# MATH 7310 Homework 7

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## Problem 1

Let $(X, \Sigma, \mu)$ be a measure space.
**(i)**: Prove that if $\mu(E_n) < +\infty$ for $n \in \mathbb{N}$ and $\mathbb{1}_{E_n} \to f$ in $L^1$, then $f$ is (a.e. equal to) the characteristic function of a measurable set.

**(ii)**: Let $\Sigma_f = \{E \in \Sigma : \mu(E) < +\infty\}$. Define an equivalence relation on $\Sigma_f$ by $E \sim F$ if $\mu(E \Delta F) = 0$. Let $\Omega = \Sigma_f / \sim$, and define a metric $\rho$ on $\Omega$ be $\rho([E], [F]) = \mu(E \Delta F)$. Show that the map $\iota : \Omega \to L^1(X, \mu)$ given by $\iota([E]) = \mathbb{1}_E$ is an isometry with closed image.

**(iii)**: Show that $(\Omega, \rho)$ is a complete metric space.

## Problem 2

If $X, Y$ are sets, and $f : X \to \mathbb{C}$, $g : Y \to \mathbb{C}$, we define $f \otimes g : X \times Y \to \mathbb{C}$ by $(f \otimes g)(x, y) = f(x)g(y)$. Fix $1 \leq p < +\infty$.

**(a)**: Let $(X, \Sigma, \mu), (Y, \mathcal{F}, \nu)$ be $\sigma$-finite measure spaces. Show that if $f \in L^p(X, \mu), g \in L^p(Y, \nu)$, then $\|f \otimes g\|_p = \|f\|_p \|g\|_p$.

**(b)**: Let $(Z, \mathcal{O}, \zeta)$ be a finite measure space. Suppose that $\mathcal{A} \subseteq \mathcal{O}$ is an algebra which generates the $\sigma$-algebra of $\mathcal{O}$. Use the monotone class lemma to show that $\{\mathbb{1}_A : A \in \mathcal{A}\}$ is dense in $\{\mathbb{1}_E : E \in \mathcal{O}\}$ in the $L^p$-norm for all $1 \leq p < +\infty$.

**(c)**: Let $(X, \Sigma, \mu), (Y, \mathcal{F}, \nu)$ be finite measure spaces. Use the previous part to show that $\{\mathbb{1}_E : E \in \Sigma \otimes \mathcal{F}\} \subseteq \overline{\text{Span}}^{\|\cdot\|_p} \{\mathbb{1}_E \otimes \mathbb{1}_F : E \in \Sigma, F \in \mathcal{F}\}$. Use this to show that $\overline{\text{Span}}^{\|\cdot\|_p} \{\mathbb{1}_E \otimes \mathbb{1}_F : E \in \Sigma, F \in \mathcal{F}\} = L^p(X \times Y, \mu \otimes \nu)$.

**(d)**: Let $(X, \Sigma, \mu), (Y, \mathcal{F}, \nu)$ be $\sigma$-finite measure spaces. Suppose that $D_X \subseteq L^p(X, \mu)$, $D_Y \subseteq L^p(Y, \nu)$ and that

$$\overline{\text{Span}}^{\|\cdot\|_p}(D_X) = L^1(X, \mu), \quad \overline{\text{Span}}^{\|\cdot\|_p}(D_Y) = L^1(Y, \nu).$$

Show that $\overline{\text{Span}}^{\|\cdot\|_p}(\{f \otimes g : f \in D_X, g \in D_Y\}) = L^p(X \times Y, \mu \otimes \nu)$.

# Problem 3

Suppose that $f \in L^p \cap L^\infty$ for some $p < +\infty$ so that $f \in L^q$ for all $q > p$. Prove that then $\|f\|_\infty = \lim_{q\to\infty} \|f\|_q$.

# Problem 4

If $f$ is a measurable function on $X$, define the *essential range* $R_f$ of $f$ to be the set of all $z \in \mathbb{C}$ such that $\{x : |f(x) - z| < \varepsilon\}$ has positive measure for all $\varepsilon > 0$.

**(a)**: Prove that $R_f$ is closed.

*Proof.* Let $z \in \overline{R_f}$. Then there exists a sequence $(z_n)_{n=1}^\infty$ in $R_f$ such that $z_n \to z$. Fix $\varepsilon > 0$. There is some $N \in \mathbb{N}$ such that $n \geq N \implies B_{\varepsilon/2}(z_n) \subseteq B_\varepsilon(z)$. Then $f^{-1}(B_{\varepsilon/2}(z_n)) \subseteq f^{-1}(B_\varepsilon(z))$, whence $0 < \mu(f^{-1}(B_{\varepsilon/2}(z_n))) \leq \mu(f^{-1}(B_\varepsilon(z)))$. Hence $z \in R_f$, so $R_f$ is closed. $\square$

**(b)**: Prove that if $f \in L^\infty$, then $R_f$ is compact and $\|f\|_\infty = \max\{|z| : z \in R_f\}$.

# Problem 5

Suppose that $1 \leq p < +\infty$ and $(f_n)_{n=1}^\infty$ in $L^p$. Prove that $(f_n)_{n=1}^\infty$ is Cauchy in the $L^p$-norm if an only if the following three conditions hold:

1. $(f_n)$ is Cauchy in measure;
2. the sequence $(|f_n|^p)_{n=1}^\infty$ is uniformly integrable
3. for every $\varepsilon > 0$ there exists $E \subseteq X$ such that $\mu(E) < +\infty$ and $\int_{E^c} |f_n|^p \, d\mu < \varepsilon$ for all $n \in \mathbb{N}$.

# Problem 6

Prove that if $E$ is a subset of a Hilbert space $\mathcal{H}$, then $(E^\perp)^\perp$ is the smallest closed subspace of $\mathcal{H}$ containing $E$.

*Claim.* If $M$ is a closed linear subspace of $\mathcal{H}$, then $(M^\perp)^\perp = M$.

*Proof of Claim.* Note that we have $\mathcal{H} = M \oplus M^\perp$. Let $y \in (M^\perp)^\perp$. Then there exist unique $x \in M$, $x^\perp \in M^\perp$ such that $y = x + x^\perp$. Noting that $M \subseteq (M^\perp)^\perp$, we have that $x^\perp = y - x \in M^\perp \cap (M^\perp)^\perp = \{0\}$, whence $x^\perp = 0$ and $y = x \in M$. Thus $M = (M^\perp)^\perp$. $\square$

*Proof.* On one hand, note that $E \subseteq \overline{\operatorname{Span}(E)} \implies (E^\perp)^\perp \subseteq (\overline{\operatorname{Span}(E)}^\perp)^\perp \overset{\text{claim}}{=} \overline{\operatorname{Span}(E)}$. On the other hand, as $(E^\perp)^\perp$ is a closed linear subspace of $\mathcal{H}$ and $E \subseteq (E^\perp)^\perp$, it follows that $\overline{\operatorname{Span}(E)} \subseteq (E^\perp)^\perp$. $\square$