Ph21 Assignment 2

Kyriacos Xanthos

January 2020

Part 1.1

Take $h(x)$ to be a delta function, ie. $h(x) = \delta(x - x_0)$. Then it can be rewritten as

$$h(x) = \sum_{k=-\infty}^{\infty} \frac{1}{L} e^{-2\pi i x f_k} \left(\int_0^L \delta\left(x - x_0\right) e^{2\pi i x f_k}\, dx \right)$$

.

By the sifting property of the delta functuon, the right hand side of the equation becomes:

$$\sum_{k=-\infty}^{\infty} \frac{1}{L} e^{2\pi i x_0 f_k} e^{-2\pi i x f_k}$$

.

$$= \sum_{k=-\infty}^{\infty} \frac{1}{L} e^{-2\pi i (x - x_0) f_k}$$

Changing now from a sum to an integral (infinitesimal change) we get:

$$\frac{1}{L} \int_{-\infty}^{\infty} e^{-\frac{2\pi i k (x - x_0)}{L}}\, dk$$

Now by using the definition of the delta function, $\delta(x - a) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ip(x-a)}\, dp$, the result just becomes

$$= \frac{1}{L} (2\pi) \frac{1}{2\pi} L \delta\left(x - x_0\right)$$

which is precisely what we started with:

$$= \delta(x - x_0)$$

Part 1.2

Using complex linear-combination coefficients, we can constract a linear combination as:

$$(\alpha + i\beta) e^{-\frac{2\pi i x}{L}} + (\alpha - i\beta) e^{\frac{2\pi i x}{L}}$$

which is precisely

$$2\alpha \cos\left(\frac{2\pi x}{L}\right) + 2\beta \sin\left(\frac{2\pi x}{L}\right)$$

Now, setting $\alpha = \frac{1}{2} A \sin(\varphi)$ and $\beta = \frac{1}{2} A \cos(\varphi)$, it can be re-written as

$$A\sin\left(\frac{2\pi x}{L}+\varphi\right)$$

. Thus, using the initial fourier terms, we can represent any function in the form of

$$A\sin\left(\frac{2\pi x}{L}+\varphi\right)$$

Part 1.3

In this part we are going to use Euler's formula. Precisely,

$$e^{\frac{2\pi i x f_k}{L}} = \cos\left(\frac{2\pi i x f_k}{L}\right) + i\sin\left(\frac{2\pi i x f_k}{L}\right)$$

and,

$$e^{\frac{-2\pi i x f_k}{L}} = \cos\left(\frac{2\pi i x f_k}{L}\right) - i\sin\left(\frac{2\pi i x f_k}{L}\right)$$

.

It can be seen that the two expressions are identical up to a minus sign in the imaginary part. It is clear then if $h(x)$ is real and , $\tilde{h_k} = \frac{1}{L}\int_0^L h(x)e^{\frac{2\pi i x f_k}{L}}$, then

$$\tilde{h}_{-k} = \tilde{h}_k$$

Part 1.4

We are given that

$$H(x) = h^{(1)}(x)h^{(2)}(x)$$

Then the fourier function is precisely:

$$\tilde{H}_k = \left(\sum_{k=-\infty}^{\infty} \tilde{h}_k^{(1)} e^{-\frac{2\pi i k x}{L}}\right) \sum_{k'=-\infty}^{\infty} \tilde{h}_{k'}^{(2)} e^{-\frac{2\pi i k' x}{L}}$$

.

$$\begin{aligned}
&\sum_{k=-\infty}^{\infty}\sum_{k'=-\infty}^{\infty}\left[\tilde{h}_{k'}^{(2)} e^{-2\pi i k' x/L}\right]\left[\tilde{h}_{k-k'}^{(1)} e^{-2\pi i (k-k')x/L}\right] \\
&= \sum_{k=-\infty}^{\infty}\sum_{k'=-\infty}^{\infty} \tilde{h}_{k-k'}^{(1)}\tilde{h}_{k'}^{(2)} e^{-2\pi i k x/L} \\
&= \sum_{k=-\infty}^{\infty} \tilde{H}_k e^{-2\pi i k x/L}
\end{aligned}$$

In order to obtain the required result, we equate the coefficients and sum over all possible k.

Graphically, the convolution of $\tilde{h}^{(1)}$ and $\tilde{h}^{(2)}$ is basically picking $\tilde{h}_{k'}^{(1)}$ and then flipping the other function on the k' axis and add an offset: $\tilde{h}_{k-k'}^{(2)}$.

As an example we can try $h^{(1)} = sin(x)$ with an offset of $h^{(2)} = sin(x-1)$

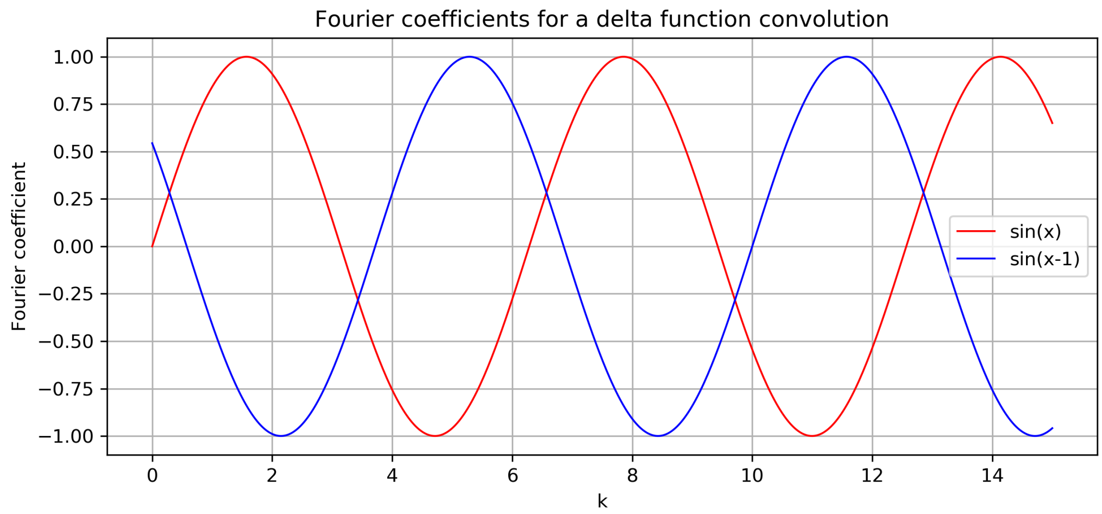

Figure 1: Graphical interpretation of convolution

Part 1.5

On the figure below, I have plotted the original cosine function with frequency 5 and amplitude 2.

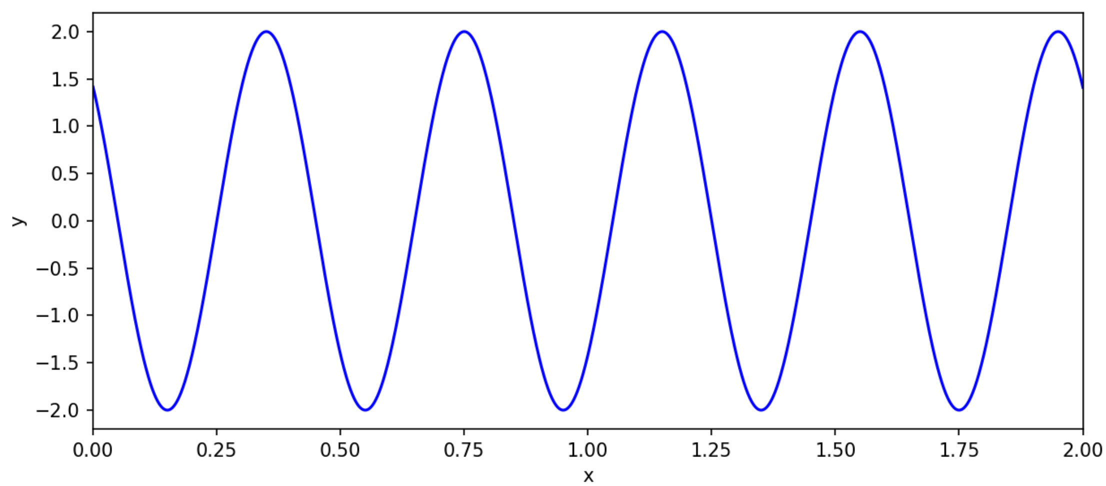

Figure 2: Cosine function with f = 5 and a = 2

Applying the fourier transform of this we can observe two peaks at 2.5 and -2.5. Inverting the fourier transform we obtain back the cosine function as expected.

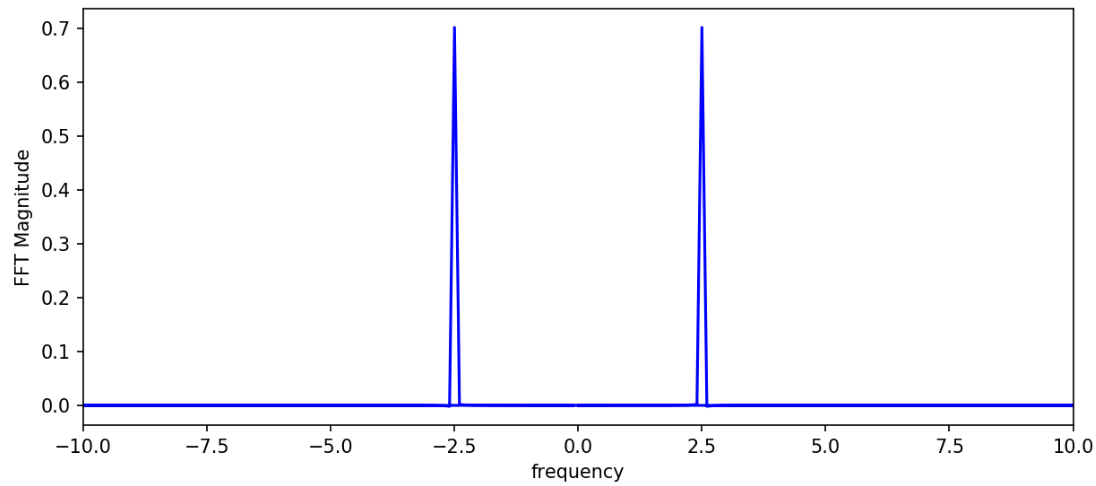

Figure 3: Fourier Transform of Cosine function with f = 5 and a = 2

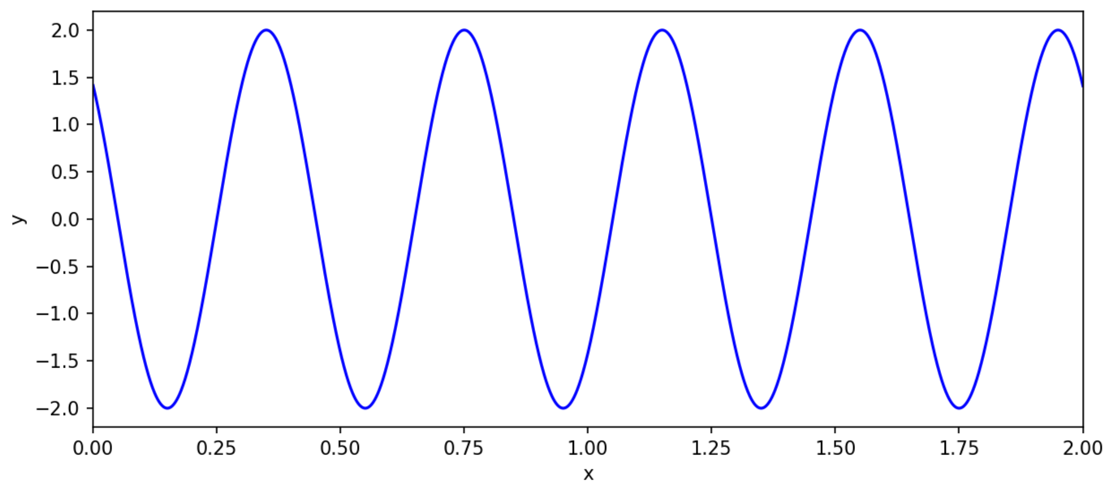

Figure 4: Inverse Fourier Transform of Cosine function with f = 5 and a = 2

Now we perform the same procedure for the Gaussian function:

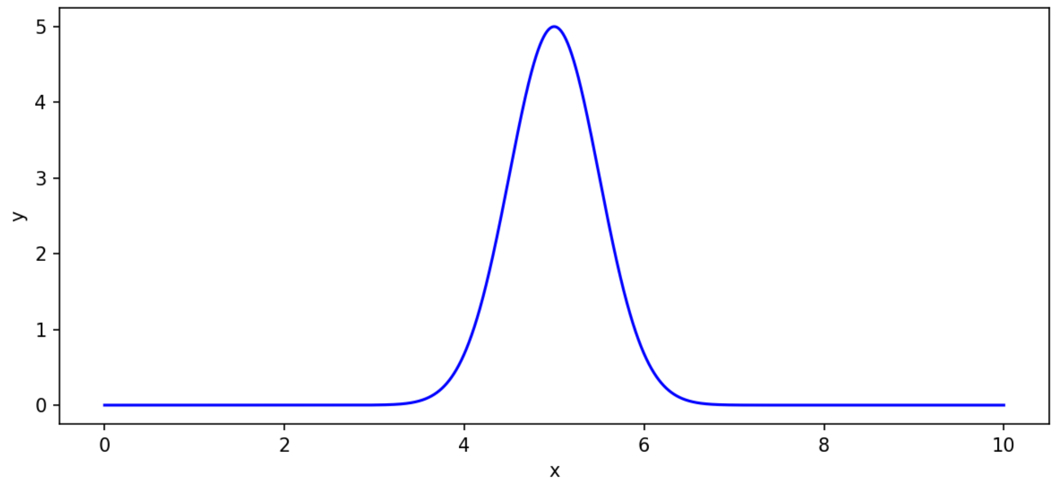

Figure 5: Gaussian function with a = 5, b=2, L=10

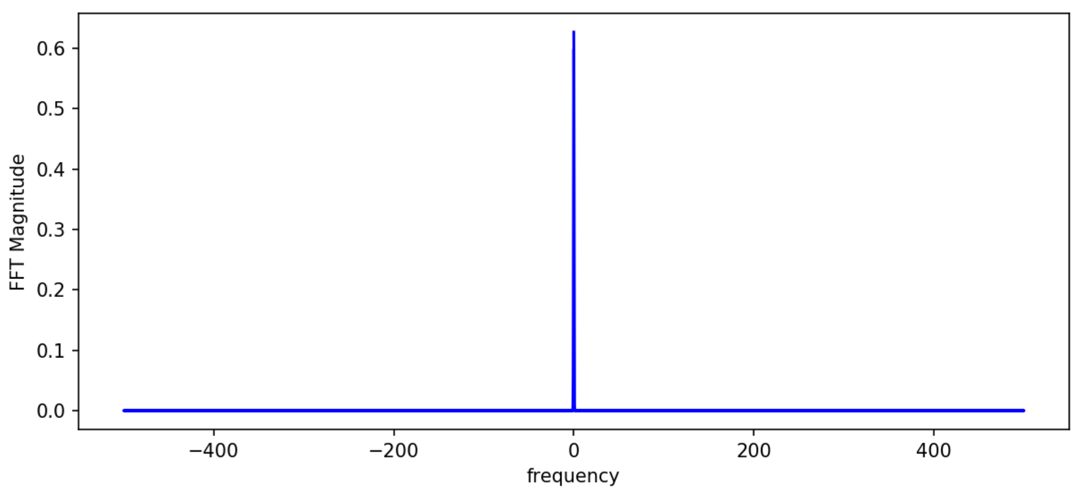

Figure 6: Fourier transform of Gaussian function with a = 5, b=2, L=10

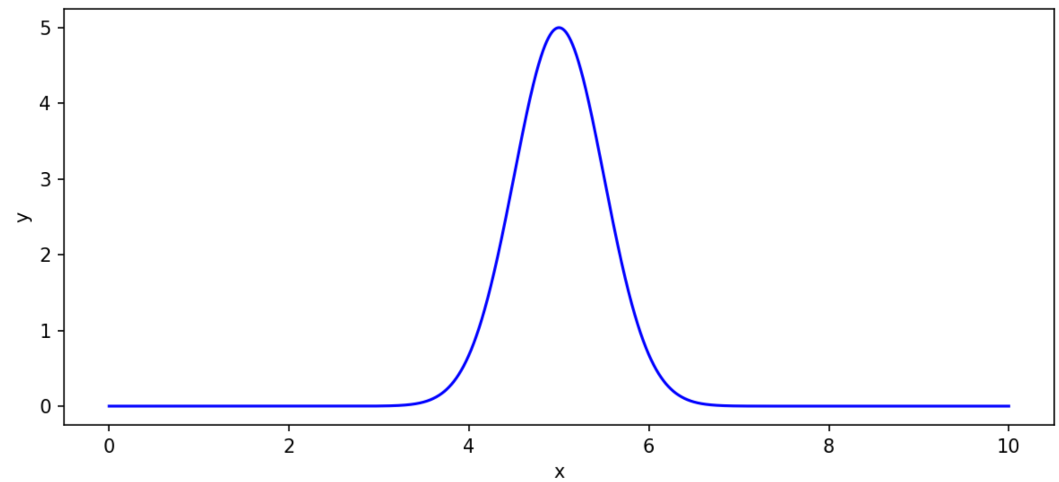

Figure 7: Inverse Fourier transform of Gaussian function with a = 5, b=2, L=10

Part 2.1

Obtaining the data of Arecibo1 and plotting them we can see a lot of noise.

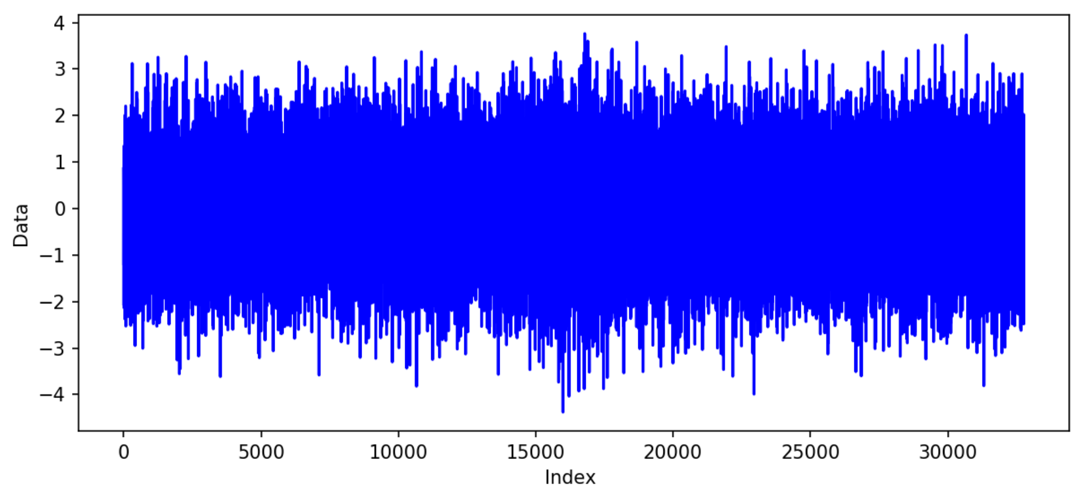

Figure 8: Arecibo1 data

Performing a fourier transform to get rid of the noise, we can clearly see two peaks close to ±150 Hz.

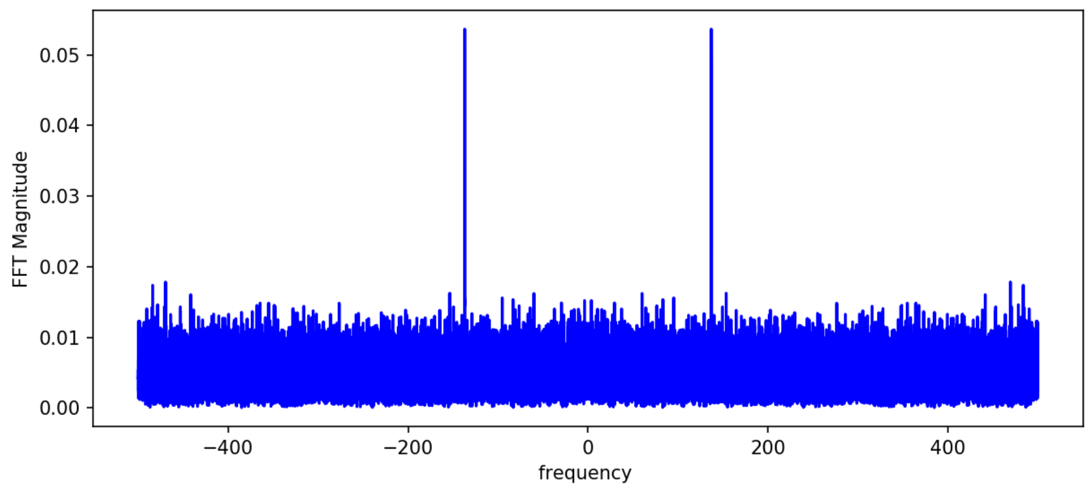

Figure 9: Fourier transform of Arecibo1 data

Zooming in and extracting the peak value we find it is at 137Hz.

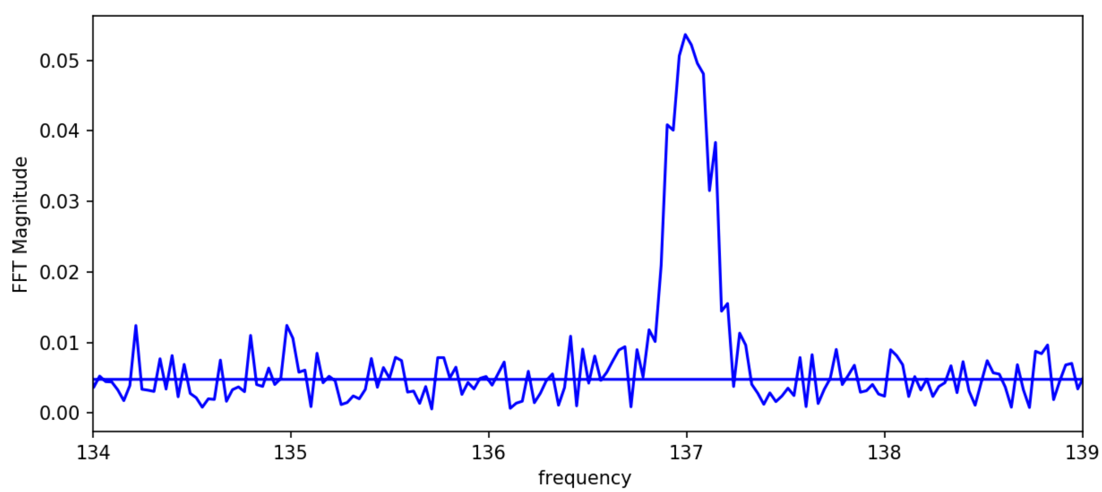

Figure 10: Fourier transform of Arecibo1 data

Part 2.2

Creating a convolution of a perfect sinusoidal together with a gaussian envelope we obtain a beautiful graph. Performing the fourier transform of this, it can be observed that it produces two peaks at roughly -150 and 150.

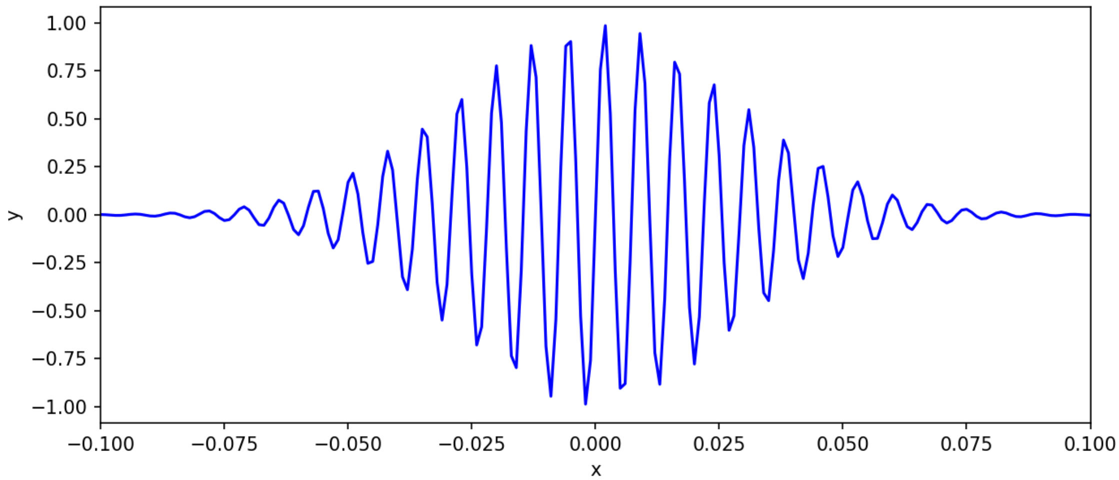

Figure 11: Guassian Envelope convoluted with a perfect sinusoidal

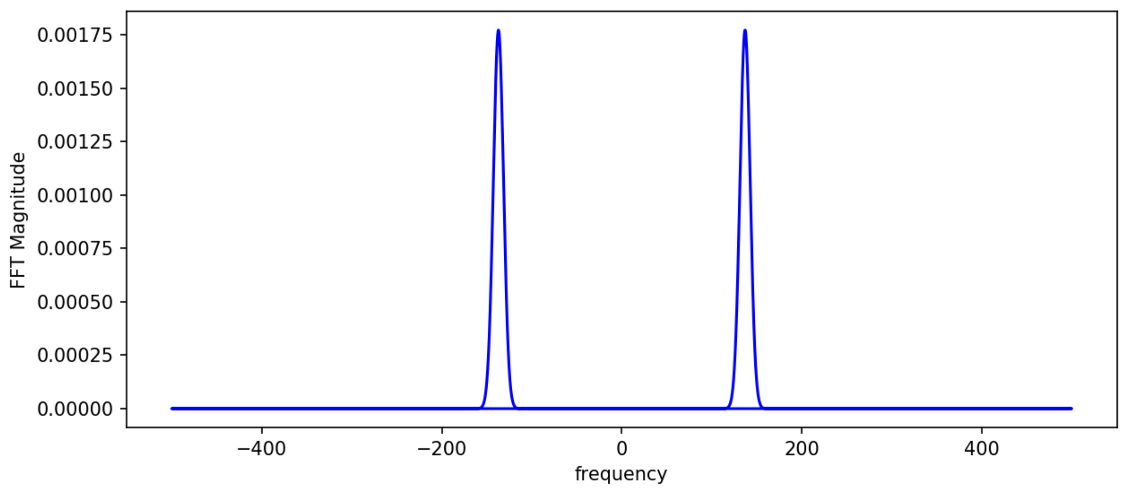

Figure 12: Fourier Transform of Guassian Envelope convoluted with a perfect sinusoidal

Trying different values of dt, and plotting the fourier transform of our convolution together with the arecibo1 data, we come to the conclusion that dt = 1 is the best rough estimate.

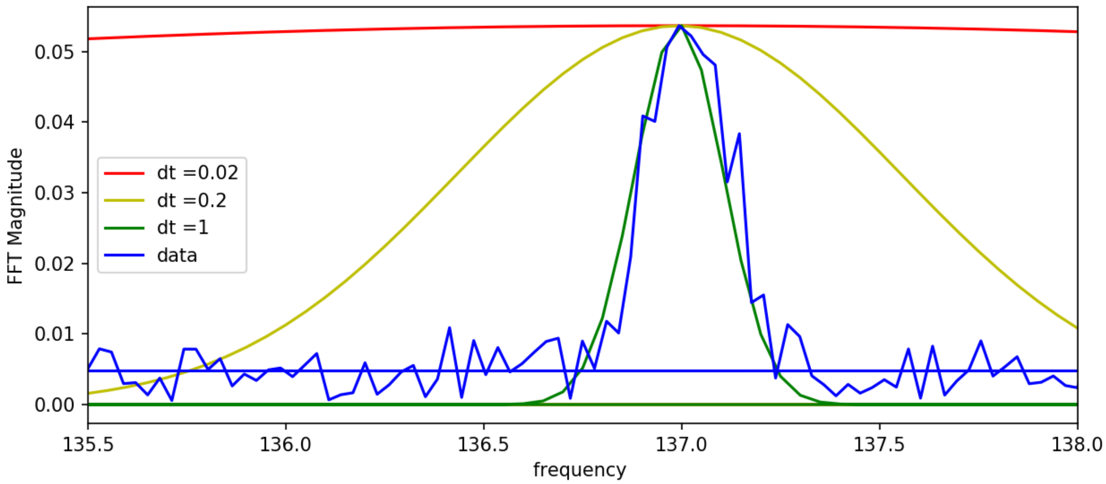

Figure 13: Varying Gaussian envelope to match Arecibo1 data

Part 3.2

Importing the Lomb-Scargle package from astropy, we can check it by applying Lomb-Scargle routine to analyze the Gaussian distribution. Performing the routine on the data from Arecibo1, it can be seen that the 137 Hz frequency is recovered again.

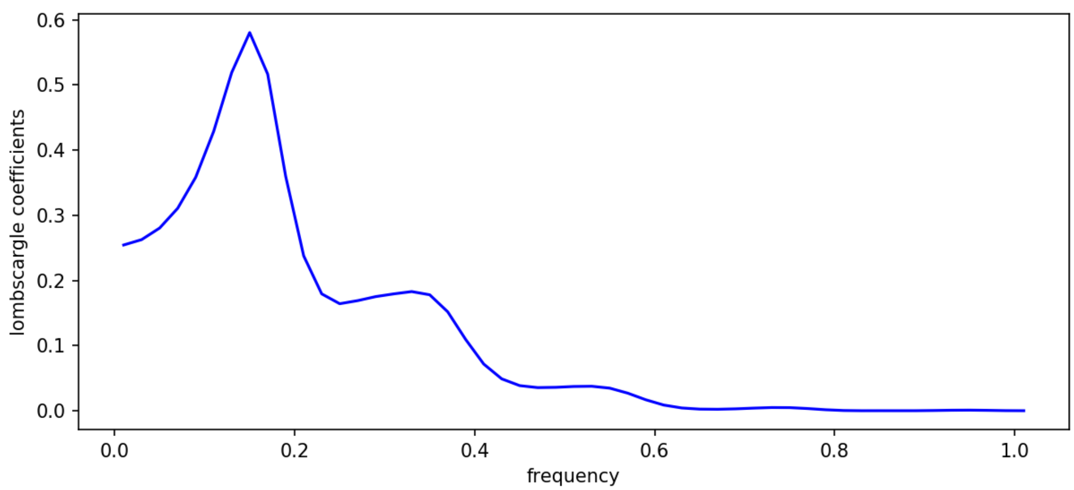

Figure 14: Lomb-Scargle routine on Gaussian

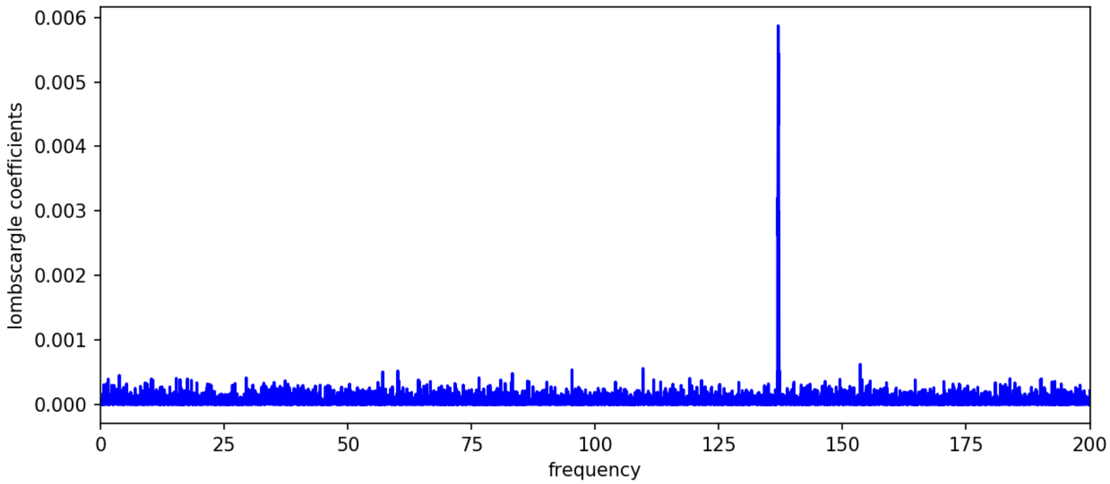

Figure 15: Lomb-Scargle routine on arecibo1 data

Part 3.3

Grabbing the data from the Catalina Real Time Survey from source Her X-1, we can perform the Lomb-Scargle routine. It can be seen from the plot that significant frequencies are roughly at 2.5, 5, 10, 12.5, 14 with MJD 1.7. Other significant frequencies can arise from cosmic background radiation, waves from supernovae explosions or cold dust. The vertical red line corresponds to the 1.7 days period where a clear peak can be seen.

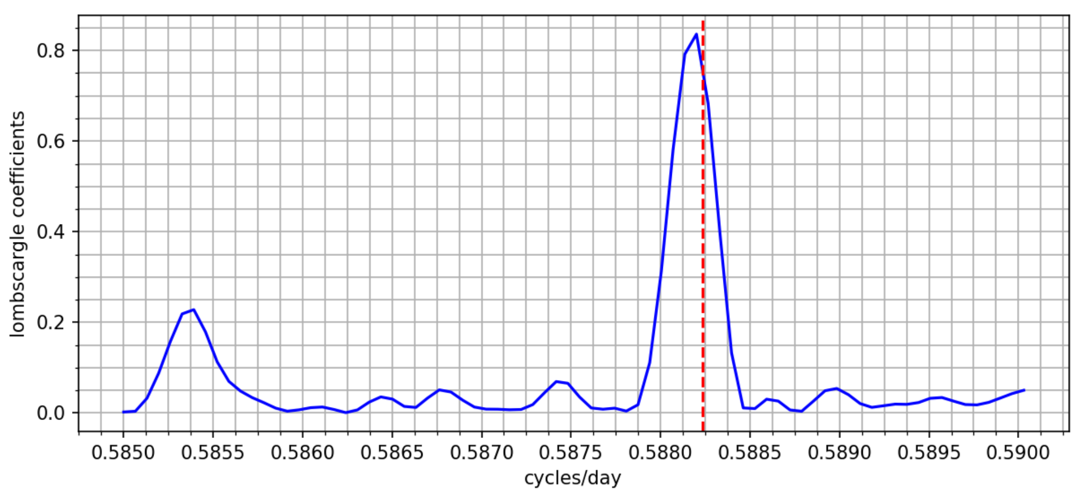

Figure 16: Lomb-Scargle routine on Her X-1 data