MATH 7310 Homework 2

James Harbour

February 7, 2022

Problem 1

Let μ be a finitely additive measure.

(a) Prove that μ is a measure if and only if it is continuous from below as in Theorem 1.8c.

Proof. Theorem 1.8c shows the forward direction so it suffices to show the reverse direction. Suppose that μ is continuous from below. Let $(E_j)_{j=1}^{\infty}$ be a sequence of disjoint elements in the sigma algebra $\mathcal{M}$ corresponding to μ. Define a new sequence $(F_n)_{n=1}^{\infty}$ in $\mathcal{M}$ by $F_n = \bigsqcup_{j=1}^{n} E_j$. Then $\bigsqcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} F_n$. As $(F_n)_{n=1}^{\infty}$ is an increasing sequence in $\mathcal{M}$, we have that

$$\mu\left(\bigsqcup_{n=1}^{\infty} E_n\right) = \mu\left(\bigcup_{n=1}^{\infty} F_n\right) = \lim_{n\to\infty} \mu(F_n) \lim_{n\to\infty} \sum_{j=1}^{n} \mu(E_j) = \sum_{j=1}^{\infty} \mu(E_j),$$

so μ is a measure. □

(b) If $\mu(X) < \infty$, prove that μ is a measure if and only if it is continuous from above as in Theorem 1.8d.

Proof. Theorem 1.8d shows the forward direction so it suffices to show the reverse direction. Suppose that μ is continuous from above. Let $(E_j)_{j=1}^{\infty}$ be a sequence of disjoint elements in $\mathcal{M}$. Define a new sequence $(F_n)_{n=1}^{\infty}$ in $\mathcal{M}$ by $F_n = \bigsqcup_{j=1}^{n} E_j$. Observe that $F_1^c \supset F_2^c \subset F_3^c \supset \cdots$ is a decreasing sequence in $\mathcal{M}$ with $\mu(F_1^c) = \mu(X) - \mu(F_1) < +\infty$. Hence, by continuity from above,

$$\mu\left(\bigsqcup_{j=1}^{\infty} E_j\right) = \mu\left(\bigcup_{n=1}^{\infty} F_n\right) = \mu\left(X \setminus \bigcap_{n=1}^{\infty} F_n^c\right) = \mu(X) - \mu\left(\bigcap_{n=1}^{\infty} F_n^c\right) = \mu(X) - \lim_{n\to\infty} (F_n^c)$$

$$= \mu(X) - \lim_{n\to\infty} \mu\left(X \setminus \bigsqcup_{j=1}^{n} E_j\right) = \mu(X) - \lim_{n\to\infty} \mu(X) - \mu\left(X \setminus \bigsqcup_{j=1}^{n} E_j\right) = \lim_{n\to\infty} \sum_{j=1}^{n} \mu(E_j) = \sum_{j=1}^{\infty} \mu(E_j),$$

so μ is a measure. □

Problem 2

Let $(X, \mathcal{M}, \mu)$ be a finite measure space.

(a) If $E, F \in \mathcal{M}$ and $\mu(E \Delta F) = 0$, then $\mu(E) = \mu(F)$.

Proof. Observe that

$$0 = \mu(E\Delta F) = \mu((E \setminus F) \sqcup (F \setminus E)) = \mu(E \setminus F) + \mu(F \setminus E).$$

As $\mu(E \setminus F), \mu(F \setminus E) \geq 0$, it follows that $\mu(E \setminus F), \mu(F \setminus E) = 0$. Then as $E = (E \setminus F) \sqcup (E \cap F)$ and $F = (F \setminus E) \sqcup (F \cap E)$, $\mu(E) = \mu(F)$. □

(b) Say that $E \sim F$ if $\mu(E\Delta F) = 0$; show that $\sim$ is an equivalence relation on $\mathcal{M}$.

Proof.
(Reflexivity): Note that $E\Delta E = E \setminus E = \emptyset \implies \mu(E\Delta E) = 0$, so $E \sim E$.
(Symmetry): Note that $E\Delta F = (E \setminus F) \sqcup (F \setminus E) = F\Delta E$, so $E \sim F \implies F \sim E$.
(Transitivity): Suppose that $E \sim F$ and $F \sim G$. Observe that

$$E \setminus G = ((E \setminus F) \sqcup (E \cap F)) \setminus G = ((E \setminus F) \setminus G) \cup ((E \cap F) \setminus G) \subseteq (E \setminus F) \cup (F \setminus G)$$
$$G \setminus E = ((G \setminus F) \sqcup (G \cap F)) \setminus E = ((G \setminus F) \setminus E) \cup ((G \cap F) \setminus E) \subseteq (G \setminus F) \cup (F \setminus E)$$

so by monotonicity and subadditivity,

$$\mu(E\Delta G) \leq \mu((E\setminus F)\cup(F\setminus G)) + \mu((G\setminus F)\cup(F\setminus E)) \leq \mu(E\setminus F) + \mu(F\setminus E) + \mu(F\setminus G) + \mu(G\setminus F) = \mu(E\Delta F) + \mu(F\Delta G) =$$

hence $E \sim G$. □

(c) For $E, F \in \mathcal{M}$, define $\rho(E, F) = \mu(E\Delta F)$. Then $\rho(E, G) \leq \rho(E, F) + \rho(F, G)$, and hence ρ defines a metric on the space $\mathcal{M}/\sim$.

Proof. Note that the inequality used in the proof of transitivity above held regardless of the assumptions that the symmetric differences were zero, whence

$$\rho(E, G) = \mu(E\Delta G) \leq \mu(E\Delta F) + \mu(F\Delta G) = \rho(E, F) + \rho(F, G).$$

□

Problem 3

Let $\mathcal{A}$ be the collection of finite unions of sets of the form $(a, b] \cap \mathbb{Q}$ where $-\infty \leq a < b \leq +\infty$.

(i) Show that $\mathcal{A}$ is an algebra on $\mathbb{Q}$. (Use Proposition 1.7.)

Proof. Let $\mathcal{E} = \{(a, b] \cap \mathbb{Q} : -\infty \leq a < b \leq +\infty\} \cup \{\emptyset\}$. By Proposition 1.7, it suffices to show that $\mathcal{E}$ is an elementary family.

Suppose $(a, b] \cap \mathbb{Q}, (c, d] \cap \mathbb{Q} \in \mathcal{E}$, with $a < b$ and $c < d$. If $b \leq c$, then $((a, b] \cap \mathbb{Q}) \cap ((c, d] \cap \mathbb{Q}) = \emptyset \in \mathcal{E}$. If $b > c$, then $((a, b] \cap \mathbb{Q}) \cap ((c, d] \cap \mathbb{Q}) = (c, b] \cap \mathbb{Q} \in \mathcal{E}$.

Lastly, suppose that $(a, b] \cap \mathbb{Q}$ with $a < b$. If $a = -\infty$ and $b = +\infty$, then $\mathbb{Q} \setminus ((-\infty, +\infty] \cap \mathbb{Q}) = \emptyset$. If $a = -\infty$ and $b \neq +\infty$, then $\mathbb{Q} \setminus ((-\infty, b] \cap \mathbb{Q}) = (b, +\infty] \cap \mathbb{Q}$. If $a \neq -\infty$ and $b = +\infty$, then $\mathbb{Q} \setminus ((a, +\infty] \cap \mathbb{Q}) = (-\infty, a] \cap \mathbb{Q}$. Finally, if $a \neq -\infty$ and $b \neq +\infty$, then $\mathbb{Q} \setminus ((a, b] \cap \mathbb{Q}) = ((-\infty, a] \cap \mathbb{Q}) \sqcup ((b, +\infty] \cap \mathbb{Q})$. So $\mathcal{E}$ is an elementary family. □

(ii) Show that the σ-algebra generated by $\mathcal{A}$ is $\mathcal{P}(\mathbb{Q})$.

Proof. As $\mathcal{A} \subseteq \mathcal{P}(\mathbb{Q})$, by minimality $\Sigma(\mathcal{A}) \subseteq \mathcal{P}(\mathbb{Q})$. Now take $q \in Q$. Observe that $(q - \frac{1}{n}, q] \cap \mathbb{Q} \in \mathcal{A}$ for all $n \in \mathbb{N}$, whence $\{q\} = \bigcap_{n=1}^{\infty}(q - \frac{1}{n}, q] \cap \mathbb{Q} \in \Sigma(\mathcal{A})$. Hence, $\Sigma(\mathcal{A})$ contains all finite and countable subsets of $\mathbb{Q}$, so countability of $\mathbb{Q}$ implies that $\mathcal{P}(\mathbb{Q}) \subseteq \Sigma(\mathcal{A})$. □

(ii) Define μ_0 on $\mathcal{A}$ by $\mu_0(\emptyset) = 0$ and $\mu_0(A) = \infty$ for $A \neq \emptyset$. Prove that μ_0 is a premeasure on $\mathcal{A}$, and that there is more than one measure on $\mathcal{P}(\mathbb{Q})$ whose restriction to $\mathcal{A}$ is μ_0.

Proof. To see that μ_0 is a premeasure, suppose that $(A_j)_{j=1}^{\infty}$ is a sequence of pairwise disjoint elements of $\mathcal{A}$ such that $\bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$. If $A_j = \emptyset$ for all $j \in \mathbb{N}$, then $\bigcup_{j=1}^{\infty} A_j = \emptyset$ whence $\mu_0(\bigcup_{j=1}^{\infty} A_j) = 0 = \sum_{j=1}^{\infty} 0 = \sum_{j=1}^{\infty} \mu_0(A_j)$. If there exists a $k \in \mathbb{N}$ such that $A_k \neq \emptyset$, then $A_k \subseteq \bigcup_{j=1}^{\infty} A_j \neq \emptyset$, so $\mu_0(\bigcup_{j=1}^{\infty} A_j) = +\infty = \sum_{j=1}^{\infty} \mu_0(A_j)$.

On one hand, we have an outer measure

$$\mu_0^*(E) = \inf \left\{ \sum_{j=1}^{\infty} \mu_0(A_j) : A_j \in \mathcal{A},\ E \subseteq \bigcup_{j=1}^{\infty} A_j \right\}$$

for $E \in \mathcal{P}(\mathbb{Q})$. Note that, $\mu_0^*(E) = 0$ if $E = \emptyset$ and $\mu_0^*(E) = +\infty$ if $E \neq \emptyset$. Moreover, this outer measure is in fact a measure on $E \in \mathcal{P}(\mathbb{Q})$ extending μ_0 by the same reasoning showing μ_0 is a premeasure, so let $\mu = \mu_0^*$.

On the other hand, consider the counting measure $\nu : \mathcal{P}(\mathbb{Q}) \to [0, +\infty]$. Note that, if $A \in \mathcal{A}$ and $A \neq \emptyset$, then A must contain infinitely many elements, whence $\nu(A) = \infty$. Hence ν agrees with μ_0 on $\mathcal{A}$. However, ν has finite, nonzero value on finite, nonempty subsets of $\mathbb{Q}$, so $\nu \neq \mu$. □

Problem 4

Let $\mathcal{A}$ be an alegbra, and let $\mu : \mathcal{A} \to [0, +\infty]$ be a finitely additive measure.

(i) Suppose $(A_j)_{j=1}^{\infty}$ are pairwise disjoint elements of $\mathcal{A}$, and that $A = \bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$. Show that

$$\mu(A) \geq \sum_{j=1}^{\infty} \mu(A_j).$$

Proof. Since μ is finitely additive, it is also finitely subadditive. Then by monotonicity, for any $n \in \mathbb{N}$,

$$\mu(A) \geq \mu\left(\bigsqcup_{j=1}^{n} A_j\right) = \sum_{j=1}^{n} \mu(A_j).$$

Hence, it follows that $\mu(A) \geq \sum_{j=1}^{\infty} \mu(A_j)$. □

(ii) Show that the following are equivalent:

1. μ is a premeasure,
2. $\mu\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} \mu(A_j)$ for any sequence $(A_j)_{j=1}^{\infty}$ with $\bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$,
3. for any increasing sequence $(E_j)_{j=1}^{\infty}$ in $\mathcal{A}$ with $\bigcup_{j=1}^{\infty} E_j \in \mathcal{A}$, we have

$$\mu\left(\bigcup_{j} E_j\right) = \lim_{n\to\infty} \mu(E_n).$$

Proof.
(2 $\implies$ 1): Suppose that $(A_j)_{j=1}^\infty$ are pairwise disjoint elements of $\mathcal{A}$ with $A = \bigcup_{j=1}^\infty A_j \in \mathcal{A}$. by part (i), $\mu(\bigsqcup_{j=1}^\infty A_j) \geq \sum_{j=1}^\infty \mu(A_j)$. On the other hand, by assumption $\mu(\bigsqcup_{j=1}^\infty A_j) \leq \sum_{j=1}^\infty \mu(A_j)$, so

$$\mu\left(\bigsqcup_{j=1}^\infty A_j\right) = \sum_{j=1}^\infty \mu(A_j).$$

Hence, μ is a premeasure.
(1 $\implies$ 3): Let $(E_j)_{j=1}^\infty$ be an increasing sequence in $\mathcal{A}$ with $\bigcup_{j=1}^\infty E_j \in \mathcal{A}$. Define a new sequence in $\mathcal{A}$ by $E_1' = E_1$ and $E_j' = E_j \setminus E_{j-1}$ for $j \geq 2$. Then,

$$\mu\left(\bigcup_{j=1}^\infty E_j\right) = \mu\left(\bigsqcup_{j=1}^\infty E_j'\right) = \sum_{j=1}^\infty \mu(E_j') = \lim_{n\to\infty} \sum_{j=1}^n \mu(E_j') = \lim_{n\to\infty} \mu(E_n).$$

(3 $\implies$ 2): Suppose that $(A_j)_{j=1}^\infty$ is a sequence in $\mathcal{A}$ with $\bigcup_{j=1}^\infty A_j \in \mathcal{A}$. Then, by finite subadditivity (which follows from finite additivity),

$$\mu\left(\bigcup_{j=1}^\infty A_j\right) = \mu\left(\bigcup_{n=1}^\infty \bigcup_{j=1}^n A_j\right) = \lim_{n\to\infty} \mu\left(\bigcup_{j=1}^n A_j\right) \leq \lim_{n\to\infty} \sum_{j=1}^n \mu(A_j) = \sum_{j=1}^\infty \mu(A_j).$$

$\square$

(iii) If $\mu(X) < +\infty$, show that μ is a premeasure if and only if for every decreasing sequence $(E_n)_{n=1}^\infty$ of sets in $\mathcal{A}$ with $\bigcap_{n=1}^\infty E_n = \emptyset$, we have

$$\lim_{n\to\infty} \mu(E_n) = 0.$$

Proof.
$\underline{\implies}$: Let $(E_n)_{n=1}^\infty$ be a decreasing sequence of sets in $\mathcal{A}$ with $\bigcap_{n=1}^\infty E_n = \emptyset$. Note that then the sequence of sets $(X \setminus E_n)_{n=1}^\infty$ is increasing, so by number 3 in part (ii) and utilizing finiteness of $\mu(X)$,

$$\mu\left(\bigcup_{n=1}^\infty X \setminus E_n\right) = \lim_{n\to\infty} \mu(X \setminus E_n) = \lim_{n\to\infty} \mu(X) - \mu(E_n).$$

Hence,

$$0 = \mu\left(\bigcap_{n=1}^\infty E_n\right) = \mu\left(X \setminus \bigcap_{n=1}^\infty (X \setminus E_n)\right) = \mu(X) - \mu\left(\bigcup_{n=1}^\infty X \setminus E_n\right) = \mu(X) - \lim_{n\to\infty}(\mu(X) - \mu(E_n)) = \lim_{n\to\infty} \mu(E_n)$$

$\underline{\impliedby}$: Let $(A_j)_{j=1}^\infty$ be a sequence of pairwise disjoint elements of $\mathcal{A}$ such that $A = \bigcup_{j=1}^\infty A_j \in \mathcal{A}$. Define a new sequence in $\mathcal{A}$ by $E_n = A \setminus \bigsqcup_{j=1}^n A_j$. Then $(E_n)_{n=1}^\infty$ is a decreasing sequence in $\mathcal{A}$ such that $\bigcap_{n=1}^\infty = 0$. Hence, by assumption,

$$0 = \lim_{n\to\infty} \mu(E_n) = \lim_{n\to\infty} \mu(A) - \mu\left(\bigsqcup_{j=1}^n A_j\right) = \mu(A) - \lim_{n\to\infty} \sum_{j=1}^n \mu(A_j) \implies \mu\left(\bigsqcup_{j=1}^\infty A_j\right) = \sum_{j=1}^\infty \mu(A_j),$$

so μ is a premeasure. $\square$

Problem 5

A *metric measure space* is a triple (X, d, μ) where (X, d) is a metric space and $\mu : \mathcal{B}_{(X,d)} \to [0, +\infty]$ is a measure. We say that $E \subseteq X$ is a *continuity set* if $\mu(\overline{E} \setminus \operatorname{Int}(E)) = 0$. For this problem, fix a metric measure space (X, d, μ).

(i) Show that the collection of continuity sets forms an algebra of sets.

Proof. Suppose that $E_1, \ldots, E_n \subseteq X$ are continuity sets. Then $\mu(\overline{E_j} \setminus \operatorname{Int}(E_j)) = 0$ for $1 \leq j \leq n$. As there are finitely many sets, the union of closures is equal to the closure of the union. Hence

$$\overline{\bigcup_{j=1}^n E_j} \setminus \operatorname{Int}\left(\bigcup_{j=1}^n E_j\right) = \bigcup_{j=1}^n \overline{E_j} \setminus \operatorname{Int}\left(\bigcup_{j=1}^n E_j\right) \subseteq \bigcup_{j=1}^n \overline{E_j} \setminus \bigcup_{j=1}^n \operatorname{Int}(E_j) = \bigcup_{j=1}^n \overline{E_j} \setminus \operatorname{Int}(E_j),$$

so by subadditivity,

$$\mu\left(\overline{\bigcup_{j=1}^n E_j} \setminus \operatorname{Int}\left(\bigcup_{j=1}^n E_j\right)\right) = \mu\left(\bigcup_{j=1}^n \overline{E_j} \setminus \operatorname{Int}(E_j)\right) \leq \sum_{j=1}^n \mu(\overline{E_j} \setminus \operatorname{Int}(E_j)) = 0$$

whence $E_1 \cup \cdots \cup E_n$ is a continuity set. Now suppose that $E \subseteq X$ is a continuity set.

$$(\overline{X \setminus E}) \setminus \operatorname{Int}(X \setminus E) = (X \setminus \operatorname{Int}(E)) \setminus \operatorname{Int}(X \setminus E) = (X \setminus \operatorname{Int}(E)) \setminus (X \setminus \overline{E}) = \overline{E} \setminus \operatorname{Int}(E)$$

so $\mu((\overline{X \setminus E}) \setminus \operatorname{Int}(X \setminus E)) = \mu(\overline{E} \setminus \operatorname{Int}(E)) = 0$, whence $X \setminus E$ is also a continuity set. □

(ii) Show that if $x \in X$, $r > 0$ and $\mu(B_r(x, d)) < +\infty$, then there is an $s \in (0, r)$ so that $B_s(x, d)$ is a continuity set.

Proof. We show the stronger statement that, for all $\varepsilon \in (0, r]$, there exists an $s \in (r - \varepsilon, r)$ so that $B_s(x, d)$ is a continuity set.

Suppose, for the sake of contradiction, that there exists an $\varepsilon \in (0, r]$ such that $\mu(\overline{B_s(x)} \setminus \operatorname{Int}(B_s(x))) \neq 0$ for all $s \in (r - \varepsilon, r)$. For $n \in \mathbb{N}$, define a set

$$A_n = \{s \in (r - \varepsilon, r) : \frac{1}{n} \leq \mu(\overline{B_s(x)} \setminus \operatorname{Int}(B_s(x))) < \frac{1}{n-1}\}$$

where $1/0 := \infty$ by convention. Then, by assumption, $(r - \varepsilon, r) = \bigcup_{n=1}^\infty A_n$ which is necessarily uncountable, so there exists an $n \in \mathbb{N}$ such that A_n is infinite. Take a countably infinite subset $\{s_1, s_2, \ldots\} \subseteq A_n$. Note that, for any fixed $t \in (0, +\infty)$, $\overline{B_t(x)} \setminus \operatorname{Int}(B_t(x)) \subseteq \{y \in X : d(x, y) = t\}$, whence the following union is disjoint:

$$\mu\left(\bigsqcup_{j=1}^\infty \overline{B_{s_j}(x)} \setminus \operatorname{Int}(B_{s_j}(x))\right) = \sum_{j=1}^\infty \mu(\overline{B_{s_j}(x)} \setminus \operatorname{Int}(B_{s_j}(x))) = \infty.$$

However, this contradicts that $\mu(B_r(x)) < \infty$. □

(iii) Suppose that (X, d) is separable and that for every $x \in X$, there is an $r > 0$ so that $\mu(B_r(x, d)) < +\infty$. Show that there is a countable basis consisting of open continuity sets. (Hint: given a countable dense $D \subseteq X$ and $x \in D$, use the preceding part to choose a countable set $J_x \subseteq (0, +\infty)$ with the property that $\inf_{t \in J_x} t = 0$ and so that $B_t(x, d)$ is a continuity set for all $t \in J_x$).

Proof. Let $D \subseteq X$ be a countable dense subset of X. Fix $x \in D$. Then there exists an $r_x > 0$ such that $\mu(B_{r_x}(x,d)) < +\infty$. Define two sets $L_x, U_x \subseteq (0, r_x)$ as follows. For each $n \in \mathbb{N}$, by part (ii) there exists a $u_n \in (r_x - \frac{1}{n}, r_x)$ and $l_n \in (0, \frac{r_x}{n})$ such that $B_{u_n}(x,d)$ and $B_{l_n}(x,d)$ are continuity sets. Let $L_x = \{l_n : n \in \mathbb{N}\}$ and $U_x = \{u_n : n \in \mathbb{N}\}$ and set $J_x = U_x \cup L_x$. Then $J_x \subseteq (0, +\infty)$ is countable such that $\inf_{t \in J_x} t = 0$, $\sup_{t \in J_x} = r_x$, and $B_t(x,d)$ is a continuity set for all $t \in J_x$. Let

$$\mathscr{J} = \bigsqcup_{x \in D} \{x\} \times J_x.$$

By the axiom of choice, $\mathscr{J}$ is countable. Let $\mathscr{B} = \{B_t(x,d) : (x,t) \in \mathscr{J}\}$. We claim that $\mathscr{B}$ is a basis for the metric topology on (X,d). As $\mathscr{B}$ covers D and D is dense in X, it is clear that $\mathscr{B}$ covers X.

Suppose that $x \in X$ and $B_t(y,d), B_{t'}(z,d) \in \mathscr{B}$ such that $x \in B_t(y,d) \cap B_{t'}(z,d)$. As $\inf_{t \in J_x} t = 0$, there exists a $t'' \in J_x$ such that $t'' \le t, t'$, whence $x \in B_{t''}(x,d) \subseteq B_t(x,d) \cap B_t(x,d)$. □

Problem 6

Let (X,d) be a metric space and μ, ν be finite Borel measures on X with $\mu(X) = \nu(X)$. Let $\mathcal{A} = \{E \in \mathcal{B}_{(X,d)} : \mu(E) = \nu(E)\}$.

(i) Show that if $F \subseteq E$ and $F, E \in \mathcal{A}$, then $E \setminus F \in \mathcal{A}$. Also show that if $(E_n)_{n=1}^{\infty}$ is an increasing sequence of elements of $\mathcal{A}$, then $\bigcup_{n=1}^{\infty} E_n \in \mathcal{A}$.

Proof. As $E, F \in \mathcal{A}$, $\mu(E) = \nu(E)$ and $\mu(F) = \nu(F)$. Then

$$\mu(E \setminus F) = \mu(E) - \mu(F) = \nu(E) - \nu(F) = \nu(E \setminus F)$$

so $E \setminus F \in \mathcal{A}$. Now suppose that $(E_n)_{n=1}^{\infty}$ is an increasing sequence of elements of $\mathcal{A}$. By continuity from above,

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n) = \lim_{n\to\infty} \nu(E_n) = \nu\left(\bigcup_{n=1}^{\infty} E_n\right),$$

so $\bigcup_{n=1}^{\infty} E_n \in \mathcal{A}$. □

(ii) Given a nonempty $F \subseteq X$ closed and $x \in X$, define $d(x,F) = \inf_{y \in F} d(x,y)$. Show that $x \mapsto d(x,F)$ is continuous and $F = \{x \in X : d(x,F) = 0\}$.

Proof. Suppose $x, y \in X$. For $z \in F$,

$$d(x,F) \le d(x,z) \le d(x,y) + d(y,z) \implies d(x,F) - d(x,y) \le d(y,z).$$

As this holds for arbitrary $z \in F$, it follows that $d(x,F) - d(x,y) \le d(y,F)$, so $d(x,F) - d(y,F) \le d(x,y)$. By symmetry, $d(y,F) - d(x,F) \le d(x,y)$, so $|d(x,F) - d(y,F)| \le d(x,y)$. Thus, the function $x \mapsto d(x,F)$ is $1-$ *Lipschitz* whence it is continuous.

Clearly $F \subseteq \{x \in X : d(x,F) = 0\}$, so it suffices to show the reverse containment. Suppose that $x \in X$ such that $d(x,F) = 0$. For all $n \in \mathbb{N}$, there exists an $f_n \in F$ such that $0 \le d(x, f_n) < \frac{1}{n}$. It follows that $d(x, f_n) \xrightarrow{n\to\infty} 0$, so $f_n \xrightarrow{n\to\infty} x$. Thus x is a limit point of F, so F being closed implies that $x \in F$. □

(iii) Show that $\{U \subseteq X : U \text{ is open}\} \subseteq \mathcal{A}$ if an only if $\{F \subseteq X : F \text{ is closed}\} \subseteq \mathcal{A}$.

Proof.

$\underline{\implies}$: Suppose that $\{U \subseteq X : U \text{ is open}\} \subseteq \mathcal{A}$. Take $F \subseteq X$ such that F is closed. Then $X \setminus F$ is open, whence by finiteness of μ and ν,

$$\mu(X) - \mu(F) = \mu(X \setminus F) = \nu(X \setminus F) = \nu(X) - \nu(F) \implies \mu(F) = \nu(F)$$

so $F \in \mathcal{A}$.

$\underline{\impliedby}$: Likewise, suppose that $\{U \subseteq X : U \text{ is closed}\} \subseteq \mathcal{A}$. Take $U \subseteq X$ such that U is open. Then $X \setminus F$ is closed, whence by finiteness of μ and ν,

$$\mu(X) - \mu(U) = \mu(X \setminus U) = \nu(X \setminus U) = \nu(X) - \nu(U) \implies \mu(U) = \nu(U)$$

so $U \in \mathcal{A}$.

$\square$