Derivation of equations for constant acceleration

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September 4, 2024

Velocity at time t

The change in velocity, $d\nu$, is equal to the product of acceleration, a, and the change in time dt.

$$d\nu = adt$$

and we integrate to find the equation to solve for velocity;

$$\begin{aligned}
d\nu &= adt \\
\int d\nu &= \int adt \\
\int d\nu &= a\int dt \\
\nu &= at + c \\
\nu_0 &= a(0) + c = c \\
\nu &= at + \nu_0
\end{aligned}$$

Note that there is no constant c for the integration of $d\nu$ because it is more a symbolic recovering of the velocity function and any constant would be inherently included in the velocity function.

Position at time t

Similar to acceleration we integrate velocity to find the equation to solve for position.

$$\begin{aligned}
dx &= \nu dt \\
\int dx &= \int \nu dt \\
&= \int (\nu_0 + at)\, dt \\
&= \nu_0 \int dt + a\int tdt \\
x &= \nu_0 t + \frac{1}{2}at^2 + c \\
x - x_0 &= \nu_0 t + \frac{1}{2}at^2
\end{aligned}$$

An equation without time

We can solve problems for various circumstances where one of these variables are missing from the problem entirely using the two previous derivations, starting with an equation without time. Given

$$x - x_0 = \nu_0 t + \frac{1}{2}at^2$$

and

$$\nu = \nu_0 + at$$

We start with the second equation to solve for time as a function of initial velocity, velocity and acceleration;

$$\frac{\nu - \nu_0}{a} = t$$

Furthermore, because acceleration is constant we know that average velocity, $\bar{\nu}$, is

$$\bar{\nu} = \frac{\nu + \nu_0}{2}$$

as well as

$$x = \bar{\nu}t + x_0$$

substituting our equations for time and average velocity;

$$\left(\frac{\nu + \nu_0}{2}\right)\left(\frac{\nu - \nu_0}{a}\right) + x_0 = x$$
$$\frac{\nu^2 - \nu_o^2}{2a} + x_0 = x$$
$$\nu^2 = 2a\left(x - x_0\right) + \nu_0^2$$

An equation without acceleration

Given

$$x = x_o + \bar{\nu}t$$

and

$$\bar{\nu} = \frac{\nu + \nu_0}{2}$$

we obtain

$$x - x_0 = \frac{\nu + \nu_0}{2}t$$

An equation without initial velocity

Given

$$\nu_0 = \nu - at$$

We substitute this into the equation for change in position

$$x - x_0 = \frac{\left(\nu + \nu - at\right)t}{2}$$

$$x - x_0 = \frac{2\nu t - at^2}{2}$$

$$x - x_0 = \nu t - \frac{at^2}{2}$$