Vectors and Kinematics

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1.1 Vector algebra 1

Given two vectors $\mathbf{A} = (2\,\hat{\mathbf{i}} - 3\,\hat{\mathbf{j}} + 7\,\hat{\mathbf{k}})$ and $\mathbf{B} = (5\,\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\,\hat{\mathbf{k}})$ find:
(a) $\mathbf{A} + \mathbf{B}$; *(b)* $\mathbf{A} - \mathbf{B}$; *(c)* $\mathbf{A} \cdot \mathbf{B}$; *(d)* $\mathbf{A} \times \mathbf{B}$.

solution

(a)

$$\begin{aligned}\mathbf{A} + \mathbf{B} &= (2+5)\,\hat{\mathbf{i}} + (-3+1)\,\hat{\mathbf{j}} + (7+2)\,\hat{\mathbf{k}} \\ &= 7\,\hat{\mathbf{i}} - 2\,\hat{\mathbf{j}} + 9\,\hat{\mathbf{k}}\end{aligned}$$

(b)

$$\begin{aligned}\mathbf{A} - \mathbf{B} &= (2-5)\,\hat{\mathbf{i}} + (-3-1)\,\hat{\mathbf{j}} + (7-2)\,\hat{\mathbf{k}} \\ &= -3\,\hat{\mathbf{i}} - 4\,\hat{\mathbf{j}} + 5\,\hat{\mathbf{k}}\end{aligned}$$

(c)

$$\begin{aligned}\mathbf{A} \cdot \mathbf{B} &= (2\cdot 5) + (-3\cdot 1) + (7\cdot 2) \\ &= 10 - 3 + 14 \\ &= 21\end{aligned}$$

(d)

$$\begin{aligned}\mathbf{A} \times \mathbf{B} &= \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & -3 & 7 \\ 5 & 1 & 2\end{vmatrix} \\ &= (-6-7)\,\hat{\mathbf{i}} - (4-35)\,\hat{\mathbf{j}} + (2+15)\,\hat{\mathbf{k}} \\ &= -13\,\hat{\mathbf{i}} + 31\,\hat{\mathbf{j}} + 17\,\hat{\mathbf{k}}\end{aligned}$$

1.2 Vector algebra 2

Given two vectors $\mathbf{A} = (3\,\hat{\mathbf{i}} - 2\,\hat{\mathbf{j}} + 5\,\hat{\mathbf{k}})$ and $\mathbf{B} = (6\,\hat{\mathbf{i}} - 7\,\hat{\mathbf{j}} + 4\,\hat{\mathbf{k}})$ find:
(a) $\mathbf{A}^2$; *(b)* $\mathbf{B}^2$; (c) $(\mathbf{A} \cdot \mathbf{B})^2$.

1.3 Cosine and sine by vector algebra

Find the cosine and sine of the angle between $\mathbf{A} = (3\,\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$ and $\mathbf{B} = (-2\,\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$.

1.4 Direction cosines

The direction cosines of a vector are the cosines of the angles it makes with the coordinate axes. The cosines of the angles between the vector and the x, y, and z axes are usually called, in turn, α, β, and γ. Prove that $\alpha^2 + \beta^2 + \gamma^2 = 1$, using either geometry or vector algebra.

1.5 Perpendicular vectors

Show that if $|\mathbf{A} - \mathbf{B}| = |\mathbf{A} + \mathbf{B}|$, then $\mathbf{A}$ and $\mathbf{B}$ are perpendicular.

1.6 Diagonals of a parallelogram

Show that the diagonals of an equilateral parallelogram are perpendicular.

1.7 Law of sines

Prove the law of sines using the corss product. It should only take a couple of lines.

1.8 Vector proof of a trigonometric identity

Let $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ be unit vectors in the x-y plane making angles θ and ϕ with the x axis, respectively. Show that $\hat{\mathbf{a}} = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}$, $\hat{\mathbf{b}} = \cos\phi\,\hat{\mathbf{i}} + \sin\phi\,\hat{\mathbf{j}}$, and using vector algebra prove that

$$\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi\ .$$

1.9 Perpendicular unit vector

Find a unit vector perpendicular to $\mathbf{A} = (\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}})$ and $\mathbf{B} = (2\,\hat{\mathbf{i}} + \hat{\mathbf{j}} - 3\,\hat{\mathbf{k}})$.

1.10 Perpendicular unit vectors

Given vector $\mathbf{A} = 3\,\hat{\mathbf{i}} + 4\,\hat{\mathbf{j}} - 4\,\hat{\mathbf{k}}$,
(a) find a unit vector $\hat{\mathbf{B}}$ that lies in the x-y plane and is perpendicular to $\mathbf{A}$.
(b) find a unit vector $\hat{\mathbf{C}}$ that is perpendicular to both $\mathbf{A}$ and $\hat{\mathbf{B}}$.
(c) Show that $\mathbf{A}$ is perpendicular to the plane defined by $\hat{\mathbf{B}}$ and $\hat{\mathbf{C}}$.