MATH 7752 Homework 6

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Problem 1

(a) Prove that two 3×3 matrices over some field F are similar if and only if they have the same minimal and characteristic polynomials. Is the same true for 4×4 matrices?

(b) A matrix A is called idempotent if $A^2 = A$. Prove that two idempotent $n \times n$ matrices are similar if and only if they have the same rank. **Hint:** What is the minimal polynomial of an idempotent matrix? How does rank relate to eigenvalue 0?

Problem 2

Let F be an algebraically closed field and V a finite dimensional F-vector space.

(a) Let $S, T \in \mathcal{L}(V)$ such that $ST = TS$. Let λ be an eigenvalue of S and $E_\lambda(S) \leq V$ be the corresponding eigenspace of S. Prove that $E_\lambda(S)$ is a T-invariant subspace.

Proof. Let $v \in E_\lambda(S)$, so $Sv = \lambda v$. Then

$$S(Tv) = (ST)(v) = (TS)(v) = T(Sv) = T(\lambda v) = \lambda \cdot (Tv)$$

so $Tv \in E_\lambda(S)$. Thus $T(E_\lambda(S)) \subseteq E_\lambda(S)$. $\square$

(b) Assume that $T \in \mathcal{L}(V)$ is diagonalizable and let $W \leq V$ be a T-invariant subspace. Prove that $T|_W \in \mathcal{L}(W)$ is also diagonalizable.

Proof. Since T is diagonalizable, $E_\lambda(T) = V_\lambda(T)$ for all $\lambda \in \operatorname{Spec}(T)$. Let $\lambda \in \operatorname{Spec}(T|_W) \subseteq \operatorname{Spec}(T)$ and $w \in W_\lambda(T|_W)$. Then, for some $k \in \mathbb{N}$, $(T - \lambda I)^k(w) = (T|_W - \lambda I|_W)^k(w) = 0$. Hence $w \in V_\lambda(T) = E_\lambda(T)$. But $w \in W$ so then $w \in E_\lambda(T|_W)$ whence $E_\lambda(T|_W) = W_\lambda(T|_W)$. So $T|_W$ is diagonalizable. $\square$

(c) Assume again that $S, T \in \mathcal{L}(V)$ such that $ST = TS$. Prove that there exists a basis Ω of V such that $[T]_\Omega$, and $[S]_\Omega$ are both diagonal.

Proof. I am quite sure that this claim is false as stated, so I will add the assumption that S, T are both diagonalizable.

Then as S is diagonalizable,

$$V = \bigoplus_{\lambda \in \operatorname{Spec}(S)} V_\lambda(S) = \bigoplus_{\lambda \in \operatorname{Spec}(S)} E_\lambda(S).$$

Fix $\lambda \in \operatorname{Spec}(S)$. By part (a), $E_\lambda(S)$ is T-invariant whence part (b) implies that $T|_{E_\lambda(S)}$ is diagonalizable. So

$$E_\lambda(S) = \bigoplus_{\delta \in \operatorname{Spec}(T|_{E_\lambda(S)})} E_\delta(T|_{E_\lambda(S)}).$$

Now we write

$$V = \bigoplus_{\lambda \in \operatorname{Spec}(S)} \bigoplus_{\delta \in \operatorname{Spec}(T|_{E_\lambda(S)})} E_\delta(T|_{E_\lambda(S)}).$$

Since this sum is direct, we may form a basis for V from bases for $E_\delta(T|_{E\lambda(S)})$ over this double direct sum, whence this basis is both an eigenbasis for T and S. $\square$

(d) Give an example of a vector space V with $\dim_F(V) \geq 3$ and two commuting linear transformations $S, T \in \mathcal{L}(V)$ such that NO basis Ω of V exists such that both $[T]_\Omega$, and $[S]_\Omega$ are in JCF.

Problem 3

Find the number of distinct conjugacy classes in the group $GL_3(\mathbb{Z}/2\mathbb{Z})$, and specify one element in each conjugacy class.

Problem 4

Let V be an n-dimensional vector space over an algebraically closed field and $T \in \mathcal{L}(V)$. Assume that T has just one eigenvalue λ and just one Jordan block. Let $S = T - \lambda I$.

(a) Prove that $\operatorname{rk}(S^k) = n - k$, for all $0 \leq k \leq n$. Deduce that $\operatorname{Im}(S^k) = \ker(S^{n-k})$, for all $0 \leq k \leq n$.

Proof. Note that $n_T(k, \lambda) = 1$ for $0 \leq k \leq n$ by assumption.

We induct on $0 \leq k \leq n$. For $k = 0$, $S^0 = I$ so $\operatorname{rk}(S^0) = n = n - 0$.

Now suppose $0 < k \leq n$ and that the claim holds for $k - 1$. On one hand, by the induction hypothesis $\operatorname{rk}(S^{k-1}) = n - (k - 1)$. On the other hand

$$1 = n_T(k, \lambda) = \operatorname{rk}((T - \lambda I)^{k-1}) - \operatorname{rk}((T - \lambda I)^k) = \operatorname{rk}(S^{k-1}) - \operatorname{rk}(S^k) = n - k + 1 - \operatorname{rk}(S^k) \implies \operatorname{rk}(S^k) = n - k.$$

To see that $\operatorname{Im}(S^k) = \ker(S^{n-k})$, note that by the rank nullity theorem we have

$$\dim \ker(S^{n-k}) = n - \operatorname{rk}(S^{n-k}) = n - (n - (n - k)) = n - k = \operatorname{rk}(S^k) = \dim \operatorname{Im}(S^k),$$

so it suffices to show that $\operatorname{Im}(S^k) \subseteq \ker(S^{n-k})$.

Take $w \in \operatorname{Im}(S^k)$. Then $w = S^k v$ for some $v \in V$. Noting that $\operatorname{rk}(S^n) = 0 \implies S^n = \mathrm{O}$, we have that $S^{n-k} w = S^{n-k}(S^k v) = S^n v = \mathrm{O}\mathrm{v} = 0$, so $w \in \ker(S^{n-k})$. $\square$

(b) Let $v \in V$ be any vector which lies outside of $\operatorname{Im}(S) = \ker(S^{n-1})$. Prove that $\{S^{n-1}v, \ldots, Sv, v\}$ is a Jordan basis for T.

Problem 5

Assume again that V is an n-dimensional vector space over an algebraically closed field F and $T \in \mathcal{L}(V)$.

(a) Assume that T has unique eigenvalue 0 and two Jordan blocks: a 1×1 block and a 2×2 block (so $n = 3$ in this case). Justify the following algorithm for computing a Jordan basis for T: Take any $v \in V \setminus \ker(T)$ and choose $w \in \ker(T)$ such that $\{w, Tv\}$ is a basis for $\ker(T)$ (why is this possible?); then $\{w, Tv, v\}$ is a Jordan basis for T.

(b) Assume that T has unique eigenvalue 0 and two Jordan blocks, both of which are 2×2 (so $n = 4$). State an algorithm for finding a Jordan basis similar to the one in (a).

(c) Assume that for each $\lambda \in \mathrm{Spec}(T)$ there is only one Jordan λ-block in $JCF(T)$. Describe an algorithm for computing a Jordan basis of T. **Hint:** You just need a minor generalization of the algorithm in the previous problem.

Problem 6

Compute the Jordan canonical form and a Jordan basis for each of the following matrices over $\mathbb{Q}$:

(a) $\begin{pmatrix} -1 & 3 & 0 \\ 0 & 2 & 0 \\ 2 & 1 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 0 \end{pmatrix}$.

Problem 7

Let $F = \mathbb{F}_3$ be the field with 3 elements and let $A \in M_{12}(\mathbb{F}_3)$. Suppose that A satisfies all the following assumptions:

- $\mathrm{rk}(A) = 10, \quad \mathrm{rk}(A^2) = 9, \quad \mathrm{rk}(A^3) = 9.$
- $\mathrm{rk}(A - I) = 12.$
- $\mathrm{rk}(A - 2I) = 9, \quad \mathrm{rk}((A - 2I)^2) = 7, \quad \mathrm{rk}((A - 2I)^3) = 6.$

(a) Assume in addition that the characteristic polynomial $\chi_A(x)$ splits completely over F (i.e. it splits into linear factors in $F[x]$). Find the Jordan canonical form of A.

(b) Find all possible RCF's of matrices A satisfying all the bullet assumptions, but not necessarily the extra assumption in (a).