

Fundamental Symmetries in Classical Field Theories

Aaron W. Tarajos

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1 Theorem

We will formally show that invariance of action implies covariance of the field equations by two proofs using functional derivatives of the action, including; a finite proof for discrete and continuous quasi-symmetry and infinitesimal proof for a continuous quasi-symmetry from the infinitesimal transformation. In addition, we will explore a proof of the infinitesimal transformation without functional derivatives using higher order partial field derivatives. Theorem; If a local action functional $S_V[\phi]$ has a quasi-symmetry transformation

$$\phi^\alpha(x) \rightarrow \phi'^\alpha(x'), \quad x^\mu \rightarrow x'^\mu, \quad (1)$$

then the equations of motion

$$e_\alpha(\phi(x), \partial\phi(x), \dots; x) := \frac{\delta S_V[\phi]}{\delta \phi^\alpha(x)} \approx 0 \quad (2)$$

must have a symmetry with respect to the same transformation;

$$e_\alpha(\phi'(x'), \partial'\phi'(x'), \dots; x') \approx e_\alpha(\phi(x), \partial\phi(x), \dots; x). \quad (3)$$

1.1 Invariance and Covariance

An object is said to be invariant if under some transformation the object remains unchanged. That is for an arbitrary transformation of a field ϕ ;

$$\phi'^x = \phi^x$$

Similarly, an object is covariant if its form is preserved when the fields and coordinates are transformed.

1.2 Quasi-symmetry

We define an action functional $S_V[\phi]$ as the integral of the n -form Lagrangian \mathbb{L} over a region of spacetime V ;

$$S_V[\phi] := \int_V \mathbb{L}, \quad \mathbb{L} := \mathcal{L} d^n x. \quad (4)$$

Where \mathcal{L} is the Lagrangian density in n -dimensional space. Then, the action functional $S_V[\phi]$ has a quasi-symmetry if it changes by a boundary integral such that the transformed action functional is equal to the original action functional plus the same boundary integral over the transformed spacetime region V' :

$$S'_V[\phi'] + \int_{\partial V'} d^{n-1}(\dots) = S_V[\phi] + \int_{\partial V} d^{n-1}(\dots) \quad (5)$$

2 Equations of Motion

We have defined the action functional to be invariant under infinitesimal variations in the field $\delta\phi^\alpha(x)$ and therefore;

$$\frac{\delta S_V[\phi]}{\delta \phi^\alpha(x)} = 0,$$

giving us the equations of motion;

$$e_\alpha(\phi(x), \partial\phi(x), \dots; x) := \frac{\delta S_V[\phi]}{\delta \phi^\alpha(x)} \approx 0. \quad (6)$$

3 Formal Finite Proof

Starting with the equations of motion;

$$e_\alpha(\phi(x), \partial\phi(x), \dots; x) = \frac{\delta S_V[\phi]}{\delta\phi^\alpha(x)}$$

by (5) we can say that

$$\delta S_{V'}[\phi'] = \delta S_V[\phi]$$

and therefore

$$e_\alpha(\phi(x), \partial\phi(x), \dots; x) = \frac{\delta S_{V'}[\phi']}{\delta\phi^\alpha(x)}$$

by the chain rule;

$$\begin{aligned} \frac{\delta S_{V'}[\phi']}{\delta\phi^\alpha(x)} &= \int_{V'} d^n x' \frac{\delta S_{V'}[\phi']}{\delta\phi'^\alpha(x')} \frac{\delta\phi'^\alpha(x')}{\delta\phi^\alpha(x)} \\ \frac{\delta S_{V'}[\phi']}{\delta\phi^\alpha(x)} &= \int_{V'} d^n x' e_\alpha(\phi'(x'), \partial'\phi'(x'), \dots; x') \frac{\delta\phi'^\alpha(x')}{\delta\phi^\alpha(x)} \\ e_\alpha(\phi(x), \partial\phi(x), \dots; x) &= \int_{V'} d^n x' e_\alpha(\phi'(x'), \partial'\phi'(x'), \dots; x') \frac{\delta\phi'^\alpha(x')}{\delta\phi^\alpha(x)} \\ e_\alpha(\phi(x), \partial\phi(x), \dots; x) &= e_\alpha(\phi'(x'), \partial'\phi'(x'), \dots; x') \quad \blacksquare \end{aligned}$$

The final step is justified because for the integral to equal zero for arbitrary variations—satisfying the equations of motion—the integrand must vanish pointwise.

4 Formal Infinitesimal Proof

From (3) we can say

$$\begin{aligned} \delta\phi^\alpha(x) &:= \phi'^\alpha(x') - \phi^\alpha(x), \\ \delta x^\mu &:= x'^\mu - x^\mu, \end{aligned}$$

which we use to define a vertical transformation

$$\begin{aligned} \delta_0\phi^\alpha(x) &:= \phi'^\alpha(x) - \phi^\alpha(x) \\ &= \delta\phi^\alpha(x) - \delta x^\mu d_\mu\phi^\alpha(x), \\ d_\mu &:= \frac{d}{dx^\mu}. \end{aligned} \tag{7}$$

Then the change in the Lagrangian density is;

$$\begin{aligned} \delta\mathbb{L} &= d_\mu f^\mu d^n x \\ \delta_0\mathbb{L} &= d_\mu (f^\mu - \mathcal{L}\delta x^\mu) d^n x \end{aligned}$$

where $d_\mu f^\mu$ is a boundary term which represents the total derivative of f^μ and $\delta_0\mathbb{L}$ is the vertical variation of the Lagrangian. We use these to construct the infinitesimal transformation of the equations of motion which are assumed to be of second order;

$$\delta e_\alpha(x) = \delta_0 e_\alpha(x) + \delta x^\mu d_\mu e_\alpha(x)$$

where $d_\mu e_\alpha(x) \rightarrow 0$ because the equations of motion are on-shell. We expand $\delta_0 e_\alpha(x)$ in terms of field variations;

$$\begin{aligned} \delta e_\alpha(x) &\approx \delta_0 e_\alpha(x) \\ &= \frac{\partial e_\alpha(x)}{\partial\phi^\beta(x)} \delta_0\phi^\beta(x) + \sum_\mu \frac{\partial e_\alpha(x)}{\partial(\partial_\mu\phi^\beta(x))} d_\mu\delta_0\phi^\beta(x) + \sum_{\mu \leq \nu} \frac{\partial e_\alpha(x)}{\partial(\partial_\mu\partial_\nu\phi^\beta(x))} d_\mu d_\nu\delta_0\phi^\beta(x) \end{aligned}$$

and then re-write the field variations as functional derivatives;

$$\delta e_\alpha(x) = \int_V d^n y \delta_0 \phi^\beta(y) \frac{\delta e_\alpha(x)}{\delta \phi^\beta(y)}$$

because $e_\alpha(x) = \frac{\delta S_V[\phi]}{\delta \phi^\alpha(x)}$ we obtain;

$$\delta e_\alpha(x) = \int_V d^n y \delta_0 \phi^\beta(y) \frac{\delta^2 S_V[\phi]}{\delta \phi^\beta(y) \delta \phi^\alpha(x)}$$

Using the fact that functional derivatives can be interchanged meaning;

$$\frac{\delta^2 S_V[\phi]}{\delta \phi^\beta(y) \delta \phi^\alpha(x)} = \frac{\delta^2 S_V[\phi]}{\delta \phi^\alpha(x) \delta \phi^\beta(y)}$$

we can write

$$\begin{aligned} \int_V d^n y \delta_0 \phi^\beta(y) \frac{\delta^2 S_V[\phi]}{\delta \phi^\beta(y) \delta \phi^\alpha(x)} &= \int_V d^n y \delta_0 \phi^\beta(y) \frac{\delta^2 S_V[\phi]}{\delta \phi^\alpha(x) \delta \phi^\beta(y)} \\ &= \frac{\delta}{\delta \phi^\alpha(x)} \int_V d^n y \delta_0 \phi^\beta(y) \frac{\delta S_V[\phi]}{\delta \phi^\beta(y)} - \int_V d^n y \frac{\delta(\delta_0 \phi^\beta(y))}{\delta \phi^\alpha(x)} \frac{\delta S_V[\phi]}{\delta \phi^\beta(y)} \\ &= \frac{\delta(\delta_0 S_V[\phi])}{\delta \phi^\alpha(x)} - \int_V d^n y \frac{\delta(\delta_0 \phi^\beta(y))}{\delta \phi^\alpha(x)} e_\beta(y) \end{aligned}$$

Recognizing that the second term includes the equations of motion which must be satisfied and therefore equal zero, we obtain;

$$\delta e_\alpha(x) = \frac{\delta(\delta_0 S_V[\phi])}{\delta \phi^\alpha(x)}$$

In the last step, we see that because the variation of the action under the vertical transformation is a boundary term;

$$\begin{aligned} \delta_0 S_V[\phi] + \int_V d^n x d_\mu (\mathcal{L} \delta x^\mu) &= \delta S_V[\phi] \\ &= \int_{\partial V} d^{n-1} x (\dots) \end{aligned}$$

5 Formal Infinitesimal Proof for x-locally

Using the same infinitesimal transformation of the equations of motion;

$$\delta e_\alpha(x) = \delta_0 e_\alpha(x) + \delta x^\mu d_\mu e_\alpha(x)$$

we can prove it x-locally

$$\delta_0 e_\alpha(x) = \underbrace{E_{\alpha(0)} d_\mu (f^\mu(x) - \mathcal{L}(x) \delta x^\mu)}_{=0} - \sum_{k \geq 0} d^k \left(\underbrace{e_\beta(x)}_{\approx 0} \cdot P_{\alpha(k)} \delta_0 \phi^\beta(x) \right) \approx 0$$

where $P_{\alpha(k)}$ are higher order partial field derivatives;

$$P_{\alpha(k)} := \frac{\partial}{\partial \phi^{\alpha(k)}}, \quad k \in \mathbb{N}_0^n,$$

and $E_{\alpha(k)}$ is an Euler operator;

$$E_{\alpha(k)} := \sum_{m \geq k} \binom{m}{k} (-d)^m P_{\alpha(m)}$$

Starting with a variation in the field $\delta \phi^\beta(y)$. The corresponding variation in the equations of motion are;

$$\delta e_\alpha(x) = \int d^n y \frac{\delta e_\alpha(x)}{\delta \phi^\beta(y)} \delta \phi^\beta(y)$$

using the chain rule,

$$\begin{aligned}\delta e_\alpha(x) &= \int d^n y \frac{\delta e_\alpha(x)}{\delta \phi^\beta(x)} \delta \phi^\beta(x) \\ &= \int d^n y \sum_{|k|=0}^N \frac{\partial e_\alpha(x)}{\partial \phi^{\beta(k)}(x)} \delta \phi^{\beta(k)}(x) \delta(x-y)\end{aligned}$$

Where the variation of the k -th derivative of the field at x is related to $\delta_0 \phi^\beta(y)$ by

$$\delta \phi^{\beta(k)}(x) = D_k \delta_0 \phi^\beta(x)$$