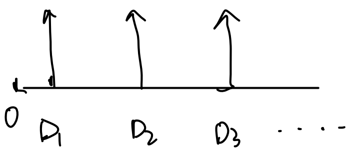
Question 1:

a.



$D_{t+1} = (1+g) D_t$

$D_2 = (1+g) D_1$

$D_3 = (1+g) D_2 = (1+g)^2 D_1$

① $r > g$, return on investment > growth of divident

$$P = NPV = \sum_{t=1}^{\infty} \frac{(1+g)^{t-1} D_1}{(1+r)^t}$$

first term $= \frac{D_1}{(1+r)}$

common ratio $= \frac{1+g}{1+r}$

$PV(t) = \frac{(1+g)^{t-1} D_1}{(1+r)^t}$

$$= \frac{D_1}{(1+r)} \sum_{t=1}^{\infty} \left(\frac{1+g}{1+r}\right)^{t-1} \quad (1)$$

$$= \frac{D_1}{(1+r)} \cdot \frac{1}{1 - \frac{1+g}{1+r}}$$

as $\left|\frac{1+g}{1+r}\right| < 1$ because $g < r$

$$= \frac{D_1}{1+r} \cdot \frac{1}{\frac{\cancel{1}+r-\cancel{1}-g}{1+r}}$$

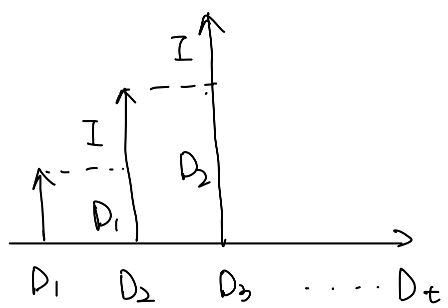
$$= \frac{D_1}{1+r} \cdot \frac{1+r}{r-g} = \frac{D_1}{r-g}$$

$\therefore$ the price of the stock is $\frac{D_1}{r-g}$

② $r < g$, $r > 0$, $g > 0$

If $g > r$, the equation (1) doesn't converge. This doesn't seem like a valid valuation because the r is underestimated and g is overestimated.
In reality, company would always want their return of investment r is higher than the growth of dividend g.

b.



$D_2 = D_1 + I$

$D_3 = D_2 + I = D_1 + 2I$

$D_t = D_{t-1} + I = D_1 + (t-1) I$

$$PV(t) = \frac{D_1 + (t-1)I}{(1+r)^t}$$ present value of each year

price of the stock P:

$$P = \sum PV(t) = \sum_{t=1}^{\infty} \frac{D_1 + (t-1)I}{(1+r)^t}$$

$$= \underbrace{\sum_{t=1}^{\infty} \frac{D_1}{(1+r)^t}}_{①} + \underbrace{\sum_{t=1}^{\infty} \frac{(t-1)I}{(1+r)^t}}_{②}$$

To solve ①:

$$\sum_{t=1}^{\infty} \frac{D_1}{(1+t)^t} = \frac{\frac{D_1}{1+r}}{1-\frac{1}{1+r}} = \frac{\frac{D_1}{1+r}}{\frac{\cancel{1}+r\cancel{-1}}{1+r}} = \frac{D_1}{\cancel{1+r}} \times \frac{\cancel{1+r}}{r} = \frac{D_1}{r} \quad ③$$

first item $= \frac{D_1}{1+r}$

ratio $= \frac{1}{1+r}$, $\left|\frac{1}{1+r}\right| < 1$

To solve ②:

$$\sum_{t=1}^{\infty} \frac{(t-1)I}{(1+r)^t} = I \cdot \sum_{t=1}^{\infty} (t-1) \cdot \frac{1}{(1+r)^t}$$

we know:

$$\sum_{t=1}^{\infty} x^t = \frac{x}{1-x} \quad (1)$$

first item $= x$

ratio $= x$ $\quad |x| < 1$

and

$$t x^t = x \cdot \frac{d}{dx}(x^t)$$

$$\therefore \sum_{t=1}^{\infty} t x^t = \sum_{t=1}^{\infty} x \cdot \frac{d}{dx}(x^t) \qquad \therefore I \cdot \sum_{t=1}^{\infty} \underbrace{(t-1)}_{t} \cdot \underbrace{\frac{1}{(1+r)^t}}_{x^t}$$

$$= x \cdot \frac{d}{dx} \sum_{t=1}^{\infty} (x^t) \qquad = I \cdot$$

$$= x \cdot \frac{d}{dx}\left(\frac{x}{1-x}\right) \text{ by using (1)}$$

$$= x \cdot \frac{(1-x)\cdot(x)' - x \cdot (1-x)'}{(1-x)^2}$$

$$= \frac{x \cdot (1\cancel{-x}\cancel{-x})}{(1-x)^2}$$

$$= \frac{x}{(1-x)^2} \quad (3)$$

when $x = \frac{1}{1+r}$, we can solve ②:

↓

$$I \cdot \sum_{t=1}^{\infty} \overbrace{(t-1)}^{t} \cdot \overbrace{\frac{1}{(1+r)^t}}^{x^t} = I \cdot \frac{\frac{1}{1+r}}{\left(1-\frac{1}{1+r}\right)^2}$$

$$= \frac{I}{1+r} \cdot \frac{1}{\left(\frac{\cancel{1+r-1}}{1+r}\right)^2}$$

$$= \frac{I}{\cancel{1+r}} \cdot \frac{(1+r)^{\cancel{2}}}{r^2} = \frac{I \cdot (1+r)}{r^2} \quad (4)$$

By combining (3) and (4)

$$P = \frac{D_1}{r} + \frac{I \cdot (1+r)}{r^2}$$

$\therefore$ The price of stock is $\frac{D_1}{r} + \frac{I \cdot (1+r)}{r^2}$

Question 2:

a.

Year	1	2	Total PV
Spot rate	0.04	0.06	
discount	$\frac{1}{(1+0.04)^1}=0.962$	$\frac{1}{(1+0.06)^2}=0.890$	
Cash Flow	10	110	
PV	9.62	97.9	107.52

$F = 100$

$r_c = 10\%$

$m = 1$

$n = 2$

$\text{coupon} = C/m = \frac{F \cdot r_c}{m} = 100 \times 10\% = 10$

$F + \frac{C}{m} = 100 + 10 = 110$

$PV_1 = 10 \times \frac{1}{1+0.04} = 9.62$

$PV_2 = 110 \times \frac{1}{(1+0.06)^2} = 97.900$

Bond Prive $= PV_1 + PV_2 = 9.62 + 97.900 = 107.52$

$\therefore$ The price of this bond is approximately £107.52

b.

$$P = \frac{F}{[1+(\lambda/m)]^n} + \sum_{k=1}^{n} \frac{C/m}{[1+(\lambda/m)]^k} \quad (1)$$

$$= \frac{F}{[1+(\lambda/m)]^n} + \frac{C}{\lambda}\left\{1 - \frac{1}{[1+(\lambda/m)]^n}\right\} \quad (2)$$ by applying Geometric series

Based on the information given:

$F = 100 \quad m = 1 \quad C/m = 10 \quad P = 107.52$

The equation (2) can be written as:

$$107.52 = \frac{100}{[1+(\lambda/1)]^2} + \frac{10}{\lambda}\left\{1 - \frac{1}{[1+\lambda/1]^2}\right\}$$

$$= \frac{100}{(1+\lambda)^2} + \frac{10}{\lambda}\left(1 - \frac{1}{(1+\lambda)^2}\right)$$

$$= \frac{100}{(1+\lambda)^2} + \frac{10}{\lambda} - \frac{10}{\lambda(1+\lambda)^2}$$

$107.52 \cdot \lambda(\lambda+1)^2 = 100\lambda + 10(1+\lambda)^2 - 10$ as λ is not 0, and $\lambda(1+\lambda)^2$ is not 0

$= 100\lambda + \cancel{10} + 20\lambda + 10\lambda^2 - \cancel{10}$

$107.52(\lambda+1)^2 = 100 + 20 + 10\lambda$ by taking all λ

$107.52\lambda^2 + 2 \times 107.52\lambda + 107.52 = 120 + 10\lambda$

$107.52\lambda^2 + 205.04\lambda - 12.48 = 0$

$\lambda_1 \approx 0.05904$

$\lambda_2 \approx -1.9660$

$\because$ in this case, we assume yield λ is always positive. (zero coupon bond is not the case here)

$\therefore \lambda = 0.05904 \approx 5.9\%$

$\therefore$ the bond's yield is 5.9%

$\downarrow$

c.

$$D = \sum_{k=0}^{n} w_k t_k \text{ , and } w_k = \frac{PV(t_k)}{PV_{total}}$$

PV_{total} = Bond Price = 107.52

$$\therefore D = \frac{PV_1 \cdot 1 + PV_2 \cdot 2}{PV_{total}} = \frac{9.62 + 97.900 \times 2}{107.52} = 1.9105 \text{ year}$$

∴ The Macauley duration is around 1.91 years.

d.

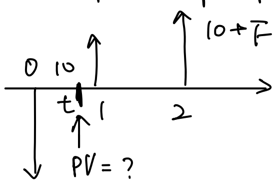
The current spot rate $S_1 = 0.04$, $S_2 = 0.06$ then the forward rate from year 1 to year 2 is calculated by:

$$S'_{j-1} = f_{1,j} = \left[\frac{(1+S_j)^j}{1+S_1}\right]^{1/(j-1)}, \ j = 2$$

$$\therefore S'_1 = f_{1,2} = \left[\frac{(1+S_2)^2}{(1+S_1)}\right]^{\frac{1}{2-1}} - 1$$

$$= \left[\frac{(1+0.06)^2}{1+0.04}\right] - 1 \approx 0.0804 = 8.04\%$$

The bond matures in Year 2, and the word "one year from now" refers to the timing t just about to receive first coupon payment, so the first coupon payment is not discounted.



$$d_{t,1} = 1$$

$$\text{discount factor } d_{1,2} = \frac{1}{(1+f_{1,2})^{2-1}} = \frac{1}{1+0.0804} = 0.9256$$

$$\therefore \text{Bond price in one year from now} = \text{coupon payment} + \frac{\text{Coupon payment + Face Value}}{1 + \text{Forward Rate from 1 to 2}}$$

$$= 10 + \frac{110}{1+8.04\%} = 111.816$$

Based on the definition of yield, which is the interest rate λ that equates the current price with the PV of the remaining cash flows. (10, 110)

The first coupon is not discounted.

$$\therefore 111.816 = \underbrace{\frac{100}{(1+\lambda)^1} + \frac{10}{(1+\lambda)}}_{\text{PV of the 2nd CF}} + \underbrace{10}_{\text{PV of the first CF}}$$

$$= \frac{110}{1+\lambda} + 10$$

$$111.816 + 111.816\lambda = 110 + 10 + 10\lambda$$

$$101.816\lambda = 8.184$$

$$\lambda \approx 0.0804 \approx 8.04\%$$

∴ The yield of this bond after a year from now is 8.04%

The price in one year from now is \$111.816.

Question 3 :

a.

$$\bar{r}_p = w_0 \cdot r_f + \sum_{i=1}^{n} \bar{r}_i \cdot w_i = w_0 \cdot r_f + w^T \cdot \bar{r} \quad \text{assuming } w_0 + \sum_{i=1}^{n} w_i = 1$$

$$\sigma^2_p = \sum_{i,j=1}^{n} w_i \sigma_{ij} w_j + 0 = w^T \Sigma w \quad \text{because risk-free has variance of } 0$$

b.

maximise $\bar{r}_p - \frac{a}{2}\sigma^2_p$

subject to $\sum_{i=1}^{n} w_i = 1$ (This is the only constraint, because allowed short-selling means weight might < 0)

$$\max(\bar{r}_p - \tfrac{a}{2}\sigma^2_p) = w_0 \cdot r_f + w^T \cdot \bar{r} - \frac{a}{2} \cdot w^T \Sigma w$$

$$= (1 - \sum_{i=1}^{n} w_i) \cdot r_f + \sum_{i=1}^{n} w_i \cdot \bar{r}_i - \frac{a}{2} \cdot w^T \Sigma w \quad \text{by eliminating constraints}$$

$$= \max(r_f + \sum_{i=1}^{n} w_i (\bar{r}_i - r_f) - \frac{a}{2} w^T \Sigma w)$$

$$f = \max(r_f + w^T \cdot (\bar{r} - r_f \cdot e) - \frac{a}{2} w^T \Sigma w) \text{ by vector notion}$$

c.

$w^T \cdot e = 1$ $\quad$ e is vector of 1s

function is part b can be written as:

$$f = \max(r_f + w^T \cdot (\bar{r} - r_f \cdot e) - \frac{a}{2} w^T \Sigma w) \quad (1)$$

Because there is no constriants after ~~part~~ b,

then:

$$\frac{\partial f}{\partial w} = \bar{r} - r_f \cdot e - a \Sigma w$$

To find the max: $\frac{\partial f}{\partial w} = 0$

$$\therefore \bar{r} - r_f \cdot e = a \Sigma w$$

$$\frac{1}{a}(\bar{r} - r_f \cdot e) = \Sigma w$$

$$\therefore w^* = \frac{1}{a} \cdot \Sigma^{-1} \cdot (\bar{r} - r_f \cdot e) \quad \text{because } \Sigma \text{ is invertible}$$

$$r_f + w^T \cdot (\bar{r} - r_f \cdot e) - \frac{a}{2} w^T \Sigma w$$

$$\therefore V^* = r_f + \frac{1}{a} \cdot (\bar{r} - r_f \cdot e)^T \cdot \Sigma^{-1} \cdot (\bar{r} - r_f \cdot e)$$

$$- \frac{a}{2} \cdot \left(\frac{1}{a} \cdot \Sigma^{-1} (\bar{r} - r_f \cdot e)\right)^T \cdot \Sigma \cdot \left(\frac{1}{a} \cdot \Sigma^{-1} (\bar{r} - r_f \cdot e)\right) \rightarrow -\frac{\cancel{a}}{2} \cdot \frac{1}{\cancel{a}} \cdot (\bar{r} - r_f \cdot e)^T \cdot \cancel{\Sigma^{-1}} \cdot \cancel{\Sigma} \cdot (\frac{1}{a} \cdot \Sigma^{-1} \cdot (\bar{r} - r_f \cdot e))$$

$$= r_f + \frac{1}{a} (\bar{r} - r_f \cdot e)^T \cdot \Sigma^{-1} \cdot (\bar{r} - r_f \cdot e) - \frac{1}{2a} \cdot (\bar{r} - r_f \cdot e)^T \cdot \Sigma^{-1} \cdot (\bar{r} - r_f \cdot e)$$

$$= r_f - \frac{1}{a} \cdot (\bar{r} - r_f \cdot e)^T \cdot \Sigma^{-1} \cdot (\bar{r} - r_f \cdot e)$$

Another way (2):

If we solve by Lagrangian $\mathcal{L}$, and take $w^T e - 1 = 0$ as the permitted constraint, then:

$$\mathcal{L}(w,\lambda) = \left(r_f + w^T\cdot(\bar{r} - r_f\cdot e) - \frac{a}{2} w^T \Sigma w\right) - \lambda\left(w^T e - 1\right)$$

while the optimal condition is:

$$\frac{\partial \mathcal{L}(w,\lambda)}{\partial w} = (\bar{r} - r_f\cdot e) - a\Sigma w - \lambda\cdot e = 0 \quad ①$$

$$\frac{\partial \mathcal{L}(w,\lambda)}{\partial \lambda} = e^T w - 1 = 0 \quad ②$$

Rearrange ①, we get:

$$a\Sigma w = \bar{r} - r_f\cdot e - \lambda\cdot e$$

$$w = \frac{1}{a}\cdot\Sigma^{-1}(\bar{r} - r_f\cdot e - \lambda\cdot e) \quad ③$$

put ③ into ②:

$$e^T\cdot\left(\frac{1}{a}\cdot\Sigma^{-1}(\bar{r} - r_f\cdot e) - \lambda e\right) = 1$$

$$a = e^T\Sigma^{-1}(\bar{r} - r_f\cdot e - \lambda e)$$

$$= e^T\Sigma^{-1}\bar{r} - e^T\Sigma^{-1} r_f\cdot e - e^T\Sigma^{-1}\lambda e$$

$$\therefore e^T\Sigma^{-1}\lambda e = e^T\Sigma^{-1}\bar{r} - e^T\Sigma^{-1} r_f\cdot e - a$$

$$\lambda = \frac{e^T\Sigma^{-1}\bar{r} - e^T\Sigma^{-1} r_f\cdot e - a}{e^T\cdot\Sigma^{-1}\cdot e} \quad ④$$

put ④ into ③:

$$w^* = \frac{1}{a}\cdot\Sigma^{-1}\left(\bar{r} - r_f\cdot e - \frac{e^T\Sigma^{-1}\bar{r} - e^T\Sigma^{-1} r_f\cdot e - a}{e^T\cdot\Sigma^{-1}\cdot e}\cdot e\right)$$

$$= \frac{1}{a}\cdot\Sigma^{-1}\cdot\bar{r} - \frac{1}{a}\cdot\left(r_f + \frac{e^T\Sigma^{-1}\bar{r} - e^T\Sigma^{-1} r_f\cdot e - a}{e^T\cdot\Sigma^{-1}\cdot e}\right)\cdot\Sigma^{-1}\cdot e$$

$$= \frac{1}{a}\left[\Sigma^{-1}\cdot\bar{r} - \left(r_f + \frac{e^T\Sigma^{-1}\bar{r} - e^T\Sigma^{-1} r_f\cdot e - a}{e^T\cdot\Sigma^{-1}\cdot e}\right)\cdot\Sigma^{-1}\cdot e\right]$$

$$\therefore V^* = \left(r_f + w^T\cdot(\bar{r} - r_f\cdot e) - \frac{a}{2} w^T\Sigma w\right)$$

$$= r_f + \frac{1}{a}\left[\Sigma^{-1}\cdot\bar{r} - \left(r_f + \frac{e^T\Sigma^{-1}\bar{r} - e^T\Sigma^{-1} r_f\cdot e - a}{e^T\cdot\Sigma^{-1}\cdot e}\right)\cdot\Sigma^{-1}\cdot e\right]^T\cdot(\bar{r} - r_f\cdot e)$$

$$-\frac{1}{2}\cdot\left[\Sigma^{-1}\cdot\bar{r} - \left(r_f + \frac{e^T\Sigma^{-1}\bar{r} - e^T\Sigma^{-1} r_f\cdot e - a}{e^T\cdot\Sigma^{-1}\cdot e}\right)\cdot\Sigma^{-1}\cdot e\right]^T\cdot\Sigma\cdot\left[\Sigma^{-1}\cdot\bar{r} - \left(r_f + \frac{e^T\Sigma^{-1}\bar{r} - e^T\Sigma^{-1} r_f\cdot e - a}{e^T\cdot\Sigma^{-1}\cdot e}\right)\cdot\Sigma^{-1}\cdot e\right]$$

$\therefore$ I don't prefer to use Lagrange in this case.

d.

Based on f in part c, if $a \to +\infty$, investor is going to avoid any risk. the $\frac{a}{2} w^T \Sigma w$ is going to be very large, dominating the f.

$\therefore$ the optimal object value is going to be rf. No investment on risky assets.

$\therefore$ the optimal portfolio is that $W_0 = 1$ and w is a all 0s vector, because when $a \to \infty$, the investor will choose all free-risk assets. The variance will be 0, so $W^T \Sigma W = 0$.

e.

From questions above, we get objective function:

$$\max \left(rf + w^T \cdot (\bar{r} - rf \cdot e) - \frac{a}{2} w^T \cdot \Sigma \cdot w \right)$$

By using $\hat{r}$, the equation become:

$$\max \left(rf + w^T \cdot (\hat{r} - rf \cdot e) - \frac{a}{2} w^T \cdot \Sigma \cdot w \right)$$

representing $\hat{r}$ as X, then:

$$f(X) = rf + w^T \cdot (X - rf \cdot e) - \frac{a}{2} w^T \cdot \Sigma \cdot w$$

$\because E(\hat{r}) = \bar{r}$

when $X = \hat{r}$. $f[E(\hat{r})] = f(\bar{r})$

$\therefore f[E(\hat{r})] = f(\bar{r}) = rf + w^T \cdot (\bar{r} - rf \cdot e) - \frac{a}{2} w^T \cdot \Sigma \cdot w$ ①

AND

$E[f(x)] = E[(rf + w^T \cdot (X - rf \cdot e) - \frac{a}{2} w^T \cdot \Sigma \cdot w)]$

$E[f(\hat{r})] = E[(rf + w^T \cdot (\hat{r} - rf \cdot e) - \frac{a}{2} w^T \cdot \Sigma \cdot w)]$ ②

Based on Jensen's inequality, $E[f(x)] \geq f[E(x)]$ ② $\geq$ 1

$\therefore E[f(\hat{r})] \geq \underline{f(E(\hat{r}))} = f(\bar{r})$

$\therefore E[(rf + w^T \cdot (\hat{r} - rf \cdot e) - \frac{a}{2} w^T \cdot \Sigma \cdot w)] \geq rf + w^T \cdot (\bar{r} - rf \cdot e) - \frac{a}{2} w^T \cdot \Sigma \cdot w$

This shows that the optimal value estimated by $\hat{r}$ is greater than the optimal value returned by using $\bar{r}$. Therefore, using $\hat{r}$ to estimate $\bar{r}$ will on average - overestimate the optimal value of the investment problem.