

# MATH 7752 Homework 3

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## Problem 1

Let  $R$  be a commutative domain, and let  $M$  be a free  $R$ -module with basis  $X = \{e_1, \dots, e_k\}$ , with  $k \geq 2$ . Prove that the element  $e_1 \otimes e_2 + e_2 \otimes e_1$  cannot be written as simple tensor  $m \otimes n$ , for some  $m, n \in M$ .

*Proof.* Suppose, for the sake of contradiction, that there exist  $m, n \in M$  such that  $m \otimes n = e_1 \otimes e_2 + e_2 \otimes e_1$ . Write  $m = \sum_{i=1}^n r_i e_i$  and  $n = \sum_{j=1}^n s_j e_j$  for some  $r_i, s_j \in R$ . Then

$$e_1 \otimes e_2 + e_2 \otimes e_1 = \left( \sum_{i=1}^n r_i e_i \right) \otimes \left( \sum_{j=1}^n s_j e_j \right) = \sum_{i,j} r_i s_j e_i \otimes e_j$$

As  $M \otimes M$  is free with basis  $\{e_i \otimes e_j\}_{i,j=1}^n$ , it follows that  $r_1 s_2 = 1$ ,  $r_2 s_1 = 1$ , and  $r_i s_j = 0$  for all  $(i, j) \neq (1, 2), (2, 1)$ . However, then  $r_1 s_1 = 0$  whence  $r_1 = 0$  or  $s_1 = 0$ , contradicting  $r_1, s_1 \in R^\times$ . □

## Problem 2

Let  $R$  be a commutative ring (with 1) and  $n, m \in \mathbb{N}$ . Prove that there is an isomorphism of  $R$ -algebras  $R^n \otimes R^m \simeq R^{nm}$ . (Here by  $R^n$  we mean the direct sum  $\underbrace{R \oplus \dots \oplus R}_n$ .)

*Proof.* As  $R$ -modules, we have the following isomorphisms:

$$R^n \otimes R^m \cong (R^n \otimes R)^m \cong ((R \otimes R)^n)^m \cong (R \otimes R)^{nm} \cong R^{nm}$$

$$(r_1, \dots, r_n) \otimes (r'_1, \dots, r'_m) \mapsto$$

□

## Problem 3

(a) Let  $V$  be a finite-dimensional  $\mathbb{C}$ -vector space. Then  $V$  can be considered as a vector over  $\mathbb{R}$  (by restriction of scalars), and it holds  $\dim_{\mathbb{R}} V = 2 \dim_{\mathbb{C}} V$ . Prove that  $V \otimes_{\mathbb{C}} V$  is not isomorphic to  $V \otimes_{\mathbb{R}} V$  as  $\mathbb{R}$ -vector spaces, and compute their dimensions over  $\mathbb{R}$ .

*Proof.* Let  $k = \dim_{\mathbb{C}} V$ . Then  $\dim_{\mathbb{R}} V = 2k$ . As a  $\mathbb{C}$ -vector space,  $\dim_{\mathbb{C}} V \otimes_{\mathbb{C}} V = (\dim_{\mathbb{C}} V)^2 = k^2$ , whence by restriction of scalars  $\dim_{\mathbb{R}} V \otimes_{\mathbb{C}} V = 2 \dim_{\mathbb{C}} V \otimes_{\mathbb{C}} V = 2k^2$ .

On the other hand  $\dim_{\mathbb{R}} V \otimes_{\mathbb{R}} V = (\dim_{\mathbb{R}} V)^2 = (2k)^2 = 4k^2$ . Hence

$$\dim_{\mathbb{R}} V \otimes_{\mathbb{C}} V = 2k^2 \neq 4k^2 = \dim_{\mathbb{R}} V \otimes_{\mathbb{R}} V$$

so  $V \otimes_{\mathbb{C}} V$  and  $V \otimes_{\mathbb{R}} V$  are not isomorphic as  $\mathbb{R}$ -vector spaces. □

**(b)** Let  $R$  be an integral domain (commutative), and let  $K$  be its fraction field. Prove that there is an isomorphism of  $F$ -modules,  $F \otimes_R F \simeq F \otimes_F F \simeq F$ , where the  $F$ -module structure on  $F \otimes_R F$  is given by **extension of scalars** (i.e. tensor product of Type I).

## Problem 4

The purpose of this problem is to classify all 2-dimensional  $\mathbb{R}$ -algebras (where  $\mathbb{R}$  are the real numbers). That means, to classify (up to algebra isomorphism) those  $\mathbb{R}$ -algebras that are 2-dimensional  $\mathbb{R}$  vector spaces. Let  $A$  be a 2-dimensional  $\mathbb{R}$ -algebra (with 1).

**(a)** Let  $u \in A$  be any element that is  $\mathbb{R}$ -linearly independent from 1. Prove that

- (i)  $u$  generates  $A$  as an  $\mathbb{R}$ -algebra. That is, the minimal  $\mathbb{R}$ -subalgebra of  $A$  containing  $u$  and 1 is  $A$  itself.
- (ii) The element  $u$  satisfies a quadratic equation  $au^2 + bu + c = 0$ , for some  $a, b, c \in \mathbb{R}$  with  $a \neq 0$ . Conclude that  $A$  is necessarily commutative.

*Proof.* Noting that the subalgebra generated by  $u$  contains  $\text{span}_{\mathbb{R}}(\{1, u\})$  which has dimension 2 as an  $\mathbb{R}$ -vector space, it follows that the subalgebra generated by  $u$  is in fact  $A$ .

Since the subalgebra generated by  $u$  is  $A$ , it follows that there exist  $a, b \in \mathbb{R}$  such that  $u^2 = au + b1$ , whence  $u^2 - au - b = 0$ . This implies the algebra  $A$  is commutative as multiplication is hence defined by the relations  $u \cdot 1 = u = 1 \cdot u$  and  $1 = 1 \cdot 1$ , which are all commutative. □

**(b)** Show that there exists some  $v \in A$  which is  $\mathbb{R}$ -linearly independent from 1 and is such that  $v^2 = -1$ , or  $v^2 = 1$ , or  $v^2 = 0$ .

*Proof.* □

**(c)** Deduce from part (b) that  $A$  is isomorphic as an  $\mathbb{R}$ -algebra to one of the following:  $\mathbb{R}[x]/(x^2 + 1)$ , or  $\mathbb{R}[x]/(x^2 - 1)$ , or  $\mathbb{R}[x]/(x^2)$ .

**(d)** Prove that the algebras  $\mathbb{R}[x]/(x^2 + 1)$ ,  $\mathbb{R}[x]/(x^2 - 1)$ , and  $\mathbb{R}[x]/(x^2)$  are pairwise non-isomorphic. **Hint:** This can be shown with almost no computation.

## Problem 5

The purpose of this problem is to prove the following theorem: Let  $D$  be a finite dimensional division algebra over  $\mathbb{R}$ . Then  $D$  is isomorphic to  $\mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$  (the quaternions). One way to proceed is to use the following steps:

**(a)** Let  $\alpha \in D$  be an element  $\mathbb{R}$ -linearly independent from 1. Show that  $\alpha$  satisfies a quadratic irreducible polynomial  $p_{\alpha}(x) = x^2 + ax + b \in \mathbb{R}[x]$ .

*Proof.* Since  $D$  is finite-dimensional over  $\mathbb{R}$ , there exists an  $n \in \mathbb{N}$  such that the set  $\{1, \alpha, \dots, \alpha^n\}$  is  $\mathbb{R}$ -linearly dependent. Hence  $\alpha$  is algebraic over  $\mathbb{R}$ , so the set  $I_\alpha = \{f(x) \in \mathbb{R}[x] : f(\alpha) = 0\}$ .

As  $I_\alpha$  is an ideal and  $\mathbb{R}[x]$  is a PID, there exists a (without loss of generality) monic polynomial  $p_\alpha(x) \in \mathbb{R}[x]$  such that  $I_\alpha = (p_\alpha)$ . As  $\alpha$  is algebraic,  $p_\alpha \neq 0$ . Moreover,  $p_\alpha$  is nonconstant by  $p_\alpha(\alpha) = 0$ . Hence,  $p_\alpha$  is not a unit in  $\mathbb{R}[x]$ . If  $f \in I_\alpha = (p_\alpha)$  is irreducible, then in writing  $f = gp_\alpha$  for some  $g \in \mathbb{R}[x]$ , irreducibility implies that  $(f) = (p_\alpha) = I_\alpha$ . Moreover, this implies that  $\deg(f) = \deg(p_\alpha)$ , so  $p_\alpha$  being monic implies that  $p_\alpha$  is the unique irreducible monic element of  $I_\alpha$ .

As  $\alpha \notin \mathbb{R} \cdot 1$ ,  $\deg(p_\alpha) \geq 2$ . By the Fundamental Theorem of Algebra, it follows then that  $p_\alpha$  must be quadratic, so there exist  $a, b \in \mathbb{R}$  such that  $p_\alpha(x) = x^2 + ax + b$ . □

**(b)** Let  $V = \{\alpha \in D : \alpha^2 \in \mathbb{R}_{\leq 0}\}$ . Show that  $V$  is an  $\mathbb{R}$ -linear subspace of  $D$ . **Hint:** Show there is an  $\mathbb{R}$ -linear map  $f : D \rightarrow \mathbb{R}$  with kernel  $V$ .

*Proof.* For  $\alpha \in D$ , define an  $\mathbb{R}$ -endomorphism  $T_\alpha$  of  $D$  via left multiplication by  $\alpha$ . This furnished a linear map  $D \rightarrow \text{End}_{\mathbb{R}}(D)$ . We claim that  $V$  is the kernel of the composition of the  $\mathbb{R}$ -linear maps

$$D \rightarrow \text{End}_{\mathbb{R}}(D) \xrightarrow{\text{Tr}} \mathbb{R}.$$

Fix  $\alpha \in D$  such that  $\alpha \notin \mathbb{R} \cdot 1$ . Then by part (a) there exist  $a, b \in \mathbb{R}$  such that  $\alpha$  satisfies a quadratic irreducible polynomial  $p_\alpha(x) = x^2 + ax + b$ . Observe that, for  $v \in D$ ,

$$p_\alpha(T_\alpha)(v) = T_\alpha^2(v) + aT_\alpha(v) + b(v) = \alpha^2 v + a\alpha v + bv = (\alpha^2 + a\alpha + b\alpha)(v) = 0$$

so  $p_\alpha(T_\alpha) = 0 \in \text{End}_{\mathbb{R}}(D)$ . Irreducibility of  $p_\alpha$  then implies that  $p_\alpha$  is the minimal polynomial for the operator  $T_\alpha$ . Let  $\chi_\alpha(x)$  be the characteristic polynomial for  $T_\alpha$ . Then  $p_\alpha(x) \mid \chi_\alpha(x)$  and there exists a  $k \in \mathbb{N}$  such that  $\chi_\alpha(x) = (p_\alpha(x))^k$ . As  $\chi_\alpha$  is monic and  $p_\alpha$  is irreducible, there exists an  $l \in \mathbb{N}$  such that  $\chi_\alpha(x) = (p_\alpha(x))^l$ . By multinomial expansion,

$$\chi_\alpha(x) = (p_\alpha(x))^l = \sum_{\substack{n_1+n_2+n_3=l \\ n_1, n_2, n_3 \geq 0}} \binom{l}{n_1, n_2, n_3} x^{2n_1+n_2} a^{n_2} b^{n_3}$$

This polynomial has  $x^{2l-1}$  coefficient

$$\binom{l}{l-1, 1, 0} a = l \cdot a$$

However, the  $x^{2l-1}$  coefficient of  $\chi_\alpha$  is also  $\pm \text{Tr}(T_\alpha)$ , so  $\pm \text{Tr}(T_\alpha) = l \cdot a$ . Moreover, as  $p_\alpha(x)$  is irreducible,  $a^2 - 4b < 0 \implies b > \frac{a^2}{4} \geq 0$ . Hence, if  $\alpha$  is such that  $\text{Tr}(\alpha) = 0$ , then  $a = 0$  whence  $0 = p_\alpha(\alpha) = \alpha^2 + b \implies \alpha^2 = -b \leq 0$ , i.e.  $\alpha \in V$ . Conversely, suppose that  $\alpha \in D \setminus \{0\}$  is such that  $\alpha^2 < 0$ . Then  $\alpha$  is linearly independent from 1, so there exist  $a, b \in \mathbb{R}$  such that  $\alpha^2 + a\alpha + b = 0$ . Note that, as  $\alpha^2 \in \mathbb{R}$ , linear independence of  $\alpha$  from 1 implies that  $a = 0$  and  $\alpha^2 + b = 0$ . Then,  $\text{Tr}(T_\alpha) = 0$ , as desired. □

**(c)** Define  $B : V \times V \rightarrow \mathbb{R}$ ,  $B(\alpha, \beta) := -\frac{\alpha\beta + \beta\alpha}{2}$ . Show that  $B$  defines an inner product on  $V$  (i.e.  $B$  is a symmetric, positive definite bilinear form on  $V$ ).

*Proof.* Observe that, for  $\alpha, \beta \in V$ ,  $\alpha^2, \beta^2, (\alpha + \beta)^2 \in \mathbb{R}$ , whence  $\alpha\beta + \beta\alpha = (\alpha + \beta)^2 - \alpha^2 - \beta^2 \in \mathbb{R}$ , so  $B$  is in fact real-valued.

Fix  $\alpha, \alpha', \beta, \beta' \in V$  and  $\lambda \in \mathbb{R}$ . Then

$$\begin{aligned} B(\alpha, \beta) &= -\frac{\alpha\beta + \beta\alpha}{2} = -\frac{\beta\alpha + \alpha\beta}{2} = B(\beta, \alpha) \\ B(\alpha + \lambda\alpha', \beta) &= -\frac{(\alpha + \lambda\alpha')\beta + \beta(\alpha + \lambda\alpha')}{2} = -\frac{\alpha\beta + \beta\alpha}{2} - \lambda\frac{\alpha'\beta + \beta\alpha'}{2} = B(\alpha, \beta) + \lambda B(\alpha', \beta). \\ B(\alpha, \beta + \lambda\beta') &= -\frac{\alpha(\beta + \lambda\beta') + (\beta + \lambda\beta')\alpha}{2} = -\frac{\alpha\beta + \beta\alpha}{2} - \lambda\frac{\alpha\beta' + \beta'\alpha}{2} = B(\alpha, \beta) + \lambda B(\alpha, \beta'), \end{aligned}$$

so  $B$  is a symmetric bilinear form. Moreover, as  $\alpha \in V \setminus \{0\}$  implies that  $\alpha^2 \in \mathbb{R}_{<0}$ , we have then that

$$B(\alpha, \alpha) = -\frac{\alpha\alpha + \alpha\alpha}{2} = -\alpha^2 > 0$$

so  $B$  is also positive-definite. □

**(d)** Let  $W$  be a linear subspace of  $V$  that generates  $D$  as an  $\mathbb{R}$ -algebra. Let  $n = \dim_{\mathbb{R}} W$ . Choose an orthonormal basis of  $W$ , i.e. a basis  $\{e_i\}$  of  $W$  such that  $B(e_i, e_i) = 1$  for all  $i$  and  $B(e_i, e_j) = 0$  for all  $i \neq j$  (such a basis always exists). Using this orthonormal basis show that if  $n \geq 2$ , then  $D$  has a subalgebra isomorphic to  $\mathbb{H}$ .

*Proof.* We must first show that such a subspace  $W$  of  $V$  with the prescribed property actually exists. Let  $\psi : D \rightarrow \mathbb{R}$  denote the linear map constructed in part (b), i.e.  $\psi(\alpha) = \text{Tr}(T_\alpha)$  where  $T_\alpha$  is the left multiplication by  $\alpha$  operator. We claim that  $\text{im } \psi \neq 0$ , whence  $\text{im } \psi = \mathbb{R}$ . To show this, consider  $\lambda \in R \setminus \{0\}$ . With respect to any  $\mathbb{R}$ -basis of  $D$ , the matrix of  $T_\lambda$  is diagonal with nonzero entries  $\lambda$  along the diagonal, so  $\text{Tr}(T_\lambda) = \lambda \dim_{\mathbb{R}} D \in \mathbb{R} \setminus \{0\}$ , as desired.

By rank-nullity theorem,

$$\dim D = \dim \ker \psi + \dim \text{im } \psi = \dim V + \dim \mathbb{R} = \dim V + 1$$

so  $\dim V = \dim D - 1$ . The remaining direct summand of  $D$  is spanned by 1 and is thus  $\mathbb{R}$ , so as the subalgebra generated by  $V$  contains 1 and  $V$ , it must be all of  $D$ .

Now suppose that  $W$  is a linear subspace of  $V$  that generates  $D$  as an  $\mathbb{R}$ -algebra. Let  $n = \dim_{\mathbb{R}} W$ . Choose an orthonormal basis  $\{e_i\}_{i=1}^n$  of  $W$  with respect to  $B$ . Then, for  $i \in \{1, \dots, n\}$ ,

$$1 = B(e_i, e_i) = -\frac{e_i^2 + e_i^2}{2} = -e_i^2 \implies e_i^2 = -1.$$

Also, for  $i, j \in \{1, \dots, n\}$ ,

$$0 = B(e_i, e_j) = -\frac{e_i e_j + e_j e_i}{2} \implies e_i e_j = -e_j e_i.$$

Let  $A$  be the subalgebra of  $D$  generated by  $\{1, e_1, e_2\}$ . We will show that  $A \cong \mathbb{H}$  as  $\mathbb{R}$ -algebras. □

**(e) Bonus:** Suppose  $n \geq 2$ . Prove that  $A = H$ . **Hint:** One way to proceed is to show that if  $n > 2$ , then the multiplication in  $D$  cannot be associative.