Derivation of Equations for Constant Acceleration

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Velocity at time t

The change in velocity, $d\nu$, is equal to the product of acceleration, a, and the change in time dt.

$$d\nu = adt$$

and we integrate to find the equation to solve for velocity;

$$\begin{aligned} d\nu &= adt \\ \int d\nu &= \int adt \\ \int d\nu &= a\int dt \\ \nu &= at + c \\ \nu_0 &= a(0) + c = c \\ \nu &= at + \nu_0 \end{aligned}$$

Position at time t

Similar to accceleration we integrate velocity to find the equation to solve for position.

$$\begin{aligned} dx &= \nu dt \\ \int dx &= \int \nu dt \\ &= \int (\nu_0 + at)\, dt \\ &= \nu_0 \int dt + a\int tdt \\ x &= \nu_0 t + \frac{1}{2}at^2 + c \\ x - x_0 &= \nu_0 t + \frac{1}{2}at^2 \end{aligned}$$

An equation without time

We can solve problems for a various circumstances where one of these variables are missing from the problem entirely using the two previous derivations, starting with an equation without time. Given

$$x - x_0 = \nu_0 t + \frac{1}{2}at^2$$

and

$$\nu = \nu_0 + at$$

We start with the second equation to solve for time as a function of initial velocity, velocity and acceleration;

$$\frac{\nu - \nu_0}{a} = t$$

Furthermore, because acceleration is constant we know that average velocity, $\bar{\nu}$, is

$$\bar{\nu} = \frac{\nu + \nu_0}{2}$$

as well as

$$x = \bar{\nu}t + x_0$$

substituting our equations for time and average velocity;

$$\left(\frac{\nu + \nu_0}{2}\right)\left(\frac{\nu - \nu_0}{a}\right) + x_0 = x$$
$$\frac{\nu^2 - \nu_o^2}{2a} + x_0 = x$$
$$\nu^2 = 2a\left(x - x_0\right) + \nu_0^2$$

An equation without acceleration

Given

$$x = x_o + \bar{\nu}t$$

and

$$\bar{\nu} = \frac{\nu + \nu_0}{2}$$

we obtain

$$x - x_0 = \frac{\nu + \nu_0}{2}t$$

An equation without initial velocity

Given

$$\nu_0 = \nu - at$$

We substitute this into the equation for change in position

$$x - x_0 = \frac{\left(\nu + \nu - at\right)t}{2}$$

$$x - x_0 = \frac{2\nu t - at^2}{2}$$

$$x - x_0 = \nu t - \frac{at^2}{2}$$