MATH 7752 - HOMEWORK 3
DUE WEDNESDAY 02/05/20

(1) Let R be a commutative domain, and let M be a free R-module with basis $X = \{e_1, \dots, e_k\}$, with $k \geq 2$. Prove that the element $e_1 \otimes e_2 + e_2 \otimes e_1$ cannot be written as simple tensor $m \otimes n$, for some $m, n \in M$.

(2) Let R be a commutative ring (with 1) and $n, m \in \mathbb{N}$. Prove that there is an isomorphism of R-**algebras** $R^n \otimes R^m \simeq R^{nm}$. (Here by R^n we mean the direct sum $\underbrace{R \oplus \cdots \oplus R}_{n}$.)

(3) (a) Let V be a finite-dimensional $\mathbb{C}$-vector space. Then V can be considered as a vector over $\mathbb{R}$ (by restriction of scalars), and it holds $\dim_{\mathbb{R}} V = 2\dim_{\mathbb{C}} V$. Prove that $V \otimes_{\mathbb{C}} V$ is not isomorphic to $V \otimes_{\mathbb{R}} V$ as $\mathbb{R}$-vector spaces, and compute their dimensions over $\mathbb{R}$.

(b) Let R be an integral domain (commutative), and let K be its fraction field. Prove that there is an isomorphism of F-modules, $F \otimes_R F \simeq F \otimes_F F \simeq F$, where the F-module structure on $F \otimes_R F$ is given by **extension of scalars** (i.e. tensor product of Type I).

(4) The purpose of this problem is to classify all 2-dimensional $\mathbb{R}$-algebras (where $\mathbb{R}$ are the real numbers). That means, to classify (up to algebra isomorphism) those $\mathbb{R}$-algebras that are 2-dimensional $\mathbb{R}$ vector spaces.

Let A be a 2-dimensional $\mathbb{R}$-algebra (with 1).

(a) Let $u \in A$ be any element that is $\mathbb{R}$-linearly independent from 1. Prove that
   (i) u generates A as an $\mathbb{R}$-algebra. That is, the minimal $\mathbb{R}$-subalgebra of A containing u and 1 is A itself.
   (ii) The element u satisfies a quadratic equation $au^2 + bu + c = 0$, for some $a, b, c \in \mathbb{R}$ with $a \neq 0$. Conclude that A is necessarily commutative.

(b) Show that there exists some $v \in A$ which is $\mathbb{R}$-linearly independent from 1 and is such that $v^2 = -1$, or $v^2 = 1$, or $v^2 = 0$.

(c) Deduce from part (b) that A is isomorphic as an $\mathbb{R}$-algebra to one of the following: $\mathbb{R}[x]/(x^2+1)$, or $\mathbb{R}[x]/(x^2-1)$, or $\mathbb{R}[x]/(x^2)$.

(d) Prove that the algebras $\mathbb{R}[x]/(x^2+1)$, $\mathbb{R}[x]/(x^2-1)$, and $\mathbb{R}[x]/(x^2)$ are pairwise non-isomorphic. **Hint:** This can be shown with almost no computation.

(5) The purpose of this problem is to prove the following theorem: Let D be a finite dimensional division algebra over $\mathbb{R}$. Then D is isomorphic to $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ (the quaternions). One way to proceed is to use the following steps:

(a) Let $\alpha \in D$ be an element $\mathbb{R}$-linearly independent from 1. Show that α satisfies a quadratic irreducible polynomial $p_\alpha(x) = x^2 + ax + b \in \mathbb{R}[x]$.

(b) Let $V = \{\alpha \in D : \alpha^2 \in \mathbb{R}_{<0}\}$. Show that V is an $\mathbb{R}$-linear subspace of D. **Hint:** Show there is an $\mathbb{R}$-linear map $f : D \to \mathbb{R}$ with kernel V.

(c) Define $B : V \times V \to \mathbb{R}$, $B(\alpha, \beta) := -\frac{\alpha\beta + \beta\alpha}{2}$. Show that B defines an inner product on V (i.e. B is a symmetric, positive definite bilinear form on V).

(d) Let W be a linear subspace of V that generates D as an $\mathbb{R}$-algebra. Let $n = \dim_{\mathbb{R}} W$. Choose an orthonormal basis of W, i.e. a basis $\{e_i\}$ of W such that $B(e_i, e_i) = 1$ for all i and $B(e_i, e_j) = 0$ for all $i \neq j$ (such a basis always exists). Using this orthonormal basis show that if $n \geq 2$, then D has a subalgebra isomorphic to $\mathbb{H}$.

(e) **Bonus:** Suppose $n \geq 2$. Prove that $A = H$. **Hint:** One way to proceed is to show that if $n > 2$, then the multiplication in D cannot be associative.

Problems for extra Practice (not due)

(1) Let I and J be ideals of a commutative ring R. Let $\pi_I : R \to R/I$ and $\pi_J : R \to R/J$ be the canonical projections.
  (a) Prove that every element of $R/I \otimes_R R/J$ can be written as a simple tensor.
  (b) Prove that there is an isomorphism of R-modules, $R/I \otimes_R R/J \simeq R/(I + J)$.
  (c) Show that there is a surjective R-module homomorphism $\Phi : I \otimes_R J \to IJ$ such that $i \otimes j \mapsto ij$.
  (d) Give an example where the homomorphism Φ of part (c) is not an isomorphism.

(2) Let R, S be commutative rings (with 1). Let $f : R \to S$ be a ring homomorphism such that $f(1_R) = 1_S$, so that f induces an R-module structure on S. Let M be an S-module and N an R-module. Prove that there is an isomorphism of S-modules, $M \otimes_R N \simeq M \otimes_S (S \otimes_R N)$.