Homework 3

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Problem 1

Given the three vectors $\vec{\mathbf{A}} = 1.00\hat{\mathbf{i}} - 4.00\hat{\mathbf{j}}$, $\vec{\mathbf{B}} = 3.00\hat{\mathbf{i}}$, and $\vec{\mathbf{C}} = -2.00\hat{\mathbf{j}}$ evaluate the following expressions if they are allowed mathematically: (a) $\vec{\mathbf{C}} \cdot \left(\vec{\mathbf{A}} + \vec{\mathbf{B}}\right)$; (b) $\vec{\mathbf{C}} \cdot \left(\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}\right)$; (c) $C + \vec{\mathbf{A}} \cdot \vec{\mathbf{B}}$.

Solution

Part a:

$$\begin{aligned}
\vec{\mathbf{C}} \cdot \left(\vec{\mathbf{A}} + \vec{\mathbf{B}}\right) &= \vec{\mathbf{C}} \cdot \left((1.00\hat{\mathbf{i}} - 4.00\hat{\mathbf{j}}) + (3.00\hat{\mathbf{i}})\right) \\
&= (0.00\hat{\mathbf{i}} - 2.00\hat{\mathbf{j}}) \cdot \left(4.00\hat{\mathbf{i}} - 4.00\hat{\mathbf{j}}\right) \\
&= 0.00 + 8.00 \\
&= 8.00
\end{aligned}$$

Part b:

The expression is not mathematicall valid because the dot product of $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ will always be a scalar and you cannot take the dot product of a scalar and a vector.

Part c:

$$\begin{aligned}
\vec{\mathbf{C}} + \vec{\mathbf{A}} \cdot \vec{\mathbf{B}} &= 2.00 + (1.00\hat{\mathbf{i}} - 4.00\hat{\mathbf{j}}) \cdot (3.00\hat{\mathbf{i}} + 0.00\hat{\mathbf{j}}) \\
&= 2.00 + 3.00 + 0.00 \\
&= 5.00
\end{aligned}$$

Problem 2

Given three vectors, $\vec{\mathbf{A}} = 2.00\hat{\mathbf{i}} - 5.00\hat{\mathbf{j}}$, $\vec{\mathbf{B}} = 4.00\hat{\mathbf{j}}$, and $\vec{\mathbf{C}} = 3.00\hat{\mathbf{i}}$, evaluate the following expressions if they are mathematically allowed: (a) $C\left(\vec{\mathbf{A}} \times \vec{\mathbf{B}}\right)$; (b) $\vec{\mathbf{C}} \cdot \left(\vec{\mathbf{A}} \times \vec{\mathbf{B}}\right)$; (c) $\vec{\mathbf{C}} \times \left(\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}\right)$.

Solution

Part a:

$$\begin{aligned}
C\left(\vec{\mathbf{A}} \times \vec{\mathbf{B}}\right) &= 3.00 * \det\left(\begin{bmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2.00 & -5.00 & 0.00 \\ 0.00 & 4.00 & 0.00 \end{bmatrix}\right) \\
&= 3.00\left(\begin{bmatrix} -5.00 & 0.00 \\ 4.00 & 0.00 \end{bmatrix}\hat{\mathbf{i}} - \begin{bmatrix} 2.00 & 0.00 \\ 0.00 & 0.00 \end{bmatrix}\hat{\mathbf{j}} + \begin{bmatrix} 2.00 & -5.00 \\ 0.00 & 4.00 \end{bmatrix}\hat{\mathbf{k}}\right) \\
&= 3.00\left(0.00\hat{\mathbf{i}} - 0.00\hat{\mathbf{j}} + 8.00\hat{\mathbf{k}}\right) \\
&= 24.00\hat{\mathbf{k}}
\end{aligned}$$

Part b:

We already know that $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$ is $\left(0.00\hat{\mathbf{i}} - 0.00\hat{\mathbf{j}} + 8.00\hat{\mathbf{k}}\right)$, therefore;

$$\begin{aligned}\vec{\mathbf{C}} \cdot \left(\vec{\mathbf{A}} \times \vec{\mathbf{B}}\right) &= \left(3.00\hat{\mathbf{i}} + 0.00\hat{\mathbf{j}} + 0.00\hat{\mathbf{k}}\right) \cdot \left(0.00\hat{\mathbf{i}} - 0.00\hat{\mathbf{j}} + 8.00\hat{\mathbf{k}}\right) \\ &= 0.00 + 0.00 + 0.00 \\ &= 0.00\end{aligned}$$

Part c:

The expression is not mathematically valid because $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}$ is a scalar and you cannot take the cross product of a vector and a scalar.

Problem 3

Consider two vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ where:

$$\begin{aligned}\vec{\mathbf{A}} &= -6.00\hat{\mathbf{i}} + 3.00\hat{\mathbf{j}} + 3.00\hat{\mathbf{k}} \\ \vec{\mathbf{B}} &= 6.00\hat{\mathbf{i}} - 8.00\hat{\mathbf{j}} + 4.00\hat{\mathbf{k}}\end{aligned}$$

If we want to find the angle between these two vectors, we have two possible options: we can use the magnitude of the dot product, or the magnitude of the cross product.

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = AB\cos\theta$$

$$\left|\vec{\mathbf{A}} \times \vec{\mathbf{B}}\right| = AB\sin\theta$$

However, these approaches give conflicting answers for the value of θ.
(a) What is the correct value of theta?
(b) Why does the other formula give the wrong answer?

Solution

The dot product gives the correct value of theta;

$$\begin{aligned}\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} &= AB\cos\theta \\ \frac{\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}}{AB} &= \cos\theta \\ \theta &= \arccos\left(\frac{\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}}{AB}\right)\end{aligned}$$

Then we plugin our values,

$$\arccos\left(\frac{(-36.00 - 24.00 + 12.00)}{\sqrt{-6.00^2 + 3.00^2 + 3.00^2}\sqrt{6.00^2 - 8.00^2 + 4.00^2}}\right) = 127.34^\circ \quad .$$

The reason that the cross product method is wrong in this situation is because the smaller angle between $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ is not theta relative to the origin so the actual angle for the cross product method should be $\sin(360 - \theta)$.

Problem 4

The position of an object as a function of time is given by $r(t) = \left(3.00t^2 - 2.00t\right)\hat{\mathbf{i}} - 1.00t^3\hat{\mathbf{j}}$ m. Find (a) its velocity at $t = 2.00$ s; (b) its acceleration at $t = 4.00$ s; (c) its average acceleration between $t = 1.00$ s and $t = 3.00$ s.

Solution

Part a:

Velocity is the derivative of position with respect to time so we differentiate the position function and evaluate it at $t = 2.00$;

$$
\begin{aligned}
r'(t) &= (6.00t - 2.00)\ \hat{\mathbf{i}} - 3.00t^2\ \hat{\mathbf{j}} \\
r'(2) &= (6.00(2.00) - 2.00)\ \hat{\mathbf{i}} - 3.00(2.00)^2\ \hat{\mathbf{j}} \\
&= 10.00\,\hat{\mathbf{i}} - 12.00\,\hat{\mathbf{j}}
\end{aligned}
$$

Part b:

Similarly, acceleration is the derivative of velocity with respect to time so we differentiate the velocity function and evaluate it at $t = 4.00$

$$
\begin{aligned}
r''(t) &= (6.00)\ \hat{\mathbf{i}} - 6.00t\,\hat{\mathbf{j}} \\
r''(4) &= (6.00)\ \hat{\mathbf{i}} - 6.00(4.00)\,\hat{\mathbf{j}} \\
&= 6.00\,\hat{\mathbf{i}} - 24.00\,\hat{\mathbf{j}}
\end{aligned}
$$

Part c:

To find the average velocity we evaluate the velocity at the respective times and take the difference over change in time.

$$\vec{\mathbf{v}}_1 = (6.00(1.00) - 2.00)\,\hat{\mathbf{i}} - 3.00(1.00^2)\,\hat{\mathbf{j}} = 4.00\,\hat{\mathbf{i}} - 3.00\,\hat{\mathbf{j}}$$

$$\vec{\mathbf{v}}_2 = (6.00(3.00) - 2.00)\,\hat{\mathbf{i}} - 3.00(3.00^2)\,\hat{\mathbf{j}} = 16.00\,\hat{\mathbf{i}} - 27.00\,\hat{\mathbf{j}}$$

Then

$$\frac{\left(16.00\,\hat{\mathbf{i}} - 27.00\,\hat{\mathbf{j}}\right) - \left(4.00\,\hat{\mathbf{i}} - 3.00\,\hat{\mathbf{j}}\right)}{2.00} = 6.00\,\hat{\mathbf{i}} - 12.00\,\hat{\mathbf{j}}$$