Homework 12

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11.10 Question 25

Find the Taylor series for $f(x)$ centered at the given value of a. [Assume that f has a power series expansion. Do not show that $R_n(x) \to 0$]. Also find the associated radius of convergence.

$$f(x) = \sin x, \quad a = \pi$$

Solution

Firs we write out the first couple of derivates for $f(x)$ and solve for $x = \pi$

$$\begin{aligned} f'(x) &= \cos x \\ f''(x) &= -\sin x \\ f'''(x) &= -\cos x \\ f^{(4)}(x) &= \sin x \end{aligned}$$

and we see that only the odd terms will construct the series because even derivatives evaluated at π are zero; so we can write the odd terms of the taylor series as;

$$\begin{aligned} n = 1 &\to \frac{1}{1!}(x-\pi) \\ n = 3 &\to \frac{1}{3!}(x-\pi)^3 \\ n = 5 &\to \frac{-1}{5!}(x-\pi)^5 \end{aligned}$$

which we use to construct the general term of the series;

$$f(x) = -\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(x-\pi)^{2n+1} \tag{1}$$

and then find the radius of convergene using the ration test;

$$\begin{aligned} L &= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\ &= \lim_{n\to\infty} \left| \frac{(x-\pi)^{2(n+1)+1}/(2(n+1)+1)!}{(x-\pi)^{2n+1}/(2n+1)!} \right| \\ &= \lim_{n\to\infty} \left| \frac{(x-\pi)^2}{(2n+3)(2n+2)} \right| = 0 \end{aligned}$$

because $L = 0$ the series converges for all $x \in \mathbb{R}$