\infty]$) with $0 \leq \phi_1 \leq \phi_2 \leq \cdots \leq f$ such that $\phi_n \to f$ pointwise. For $n \in \mathbb{N}$, define a new simple function ϕ'_n by

$$\phi'_n = \phi_n \mathbb{1}_{X\setminus\mathcal{N}} + n \cdot \mathbb{1}_{\mathcal{N}}.$$

Note that, as $\phi_n \equiv \phi'_n$ on $X \setminus \mathcal{N}$ and $\phi'_n \leq f$ on $\mathcal{N}$, it follows that $0 \leq \phi'_1 \leq \phi'_2 \leq \cdots \leq f$ as well. Moreover, for $n \in \mathbb{N}$, as $\phi'_n \geq n \cdot \mathbb{1}_{\mathcal{N}}$, we have that

$$\int f \, d\mu \geq \int \phi'_n \, d\mu \geq \int n \cdot \mathbb{1}_{\mathcal{N}} \, d\mu = n \cdot \mu(\mathcal{N}) \to \infty \text{ as } n \to \infty.$$

Thus, $\int f \, d\mu = +\infty$, contradicting the assumption.

Let $X = \{x : f(x) > 0\}$ and consider the sets $\{A_n\}_{n=0}^{\infty}$ given by $A_0 = f^{-1}(\{\infty\})$, $A_n = f^{-1}([\frac{1}{n}, \frac{1}{n-1}))$ for $n \geq 1$. Then

$$X = \bigsqcup_{n=0}^{\infty} A_n$$

Suppose, for the sake of contradiction, that X is not σ-finite. Then, as $\mu(A_0) = 0$, some A_k for $k \geq 1$ must have infinite measure. As $f \geq f \cdot \mathbb{1}_{A_k} \geq \frac{1}{n}\mathbb{1}_{A_k}$, it follows that

$$\int f \, d\mu \geq \int f \cdot \mathbb{1}_{A_k} \, d\mu \geq \int \frac{1}{n}\mathbb{1}_{A_k} \, d\mu = \frac{1}{n}\mu(A_k) = \infty,$$

contradicting the assumption that $\int f \, d\mu < \infty$. $\square$

Problem 5

If $f \in L^+$, let $\lambda(E) = \int_E f \, d\mu$ for $E \in \Sigma$. Prove that λ is a measure on Σ, and that for any $g \in L^+$, $\int g \, d\lambda = \int fg \, d\mu$.

Proof. We first show that λ is a measure. Note that $\mathbb{1}_\emptyset$ is the zero function on X, so $\lambda(\emptyset) = \int_\emptyset f\, d\mu = \int f \mathbb{1}_\emptyset\, d\mu = 0$. If $E, F \in \Sigma$ are such that $E \subseteq F$, then $\mathbb{1}_E \leq \mathbb{1}_F \implies f\mathbb{1}_E \leq f\mathbb{1}_F$, so by monotonicity,

$$\lambda(E) = \int f\mathbb{1}_E\, d\mu \leq \int f\mathbb{1}_F\, d\mu = \lambda F.$$

Lastly, suppose that $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of disjoint elements of Σ. Set $A = \bigsqcup_{i=1}^\infty A_i$. Let $f_n = f \cdot \mathbb{1}_{\bigsqcup_{i=1}^n A_i}$. Then $0 \leq f_1 \leq f_2 \leq \cdots \leq f \cdot \mathbb{1}_A$ and $f_n \to f\mathbb{1}_A$ pointwise. By the monotone convergence theorem,

$$\lambda(A) = \int f\mathbb{1}_A\, d\mu = \lim_{n\to\infty} \int f\mathbb{1}_{\bigsqcup_{i=1}^n A_i}\, d\mu = \lim_{n\to\infty} \sum_{i=1}^n \int f\mathbb{1}_{A_i}\, d\mu = \sum_{i=1}^\infty \lambda(A_i).$$

Suppose that g is a simple function. Write $g = \sum_{i=1}^n c_i \mathbb{1}_{E_i}$ where $E_i \in \Sigma$ and $c_i \in [0,\infty)$. By definition,

$$\int g\, d\lambda = \sum_{i=1}^n c_i \lambda(E_i) = \sum_{i=1}^n c_i \int f\mathbb{1}_{E_i}\, d\mu = \int f \left(\sum_{i=1}^n c_i \mathbb{1}_{E_i} \right) d\mu = \int fg\, d\mu\,.$$

Now suppose that $g \in L^+$ is arbitrary. Then there exist a sequence of simple functions $0 \leq \phi_1 \leq \phi_2 \leq \cdots \leq g$ such that $\phi_n \to g$ pointwise. Then $0 \leq f\phi_1 \leq f\phi_2 \leq \cdots \leq fg$ and $f\phi_n \to fg$ pointwise. By applying the monotone convergence theorem twice, we see that

$$\int g\, d\lambda = \lim_{n\to\infty} \int \phi_n\, d\lambda = \lim_{n\to\infty} \int f\phi_n\, d\mu = \int fg\, d\mu\,.$$

□

Problem 6

If $f \in L^+$ and $\int f\, d\mu < \infty$, show that for every $\varepsilon > 0$ there exists an $E \in \Sigma$ such that $\mu(E) < \infty$ and $\int_E f\, d\mu > (\int f\, d\mu) - \varepsilon$.

Proof. Let $\varepsilon > 0$. By definition, there exists a simple ϕ with $0 \leq \phi \leq f$ such that $\int \phi\, d\mu > (\int f\, d\mu) - \varepsilon$. Write ϕ as $\phi = \sum_{i=1}^n c_i \mathbb{1}_{E_i}$ for some $E_i \in \Sigma$ and $c_i \in [0,\infty)$. Note that, as $\sum_{i=1}^n c_i \mu(E_i) = \int \phi\, d\mu \leq \int f\, d\mu < \infty$, we have that $\mu(E_i) < \infty$ for all i. Set $E = \bigcup_{i=1}^n E_i$.

Noting that $\phi = \phi\mathbb{1}_E \leq f\mathbb{1}_E$, it follows that

$$\int_E f\, d\mu \geq \int f\mathbb{1}_E\, d\mu \geq \int \phi\, d\mu > (\int f\, d\mu) - \varepsilon$$

with $\mu(E) \leq \sum_{i=1}^n \mu(E_i) < \infty$ as desired. □