Fundamental Symmetries in Classical Field Theories

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November 21, 2024

1 Theorem

We will formally show that invariance of action implies covariance of the field equations by two proofs using functional derivatives of the action, including; a finite proof for discrete and continuous quasi-symmetry and infinitesimal proof for a continouous quasi-symmetry from the infinitesimal transformation[1]. In addition, we will explore a proof of the infinitesimal transfromation without functional derivatives using higher order partial field derivatives. Theorem; If a local action functional $S_V[\phi]$ has a quasi-symmetry transformation

$$\phi^\alpha(x) \to \phi'^\alpha(x'), \quad x^\mu \to x'^\mu, \tag{1}$$

then the equations of motion

$$e_\alpha(\phi(x), \partial\phi(x), \ldots; x) \ := \ \frac{\delta S_V[\phi]}{\delta\phi^\alpha(x)} \ \approx \ 0 \tag{2}$$

must have a symmetry with respect to the same transformation;

$$e_\alpha(\phi'(x'), \partial'\phi'(x'), \ldots; x') \ \approx \ e_\alpha(\phi(x), \partial\phi(x), \ldots; x). \tag{3}$$

1.1 Invariance and Covariance

An object is said to be invariant if under some transformation the object remains unchanged. That is for an arbitrary transformation of a field ϕ;

$$\phi'^x = \phi^x$$

Similarly, an object is covariant if its form is preserved when the fields and coordinates are transformed.

1.2 Quasi-symmetry

We define an action functional $S_V[\phi]$ as the integral of the n-form Lagrangian $\mathbb{L}$ over a region of spacetime V;

$$S_V[\phi] := \int_V \mathbb{L}, \quad \mathbb{L} := \mathcal{L}\, d^n x\,. \tag{4}$$

Where $\mathcal{L}$ is the Lagrangian density in n-dimensional space. Then, the action functional $S_V[\phi]$ has a quasi-symmetry if it changes by a boundary integral such that the transformed action functional is equal to the original action functional plus the same boundary integral over the transformed spacetime region V':

$$S'_V[\phi'] + \int_{\partial V'} d^{n-1}(\ldots) = S_V[\phi] + \int_{\partial V} d^{n-1}(\ldots) \tag{5}$$

2 Equations of Motion

We have defined the action functional to be invariant under infinitesimal variations in the field $\delta\phi^\alpha(x)$ and therefore;

$$\frac{\delta S_V[\phi]}{\delta\phi^\alpha(x)} = 0\,,$$

giving us the equations of motion;

$$e_\alpha(\phi(x), \partial\phi(x), \ldots; x) := \frac{\delta S_V[\phi]}{\delta\phi^\alpha(x)} \approx 0\,. \tag{6}$$

[1]These proofs are sourced from Physics Stack Exchange and have been re-written and explained here as an excercise for my own personal use.

3 Formal Finite Proof

Starting with the equations of motion;

$$e_\alpha(\phi(x), \partial\phi(x), \ldots; x) = \frac{\delta S_V[\phi]}{\delta\phi^\alpha(x)}$$

by (5) we can say that

$$\delta S_{V'}[\phi'] = \delta S_V[\phi]$$

and therefore

$$e_\alpha(\phi(x), \partial\phi(x), \ldots; x) = \frac{\delta S_{V'}[\phi']}{\delta\phi^\alpha(x)}$$

by the chain rule;

$$\begin{aligned}
\frac{\delta S_{V'}[\phi']}{\delta\phi^\alpha(x)} &= \int_{V'} d^n x' \frac{\delta S_{V'}[\phi']}{\delta\phi'^\alpha(x')} \frac{\delta\phi'^\alpha(x')}{\delta\phi^\alpha(x)} \\
\frac{\delta S_{V'}[\phi']}{\delta\phi^\alpha(x)} &= \int_{V'} d^n x' e_\alpha\left(\phi'(x'), \partial'\phi'(x'), \ldots; x'\right) \frac{\delta\phi'^\alpha(x')}{\delta\phi^\alpha(x)} \\
e_\alpha(\phi(x), \partial\phi(x), \ldots; x) &= \int_{V'} d^n x' e_\alpha\left(\phi'(x'), \partial'\phi'(x'), \ldots; x'\right) \frac{\delta\phi'^\alpha(x')}{\delta\phi^\alpha(x)} \\
e_\alpha(\phi(x), \partial\phi(x), \ldots; x) &= e_\alpha\left(\phi'(x'), \partial'\phi'(x'), \ldots; x'\right) \quad \blacksquare
\end{aligned}$$

The final step is justified because for the integral to equal zero for arbitrary variations–satifying the equations of motion–the integrand must vanish pointwise.

4 Formal Infinitesimal Proof

From (3) we can say

$$\begin{aligned}
\delta\phi^\alpha(x) &:= \phi'^\alpha(x') - \phi^\alpha(x)\ , \\
\delta x^\mu &:= x'^\mu - x^\mu\ ,
\end{aligned}$$

which we use to define a vertical transformation[2];

$$\begin{aligned}
\delta_0\phi^\alpha(x) &:= \phi'^\alpha(x) - \phi^\alpha(x) \\
&= \delta\phi^\alpha(x) - \delta x^\mu\, d_\mu\phi^\alpha(x), \\
d_\mu &:= \frac{d}{dx^\mu}.
\end{aligned} \tag{7}$$

Then the change in the Lagrangian density is;

$$\delta\mathbb{L} = d_\mu f^\mu d^n x$$

$$\delta_0\mathbb{L} = d_\mu\left(f^\mu - \mathcal{L}\delta x^\mu\right)\, d^n x$$

where $d_\mu f^\mu$ is a boundary term which represents the total derivative of f^μ and $\delta_0\mathbb{L}$ is the vertical variation of the Lagrangian. We use these to construct the infinitesimal transformation of the equations of motion which are assumed to be of second order;

$$\delta e_\alpha(x) = \delta_0 e_\alpha(x) + \delta x^\mu d_\mu e_\alpha(x)$$

[2] I believe the author is defining this vertical transformation along the fiber because because he is taking a comparison of two fields related via a symmetry at the same point.

where $d_\mu e_\alpha(x) \to 0$ because the equations of motion are on-shell. We expand $\delta_0 e_\alpha(x)$ in terms of field variations;

$$\begin{aligned}\delta e_\alpha(x) &\approx \delta_0 e_\alpha(x)\\ &= \frac{\partial e_\alpha(x)}{\partial \phi^\beta(x)}\delta_0\phi^\beta(x) + \sum_\mu \frac{\partial e_\alpha(x)}{\partial(\partial_\mu \phi^\beta(x))} d_\mu \delta_0 \phi^\beta(x) + \sum_{\mu\leq\nu} \frac{\partial e_\alpha(x)}{\partial(\partial_\mu\partial_\nu \phi^\beta(x))} d_\mu d_\nu \delta_0\phi^\beta(x)\end{aligned}$$

and then we take the first order term and re-write the field variation as a functional derivative;

$$\delta e_\alpha(x) = \int_V d^n y\, \delta_0\phi^\beta(y) \frac{\delta e_\alpha(x)}{\delta\phi^\beta(y)}$$

because $e_\alpha(x) = \frac{\delta S_V[\phi]}{\delta\phi^\alpha(x)}$ we obtain;

$$\delta e_\alpha(x) = \int_V d^n y\, \delta_0\phi^\beta(y) \frac{\delta^2 S_V[\phi]}{\delta\phi^\beta(y)\delta\phi^\alpha(x)}$$

Using the fact that funcational derivatives can be interchanged meaning;

$$\frac{\delta^2 S_V[\phi]}{\delta\phi^\beta(y)\delta\phi^\alpha(x)} = \frac{\delta^2 S_V[\phi]}{\delta\phi^\alpha(x)\delta\phi^\beta(y)}$$

we can write

$$\begin{aligned}\int_V d^n y\, \delta_0\phi^\beta(y) \frac{\delta^2 S_V[\phi]}{\delta\phi^\beta(y)\delta\phi^\alpha(x)} &= \int_V d^n y\, \delta_0\phi^\beta(y) \frac{\delta^2 S_V[\phi]}{\delta\phi^\alpha(x)\delta\phi^\beta(y)}\\ &= \frac{\delta}{\delta\phi^\alpha(x)} \int_V d^n y\, \delta_0\phi^\beta(y) \frac{\delta S_V[\phi]}{\delta\phi^\beta(y)} - \int_V d^n y\, \frac{\delta(\delta_0\phi^\beta(y))}{\delta\phi^\alpha(x)} \frac{\delta S[\phi]}{\delta\phi^\beta(y)}\\ &= \frac{\delta(\delta_0 S_V[\phi])}{\delta\phi^\alpha(x)} - \int_V d^n y\, \frac{\delta(\delta_0\phi^\beta(y))}{\delta\phi^\alpha(x)} e_\beta(y)\end{aligned}$$

Recognizing that the second term includes the equations of motion which must be satisfied and therefore equal zero, we obtain;

$$\delta e_\alpha(x) = \frac{\delta(\delta_0 S_V[\phi])}{\delta\phi^\alpha(x)}$$

In the last step, we see that because the variation of the action under the vertical transformation is a boundary term;

$$\begin{aligned}\delta_0 S_V[\phi] + \int_V d^n x\, d_\mu\,(\mathcal{L}\,\delta x^\mu) &= \delta S_V[\phi]\\ &= \int_{\partial V} d^{n-1} x\,(\ldots)\end{aligned}$$

5 Formal Infinitesimal Proof for x-locally

Using the same infinitesimal transformation of the equations of motion;

$$\delta e_\alpha(x) = \delta_0 e_\alpha(x) + \delta x^\mu d_\mu e_\alpha(x)$$

we can prove the symmetry x-locally

$$\delta_0 e_\alpha(x) = \underbrace{E_{\alpha(0)} d_\mu}_{=0}\,(f^\mu(x) - \mathcal{L}(x)\,\delta x^\mu) - \sum_{k\geq 0} d^k \left(\underbrace{e_\beta(x)}_{\approx 0} \cdot P_{\alpha(k)}\delta_0\phi^\beta(x)\right) \approx 0$$

where $P_{\alpha(k)}$ are higher order partial field derivatives;

$$P_{\alpha(k)} := \frac{\partial}{\partial\phi^{\alpha(k)}}, \quad k \in \mathbb{N}_0^n,$$

and $E_{\alpha(k)}$ is an Euler operator;

$$E_{\alpha(k)} := \sum_{m\geq k} \binom{m}{k} (-d)^m P_{\alpha(m)}$$