Homework 11

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11.9 Question 25

Evaluate the indefinite integral as a power series. What is the radius of convergence?

$$\int \frac{t}{1-t^8}\,dt$$

0.1 Solution

The integrand can be expanded using the geometric series where

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \tag{1}$$

and so we have

$$\begin{aligned}
\int \frac{t}{t-t^8} &= \int t \cdot \sum_{n=0}^{\infty} t^{8n} \\
&= \int \sum_{n=0}^{\infty} t^{8n+1} \\
&= \sum_{n=0}^{\infty} \int t^{8n+1} \\
&= \sum_{n=0}^{\infty} \frac{t^{8n+2}}{8n+2} + C
\end{aligned}$$

because this is a geometric series, the radius of convergence is 1. We can show this using the ratio test where the series converges if;

$$L = \lim_{n\to\infty} \left| \frac{a_n + 1}{a_n} \right| < 1$$

and diverges if $L > 1$. Let $a_n = \frac{t^{8n+2}}{8n+2}$, then;

$$\begin{aligned}
\lim_{n\to\infty} \frac{\left|\frac{t^{8n102}}{8n+10}\right|}{\left|\frac{t^{8n+2}}{8n+2}\right|} &= \lim_{n\to\infty} \left| \frac{t^{8n+10}}{8n+10} \frac{8n+2}{t^{8n+2}} \right| \\
&= \lim_{n\to\infty} \left| t^{8n+10-8n-2} \frac{8n+2}{8n+10} \right| \\
&= \lim_{n\to\infty} \left| t^8 \frac{8n+2}{8n+10} \right| \\
&= \left| t^8 \cdot 1 \right| \\
&= \left| t^8 \right|
\end{aligned}$$

then

$$\left|t^8\right| < 1 \implies |t| < 1$$

therefore the radius of convergence is $R = 1$