Final Exam Notes

Trig angles

Function	$\frac{\pi}{4}(45)$	$\frac{\pi}{3}(60)$	$\frac{\pi}{6}(30)$
$\sin\theta$	$\frac{2}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\cos\theta$	$\frac{2}{\sqrt{2}}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\tan\theta$	1	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$

Work

$$W = \int f(x)\,dx \quad OR \quad W = \int \rho\cdot d\cdot A\,dy$$

Integration by parts

$$\int f(x)g'(x)\,dx = \int f(x)g(x) - \int g(x)f'(x)\,dx$$

Partial Fraction Decomposition

Given some rational function
$$\frac{x+1}{(x+1)(x+3)} = \frac{A}{x+1} + \frac{B}{x+3} \implies x+1 = A(x+3) + B(x+1)$$
Solve for A and B using a system of equations then integrate. Note if we have a repeat term like $(x+1)^2$ then;
$$\frac{x+1}{(x+1)^2(x+3)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+3}$$
Then solve as normal

Volumes by slices

$$V = \int_{x_1}^{x_2} \pi r^2(x)\,dx \quad \text{OR} \quad V = \int_{x_1}^{x^2} \pi\left(r_1^2 - r_2^2\right)\,dx$$

Volumes by shells

$$V = \int_{x_1}^{x_2} 2\pi x f(x)\,dx$$
Note: be careful about the integration bounds and radius, for example, a slice with r_1 of $y = 3$ and r_2 of $\sec(x) + 1$ rotated about $y = 1$ would be
$A = \pi\left((3-1)^2 - (\sec(x) + 1 - 1)^2\right)$.

Trig sub

Expression	Substitution
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$

1. Draw a triangle
2. Define the edge lengths using pythagorean theorem
3. Define trig functions for the triangle
4. Sovle for x in terms of a trig function and dx. Ex;

$$x = \sin\theta(x)$$

$$dx = cos\theta\,d\theta$$

Arc length

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

Derived from pathagorean theorem for the infinite sum of secant lines i.e. $\sqrt{\Delta x^2 + \Delta y^2} = \sqrt{\Delta x^2 + (f'(x)\Delta x)^2}$ factor out Δx to obtain the equation.

Some other useful antiderivates

$$\int \frac{1}{\sqrt{1-x^2}} = \arcsin x + C$$
$$\int -\frac{1}{\sqrt{1-x^2}} = \arccos x + C$$
$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$$
$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right|$$

Surface area

$$S = 2\pi\int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

Hydrostatic force

Pressure is given by

$$P = \int_{\text{deepest point}}^{\text{shallow point}} \rho\cdot area\cdot depth$$

where area is width as a function of y and height is dy and ρ is the density.

Trig integral tip

If encountering a form like

$$\int \sin^5 x\cos^4 x\,dx$$

let u be the lower order term and keep one of the higher order terms from the other then convert the rest to match using $1 = \cos^2 x + \sin^2 x$

Problems that were implied to be on the exam

Separable differential equation with PFD

$$\frac{dy}{dx} = y \cdot \frac{x+1}{(x-1)^2(x+2)}$$
$$\frac{1}{y}\frac{dy}{dx} = \frac{x+1}{(x-1)^2(x+2)}$$
$$\int \frac{1}{y}\frac{dy}{dx}\,dx = \int \frac{x+1}{(x-1)^2(x+2)}\,dx$$
by ROS
$$\int \frac{1}{y}\,dy = \int \frac{x+1}{(x-1)^2(x+2)}\,dx$$

handling the fraction;

$$\begin{aligned}\frac{x+1}{(x-1)^2(x+2)} &= \frac{A}{(x-1)^2} + \frac{B}{x+2} + \frac{C}{x-1}\\ x+1 &= A(x+2) + B(x-1)^2 + C(x-1)(x+2)\\ x+1 &= Ax + 2A + Bx^2 - 2Bx + B + Cx^2 + Cx - 2C\\ x+1 &= x(A - 2B + C) + x^2(B+C) + (2A + B - 2C)\end{aligned}$$

then we have the system of equations;

$$\begin{aligned}0 &= B + C\\ 1 &= 2A + B - 2C\\ 1 &= A - 2B + C\end{aligned}$$

$$2 = 3A \implies A = \frac{2}{3}$$

Solve for B and C we obtain the following

$$\int \frac{1}{y}\,dy = \int \left(\frac{2/3}{(x-1)^2} + \frac{-1/9}{x+2} + \frac{1/9}{x-1}\right)dx$$

Use u-sub on the first term where $u = x - 1$;

$$\frac{2}{3}\int \frac{1}{u^2} = -\frac{2}{3u}$$

so we have;

$$\ln|y| = \frac{2}{3x-3} + \frac{1}{9}\ln|x-1| - \frac{1}{9}\ln|x+2| + C$$

Improper integrals

$$\begin{aligned}\int_{-3}^{2} \frac{1}{6+2x}\,dx &= \lim_{a\to-3}\int_a^2 \frac{1}{6+2x}\,dx\\ &= \lim_{a\to-3}\left[\frac{\ln|2x+6|}{2}\right]_a^2\\ &= \lim_{a\to-3}\frac{\ln|2(2)+6|}{2} - \lim_{a\to-3}\frac{\ln|2(a)+6|}{2}\end{aligned}$$

the limit is undefined therefore the integral is divergent.

Revolution

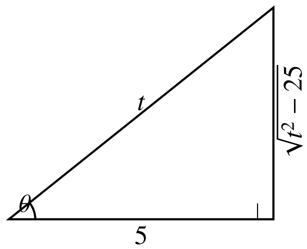
Find the volume of the region bounded by

$$y = x^2 - 6x + 9 \quad \text{and} \quad y = -x^2 + 6x - 1$$

rotated about $x = 8$. These are two parabolas, find the horizontal integration bounds by setting them equaal to eachother and solving for x. Then find the vertex by finding the critical points by setting derivative to zero and solving. Height as a function of x is the difference between the two functions, radius is $8 - x$.

Trig sub + u-sub

$$\int_{-7}^{4} \frac{1}{t^4\sqrt{t^2-25}}\,dt$$



$$\begin{aligned}\sec\theta &= \frac{t}{5}\\ \tan\theta &= \frac{\sqrt{t^2-25}}{5}\\ \frac{d}{dt}5\sec\theta &= \frac{d}{dt}t \implies 5\sec\theta\tan\theta\frac{d\theta}{dt} = 1\end{aligned}$$

then make our substitutions;

$$\begin{aligned}&\int_{-7}^{4} \frac{1}{5^4\sec^4\theta}\cdot\frac{1}{5\tan\theta}5\sec\theta\tan\theta\frac{d\theta}{dt}\,dt\\ &= \frac{1}{5^4}\cdot\frac{1}{sec^3\theta}\frac{d\theta}{dt}\,dt\\ &\text{by ROS}\\ &= \frac{1}{5^4}\int_A^B \cos^3\theta\,d\theta\\ &= \frac{1}{5^4}\int_A^B (1-\sin^2\theta)\cos\theta\,d\theta\\ &\text{let } u = \sin\theta\\ &\ldots\end{aligned}$$

Taylor series

Find $\mathrm{T}_4(x)$ for x near $x = 1$ for the function

$$f(x) = \ln(1 + 2x)$$

$$
\begin{aligned}
f'(x) &= \frac{1}{2x+1}\cdot 2\Big|_{x=1} = \frac{2}{3} \\
f''(x) &= -\frac{2^2}{(2x+1)^2}\Big|_{x=1} = \frac{2^2}{3^2} \\
f'''(x) &= \frac{2^4}{(2x+1)^3}\Big|_{x=1} = \frac{2^4}{3^3} \\
f''''(x) &= -\frac{2^5\cdot 3}{(2x+1)^4}\Big|_{x=1} = \frac{2^5}{3^3}
\end{aligned}
$$

The use Talor's formula;

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

Power Series Integral

Evaluate the following integral using a power series;

$$\int \frac{1}{1+x^6}\,dx$$

We know that

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

therefore

$$1 + (-x^6) + (-x^6)^2 + (-x^6)^3 + \cdots = \frac{1}{1+x^6}$$

and

$$
\begin{aligned}
\int 1 - x^6 + x^{12} - x^{18} + \dots\,dx &= x - \frac{x^7}{7} + \frac{x^13}{13} - \frac{x^19}{19} + \dots \\
&= \sum_{n=0}^{\infty}(-1)^n\frac{x^{6n+1}}{6n+1} + C
\end{aligned}
$$

Convergence tests

Is the series absolutely convergent, conditionally convergent, or divergent;

$$\sum_{n=1}^{\infty}(-1)^n\frac{\ln n}{\sqrt{n}}$$

absolute convergence:

$$\sum_{n=1}^{\infty}\left|(-1)^n\frac{\ln n}{\sqrt{n}}\right| = \sum_{n=1}^{\infty}\frac{\ln n}{\sqrt{n}}$$

Comparison test for power series it converges for $p > 1$ and diverges for $p \le 1$. We know that $\frac{lnn}{\sqrt{n}} > \frac{1}{\sqrt{n}}$ for $n > 3$ so we can check

$$\sum = \frac{1}{n^p}$$

and because

$$\sum_{n=1}^{\infty} = \frac{1}{n^{1/2}}$$

we say that by the comparison test the series is divergent. Now check conditional convergence, alternating series test; is the function decreasing?

$$f'(x) = \frac{\sqrt{x}/x - \ln x\frac{1}{2\sqrt{x}}}{\sqrt{x^2}} = \frac{2-\ln x}{2x^{3/2}}$$

which is decreasing for $x > e^2$. Now we check that the limit approaches 0.

$$\lim_{n\to\infty}\frac{\ln n}{\sqrt{n}} \quad \text{idet. } \frac{\infty}{\infty}$$

Using L'H;

$$\lim_{n\to\infty}\frac{2}{\sqrt{n}} = 0$$

Therefore the series converges conditionally

Notable tricks if you're stuck

1. You can integrate by parts with $g(x) = 1$
2. $$\frac{x^2}{x^2+1} = \frac{x^2+1-1}{x^2+1} = 1 - \frac{1}{x^2+1}$$
3. related to (2) find creative ways to multiply by 1 to create substitutable problems. i.e. $\int \sec\theta$ see below
4. when doing u sub you can write x as a function of u and sub out remaining x that are unresolved by $\frac{du}{dx}$
5. for absolute value setup as piecewise and integrate both sides of zero.

$$
\begin{aligned}
\int \sec\theta\,d\theta &= \int \sec\theta\cdot\frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}\,d\theta \\
&= \int \frac{sec^2\theta + \sec\theta\tan\theta}{\sec\theta+\tan\theta}\,d\theta \\
&= \int \frac{1}{u}\,du \\
&= \ln|u| \\
&= \ln|\sec\theta+\tan\theta| + C
\end{aligned}
$$

Series Tests

Alternating series;

$$\sum(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + \dots$$

if $b_{n+1} > n$ for all n and if $\lim_{n\to\infty} b_n = 0$ then the series is convergent.
Ration test;

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

if the limit is < 1 absolutely convergent if > 1 divergent.
Root test;

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}$$

same conditions as ratio test.