MATH 7310 Homework 3

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Problem 2

Let (X, Σ, μ) be a measure space. We say that $E \subseteq X$ is *an atom* if

- $E \in \Sigma$,
- $\mu(E) > 0$,
- $\{\mu(F) : F \subseteq E, F \in \Sigma\} = \{0, \mu(E)\}$.

We say the μ is *diffuse* if it has no atoms.

(a) Let (X, d, μ) be a metric measure space. Assume that μ is outer regular, and that

$$\mu(E) = \sup\{\mu(K) : K \subseteq E \text{ compact}\} \text{ for all Borel } E \subseteq X.$$

If $\mu(\{p\}) = 0$ for all $p \in X$, show that μ is diffuse.

Proof. Suppose, for the sake of contradiction, that μ is not diffuse. Then there exists an atom $E \subseteq X$. $\square$

(b) Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing, right-continuous function. Show that for $p \in \mathbb{R}$ we have that $\{p\}$ is an atom of μ_F if and only if F is discontinuous at p. Show that μ_F is diffuse if and only if F is continuous.

Problem 3

Let (X, Σ, μ) be a σ-finite measure space.
(i) Suppose that $(E_j)_{j\in J}$ is a collection of sets with $E_j \in \Sigma$ for all $j \in J$ and with $\mu(E_j) > 0$ for all $j \in J$, and so that $\mu(E_j \cap E_k) = 0$ for all $j \neq k$ in J. Show that J is countable.

Proof. Without loss of generality, assume that $X = \bigsqcup_{n=1}^{\infty} X_n$ where $X_n \in \Sigma$, $\mu(X_n > 0)$ for all $n \in \mathbb{N}$, and $X_i \cap X_j = \emptyset$ for $i \neq j$.

Suppose, for the sake of contradiction, that J is uncountable. For $j \in J$, note that

$$0 < \mu(E_j) = \sum_{n=1}^{\infty} \mu(E_j \cap X_n),$$

whence there exists an $n_j \in \mathbb{N}$ such that $\mu(E_j \cap X_n) > 0$. As there can only be countably many such n_j's and there are uncountably many E_j's, there exists a $k \in \mathbb{N}$ and $J_0 \subseteq J$ uncountable such that $\mu(E_j \cap X_k) > 0$ for all $j \in J_0$. By a pigeonhole argument, there exists a $b > 0$ and an infinite $J_0' \subseteq J_0$ such that $\mu(E_j \cap X_k) > b$

for all $j \in J_0'$.

Choose a countable sequence $(j_l)_{l=1}^{\infty}$ in J_0 such that $j_l \neq j_s$ for $l \neq s$. For $n \in \mathbb{N}$, set $F_n = E_j \cap X_k$. Note that, for $l \neq s$, we have that $\mu(F_l \cap F_s) = 0$.

We claim that $\mu(\sum_{l=1}^{n} F_l) = \sum_{l=1}^{n} \mu(F_l)$ for all $n \in \mathbb{N}$ by induction. Observe that

$$\mu(F_1 \cup F_2) = \mu(F_1) + \mu(F_2) - \mu(F_1 \cap F_2) = \mu(F_1) + \mu(F_2).$$

Now fix $n > 2$ and suppose that $\mu(\sum_{l=1}^{n-1} F_l) = \sum_{l=1}^{n-1} \mu(F_l)$. Observe that

$$\mu\left(F_n \cap \bigcup_{j=1}^{n-1} F_j\right) = \mu\left(\bigcup_{j=1}^{n-1} F_n \cap F_j\right) \leq \sum_{j=1}^{n-1} \mu(F_n \cap F_j) = 0,$$

so

$$\mu\left(\bigcup_{j=1}^{n} F_j\right) = \mu\left(F_n \cup \bigcup_{j=1}^{n-1} F_j\right) = \mu(F_n) + \sum_{j=1}^{n-1} \mu(F_j) - \mu\left(F_n \cap \bigcup_{j=1}^{n-1} F_j\right) = \sum_{j=1}^{n} \mu(F_j).$$

By induction, the claim holds for all $n \in \mathbb{N}$.

Observe that, for all $n \in \mathbb{N}$,

$$\mu\left(\bigcup_{j=1}^{\infty} F_j\right) \geq \mu\left(\bigcup_{j=1}^{n} F_j\right) = \sum_{j=1}^{n} \mu(F_j) \geq n \cdot b.$$

Hence $\mu\left(\bigcup_{j=1}^{\infty} F_j\right) = +\infty$, contradicting the fact that $\mu(X_k) < +\infty$.

$\square$

(ii) Let (Ω, ρ) be the metric space defined in Problem 12 of Chapter 1 of Folland. For $E \in \Sigma$, let $[E]$ be its equivalence class in Ω. Show that

$$\{[E] : E \subseteq X \text{ is an atom}\},$$

is countable.

Proof. Let $\sim$ be the equivalence relation $E \sim F \iff \mu(E \Delta F) = 0$. Let $\mathcal{E} = \{E : E \subseteq X \text{ is an atom}\}$. Let $\pi : \mathcal{E} \to \mathcal{E}/\sim$ be the canonical surjection. By the axiom of choice, there exists a section $s : \mathcal{E}/\sim \to \mathcal{E}$ such that $\pi \circ s = id_{\mathcal{E}/\sim}$. Take $E \neq F \in s(\mathcal{E}/\sim)$. Then, as s is injective, $[E] \neq [F]$, so $\mu(E_i \Delta E_j) > 00$.

Suppose, without loss of generality, that $\mu(E_i \setminus E_j) > 0$. Then, as E_i is an atom, $\mu(E_i \setminus E_j) = \mu(E_i)$. Hence

$$\mu(E_i) = \mu(E_i \setminus E_j) + \mu(E_i \cap E_j) = \mu(E_i) + \mu(E_i \cap E_j) \implies \mu(E_i \cap E_j) = 0$$

Hence, $s(\mathcal{E}/\sim)$ has the properties of the collection in part (i), so $s(\mathcal{E}/\sim)$ is countable whence injectivity implies that $\mathcal{E}/\sim$ is countable. $\square$

Problem 4

Let (X, Σ, μ) be a diffuse σ-finite measure space. For $A \in \Sigma$, show that:

$$\{\mu(B) : B \subseteq A, B \in \Sigma\} = [0, \mu(A)].$$

Suggestions: Reduce to the finite case. It might be helpful to first show that for every $E \in \Sigma$ with $\mu(E) > 0$, we have $0 = \inf\{\mu(B) : B \subseteq E \text{ and } \mu(B) > 0\}$.

Proof.
(*reduction to finite case*): Write $X = \bigcup_{i=1}^{\infty} X_i$ where $X_i \in \Sigma$ and $\mu(X_i) < +\infty$.

Suppose that $E \in \Sigma$ with $\mu(E) > 0$. Since μ is diffuse, there exists a $B_1 \subseteq E$ such that $B_1 \in \Sigma$ and $0 < \mu(B_1) < \mu(E)$. Note that either $\mu(B_1)$ or $\mu(E \setminus B_1)$ is less than $2^{-1}\mu(E)$, so without loss of generality assume that $\mu(B_1) < 2^{-1}\mu(E)$. Now, again as μ is diffuse, there exists a $B_2 \subseteq B_1$ such that $B_2 \in \Sigma$ and $0 < \mu(B_2) < \mu(B_1) < \mu(E)$. Again, we may assume without loss of generality that $\mu(B_2) < 2^{-1}\mu(B_1) < 2^{-2}\mu(E)$. Continuing as such, we obtain a decreasing sequence of sets $E \supset B_1 \supset B_2 \supset \cdots$ such that $0 < \mu(B_n) < 2^{-n}\mu(E)$. It follows that

$$0 = \inf\{\mu(B) : B \subseteq E \text{ and } \mu(B) > 0\}. \tag{1}$$

Suppose, for the sake of contradiction, that the claim is false. Then there exists an $A \in \Sigma \setminus \{\emptyset\}$ and $b \in (0, \mu(A))$ such that $\mu(B) \neq b$ for all $B \subseteq A$ with $B \in \Sigma$.
We proceed via transfinite induction on following statement:

$$P(\alpha): \quad \exists\ (B_\eta)_{\eta\in\alpha} \text{ in } \Sigma, \text{ pairwise disjoint subsets of } A, \text{ such that}$$
$$0 \notin \mu(\{B_\eta : \eta \in \alpha\}), \quad \bigsqcup_{\eta\in\alpha} B_\eta \in \Sigma, \text{ and } b - \mu\left(\bigsqcup_{\eta\in\alpha} B_\eta\right) > 0$$

First, note that we may choose B_0 such that $0 < \mu(B_0) < b$, so $P(0)$ holds. Suppose now that α is an ordinal and $P(\alpha)$ is true. Then there is a collection of pairwise disjoint elements $(B_\eta)_{\eta\in\alpha}$ of Σ which are subsets of A such that $\mu(B_\eta) > 0$ for all $\eta \in \alpha$, $\bigsqcup_{\eta\in\alpha} B_\eta \in \Sigma$, and $b - \mu\left(\bigsqcup_{\eta\in\alpha} B_\eta\right) > 0$. By (1), there exists a $B_\alpha \in \Sigma$ with $B_\alpha \subseteq A \setminus \bigsqcup_{\eta\in\alpha} B_\eta$ such that

$$0 < \mu(B_\alpha) < b - \mu\left(\bigsqcup_{\eta\in\alpha} B_\eta\right) \implies b - \mu\left(\bigsqcup_{\eta\in\alpha+1} B_\eta\right) > 0$$

and $B_\alpha \sqcup \bigsqcup_{\eta\in\alpha} B_\eta \in \Sigma$. Hence, $P(\alpha+1)$ holds.

Now, suppose that δ is a limit ordinal. $\square$