

MATH 7310 Homework 1

James Harbour

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Problem 1

A family of sets $\mathcal{R} \subseteq \mathcal{P}(X)$ is called a *ring* if it is closed under finite unions and differences. A ring that is closed under countable unions is called a *σ -ring*.

(a) Rings (resp. σ -rings) are closed under finite (resp. countable) intersections.

Proof. Let \mathcal{R} be ring (resp. σ -ring). Let $(E_i)_{i \in I}$ be a finite (resp. countable) family of elements of \mathcal{R} with finite (resp. countable) indexing set I . Let $j \in I$ be arbitrary. For any set $F \subseteq X$, note that

$$E_j \setminus (E_j \setminus F) = (E_j \cap F) \cup (E_j \setminus E_j) = E_j \cap F$$

Hence, utilizing De Morgan's laws, we write

$$\bigcap_{i \in I} E_i = E_j \cap \bigcap_{i \in I \setminus \{j\}} E_i = E_j \setminus \left(E_j \setminus \left(\bigcap_{i \in I \setminus \{j\}} E_i \right) \right) = E_j \setminus \left(\bigcup_{i \in I \setminus \{j\}} (E_j \setminus E_i) \right) \in \mathcal{R}$$

as desired. □

(b) If \mathcal{R} is a ring (resp. σ -ring), then \mathcal{R} is an algebra (resp. σ -algebra) if and only if $X \in \mathcal{R}$.

Proof.

\implies : Let $E \in \mathcal{R}$. Then $\emptyset = E \setminus E \in \mathcal{R}$. As \mathcal{R} is an algebra (resp. σ -algebra), it follows that $X = \emptyset^c \in \mathcal{R}$.

\impliedby : Suppose that $X \in \mathcal{R}$. As \mathcal{R} is already a ring (resp. σ -algebra), to show \mathcal{R} is an algebra (resp. σ -algebra) it suffices to show that \mathcal{R} is closed under complements. Suppose $E \in \mathcal{R}$. As $X \in \mathcal{R}$ and \mathcal{R} is closed under differences, it follows that $E^c = X \setminus E \in \mathcal{R}$. □

(c) If \mathcal{R} is a σ -ring, then $\{E \subseteq X : E \in \mathcal{R} \text{ or } E^c \in \mathcal{R}\}$ is a σ -algebra.

Proof. Let $\Sigma = \{E \subseteq X : E \in \mathcal{R} \text{ or } E^c \in \mathcal{R}\}$. By construction, Σ is closed under complements. Suppose that $(E_i)_{i \in I}$ is a countable family of elements of Σ , where I is an indexing set. Let $I_0 = \{i \in I : E_i \in \mathcal{R}\}$. Note that $\bigcup_{i \in I_0} E_i \in \mathcal{R} \subseteq \Sigma$. Moreover

$$\left(\bigcup_{j \in I \setminus I_0} E_j \right)^c = \bigcap_{j \in I \setminus I_0} E_j^c \in \mathcal{R},$$

so $\bigcap_{j \in I \setminus I_0} E_j \in \Sigma$. Hence

$$\bigcup_{i \in I} E_i = \bigcup_{i \in I_0} E_i \cup \bigcup_{j \in I \setminus I_0} E_j \in \Sigma,$$

so Σ is a σ -algebra. □

(d) If \mathcal{R} is a σ -ring, then $\{E \subseteq X : E \cap F \in \mathcal{R} \text{ for all } F \in \mathcal{R}\}$ is a σ -algebra.

Proof. Let $\Sigma = \{E \subseteq X : E \cap F \in \mathcal{R} \text{ for all } F \in \mathcal{R}\}$. Suppose $(E_i)_{i \in I}$ is a countable family of elements of Σ . For any $F \in \mathcal{R}$,

$$F \cap \bigcup_{i \in I} E_i = \bigcup_{i \in I} F \cap E_i \in \mathcal{R}$$

by definition, so $\bigcup_{i \in I} E_i \in \Sigma$. Now suppose $E \in \Sigma$. Let $F \in \mathcal{R}$. Observe that, as $E \cap F \in \mathcal{R}$ and \mathcal{R} is closed under differences,

$$E^c \cap F = (X \setminus E) \cap F = X \cap (F \setminus E) = F \setminus E = (F \setminus E) \cup (F \setminus F) = F \setminus (E \cap F) \in \mathcal{R}.$$

Thus, $E^c \in \Sigma$, so Σ is a σ -algebra. □

Problem 2

An algebra \mathcal{A} is a σ -algebra if and only if \mathcal{A} is closed under countable increasing unions.

Proof. The forward direction follows *a fortiori* by the definition of a σ -algebra. For the reverse direction, suppose that \mathcal{A} is closed under countable increasing unions. As \mathcal{A} is already an algebra, it suffices to show that \mathcal{A} is closed under arbitrary countable unions. Let $(E_i)_{i=1}^\infty$ be a countable family of elements of \mathcal{A} . Define a new sequence $(E'_j)_{j=1}^\infty$ by $E'_j := E_1 \cup \cdots \cup E_j$. Note that, as \mathcal{A} is closed under finite unions, $E'_j \in \mathcal{A}$ for all $j \in \mathbb{N}$. Moreover, $(E'_j)_{j=1}^\infty$ is an increasing sequence of sets by construction, so

$$\bigcup_{n=1}^\infty E_n = \bigcup_{n=1}^\infty \bigcup_{i=1}^n E_i = \bigcup_{n=1}^\infty E'_n \in \mathcal{A}.$$

□

Problem 3

Let X, Y be sets and $f : X \rightarrow Y$.

(i) If $\Sigma \subseteq \mathcal{P}(Y)$ is a σ -algebra, show that

$$\{E \subseteq X : f^{-1}(E) \in \Sigma\}$$

is a σ -algebra.

(ii) If $\Sigma \subseteq \mathcal{P}(X)$ is a σ -algebra, show that

$$\{E \subseteq X : E = f^{-1}(F) \text{ for some } F \in \Sigma\}$$

is a σ -algebra.

(iii) If Y is a countable, show that

$$\{E \subseteq X : E = f^{-1}(F) \text{ for some } F \in \Sigma\}$$

is the σ -algebra generated by $\{f^{-1}(\{y\}) : y \in Y\}$.

Problem 4

(i) Suppose that X is a set. Let J be a countable set and suppose that $X = \bigsqcup_{j \in J} A_j$ with $A_j \neq \emptyset$ for all $j \in J$. Show that the σ -algebra generated by $(A_j)_{j \in J}$ is $\{\bigcup_{j \in J_0} A_j : J_0 \subseteq J\}$.

Proof. Let $\mathfrak{A} = \{\bigcup_{j \in J_0} A_j : J_0 \subseteq J\}$ and $\mathcal{A} = \{A_j : j \in J\}$. We show first that \mathfrak{A} is indeed a σ -algebra. Suppose that $(J_n)_{n \in \mathbb{N}}$ is a sequence of subsets of J and let $I = \bigcup_{n=1}^{\infty} J_n \subseteq J$. Then

$$\bigcup_{n=1}^{\infty} \bigcup_{j \in J_n} A_j = \bigcup_{j \in I} A_j \in \mathfrak{A}.$$

Now let $J_0 \subseteq J$. Then

$$\left(\bigcup_{j \in J_0} A_j \right)^c = X \setminus \bigcup_{j \in J_0} A_j = \bigcup_{j \in J \setminus J_0} A_j \in \mathfrak{A},$$

so \mathfrak{A} is a σ -algebra. Clearly $\mathcal{A} \subseteq \mathfrak{A}$, so by minimality $\Sigma(\mathcal{A}) \subseteq \mathfrak{A}$.

On the other hand, as J is countable, each element of \mathfrak{A} is an at most countable union of elements of \mathcal{A} , so $\mathfrak{A} \subseteq \Sigma(\mathcal{A})$. \square

(ii) Show that the σ -algebra generated by a finite collection of sets is finite.

Problem 5

Let (X, d_X) be a metric space. A collection \mathcal{B} of open sets in X is said to be a *basis* if for every open $U \subseteq X$ and every $x \in U$, there is a $B \in \mathcal{B}$ so that $x \in B \subseteq U$. If \mathcal{B} is a *countable* basis of open sets in X , show that the σ -algebra generated by \mathcal{B} coincides with the Borel sets in X .

Proof. Let \mathfrak{B} be the collection of Borel sets in X . This collection is a σ -algebra by definition. As \mathcal{B} is a collection of open sets in X , $\mathcal{B} \subseteq \mathfrak{B}$. Hence, by minimality, $\Sigma(\mathcal{B}) \subseteq \mathfrak{B}$.

Suppose that $U \subseteq X$ is open. For all $x \in U$, there exists a $B_x \in \mathcal{B}$ such such that $x \in B_x \subseteq U$. As \mathcal{B} is countable, the union $U = \bigcup_{x \in U} B_x$ is in fact countable, so $U = \bigcup_{x \in U} B_x \in \Sigma(\mathcal{B})$. Thus, $\Sigma(\mathcal{B})$ contains all open sets in X , whence by minimality of \mathfrak{B} , $\Sigma(\mathcal{B}) \subseteq \mathfrak{B}$. \square