Midterm Notes

Trig angles

Function	$\frac{\pi}{4}(45)$	$\frac{\pi}{3}(60)$	$\frac{\pi}{6}(30)$
$\sin\theta$	$\frac{2}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\cos\theta$	$\frac{2}{\sqrt{2}}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\tan\theta$	1	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$

Work

$$W = \int f(x)\,dx \quad OR \quad W = \int \rho \cdot d \cdot A\,dy$$

Integration by parts

$$\int f(x)g'(x)\,dx = \int f(x)g(x) - \int g(x)f'(x)\,dx$$

Partial Fraction Decomposition

Given some rational function

$$\frac{x+1}{(x+1)(x+3)} = \frac{A}{x+1} + \frac{B}{x+3} \implies x+1 = A(x+3) + B(x+1)$$

Solve for A and B using a system of equations then integrate. Note if we have a repeat term like $(x+1)^2$ then;

$$\frac{x+1}{(x+1)^2(x+3)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+3}$$

Then solve as normal

Volumes by slices

$$V = \int_{x_1}^{x_2} \pi r^2(x)\,dx \quad \text{OR} \quad V = \int_{x_1}^{x^2} \pi\left(r_1^2 - r_2^2\right)\,dx$$

Volumes by shells

$$V = \int_{x_1}^{x_2} 2\pi x f(x)\,dx$$

Note: be careful about the integration bounds and radius, for example, a slice with r_1 of $y = 3$ and r_2 of $\sec(x) + 1$ rotated about $y = 1$ would be
$A = \pi\left((3-1)^2 - (\sec(x) + 1 - 1)^2\right)$.

Trig sub

Expression	Substitution
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$

1. Draw a triangle
2. Define the edge lengths using pythagorean theorem
3. Define trig functions for the triangle
4. Sovle for x in terms of a trig function and dx. Ex;

$$x = \sin\theta(x)$$

$$dx = cos\theta\,d\theta$$

Arc length

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

Derived from pathagorean theorem for the infinite sum of secant lines i.e. $\sqrt{\Delta x^2 + \Delta y^2} = \sqrt{\Delta x^2 + (f'(x)\Delta x)^2}$ factor out Δx to obtain the equation.

Some other useful antiderivates

$$\int \frac{1}{\sqrt{1-x^2}} = \arcsin x + C$$
$$\int -\frac{1}{\sqrt{1-x^2}} = \arccos x + C$$
$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$$
$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right|$$
$$\int f^{-1}(x)\,dx = xf^{-1}(x) - (F \circ f^{-1})(x) + C$$

Surface area

$$S = 2\pi \int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

Hydrostatic force

Pressure is given by

$$P = \int_{\text{deepest point}}^{\text{shallow point}} \rho \cdot area \cdot depth$$

where area is width as a function of y and height is dy and ρ is the density.

Trig integral tip

If encountering a form like

$$\int \sin^5 x \cos^4 x\,dx$$

let u be the lower order term and keep one of the higher order terms from the other then convert the rest to match using $1 = \cos^2 x + \sin^2 x$

Problems that were implied to be on the exam

Work

A tank is full of water. Find the work required to pump the water out of the spout. In Exercises 25 and 26 use the fact that water weighs 62.5 lb/ft^3.

Draw a circle and use the slice method. The function of the circle is $x^2 + y^2 = 9$ and the volume of each slice is $V = \pi x^2 \Delta y = \pi(9 - y^2)\Delta y$. Then the distance is $(4 - y)$

$$W = \pi \int_{-3}^{3} (9 - y^2)(4 - y)\, dy$$

Note that its in meters so we can just use 1000 kg/m^3 for density of water don't forget to convert mass to weight if using this approach, lb are a unit of weight.

Integration by parts

$$\int e^{2x} \cos(3x)\, dx$$

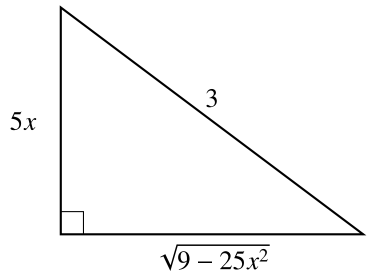
$$\begin{aligned}
\int e^{2x}\cos(3x)\,dx &= \frac{e^{2x}\cos(3x)}{2} - \int -\frac{e^{2x}3\sin(3x)}{2}\,dx \\
&= \frac{e^{2x}\cos(3x)}{2} + \frac{3}{2}\int e^{2x}\sin(3x)\,dx \\
&= \frac{e^{2x}\cos(3x)}{2} + \frac{3}{2}\left[\frac{e^{2x}\sin(3x)}{2} - \frac{3}{2}\int e^{2x}\cos(3x)\,dx\right] \\
&= \frac{e^{2x}\cos(3x)}{2} + \frac{3e^{2x}\sin(3x)}{4} - \frac{9}{4}\int e^{2x}\cos(3x)\,dx
\end{aligned}$$

Then

$$\begin{aligned}
\frac{13}{4}\int e^{2x}\cos(3x)\,dx &= \frac{e^{2x}\cos(3x)}{2} + \frac{3e^{2x}\sin(3x)}{4} \\
&= \boxed{\frac{2e^{2x}\cos(3x)}{13} + \frac{3e^{2x}\sin(3x)}{13}}
\end{aligned}$$

Trig sub

$$\int \frac{x^2}{\sqrt{9 - 25x^2}}\, dx$$



Let $x = \frac{3}{5}\sin\theta$, $dx = \frac{3}{5}\cos\theta\, d\theta$, and $\sec\theta = \frac{3}{\sqrt{9-25x^2}}$, then;

$$\begin{aligned}
\int \frac{x^2}{\sqrt{9-25x^2}}\,dx &= \int \left(\frac{3\sin\theta}{5}\right)^2 \cdot \frac{\sec\theta}{3} \cdot \frac{3}{5}\cos\theta\,d\theta \\
&= \int \frac{9}{125}\sin^2\theta\,d\theta \\
&= \frac{9}{125}\int \frac{1-\cos(2\theta)}{2}\,d\theta \\
&= \frac{9}{250}\int 1-\cos(2\theta)\,d\theta \\
&= \frac{9}{250}\left[\theta - \frac{\sin 2\theta}{2}\right] + C \\
&= \frac{9}{250}\left[\theta - 2\sin\theta\cos\theta\right] + C \\
&= \frac{9}{250}\left[arcsin\left(\frac{5x}{3}\right) - 2\left(\frac{5x}{3}\right)\left(\frac{\sqrt{9-25x^2}}{3}\right)\right] + C
\end{aligned}$$

Notable tricks if you're stuck

1. You can integrate by parts with $g(x) = 1$
2. $$\frac{x^2}{x^2+1} = \frac{x^2+1-1}{x^2+1} = 1 - \frac{1}{x^2+1}$$
3. related to (2) find creative ways to multiply by 1 to create substitutable problems. i.e. $\int \sec\theta$ see below
4. when doing u sub you can write x as a function of u and sub out remaining x that are unresolved by $\frac{du}{dx}$

$$\begin{aligned}
\int \sec\theta\,d\theta &= \int \sec\theta \cdot \frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta}\,d\theta \\
&= \int \frac{sec^2\theta + \sec\theta\tan\theta}{\sec\theta + \tan\theta}\,d\theta \\
&= \int \frac{1}{u}\,du \\
&= \ln|u| \\
&= \ln|\sec\theta + \tan\theta| + C
\end{aligned}$$