Reading:

- For this homework: 5.5/3.1-3.2
- For Wedneday, March 30: 3.2-3.4
- For Monday, April 4: 3.4-3.5

Problem 1.

Folland, Chapter 3, Problem 20

Problem 2.

Folland, Chapter 3, Problem 21.

Problem 3.

(a) Let (X, Σ) be a measurable space. Let $M(\Sigma)$ be the vector space of complex measures on Σ with the total variation norm $\|\mu\| = |\mu|(X)$. Show that $M(\Sigma)$ is a Banach space.

Suggestion: it may be helfpul to use that for $\mu \in M(\Sigma)$ we have

$$\sum_{n=1}^{\infty} |\mu(E_n)| \leq \|\mu\|,$$

where $(E_n)_{n=1}^{\infty}$ is a sequence of pairwise disjoint elements of Σ (this is a consequence of a prior problem on this homework).

(b) Fix a positive, σ-finite measure μ on Σ. Show that the map $J\colon L^1(X, \mu) \to M(\Sigma)$ given by $J(f) = f\, d\mu$ is a linear isometry with closed image.

(c) Suppose that $\mu, \nu \in M(\Sigma)$, and let $d\nu = f\, d\mu + d\lambda$ with $\lambda \perp \mu$ be the Lebesgue-Radon-Nikodym decomposition. Show that

$$\|\mu - \nu\| = \|1 - f\|_{L^1(\mu)} + \|\lambda\|.$$

Remark: intuitively, this says two measures are "close" in total variation distance if the singular part of one with respect to the other is "small" and the Radon-Nikodym derivaive of one with respect to the other is "close to 1."

Problem 4.

Folland, Chapter 3, Problem 25.

Problem 5.

Let $\psi\colon \mathbb{R} \to \mathbb{R}$ be given as $\psi = 1_{[0,1/2)} - 1_{[1/2,1]}$. For $n, k \in \mathbb{Z}$ define $h_{n,k}(t) = 2^{n/2}\psi(2^n t - k)$. Show that $\mathcal{E} = \{1\} \cup \{h_{n,k} : n \in \mathbb{N} \cup \{0\}, 0 \leq k < 2^n\}$. is an orthonormal basis for $L^2([0,1])$.

Suggestion: Try showing that if $f \in L^2([0,1])$ is orthogonal to $\mathcal{E}$, then $f = 0$. It may be helfpul to find $\int_{[k2^{-n},(k+1/2)2^{-n})} f(x)\, dx$, $\int_{[(k+1/2)2^{-n},(k+1)2^{-n})} f(x)\, dx$ for such f starting from the fact that $\int_0^1 f(x)\, dx = 0$, and then apply the Lebesgue differentiation theorem.

Consider drawing the graphs of $h_{n,k}$ for $n = 0, 1, 2, \cdots$.

Problem 6.

Fix $n \in \mathbb{N}$, and $1 \leq p < +\infty$. For $y \in \mathbb{R}^n$, define $\tau_y\colon L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ by $\tau_y(f)(x) = f(x - y)$.

Show that if $f \in L^p(\mathbb{R}^n)$, then

$$\|\tau_y f\|_p = \|f\|_p. \tag{1}$$

$$\lim_{y \to 0} \|\tau_y f - f\|_p = 0. \tag{2}$$

Hint: use (1) to show that the set of f's for which (2) is true is a closed, linear subspace of $L^p(\mathbb{R}^n)$. Then check (2) on a dense set of f's where (2) is easier to see.