MATH 7310 Homework 3

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Problem 2

Let (X, Σ, μ) be a measure space. We say that $E \subseteq X$ is *an atom* if

- $E \in \Sigma$,
- $\mu(E) > 0$,
- $\{\mu(F) : F \subseteq E, F \in \Sigma\} = \{0, \mu(E)\}$.

We say the μ is *diffuse* if it has no atoms.

(a) Let (X, d, μ) be a metric measure space. Assume that μ is outer regular, and that

$$\mu(E) = \sup\{\mu(K) : K \subseteq E \text{ compact}\} \text{ for all Borel } E \subseteq X.$$

If $\mu(\{p\}) = 0$ for all $p \in X$, show that μ is diffuse.

Proof. Suppose, for the sake of contradiction, that μ is not diffuse. Then there exists an atom $E \subseteq X$. $\square$

(b) Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing, right-continuous function. Show that for $p \in \mathbb{R}$ we have that $\{p\}$ is an atom of μ_F if and only if F is discontinuous at p. Show that μ_F is diffuse if and only if F is continuous.

Problem 4

Let (X, Σ, μ) be a diffuse σ-finite measure space. For $A \in \Sigma$, show that:

$$\{\mu(B) : B \subseteq A, B \in \Sigma\} = [0, \mu(A)].$$

Suggestions: Reduce to the finite case. It might be helpful to first show that for every $E \in \Sigma$ with $\mu(E) > 0$, we have $0 = \inf\{\mu(B) : B \subseteq E \text{ and } \mu(B) > 0\}$.

Proof.
(*reduction to finite case*): Write $X = \bigcup_{i=1}^{\infty} X_i$ where $X_i \in \Sigma$ and $\mu(X_i) < +\infty$.

Suppose that $E \in \Sigma$ with $\mu(E) > 0$. Since μ is diffuse, there exists a $B_1 \subseteq E$ such that $B_1 \in \Sigma$ and $0 < \mu(B_1) < \mu(E)$. Note that either $\mu(B_1)$ or $\mu(E \setminus B_1)$ is less than $2^{-1}\mu(E)$, so without loss of generality assume that $\mu(B_1) < 2^{-1}\mu(E)$. Now, again as μ is diffuse, there exists a $B_2 \subseteq B_1$ such that $B_2 \in \Sigma$ and $0 < \mu(B_2) < \mu(B_1) < \mu(E)$. Again, we may assume without loss of generality that $\mu(B_2) < 2^{-1}\mu(B_1) <$

$2^{-2}\mu(E)$. Continuing as such, we obtain a decreasing sequence of sets $E \supset B_1 \supset B_2 \supset \cdots$ such that $0 < \mu(B_n) < 2^{-n}\mu(E)$. It follows that

$$0 = \inf\{\mu(B) : B \subseteq E \text{ and } \mu(B) > 0\}. \tag{1}$$

Suppose, for the sake of contradiction, that the claim is false. Then there exists an $A \in \Sigma \setminus \{\emptyset\}$ and $b \in (0, \mu(A))$ such that $\mu(B) \neq b$ for all $B \subseteq A$ with $B \in \Sigma$.
We proceed via transfinite induction on following statement:

$$P(\alpha): \quad \exists\ (B_\eta)_{\eta\in\alpha} \text{ in } \Sigma, \text{ pairwise disjoint subsets of } A, \text{ such that}$$
$$0 \notin \mu(\{B_\eta : \eta \in \alpha\}), \quad \bigsqcup_{\eta\in\alpha} B_\eta \in \Sigma, \text{ and } b - \mu\left(\bigsqcup_{\eta\in\alpha} B_\eta\right) > 0$$

First, note that we may choose B_0 such that $0 < \mu(B_0) < b$, so $P(0)$ holds. Suppose now that α is an ordinal and $P(\alpha)$ is true. Then there is a collection of pairwise disjoint elements $(B_\eta)_{\eta\in\alpha}$ of Σ which are subsets of A such that $\mu(B_\eta) > 0$ for all $\eta \in \alpha$, $\bigsqcup_{\eta\in\alpha} B_\eta \in \Sigma$, and $b - \mu\left(\bigsqcup_{\eta\in\alpha} B_\eta\right) > 0$. By (1), there exists a $B_\alpha \in \Sigma$ with $B_\alpha \subseteq A \setminus \bigsqcup_{\eta\in\alpha} B_\eta$ such that

$$0 < \mu(B_\alpha) < b - \mu\left(\bigsqcup_{\eta\in\alpha} B_\eta\right) \implies b - \mu\left(\bigsqcup_{\eta\in\alpha+1} B_\eta\right) > 0$$

and $B_\alpha \sqcup \bigsqcup_{\eta\in\alpha} B_\eta \in \Sigma$. Hence, $P(\alpha+1)$ holds.

Now, suppose that δ is a limit ordinal. $\square$